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An asymmetric dynamic struggle between pirates and producers

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Abstract

The purpose of this paper is to contribute to our understanding of the dynamics of struggles over resources by studying a game between a producer that can guard and buy fortifications and a pirate. It is assumed that the returns from defence and raiding depends on the ratio of the resources spent on each activity and that all produced goods can be stolen. It attempts to characterise the trajectory of the resources and the defence and raiding activities of the pirate and producer. I show, among other things, that the pirate's strategy is to farm the producer and that the pirate's raiding activities and resources will decline as the productive capacity of the producer increases. I also show that a flexible guarding strategy may be preferred to fixed fortifications if the producer's resources are low at any time.

Key words: resource struggles, piracy, differential games, optimal control.

JEL classification: C61, C72, P14, Doo.

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1 Introduction

Social scientists have begun to pay increasing attention to the problem of understanding struggles over resources and to the effect of these struggles on the development and growth of economies and firms.[1] The most straightforward of these struggles is that between producers and pirates or raiders. In the real world, of course, many struggles, such as those resulting from predators attacking firms in order to steal goods, or market shares, or returns from innovation tend to be more complicated than simple raiding. It might be the case, for example, that the pirate piggy backs on the producer and, at some point, begins to simultaneously develop its own production or sales capacity. Nonetheless, an element of direct piracy might be common to all these, more complicated, cases. It might be hoped, therefore, that a better understanding of questions about the dynamics of raiding might throw some light on a broader range of situations. This paper deals with some of these questions.

The problem is that, even though the existing theoretical literature gives some help in understanding raiding and other forms of struggle over resources, and we have some idea about how they might be modelled, there is still a way to go in trying to completely understand the dynamics of these conflicts. The literature has so far mostly concentrated on either static or discrete models in which strategies are not fully dynamic. This gap has often been noted [8], [2], [13]. In response there have been some recent attempts to develop more dynamic approaches [6] [13]. For reasons discussed below, these leave questions about the dynamics of raiding unanswered.

The specific purpose of the paper is to make a start on filling this gap by analyzing the dynamics of a simple model of struggles between a producer and a pirate in which all produced goods can be stolen. I treat the amount stolen as a function of the ratio between stealing and defence.

The idea that the amount stolen depends on the ratio between resources devoted to stealing and effort devoted to defence has already been extensively developed in the literature in the well known Hirshleifer type approach [8] [2],[13]. It is discussed and justified in general terms by Hirshleifer [8]. This paper

[1] See the bibliography for some selected works

departs from Hirschleifer's in allowing all goods to be stolen. Some of the other differences between my paper and other work that has sought to develop the dynamics of this approach are noted. Maxwell and Reuveny's [13] work on the dynamics of a Hirschleifer type of model stays close to the original and deals with questions about struggles between producers with a common good rather than with the case of raiding by pirates and a situation where all goods can be stolen. In addition their analysis is not fully dynamic since players are assumed to be myopic whereas I allow player to optimize across the complete time period. Hausken [6] has also written a dynamic model that follows this approach. This looks at the dynamics produced by the effect of population change on fighting ability and rather than the resources put into raiding and defence. A paper which also uses proportional returns and asks similar questions to these is that of Grossman and Kim [2]. This also uses but differs from the present work in using a two stage static framework.

Although not related to the proportional returns approach, a paper by Sethi [18] comes closer to the raiding problem by considering the problem of a continuous dynamic thief. It differs from my approach in that it does not involve strategic interaction.

One drawback in modelling conflict in terms of proportional returns to effort is that it gives a non-linear differential game which is difficult to analyse without simplifying in other respects. In fact one of the barriers to work in this area is the analytical difficulty of the problem.[2] Although it would be possible to specify the game in a more tractable form, I believe that the analysis of this type of return function is sufficiently worthwhile to justify any other losses.

Among the simplifications is the assumption that consumption is not a strategic variable and that the producer does not steal from the pirate. Non-stealing can be justified on the grounds that the raider does not have anything to steal. This point is made, for example, in Mann's study of the conflict between the Huns and the Roman Empire and can be generalized across the range of conflicts that are of interest [12]. Treating consumption as a fixed portion of resources greatly simplifies matters and is not implausible. A less satisfactory simplification is that the analysis is restricted to open loop information. I do not see a practical way of dealing with closed loop information in asymmetric models based on proportional returns.

Moving in the direction of more complexity, the defensive activity of the producer is treated as a combination of resources spent on guarding and of expenditure on fortifications such as walls, moats, and electronic surveillance devices. Although the model is not intended to be realistic, it should somehow capture essential elements of struggles and this complication is justified on the grounds that fortification is a common and important element in real attempts to defend possessions. Fortifications have also been studied in [2] in a static setting. I deal with technological redundancy in fortifications by using a fixed time horizon.

Among the specific questions of concern are the following. (i). What are the dynamics of the reources for the producer and pirate? (ii). What is the relation between productivity, the resources put into

[2]Hirsch says, for example, that the dynamic case poses a 'fearful analytical problem' ([8], 31).

defence and the activities of pirates? (*iii*). What is the relation between expenditure on defence and raiding? (*iv*). How does expenditure on fortifications change with resources? (*v*). What happens to expenditure on fortification and guarding as the price of fortification changes? These parallel some of the questions in ([2], 1286-7).

Briefly, the overall story is that the pirate optimizes by farming the producer. The pirate takes fewer resources early and increases its take as the game continues. If the producer is not very productive, the pirate's resources increase over time while the producer's decline. If the producer is sufficiently productive and wealthy, however, the pirate's raiding activities and resources will decline as productivity increases. This is the reverse of an expected rob from the rich strategy. I also show that a flexible guarding strategy may be preferred to fixed fortifications if the producer's resources are low at any time. I give a more detailed summary in the conclusion.

I set out the paper as follows. The model is constructed in §.2 and the dynamics of the system are analysed in §.3. I consider the special case where the parameters of production and stealing are equal in §.3 since this can be fully analysed. I consider the case where they are unequal in §.4. I conclude in §.5.

2 The Model.

Set up for the model.
The system is made up of a society, or a firm, that produces a material good that is an input to its own production and a pirate or exploiter that relies on raiding and each player derives a payoff from the amount of the good it holds over time. The payoff might be thought of as coming from consumption of resources left over after costs of defence or stealing. Since the main aim is to study the dynamics of this system under the optimal choice of effort devoted to buying fortifications, guarding and raiding, it is assumed that consumption is a constant proportion of the good. To allow concentration on essentials, the fixed proportion of consumption is ignored without loss of generality. It can be re-introduced without difficulty. The producer has two choices. It can spend an amount of its initial endowment of the good on buying fortifications such as walls, forts electronic devices and the like. It can also devote some of its resources to guarding. The pirate decides on the amount of resource to devote to raiding. It is assumed that resources and effort put into guarding and raiding consume some of the good and also require other inputs such as extra effort, increased risk and reduction of leisure. It follows that guarding and raiding reduce growth and the payoffs to the players.

The game is given a finite time horizon. This allows us to take into account the fact that changes in the technology of fortifications will bring about redundancies For example, medieval high walled forts were made redundant by cannons and cannon forts by long range artillery and so on. If the time period were infinite we would have to complicate the issue by allowing continual reinvestment. A fixed time period can also be thought of in other ways. Each generation might make decisions about expenditure on fortifications and guarding and attach some value on what it bequeaths to subsequent generations. The pirate may also face redundancy in equipment or may wish to either pass on the enterprise or sell it as a going concern. Alternatively pirates might only raid for a finite time and shift targets in order to avoid provoking an extermination campaign. Consideration of time also raises questions of discounting. For simplicity I consider the game without discounting since the qualitative properties are the same when this is taken into account.

The produced good held at t is written $x_1(t)$ and the good held by the pirate is written $x_2(t)$. Reference to time is dropped to simplify notation unless required. Since the model implicitly has some notion of handing goods over after terminal time the amount of good left at T has a scrap value. For simplicity this is assumed to be $x_i(T)$ for $i = 1, 2$. The amount of effort devoted to guarding and raiding at time t is written $u_1(t)$ and $u_2(t)$ respectively. These are sometimes referred to as the players controls. The price of b units of fortification is written $p(b)$ where p is the price function and is increasing and differentiable in b with $\frac{dp}{db} > 0$ and $\frac{d^2p}{db^2} > 0$.

Fortification increases the effectiveness of guarding although the precise details are hard to capture because of possible discontinuities. A slightly higher wall, or better electronic encryption system may produce large changes in the level of protection. For present purposes I assume that returns are continuous and that total defensive effort takes the form $\bar{u_1} = u_1 + b$. An additive function has been chosen instead of, say, a function where u_1 and b are multiplied in some way because it is reasonable to assume that guarding is effective even in the absence of fortification.

A difficulty in expressing guarding and fortification as additive is that it is possible to set a level of $\bar{u}_1 > 0$ and b such that $u_1 \leq 0$ and the model needs to specified to prevent this. It would be tempting, but wrong, to introduce a constraint $u_1 \geq 0$, but this would force player one to compensate for b being cheap by increasing u_1. The real difficulty is that, although fortification is a substitute up to a point for positive guarding, it is not reasonable to think that there is any technology of fortification at which player one can buy enough to set guarding negative. The appropriate way to capture this is to put the constraint on the amount that can be purchased. It is assumed that

$$q - b \geq 0 \tag{1}$$

where $q = q : min\ u_1 = \epsilon > 0$ where ϵ is some small number. It is important to note that this is an exogenous constraint on technology and is telling us something about the nature of fortifications. It is not part of the producer's optimization programme with respect to u_1.

The producer's good grows at a rate that depends on the amount left over for investment, after subtracting the amount used for guarding and the amount stolen. The amount stolen is a function of the ratio of the resources devoted to stealing and guarding including fortifications. Remembering that we have suppressed the constant for consumption to simplify the notation, the rate of growth is written

$$\dot{x}_1 = kx_1 - u_1 - s(\frac{u_2}{u_1 + b})^a x_1 \tag{2}$$

where $a < 1$ is a positive constant, $k > 0$ is a constant that represents the technology of production and $s < 1$ is a positive constant that represents the technology of stealing. The constants k and s are referred to as parameters, or coefficeints, of production or stealing, respectively. The form $kx_1 - u_1$ has been used instead of $k(x_1 - u_1)$ to simplify the analysis. It must be remembered that equation (2) is the rate of change in x_1. A situation where $s(\frac{u_2}{u_1+b})^a$ is large is simply interpreted as a rapid run down of the available resource. It appears reasonable however to require $s \leq 1$ to cover dead weight losses to stealing.

Producer's and pirate's problems.

It is now possible to specify the producer's and the pirate's problems in more detail. Since the payoff depends on resources after costs of defensive activities, the producer's problem is to choose a piecewise continuous u_1 and a b in order to maximize

$$J_1 = \int_0^T (x_1 - u_1)dt - p(b) + x_1(T) \tag{3}$$

subject to the constraint in equation (1) and the dynamics of the system given in equation (2).

Because the pirate does not produce, the total amount of the good it possesses is the sum of the amount it has stolen less the amount it has invested in stealing over the time interval and the rate of growth is

$$\dot{x}_2 = s(\frac{u_2}{\bar{u}_1})^a x_1 - u_2 \tag{4}$$

It follows in a similar manner that the pirate wants to choose a piecewise continuous u_2 to maximize

$$J_2 = \int_0^T (x_2 - u_2)dt + x_2(T) \tag{5}$$

subject to equation (4).

It is assumed that the producer and the pirate have complete knowledge of initial resource holdings, the dynamics of the system and the payoff functions. They use this to formulate their guarding and raiding strategies at the beginning of the game and stick to these strategies for all time.

3 The general case.

1. Necessary conditions for guarding and raiding

The optimal expenditures on defence for the producer and raiding for the pirate are established in Appendix 1 for the case where players' use their Nash equilibrium strategies. These are

$$\bar{u}_1 = asx_1\bar{\alpha}_1(\frac{\bar{\alpha}_2}{\bar{\alpha}_1})^a \quad \text{and} \quad u_2 = asx_1\bar{\alpha}_2(\frac{\bar{\alpha}_2}{\bar{\alpha}_1})^a \tag{6}$$

where $\alpha_i(t)$ for $i = 1, 2$ are costate variables for the problems in equations (3) and (5) and $\bar{\alpha}_1 = \frac{\alpha_1}{1+\alpha_1}$ and $\bar{\alpha}_2 = \frac{\alpha_2}{1+\alpha_2}$. This means that

$$\frac{u_2}{\bar{u}_1} = \frac{\bar{\alpha}_2}{\bar{\alpha}_1}$$

It is also necessary that

$$\dot{\alpha}_1 = -1 - \alpha_1(k - s(\frac{u_2}{\bar{u}_1})^a) \quad \text{and} \quad \dot{\alpha}_2 = -1 \tag{7}$$

These conditions allow us to establish some of the general properties of the differential game. From equations (6) and (7) it follows that α_i does not depend on x_1. This means that

$$\frac{\partial(u_2/\bar{u}_1)}{\partial x_1} = 0 \tag{8}$$

and hence u_1 and u_2 both increase for an exogenous increase in x_1 with everything else constant.

It follows from equations (4) and (6) that $\dot{x}_2 = s(\frac{\bar{\alpha}_2}{\bar{\alpha}_1})^a x_1(1 - a\bar{\alpha}_2)$. This means that

$$\dot{x}_2 > 0 \tag{9}$$

and the pirate's resources are always increasing for $x_1 > 0$ since $a < 1$ and $\bar{\alpha}_2 < 1$.

This seems to indicate the pirate more or less farms the producer and that it pays it to leave resources in the early period of the game in order to be able to take more later on. Note that the result is independent of the scrap value since $\bar{\alpha}_2 < 1$ always.

2. Fortifications.

The necessary condition for the optimal level of fortification is, from Appendix 1, that

$$\frac{\partial p}{\partial b} = \int_0^T as\alpha_1 x_1 u_2^a \bar{u}_1^{-(1+a)} dt - \eta \tag{10}$$

and this immediately gives us the following proposition

Proposition 1. *The optimal level of fortification depends only on p, k, s, and T when $\eta = 0$.*

Proof. Substituting the results from equation (6) into the term under the integral in (10) gives

$$\frac{\partial p}{\partial b} = \int_0^T (1 + \alpha_1)$$

where α_1 only depends on the parameters k, s and T from equation (7).
□

Proposition 2 says that the amount of fortification purchased, when the upper limit constraint is not active, is the same for all values of $x_1(0)$. This is something of a surprise since it might have been expected that fortification would increase if there were more to guard and it can be shown that resources increase as $x_1(0)$ increases. It will be observed that from the proof of this proposition and Proposition 2, however, that more fortifications will be purchased as the coefficient of production increases for s constant. This because α_1 increases as k increases. It is also shown that guarding increases with an increase in resources. It seems, then, that the producer's optimal purchase of fortifications depends on productivity and that differences in resources, for the same level of productivity are compensated for by adjusting guarding.

This proposition also tells us that, the longer the time horizon for the technology, the more fortification is purchased. This time horizon effect may, however, be offset by the possibility that, if the producer's resources are decreasing, the inequality constraint may become active and reduce the amount of fortification that can be purchased. This point is discussed further below.

For any set of parameters, or initial values of the resource, such that the constraint on fortifications is active the amount purchased is $b = q$. For given parameters it follows from (6) that this will occur when x_1 is sufficiently small. This might happen if either $x_1(0)$ is small or x_1 becomes small at some time as the result of raiding. In the second case the combination of fortification and guarding is insufficient to protect the resources and the producer does better by putting most effort into guarding.

3. Conditions on the costate.
Before looking at the details of different cases it is useful to establish the following facts about the costate.

Proposition 2. [a] Suppose $k = s$. Then $\alpha_i = T + 1 - t$ for $i = 1, 2$;
[b]. $\frac{d\alpha_1}{dk} \geq 0$.

Proof. See Appendix 2.

This means that for $k > s$ we have $\alpha_1(k,t) > \alpha_1(s,t)$. It also follows from this proposition and from equation (7) that, for $k - s > -\delta$ for some small $\delta > 0$

$$\dot{\alpha}_i < 0 \quad \text{and} \quad \alpha_i > 0$$

for and all $t \in [0, 1]$ where $i = 1, 2$. I deal with this in more detail in the remark to Appendix 2.

I start with the case where $k = s$ since this provides a complete analytical solution. From the constraint on feasible values of s we have $k, s \leq 1$. I will then concentrate on the case where $k > s$ since this is the most plausible and the most interesting.

4 Parameters of stealing and production equal. $k = s$.

1. Trajectories.
The rate of growth from production and the loss to stealing are always equal along the optimal trajectory, in this case, regardless of the initial values of the resource, and the amount of resources in the system decreases for all time. Since the producer's resource is constantly decreasing the amount spent on fortifications must become negligible for T sufficiently large. This offsets the tendency to invest more in

fortification for a larger T noted in the discussion of Proposition 1.

In order to support these assertions, the trajectory for the produced good and the relation between fortification guarding and raiding are obtained by substituting the results from Proposition 2 into equations (2) and (6). This gives

$$\dot{x}_1 = -u_1$$

where $u_1 = asx_1\bar{\alpha}_1 - b \geq 0$ and this equation is solved in Appendix 3 to get the explicit trajectory for x_1. See fig. 1. (a). It follows immediately that

$$\dot{x}_1 \leq 0 \quad \text{and} \quad \ddot{x} \geq 0$$

where the second inequality comes from the sign on $\dot{u}_1$. Hence x_1 tends to zero for t sufficiently large but at a decreasing rate. It can also be shown by differentiating the equation for x_1 in Appendix 3 that as T becomes larger x_1 converges for different initial values $x_1(0)$

To get the sign on $\dot{u}_1$ note that from equation (6) and Proposition 2 we have $u_2 = \bar{u}_1$ and differentiating $\bar{u}_1$ gives

$$\dot{u}_i \leq 0$$

for $i = 1, 2$ for all t. It also follows from the fact that $\bar{u}_1 = u_1 + b$ that $u_1 < u_2$ for a price p such that $b > 0$ along an optimum path.

Since losses to the producer are gains to the pirate there is no conclusion about the trajectory of x_2 that follows automatically from the fact that $\dot{x}_1 < 0$. We know from equation (9) however that $\dot{x}_2 > 0$ and rewriting $\dot{x}_2 = sx_1(1 - a\bar{\alpha}_1)$ gives the additional information

$$\ddot{x}_2 > 0$$

for all t. This gives a trajectory in phase space something like fig. 1. (b).

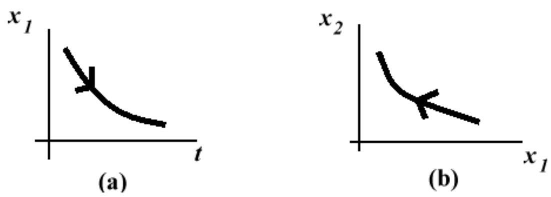

Figure 1. Trajectory of resources. (a) For producer. (b) For producer and pirate.

An interesting property of this trajectory is that, since x_2 is increasing for all time at an increasing rate, it never pays the pirate to raid at such a high level that it drives the producer's resources too small. It will be noted that raiding activity decreases with time. This might be interpreted as something like the pirate farming the producer.

What seems to be happening in this case is that the producer's productivity is so low that it is too poor to defend itself sufficiently to stop its resources declining. I consider what happens when the producer is wealthier and productivity increases in the discussion of $k > s$.

2. Fortifications.
The fact that x_1 is decreasing means that the amount of produced goods will eventually become small. It follows that the amount of effort put into guarding and raiding also becomes small since $x_1 \to 0$ means that $u_1 \to 0$. Because of the technological constraint on fortifications it follows that $\bar{u}_1 \to 0$ for any price function p. This means that $x_1 > 0$ for all time regardless of the initial value of x_1. It also means that, for T sufficiently large, the amount of fortifications purchased by the settler becomes negligibly small.

There are at least two ways to look at this result. The first is that, if the initial value of the resource is low there isn't much to protect and the settler can't afford much fortification. The second adds to our intuition about the technological constraint. It is that, for T large, the amount of the resource becomes small for some period of time and also $u_2 \to 0$. This means that there is a period of time during which the fixed fortification becomes useless and no longer justifies the initial expense. It follows that the settler is better off investing in temporary guarding costs which can be wound down as the resource becomes smaller.

When the technological constraint on fortification is not active the price affects the strategies of both the settler and the pirate. To understand this the equation for x_1 with respect to the price of fortifications along the optimal path is differentiated in Appendix 3. Using the fact that $\frac{\partial b}{\partial p} < 0$ from equation (10) gives

$$\frac{\partial x_1}{\partial p} \leq 0 \tag{11}$$

It follows that the producer has less resources at any time if the price of fortifications increases, as might be expected.

Changes in effort devoted to fortification and guarding are slighly less obvious. It follows from equation (11) that

$$\frac{\partial \bar{u}_1}{\partial p} \leq 0 \tag{12}$$

for $i = 1, 2$ and hence the total amount of resources devoted to defending and to stealing decrease as the price of fortifications increases.

In the case of resources devoted to guarding the sign on $\frac{\partial u_1}{\partial p}$ is given in Appendix 3 by the sign on $(-1 - \frac{\bar{\alpha}_1}{\bar{\alpha}_1(0)} \frac{\partial b}{\partial p})$. This gives

$$\frac{\partial u_1}{\partial p}\left|\begin{array}{llll} <0 & \text{for all} \quad t & \text{for} \quad \frac{\partial b}{\partial p} & \text{sufficiently small} \\ >0 & \text{for} \quad t \in [0,s) & \text{and} \quad <0 & \text{for} \quad t \in (s,1] \\ >0 & \text{for all} \quad t & \text{for} \quad \frac{\partial b}{\partial p} & \text{sufficiently large} \end{array}\right.$$

What the first inequality, and equation (12), tells us is that, if the optimal level of fortifications is not very sensitive to price, the producer reduces expenditure on fortification and also guards less. In this case the producer is trying to minimize the loss of fortification by shifting expenditure out of guarding. Note that, from equation (10), the optimal level of fortifications will not be sensitive to an increase in price when the quantity of fortifications purchased is not sensitive to price.

In the second and third inequalities the producer is partly compensating for a reduction in the amount of fortifications purchased by increasing guarding. This also tells us that, if guarding cannot be increased for all time, an increase near the beginning will have the greatest overall impact on resources.

In order to generalize the analysis, and deal with the case where the coefficient of production changes relative to that of sstealing, I now consider the case $k > s$. For $k < s$ it is obvious that the system contracts for all t.

5 Unequal parameters. $k > s$.

1. Trajectories.

The rate of growth for the produced good might be expected to increase for $k > s$, but this is not guaranteed since the pirate may respond with a greater stealing effort and swamp the gains. Since there isn't much hope of getting an explicit solution for α_1 in terms of t we have to use the equations in (6) to analyse the trajectory of the good. Rewriting equation (2) gives

$$\dot{x}_1 = \varphi(t)x_1 + b \tag{13}$$

where $\varphi = k - s(\frac{\bar{\alpha}_2}{\bar{\alpha}_1})^a(1 + a\bar{\alpha}_1)$ and this is solved explicitly to give

$$x_1(t) = be^{\int_0^t \varphi(s)}\left(\int_0^t e^{-\int_0^s \varphi(q)} ds + x_1(0) - p(b)\right) \tag{14}$$

The effect of a change in the parameter of production on φ is calculated in in Appendix 4 (a) to give

$$\frac{\partial \varphi}{\partial k} > 0$$

It is not possible to get a sign for $\dot{\varphi}$ but it is bounded with $\mid \dot{\varphi} \mid < \delta(\frac{1}{2} \pm \epsilon\dot{\alpha_1})$ for $\delta < 1$ and some small ϵ. See Appendix 4 (b).

The possible trajectories for x can now be expressed in terms of k for s fixed as follows.

(i). $k \geq k_1 : \varphi > -\epsilon$ for some $\epsilon > 0$ for all t. Then $\dot{x}_1 > 0$ for all t.

(ii). $k \leq k_2 : \varphi < -\delta$ for all t for some $\delta > \epsilon$. Then for $x_1(0)$ sufficiently large $\dot{x}_1 < 0$ for all t and for an $\epsilon : x_1(0) < \frac{-b}{\varphi}$ it is possible $\dot{x}_1 > 0$. See fig. 2 for an example. The slope of x_1 comes from $\ddot{x}_1 = \varphi\dot{x}_1 + \dot{\varphi}x_1$ where $\dot{\varphi}$ is assumed small.
(iii). $k_2 < k < k_3 : \dot{x}_1$ switches signs.

It might help to set out the connection between case (ii) and (iii). Remember from Proposition 2 that b will depend on k through the inequality constraint $u_1 - b \geq \epsilon$ for $\epsilon > 0$ some small number. There are two possibilities:
(a). k decreasing in case (ii). This means $\frac{-b}{\varphi}$ becomes smaller since φ becomes smaller and the line $\frac{-b}{\varphi}$ in fig. 2 moves down. At some point, the condition $x_1(0) < \frac{-b}{\varphi}$ cannot be met and $\dot{x} < 0$ always. It will be noted from the previous section that this occurs when $k = s$. In this case there is no starting point below the line $\frac{-b}{\varphi}$ because there is no $x_1(0) < \frac{-b}{\varphi}$ such that $u_1 = asx_1\alpha_1 - b > 0$.
(b). k increasing. This means $\mid \varphi \mid \rightarrow 0$ and $\frac{-b}{\varphi}$ becomes arbitrarily large. This means that for $b > 0$ and any $x_1(0)$ it is possible to find a $\varphi < \epsilon$ such that $\dot{x} > 0$ always.

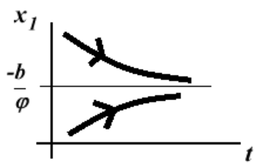

Figure 2. Example of dynamics for the produced good for $\varphi < 0$ and $k > s$.

Case (iii) takes care of the indeterminacy for $\dot{x}$ in the intermediate region $k_2 < k < k_3$. Since we do not have a sign for $\dot{\varphi}$ it is possible that this is either increasing or decreasing. This can be thought of as less general and, since $\mid \dot{\varphi} \mid$ is small, less important than cases (i) and (ii). I have listed the possibilities in Appendix 4 (c).

It is more difficult to analyze the trajectories of resources devoted to stealing and raiding than in the previous section because these depend on the value of x_1. In Appendix 4 the equation in (6) is differentiated to give

$$sign\ \dot{u}_i = \ sign\ (\frac{\dot{x}_1}{x_1}\bar{\alpha}_i + \dot{\alpha}_i \mp \frac{\tau}{x_1})$$

for some small τ, for $i = 1, 2$. It is immediate that, if $\dot{x}_1 < 0$ we have $\dot{u}_i < 0$, although it is, of course, more accurate to say that the trajectory $\dot{x} < 0$ is the optimal path produced by the optimal contols $\dot{u}_i < 0$. If $\dot{x}_1 > 0$ and $\frac{\dot{x}_1}{x_1}$ is sufficiently large $\dot{u}_i > 0$.

2. Fortifications.
The effect of a change in the price of fortifications on the trajectory of the produced good can be obtained by straightforward differentiation of equation (14). This gives

$$\frac{\partial x_1}{\partial p} \leq 0$$

and hence from (6) for defence and raiding

$$\frac{\partial \bar{u}_1}{\partial p} \leq 0 \quad \text{and} \quad \frac{\partial u_2}{\partial p} \leq 0$$

which gives similar trajectories to the case where $s = k$. We also get similar possibilities to the case $k = s$ for changes in the trajectory of guarding.

Increase in the parameter of production.

The effect of an increase in the coefficient of production is to give the producer more resources and to increase expenditure on defence for all time. This follows from the equation (14), the fact that $\frac{\partial \varphi}{\partial k} > 0$ and from the first equation in (6) which give

$$\frac{\partial x_1}{\partial k} > 0 \quad \text{and} \quad \frac{\partial \bar{u}_1}{\partial k} > 0$$

and since $\frac{\partial (\bar{\alpha}_2/\bar{\alpha}_1)^a}{\partial k} < 0$ it also follows that the amount of effort put into raiding relative to the amount of effort put into guarding and fortification decreases as productivity increases. This means that the proportion of the produced good lost to stealing decreases as productivity increases.

It is more difficult to analyse the effect of an increase in productivity on the pirate's resources and on raiding. This is because the pirate gains from having more to steal, but loses because of the increased effort put into fortifications and guarding by the producer and the outcome depends on the ratio of the gains from the extra goods to steal and the losses from extra defence. In addition, even if the pirate increases stealing, it may get less goods owing to the increase in defensive activity.

To begin with it can be shown that the pirate's resources always increase whenever it is optimal for it to put more effort into raiding. This is an interesting result because it rules out the possibility that an increase in raiding is sometimes a purely defensive move designed to reduce losses. It follows from the fact that the signs on $\frac{\partial u_2}{\partial k}$ and $\frac{\partial x_2}{\partial k}$ are given by the sign on

$$\frac{\partial x_1}{\partial k} + a x_1 \frac{\bar{\alpha}_1}{\bar{\alpha}_2} \frac{\partial (\bar{\alpha}_2/\bar{\alpha}_1)^a}{\partial k} \tag{15}$$

It will be noted, from equations (7) and (14) that $\frac{\partial \bar{\alpha}_1}{\partial k}$ is independent of x_1 and $\frac{\partial x_1}{\partial k}$ only depends on $x_1(0)$ but is positive for $x_1(0) = 0$. This means that we can make rewrite equation (15) and alter $\frac{\partial x_1/\partial k}{x_1}$ without changing $\frac{\partial \bar{\alpha}_1}{\partial k}$. For $\frac{\partial x_1/\partial k}{x_1}$ sufficiently large

$$\frac{\partial u_2}{\partial k} > 0 \quad \text{and} \quad \frac{\partial x_2}{\partial k} > 0$$

with the reverse inequalities for $\frac{\partial x_1/\partial k}{x_1}$ sufficiently small. This says roughly that, for any period during which the rate of change in the amount of the resource, relative to the existing amount is large, the pirate does better by increasing its stealing effort.

Since $\frac{\partial x_1}{\partial k} > 0$ always there is some x_1 sufficiently small that $\frac{\partial u_2}{\partial k} > 0$. Consider what happens as k increases. For k sufficiently large it is possible that there is some $k = \kappa$ such that raiding would always decrease for $k > \kappa$. In this case we could have the interesting possibility that the pirate initially becomes better-off as the producer gets more productive and wealthier, but at some point starts to becomes poorer. See fig. 3 for an example. Note from equation (9), however, that, even though the pirate's take decreases with k, it continues to increase throughout the game.

How is a decrease in raiding and in the pirate's resources with an increase in the wealth of the producer explained? There are two intuitions at work in this case. The first is that a fatter target would attract more, rather than less, piracy because there is more to steal. The second is that a fatter target can afford more defence and has sufficient resources to make raiding prohibitively expensive. In this regard, there are many historical examples where pirates prevail when their targets are poor but get put out of business when their targets can afford enough deterrence. A good illustration of this is the demise of piracy and white slavery in the Atlantic and Mediterranean [14]. A more contemporary example comes from individuals or firms that are able to use threats of litigation to prevent raiders moving in on established markets.

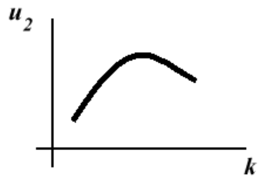

Figure 3. Relation between productivity and resources put into raiding.

Although the second intuition gives some of the explanation it is important to emphasise that simply being rich isn't enough. An increase in wealth alone will not explain the change in raiding because it was shown in equation (8) that raiding and defence increase in proportion for an exogenous increase in the producer's resources. It is wealth together with an increase in productivity that makes it optimal for the producer to spend enough to force the pirate to reduce raiding.

6 Conclusion.

This paper has studied the dynamics of struggles over resources in a model of producers and pirates that may go some way to explaining the dynamics of a firm, or economy, under continuous attack from a raider and help in understanding some of the strategic issues. The specific results are summarised with reference to the questions in the introduction.

(i). The pirate's optimum strategy is to more or less farm the producer so that it takes less near the beginning of the game and increases its take continually. This dynamic is maintained even when the take per unit of time is reduced owing to an increase in productivity. This sort harvesting relationship is probably typical of a far-sighted predator. It indicates that a producer pirate system based on optimizing behaviour does not cycle in the same manner as the Lotka-Volterra predator-prey relation. The trajectory of the producer's resources depend on its productivity and the initial value of its resources.

(ii). If the parameter for production is smaller than the parameter for stealing the producer's resources decline and resources put into defence and raiding decrease. If the producer's parameter of production is increasing, however, and it has sufficient wealth, it can buy enough defence to force the pirate's raiding and resources to decline as productivity increases. These results extend the argument in [2] that security of possessions increases as the ratio of the effectiveness of stealing to effectiveness of guarding declines. In addition to putting the problem in a dynamic setting, I derive the relative effectiveness of stealing and defence from changes in productivity, rather than assuming them directly.

(iii). Resources put into defence are almost always greater than resources put into raiding. The amount of fortifications depends on the parameters of production and stealing and is either constant for all initial values of the resource, or is determined by the lowest level of resources.

(iv). Investment in fortifications may become negligible if there is only a small amount of resources available at any time. One example of this would be where resources are declining and there is a large time horizon. In this case it pays to invest in the flexible guarding technology that can be wound down as resources decrease. This explains why groups that are not productive enough to ensure resource growth in the long run may do better by choosing a flexible guarding strategy. It is not that they cannot afford fortifications, but that the cost does not repay the return.

(v) As the price of fortification increases, less resources are put into fortification and the producer is worse off as expected. Resources put into guarding will also decrease if fortification is not very sensitive to price, but may otherwise increase.

Appendix 1.

The Lagrangian for the producer's problem is

$$L = x_1 - u_1 + \alpha_1(kx_1 - u_1 - (s\frac{u_2}{u_1+b})^a x_1) + \eta(q-b)$$

where $\alpha_1(t)$ is the costate variable and $\eta(t)$ is a multiplier with $\eta(t) \geq 0$ and $\eta(t)(q-b) = 0$. The Hamiltonian for the pirate problem is

$$H_2 = x_2 - u_2 + \alpha_2(s(\frac{u_2}{u_1+b})^a x_1 - u_2)$$

where $\alpha_2(t)$ is the costate variable. From the Pontryagin principle, the necessary conditions for the existence of a set of Nash equilibrium controls are that there exist piecewize continuous u_1 and u_2 that satisfy $\bar{u}_1^{(1+a)} = asu_2^a x_1 \bar{\alpha}_1$ and $u_2^{(1-a)} = as\frac{x_1}{\bar{u}_1^a}\bar{\alpha}_2$ where $\bar{\alpha}_1 = \frac{\alpha_1}{1+\alpha_1}$ and $\bar{\alpha}_2 = \frac{\alpha_2}{1+\alpha_2}$.
For the level of fortification to be an internal optimum we require $\frac{\partial(L-p)}{\partial b} = 0$.

Appendix 2.

Proof of Proposition 2. $[a]$. Suppose $\alpha_i(t) = \psi(t)$ for $i = 1, 2$. Now substitute for $(\frac{u_2}{\bar{u}_1})^a$ expressed in terms of α_1 and α_2 into (7) to give $\dot{\alpha}_1 = -1 = \dot{\alpha}_2$ and solving this gives $\psi_(t) = 2 - t$ and this is sufficient for the proof from the uniqueness of the solution to the differential equation.
$[b]$. The idea is to start at $t = 1$ for $\alpha_1(1) = 1$ and then complete the proof for all t by backward induction. Let $\acute{k} > k$. The rhs derivative at $t = 1$ is just $\frac{d\dot{\alpha}_1}{dk} = -\alpha_1 < 0$. This means that $\dot{\alpha}_1$ is becoming more negative and $\alpha_1(\acute{k}) > \alpha_1(k)$ in some interval $(1-\epsilon, 1]$ by the continuity of $\dot{\alpha}_1$. Hence $\frac{d\alpha_1}{dk} > 0$ in $(1-\epsilon, 1]$ and . Now differentiate $\dot{\alpha_1}$ at $t = 1 - \epsilon$. This gives

$$\frac{d\dot{\alpha}_1}{dk} = -\frac{d\alpha_1}{dk}(k - s(\frac{\bar{\alpha}_2}{\bar{\alpha}_1})^a) - \alpha_1(1 + as(\frac{\bar{\alpha}_2}{\bar{\alpha}_1})^{a-1}\frac{\frac{d\bar{\alpha}_1}{dk}\bar{\alpha}_2}{\bar{\alpha}_1^2})$$

where $\frac{d\bar{\alpha}_1}{dk} = \frac{\frac{d\alpha_1}{dk}}{(1+\alpha_1)^2} > 0$ since $\alpha_1(\acute{k}) > \alpha_1(k)$ at $1-\epsilon$. This means the second term on the rhs is less than zero. Since $\frac{d\alpha_1}{dk} > 0$, $k > s$ and $\frac{\bar{\alpha}_2}{\bar{\alpha}_1} < 1$ at $1-\epsilon$ the first term on the rhs is also less than zero. This means that $\frac{d\dot{\alpha}_1}{dk} < 0$ evaluated at $1-\epsilon$ and hence $\alpha_1(\acute{k}) > \alpha_1(k)$ in some interval $(1-2\epsilon, 1]$ and $\frac{d\alpha_1}{dk} > 0$ in this interval. This is all that is needed to complete the induction.

Remark. It might be expected that $\dot{\alpha}_1 < 0$ always. To see that this cannot be guaranteed consider $k = 0$. This gives $\dot{\alpha}_1(1) = -1 + \alpha_1 s(\frac{u_2}{\bar{u}_1})^a = -1 + s < 0$ since $\alpha_1(1) = 1$ for $i = 1, 2$ and hence $(\frac{u_2}{\bar{u}_1})^a) = 1$. For $t < 1$ we have $\alpha_i(1) > 1$ and $(\frac{u_2}{\bar{u}_1})^a > 1$. This means that it is not possible to exclude $\dot{\alpha}_1 \geq 0$ for some s sufficiently large at $t < 1$.

Appendix 3

Solving $\dot{x}_1 = -u_1$ gives $x_1 = \frac{b}{as\bar{\alpha}_1} + ke^{-as\int^t \bar{\alpha}_1}$ where $\bar{\alpha}_1 = \frac{T+1-t}{T+2-t}$ and $k = (x_1(0) - p(b) - \frac{b}{as\bar{\alpha}_1(0)})(-T-2)^{as}$. This gives

$$x_1(t) = \frac{b}{as\bar{\alpha}_1(t)} + (x(0) - p(b) - \frac{b}{as\bar{\alpha}_1(0)})e^{-ast}(\frac{(T+2)}{(T+2-t)})^{as}$$

Differentiation gives $\frac{\partial x_1}{\partial p} = \frac{\partial b}{\partial p}(\frac{1}{as\bar{\alpha}_1} - \frac{1}{as\bar{\alpha}_1(0)}q) - q$ where $q = e^{-ast}(\frac{(T+2)}{(T+2-t)})^{as}$. Since $\dot{\bar{\alpha}}_1 < 0$ and hence $max\ \bar{\alpha}_1 = \bar{\alpha}_1(0)$ we have $\frac{1}{as\bar{\alpha}_1} \geq \frac{1}{as\bar{\alpha}_1(0)}$. In addition $\frac{dq}{dt} < 0$ and $max\ q = q(0) = 1$. From equation (10)

$\frac{\partial b}{\partial p} < 0$ in equilibrium. Hence $\frac{\partial x_1}{\partial p} \leq 0$.

Appendix 4

(a). $\frac{\partial \varphi}{\partial k} = 1 + as((\frac{\bar{\alpha}_2}{\bar{\alpha}_1})^{a-1}\bar{\alpha}_2(1 + as\bar{\alpha}_1) - s(\frac{\bar{\alpha}_2}{\bar{\alpha}_1})^a)\frac{\partial \alpha_1/\partial k}{(1+\alpha_1)^2}$ where $\frac{\partial \alpha_1}{\partial k} > 0$ from proposition 2. Multiplying the term in brackets through by $(\frac{\bar{\alpha}_2}{\bar{\alpha}_1})^{-a}$ gives $(\bar{\alpha}_1)^{-1}(1 + as\bar{\alpha}_1) - s > 0$ since $s < 1$ and $\bar{\alpha}_1^{-1} > 1$ and it follows that $\frac{\partial \varphi}{\partial k} > 0$.

(b). $\dot{\varphi} = -as(\frac{\bar{\alpha}_2}{\bar{\alpha}_1})^a((\frac{\dot{\bar{\alpha}}_2\bar{\alpha}_1 - \dot{\bar{\alpha}}_1\bar{\alpha}_2}{\bar{\alpha}_2\bar{\alpha}_1})(1 + as\bar{\alpha}_1) + \dot{\bar{\alpha}}_1)$ where $as(\frac{\bar{\alpha}_2}{\bar{\alpha}_1})^a < 1$. A little work gives $\mid (\frac{\dot{\bar{\alpha}}_2\bar{\alpha}_1 - \dot{\bar{\alpha}}_1\bar{\alpha}_2}{\bar{\alpha}_2\bar{\alpha}_1})) \mid < \delta \mid -1 - \dot{\alpha} \mid$ for some small $\delta > 0$ and the whole term inside the brackets following $as(\frac{\bar{\alpha}_2}{\bar{\alpha}_1})^a$ can be approximated by $q \pm \epsilon \dot{\alpha}_1$ for $q < \frac{1}{2}$ and ϵ small.

(c). The possibilities for $\mid \dot{\varphi} \mid \neq 0$ are:

$\varphi(0) < 0$

$\dot{\varphi} > 0$ it is possible that $\dot{x}_1$ changes sign from positive to negative.

$\dot{\varphi} < 0$ it is possible that $\dot{x}_1$ changes sign from negative to positive.

$\varphi(0) > 0$

$\dot{\varphi} > 0$ it is possible that φ changes signs from positive to negative and $\dot{x}_1$ changes sign from positive to negative.

$\dot{u}_1 = \dot{\bar{u}}_1$ since b is fixed. Note that $\dot{u}_1$ and $\dot{u}_2$ will be similar except for the α_i for $i = 1, 2$ terms and their signs will be covered by the same arguments. $\dot{\bar{u}}_1 = as(\frac{\bar{\alpha}_2}{\bar{\alpha}_1})^a(\dot{x}_1\bar{\alpha}_1 + \dot{\alpha}_1 x_1 + \alpha_1 x_1 a(\frac{\bar{\alpha}_1}{\bar{\alpha}_2})(\frac{\dot{\bar{\alpha}}_2\bar{\alpha}_1 - \dot{\bar{\alpha}}_1\bar{\alpha}_2}{\bar{\alpha}_2\bar{\alpha}_1}))$. Dividing by x_1 and remembering that the last term in the expression is small by the previous argument gives sign $\dot{u}_1 =$ sign $\frac{\dot{x}_1}{x_1}\bar{\alpha}_1 + \dot{\alpha}_1 \mp \tau$ for some small τ.

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