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# DEPARTMENT OF ECONOMICS

## Working Paper

**Long waves and short cycles in a model of endogenous  
financial fragility**

by

Soon Ryoo

Working Paper 2009-03



**UNIVERSITY OF MASSACHUSETTS  
AMHERST**

# Long waves and short cycles in a model of endogenous financial fragility

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## Abstract

This paper presents a stock-flow consistent macroeconomic model in which financial fragility in firm and household sectors evolves endogenously through the interaction between real and financial sectors. Changes in firms' and households' financial practices produce long waves. The Hopf bifurcation theorem is applied to clarify the conditions for the existence of limit cycles, and simulations illustrate stable limit cycles. The long waves are characterized by periodic economic crises following long expansions. Short cycles, generated by the interaction between effective demand and labor market dynamics, fluctuate around the long waves.

*Key words.* cycles, long waves, financial fragility, stock-flow consistency

*JEL classification.* E12, E32, E44

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## 1. Introduction

Financial crisis hit the U.S and world economy in 2008. Giant financial institutions have collapsed. Stock markets have tumbled, and exchange rates are in turmoil. Governments and central banks around the world have responded by implementing bailout plans for troubled financial institutions and cutting interest rates to contain the financial panic, and expansionary fiscal packages are being pushed through to prop up aggregate demand. Hyman Minsky's Financial Instability Hypothesis offers an interesting perspective on these developments,

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which came after a long period of financial deregulation, rapid securitization and the development of a range of new financial instruments and markets.<sup>2</sup>

According to Minsky's financial instability hypothesis, a capitalist economy cannot lead to a sustained full employment equilibrium and serious business cycles are unavoidable due to the unstable nature of the interaction between investment and finance (Minsky, 1986, 173). An initially robust financial system is endogenously turned into a fragile system as a prolonged period of good years induces firms and bankers to take riskier financial practices. During expansions, an investment boom generates a profit boom but this induces investors and banks to adopt more speculative financial arrangements. This is typically reflected in rising debt finance, which eventually turns out to be unsustainable because the rising debt changes cash flow relations and leads to various types of financial distress. Minsky suggests that this kind of endogenous change in financial fragility can generate debt-driven long expansions followed by deep depressions (Minsky 1964, 1995). In Minsky's theory of long waves, short cycles fluctuate around the long waves produced by endogenous changes in financial structure. Thus, the distinction between short cycles and long waves is an important characteristic of Minsky's cycle theory.

In spite of difficulties inherent in the formalization of Minsky's theories, Minsky's financial instability hypothesis has inspired a number of researchers to model the dynamic interaction between real and financial sectors. Taylor and O'Connell (1985), Foley (1986), Semmler (1987), Jarsulic (1989), Delli Gatti and Gallegati (1990), Skott (1994), Dutt (1995), Keen (1995) and Flaschel, Franke and Semmler (1998, Ch.12) are early contributions. Recent studies include Setterfield (2004), Nasica and Raybaut (2005), Lima and Meirelles (2007), and Fazzari et al. (2008).

This paper presents a stock-flow consistent model where firms' and households' financial practices evolve endogenously through the interaction between real and financial sectors. The interaction between changes in firms' and households' financial practices produces long waves. The resulting long waves are characterized by periodic economic crises following long expansions. Short cycles, generated by the interaction between effective demand and labor market dynamics, fluctuate around the long waves.

Compared to the previous literature, this paper has three distinct features:

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<sup>2</sup>Wray (2008), Cynamon and Fazzari (2008) and Crotty (2008), among others, provide perspectives on how shaky are the foundations of these 'sophisticated' developments in financial markets.

First, the model in this paper is stock-flow consistent.<sup>3</sup> Financial stocks are explicitly introduced and their implications for income and financial flows are carefully modeled. In particular, unlike the previous studies listed above, capital gains from holding stocks are not assumed away and enter the definition of the rate of return on equity.<sup>4</sup> The rate of return on equity defined in this way provides a basis of households' portfolio decision. Firms' and households' financial decisions jointly determine stock prices and the rate of return on equity in equilibrium. Thus, stock markets receive a careful treatment in this model and play a central role in producing cycles.

Second, this paper pays attention to both firms' and households' financial decisions. Minsky's own account of financial instability tends to privilege the firm sector as a source of fragility.<sup>5</sup> Most previous studies follow this tradition and tend to neglect the role of households' financial decisions in creating instability and cycles. Some of the previous studies, including Taylor and O'Connell (1985), Delli Gatti and Gallegati (1990), and Flaschel, Franke and Semmler (1998, Ch.12), do not suffer from this kind of limitation but analyze households' portfolio decision as well. However, their neglect of the role of capital gains in households' portfolio decision makes it difficult to analyze the implication of households' financial decisions and stock market behavior for instability and cycles. In contrast to these models, the model in this paper analyzes both households' and firms' financial decisions. Capital gains and stock markets are considered explicitly in a stock-flow consistent framework. The interactions between households and firms turn out to be critical to the behavior of the system. The model consists of two subsystems: firms' debt dynamics and households' portfolio dynamics. One interesting result of our analysis is that two stable subsystems can be combined to produce instability and cycles in the whole system (See section 3). Thus, the resulting instability and cycles are genuinely attributed to the interaction between sectors rather than characteristics of one particular sector.

Lastly, existing Minskian models do not distinguish long waves from short

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<sup>3</sup>See Skott (1981), Godley and Cripps (1983) and Taylor (1985) for early introductions of explicit stock-flow relations in a post-Keynesian / structuralist context. Simulation exercises based on the stock-flow consistent framework have been flourishing since Lavoie and Godley (2001-2).

<sup>4</sup>Empirically, the movements of capital gains explain most of cyclical movements of the rate of return on equity.

<sup>5</sup>Minsky's neglect of the household sector is explained by his observation that "[H]ousehold debt-financing of consumption is almost always hedge financing." (1982, p. 32) This position, however, has been challenged by some Minskian explanations of the sub-prime mortgage crisis. (e.g. Wray(2008) and Kregel (2008))

cycles and the periodicity of cycles in those models is ambiguous. My model is explicit in this matter. It produces two distinct cycles: long waves and short cycles. Long waves are produced by the interaction between firms' and households' financial decisions, while short cycles are generated by the interaction between effective demand and labor market dynamics. In this framework, Minsky's financial instability hypothesis is seen as a basis of long waves.<sup>6</sup> To the best of my knowledge, my model is the first to integrate an analysis of Minskian long waves with that of short cycles.

The analysis of the implications of financial behavior for instability and cycles in this paper complements previous studies on financialization and financed growth in Skott and Ryo (2008) where the emphasis is on the effects of changes in financial behavior on long-run steady growth path with little attention to questions of stability and fluctuations.

The rest of the paper is structured as follows. Section 2 sets up a stock-flow consistent model. Section 3 analyzes how the interaction between firms' and households' financial practices produces long waves. Section 4 briefly introduces a model of short cycles into the current context. Section 5 combines our model of long waves with the short-cycle model and provides simulation results. Section 6, finally, offers some concluding remarks.

## 2. Model

This section presents a model. Firms make decisions concerning accumulation, financing, and pricing/output; households make consumption and portfolio decisions; banks accept deposits and make loans. It is assumed that there are only two types of financial assets - equity and bank deposits - and banks are the only financial institution. It is assumed that the available labor force grows at a constant rate<sup>7</sup> and long run growth is constrained by the availability of labor.

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<sup>6</sup>Minsky's two papers (Minsky, 1964, 1995) provide a strong support for this view. In these two papers, Minsky argues that there exists a mechanism in a capitalist economy that generates a 'long swing': the "mechanism which has generated the long swings centers around the cumulative changes in financial variables that take place over the long-swing expansions and contractions." (Minsky, 1964). "The more severe depressions of history occur after a period of good economic performance, with only minor cycles disturbing a generally expanding economy." (Minsky, 1995, p.85) During this long expansion, an initially robust financial structure is transformed to a fragile structure.

<sup>7</sup>We assume that there is no technical progress but the model can easily accommodate Harrod neutral technical progress

## 2.1. Firms

### 2.1.1. The finance constraint

Firms have three sources of funds in our framework: profits, new issue of equity and debt finance. Using these funds, firms make investments in real capital, pay out dividends and make interest payments. Algebraically,

$$pI + Div + iM = \Pi + v\dot{N} + \dot{M} \quad (1)$$

where  $I$ ,  $\Pi$ ,  $Div$ ,  $M$ , and  $N$  are real gross investment, gross profits, dividends, bank loans and the number of shares, respectively. Bank loans carry the nominal interest rate ( $i$ ).  $p$  represents the price of investment goods as well as the general price of output in this one-sector model. All shares are assumed to have the same price  $v$ .<sup>8</sup>

We assume that firms' dividend payout is determined as a constant fraction of profits net of depreciation and real interest payments. The dividend payout rate is denoted as  $1 - s_f$  and, consequently,  $s_f$  represents firms' retention rate. Thus, we have

$$Div = (1 - s_f)(\Pi - \delta pK - rM) \quad (2)$$

where  $K$  and  $\delta$  are real capital stock and the rate of depreciation of real capital.  $r$  represents the real interest rate,  $r = i - \hat{p}$ .<sup>9</sup> Lavoie and Godley (2001-2002) and Dos Santos and Zezza (2007), among others, use the specification (2) regarding firms' retention policy.

New equity issue can be represented by the growth of the number of shares ( $\hat{N}$ ) or by the share of investment financed by new issues denoted as  $x$ . Skott (1989) and Foley and Taylor (2004) use the former and Lavoie and Godley (2001-2002) the latter. Two measures, however, are related to each other in the following manner.

$$vN\hat{N} = x p I \quad (3)$$

Substituting (2) into (1), we get

$$pI - \delta pK = s_f(\Pi - \delta pK - rM) + vN\hat{N} + M(\hat{M} - \hat{p}) \quad (4)$$

Scaling by the value of capital stock ( $pK$ ), we have

$$\hat{K} \equiv g = s_f(\pi u \sigma - \delta - r m) + x(g + \delta) + \hat{m} + g m \quad (5)$$

<sup>8</sup>A dot over a variable refers to a time derivative ( $\dot{y} = dy/dt$ ).

<sup>9</sup>A hat over a variable is used to denote a growth rate of the variable, for instance,  $\hat{y} = (1/y)(dy/dt)$

where  $\pi$ ,  $u$ , and  $m$  is the profit share ( $\pi \equiv \frac{\Pi}{pY}$ ), the utilization rate ( $u \equiv \frac{Y}{Y_F}$ ,  $Y_F$  is full capacity output) and the debt-capital ratio ( $m \equiv \frac{M}{pK}$ ). The technical output/capital ratio,  $\sigma \equiv \frac{Y_F}{K}$ , is assumed to be fixed.  $\delta$  is the depreciation rate. Equation (5) has a straightforward interpretation: firms' investment ( $g$ ) is financed by three sources: retained earnings,  $s_f(\pi u \sigma - \delta - rm)$ , new equity issue,  $x(g + \delta)$  and bank loans,  $\dot{m} + gm$ . Given this finance constraint, firms' financial behavior is characterized by  $s_f$ ,  $x$  (or  $\hat{N}$ ) and  $m$ . Most theories treat the rates of firms' retention and equity issue as parameters and debt finance as an accommodating variable (Skott 1989, Lavoie and Godley 2001-2002 and Dos Santos and Zezza 2007). This paper assumes that the retention rate ( $s_f$ ) is exogenous as in the above literature but both the rate of equity issue ( $x$  or  $\hat{N}$ ) and the leverage ratio  $m$  are endogenous. However, our way of treating equity finance and debt finance is not symmetric.

Debt finance evolves through endogenous changes in firms' and banks' financial practices which are directly influenced by the relationship between firms' profitability and leverage ratio (see section 2.1.2 below). With debt finance determined in this way, equity finance ( $x$ ) serves as a buffer in the sense that once the other sources of finance – the retention and debt finance policies – and investment plans are determined, equity issues fill the gap between the funds needed for the investment plans and the funds available from retained earnings and bank loans. In this regard, equity finance is seen as a pure residual of firms' financing constraint. Formally, for a given set of parameters  $s_f$ ,  $\sigma$ ,  $\delta$  and  $r$ , the trajectories of endogenous variables  $g$ ,  $\pi$ ,  $u$ ,  $m$  and  $\dot{m}$  determine the required ratio of equity finance to gross investment:

$$x = \frac{g - s_f(\pi u \sigma - \delta - rm) - \dot{m} - gm}{g + \delta} \quad (6)$$

The treatment of equity finance as a residual may appear to be unsatisfactory from a point of view that emphasizes substantial difficulty involved in raising capital in equity markets compared to the other methods of finance. However, as Figure 1 shows, the degree of flexibility in issuing equities was historically very large. This was even more prominent when there were significant stock buybacks, i.e. the rate of net issue of equity was negative ( $x < 0$ ). For instance, the share of fixed investment financed by equity issues was nearly zero in 1982 but reached -42% in 1985. It then bounced back to a positive rate, 4.3% in 1991, and hit the historical low, -71.5% in 2007.

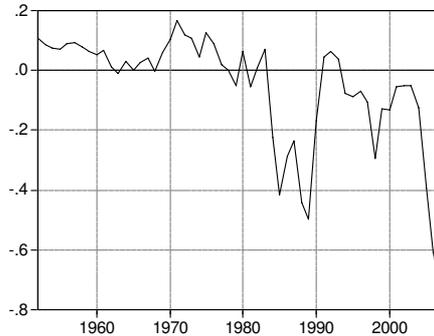


Figure 1: The Ratio of Net Issues of Equities to Fixed Investment (1952-2007)  
*Notes:* Net issues of nonfinancial corporate equities divided by nonfarm nonfinancial corporate (gross) fixed investment  
*Sources:* Federal Reserve Board, *Flow of Funds Accounts of the United States*, Table F.213 and Table F.102. Author's calculation.

### 2.1.2. Endogenous changes in firms' liability structure

Endogenous changes in firms' liability structure, which are captured by changes in firms' debt-capital ratio ( $m$ ), are central in this paper, and a Minskyan perspective suggests that the debt-capital ratio evolves according to sustained changes in firms' profitability relative to their payment obligations on debt. Changes in profitability that are perceived as highly temporary have only limited effects on desired leverage. I, therefore, distinguish cyclical movements in profitability from the trend in average profitability and assume that changes in liability structure are determined as the trend of profitability.<sup>10</sup>

The perception of strong profitability relative to payment commitments during good years, Minsky argues, induces bankers and businessmen to adopt riskier financial practices which typically results in increases in the leverage ratio. Following Minsky's idea (Minsky, 1982, 1986), I assume that changes in the ratio of profit to debt service commitments drive changes in the debt structure. Formally,

$$\dot{m} = \tau\left(\frac{\rho_T}{rm}\right); \quad \tau'(\cdot) > 0 \quad (7)$$

where  $\rho_T$  represents the *trend* rate of profit<sup>11</sup> and  $\tau$  is an increasing function. During a period of good years when the level of profit is sufficiently high com-

<sup>10</sup>See section 3.1 for more discussion.

<sup>11</sup>A definition of the trend rate of profit will be provided in section 3.

pared to interest payment obligations, firms' and bankers' optimism, reinforced by their success, tends to make them adopt riskier financial arrangements which involve higher leverage ratios. Moreover, a high profit level compared to debt servicing is typically associated with a low probability of default which helps bankers maintain their optimism. The opposite is true when the ratio of profit to interest payments is low. Firms' failure to repay debt obligations - defaults and bankruptcies in the firm sector - put financial institutions linked to those firms in trouble as well. This situation, which is often manifested in a system-wide credit crunch, tends to force firms and bankers to reduce firms' indebtedness.

### 2.1.3. Accumulation

In general, capital accumulation is affected by several factors including profitability, utilization, Tobin's  $q$ , the level of internal cash flows, the real interest rates and the debt ratio, but there is no consensus among theorists concerning the sensitivity of firms' accumulation behavior to changes in the various arguments. This paper follows a Harrodian perspective in which capacity utilization has foremost importance in firms' accumulation behavior (Harrod, 1939). The perspective assumes that firms have a desired rate of utilization. In the short run, the actual rate of utilization may deviate from the desired rate since firms' demand expectations are not always met and capital stocks slowly adjust. If the actual rate exceeds the desired rate, firms will accelerate accumulation to increase their productive capacity and if the actual rate is smaller than the desired rate, they will slow down accumulation to reduce the undesired reserve of excess productive capacity. However, in the long run, it is not reasonable to assume that the actual rate can persistently deviate from the desired rate because capital stocks can flexibly adjust to maintain the desired rate. This perspective naturally distinguishes the short-run accumulation function from the long-run accumulation function.<sup>12</sup>

A simple version of the long-run accumulation function can be written as

$$u = u^* \tag{8}$$

where  $u^*$  is an exogenously given desired rate of utilization. (8) represents the idea that in the long run, the utilization rate must be at what firms want it to be and capital accumulation is perfectly elastic so as to maintain the desired rate. The strict exogeneity of the desired rate in (8) may exaggerate reality but tries to capture mild variations of the utilization rate in the long-run. For

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<sup>12</sup>This Harrodian perspective is elaborated in Skott (1989, 2008a, 2008b) in greater detail.

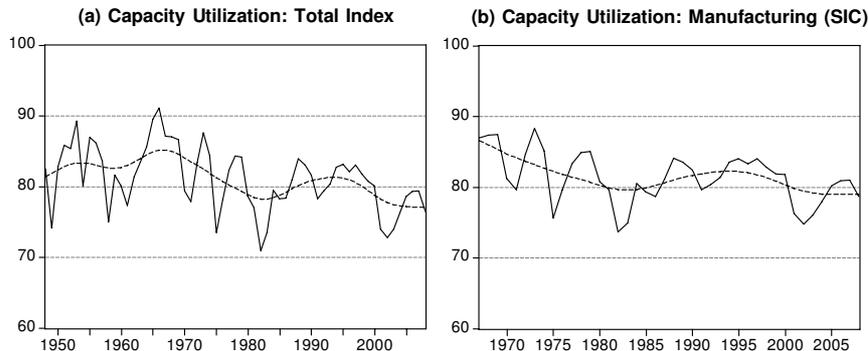


Figure 2: Capacity Utilization. U.S (1948-2008)

Sources: Federal Reserve Board, *Industrial Production and Capacity Utilization*

instance, Figure 2 (a) and (b) plot the rate of capacity utilization in the U.S. for the industrial sector and the manufacturing sector, respectively. The Hodrick-Prescott filtered series (dotted lines) are added to capture the long-run variations in the utilization rate. The figures show that the degree of capacity utilization is subject to significant short-run variations but exhibits only mild variations around 80% in the long-run.

In this paper, we use the long run accumulation function (8) to analyze long waves: as long as we are interested in cycles over a fairly long period of time, the assumption that the actual utilization rate is on average at the desired rate is a reasonable approximation.

For the analysis of short cycles, however, the accumulation function (8) cannot be an appropriate specification because the deviation of the actual from the desired rate normally occurs in the short run. Thus, we will use the following specification (9) to describe accumulation behavior during a course of short cycles in section 4.

$$\hat{K} \equiv g = \phi(u); \quad \phi'(u) \gg 0, \quad \phi(u^*) = n \quad (9)$$

The strong positive effect of utilization on accumulation in (9) embodies the Harrodian accelerator principle and the function  $\phi$  is configured so that the desired rate of utilization is consistent with steady growth at a natural rate.<sup>13</sup>

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<sup>13</sup>The specification (9) is clearly an oversimplification since it leaves out other determinants of investment. For instance, it does not capture the direct impact of financial variables such as cash flow and asset prices which are highly emphasized by Minsky (1975, 1982, 1986) and

## 2.2. Banks

In the model, banks' active role in shaping firms' financial structure is represented by equation (7) which reflects both firms' and banks' behavior. For a given profit-interest ratio, equation (7) determines the trajectory of the debt-capital ratio  $m$ . At any moment, the amount of loans supplied to firms will be  $M = mpK$ . I assume that neither households nor firms hold cash, the loan and deposit rates are equal and there are no costs involved in banking. With these assumptions, the amount of loans to the firm sector must equal the total deposits of the household sector.

$$M = M^H \quad (10)$$

where  $M^H$  represents households' deposit holdings.

Banks set the nominal interest rate  $i$ , which is typically affected by inflation. To simplify the analysis, I assume that banks effectively control the real interest rate  $r$ .

## 2.3. Households

Households receive wage income, dividends in return for their stock holdings and interest income. Thus, household real disposable income denoted as  $Y^H$  is given as:  $Y^H = \frac{W + Div + rM^H}{p}$ .

Households hold stocks and deposits and household wealth is denoted as  $NW^H$ , where  $NW^H = \frac{vN^H + M^H}{p}$ . Based on their income and wealth, households make consumption and portfolio decisions. We adopt a conventional specification of consumption function. (e.g. Ando and Modigliani, 1963)

$$C = C(Y^H, NW^H); C_{Y^H} > 0, C_{NW^H} > 0 \quad (11)$$

For simplification, we assume that the function takes a linear form. We then have, after normalizing by capital stock and simple manipulations,

$$\frac{C}{K} = c_1[u\sigma - s_f(\pi u\sigma - \delta - rm)] + c_2q \quad (12)$$

where  $u\sigma - s_f(\pi u\sigma - \delta - rm)$  is household income scaled by capital stock and Tobin's  $q$  captures household wealth.  $c_1$  and  $c_2$  are household propensities to consume out of income and wealth.

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Tobin (1969), as well as current New Keynesian economics (Fazzari et al.(1988) and Bernanke, Gertler and Gilchrist (1996), among others). However, equation (9) can be easily extended to accommodate the effect of those variables without affecting major results of this study. In fact, the effect of cash flow and Tobin's  $q$  on accumulation, it can be shown, reinforces the utilization effect on accumulation embodied in (9). The merit of simple specification in equation (9) is that it shows the underlying mechanisms in a transparent way.

In addition to consumption/saving decisions, households make portfolio decisions. We denote the equity-deposit ratio as  $\alpha$ , where  $\alpha \equiv \frac{vN^H}{MH}$ .

I assume that the composition of households' portfolio is affected by their views on stock market performance. Applying a Minskian hypothesis to household behavior, it is assumed that during good years, households tend to hold a greater proportion of financial assets in the form of riskier assets. In our two-asset framework, equity represents a risky asset and deposits a safe asset. Thus, a rise in fragility during good years is captured by a rise in  $\alpha$ . We introduce a new variable  $z$  to represent the degree of households' optimism about stock markets. We can normalize the variable  $z$  so that  $z = 0$  corresponds to the state where households' perception of tranquility is neutral and there is no change in  $\alpha$ . Given this framework, the evolution of  $\alpha$  is determined by an increasing function of  $z$ .

$$\dot{\alpha} = \zeta(z); \quad \zeta(0) = 0, \quad \zeta'(z) > 0 \quad (13)$$

The next question is what determines households' views about stock markets,  $z$ . It is natural to assume that household portfolio decisions, the division of their wealth into stocks and deposits, will be affected by the difference between the rates of return on stocks and deposits.

Our specification of the process in which households form their views on stock markets emphasizes historical elements in financial markets. Thus, the past trajectories of rates of return on assets as well as those of  $\alpha$  matter in the formation of  $z$ . As a crude approximation of this perception formation process, the following exponential decay specification is introduced:

$$z = \int_{-\infty}^t \exp[-\lambda(t - \nu)] \kappa(r_\nu^e - r, \alpha_\nu) d\nu \quad (14)$$

where  $r^e$  is the real rate of return on equity,  $\kappa_{r^e} \equiv \frac{\partial \kappa(r^e - r, \alpha)}{\partial r^e} > 0$  and  $\kappa_\alpha \equiv \frac{\partial \kappa(r^e - r, \alpha)}{\partial \alpha} < 0$ . In expression (14),  $\kappa(r_\nu^e - r, \alpha_\nu)$  represents the information regarding the state of asset markets at time  $\nu$ . The higher the rate of return on equity relative to the deposit rate of interest, the more optimistic households' view on stock markets becomes ( $\kappa_{r^e} > 0$ ). However, other things equal, a higher proportion of their financial wealth in the form of stock holdings (high  $\alpha$ ) tempers the desire of further increases in equity holdings, i.e.  $\kappa_\alpha < 0$ .

Information on asset markets at different times enters in the formation of  $z$  with different weights. The term,  $\exp[-\lambda(t - \nu)]$ , represents these weights, implying that a more remote past receives a smaller weight in the formation of households' perception of tranquility. Thus,  $\lambda$  may be seen as the rate of loss

of relevance or loss of memory of past events. The higher  $\lambda$ , the more quickly eroded is the relevance of past events.<sup>14</sup>

Differentiation of (14) with respect to  $t$  yields the following differential equation:

$$\dot{z} = \kappa(r^e - r, \alpha) - \lambda z \quad (15)$$

Two dynamic equations (13) and (15), along with the equation describing the evolution of firms' liability structure, (7), are essential building blocks for our model of long waves. To proceed, we need to see how the rate of return on equity,  $r^e$ , is determined.  $r^e$  is defined as follows:

$$r^e \equiv \frac{Div + \Gamma}{vN^H} = \frac{(1 - s_f)(\Pi - \delta pK - rM) + (\hat{v} - \hat{p})vN^H}{vN^H} \quad (16)$$

where  $\Gamma$  is capital gains adjusted for inflation ( $\Gamma \equiv (\hat{v} - \hat{p})vN^H$ ).

The rate of return on equity is determined by stock market equilibrium. Stock market equilibrium requires that the number of shares supplied by firms equals that of shares held by households,  $N = N^H$ , which implies  $\dot{N} = \dot{N}^H$  in terms of the change in the number of shares. Firms issue new shares whenever retained earnings and bank loans fall short of the funds needed to carry their investment plans. Thus firms' finance constraint (1) implies that:

$$\dot{N} = \frac{1}{v}[pI + Div + iM - \Pi - \dot{M}] \quad (17)$$

Simple algebra shows that capital gains can be expressed as follows:

$$\Gamma = (\hat{v} - \hat{p})vN^H = (\hat{\alpha} + \hat{m} + \hat{K})vN^H - v\dot{N}^H \quad (18)$$

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<sup>14</sup>An alternative specification to (13) and (14) is possible. Consider the following specification.

$$\dot{\alpha} = \zeta(\alpha^* - \alpha) \quad (13a)$$

$$\alpha^* = \int_{-\infty}^t \exp[-\lambda(t - \nu)] \bar{\kappa}(r_\nu^e - r) d\nu \quad (14a)$$

where  $\bar{\kappa}'(\cdot) > 0$  and  $\alpha^*$  is the desired equity-deposit ratio. (14a) tells us that households' desired portfolio is determined by the trajectory of the difference between the rates of return on equity and deposit. This desired ratio may not be instantaneously attained so that the adjustment of the actual to the desired ratio takes time. (13a) represents this kind of lagged adjustment of the actual equity-deposit ratio toward the desired ratio. In spite of different interpretations, the two specifications, (13)-(14) and (13a)-(14a), are qualitatively similar. To see this, let  $z \equiv \alpha^* - \alpha$ . Then  $\dot{z} = \dot{\alpha}^* - \dot{\alpha}$ . Differentiating (14a) with respect to  $t$ , we have  $\dot{\alpha}^* = \bar{\kappa}(r^e) - \lambda\alpha^* = \bar{\kappa}(r^e) - \lambda(\alpha + z)$ . Therefore, we can rewrite (13a) and (14a) to:

$$\dot{\alpha} = \zeta(z) \quad (13b)$$

$$\dot{z} = \bar{\kappa}(r^e) - \lambda\alpha - \zeta(z) - \lambda z \quad (15a)$$

One may want to compare (13b)-(15a) with (13)-(15).

$(\hat{\alpha} + \hat{m} + \hat{K})vN^H$  represents the total increase in the real value of stock market wealth<sup>15</sup> but some of the increase is attributed to the increase in the number of shares ( $= v\dot{N}^H$ ). To get the measure of capital gains, the latter should be deducted from the total increase.

Using  $N = N^H$ , substituting (20) in (21) and plugging this result in (19), we get the new expression for  $r^e$ :

$$r^e = \frac{\Pi - iM + \dot{M} + (\hat{\alpha} + \hat{m} + \hat{K})vN^H - pI}{vN^H} \quad (19)$$

Normalizing by  $pK$ , we get the expression for  $r^e$  as a function of  $\pi$ ,  $u$ ,  $m$ ,  $\dot{m}$ ,  $\alpha$  and  $\dot{\alpha}$ :

$$r^e = \frac{\pi u \sigma - \delta - r m + (1 + \alpha)[\dot{m} + m \phi(u)] + \dot{\alpha} m - \phi(u)}{\alpha m} \quad (20)$$

$$\equiv r^e(\pi, u, m, \alpha, \dot{m}, \dot{\alpha}) \quad (21)$$

Substituting this expression in the dynamic equation (15), we have:

$$\dot{z} = \kappa [r^e(\pi, u, m, \alpha, \dot{m}, \dot{\alpha}) - r, \alpha] - \lambda z \quad (22)$$

(22) shows that households' views of tranquility are affected by a number of variables and the relationship is complex. We consider several cases according to the property of (22) in section 3.

#### 2.4. Goods market equilibrium

The equilibrium condition for the goods market is that  $\frac{C}{K} + \frac{I}{K} = \frac{Y}{K}$ , and the definition of  $q$  implies that  $q = (1 + \alpha)m$ . Using these, the equilibrium condition for the goods market can be written as:

$$c_1[u\sigma - s_f(\pi u \sigma - \delta - r m)] + c_2(1 + \alpha)m + \phi(u) + \delta = u\sigma \quad (23)$$

We take the profit share ( $\pi$ ) as endogenous and the equilibrium value of  $\pi$  can be found for given  $u$ ,  $m$  and  $\alpha$ . Explicitly, we have:

$$\pi = \frac{\phi(u) + \delta - (1 - c_1)u\sigma + c_2(1 + \alpha)m + c_1 s_f(\delta + r m)}{c_1 s_f u \sigma} \quad (24)$$

$$\equiv \pi(u, m, \alpha) \quad (25)$$

As  $u$ ,  $m$  and  $\alpha$  evolve over time, the profit share changes as well. The Harrodian investment function adopted in this paper emphasizes a high sensitivity of investment to changes in the utilization. Specifically, it assumes that investment

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<sup>15</sup>Note that  $\hat{\alpha} + \hat{m} + \hat{K} = \hat{v} + \hat{N} - \hat{p}$ .

risers much faster than saving as the utilization rate changes. This Harroddian assumption has an implication for the effect of changes in utilization on profitability: utilization has a positive effect on the profit share and the magnitude will be quantitatively large.<sup>16</sup> The large effect of changes in utilization on the profit share plays an important role in generating short cycles. (See section 4)

It is also readily seen that changes in the debt ratio and the equity-deposit ratio positively affect the profit share. Increases in the debt ratio or the equity-deposit ratio raise consumption demand though changes in disposable income or wealth, thereby increases the profit share.<sup>17</sup>

### 3. Long Waves

This section shows how endogenous changes in firms' and households' financial practices generate long waves. Our model of long waves consists of two subsystems: one describes changes in firms' liability structure and the other specifies changes in households' portfolio composition. Section 3.1 analyzes the evolution of firms' liability structure, assuming households' portfolio composition is frozen. Section 3.2 examines households' portfolio dynamics, given the assumption that firms' liability structure does not change. Section 3.3 combines two subsystems and shows how long waves emerge from the interaction between two subsystems.

#### 3.1. Long-Run Debt Dynamics

This section analyzes the long-run evolution of firms' debt structure. For convenience, I reproduce equation (7).

$$\dot{m} = \tau \left( \frac{\rho_T}{rm} \right) \quad \text{where } \tau'(\cdot) > 0 \quad (7)$$

Regarding the shape of  $\tau$  in (7), Minsky's discussion suggests that the prosperity during tranquil years tends to induce firms and bankers to *gradually* raise the leverage ratio; the rise in the leverage ratio, however, cannot sustain because it worsens the profit/interest relation. Minsky points out that the financial system is prone to crises as the ratio of profit to interest traverses a critical level (Minsky, 1995). The resulting systemic crisis may prompt a rapid de-leveraging process. To capture this idea, we assume that  $\tau'(\cdot)$  takes relatively small positive values within a narrow bound when  $\frac{\rho_T}{rm}$  is above a threshold level (good years),

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<sup>16</sup>If  $\frac{\partial(I/K)}{\partial u} = \phi'(u) > (1-c_1)\sigma + c_1 s_f \pi \sigma = \frac{\partial(S/K)}{\partial u}$ , then  $\frac{\partial \pi}{\partial u} = \frac{\phi'(u) - (1-c_1)\sigma - c_1 s_f \pi \sigma}{c_1 s_f u \sigma} > 0$ .

<sup>17</sup> $\frac{\partial \pi}{\partial m} = \frac{c_1 s_f r + c_2(1+\alpha)}{c_1 s_f u \sigma} > 0$  and  $\frac{\partial \pi}{\partial \alpha} = \frac{c_2 m}{c_1 s_f u \sigma} > 0$

whereas it takes relatively large negative values when  $\frac{\rho_T}{rm}$  is below the threshold level (bad years). When falling profit/interest ratio passes through the threshold level,  $\dot{m}$  sharply falls reflecting a rapid del-everaging process. Thus,  $\tau'(\cdot)$  is likely to be very large when  $\frac{\rho_T}{rm} = \tau^{-1}(0)$ . Figure 3 reflects this assumption.

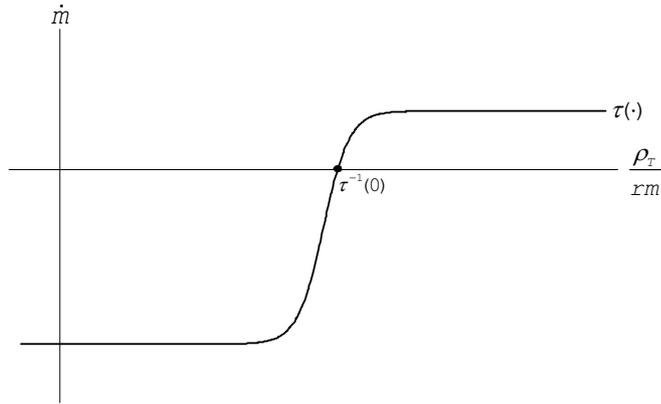


Figure 3: **Debt-Capital Ratio and Profit-Interest Ratio**

As briefly discussed in section 2.1.2, we use the trend rate of profit  $\rho_T$  as a basis of the evolution of firms' liability structure. Behind equation (7) is the idea that firms' liability structure evolves endogenously over time and that the key determinant of the evolution is firms' and banks' perception of tranquility. The level of firms' profit relative to payment commitments on liabilities is an indicator of firms' performance and solvency status. Movements of the profit rate in general include both trend and cyclical components. It seems reasonable to assume that the long-run evolution of firms' liability structure is primarily determined by the trend of the profit rate rather than the current profit rate.<sup>18</sup>

The driving force of the short-run cyclical movements in the current profit rate is changes in capacity utilization while the desired rate,  $u^*$ , provides a good approximation of the long-run average of actual rates of utilization. Thus setting the utilization rate at the desired rate, the short-run cyclical component

<sup>18</sup>This perspective is in line with Minsky's statement that "[T]he inherited debt reflects the history of the economy, which includes a period in the not too distant past in which the economy did not do well. Acceptable liability structures are based on some margin of safety so that expected cash flows, even in periods when the economy is not doing well, will cover contractual debt payments" (Minsky, 1982, 65).

in the profit rate is effectively eliminated, and we have

$$\begin{aligned}\rho_T &= \pi(u^*, m, \alpha)u^*\sigma \\ &= \frac{n + \delta - (1 - c_1)u^*\sigma + c_2(1 + \alpha)m + c_1s_f(\delta + rm)}{c_1s_f}\end{aligned}\quad (26)$$

The trend rate of profit defined as (26) depends positively on the debt-capital ratio  $m$  and the equity-deposit ratio  $\alpha$  ( $\frac{\partial \rho_T}{\partial m} > 0$  and  $\frac{\partial \rho_T}{\partial \alpha} > 0$ ). The profit-interest ratio, the key determinant of the liability structure, is written as

$$\frac{\rho_T}{rm} = \frac{n + \delta - (1 - c_1)u^*\sigma + c_2(1 + \alpha)m + c_1s_f(\delta + rm)}{c_1s_frm}\quad (27)$$

(27) implies that for a given value of  $\alpha$ , the profit-interest ratio is uniquely determined by the debt-capital ratio  $m$ . Minsky's implicit assumption that a rising debt ratio deteriorates the profit/commitment relation can be written as:

$$n + \delta - (1 - c_1)u^*\sigma + c_1s_f\delta > 0\quad (28)$$

The average gross saving rate is typically greater than household marginal propensity to save out of disposable income, and this condition ensures that (28) will be met: if  $\frac{S}{Y} = \frac{I}{Y} = \frac{n+\delta}{u^*\sigma} > (1 - c_1)$ , then  $n + \delta - (1 - c_1)u^*\sigma + c_1s_f\delta > 0$ . Thus, we assume that this condition is satisfied.<sup>19</sup>

Using (7) and (27),  $\dot{m}$  can be written as a function of  $m$  and  $\alpha$ .

$$\dot{m} = \tau \left( \frac{n + \delta - (1 - c_1)u^*\sigma + c_2(1 + \alpha)m + c_1s_f(\delta + rm)}{c_1s_frm} \right) \equiv F(m, \alpha)\quad (29)$$

(29), along with the condition (28), implies that for any value of  $\alpha$ , (i)  $F$  is decreasing in  $m$ , (ii) there exists a unique value of the debt ratio  $m^*(\alpha)$  such that if  $m = m^*(\alpha)$ ,  $\dot{m} = 0$ , and (iii)  $m^*(\alpha)$  depends positively on  $\alpha$ , i.e.  $m^{*\prime}(\alpha) > 0$ . By setting  $\dot{m}$  to zero and solving for  $m$ , we obtain the algebraic expression for  $m^*(\alpha)$ :

$$m^*(\alpha) \equiv \frac{n + \delta - (1 - c_1)u^*\sigma + c_1s_f\delta}{[\tau^{-1}(0) - 1]c_1s_f r - c_2(1 + \alpha)}\quad (30)$$

It is straightforward from properties (i), (ii) and (iii) that (assuming  $\alpha$  constant) our dynamic specification of Minsky's financial instability hypothesis implies that firms' debt structure monotonically converges to a stable fixed point  $m^*$ . The intuition is simple. When the actual debt ratio ( $m$ ) is lower than  $m^*(\alpha)$ , the corresponding profit-interest ratio is greater than the threshold level

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<sup>19</sup>Otherwise, an increase in the debt ratio will raise the profit-interest ratio which leads to a self-repelling process of debt ratio without any ceiling.

at which the debt ratio does not change. This will induce firms to raise the debt ratio. The same kind of event will happen as long as  $m < m^*(\alpha)$ :  $m$  will eventually converge to  $m^*(\alpha)$ . The opposite will happen when the debt ratio is greater than the critical level ( $m > m^*(\alpha)$ ).

Given assumption (28), a *stable* dynamics is inevitable in a one-dimensional continuous time framework. Moving from continuous to discrete time framework may change the picture so that firms' debt dynamics alone can produce long-run cyclical movements. In this paper, however, I explore another avenue toward long waves by integrating firms' debt dynamics into households' portfolio dynamics.

### 3.2. Household Portfolio Dynamics

The other subsystem of our model of long waves, which describes households' portfolio dynamics, consists of two dynamic equations:

$$\dot{\alpha} = \zeta(z) \quad (13)$$

$$\dot{z} = \kappa(r^e - r, \alpha) - \lambda z \quad (15)$$

Analogously to the analysis of firms' debt dynamics, we are interested in the long-run evolution of household portfolio decisions and, to simplify the analysis abstracts from the effect of short-run variations in capacity utilization. The rate of return on equity evaluated at  $u = u^*$  equals

$$r^e|_{u=u^*} = \frac{\rho_T(m, \alpha) - \delta - rm + (1 + \alpha)[F(m, \alpha) + mn] + \zeta(z)m - n}{\alpha m} \quad (31)$$

Given this expression for  $r^e$ , equation (15) becomes

$$\dot{z} = \kappa(r^e|_{u=u^*} - r, \alpha) - \lambda z \equiv G(m, \alpha, z) \quad (32)$$

(13), (29), and (32) constitute a three-dimensional dynamical system. To better understand the mechanics of this three dimensional system, let us take a look at the subsystem (13) and (32), assuming that  $m$  is fixed. By differentiating (32) with respect to  $\alpha$  and  $z$ , the effects of  $\alpha$  and  $z$  on  $\dot{z}$  are given by:

$$G_\alpha = \kappa_{r^e} \frac{\partial r^e}{\partial \alpha} + \kappa_\alpha \begin{matrix} \leq \\ \geq \end{matrix} 0 \quad (33)$$

$$G_z = \kappa_{r^e} \frac{\partial r^e}{\partial z} - \lambda = \kappa_{r^e} \frac{\zeta'}{\alpha} - \lambda \begin{matrix} \leq \\ \geq \end{matrix} 0 \quad (34)$$

The effect of changes in  $\alpha$  on  $z$ ,  $G_\alpha$  in (43), is decomposed into two parts. First, changes in  $\alpha$  affect the rate of return on equity, which influences households'

views on stock markets,  $\kappa_{r^e} \frac{\partial r^e}{\partial \alpha}$ . The effect of an increase in  $\alpha$  on  $r^e$ ,  $\frac{\partial r^e}{\partial \alpha}$ , can be negative or positive in the steady state. Second, an increase in  $\alpha$  mitigates the desire for further increases in equity holdings ( $\kappa_\alpha < 0$ ). Thus, the overall effect depends on the precise magnitude of these two effects.

The effect of  $z$  on  $\dot{z}$  is also unclear. On the one hand, an increase in households' optimism about stock markets accelerates stock holdings, which raises capital gains and the rate of return on equity. The increase in  $r^e$  reinforces their optimism ( $\kappa_{r^e} \frac{\partial r^e}{\partial z} > 0$ ). On the other hand, the degree of optimism will erode at a speed of  $\lambda$ , holding  $r^e$  and  $\alpha$  constant. Thus, the net effect is ambiguous.

Let  $J^H$  be the Jacobian matrix evaluated at the fixed point of (39) and (42). The ambiguity of the signs of  $G_\alpha$  and  $G_z$  yields four cases. Table 1 summarizes it.

Table 1: Classifying Fixed Points

	$G_z < 0$	$G_z > 0$
$G_\alpha < 0$	<b>Case I</b> Stable $\text{Tr}(J^H) < 0$ and $\text{Det}(J^H) > 0$	<b>Case II</b> Unstable $\text{Tr}(J^H) > 0$ and $\text{Det}(J^H) > 0$
$G_\alpha > 0$	<b>Case III</b> Saddle $\text{Tr}(J^H) < 0$ and $\text{Det}(J^H) < 0$	<b>Case IV</b> Saddle $\text{Tr}(J^H) > 0$ and $\text{Det}(J^H) < 0$

A locally stable steady state in the subsystem is obtained when  $G_z$  and  $G_\alpha$  are both negative (Case I). In this case,  $\lambda$  is large relative to  $\kappa_{r^e} \frac{\partial r^e}{\partial z}$ , and  $\kappa_{r^e} \frac{\partial r^e}{\partial \alpha}$  is negative or, if positive, relatively small compared to the absolute value of  $\kappa_\alpha$ . Thus, to get a local stable steady state for households' portfolio dynamics, the positive effect of changes in  $\alpha$  and  $z$  on  $\dot{z}$  via the rate of return on equity needs to remain relatively small in the neighborhood of the steady state.

Moving from Case I, as  $\lambda$  gets smaller than  $\kappa_{r^e} \frac{\partial r^e}{\partial z}$  ( $G_z > 0$ ), keeping the condition  $G_\alpha < 0$ , the steady state becomes locally unstable, yielding Case II. In this case, a high optimism further boosts households' optimistic views on stock markets, creating destabilizing forces. The locally unstable steady state, along with nonlinearities of (13) and (32), can produce limit cycles as long as  $\lambda$  is not too small. Thus, in this case, households' portfolio dynamics alone can generate persistent long waves.

If  $G_\alpha > 0$ , i.e.  $\kappa_{r^e} \frac{\partial r^e}{\partial \alpha}$  is larger than  $|\kappa_\alpha|$ , then the fixed point of the households' portfolio dynamics becomes saddle, regardless of the sign of  $G_z$  (Case III and IV). In both Case III and IV, a high level of equity holdings creates increasing optimism ( $G_\alpha > 0$ ), making the steady state a saddle point. However, Case

IV is distinguished from Case III because it is an exceptional case: it turns out that the destabilizing force in Case IV is too strong to produce a limit cycle for the three dimensional full system ((13), (29), and (32)), whereas, in all other three cases I, II, and III, an appropriate choice of parameter values can produce a limit cycle for the full system. The next section analyzes the full system of long waves.

### 3.3. Full Dynamics: Long Waves

We now put together firms' debt and households' portfolio dynamics and obtain the following three dimensional dynamical system:

$$\dot{m} = F(m, \alpha) \quad (29)$$

$$\dot{\alpha} = \zeta(z) \quad (13)$$

$$\dot{z} = G(m, \alpha, z) \quad (32)$$

Let us first consider the Jacobian matrix of the system evaluated in the steady state.

$$\mathbf{J} = \begin{bmatrix} F_m & F_\alpha & 0 \\ 0 & 0 & \zeta' \\ G_m & G_\alpha & G_z \end{bmatrix} = \begin{bmatrix} - & + & 0 \\ 0 & 0 & + \\ - & +/- & +/- \end{bmatrix} \quad (35)$$

$G_\alpha$  and  $G_z$  are ambiguously signed but the partial derivative of  $G$  with respect to  $m$  is likely to be negative:

$$G_m = \kappa_{r^e} \frac{\partial r^e}{\partial m} \quad (36)$$

where

$$\frac{\partial r^e}{\partial m} = \frac{\left[ \frac{\partial \rho_T}{\partial m} m - \rho_T \right] + (1 + \alpha)mF_m + n + \delta}{\alpha m^2} \quad (37)$$

in the steady state. The sign of (37) may appear to be indeterminate: while  $\frac{\partial \rho_T}{\partial m} m - \rho_T$  is negative due to assumption (28) and  $(1 + \alpha)mF_m$  is negative since  $F_m < 0$ ,  $n + \delta$  is positive. The discussion on the shape of  $\tau(\cdot)$  in section 3.1, however, suggests that  $F_m$  is large in magnitude at the steady state growth path.<sup>20</sup> Thus, at the steady state, the negative terms in the numerator in (37) dominate, and the rate of return on equity will decrease as firms' indebtedness increases in the neighborhood of the steady state. Thus, we have  $G_m = \kappa_{r^e} \frac{\partial r^e}{\partial m} < 0$ .

<sup>20</sup>If  $\tau'(\cdot)$  is large at  $\frac{\rho_T}{\tau m} = \tau^{-1}(0)$ , the derivative of  $F(m, \alpha)$  with respect to  $m$  is strongly negative at  $m = m^*(\alpha)$ , i.e.  $|F_m|$  is large. In a limiting case where the de-leveraging process is instantaneous at  $m^*(\alpha)$ ,  $F_m \rightarrow -\infty$ .

We are interested in the conditions under which the system exhibits limit cycle behavior. As 3.1 and 3.2 showed, the specification of firms' financial decisions, (29), leads to asymptotically stable dynamics, whereas households' portfolio dynamics ((13) and (32)) produces several cases in Table 1. Our analytic result suggests that if households' portfolio dynamics is neither strongly stabilizing nor strongly destabilizing, our baseline system of (13), (29) and (32) tends to generate limit cycles. Our analysis of limit cycles is based on the Hopf bifurcation theorem. The Hopf bifurcation occurs if the nature of the system experiences the transition from stable fixed point to stable cycle as we gradually change a parameter value of a dynamical system (Medio, 1992, section 2.7). I will use  $\lambda$  as the parameter for the analysis of bifurcation.<sup>21</sup> Proposition 1<sup>22</sup> provides the main results of our analysis of long waves:

**Proposition 1.** *Consider the three dimensional system of (12), (29) and (32) and the Jacobian matrix (35) where the partial derivatives are taken at the steady state values. Let*

$$b \equiv \frac{(|F_m|^2 - \zeta'G_\alpha) - \sqrt{(|F_m|^2 - \zeta'G_\alpha)^2 + 4\zeta'|F_m||G_m|F_\alpha}}{2|F_m|} < 0$$

- (I) (**Case I and Case II**) *Suppose that  $G_z < \min\left\{|F_m|, \frac{\zeta'|G_\alpha|}{|F_m|}\right\}$ <sup>23</sup> and  $G_\alpha < 0$ . Then a Hopf bifurcation occurs at  $\lambda = \lambda^* \equiv \kappa_{r,e} \frac{\partial r^e}{\partial z} + |b|$ . As  $\lambda$  falls passing through  $\lambda^*$ , the system with a stable steady state loses its stability, giving rise to a limit cycle.*
- (II) (**Case III**) *Suppose that  $G_z < 0$  and  $0 < G_\alpha < \min\left\{\frac{|F_m||G_z|}{\zeta'}, \frac{F_\alpha|G_m|}{|F_m|}\right\}$ . Then a Hopf bifurcation occurs at  $\lambda = \lambda^* \equiv \kappa_{r,e} \frac{\partial r^e}{\partial z} + |b|$ . As  $\lambda$  falls passing through  $\lambda^*$ , the system with a stable steady state loses its stability, giving rise to a limit cycle.*
- (III) (**Case IV**) *Suppose that  $G_\alpha > 0$  and  $G_z > 0$ . Then the steady state is unstable. There exists no limit cycle by way of Hopf bifurcation.*

Part (I) in the proposition suggests that the existence of a limit cycle requires at least three conditions: first, the mitigation effect of a high proportion

<sup>21</sup> $\lambda$  is particularly useful for the analysis not only because it is of obvious behavioral importance but also because it provides analytic tractability due to the fact that changes in  $\lambda$  do not affect steady state values.

<sup>22</sup>The proof of Proposition I is found in Appendix A but the proof is concerned about only the existence of a limit cycle. The computation of the coefficient that shows whether the limit cycle is stable is very complicated and hard to interpret. Therefore, we extensively use simulation exercises to observe the stability of cycles.

<sup>23</sup>Note that Case I automatically satisfies the second condition since  $G_z < 0$  in Case I.

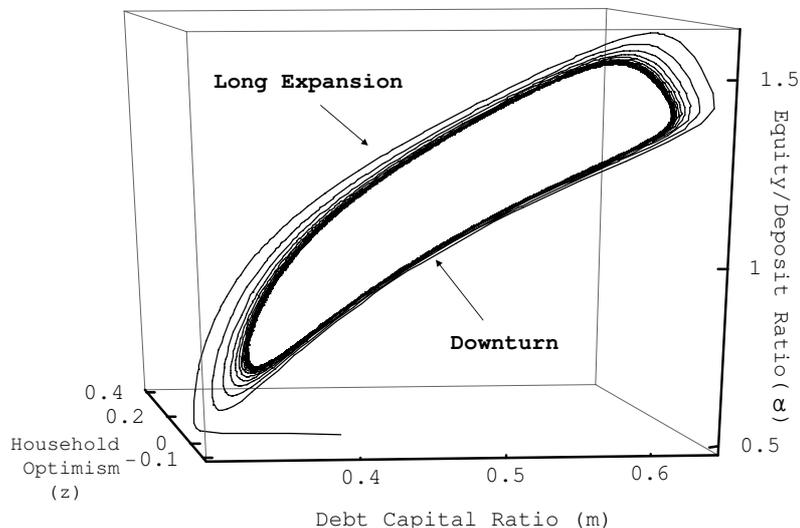


Figure 4: A Limit Cycle in the Model of Long Waves

of equity holdings on increasing optimism ( $|\kappa_\alpha|$ ) is sufficiently large so that  $G_\alpha < 0$ <sup>24</sup>; second, households' optimistic or pessimistic view on stock markets is not excessively persistent ( $G_z < \min \left\{ |F_m|, \frac{\zeta' |G_\alpha|}{|F_m|} \right\}$ ); third, the rate of loss of relevance of past events ( $\lambda$ ) should not be too large ( $\lambda < \lambda^*$ ).<sup>25</sup> The second and third conditions imply that for the existence of a limit cycle,  $\lambda$  should be of appropriate magnitude:

$$\kappa_{r^e} \frac{\partial r^e}{\partial z} - \min \left\{ |F_m|, \frac{\zeta' |G_\alpha|}{|F_m|} \right\} < \lambda < \kappa_{r^e} \frac{\partial r^e}{\partial z} + |b| \quad (38)$$

All of these conditions imply that to get a limit cycle, households' portfolio dynamics should be neither strongly stabilizing nor strongly destabilizing.

One interesting aspect of Part (I) in Proposition I is that the interaction between two *stable* subsystems - firms' debt and households' portfolio dynamics - can generate an unstable steady state and a limit cycle (Case I). Thus, in this case, the source of the resulting long waves lies purely in the *interaction* between both firm and household sectors. Figure 4 depicts the emergence of a limit cycle

<sup>24</sup>Or the positive effect of changes in  $\alpha$  on  $\dot{z}$  via its effect on the rate of return on equity should not be too large.

<sup>25</sup>If  $\lambda$  exceeds  $\lambda^*$ , then the system will be stabilized.

in this case in a three dimensional space. Figure 5 shows the trajectories of the debt-capital ratio and the equity-deposit ratio in this case.<sup>26</sup>

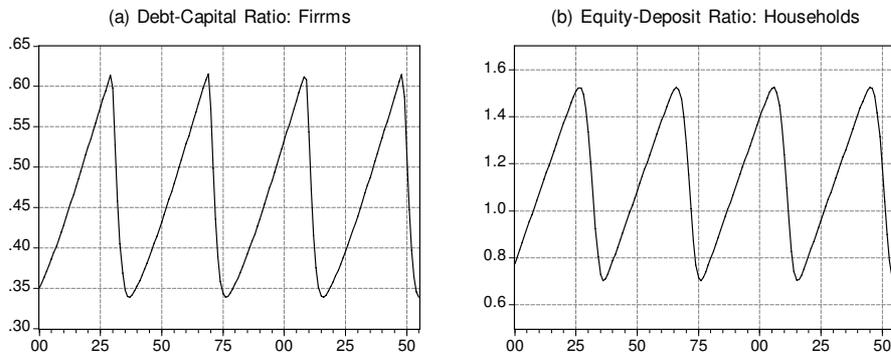


Figure 5: **Long Waves**

The debt-capital ratio and the equity-deposit ratio steadily increase during a long boom.<sup>27</sup> This expansion, however, is followed by a sharp fall in  $m$  and  $\alpha$ , which have significant negative impacts on effective demand and trigger an abrupt downturn in the real sector (See section 4 below).

Part (I) also covers Case II where the subsystem of households' portfolio dynamics is unstable. As shown in 3.2, in Case II, portfolio dynamics alone can create a limit cycle. Part (I) in the proposition suggests that the system can still have a limit cycle when the portfolio dynamics is combined with firms' debt dynamics. Then what is the implication of introducing the debt dynamics into portfolio dynamics? The qualitative analysis does not tell much about the answer to this question. Numerical experiments, however, provide a case in which the amplitude and period of long waves get significantly larger as we

<sup>26</sup>The functions and parameter values for this simulation, which are also used for the simulation in section 5, are found in Appendix B. A sufficiently long period of time (from  $t = 0$  to  $t = 30000$ ) is taken in all simulation exercises in this paper.

<sup>27</sup>Figure 5 (b) shows the steady increase in the equity-deposit ratio during a long expansion. This implies that firms' debt/equity ratio *steadily falls* during a expansion (Note that firms' stock of debt is always equal to household deposits in this model. Thus, firms' debt/equity ratio is given by  $1/\alpha$ ). Minsky often uses the debt/equity ratio to refer to the degree of indebtedness. The result in this paper, however, shows that rising indebtedness, measured by the debt-capital ratio ( $m$ ), is consistent with falling debt-equity ratio ( $1/\alpha$ ). Interestingly, Lavoie and Seccareccia (2001) question the empirical relevance of Minsky's Financial Instability Hypothesis based on their finding that the debt-equity ratio is not procyclical. The result of this paper suggests that Minsky's Instability Hypothesis does not necessarily imply the procyclical movement of debt-equity ratio.

move from the 2D subsystem of portfolio dynamics to the full 3D system.

Part (II) in the proposition concerns Case III where the household portfolio subsystem yields a saddle point steady state. Thus, this part of Proposition 1 shows how stabilizing debt dynamics and households' portfolio dynamics with saddle property are combined to produce a limit cycle. Not surprisingly, not all saddle cases can generate a limit cycle. First, the destabilizing effect that makes the fixed point in the 2D household subsystem saddle – the magnitude of  $G_\alpha$  – should be mild:  $G_\alpha < \min \left\{ \frac{|F_m||G_z|}{\zeta}, \frac{F_\alpha|G_m|}{|F_m|} \right\}$ . Second,  $G_z$  should be negative. If it is positive ( $G_z > 0$ ), the condition for the saddle point,  $G_\alpha > 0$ , eliminates the possibility of the emergence of a limit cycle *a la* the Hopf bifurcation. Proposition 1-(III) makes this point. Intuitively, if *both*  $G_\alpha > 0$  and  $G_z > 0$  (Case IV), the portfolio dynamics in the household sector is excessively destabilizing in the sense that stabilizing forces in firms' debt dynamics cannot contain such a strong destabilizing effect.

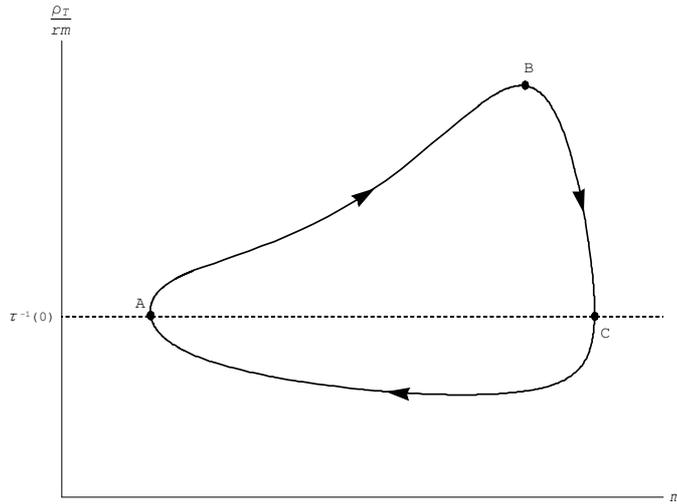


Figure 6: **The relationship between the debt-capital ratio and the profit-interest ratio**

To understand the mechanism behind the long waves, it is illuminating to compare the full system with the subsystem of debt dynamics. As seen in section 3.1, with households' portfolio composition ( $\alpha$ ) fixed, the debt-capital ratio ( $m$ ) monotonically converges to its steady state value  $m^*(\alpha)$ . The main reason for this convergence was the inverse relation between  $m$  and  $\frac{\rho_T}{r m}$ : a rising debt-capital ratio deteriorates firms' profit-interest ratio for any given  $\alpha$ . However,

once households' portfolio composition evolves endogenously, this kind of strict inverse relationship breaks down because changes in  $\alpha$  also affect  $\frac{\rho\tau}{rm}$ .

Figure 6 illustrates this point, where the horizontal dotted line represents the threshold level ( $= \tau^{-1}(0)$ ) of the profit-interest ratio that makes  $\dot{m}$  zero. In the area above the horizontal line, the debt-capital ratio increases and in the area below the line, it decreases. With  $\alpha$  held fixed, the movement along the curve AB is not possible since for any given  $\alpha$ , a rise in  $m$  is incompatible with a rise in  $\frac{\rho\tau}{rm}$ . However, increases in  $\alpha$  fueled by households' optimism during an expansion have a positive effect on the profit-interest ratio by raising aggregate demand. Thus, from A to B, the economy experiences increases in both  $\alpha$  and  $m$ .<sup>28</sup> However, households' optimistic views on stock markets eventually fade as both  $m$  and  $\alpha$  increase. As a result, the negative effect of a rise in the debt ratio starts to be dominant at some point and the profit-interest ratio begins falling (point B). Because the profit-interest ratio is still above the threshold level, the debt ratio keeps increasing and the profit-interest ratio falls along the curve BC. When the profit-interest ratio passes through point C, the debt-capital ratio starts to fall. When the economy reaches point A, a new cycle begins.

Figure 7 depicts the same story from a slightly different angle. The solid line plots a trajectory of the actual debt-capital ratio over time and the dotted line a trajectory of the desired debt ratio ( $m^* \equiv m^*(\alpha)$  in (30)). For a given value of  $\alpha$ , the debt dynamics, (29), implies that the actual debt ratio  $m$  tends to gravitate toward the desired ratio  $m^*(\alpha)$ . However, when  $\alpha$  changes, the desired ratio becomes a *moving* target of the actual ratio. From this view, a period of expansion (contraction) is the time when the actual ratio is below (above) the desired ratio, i.e.  $m < m^*$  ( $m > m^*$ ) and consequently the actual debt ratio is increasing (decreasing). In words, a stock market boom (rising  $\alpha$ ) tends to raise the tolerable level of the debt-capital ratio which the actual ratio is chasing. When the relation between  $m$  and  $m^*$  is reversed, a long downturn begins (See point C in Figure 7).

#### 4. A Model of Short Cycles

The model of long waves in section 3.3 can be combined with a model of short cycles. In our analysis of long waves, the degree of capacity utilization is set at its long run average. However, when it comes to short cycles, the

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<sup>28</sup>The positive effect of the rise in  $\alpha$  on the profit-interest ratio dominates the negative effect of the rise in  $m$  and consequently the profit-interest ratio also increases during this period.

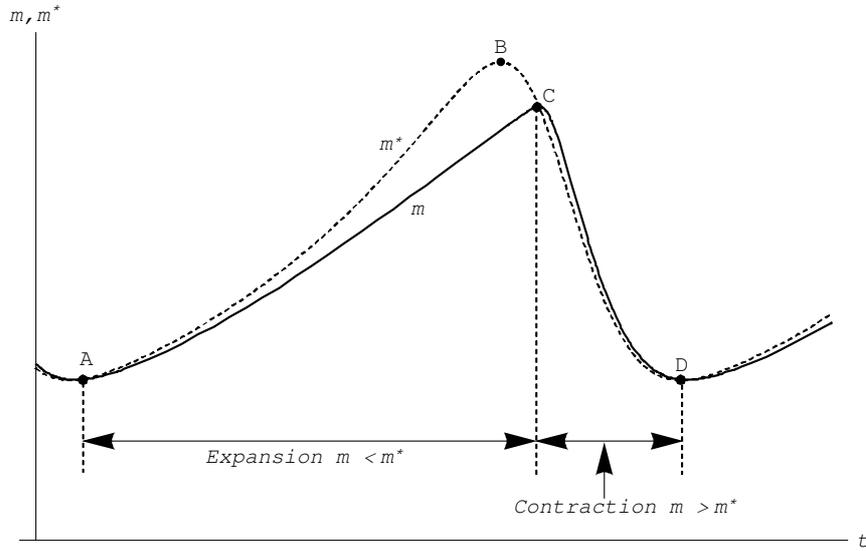


Figure 7: **Actual and Desired Debt Ratios**

utilization rate can deviate from the desired rate due to falsified demand expectations and slow adjustment of capital stocks. Thus, we use equation (9) for our analysis of short cycles. In 2.4, using this accumulation function (9) and the consumption function (12), we derived the profit share that ensures the goods market equilibrium, which depends positively on  $u$ ,  $m$ , and  $\alpha$ . (See (24))

Regarding firms' pricing/output decisions, this paper adopts a Marshallian approach elaborated in Skott (1989). The Keynesian literature often assumes that prices are sticky while output adjusts instantaneously and costlessly to absorb demand shocks but the Marshallian approach assumes the opposite. Output does not adjust instantaneously due to a production lag and substantial adjustment costs.<sup>29</sup> In this framework, fast adjustments in prices and the profit share establish product market equilibrium for a given level of output. In a continuous-time setting, sluggish output adjustment can be approximated by assuming that output is predetermined at each moment and that firms choose the rate of growth of output, rather than the level of output. Then output growth is determined by comparing the costs and benefits involved in the output

<sup>29</sup>For instance, increases in production and employment require substantial search, hiring and training costs. Hiring or layout costs include not only explicit costs but also hidden costs such as a deterioration in industrial relations and morale.

adjustment which in turn are determined by the labor market conditions and the profit signal in the goods market, respectively. Thus we can formulate:

$$\hat{Y} = h(\pi, e); \quad h_\pi > 0, \quad h_e < 0 \quad (39)$$

where  $e$  is the employment rate. A higher profitability induces firms to expand output more rapidly whereas the tightened labor market gives firms negative incentives to expand production.<sup>30</sup> Assuming a fixed-coefficient Leontief technology,  $Y = \min\{\sigma K, \nu L\}$ , the employment rate can be expressed as:  $e = \frac{Y/\nu}{\bar{L}}$ , where  $\nu$  is constant labor productivity and  $\bar{L}$  is available labor force which exponentially grows at a constant natural rate  $n$ . From this definition,

$$\hat{e} = \hat{Y} - n \quad (40)$$

The definition of  $u$  yields:

$$\hat{u} = \hat{Y} - \hat{K} \quad (41)$$

Putting together (9), (24), (39), (40) and (41), we get the following system of short cycles.

$$\hat{u} = h(\pi(u, m, \alpha), e) - \phi(u) \quad (42)$$

$$\hat{e} = h(\pi(u, m, \alpha), e) - n \quad (43)$$

When  $m$  and  $\alpha$  are fixed, the system of (42) and (43) exhibits essentially the same dynamic properties as Skott (1989). As Skott shows, under plausible assumptions, the system of (42) and (43) ensures the existence of a steady growth equilibrium and the steady state is locally asymptotically unstable unless the negative effect of employment on output expansion is implausibly large. Once the boundedness of the trajectories is proved, the system (42) and (43) will generate a limit cycle *a la* the Poincare-Bendixson theorem (See Skott 1989, Appendix 6C for the proof).

## 5. Putting all together: Long Waves and Short Cycles

This section puts all elements together in order to integrate long waves with short cycles and presents our simulation results.<sup>31</sup> Our full model of long waves and short cycles is a five dimensional dynamical system that consists of (12),

<sup>30</sup>For more details about the behavioral foundation of (39), see Skott (1989, Ch.4).

<sup>31</sup>Parameter values and functions used for this simulation are available in Appendix B. The simulation in this section is based on Case I in Table 1. Simulation results in other cases are available upon request.

(29), (32), (42), and (43). We have seen that (12), (29), and (32) provide a model of long waves, whereas (42) and (43) generate a mechanism of short cycles.<sup>32</sup>

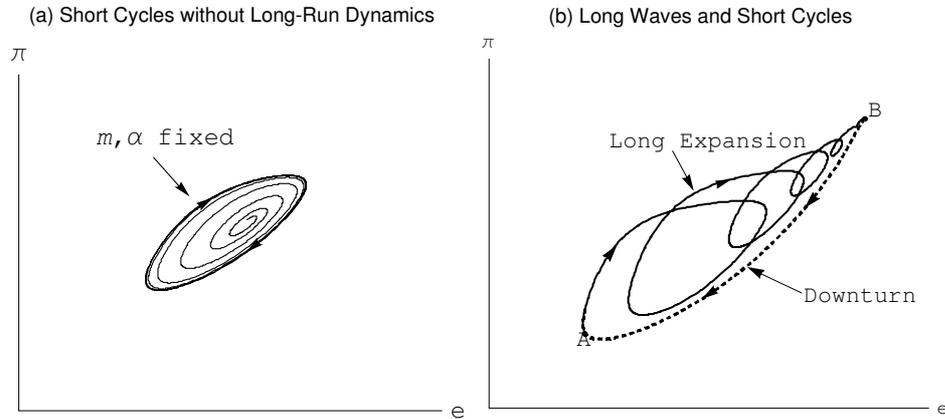


Figure 8: **Comparison**

As seen in section 4, if  $m$  and  $\alpha$  are fixed, (42) and (43) produce a limit cycle under plausible conditions. It can be shown that the resulting limit cycle exhibits a clockwise movement on the  $e-u$  space, or alternatively, the  $e-\pi$  space. Figure 8 (a) presents an example of the limit cycle on the  $e-\pi$  space. The system of (12), (29) and (32), however, generates long waves of the debt-capital ratio ( $m$ ) and the equity-deposit ratio ( $\alpha$ ), which are represented in Figure 5. As  $m$  and  $\alpha$  change endogenously, the limit cycle in Figure 8 (a) breaks down and the clockwise movement of  $e$  and  $\pi$  spirals up to the northeast or down to the southwest, depending on the direction of changes in  $m$  and  $\alpha$ . Figure 8 (b) illustrates this. The upward spiral from A to B represents a long expansion driven by increases in the debt-capital ratio and the equity-deposit ratio, whereas the downward spiral from B to A an economic downturn prompted by sharp decreases in  $m$  and  $\alpha$ .

During each long expansion, the profit share exhibits a strong upward movement with mild cyclical fluctuations around the trend (Figure 9 (a)). The similar

<sup>32</sup>By using (26) as our definition of trend profitability based on  $u = u^*$ , the system of long waves becomes independent of that of short cycles, while the latter depends on the former. This kind of unilateral dependence can be relaxed by adopting an alternative formulation of trend profitability, without affecting the qualitative results. For instance, we can use a weighted moving average of current profit rates as a measure of the trend rate of profit (See Appendix C).

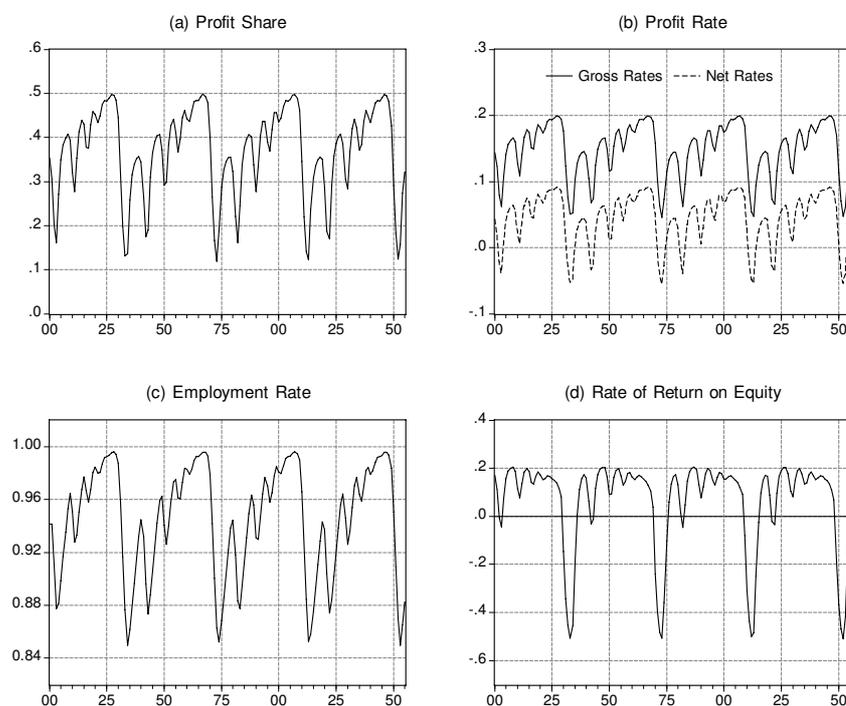


Figure 9: **Simulation Paths I**

pattern characterizes the movements in the profit rates (Figure 9 (b)). During crises, the rate of profit net of depreciation and interest payment ( $\pi u \sigma - \delta - r m$ ) tumbles even to negative rates. A change in the debt structure have large impacts on the real sector performance through its effect on the profitability. This is prominently shown in the behavior of the employment rate (Figure 9 (c)). Figure 9 (d) depicts a trajectory of the rate of return on equity. During long booms, the rate of return on equity is strong and sound on average but during crises, it suddenly drops to significantly negative rates.

Figure 10 (b) shows the growth rate of output where the Hodrick-Prescott filtered trend is added.<sup>33</sup> A financial sector induced crisis triggers a deep recession in the real sector which is reflected in the negative growth rates during

<sup>33</sup>The filtered series is only for illustrative purpose since it simply smoothes the original series and it does not adequately capture asymmetric features and structural breaks in the original series.

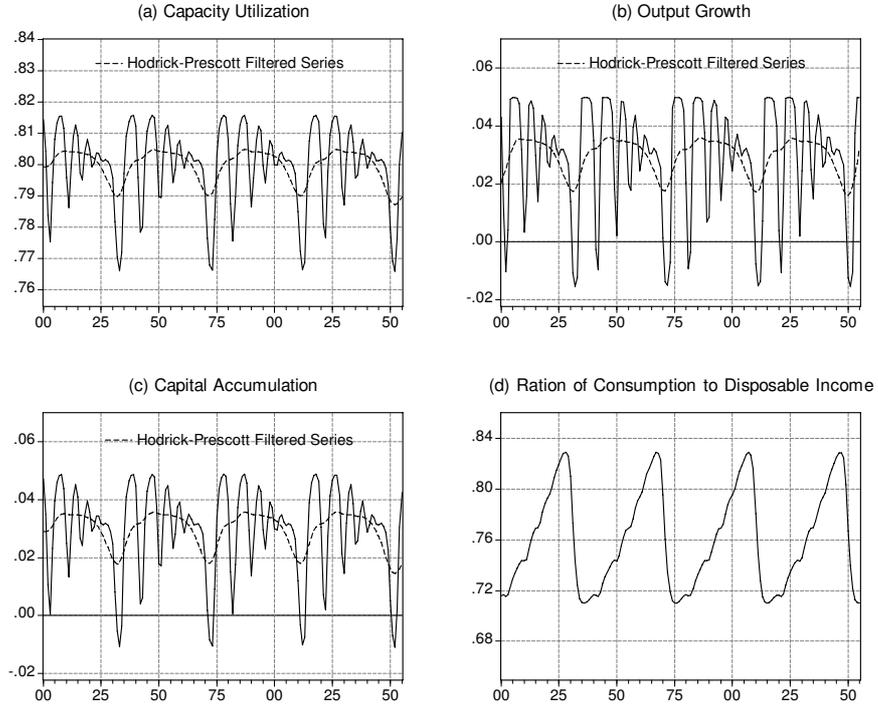


Figure 10: **Simulation Paths II**

periodic deep downturns. Capacity utilization and capital accumulation follow the pattern similar to that of output growth (Figure 10 (a) and (c)). Figure 10 (d), finally, plot the ratio of consumption to household income. The series follows the basic long waves/short cycles pattern as shown in the profit share and the employment rate but the movement in the consumption/income ratio is noticeably smooth compared to other simulated series.<sup>34</sup>

## 6. Conclusion

The U.S. economy is going through a deep recession triggered by the biggest financial crisis since the Great Depression. A Minskian perspective suggests

<sup>34</sup>The long run behavior of consumption is closely related to the movement in household net worth to income ratio:  $\frac{C}{Y^H} = \frac{c_1 Y^H + c_2 NW^H}{Y^H} = c_1 + c_2 \frac{NW^H}{Y^H}$  where  $\frac{NW^H}{Y^H} = \frac{(1+\alpha)m}{u\sigma - s_f(\pi u\sigma - \delta - r m)}$ .

that the explanation of this crisis should be found in endogenous changes in financial fragility.

This study has modeled a Minskian theory of long waves. The model clarifies the underlying mechanism of endogenous changes in financial fragility and the interaction between real and financial sectors. At a theoretical level, the study provides a promising way of integrating two types of instability principles: Minsky's Financial Instability Hypothesis and Harrod's Instability Principle. While both principles provide a source of cycles, they have distinct frequencies and amplitudes in this model. The Minskian instability hypothesis creates long waves and the Harrodian instability principle produces short cycles. The limit to the upward trend created by Minskian instability is imposed by financial crisis, while explosive trajectories implied by Harrodian instability are contained by stabilizing labor market dynamics.<sup>35</sup> When two principles are combined into a coherent stock-flow consistent framework, the proposed pattern of long waves and short cycles emerges.

A purely mathematical model of this kind may clarify the logic of interactions but clearly has many limitations. The depth of the current crisis and the time needed to initiate a new cycle depend on institutional and policy dimensions. Minsky devotes a large part of his analysis to the institutional and historical developments of financial markets and policy responses. Thus, the patterns of long waves are heavily affected by these elements. The full account of long waves and crises is possible only when one takes a serious look at these dimensions.

Disregarding the historical contingencies of actual movements, it may be useful to extend the model in a number of directions. First, it may be desirable to explicitly treat the banking sector as an active profit-seeking unit. Bankers' perception of tranquility, possibly affected by their own profitability, may not always agree with those of the firm and household sectors.<sup>36</sup> Next, this paper did not explore the implications of households' indebtedness. Instead, it has focused on an increasing share of stocks (riskier asset) in households' financial wealth as an indicator of increasing fragility in the household sector. It would be interesting to see the effect of the introduction of the evolution of household

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<sup>35</sup>The following quote from Minsky (1995, 84) is suggestive: "As reasonable values of the parameters of the endogenous interactions lead to an explosive endogenous process, and as explosive expansions and contractions rarely occur, then constraints by devices such as the relative inelasticity of finance or an inelastic labor supply need to be imposed and be effective in generating what actually happens."

<sup>36</sup>Setterfield (2004) assumes that the private sector (the aggregate of firm and household sectors) and the banking sector have different fragility functions but does not try to justify the assumed shapes of those functions.

debt into the model.<sup>37</sup> Third, the proposed model is inflation neutral in the sense that the decisions on real quantities such as investment, consumption and output expansion are made with no reference to inflation and the banking sector holds the real interest rate at a constant level. In some account of Minskian ideas (e.g see Fazzari et al., 2008), changes in the inflation rate play an important role. Finally, the assumption of a closed economy in this paper is another major limitation. Unfettered international capital flows, in contrast to the belief of its proponents, have created growing instability and global imbalances (Blecker, 1999). Several authors suggest that Minsky's theory can be extended to an international context (e.g. Wolfson, 2002), but few attempt has been made to formalize the ideas and to propose precise mechanisms behind them. Addressing these issues is left for future research.

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<sup>37</sup>To introduce this aspect, the model may have to be extended to allow heterogeneity among households as long as the household sector as a whole is in a net credit position.

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## Appendix A: Proof of Proposition 1

To prove the existence of a limit cycle for the system of (12), (29), and (32), we need to show that the Jacobian matrix (35) evaluated at  $(m(\lambda), \alpha(\lambda), z(\lambda), \lambda)$ , where  $(m(\lambda), \alpha(\lambda), z(\lambda))$  is a fixed point of the system,<sup>38</sup> should have the following properties:

- The Jacobian matrix has a pair of complex conjugate eigenvalues  $\beta(\lambda) \pm \theta(\lambda)i$  such that  $\beta(\lambda^*) = 0$ ,  $\theta(\lambda^*) \neq 0$ , and  $\beta'(\lambda^*) \neq 0$  and no other eigenvalues with zero real part exist at  $(m(\lambda^*), \alpha(\lambda^*), z(\lambda^*), \lambda^*)$

where  $\lambda^*$  is a Hopf bifurcation point.

To apply the above condition for the Hopf bifurcation to the current context, I will use the fact that the Jacobian matrix will have a negative real root and a pair of pure imaginary roots if and only if:

$$(R1) \quad \mathbf{Tr}(\mathbf{J}) = F_m + G_z < 0$$

$$(R2) \quad \mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3 = F_m G_z - \zeta' \cdot G_\alpha > 0$$

$$(R3) \quad \mathbf{Det}(\mathbf{J}) = -\zeta' \cdot (F_m G_\alpha - F_\alpha G_m) < 0$$

$$(R4) \quad -\mathbf{Tr}(\mathbf{J})(\mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3) + \mathbf{Det}(\mathbf{J}) = -(F_m + G_z)(F_m G_z - \zeta' \cdot G_\alpha) - \zeta' \cdot (F_m G_\alpha - F_\alpha G_m) = 0$$

Let us denote the eigenvalues of the Jacobian matrix as  $\mu(\lambda)$  and  $\beta(\lambda) \pm \theta(\lambda)i$ .

**Proof of (I).** Suppose that  $G_\alpha < 0$ . Then (R3) is always met. In order to satisfy (R1) and (R2), we should have  $G_z < \min\left\{|F_m|, \frac{\zeta'|G_\alpha|}{|F_m|}\right\}$ . (R4) is quadratic in  $G_z$ . (R4) can be rewritten as:

$$a_1 G_z^2 + a_2 G_z + a_3 = 0 \tag{A1}$$

where

$$\begin{aligned} a_1 &\equiv -F_m > 0 \\ a_2 &\equiv -(F_m^2 - \zeta' G_\alpha) \leq 0 \\ a_3 &\equiv \zeta' F_\alpha G_m < 0 \end{aligned}$$

Solving (A1) for  $G_z$ , we obtain one negative and one positive real roots. Let us select the negative root<sup>39</sup>, which is given as:

<sup>38</sup>Note that in our case the fixed point is independent of the value of  $\lambda$ .

<sup>39</sup>It can be shown that the positive root is irrelevant for the analysis.

$$b \equiv \frac{(|F_m|^2 - \zeta' G_\alpha) - \sqrt{(|F_m|^2 - \zeta' G_\alpha)^2 + 4\zeta'|F_m||G_m|F_\alpha}}{2|F_m|} < 0 \quad (\text{A2})$$

Since  $G_z = \kappa_{r^e} \frac{\partial r^e}{\partial z} - \lambda$ , the value of  $\lambda$  that satisfies (R4) is:  $\lambda = \kappa_{r^e} \frac{\partial r^e}{\partial z} + |b|$ . Let  $\lambda^* \equiv \kappa_{r^e} \frac{\partial r^e}{\partial z} + |b|$ . We have shown that if  $G_z < \min \left\{ |F_m|, \frac{\zeta'|G_\alpha|}{|F_m|} \right\}$  and  $\lambda = \lambda^*$ , then the Jacobian matrix has a negative real root and a pair of imaginary roots:  $\mu(\lambda^*) < 0$ ,  $\beta(\lambda^*) = 0$ , and  $\theta(\lambda^*) \neq 0$ . To prove  $\lambda^*$  is indeed the bifurcation point, we still need to show that  $\beta'(\lambda^*) \neq 0$ . To prove  $\beta'(\lambda^*) \neq 0$ , let us use the following fact:

$$\begin{aligned} \mu(\lambda) + 2\beta(\lambda) &= F_m + G_z \\ 2\mu(\lambda)\beta(\lambda) + \beta(\lambda)^2 + \theta(\lambda)^2 &= F_m G_z - \zeta' \cdot G_\alpha \\ \mu(\lambda)[\beta(\lambda)^2 + \theta(\lambda)^2] &= -\zeta' \cdot (F_m G_\alpha - F_\alpha G_m) \end{aligned}$$

Totally differentiating both sides with respect to  $\lambda$ , we get

$$\begin{bmatrix} 1 & 2 & 0 \\ 2\beta(\lambda) & 2[\mu(\lambda) + \beta(\lambda)] & 2\theta(\lambda) \\ [\beta(\lambda)^2 + \theta(\lambda)^2] & 2\mu(\lambda)\beta(\lambda) & 2\mu(\lambda)\theta(\lambda) \end{bmatrix} \begin{bmatrix} \mu'(\lambda) \\ \beta'(\lambda) \\ \theta'(\lambda) \end{bmatrix} = \begin{bmatrix} -1 \\ |F_m| \\ 0 \end{bmatrix} \quad (\text{A3})$$

The right hand side of (A3) is obtained using the fact that  $\frac{\partial G_z}{\partial \lambda} = -1$  and  $\lambda$  does not affect all other partial derivatives than  $G_z$ . Evaluating (A3) at  $\lambda = \lambda^*$ , we have:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2\mu(\lambda^*) & 2\theta(\lambda^*) \\ \theta(\lambda^*)^2 & 0 & 2\mu(\lambda^*)\theta(\lambda^*) \end{bmatrix} \begin{bmatrix} \mu'(\lambda^*) \\ \beta'(\lambda^*) \\ \theta'(\lambda^*) \end{bmatrix} = \begin{bmatrix} -1 \\ |F_m| \\ 0 \end{bmatrix}$$

Solving this for  $\beta'(\lambda^*)$ , we finally get:

$$\beta'(\lambda^*) = \frac{2\mu(\lambda^*)\theta(\lambda^*)|F_m| - 2\theta(\lambda^*)^3}{4\mu(\lambda^*)^2\theta(\lambda^*) + 4\theta(\lambda^*)^3} < 0 \quad \text{since } \mu(\lambda^*) < 0$$

Thus,  $\beta'(\lambda^*)$  is strictly negative.

**Proof of (II).** Suppose that  $G_\alpha > 0$  and  $G_z < 0$ . Then (R1) is always satisfied. To meet (R2) and (R3), we need  $G_\alpha < \min \left\{ \frac{|F_m||G_z|}{\zeta'}, \frac{F_\alpha|G_m|}{|F_m|} \right\}$ . The rest of the proof is essentially the same as that of (I).

***Proof of (III).*** Routh-Hurwitz necessary and sufficient conditions for the local stability of a three dimensional system are (R1), (R2) and (R3) with replacing the equality in (R4) by the inequality:  $-\mathbf{Tr}(\mathbf{J})(\mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3) + \mathbf{Det}(\mathbf{J}) > 0$ . Suppose that  $G_\alpha > 0$  and  $G_z > 0$ . Then (R2) is always violated and the fixed point is unstable. At the same time, since (R2) is not met, it is impossible to get a limit cycle *a la* the Hopf bifurcation.

## Appendix B: Functions and Parameter Values in Simulation

$$g = \gamma_0 + \gamma_1 u \quad (\text{B1})$$

$$\frac{I}{K} = g + \delta \quad (\text{B2})$$

$$\hat{Y} = h(\pi, e) = h_0 + \frac{h_1}{1 + \exp[-h_2(\pi + h_3 \ln(h_4 - e) + h_5)]} \quad (\text{B3})$$

$$\dot{m} = \tau \left( \frac{\rho_T}{rm} \right) = \tau_0 + \frac{\tau_1 - \tau_0}{1 + \exp[-\tau_2 \left( \frac{\rho_T}{rm} - \tau_3 \right)]} \quad (\text{B4})$$

$$\text{where } \rho_T = \pi(u^*, m, \alpha) u^* \sigma \text{ and } u^* = \frac{1}{\gamma_1} (n - \gamma_0)$$

$$\dot{\alpha} = \zeta(z) = \zeta_0 + \frac{\zeta_1 - \zeta_0}{1 + \exp[-\zeta_2(z - \zeta_3)]} \quad (\text{B5})$$

$$\dot{z} = \kappa(r^e|_{u=u^*} - r, \alpha) - \lambda z = \kappa_0 + \kappa_1(r^e|_{u=u^*} - r) - \kappa_2 \alpha - \lambda z \quad (\text{B6})$$

$$\text{where } r^e|_{u=u^*} = \frac{\rho_T - \delta - rm + (1 + \alpha)(\dot{m} + mn) + \dot{\alpha}m - n}{\alpha m}.$$

Table 2: Parameter Values

$\gamma_0$	$\gamma_1$	$h_0$	$h_1$	$h_2$	$h_3$	$h_4$	$h_5$	$\sigma$
-0.93	1.2	-0.02	0.07	50	0.4	1.1	0.423	0.5
$n$	$\delta$	$s_f$	$r$	$c_1$	$c_2$	$\kappa_0$	$\kappa_1$	$\kappa_2$
0.03	0.09	0.7	0.03	0.65	0.04	0.093	0.03	0.08
$\lambda$	$\zeta_0$	$\zeta_1$	$\zeta_2$	$\zeta_3$	$\tau_0$	$\tau_1$	$\tau_2$	$\tau_3$
0.04	-0.24	0.03	30	-0.069	-0.135	0.01	20	10.4

## Appendix C: Alternative Measure of the Trend Rate of Profit

A weighed moving average specification may provide an alternative measure of the trend rate of profit:

$$\rho_T = \int_{-\infty}^t \eta \exp[-\eta(t - \nu)] \rho_\nu d\nu \text{ where } \mu > 0 \quad (\text{C1})$$

where  $\rho_\nu$  is the current rate of profit at each moment of time  $\nu \in (-\infty, t]$  and  $\eta \exp[-\eta(t - \nu)]$  represents the weight attached to  $\rho_\nu$  in the calculation of the

trend rate of profit at time  $t$ , which exponentially decreases as  $\nu$  gets further back to the past. This specification implies that the trend profit rate is constantly updated based on the following averaging process.

$$\dot{\rho}_T = \eta(\rho - \rho_T)$$

where  $\rho = \pi(u, m, \alpha)u\sigma$ . Note that the expression for the current profit rate  $\rho$  includes capacity utilization ( $u$ ) as well as the debt ratio ( $m$ ) and the equity-deposit ratio ( $\alpha$ ). Thus, the system of short cycles and that of long waves become interdependent.

The two specifications, (26) and (C1), produce qualitatively similar results. The basic idea behind both specifications is to smooth actual profitability and get a measure of the long-run trend of profitability and one would expect the two specifications to produce qualitatively similar results. Simulations confirm that this is indeed the case (Simulation results based on (C1) is available upon request.) Analytically, the specification (26) is more tractable and the analysis in this paper has been based on (26).