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The dynamics of resource spending in a competition between political parties: general notes on the Red Queen effect

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by

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Abstract

Competition between political parties is a process that unfolds over time whereas formal theories of party competition have tended to take an essentially static, or one-shot, approach. This leaves some gaps in our understanding of the dynamics of campaigning. The aim of this paper is to make up some of this gap. This is done using a differential game theory model to analyse a situation in which support for a party depends on the amount spent on marketing relative to the expenditure of the other party. One of the main results is that, even when voters are not myopic, the logic of the competition forces parties to accelerate expenditure on campaigning during the period between elections.

Key words: party competition, dynamics, differential games.

JEL classification: C61, C72, C73, D72.

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1 Introduction

Social scientists are fully aware that competition between political parties or candidates for election is a dynamic process in which resources, such as energy and funds, are spent over time in trying to win voter support but, so far, most existing studies of campaign expenditure are static and we have very little by way of a theoretical understanding of these dynamics. This is, in fact, part of a larger shortcoming in the analysis of the dynamics of electoral competition.[1] This paper attempts to help fill some of the gap in our formal theories. It is concerned with the dynamics of resources spending in the period between elections. I fill out the term 'resources' subsequently.

The types of questions that are of concern might be illuminated, in a preliminary way, by considering some casual observations about the way in which parties, or candidates, spend resources in trying to market themselves to voters.

(i). The first is that, for political parties, campaigning seems to be a constant activity throughout the period between elections and that the effort spent on trying to sell a favourable image of candidates and policies tends to increase as the election approaches [17], ([39], 61). Whitely and Seyd [40] describe the underlying pattern by dividing the campaign into three periods. These are the long period which starts at the end of the last election and lasts throughout followed by a subsequent medium and then a short term campaign. Similarly, in presidential elections candidates seem to accelerate expenditure as election day approaches ([21], 77). Although the observation that expenditure increases through time might seem too obvious to warrant consideration, it is only one possible trajectory. The question is, what explains this observations, or in fact any other trajectory? Is there any reason why expenditure should increase over the electoral period? Why not spend all resources at the beginning, or the end?

(ii). Secondly, it has been noted that here sometimes seems to be a correlation between the expenditure of candidates. This observation is again fairly casual. It has recently been tested in some work by Box-Steffensmeier [6] for the 2000 presidential election. This shows that Bush's expenditure increased in response to increases in expenditure by Gore. Why should the fact that one candidate has increased its expenditure cause the other candidate to increase?

The specific purpose of this paper is to help answer some of these questions. This is done by examining a fairly stripped down model of the dynamics of campaign expenditure in a competition between two parties or candidates.

In order to concentrate on the influence of competition on the dynamics of resource expenditure the model is constructed to abstract from the impact of personalities and other unique events. It also assumes that the determinant of support has two components. The first is fixed by partisan identification or perceptions of the economy or some other factor that does not change during the period. The second is the variable component which increases with the amount of resources spent on campaigning for a fixed level

[1] The comment is often made, for example, that most studies of campaign expenditure are static or ignore the temporal properties of data when it is available [6], [7], [4] and standard texts on party competition such as [27] and [?] do not mention time. This situation is changing with the increasing availability of data and popularisation of techniques to deal with time series analysis [3], [15], [8].

of expenditure by the opposition. This is consistent with the substantial body of work that indicate that many voters are responsive to advertising and that support increases with campaign effort [6], [32], [18],[29].[2]

In one sense the level of abstraction in this paper is higher than in some existing static models because of the complexity of the dynamic analysis. As with much of life, gains in one direction are bought at a cost in some other. It might be hoped that some of these abstractions could be reduced in future developments.

Although the model deals with competition for support in general, the reference in what follows will usually be to parties in order to save repetition. The findings hold for candidates in other systems, with obvious local modification, and some of the empirical data on support for parties, or candidates, is from work done on these systems.

The main result of the analysis is, roughly, that, under most of the conditions studied, the government and the opposition increase their rate of resource expenditure on marketing through the entire period between elections and the opposition spends more than the government in every time interval. The case which strikingly sums up the most important lesson of the analysis is where parties start out with equal support and the cost per unit of impact on the electorate is the same. In a simultaneous move game both parties have to spend more and more at each instant in order to retain their initial levels of support. This is often known as the Red Queen effect.[3] This stands in sharp contrast to the hypothetical one horse race. Where only a single party tries to buy votes, and the electorate is not myopic, it spends at a constant rate for the entire time period.

It is also shown that changes in the level of support for the parties depend, roughly, on the cost per unit of impact. If the cost is high for the government, relative to the opposition, the government allows its support to decrease. If this is low it may pay it to spend resources in order to increase its support through the entire interval.

In addition it is shown that when the cost per unit of impact is the same for both parties it pays the opposition to try harder and devote more resources to building support than the government. Although results are given for all cost structures, the case where the relative cost of impact for the government is higher than that of the opposition is of most interest since it is frequently argued in the literature that this is the usual situation [29] [17] [20]. In this case it pays the government to increase its spending at a faster rate than the opposition.

A rough intuitive interpretation of these results goes as follows. A single party will spend at a constant rate to avoid the costs associated with trying to change support rapidly in any period. If there were two parties and party one spends evenly across the entire period, or more at the beginning party two is better off, and party one is worse off, when party two spends little during the early part of the campaign and increasing

[2]See [17] for a discussion of some of the problems in an empirical test of the relation between spending and support. For a discussion of the continuing relevance of economic factors and party support see [31]

[3]It gets its name from the Red Queen's comment to Alice that, 'here, you see, it takes all the running you can do to keep in the same place.' [10].

amounts on converting the electorate towards the end. If, on the other hand, party one were to defer all its expenditure until near the end of the campaign its opponent could do better by building up support at an increasing rate. Party one would then find that it would cost more to reduce its opponent's support in the final period than if it had been spending earlier. If party one spends resources on building support at an increasing rate throughout the entire period, the opposition can do no better than follow a similar strategy.

As far as I am aware there are no similar studies in the political economy literature. There is some work on competitions between oligopolies in discrete time such as [37] and [34] and in continuous time such as [19] and [11] that have some parallels. Of these, this paper is closest to [19], but differs in assumptions and approach.[4]

I set out the paper as follows. In section §.2 the model of party competition is specified. This is analysed in sections §.3 for one party attempting to buy support. It is analysed for competition between the parties in §.4 – §.6 under different assumptions about information and strategies of the parties.

2 The model and discussion of the problem.

Suppose there is a political system with two parties and that each spends an all purpose resource on marketing its image over a continuous time period running up to an election in order to maximize its support at this election.[5] This time period covers the entire duration of campaigning activity and is not restricted to the final few weeks running up to an election. The resource that is allocated includes finances and the time and effort of the candidates, volunteers, and party members. The activities on which resources are spent are things like appearances at public functions and talk shows, low level advertising, consultation with groups, research, selling policies to various constituencies, attending cake stalls, and listening to complaints. Time spent on these activities is costly in the sense that there are other opportunities lost, or leisure and private income sacrificed. It is assumed, for simplicity, that these different types of resource expenditures can be expressed in a common monetary scale. There is a cost attached to spending resources. There is no limit on the amount that can be spent, however, because it seems reasonable to assume that parties can run up debts in money and goodwill.[6] It is also assumed that the image of the parties and, to a lesser extent, their policies are fixed in the period running up to an election. This simplification is intended to make the analysis tractable. Although party positions and policies may vary from one election to the next [9], [1] it is assumed that there is sufficient consistency in views of what the parties stand for, and their policies, to make this a reasonable first approximation for any one period between elections. In order to accommodate policy changes of sufficient magnitude to alter a party image it might be imagined that they take place prior to the campaign period.[7]

[4] Among the main differences are that I study the situation where there is a finite termination time and produce results for different information structures.

[5] It would be possible to see parties as firms with the main activity of marketing an image of themselves which they have produced at no cost.

[6] See ([39], 208-9) for a discussion of the importance of the activities of members and the role of voluntary activity as substitutes for other forms of campaign expenditure.

[7] See Budge [9] for a discussion of long run consistency in party images and variations in policy in individual elections from one election to the next.

The assumption that parties wish to maximize support at the time of the election captures the idea that the level of representation in the current legislative period is of value for itself and for its contribution to credibility and future elections. This assumption is consistent with early work on modelling electoral competition by Stigler and by Crain [33], [12]. It gives a continuous payoff function rather than the discontinuous payoff associated with the assumption that getting a majority is all that matters.

It is assumed that support for a party increases at a declining marginal rate with the amount of resources spent on marketing, doing good deeds and other forms of promotion, everything else constant. It might be imagined, for example, that each party has a natural constituency and that the closer a voter is to this constituency the more easily it is won over by the efforts of that party. As a voter gets further away, more effort is required to gain support. This assumption combines some elements of the Michigan model, which assumes that party identification is largely fixed, and the more recent literature which claims that many voters are responsive to advertising and that support for a party increases with campaign effort as required for my model [6], [32] [18].[8] In this case fixed party affiliations are replaced with the idea that it is increasingly costly to dislodge voters.

The strategies available to the parties depend on the information available and this introduces some subtle points to do with observation and responses over time that don't feature in the analysis of static games.[9] It is also possible that parties may formulate their strategies at different times. The following possibilities seem the most interesting.

*A*1. The parties formulate their expenditure programmes for the entire period at time zero. This might be explained by the long lead times needed to map out legislative programmes, to organize public engagements, to negotiate and meet with pressure groups, to set up public events, and to mobilize support from members. It might also be explained by uncertainties about the value of the information received during the competition. This gives an open loop competition.

*A*2. Parties might change their strategies at each instant in response to information on the effect of their actions, and those of their rival, on the level of support. This requires considerable flexibility. It gives a closed loop memoryless state feedback competition.

*A*3. One party might delay its planning until the other has formulated its campaign. In this case one party is the leader and decides on its expenditure of resources knowing that the other will then choose a pattern of expenditure to get the greatest share of support for itself. This gives an open loop Stackelberg game. It seems natural, although not necessary, to assume that the government, or incumbent, moves first.

In order to get a complete analysis we need to look at each of these. It is not possible to tell, in advance, how the trajectories will vary in each case.

The most natural way to analyse the dynamics of this situation, and the one that has considerable advantage from the viewpoint of mathematical elegance, is as a differential game in continuous time. Even if

[8] See fish for a discussion of some of the problems in an empirical test of this.
[9] See ([2], 227-8) for a detailed discussion of information.

parties to adjust their expenditure at discrete intervals there may be so many of these that they are best approximated continuously. On the other hand, many political economists will be more familiar with static or discrete time optimization problems and it may be useful to give some pointers about what is happening in these terms.[10]

In order to specify the model, let subscripts 1 and 2 refer to the government and opposition respectively. Time is written t for $t = 0, 1$. The resource expenditure of the government at time t is $u_1(t)$ and of the opposition is $u_2(t)$ and the cost of expenditure is $f_i(u_i)$ for $i = 1, 2$. The natural constituency of party i is written q_i and, for simplicity, let $q_1 = q_2 = q$. Without loss of generality change coordinates to set $q = 0$. The number of supporters for the government at time t is written $z_1(t)$ and for the opposition $z_2(t)$. The fraction of support is $x(t) = \frac{z_1(t)}{n}$ and $y(t) = \frac{z_2(t)}{n}$. Assume that n is sufficiently large that $x(t)$ can be approximated as closely as we wish by a smoothly differentiable function. This is also written $x(t)$ to save notation. Since gains for the opposition are losses to the government $y(t) = (1 - x(t))$. For the simple two period model the dynamics of support for party one could be written as

$$x(t+1) - x(t) = k_1(1 - x(t))u_1(t) - k_2 x(t)u_2(t) \tag{1}$$

where the parameters $k_1, k_2 > 0$ capture the idea that the voters may respond differently to expenditure by each party [29], [17], [20].

In this case, the problem for party one is to maximize $x(2)$ and for player two to minimize $x(2)$. Note that the firms are only interested in the payoff at the terminal time but that the costs of expenditure will be incurred across the entire time period. This means that, since $x(2)$ is the support for party one in the final period the problem for each player is to maximize

$$J^i = \sum_{t=0}^{1} f_i(u_i(t)) + \theta_i x(2) \tag{2}$$

where $i = 1, 2$ and $\theta = 1$ for $i = 1$ and $\theta = -1$ for $i = 2$.

In order to solve this game we proceed backward from the values of $u_i(1)$ that maximize $x(2)$ to obtain the maximum expenditure at each step.

The continous time game can be thought of as an extension of this idea across an arbitrarily large number of time steps. In order to specify it we take advantage of the fact that x is smoothly differentiable and write the effect of spending by the government and opposition on the rate of change in support as[11]

$$\dot{x} = k_1(1 - x)u_1 - k_2 x u_2 \tag{3}$$

where $\dot{x} = \frac{dx}{dt}$. Reference to time arguments are supressed unless they are needed to simplify the notation.

[10] I owe the idea of discussing the model in this stripped down form to John Roemer.

[11] The equation for the dynamics could easily be extended to take into account things like a natural rate of growth in support for one of the parties. As it stands it is similar to the Lanchester model. See [19] for references and a discussion.

For tractability the payoff functions, f^i, are simplified and the sum of costs is dealt with in continuous time by writing it in integral form. It is assumed that the parties attempt to maximize payoff functions of the form

$$J^i = -\int_0^1 \frac{c_i}{2} u_i^2 \; dt \; + \theta_i x(1) \tag{4}$$

for $i = 1, 2$ subject to the dynamics in equation (3). $c_i > 0$ is a paramater that captures any differences in the cost to each party of spending resources and this may depend on such things as the level of donations, availability of voluntary labour and so on. In many cases information on costs will be coarse owing to the different influences on cost. In addition it will also be interesting for analytical purposes to assume $c_1 = c_2$. Since we are dealing with time as an interval, and not units, the period between elections has been normalized and set at one.

It is assumed that a winning majority is $\bar{m} \geq \frac{n}{2} + 1$ and hence that $x(0) \geq \frac{1}{2}$ where equality ocurs when support is tied. Party support, as opposed to actual votes, could be tied for any number of reasons. Maybe a supporter of the opposition forgot to vote, or submitted a spoiled ballot, for example.

Before analysing the competition between the parties I consider the dynamics for the one horse, or single party, race. This is, of course, not meant to capture any actual situation. It is meant to provide a point of comparison to help answer the question, how much of the trajectory of party expenditure is produced by the dynamics of competition?

3 The optimal expenditure without competition.

The election is contested by a single party. In this case the optimal use of resources is to spend at a constant rate throughout the entire period. Any deviation from this can be attributed to the effects of competition between the parties. In order to show this the simple two step model is first analysed. I then prove it for the more general continuous time case.

The dynamics of the two step model for the single player are obtained for the closed loop case by modifying Equation (1) to get

$$x(2) = x(1) + k_1(1 - x(1))u_1(1) \quad \text{and} \qquad x(1) = x(0) + k_1(1 - x(0))u_1(0)$$

and using the appropriately modified specification for the payoff function in Equation (4) the problem is to maximize

$$J^1 = -\frac{u_1(0)^2}{2} - \frac{u_1(1)^2}{2} + x(2)$$

Substituting from the second equation for the dynamics into the first and using the known value of $x(0) = \beta$ gives

$$x(2) = \beta + k_1(1-\beta)u_1(0) + k_1(1 - k_1(1-\beta)u_1(0))u_1(1)$$

Differentiating J^1 with respect to the controls gives

$$u_1(\rho) = k_1(1-\beta) - k_1^2(1-\beta)u_1(\bar{\rho})$$

for $\rho, \bar{\rho} \in \{0, 1\}$ and $\rho \neq \bar{\rho}$ and hence

$$u_1(0) = u_1(1)$$

as required for constant expenditure in each period.

In the continuous time case the dynamics are given by

$$\dot{x} = k_1(1-x)u_1$$

and the payoff function is J^1 in Equation (4). In order to deal with this either the Pontryagin principle for the open loop game in $A1$ or the Hamilton-Jacobi-Bellman equation for the closed loop game in $A2$ will be used. Since the solutions will be the same, in this example, for either of these approaches the open loop case is solved for illustration.

The Pontryagin principle gives the necessary conditions that a piecewise differential optimal path of expenditure must follow in continuous time [24]. It can be thought of as a means of incorporating the effect of a change in expenditure at each instant on the value at that instant and on the value at all future times. In order to apply this we need to form the Hamiltonian

$$H = -\frac{u_1^2}{2} + \alpha k_1(1-x)u_1$$

where $\alpha(t)$ is a costate variable and is required to be piecewise continuously differentiable. It can be loosely thought of as the marginal value of an increase in support at time t. The necessary conditions for an internal solution are

$$u_1 = \alpha k_1(1-x)$$

and

$$\dot{\alpha} = \alpha k_1 u_1$$

Differentiating the first of these equations with respect to time gives

$$\dot{u}_1 = \dot{\alpha} k_1(1-x) - \alpha k_1 \dot{x}$$

and substituting gives

$$\dot{u}_1 = 0$$

as required.

It is worthwhile noting that this result is not limited to the specific assumptions made about the dynamics of support. It can be generalised to any specification of the dynamics of the form $\dot{x} = g(x)u$.

With this result in mind we can now consider what happens in the competition for support.

4 A 1. The open loop competition.

The government and the opposition formulate their strategies at the beginning of the game and the optimal programme for each party is to accelerate its expenditure for the entire time period. This gives an immediate, and important, difference between the dynamics of the two party competition and the optimal programme for a single party. The way in which the trajectory of support changes along an optimal path depends on a parameter that captures the relative cost of influencing voter support, called the impact cost index. When the cost index and support are equal we get the Red Queen effect. If the government's relative cost is sufficiently low support increases. In what seems to be the more usual case, where the government's cost per unit of impact is high relative to that of the opposition, support for the government is decreasing throughout the entire period. It is interesting to see that the government accelerates its expenditure faster than the opposition. This tells us that, from the government's perspective, it is better to spend relatively more near the end of the campaign. What needs to be done is to prove all this.

In order to use the Pontryagin principle in this case we form the Hamiltonians

$$H_1 = -\frac{c_1}{2}u_1^2 + \alpha_1(\ k_1(1-x)u_1 - k_2xu_2) \quad \text{and} \quad H_2 = -\frac{c_2}{2}u_2^2 + \alpha_2(\ k_1(1-x)u_1 - k_2xu_2) \tag{5}$$

where α_i for $i = 1, 2$ are the costates and are required to be piecewise continuously differentiable. The details of the analysis are in Appendix 1. The necessary conditions that a piecewise continuous optimum control must satisfy for an interior solution are

$$u_1 = \alpha_1 \frac{k_1}{c_1}(1-x) \quad \text{and} \quad u_2 = -\alpha_2 \frac{k_2}{c_2}x \tag{6}$$

$$\dot{\alpha}_1 = (k_1u_1 + k_2u_2)\alpha_1 \quad \text{and} \quad \dot{\alpha}_2 = (k_1u_1 + k_2u_2)\alpha_2 \tag{7}$$

where $\alpha_1 = -\alpha_2 = \alpha$ for $i = 1, 2$. Differentiating and substituting gives

$$\dot{u}_1 > 0,\ \ddot{u}_1 > 0,\ \dot{u}_2 > 0 \quad \text{and} \quad \ddot{u}_2 > 0 \tag{8}$$

which means that u_i is increasing at an increasing rate for all t for $i = 1, 2$.

Substituting Equation (6) into Equation (3) give the trajectory of support for the government as

$$\dot{x} = \alpha(\frac{k_1^2}{c_1}(1-x)^2 - \frac{k_2^2}{c_2}x^2) \tag{9}$$

along the optimal path. It follows that $\dot{x}$ has the same sign as

$$\psi = 1 - 2x + x^2(1-\sigma)$$

where

$$\sigma = \frac{k_2^2}{c_2} / \frac{k_1^2}{c_1}$$

For $x \leq 1$ the positive root for ψ is $r = \frac{1-\sqrt{\sigma}}{1-\sigma}$ $\sigma \neq 1$ with $\frac{\partial r}{\partial \sigma} < 0$. It follows that for $x < r$ we have $\dot{x} > 0$ and for $x > r$ we have $\dot{x} < 0$. What this tells us immediately is that, the fraction of the support for party one is always either increasing, decreasing or stationary. See fig. (1a). It will also be noted that $lim_{\sigma \to 1}\psi = \frac{1}{2}$ and ψ is continous.

It is possible to get a better picture of what is happening, and to set up an important parameter for the analysis, by rewriting σ as $\frac{c_1}{k_1^2}/\frac{c_2}{k_2^2}$ and interpreting $\frac{c_1}{k_1^2}$ as an index of the cost of impact of the government's expenditure on campaigning, with the analogous interpretation for the opposition. Even though we are working with k_i^2 the index will have the required properties for changes in c_i and k_i. Call this the impact cost index. This allows us to interpret σ as the ratio of the cost of impact of the government's campaign expenditure over the cost of impact of the opposition's expenditure. This may vary because voters are more receptive to opposition messages, or beause one party might deal with their advertising in a more cost effective manner than its opponent [29], [17] [20]. This gives three cases to consider.

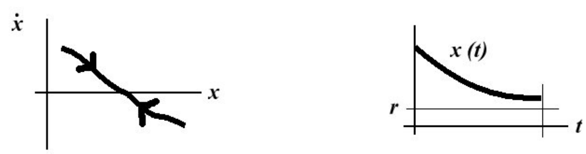

Figure 1. The dynamics of support for party one.

Case 1. Equal impact cost index. $\sigma = 1$.
(i). The parties are symmetrical in the sense that each is supported by half the population at the beginning of the electoral period. It is immediate from Equations (6) and (9) that

$$\dot{x} = 0 \quad \text{and} \quad u_i = u_2$$

which means from Equation (8) that, like Alice and the Red Queen, each party is spending more and more as time progresses in order to stand still.

(ii). The government has more initial support than the opposition and it is assumed that the coefficients in the dynamic equation are the same and $k_1 = k_2 = \bar{k}$ for $\bar{k} > 0$ a constant. In this case expenditure for the opposition is higher than that for the government for the entire period. In other words, it pays the opposition to try harder. In addition

$$\dot{x} < 0$$

for all t and the fraction of total support for the government falls for the entire period. In addition it can be shown that $\ddot{x} > 0$ to give the trajectory in fig. (1b). See Appendix 1.

To get the results for the dynamics of resource expenditure use the trajectory for x together with Equation (6) to give

$$u_1 < u_2 \quad \text{and} \quad \dot{u}_1 > \dot{u}_2$$

for all $t \in [0, 1)$ as required. See fig. 2 for an example. It is also possible to get explicit solutions for x and u_i for $i = 1, 2$ for $\sigma = 1$ for all t, if these are required.

Case 2. Impact cost index of the government less than the opposition. $\sigma < 1$**.**
The government's support increases for all t for σ sufficiently small and, for the special case $k_1 = k_2 = \bar{k}$, expenditure by the opposition is increasing faster than expenditure by the government. To see this note that for $\sigma = 0$ we have $r = 1$ and as σ increases r decreases with $r \to 0$ as $\sigma \to \infty$. It follows that, for σ sufficiently small, $x(0) < r$ and $\dot{x} > 0$ for all t. See fig. 3

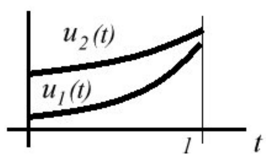

Figure 2. Trajectories of reource expenditure for equal impact cost indexes .

To get the relation between the trajectories of expenditure for the government and opposition note that, for $k_1 = k_2 = \bar{k}$, it follows immediately from equation (3) that $u_1 > u_2$ for all t. Using equation (6) and writing $\rho = u_1 - u_2$ and differentiating gives $\frac{\partial \rho}{\partial x} < 0$. This means $u_1 - u_2 < 0$ as required.

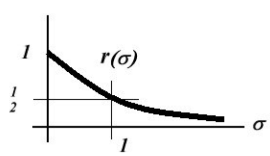

Figure 3. Example of trajectories for $r(\sigma)$.

Case 3. Impact cost index of government greater than the opposition. $\sigma > 1$.
In this, the more usual, case support for the government is decreasing and, for $k_1 = k_2 = \bar{k}$, the government is increasing its expenditure faster than the opposition. To begin note that for $\sigma \geq 1$ we get $x(0) > r$ and $\dot{x} < 0$ for all t as in fig. 3.

When $k_1 = k_2 = \bar{k}$ equation (6) gives $u_1(0) < u_2(0)$. Since $\frac{\partial \rho}{\partial x} < 0$ we now have $u_1 - u_2$ increasing.

5 A 2. The closed loop game with equal impact cost indexes. $\sigma = 1$.

The parties now adjust their advertising expenditure according to the level of support at each instant as revealed by polls and other information sources to give a closed loop game with state space information. This is rather more difficult to analyse than the previous case and results are only produced for $\frac{k_i^2}{c_i} = k$ for $i = 1, 2$. The general result in this case is that the trajectory is essentially the same as the trajectory in the open loop case. Where the government and the opposition have the same initial level of support we again see the Red Queen effect and for $k_1 = k_2 = \bar{k}$ the opposition spends more than the government as before. An unexpected feature of this case is that parties spend less on campaigning at each instant and hence across the entire time period than in the closed loop game, even thought the end results are the same. With the benefit of hindsight we can see that, if parties are able to adjust their strategies at each instant, they must be able to do at least as well as, or better than, they can if they are not able to adjust. Each party can constantly monitor the other's activities and will tend to fine tune its expenditure according to its opponent's moves at each instant. If parties have to commit themselves at the beginning of the game they do not have the possibility of this fine tuning.

In order to solve this problem it is necessary to use the Hamilton-Jacobi-Bellman equation to take into account the feedback between support at each instant and the expenditure of the parties. Write the value of the game for player i from time t and initial condition x_0 as $\omega^i(t, x_0)$ and the partial derivative of ω^i with respect to any variable z as ω^i_z. The details of the analysis are in Appendix 2. This gives us

$$-\omega^1_t = \; max \;_{u_1}(-\frac{c_1}{2}u_1^2 + \omega^1_x(k_1(1-x)u_1 - k_2 x u_2)) \tag{10}$$

with the analogous expression for ω^2_t.

The partial differential equation in (10) is solved in Appendix 2 in terms of $\varphi_i(t)$ where φ_i has the same place in the dynamics as α_i in the open loop game. This gives

$$\dot{\varphi} = \frac{3k}{2}\varphi^2 \tag{11}$$

and this can be used to get explicit solutions for x and u_i.

Comparison of open and closed loop strategies.
The general result is set out below. Paths are said to have the same profile if they have the same sign and

the same first and second derivatives for every t.[12]

Proposition 1. For $\sigma = 1$ the trajectories for x and u_i for $i = 1, 2$ in the closed loop game have the same time profile as the open loop game with $u_1 < u_2$ when $k_1 = k_2 = \bar{k}$.

Proof. Since $\varphi_1 = -\varphi_2$ we have $\omega_x^i = \varphi_i$ with the same place in the dynamics as α_i and x is bounded away and above $x = \frac{1}{2}$ for all $t \in [0, 1)$ in both games. In addition $u_1 < u_2$ in both games. In order to get the rest of the profile compare equations (17) and (11).

Since $\varphi(1) = 1$ we have $\alpha > \varphi$ from equations (17) and (11) for all t and $u_i(0)$ greater in the open loop game than in the feedback game. In order to support the statements at the beginning of the analysis of this case we look at explicit solutions to the equations for the case where $> \frac{1}{2}$. This gives
(i). Support for the government is always higher in the closed loop than in the open loop game with the highest level of support at approximately $t = \frac{1}{2}$ and the level of support roughly equal for $t = 1$ in both games. See fig. 4 where $\varpi(t) = x_c(t) - x_o(t)$ where x_c and x_o are support for the government in the closed loop and open loop games respectively.
(ii). Government expenditure is higher for all t in the open loop game than the closed loop game with the maximum difference at approximately $t = \frac{1}{2}$.
(iii). Opposition expenditure is higher for all t in the open loop game than in the closed loop game with the maximum difference at $t = 0$.

6 A 3. The open loop competition with the government as a leader.

The government formulates its strategy first and the opposition responds. This is the most difficult case to analyse, but we might hope to get the most important characteristics of the trajectories. As in case $A1$ support for the government depends on the relative cost of impact for government expenditure. If relative cost is sufficiently high support will be decreasing along the optimal path although there is a possible case where it decreases and then increases. If the relative cost is low support for the government will increase. The opposition increases its expenditure for the entire time period. It can also be shown that the government is increasing its expenditure at the beginning and end of the time period.

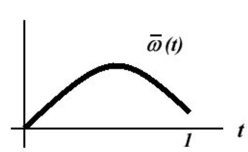

Figure 4. Difference between trajectory of support in closed loop and open loop games for $\varpi(t) = x_c(t) - x_o(t)$.

[12] A similar sort of result is obtained by Fruchter [19] for an infinite horizon duopoly market.

The solution to this problem is similar to the open loop game. In this case the Lagrangean for the government is

$$L_1 = -\frac{c_1 u_1^2}{2} + \alpha_3(\ k_1(1-x)u_1 - k_2 u_2 x) + \alpha_4(-\frac{\partial H_2}{\partial x}) + \alpha_5 \frac{\partial H_2}{\partial u_2} \tag{12}$$

with H_2 given by equation (5) and the $\alpha(t)$ terms again being costate variables with properties given in Appendix 3. This gives us the necessary conditions for an internal solution as

$$u_1 = \alpha_3 \frac{k_1}{c_1}(1-x) + k_1\alpha_2\alpha_4 \quad \text{and} \quad u_2 = -\alpha_2 \frac{k_2}{c_2} x \tag{13}$$

In a similar manner to the previous analysis of $\{x : \dot{x} = 0\}$ we get $\dot{x} < 0$ if $x > \bar{r}$ and $\dot{x} > 0$ if $x < \bar{r}$ where $\bar{r} = (1 - \sqrt{\sigma e^{\frac{k_2^2}{c_2}(1-t)}})(1 - \sigma e^{\frac{k_2^2}{c_2}(1-t)})^{-1}$.

Rewriting $\frac{k_2^2}{c_2} = \tilde{k}$ and $\bar{\sigma} = \sigma e^{\tilde{k}(1-t)}$ and doing some work on the derivative to get the sign gives $\frac{\partial r}{\partial \bar{\sigma}} < 0$ and $\frac{\partial \bar{\sigma}}{\partial t} < 0$. This gives a similar analysis to the open loop game with the additional time dynamic given by the fact that, for σ given, $\bar{r}$ increases as t increases. This means that we cannot rule out the possibility that $\dot{x}$ switches sign and support for the government starts to increase at some $t < 1$. See fig. 5 for an illustration.

This again gives three cases to analyse.

Case 1. Equal impact cost indexes. $\sigma = 1$.
The government's support initially declines along the optimum trajectory, but may increase near the end of the time period. The opposition starts by spending more than the government and increases its expenditure for all time and the government also increases its expenditure for some time interval near the beginning of the campaign. For the specific case where $\frac{k_i}{c_i} = 1$ the government also increases its expenditure in an interval near the end of the campaign.

To analyse the trajectory of support note that the positive root is now $\bar{r}(k,t) = \frac{1 - e^{\frac{\tilde{k}(1-t)}{2}}}{1 - e^{\tilde{k}(1-t)}}$. Taking the limit as $t \to 1$ gives $\bar{r}(k,t) \to \frac{1}{2}$ and since $\frac{\partial \bar{r}}{\partial t} > 0$ we have $\bar{r}(k,0) < \frac{1}{2}$. This means that $\dot{x}(0) < 0$. For some $x(0)$ sufficiently close to one half it must be the case that x is increasing in the vicinity of $x = 1$ as in fig.5.[13]

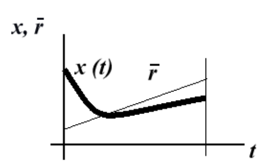

Figure 5. Example of trajectory for $\sigma = 1$.

[13]If required we could get $x(t)$ for specific parameters by solving the two point boundary value problem for equations (3), (22), (23) with the value for α given in (19). See Roberts and Shipman, [30] for example.

It is straightforward to analyse the resource expenditure of the opposition since it will follow the same path as the open loop game to give

$$\dot{u}_2 > 0 \quad \text{and} \quad \ddot{u}_2 > 0$$

for all t. It can also be shown from the analysis of the costates and Equation (13) that

$$u_2(0) > u_1(0)$$

as in the open loop game.

In order to analyse the resource expenditure for the government consider the special case where $k_i = c_i = 1$ for $i = 1, 2$. Since the relative values for the $\alpha(t)$ terms are either known or can be obtained at $t = 0$ and $t = 1$ we evaluate u_1 at these points. Differentiating u_1 and using values for the α terms gives

$$\dot{u}_1(0) > 0,\ \ddot{u}_1(0) > 0 \quad \text{and} \quad \dot{u}_1(1) > 0,\ \ddot{u}_1(1) > 0$$

This might suggest that the government is accelerating its expenditure for all t but, even though this seems plausible, I do not have a proof.

It is also noted from equation (6) that the final level of expenditure of the government in the Stackelberg game is greater than in the open loop and closed loop games for the same level of $x(1)$.

Case 2. Lower impact cost index for government. $\sigma < 1$.
Support for the government is increasing for all t for σ sufficiently small. To see this observe that we can make $\bar{r}(k, 0) > 1 - \epsilon$ for some ϵ as small as we wish by letting $\sigma \to 0$. The trajectory of expenditure by the opposition remains the same as in the previous case. It is also the same for the government when $k_i = c_i = 1$.

Case 3. Greater impact cost index for government. $\sigma > 1$.
If $\sigma > m$ for some m sufficiently large we have $r(k, t) \to \epsilon$ for any $\epsilon > 0$ and hence support for the government is decreasing. The trajectory of government and opposition expenditures are the same as in the previous case.

7 Conclusion.

The aim of this paper has been to make up some of the gap in our theories of the dynamics of competition between political parties. It was found that both parties increase the effort spent on marketing over time at an accelerating rate under open loop and closed loop information, although the picture is not so clear when the government is a leader. In the first two cases it was also found that there is a Red Queen effect. The contrast between these results and spending by a single party indicates that any observed increase in marketing effort over time can, in part be explained, by the dynamics of the struggle for support. It does not require special assumptions about memory on the part of the electorate.

It was also found that the trajectory of support depended on the relative cost of impact for the government and opposition. In the case where these costs are equal, or greater for the government, the government loses support through the entire period except, perhaps, in the case of the Stackelberg game.

It was stressed that this is only a first step in building an analysis of the dynamics of competition. Among the variants of the model that could be explored are those in which the time of the election is optimally chosen and in which the amount of resources to be spent are fixed. It would also be possible to explore the case where the cost and response functions depend on the level of support.

Appendix 1. A1. Open loop game.

The Hamiltonians are given in Equation (5) and the necessary conditions are given by Equations (6) and (7) where the terminal conditions on the costates are $\alpha_1(1) = 1$ and $\alpha_2(1) = -1$. Since the Hessian matrices for the Hamiltonians are negative semi-definite these conditions are also sufficient. The solution for the costates is

$$\alpha_1(t) = e^{-\int_t^1 k_1 u_1(t) + k_2 u_2(t)} \quad \text{and} \quad \alpha_2(t) = -e^{-\int_t^1 k_1 u_1(t) + k_2 u_2(t)} \tag{14}$$

which gives $\alpha_1 = -\alpha_2$. Writing $\alpha = \alpha_1 = -\alpha_2$ for $i = 1, 2$ gives

$$\dot{u}_1 = \frac{k_1}{c_1}\alpha k_2 u_2 > 0 \quad \text{and} \quad \dot{u}_2 = \frac{k_2}{c_2}\alpha k_1 u_1 > 0 \tag{15}$$

and this tells us immediately that

$$\ddot{u}_1 = \frac{k_1}{c_1}k_2(\dot{\alpha}u_2 + \alpha\dot{u_2}) > 0 \quad \text{and similarly} \quad \ddot{u}_2 > 0 \tag{16}$$

To get the trajectory of support for the government substitute Equation (15) into Equation (3). This gives $\dot{x} = \alpha(\frac{k_1^2}{c_1}(1-x)^2 - \frac{k_2^2}{c_2}x^2)$

Notes $\sigma = 1$

Substituting for $\frac{k_i^2}{c_i} = k$ into equation (7) gives

$$\dot{\alpha} = k\alpha^2 \tag{17}$$

and this gives

$$\ddot{x} = k(\dot{\alpha}(1-2x) - 2\alpha\dot{x}) > 0 \tag{18}$$

and substituting from equation (17) and dividing through by $1 - 2x < 0$ gives $\ddot{x} > 0$ because $\dot{\alpha} - 2k\alpha^2 < 0$. For $k_1 = k_2 = \hat{k}$ it follows from equations (6) and (8) and the trajectory for x that $u_1 < u_2$ for all $t \in [0, 1)$ and that $\dot{u}_1 > \dot{u}_2$ as required.

It is also possible to get explicit solutions for x and u_i for $i = 1, 2$ for $\sigma = 1$ for all t, if these are required. This gives more detail than is necessary for present purposes. In order to do this use equation (17) to give

$$\alpha(t) = \frac{1}{1 + k(1-t)} \tag{19}$$

and substituting into the differential equation for x and solving gives

$$x(t) = \frac{1}{2}(1 + (2x(0) - 1)(\frac{\alpha(0)}{\alpha(t)})^2) \tag{20}$$

It is now possible to get the equations for the expenditure of the parties by substituting into equation (6).

Appendix 2. A 2. Closed loop game.

Solution to the Hamilton-Jacobi-Bellman partial differential equation. (10)

Solving the optimizing problem for equation (10) for u_1 and u_2 gives us the analogous expressions to (6) with the partial differentials ω_x^1 and ω_x^2 replacing α_1 and α_2 ([**?**], 259-63). Substituting the solutions back into equation (10) and its counterpart and simplifying gives a system of two partial differential equations

$$-\omega_t^1 = k((\omega_x^1)^2\frac{(1-x)^2}{2} + \omega_x^1\omega_x^2 x^2) \quad \text{and} \quad -\omega_t^2 = k((\omega_x^2)^2\frac{x^2}{2} + \omega_x^1\omega_x^2(1-x)^2) \tag{21}$$

These equations indicate that there might be a symmetrical solution for ω. Redefine the game in equation (4) with the payoffs at $t = 1$ written $x(1) - \frac{1}{2}$ for player one and $\frac{1}{2} - x(1)$ for player two. Since the payoffs have only been altered by a constant this gives us the equivalent game. Consider solutions of the form

$$\omega^1 = \varphi(t)(x(t) - \frac{1}{2}) \quad \text{and} \quad \omega^2 = -\varphi(t)(x(t) - \frac{1}{2})$$

with the boundary conditions derived from the new specification of the game. This means $\omega^1(1) = (x(1) - \frac{1}{2})$ and $\omega^2(1) = -(x(1) - \frac{1}{2})$. This gives $\varphi(1) = 1$.

Taking the partial differentials of the proposed solution and substituting into (21) gives

$$-\dot{\varphi}(x - \frac{1}{2}) = k(\varphi^2\frac{(1-x)^2}{2} - \varphi^2 x^2) \quad \text{and} \quad \dot{\varphi}(x - \frac{1}{2}) = k(\varphi^2\frac{x^2}{2} - \varphi^2(1-x)^2)$$

and subtracting the first equation from the second and simplifying gives

$$\dot{\varphi} = \frac{3k}{2}\varphi^2$$

which means

$$\varphi(t) = \frac{2}{2 + 3k(1-t)}$$

and substituting into $\dot{x}$ in equation (9) and solving gives

$$x(t) = \frac{1}{2}(1 + (2x(0) - 1)(\frac{\varphi(0)}{\varphi(t)})^{\frac{4}{3}})$$

In order to get explicit solutions all that we need to do is substitute into $u_1(t) = \frac{k_1}{c_1}(1 - x(t))\varphi(t)$ and $u_2(t) = \frac{k_2}{c_2}x(t)\varphi(t)$.

Appendix 3. A3. Stackelberg game.

1. Analysis of costates

The Lagrangean for the problem is Equation (12) where $\alpha_4(t)$ is the costate associated with α_2 now treated as a state variable and $\alpha_5(t)$ is the multiplier for the condition that must hold for an optimum u_2. See ([2], p.410-12). This gives the necessary conditions for an internal solution as in Equation (13) and

$$\dot{\alpha}_3 = (k_1u_1 + k_2u_2)\alpha_3 + k_2\alpha_5\alpha_2 \tag{22}$$

$$\dot{\alpha}_4 = -\alpha_4(k_1u_1 + k_2u_2) + k_2\alpha_5 x \tag{23}$$

with the transversality conditions $\alpha_3(1) = 1$ because the second derivative of the terminal condition is zero and $\alpha_4(0) = 0$ from ([2], p.412). It is shown below that $\alpha_3 > 0$, $\alpha_4 < 0$ and $\dot{\alpha_3} > 0$ for all t and that $\mid \alpha_4(1) \mid \leq \frac{k_2^2}{c_2}\int_0^1 x^2$. In order to get α_5 we use the fact that $\frac{\partial L}{\partial u_2} = 0$ for an internal solution. This gives

$$\alpha_5 = \frac{k_2}{c_2}(-\alpha_3 x + \alpha_4\alpha_2) \tag{24}$$

Substituting for α_5 in equation (24) and solving equation (22) and writing $k_1u_1+k_2u_2-\frac{k_2}{c_2}\alpha_2 x = k_1u_1+2k_2u_2$ as γ to save notation gives $\alpha_3 = e^{\int_0^t \gamma dq}\frac{k_2}{c_2}\int_0^t \alpha_4(\alpha_2)^2 e^{-\int_0^s \gamma dq}ds + e^{\int_0^t \gamma ds}(e^{-\int_0^1 \gamma} - \frac{k_2}{c_2}\int_0^1 \alpha_4(\alpha_2)^2 e^{-\int_0^s \gamma dq}ds.$ Simplifying this gives

$$\alpha_3 = e^{-\int_t^1 \gamma ds} + e^{\int_0^t \gamma ds}(-\frac{k_2}{c_2}\int_t^1 \alpha_4(\alpha_2)^2 e^{-\int_0^s \gamma dq}ds) \tag{25}$$

In a similar manner we can solve for (23) to get

$$\alpha_4 = -e^{-\int_0^t \gamma ds}\frac{k_2^2}{c_2}\int_0^t \alpha_3 x^2 e^{\int_0^s \gamma dq}ds \tag{26}$$

It is also possible to get the following information on the costates.

Proposition 2. The signs on the costates are $\alpha_3 > 0$, $\alpha_4 < 0$ and $\dot{\alpha_3} > 0$ for all t for $\frac{k_2^2}{c_2} > 0$.

Proof. Since $\alpha_4(0) = 0$, we get from equation (25) that $\alpha_3(0) > 0$ and hence $\alpha_3(t) \geq 0$ in some interval $t \in [0, \epsilon)$ for some $\epsilon > 0$ and ϵ sufficiently small, since α_3 is continuously differentiable. Differentiating α_4 at ϵ gives $\dot{\alpha}_4 = \gamma e^{-\int_0^\epsilon \gamma}\epsilon\frac{k_2^2}{c_2}\alpha_3 x^2 e^{\int_0^s \gamma dq} - e^{-\int_0^\epsilon \gamma}\frac{k_2^2}{c_2}\alpha_3 x^2 e^{\int_0^s \gamma dq} < 0$ for ϵ sufficiently small. It follows that $\alpha_4(\epsilon + \delta) < 0$ for some $\delta > 0$. Let S be the set $S = \{t \in [0, w) : \alpha_4 \leq 0 \; and \; w = sup \; S\}$. It is clear that S is non-empty from what has just been said. Since $w = sup\ S$ we have $\alpha_4(w) = 0$ and this requires $\alpha_3(w) = 0$. This contradicts the fact that for $\alpha_4(w) = 0$ we have $\alpha_3(w) > 0$. It follows immediately that $\dot{\alpha_3} > 0$.

□

Proposition 3. $\mid \alpha_4(1) \mid \leq \frac{k_2^2}{c_2}\int_0^1 x^2$.

Proof. The mean value theorem for integrals gives $\mid \alpha_4(1) \mid \; = e^{-\int_0^1 \gamma ds}e^{\int_0^s \gamma ds}\frac{k_2^2}{c_2}\int_0^1 \alpha_3 x^2 ds \leq \frac{k_2^2}{c_2}\int_0^1 \alpha_3 x^2 ds.$ To complete the proof note that, from Proposition 2, $max\ \alpha_3 = 1$.

□

Resource expenditure. $\sigma = 1$.

The resource expenditures of the opposition is analysed by first noting from equation (14) and equation (25) that $\alpha(0) > \alpha_3(0)$. Since $\alpha_4(0) = 0$ we have $u_2(0) > u_1(0)$ as in the open loop game and we again have

$$\dot{u}_2 > 0 \quad \text{and} \quad \ddot{u}_2 > 0$$

for all t.

In order to analyse the resource expenditure for the government consider the special case where $k_i = c_i = 1$ for $i = 1, 2$. Differentiating u_1 gives

$$\dot{u}_1 = \dot{\alpha}_3 \frac{k_1}{c_1}(1 - x) - \dot{x}\frac{k_1}{c_1}\alpha_3 + \dot{\alpha}_2 k_1 \alpha_4 + k_1 \alpha_2 \dot{\alpha}_4 \tag{27}$$

which gives

$$\dot{u}_1(0) > 0$$

for all values of k_i and $c_1 : \sigma = 1$ from $\alpha_2 < 0$, $\alpha_4(0) = 0$, $\dot{\alpha}_4(0) < 0$ and $\dot{x}(0) < 0$. In order to get the sign for $\dot{u}_1$ at $t = 1$ substitute from the previous results into equation (27) and simplify to get

$$\dot{u}_1 = -2\alpha_2\alpha_3 x + \alpha_2^2 \alpha_4$$

and

$$\ddot{u}_1 = -2(u_1 + u_2)\alpha_2\alpha_3 x - \alpha_2^2(-\alpha_3 x + \alpha_2\alpha_4) + \alpha_2^2\alpha_4(u_1 + u_2) - 2\alpha_2\alpha_3 u_1$$

Rewrite $\dot{u}_1(1) = 2x + \alpha_4$ and use the value for $\alpha_4(1)$ to get $2x - \int_0^1 x^2 > 0$ for $x \leq 1$. This gives

$$\dot{u}_1(1) > 0 \quad \text{and} \quad \ddot{u}_1(0) > 0, \; \ddot{u}_1(1) > 0$$

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