The Singular Supports of IC Sheaves on Quasimaps' Spaces are Irreducible

THE SINGULAR SUPPORTS OF IC SHEAVES ON QUASIMAPS' SPACES ARE IRREDUCIBLE

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1. Introduction

1.1. Let C be a smooth projective curve of genus 0. Let $\mathcal{B}$ be the variety of complete flags in an n-dimensional vector space V. Given an $(n-1)$-tuple $\alpha \in \mathbb{N}[I]$ of positive integers one can consider the space $\mathcal{Q}_\alpha$ of algebraic maps of degree α from C to $\mathcal{B}$. This space is noncompact. Some remarkable compactifications $\mathcal{Q}^D_\alpha$ (Quasimaps), $\mathcal{Q}^L_\alpha$ (Quasiflags) of $\mathcal{Q}_\alpha$ were constructed by Drinfeld and Laumon respectively. In [Ku] it was proved that the natural map $\pi : \mathcal{Q}^L_\alpha \to \mathcal{Q}^D_\alpha$ is a small resolution of singularities. The aim of the present note is to study the singular support of the Goresky-MacPherson sheaf IC_α on the Quasimaps' space $\mathcal{Q}^D_\alpha$.

Namely, we prove that this singular support $SS(IC_\alpha)$ is irreducible. The proof is based on the *factorization property* of Quasimaps' space and on the detailed analysis of Laumon's resolution $\pi : \mathcal{Q}^L_\alpha \to \mathcal{Q}^D_\alpha$.

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This note is a sequel to [Ku] and [FK]. In fact, the local geometry of $\mathcal{Q}^D_\alpha$ was the subject of [Ku]; the global geometry of $\mathcal{Q}^D_\alpha$ was the subject of [FK], while the microlocal geometry of $\mathcal{Q}^D_\alpha$ is the subject of the present work. We will freely refer the reader to [Ku] and [FK].

2. Reductions of the main theorem

2.1. **Notations.**

2.1.1. We choose a basis $\{v_1, \dots, v_n\}$ in V. This choice defines a Cartan subgroup $H \subset G = SL(V) = SL_n$ of matrices diagonal with respect to this basis, and a Borel subgroup $B \subset G$ of matrices upper triangular with respect to this basis. We have $\mathcal{B} = G/B$.

Let $I = \{1, \dots, n-1\}$ be the set of simple coroots of $G = SL_n$. Let R^+ denote the set of positive coroots, and let $2\rho = \sum_{\theta \in R^+} \theta$. For $\alpha = \sum a_i i \in \mathbb{N}[I]$ we set $|\alpha| := \sum a_i$. Let X be the lattice of weights of G, H. Let $X^+ \subset X$ be the set of dominant (with respect to B) weights. For $\lambda \in X^+$ let V_λ denote the irreducible representation of G with the highest weight λ.

Recall the notations of [Ku] concerning Kostant's partition function. For $\gamma \in \mathbb{N}[I]$ a *Kostant partition* of γ is a decomposition of γ into a sum of positive coroots with multiplicities. The set of Kostant partitions of γ is denoted by $\mathfrak{K}(\gamma)$.

There is a natural bijection between the set of pairs $1 \le q \le p \le n-1$ and R^+, namely, (p,q) corresponds to $i_q + i_{q+1} + \ldots + i_p$. Thus a Kostant partition κ is given by a collection of nonnegative integers $(\kappa_{p,q})$, $1 \le q \le p \le n-1$. Following *loc. cit.* (9) we define a collection $\mu(\kappa)$ as follows: $\mu_{p,q} = \sum_{r \le q \le p \le s} \kappa_{s,r}$.

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Recall that for $\gamma \in \mathbb{N}[I]$ we denote by $\Gamma(\gamma)$ the set of all partitions of γ, i.e. multisubsets (subsets with multiplicities) $\Gamma = \{\{\gamma_1, \dots, \gamma_k\}\}$ of $\mathbb{N}[I]$ with $\sum_{r=1}^k \gamma_r = \gamma,\ \gamma_r > 0$ (see e.g. [Ku], 1.3).

The configuration space of colored effective divisors of multidegree γ (the set of colors is I) is denoted by C^γ. The diagonal stratification $C^\gamma = \sqcup_{\Gamma\in\Gamma(\gamma)} C^\gamma_\Gamma$ was introduced e.g. in *loc. cit.* Recall that for $\Gamma = \{\{\gamma_1, \dots, \gamma_k\}\}$ we have $\dim C^\gamma_\Gamma = k$.

2.1.2. For the definition of Laumon's Quasiflags' space $\mathcal{Q}^L_\alpha$ the reader may consult [La] 4.2, or [Ku] 1.4. It is the space of complete flags of locally free subsheaves

$$0 \subset E_1 \subset \dots \subset E_{n-1} \subset V \otimes \mathcal{O}_C =: \mathcal{V}$$

such that $\mathrm{rank}(E_k) = k$, and $\deg(E_k) = -a_k$.

It is known to be a smooth projective variety of dimension $2|\alpha| + \dim \mathcal{B}$.

2.1.3. For the definition of Drinfeld's Quasimaps' space $\mathcal{Q}^D_\alpha$ the reader may consult [Ku] 1.2. It is the space of collections of invertible subsheaves $\mathcal{L}_\lambda \subset V_\lambda \otimes \mathcal{O}_C$ for each dominant weight $\lambda \in X^+$ satisfying Plücker relations, and such that $\deg \mathcal{L}_\lambda = -\langle \lambda, \alpha\rangle$.

It is known to be a (singular, in general) projective variety of dimension $2|\alpha| + \dim \mathcal{B}$.

The open subspace $\mathcal{Q}_\alpha \subset \mathcal{Q}^D_\alpha$ of genuine maps is formed by the collections of line subbundles (as opposed to invertible subsheaves) $\mathcal{L}_\lambda \subset V_\lambda \otimes \mathcal{O}_C$. In fact, it is an open stratum of the stratification by the *type of degeneration* of $\mathcal{Q}^D_\alpha$ introduced in [Ku] 1.3:

$$\mathcal{Q}^D_\alpha = \bigsqcup_{\beta\le\alpha}^{\Gamma\in\Gamma(\alpha-\beta)} \mathfrak{D}^{\beta,\Gamma}_\alpha$$

We have $\mathfrak{D}_{\alpha,\emptyset} = \mathcal{Q}_\alpha$, and $\mathfrak{D}^{\beta,\Gamma}_\alpha = \mathcal{Q}_\beta \times C^{\alpha-\beta}_\Gamma$ (see *loc. cit.* 1.3.5).

The space $\mathcal{Q}^D_\alpha$ is naturally embedded into the product of projective spaces

$$\mathbb{P}_\alpha = \prod_{1\le p\le n-1} \mathbb{P}(\mathrm{Hom}(\mathcal{O}_C(-\langle\omega_p, \alpha\rangle), V_{\omega_p} \otimes \mathcal{O}_C))$$

and is closed in it (see *loc. cit.* 1.2.5). Here ω_p stands for the fundamental weight dual to the coroot i_p. The fundamental representation V_{ω_p} equals $\Lambda^p V$.

2.2. We will study the characteristic cycle of the Goresky-MacPherson perverse sheaf (or the corresponding regular holonomic D-module) IC_α on $\mathcal{Q}^D_\alpha$. As $\mathcal{Q}^D_\alpha$ is embedded into the smooth space $\mathbb{P}_\alpha$, we will view this characteristic cycle $SS(IC_\alpha)$ as a Lagrangian cycle in the cotangent bundle $T^*\mathbb{P}_\alpha$. *A priori* we have the following equality:

$$SS(IC_\alpha) = \overline{T^*_{\mathcal{Q}_\alpha}\mathbb{P}_\alpha} + \sum_{\beta<\alpha}^{\Gamma\in\Gamma(\alpha-\beta)} m^{\beta,\Gamma}_\alpha \overline{T^*_{\mathfrak{D}^{\beta,\Gamma}_\alpha}\mathbb{P}_\alpha},$$

closures of conormal bundles with multiplicities.

Theorem. $SS(IC_\alpha) = \overline{T^*_{\mathcal{Q}_\alpha}\mathbb{P}_\alpha}$ is irreducible.

In the following subsections we will reduce the Theorem to a statement about geometry of Laumon's resolution.

2.3. We fix a coordinate z on C identifying it with the standard $\mathbb{P}^1$. We denote by $\mathcal{Q}^\infty_\alpha \subset \mathcal{Q}^D_\alpha$ the open subspace formed by quasimaps which are genuine maps in a neighbourhood of the point $\infty \in C$. In other words, $(\mathcal{L}_\lambda \subset V_\lambda \otimes \mathcal{O}_C)_{\lambda \in X^+} \in \mathcal{Q}^\infty_\alpha$ iff for each λ the invertible subsheaf $\mathcal{L}_\lambda \subset V_\lambda \otimes \mathcal{O}_C$ is a line subbundle in some neighbourhood of $\infty \in C$.

Evidently, $\mathcal{Q}^\infty_\alpha$ intersects all the strata $\mathfrak{D}_{\beta,\Gamma}$. Thus it suffices to prove the irreducibility of the singular support of Goresky-MacPherson sheaf of $\mathcal{Q}^\infty_\alpha$.

There is a well-defined map of evaluation at $\infty \in C$:

$$\Upsilon_\alpha : \ \mathcal{Q}^\infty_\alpha \longrightarrow \mathcal{B}$$

It is compatible with the stratification of $\mathcal{Q}^\infty_\alpha$ and realizes $\mathcal{Q}^\infty_\alpha$ as a (stratified) fibre bundle over $\mathcal{B}$. In effect, G acts naturally both on $\mathcal{Q}^\infty_\alpha$ (preserving stratification) and on $\mathcal{B}$; the map Υ_α is equivariant, and $\mathcal{B}$ is homogeneous. We denote the fiber $\Upsilon_\alpha^{-1}(B)$ over the point $B \in \mathcal{B}$ by $\mathcal{Z}_\alpha$.

It inherits the stratification

$$\mathcal{Z}_\alpha = \bigsqcup_{\beta \le \alpha}^{\Gamma \in \Gamma(\alpha-\beta)} \mathcal{Z}\mathfrak{D}^{\beta,\Gamma}_\alpha$$

from $\mathcal{Q}^\infty_\alpha$ and $\mathcal{Q}^D_\alpha$. It is just the transversal intersection of the fiber $\Upsilon_\alpha^{-1}(B)$ with the stratification of $\mathcal{Q}^\infty_\alpha$. As in [Ku] 1.3.5 we have $\mathcal{Z}\mathfrak{D}^{\beta,\Gamma}_\alpha \xrightarrow{\sim} \mathcal{Z}_\beta \times (C-\infty)^{\alpha-\beta}_\Gamma$.

Hence it suffices to prove the irreducibility of the singular support $SS(IC(\mathcal{Z}_\alpha))$ of Goresky-MacPherson sheaf $IC(\mathcal{Z}_\alpha)$ of $\mathcal{Z}_\alpha$.

2.4. **Factorization.** The Theorem 6.3 of [FM] admits the following immediate Corollary. Let $(\phi_\beta, \gamma_1 x_1, \dots, \gamma_k x_k) = \phi_\alpha \in \mathcal{Z}_\beta \times (C-\infty)^{\alpha-\beta}_\Gamma = \mathcal{Z}\mathfrak{D}^{\beta,\Gamma}_\alpha \subset \mathcal{Z}_\alpha$. Consider also the points $(\phi_r, \gamma_r x_r) = \phi_{\gamma_r} \in \mathcal{Z}_0 \times (C-\infty)^{\gamma_r}_{\{\{\gamma_r\}\}} = \mathcal{Z}\mathfrak{D}^{0,\{\{\gamma_r\}\}}_{\gamma_r} \subset \mathcal{Z}_{\gamma_r},\ 1 \le r \le k$.

Proposition. There is an analytic open neighbourhood U_α (resp. U_β, resp. $U_{\gamma_r},\ 1 \le r \le k$) of ϕ_α (resp. ϕ_β, resp. $\phi_{\gamma_r},\ 1 \le r \le k$) in $\mathcal{Z}_\alpha$ (resp. $\mathcal{Z}_\beta$, resp. $\mathcal{Z}_{\gamma_r},\ 1 \le r \le k$) such that

$$U_\alpha \xrightarrow{\sim} U_\beta \times \prod_{1 \le r \le k} U_{\gamma_r}$$

□

Recall the nonnegative integers $m^{\beta,\Gamma}_\alpha$ introduced in 2.2. The Proposition implies the following Corollary.

Corollary. $m^{\beta,\Gamma}_\alpha = \prod_{1 \le r \le k} m^{0,\{\{\gamma_r\}\}}_{\gamma_r}$. □

Thus to prove that all the multiplicities $m^{\beta,\Gamma}_\alpha$ vanish, it suffices to check the vanishing of $m^{0,\{\{\gamma\}\}}_\gamma$ for arbitrary $\gamma > 0$.

2.5. It remains to prove that the conormal bundle $T^*_{\mathfrak{D}^{0,\{\{\gamma\}\}}_\gamma}\mathbb{P}_\alpha$ to the closed stratum of $\mathcal{Q}_\gamma$ enters the singular support $SS(IC_\alpha)$ with multiplicity 0. To this end we choose a point $(B, \gamma 0) = \phi \in \mathcal{B} \times C = \mathcal{Q}_0 \times C^\gamma_{\{\{\gamma\}\}} = \mathfrak{D}^{0,\{\{\gamma\}\}}_\gamma \subset \mathcal{Q}_\gamma \subset \mathbb{P}_\gamma$. We also choose a sufficiently generic meromorphic function f on $\mathbb{P}_\gamma$ regular around ϕ and vanishing on $\mathfrak{D}^{0,\{\{\gamma\}\}}_\gamma$. According to the Proposition 8.6.4 of [KS], the multiplicity in question is 0 iff $\Phi_f(IC_\gamma)_\phi = 0$, i.e. the stalk of vanishing cycles sheaf at the point ϕ vanishes.

To compute the stalk of vanishing cycles sheaf we use the following argument, borrowed from [BFL] §1. As $\pi : \ \mathcal{Q}^L_\gamma \longrightarrow \mathcal{Q}^D_\gamma$ is a small resolution of singularities, up to a shift, $IC_\alpha = \pi_* \underline{\mathbb{Q}}$. By the proper base change, $\Phi_f \pi_* \underline{\mathbb{Q}} = \pi_* \Phi_{f \circ \pi} \underline{\mathbb{Q}}$. So it suffices to check that $\Phi_{f \circ \pi} \underline{\mathbb{Q}}|_{\pi^{-1}(\phi)} = 0$.

Let us denote the differential of the function f at the point ϕ by ξ so that $(\phi,\xi)\in T^*_{\mathfrak{D}^{0,\{\{\gamma\}\}}_\gamma}\mathbb{P}_\gamma$. Then the support of $\Phi_{f\circ\pi}\underline{\mathbb{Q}}|_{\pi^{-1}(\phi)}$ is *a priori* contained in the *microlocal fiber* over (ϕ,ξ) which we define presently.

2.5.1. *Definition.* Let $\varpi:\ A\to B$ be a map of smooth varieties. For $a\in A$ let $d^*_a\varpi:\ T^*_{\varpi(a)}B\longrightarrow T^*_aA$ denote the codifferential, and let (b,η) be a point in T^*B. Then the *microlocal fiber* of ϖ over (b,η) is defined to be the set of points $a\in\varpi^{-1}(b)$ such that $d^*_a\varpi(\eta)=0$.

2.5.2. Thus we have reduced the Theorem 2.2 to the following Proposition.

Proposition. For a sufficiently generic ξ such that $(\phi,\xi)\in T^*_{\mathfrak{D}^{0,\{\{\gamma\}\}}_\gamma}\mathbb{P}_\gamma$, the microlocal fiber of Laumon's resolution π over (ϕ,ξ) is empty. Equivalently, the cone $\cup_{E_\bullet\in\pi^{-1}(\phi)}\mathrm{Ker}(d^*_{E_\bullet}\pi)$ is a proper subvariety of the fiber of $T^*_{\mathfrak{D}^{0,\{\{\gamma\}\}}_\gamma}\mathbb{P}_\gamma$ at ϕ.

2.6. **Piecification of a simple fiber.** The fiber $\pi^{-1}(\phi)$ was called the *simple fiber* in [Ku] §2. It was proved in *loc. cit.* 2.3.3 that $\pi^{-1}(\phi)$ is a disjoint union of (pseudo)affine spaces $\mathfrak{S}(\mu(\kappa))$ where κ runs through the set $\mathfrak{K}(\gamma)$ of Kostant partitions of γ (for the notation $\mu(\kappa)$ see 2.1.1 or [Ku] (9)). Another way to parametrize these pseudoaffine pieces was introduced in [FK] 2.11. Let us recall it here.

We define nonnegative integers $c_p, 1\le p\le n-1$, so that $\gamma=\sum_{p=1}^{n-1}c_pi_p$.

2.6.1. *Definition.* $\mathcal{D}(\gamma)$ is the set of collections of nonnegative integers $(d_{p,q})_{1\le q\le p\le n-1}$ such that

a) For any $1\le q\le p\le r\le n-1$ we have $d_{r,q}\le d_{p,q}$;

b) For any $1\le p\le n-1$ we have $\sum_{q=1}^p d_{p,q}=c_p$.

2.6.2. *Lemma.* The correspondence $\kappa=(\kappa_{p,q})_{1\le q\le p\le n-1}\mapsto(d_{p,q}:=\sum_{r=p}^{n-1}\kappa_{r,q})_{1\le q\le p\le n-1}$ defines a bijection between $\mathfrak{K}(\gamma)$ and $\mathcal{D}(\gamma)$. $\square$

2.6.3. Using the above Lemma we can rewrite the parametrization of the pseudoaffine pieces of the simple fiber as follows:

$$\pi^{-1}(\phi)=\bigsqcup_{\mathfrak{d}\in\mathcal{D}(\gamma)}\mathfrak{S}(\mathfrak{d})$$

In these terms the dimension formula of [Ku] 2.3.3 reads as follows: for $\mathfrak{d}=(d_{p,q})_{1\le q\le p\le n-1}$ we have $\dim\mathfrak{S}(\mathfrak{d})=\sum_{1\le q<p\le n-1}d_{p,q}$.

Note also that $\sum_{1\le q\le p\le n-1}d_{p,q}=\sum_{1\le p\le n-1}c_p=|\gamma|$.

2.7. **Proposition.** For arbitrary $\mathfrak{d}=(d_{p,q})_{1\le q\le p\le n-1}\in\mathcal{D}(\gamma)$ and arbitrary quasiflag $E_\bullet\in\mathfrak{S}(\mathfrak{d})\subset\pi^{-1}(\phi)$ we have $\dim\mathrm{Ker}(d_{E_\bullet}\pi)<\sum_{1\le p\le n-1}d_{p,q}+\sum_{1\le q\le p\le n-1}d_{p,q}-1$.

This Proposition implies the Proposition 2.5.2 straightforwardly. In effect, $\mathrm{codim}\mathrm{Ker}(d_{E_\bullet}\pi)=\dim\mathcal{Q}^L_\gamma-\dim\mathrm{Ker}(d_{E_\bullet}\pi)>2|\gamma|+\dim\mathcal{B}-\sum_{1\le p\le n-1}d_{p,p}-\sum_{1\le q\le p\le n-1}d_{p,q}+1=\dim\mathcal{B}+1+\sum_{1\le q<p\le n-1}d_{p,q}$. Hence the subspace $\mathrm{Ker}(d^*_{E_\bullet}\pi)\subset T^*_\phi\mathbb{P}_\gamma$ has codimension greater than $\dim\mathcal{B}+1+\sum_{1\le q<p\le n-1}d_{p,q}$. Recall that $\dim\mathfrak{D}^{0,\{\{\gamma\}\}}_\gamma=\dim\mathcal{B}+1$. Hence the codimension of $\mathrm{Ker}(d^*_{E_\bullet}\pi)\cap T^*_{\mathfrak{D}^{0,\{\{\gamma\}\}}_\gamma}\mathbb{P}_\gamma$ in the fiber of $T^*_{\mathfrak{D}^{0,\{\{\gamma\}\}}_\gamma}\mathbb{P}_\gamma$ at ϕ is greater than $\sum_{1\le q<p\le n-1}d_{p,q}=\dim\mathfrak{S}(\mathfrak{d})$. Hence the cone $\cup_{E_\bullet\in\mathfrak{S}(\mathfrak{d})}\mathrm{Ker}(d^*_{E_\bullet}\pi)$ is a proper subvariety of the fiber of $T^*_{\mathfrak{D}^{0,\{\{\gamma\}\}}_\gamma}\mathbb{P}_\gamma$ at ϕ.

The union of these proper subvarieties over $\mathfrak{d} \in \mathcal{D}(\gamma)$ is again a proper subvariety of the fiber of $T^*_{\mathfrak{D}^{0,\{\{\gamma\}\}}_\gamma}\mathbb{P}_\gamma$ at ϕ which concludes the proof of the Proposition 2.5.2.

2.8. **Fixed points.** It remains to prove the Proposition 2.7. To this end recall that the Cartan group H acts on V and hence on $\mathcal{Q}^L_\alpha$. The group $\mathbb{C}^*$ of dilations of $C = \mathbb{P}^1$ preserving 0 and ∞ also acts on $\mathcal{Q}^L_\alpha$ commuting with the action of H. Hence we obtain the action of a torus $\mathbb{T} := H \times \mathbb{C}^*$ on $\mathcal{Q}^L_\alpha$.

It preserves the simple fiber $\pi^{-1}(\phi)$ and its pseudoaffine pieces $\mathfrak{S}(\mathfrak{d}),\ \mathfrak{d} \in \mathcal{D}(\gamma)$, for evident reasons. It was proved in [FK] 2.12 that each piece $\mathfrak{S}(\mathfrak{d}),\ \mathfrak{d} = (d_{p,q})_{1\le q\le p\le n-1}$ contains exactly one $\mathbb{T}$-fixed point $\delta(\mathfrak{d}) = (E_1, \dots, E_{n-1})$. Here

$$\begin{array}{lllll} E_1 & = E_{1,1} & & & \\ E_2 & = E_{2,1} & \oplus E_{2,2} & & \\ \vdots & & \vdots & & \vdots \\ E_{n-1} & = E_{n-1,1} & \oplus E_{n-1,2} & \oplus \dots & \oplus E_{n-1,n-1} \end{array}$$

and $E_{p,q} = \mathcal{O}(-d_{p,q}) \subset \mathcal{O}v_q \subset \mathcal{V} = V \otimes \mathcal{O}_C$ with quotient sheaf $\dfrac{\mathcal{O}}{\mathcal{O}(-d_{p,q})}$ concentrated at $0 \in C$.

2.8.1. Now the $\mathbb{T}$-action contracts $\mathfrak{S}(\mathfrak{d})$ to $\delta(\mathfrak{d})$. Since the map π is $\mathbb{T}$-equivariant, and the dimension of $\mathrm{Ker}(d_{E_\bullet}\pi)$ is lower semicontinuous, the Proposition 2.7 follows from the next one.

Key Proposition. For arbitrary $\mathfrak{d} = (d_{p,q})_{1\le q\le p\le n-1} \in \mathcal{D}(\gamma)$ $(\gamma \ne 0)$ we have $\dim \mathrm{Ker}(d_{\delta(\mathfrak{d})}\pi) < \sum_{1\le p\le n-1} d_{p,p} + \sum_{1\le q\le p\le n-1} d_{p,q} - 1$.

The proof will be given in the next section.

2.8.2. *Remark.* In general, the pieces $\mathfrak{S}(\mathfrak{d})$ of the simple fiber are not equisingular, i.e. $\dim \mathrm{Ker}(d_{E_\bullet}\pi)$ is not constant along a piece. The simplest example occurs for $G = SL_3,\ \gamma = 2i_1 + 2i_2$. Then the simple fiber is a singular 2-dimensional quadric. Its singular point is the fixed point of the 1-dimensional piece $\mathfrak{S}(\mathfrak{d})$ where $d_{1,1} = 2, d_{2,1} = d_{2,2} = 1$. At this point we have $\dim \mathrm{Ker}(d_{\delta(\mathfrak{d})}\pi) = 3$ while at the other points in this piece we have $\dim \mathrm{Ker}(d_{E_\bullet}\pi) = 2$.

3. The proof of the Key Proposition

3.1. **Tangent spaces.** Let Ω be the following quiver: $\Omega = 1 \longrightarrow 2 \longrightarrow \dots \longrightarrow n-1$. Thus the set of vertices coincides with I. A quasiflag $(E_1 \hookrightarrow E_2 \hookrightarrow \dots \hookrightarrow E_{n-1} \subset \mathcal{V}) \in \mathcal{Q}^L_\gamma$ may be viewed as a representation of Ω in the category of coherent sheaves on C. If we denote the quotient sheaf $\mathcal{V}/E_p$ by $Q_p,\ 1 \le p \le n-1$, we have another representation of Ω in coherent sheaves on C, namely,

$$Q_\bullet := (Q_1 \twoheadrightarrow Q_2 \twoheadrightarrow \dots \twoheadrightarrow Q_{n-1})$$

3.1.1. *Exercise.* $T_{E_\bullet}\mathcal{Q}^L_\gamma = \mathrm{Hom}_\Omega(E_\bullet, Q_\bullet)$ where $\mathrm{Hom}_\Omega(?,?)$ stands for the morphisms in the category of representations of Ω in coherent sheaves on C.

3.1.2. Consider a point $\mathcal{L}_\bullet = (\mathcal{L}_1, \dots, \mathcal{L}_{n-1}) \in \mathbb{P}_\gamma$. Here $\mathcal{L}_p \subset V_{\omega_p} \otimes \mathcal{O}_C$ is an invertible subsheaf, the image of morphism $\mathcal{O}_C(-\langle\omega_p, \gamma\rangle) \hookrightarrow V_{\omega_p} \otimes \mathcal{O}_C$.

Exercise. $T_{\mathcal{L}_\bullet}\mathbb{P}_\gamma = \prod_{p=1}^{n-1} \mathrm{Hom}(\mathcal{L}_p, V_{\omega_p} \otimes \mathcal{O}_C/\mathcal{L}_p)$.

3.1.3. Recall that for $E_\bullet \in \mathcal{Q}^L_\gamma$ we have $\pi(E_\bullet) = \mathcal{L}_\bullet \in \mathbb{P}_\gamma$ where $\mathcal{L}_p = \Lambda^p E_p$ for $1 \le p \le n-1$.

Exercise. For $h_\bullet = (h_1, \dots, h_{n-1}) \in T_{E_\bullet}\mathcal{Q}^L_\gamma$ we have $d_{E_\bullet}\pi(h_\bullet) = (\Lambda^1 h_1, \Lambda^2 h_2, \dots, \Lambda^{n-1} h_{n-1}) \in T_{\mathcal{L}_\bullet}\mathbb{P}_\gamma$.

3.2. From now on we fix $\gamma > 0$, $\mathfrak{d} \in \mathcal{D}(\gamma)$, $\delta(\mathfrak{d}) =: E_\bullet$. To unburden the notations we will denote the tangent space $T_{E_\bullet}\mathcal{Q}^L_\gamma$ by T. Since $\mathcal{Q}^L_\gamma$ is a smooth $(2|\gamma| + \dim \mathcal{B})$-dimensional variety it suffices to find a subspace $N \subset T$ of dimension

$$2|\gamma| + \dim \mathcal{B} - \sum_{1 \le p \le n-1} d_{p,p} - \sum_{1 \le q \le p \le n-1} d_{p,q} + 1 = \sum_{1 \le q < p \le n-1} (d_{p,q} + 1) + 1$$

such that $d_{E_\bullet}\pi|_N$ is injective.

3.3. Let $N_0 = \bigoplus_{n-1 \ge p > q \ge 1} \mathrm{Hom}(\mathcal{O}(-d_{p,q}), \mathcal{O})$. We have $\dim N_0 = \sum_{n-1 \ge p > q \ge 1} (d_{p,q} + 1)$.

Recall that we have canonically $T = \mathrm{Hom}_\Omega(E_\bullet, Q_\bullet)$, where

$$Q_p = \mathcal{V}/E_p = \left(\bigoplus_{q=1}^{p} \left(\frac{\mathcal{O}}{\mathcal{O}(-d_{p,q})} \right) v_q \right) \oplus \left(\bigoplus_{q=p+1}^{n} \mathcal{O} v_q \right).$$

3.4. Let us define a map $\nu_0 : N_0 \to T$ assigning to an element $(f_{p,q}) \in N_0$ a morphism $\nu_0(f_{p,q}) := F \in \mathrm{Hom}_\bullet(E_\bullet, Q_\bullet)$ of graded coherent sheaves, where $F|_{E_{p,q}} = \bigoplus_{r=p+1}^{n} F^r_{p,q}$, and

$$F^r_{p,q} : E_{p,q} \to \mathcal{O} v_r \subset Q_p \quad \text{is defined as the composition} \quad E_{p,q} \subset E_{r,q} = \mathcal{O}(-d_{r,q}) \xrightarrow{f_{r,q}} \mathcal{O} v_r$$

3.5. **Lemma.** The map $F : E_\bullet \to Q_\bullet$ is a morphism of representations of the quiver Ω.

Proof. We need to check the commutativity of the following diagram

$$\begin{tikzcd} E_p \arrow[r] \arrow[d, "F"'] & E_{p'} \arrow[d, "F"'] \\ Q_p \arrow[r] & Q_{p'} \end{tikzcd}$$

Since E_p and $Q_{p'}$ are canonically decomposed into the direct sum it suffices to note that for any $q \le p \le p' < r$ the following diagram

$$\begin{tikzcd} E_{p,q} \arrow[r] \arrow[d, "F^r_{p,q}"'] & E_{p',q} \arrow[r] \arrow[d, "F^r_{p',q}"'] & E_{r,q} \arrow[d, "f_{r,q}"'] \\ \mathcal{O} v_r \arrow[r, equal] & \mathcal{O} v_r \arrow[r, equal] & \mathcal{O} v_r \end{tikzcd}$$

commutes and for any $q \le p < r \le p'$ the following diagram

$$\begin{tikzcd} E_{p,q} \arrow[r] \arrow[d, "F^r_{p,q}"'] & E_{r,q} \arrow[r] \arrow[d, "f_{r,q}"'] & E_{p',q} \arrow[d, "0"'] \\ \mathcal{O} v_r \arrow[r, equal] & \mathcal{O} v_r \arrow[r] & \left(\frac{\mathcal{O}}{\mathcal{O}(-d_{r,q})} \right) v_r \end{tikzcd}$$

commutes as well. $\square$

3.6. Let $N_1 = \mathbb{C}$. Let $p_0 = \min\{1 \le p \le n-1 \mid d_{p,p} > 0\}$ and pick a non-zero element $\mathfrak{f} \in \mathrm{Hom}(\mathcal{O}(-d_{p,p}), \frac{\mathcal{O}}{\mathcal{O}(-d_{p,p})})$. Define the map $\nu_1 : N_1 \to T$ by assigning to $1 \in N_1$ the element $\mathfrak{F} \in \mathrm{Hom}_\Omega(E_\bullet, Q_\bullet)$ defined on $E_{p,p}$ as the composition

$$E_{p,p} = \mathcal{O}(-d_{p,p}) \xrightarrow{\mathfrak{f}} \frac{\mathcal{O}}{\mathcal{O}(-d_{p,p})} v_p \subset Q_p$$

and with all other components equal to zero.

3.7. Let $\mathcal{M}(r, d; \mathcal{V})$ denote the space of rank r and degree d subsheaves in $\mathcal{V}$.

3.7.1. Let $E \subset \mathcal{V}$ be a rank k and degree d subsheaf in the vector bundle $\mathcal{V}$. Let $\mathcal{V}/E = \mathcal{T} \oplus \mathcal{F}$ be a decomposition of the quotient sheaf into the sum of the torsion $\mathcal{T}$ and a locally free sheaf $\mathcal{F}$. Consider the map $\det : \mathcal{M}(r, d; \mathcal{V}) \to \mathcal{M}(1, d; \Lambda^k \mathcal{V})$ sending E to $\Lambda^k E$. Then the restriction of its differential $d_E \det : T_E \mathcal{M}(r, d; \mathcal{V}) = \mathrm{Hom}(E, \mathcal{V}/E) \to \mathrm{Hom}(\Lambda^k E, \Lambda^k V/\Lambda^k E) = T_{\Lambda^k E} \mathcal{M}(1, d; \Lambda^k \mathcal{V})$ to the subspace $\mathrm{Hom}(E, \mathcal{F}) \subset \mathrm{Hom}(E, \mathcal{V}/E)$ factors as $\mathrm{Hom}(E, \mathcal{F}) \cong \mathrm{Hom}(\Lambda^k E, \Lambda^{k-1} E \otimes \mathcal{F}) \subset \mathrm{Hom}(\Lambda^k E, \Lambda^k V/\Lambda^k E)$. Therefore it is injective.

3.7.2. Let $E = \mathcal{O}^{\oplus(r-1)} \oplus \mathcal{O}(-d)$ be a subsheaf in $\mathcal{V} = \mathcal{O}^{\oplus n}$. Then the restriction of differential $d_E \det$ to the subspace $\mathrm{Hom}(E, \mathcal{T}) \subset \mathrm{Hom}(E, \mathcal{V}/E)$ is injective.

This immediately follows from the following fact. Let $\tilde{E} = \mathcal{O}^{\oplus r} \subset \mathcal{V}$ be the normalization of E in $\mathcal{V}$, that is, the maximal vector subbundle $\tilde{E} \subset \mathcal{V}$ such that $\tilde{E}/E$ is torsion. Then $\mathcal{T} = \tilde{E}/E \cong \Lambda^k \tilde{E}/\Lambda^k E \subset \Lambda^k \mathcal{V}/\Lambda^k E$.

3.7.3. Clearly, the subsheaves $\Lambda^{k-1} E \otimes \mathcal{F} \subset \Lambda^k \mathcal{V}/\Lambda^k E$ and $\mathcal{T} \subset \Lambda^k \mathcal{V}/\Lambda^k E$ do not intersect.

3.7.4. It follows from 3.7.1, 3.7.2 and 3.7.3 that the composition $d_{E_\bullet} \pi \circ (\nu_0 \oplus \nu_1) : N_0 \oplus N_1 \to T_{\pi(E_\bullet)} \mathbb{P}_\gamma$ is injective, hence $N := (\nu_0 \oplus \nu_1)(N_0 \oplus N_1) \subset T_{E_\bullet} \mathcal{Q}^D_\gamma$ enjoys the desired property. Namely, $d_{E_\bullet} \pi|_N$ is injective, and $\dim N = \sum_{1 \le q < p \le n-1} (d_{p,q} + 1) + 1$.

This completes the proof of the Key Proposition 2.8.1 along with the Main Theorem 2.2. □

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