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DEPARTMENT OF ECONOMICS

Working Paper

Prosperity and Stagnation in Capitalist Economies

by

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Prosperity and Stagnation in Capitalist Economies

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Abstract

The KMG growth dynamics in Chiarella and Flaschel (2000) assume that wages, prices and quantities adjust sluggishly to disequilibria in labor and goods markets. This paper modifies the KMG model by introducing Steindlian features of capital accumulation and income distribution. The resulting KMGS(teindl) model replaces the neoclassical medium- and long-run features of the original KMG model by a Steindlian approach to capital accumulation, as developed in a paper by Flaschel and Skott (2005). The model is of dimension 4 or 5, depending on the specification of the labor supply. We prove stability assertions and show that loss of stability always occurs by way of Hopf-bifurcations. When global stability gets lost, a nonlinear form of the Steindlian reserve army mechanism can ensure bounded dynamics. These dynamics are studied numerically and shown to exhibit long phases of prosperity, but also long phases of stagnant growth.

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KEYWORDS: KMGS dynamics, accelerating growth, stagnant growth, normal / adverse income shares adjustment, reserve army mechanisms.

1 Steindlian views on prosperity and stagnation in integrated models of KMG growth

Building on Steindl's contributions, Flaschel and Skott (2005) derive a hierarchically structured sequence of Steindlian models. The present paper combines elements of these Steindlian models with the Keynes-Metzler-Goodwin (KMG) approach, developed in Chiarella and Flaschel (2000). Unlike the models in Flaschel and Skott, which assume goods-market (IS) equilibrium, the KMG approach includes explicit disequilibrium adjustment processes in goods and labor markets. The resulting KMGS(teindl) model has five markets: labor, goods, money, bonds and equities and three sectors: households (workers and asset holders, with differentiated saving habits), firms, and the fiscal and monetary authority. All budget equations are fully specified and stock-flow consistency is satisfied.

The KMG approach includes sluggish wage and price adjustment (by means of separate structural equations) as well as sluggish output adjustment via a Metzlerian process of active inventory adjustment.[1] Replacing goods market or IS-equilibrium by this process of delayed responses may seem a minor improvement of the specification of the goods market. Mathematically, however, it is a big step, since it replaces an algebraic equation by two differential equations. Moreover, even when adjustments are fast, the properties of systems that incorporate these explicit adjustment mechanisms need not, in general, generate outcomes that are similar to those characterizing systems with instantaneous adjustment. Thus, the simpler system of instantaneous IS-equilibrium may be misleading even if adjustment speeds are high. Explicit adjustment mechanisms increases the dimension of the dynamic system and this increase complicates the analysis. These complications, however, are partly offset by the simplifications of avoiding certain nonlinearities that arise in a static equilibrium formulation of the relationship between income distribution and capacity utilization.

While allowing for disequilibrium in real markets, the KMG approach assumes that all financial markets are in equilibrium.[2] The present paper simplifies the financial side and its interaction with the real side even further by assuming an interest rate peg (of the nominal rate) by the central bank and by excluding the conventional influence of the real rate of interest on aggregate demand. Thus, the emphasis is on the real markets and their disequilibrium adjustment processes.

A key difference between the present model and the KMG approach of Chiarella and Flaschel (2000) concerns the determination of trend growth (steady state growth). Chiarella and Flaschel assume that the trend rate of growth is determined by the natural growth rate of the population. Changing participation rates, migration and the existence of hidden unemployment in backward sectors of the economy make this neoclassical closure of the model implausible. The supply of labor (to the modern or capitalist sector) may not be perfectly elastic in the short run but the growth rate of the labor supply

[1] See Skott (1989a,b) for an alternative approach to the modelling of short-run disequilibrium in a process of cyclical growth.

[2] The KMG model analyses the financial markets by way of a simple liquidity preference approach to the money market and the assumption of perfect substitution among all other financial assets. Extensions of this formulations are pursued in Köper and Flaschel (2000).

is affected by the demand for labor. This Steindlian position motivates the alternative specification in this paper. We assume that investment determines the trend growth rate of the economy and that the labor supply adjusts endogenously to this trend. The specification of the growth rate of the labor supply in this paper gives rise to 5D dynamics. An alternative specification yields a simpler 4D system which may be used if one wants to consider the dynamics without fluctuations in the labor-capital ratio; we reserve the analysis of this simpler system for future investigation.

The KMGS model may generate long waves of prosperity and stagnation. The Metzlerian inventory adjustment process can lead to local instability of the steady state, and in this case the destabilizing forces eventually drive the economy out of a neighborhood of the steady state. When this happens in an upward direction, high and increasing employment gradually undermines 'animal spirits' and generates a progressively decreasing trend in the investment function. This declining trend in turn implies that the model may produce recurrent phases of prolonged prosperity and subsequent stagnation. These results, which are confirmed by numerical simulation of the KMGS model, mirror Steindl's (1979) view of prolonged full employment in the 1950's and 1960s as a key factor behind the subsequent stagnation. .

The KMGS model is introduced in detail in the next section. In section 3 we present the reduced-form 5D version of the model and discuss its feedback structures: the dynamics of income distribution, the accelerator mechanism that characterize inventory adjustments, and the accelerator and reserve army mechanisms that are involved in the determination of the rate of accumulation. Section 4 provides the sketch of a proof of local asymptotic stability of the steady state (and its loss of local stability by way of Hopf bifurcations) in the special case of classical saving habits. This section shows how common components in the laws of motion can be manipulated or temporarily frozen in order to simplify the stability calculations in this case. The general case is considered in section 5. In the main text of section 5, we study eigenvalue diagrams that characterize the loss of stability from a numerical perspective and then show – also numerically – that the Kalecki/Steindl reserve army mechanism (whereby accumulation slackens systematically when the economy operates close to its full employment ceiling) can generate bounded fluctuations in a 5D dynamic system that is locally repelling in nature. A rigorous proof of local stability and uniquely existing Hopf-bifurcation points that accompany the loss of stability for faster disequilibrium adjustment processes is relegated to a mathematical appendix. Section 6 concludes the paper.

2 KMGS accumulation dynamics

In this section we start from of a standard K(eynes)M(etzler)G(oodwin) approach to Keynesian monetary macrodynamics.[3] We then extend the model by introducing Steindlian aspects of capitalist accumulation dynamics, including in particular a new non-neoclassical determination of the steady state of the model.

[3] A detailed introduction to the baseline KMG approach and its significance within the conventional literature on Keynesian monetary growth models is given in Chiarella and Flaschel (2000). A thorough analytical, numerical and empirical investigation is provided in Chiarella, Flaschel and Franke (2005).

The model has three types of agent: households (in their capacity as workers and asset holders), firms (that perform the role of production units and investors) and government (acting as the fiscal and monetary authority). These agents interact on markets for goods, labor and financial assets. We have sluggish wage adjustment on the labor market, and sluggish price and quantity adjustment on the market for goods. The price and wage adjustments are related to the employment rate and the degree of utilization of the capital stock, respectively,[4] while quantity adjustment takes place in response to goods market disequilibrium. The financial part of the model is simple and traditional. We use a conventional LM curve and assume that the central bank keeps the nominal interest rate constant.[5] The assumption of a constant interest rate, which is in line with other post Keynesian work, implies that the money supply is fully endogenous and that the demand for money becomes proportional to aggregate income. Government bonds and equities of firms are perfect substitutes as in Sargent (1987), and standard specifications of tax and government expenditure policy allow us to characterize fiscal policy in terms of constant parameters on the intensive-form level of the model.

According to Steindl, deviations of the rate of capacity utilization from the desired level lead to changes in the markup. Unwanted excess capacity, he argued, exerted downward pressure on the markup. The KMG model, as introduced in Chiarella and Flaschel (2000), employs wage-price equations that can generate this Steindlian dynamics of the profit share. Thus, despite the different theoretical origins of its wage-price dynamics, we need not extend the KMG model in this respect.

When it comes to the investment behavior of firms, however, the Steindlian perspective introduces a significant change in the causal structure of the model. The trend rate of growth in the KMGS model is determined by the investment habits of firms, and not, as in Chiarella and Flaschel, by labor force growth. Furthermore, the long-run behavior of firms' investment decisions is influenced by two reserve army mechanisms: a traditional Goodwin growth cycle mechanism via the distribution of income and an additional, direct mechanism. The second mechanism can be characterized by the Kaleckian phrase that 'bosses do not like full employment', see Kalecki's (1943) essay on the ' political aspects of full employment' for details. This Kaleckian mechanism is modelled as a direct but gradual effect of the rate of employment on the trend term in the accumulation function. The Kaleckian mechanism helps to ensure the boundedness of the fluctuations generated by the model and thus the economic viability of the model.

In the KMGS specification investment determines the long-run trend in the growth of capital and output. Natural growth must therefore be endogenized in order to get steady state solutions around which the economy is fluctuating. This endogeneity of labor supply can be obtained in various ways, and we consider two specifications. The specifications are chosen with an eye to getting dynamic systems that are relatively simple and analytically tractable.

With respect to saving and consumption, finally, we follow Steindl and the post Keynesian tradition and allow for differentiated saving habits.

[4]See Chiarella, Flaschel, Groh and Semmler (2000, Ch.5) for an analysis that also allows variations in the utilization rate of the employed part of the labor force.

[5]The standard KMG model linearizes the LM curve. This linearization is unnecessary in the present setting since we assume a constant rate of interest.

Turning now to the structural equations of the K(eynes)-M(etzler)-G(oodwin)-S(teindl) model, we first introduce some definitions:

$$
\begin{aligned}
1. \quad & \underline{\text{Definitions (remunerations and wealth):}} \\
\omega &= w/p, \quad \rho^e = (Y^e - \omega L^d)/K, && (1)\\
W &= (M + B + p_e E)/p, \quad p_b = 1. && (2)
\end{aligned}
$$

This set of equations define the real wage ω, the expected rate of profit on real capital ρ^e (based on expected sales Y^e) and real wealth W. Wealth is composed of money M, fixed price bonds B ($p_b = 1$) and equities E as in Sargent (1987). A full list of notation is given at the end of the paper.

Household behavior is described by the following set of equations:

$$
\begin{aligned}
2. \quad & \underline{\text{Households (workers and asset-holders):}} \\
W &= (M^d + B^d + p_e E^d)/p, \; M^d = h_1 pY, && (3)\\
C &= (1 - s_w)(\omega L^d + rB_w/p - T_w) + (1 - s_c)[(1 - s_f)\rho^e K + rB_p/p - T_p], && (4)\\
S_p &= s_w(\omega L^d + rB_w/p - T_w) \\
&+ s_c[(1 - s_f)\rho^e K + rB_p/p - T_p] = (\dot{M}^d + \dot{B}^d + p_e \dot{E}^d)/p, && (5)\\
\hat{L} &= n = a \quad [\text{or} \quad n = a + i(u - \bar{u}) = \frac{I}{K}], \quad \text{see module 3} && (6)
\end{aligned}
$$

We here start from Walras' Law of Stocks which states that real wealth W, in equation (3), is the constraint for the stock demand for real money balances and real bond and equity holdings at each moment in time. Money demand is specified as a linear function of nominal income pY. The coefficient h_1 will, in general, depend on the interest rate, $h_1 = h(r)$, but by assumption the monetary authorities keep the interest rate constant.[6]

Consumption C, in equation (4), is based on Kaldorian saving habits. It is assumed that workers' saving rate is lower than that of pure asset holders. Note that dividend payments to asset holders are given by a fraction $(1 - s_f)$ of expected profits. For simplicity, we assume that the real taxes T_w, T_p of workers and asset holders are paid in a lump-sum fashion and that taxes net of interest payments on government debt are proportional to the capital stock (see the government module below). These, or similar, simplifying assumptions are common when feedbacks from the government budget constraint and wealth accumulation are excluded from consideration and fiscal policy is treated in a parametric fashion.

Equation (5) provides the definition of aggregate personal saving S_p of workers and pure asset holders (in real terms). Personal saving, by definition, is equal to the sum of changes in the stock of money, bonds and equities held by the personal sector. The desired portfolio compositions and the composition of the changes in holdings of the different assets may be different for the two groups. We assume in line with earlier

[6] See Chiarella, Flaschel, Groh and Semmler (2000, part III) for the treatment of much more general situations.

work on KMG modeling that workers only accumulate financial wealth in the form of short-term government bonds; we also use specific tax collection rules (see module 4.) that allow us to ignore feedbacks from interest income. [7]

Equation (6), finally, describes the growth of the labor supply. One specification equates the growth of L to the trend rate of growth of the capital stock; the other specification equates it to the actual rate accumulation I/K. In the latter case the rate of capacity utilization u and the rate of employment e will be strictly proportional – a particularly simple formulation of Okun's law in the case of a fixed proportions technology - and the labor-capital ratio $l = L/K$ is constant over time. A more satisfactory specification of the labor supply relates the growth of L to the employment rate e. This specification, used in Flaschel and Skott (2005), would increase the complexity of the system considerably, since growth in labor supply and the capital stock then have to interact in a way that allows for a common rate in the steady state. We therefore leave an examination of this case in a KMG context for future research.[8] Next, the behavior of the production sector of the economy is described by the following set of equations:

3. Firms (production-units and investors):

$$Y^p = y^p K,\ y^p = \text{ const. }, \quad u = Y/Y^p = y/y^p \ \ (y = Y/K), \tag{7}$$

$$L^d = Y/x,\ \hat{x} = 0, \quad e = L^d/L = Y/(xL), \tag{8}$$

$$I/K = i(u - \bar{u}) + a, \tag{9}$$

$$\hat{a} = \alpha_1(u - \bar{u}) - \alpha_2(\omega - \bar{\omega}) - \alpha_3(e - \bar{e}) \tag{10}$$

$$S_f = Y - Y^e + s_f \rho^e K = \mathcal{I} + s_f \rho^e K, \tag{11}$$

$$p_e \dot{E}/p = I + \dot{N} - \mathcal{I} - s_f \rho^e K = I + \dot{N} - S_f \tag{12}$$

$$\hat{K} = I/K \tag{13}$$

$$I^a = I + \dot{N} \tag{14}$$

According to equations (7) and (8), firms produce commodities using is a fixed proportions technology characterized by the potential output/capital ratio $y^p = Y^p/K$ and a fixed ratio x between actual output Y and labor L^d. This simple concept of technology allows for a straightforward definition of the rate of utilization of capital u and the rate of employment e.[9]

In equation (9) investment per unit of capital I/K is driven by two forces, the trend accumulation term and the deviation of the actual rate of capacity utilization from the

[7] See again Chiarella, Flaschel, Groh and Semmler (2000, part III) for the treatment of much more general situations and note that the consideration of the allocation of the flow of savings to specific assets is here only presented for completeness and for clarity with respect to future extensions of the model.

[8] Sargent (1987) avoided this (Harrodian) problem by assuming that trend growth term in investment function is given by the natural rate of growth, while this paper takes to some extent the opposite view that labor supply growth (not necessarily natural growth) is adjusting perfectly to the trend growth rate a of the capital stock. Both assumptions relegate the treatment of the question what in fact synchronizes these two growth rates for future investigation.

[9] Chiarella and Flaschel (2000, Ch.5) show that the approach can be extended to the case of smooth factor substitution without much change to the qualitative behavior of the model. See also Skott (1989a) for an analysis of the choice of technique in the context of a Keynesian / neo-Marxian theory of cyclical growth.

rate of utilization. Unlike in the original KMG model, the trend term may change endogenously: the growth rate of a is determined in equation (10) by capacity utilization, the wage rate (which determines the profit share) and the employment rate. This specification of investment implies that aggregate demand is wage led in the short-run, since consumption depends directly and positively on the real wage ω while the negative response of investment occurs with a time delay (through changes in the variable a).

Firms' saving, equation (11), is equal to the excess of output over expected sales (caused by planned inventory changes) plus retained profits. It follows, as expressed in equation (12), that the total amount of new equities issued by firms (their only source of external finance) must be equal to the sum of intended fixed capital investment and unexpected inventory changes minus retained earnings. (This specification may be compared with the formulation of the inventory adjustment mechanism in module 6.)

The next equation (13) states that firms' fixed investment plans are always realized. Output is predetermined in this model (cf. module 6 below) but accommodating changes in inventories make the realization of investment plans possible. Equation (14), finally, describes actual investment I^a as the sum of (actual=intended) fixed investment and actual changes in inventories.

We now turn to a brief description of the government sector:

4. <u>Government (fiscal and monetary authority):</u>

$$T = T_w + T_p, \quad \text{such that} \quad t_w = \frac{T_w - rB_w/p}{K}, t_p = \frac{T_p - rB_p/p}{K} \quad \text{both const.} \tag{15}$$

$$G = \gamma K, \quad \gamma = \text{const.} \tag{16}$$

$$S_g = T - rB/p - G \quad [= -(\dot{M} + \dot{B})/p, \text{ see below}], \tag{17}$$

$$r = r_o, \quad \dot{M} = \dot{M}^d = \hat{p} + \hat{Y}, \tag{18}$$

$$\dot{B} = pG + rB - pT - \dot{M}. \tag{19}$$

This part of the model is kept as simple as possible. In equation (15), taxes net of interest payments are assumed proportional to the stock of capital; similar assumptions have been employed by Sargent (1987, part I) and Rødseth (2000). Real government expenditures are also taken to be constant per unit of capital (equation (16)). Given these proportionality assumptions, fiscal policy is represented by simple parameters in the intensive form of the model. The definition of government saving, equation (17), S_g is the obvious one. Money supply is used to peg the nominal rate of interest, and the growth of money supply μ is driven by the growth rate of output plus the price inflation rate, cf equation (18). The new issue of bonds by the government, finally, is determined residually via equation (19) which states that money and bond financing must exactly cover the deficit in government expenditure financing. Note that the bond financing of government expenditure generates no feedback effects on the real part of the private sector of the economy (since interest rate effects are neutralized and wealth effects are excluded by the formulation of the model).

Our model of disequilibrium dynamics retains some static equilibrium conditions for the financial markets. These equilibrium conditions are given by:[10]

5. Equilibrium conditions (asset-markets):

$$M = M^d = h_1 pY \quad [B = B^d, E = E^d], \tag{20}$$

$$r_o = \rho^e pK/[p_e E] + \hat{p}_e, \tag{21}$$

$$\dot{M} = \dot{M}^d, \ \dot{B}^d = \dot{B} \quad \text{[which implies: } \dot{E} = \dot{E}^d, \quad \text{see appendix 2]}. \tag{22}$$

Asset markets are assumed to clear at all times. Equation (20) describes this assumption for the money market, given the constant interest rate. Bonds and equities are perfect substitutes, and we assume perfect share price expectations; these assumptions are captured by equation (21) which determines the evolution of the price of shares. In the absence of wealth effects, however, share price expectations and the evolution of share prices have no influence of the rest of the model; equation (21) is given for completeness. Given the perfect substitutability between equities and bonds and Walras' Law of stocks, the clearing of the money market implies that the bond and equity market are then cleared as well. With perfect substitutability, the stock demands B^d and E^d are not unique when the expected returns are equal; asset holders are happy with any portfolio composition and the composition of demand for bonds and stocks simply accommodates to the composition of supply.

Note that this discussion concerns secondary asset markets (existing assets) and does not yet guarantee that the new asset demand generated by the flow of savings of households matches with the supply of new bonds issued by the government and the issue of new equities by firms on the primary markets which are separated from secondary ones in continuous time, see Sargent (1987, Ch. II.7) on this matter. Stock market equilibria are thus independent from the extent of the savings decision of households in this continuous time framework. Furthermore, due to the interest rate peg of the central bank, the stock demand for money of households is always fulfilled by the Central Bank and the growth in money supply therefore just given by the sum of current output growth and inflation. In equation (22), it is finally assumed that wealth owners accept the new bond issue by the government for the current period, reallocating them only in the 'next period' by readjusting their portfolios then in view of a changed situation. It is easy to check by means of the saving relationships (budget equations) that the assumed (ex post) consistency between flow supplies and flow demands of money and bonds implies the consistency of the flow supply and demand for equities. These implied flow equalities only represent a consistency condition, implied by the budget constraints for households, firms and the government, but do not explain the forces that lead to their fulfillment on the primary or flow markets for financial assets. The working of these primary markets is not explained in the present approach to KMGS growth. Since flows are 'small' as compared to stocks, however, one may assume for simplicity that the rates of return on financial assets that clear the stock market are sufficient to clear the flow markets as well.

[10]Note that money demand could have been specified as follows: $M^d = h_1 pY + h_2(r_o - r)W)$, by including interest and wealth effects into it, an extension that is irrelevant here due to the assumed interest rate peg.

As financial markets are formulated here they do not matter at all for the evolution of the economy. Our formulations however show how they relate to the original KMG approach and its portfolio extension in Köper and Flaschel (2000) and how their working may be added to future extensions of the KMGS dynamics of this paper.

The goods-market adjustments are described by the following six equations:

6. Disequilibrium situation (goods-market adjustments):

$$S = S_p + S_g + S_f = I + \dot{N}, \tag{23}$$

$$Y^d = C + I + G, \tag{24}$$

$$N^d = \alpha_{n^d} Y^e, \;\; \mathcal{I} = aN^d + \beta_n(N^d - N), \tag{25}$$

$$Y = Y^e + \mathcal{I}, \tag{26}$$

$$\dot{Y}^e = aY^e + \beta_{y^e}(Y^d - Y^e), \tag{27}$$

$$\dot{N} = Y - Y^d = S - I \tag{28}$$

Equation (23) describes the ex post identity between saving S and investment I^a in a closed economy. Equation (24) defines aggregate demand Y^d which is never constrained in the present model. In equation (25), desired inventories N^d are assumed to be a constant proportion of expected sales Y^e. Intended inventory investment $\mathcal{I}$ is determined on this basis via the adjustment speed β_n multiplied by the current gap between intended and actual inventories $(N^d - N)$ and augmented by a growth term to capture the fact that this inventory adjustment rule is operating in a growing economy. Firms' output Y in equation (26) is the sum of expected sales and planned inventory adjustments. Sales expectations are formed in a purely adaptive way, again augmented by a growth trend as shown by equation (27). The trend terms in both expected sales and planned inventory dynamics are both given by a, the trend term in investment behavior and not by the natural rate of growth n as in the standard KMG model and the neoclassical approach to economic growth. Finally, in equation (28), actual inventory changes are given by the discrepancy between output Y and actual sales Y^d, or alternatively, by the difference between total savings S and fixed business investment I.

The disequilibrium adjustment in the goods market is central for the dynamics of the economy. It is our conjecture that the implications of the disequilibrium formulation are different than those of a static IS-equilibrium treatment of the goods market; in future research we shall try to prove that the use of an algebraic condition (IS-equilibrium) in the place of the sluggish adjustment described by differential equations leads to quite different types of behavior and that the IS-equilibrium situation cannot be conceived as the continuous limit of the quantity dynamics.

The final module of the model describes the wage-price spiral:

7. Wage-Price-Sector (adjustment equations):

$$\hat{w} = \beta_{w_1}(e - \bar{e}) - \beta_{w_2}(\omega - \bar{\omega}) + \kappa_w \hat{p} + (1 - \kappa_w)\hat{p}^c, \tag{29}$$

$$\hat{p} = \beta_{p_1}(u - \bar{u}) + \beta_{p_2}(\omega - \bar{\omega}) + \kappa_p \hat{w} + (1 - \kappa_p)\hat{p}^c, \tag{30}$$

$$\dot{\pi}^c = \beta_{p^c}(\hat{p} - \hat{p}^c). \tag{31}$$

These adjustment equations are based on symmetric assumptions concerning the causes of wage- and price-inflation. Our specification follows Rose (1967, 1990) and assumes

that two Phillips Curves (PC's) describe wage and price dynamics separately. The two equations include measures of demand pressure $e - \bar{e}$ and $u - \bar{u}$, in the markets for labor and for goods, respectively, and a feedback effect from the level of the real wage ω (or the wage share) as obtained in Blanchard and Katz (1999) and Chiarella, Flaschel and Franke (2005). Both Phillips curves are expectations augmented to allow for expected changes in cost. We assume a weighted average of myopic perfect foresight and a backward looking measure of the prevailing inflationary climate, symbolized by $\hat{p}^c$. The inflationary climate - measure of expected inflation in the medium term - is determined by adaptive expectations in equation 31. The wage Phillips curve uses a weighted average of the current price inflation $\hat{p}$ and a longer-run concept of price inflation, $\hat{p}^c$, based on past observations. Similarly, cost pressure perceived by firms is given as a weighted average of the current wage inflation $\hat{w}$ and a measure of the inflationary climate in which the economy is operating. Thus, we have two PC's with very similar building blocks; the associated wage-price dynamics or wage-price spirals are discussed and estimated for the US Economy in Flaschel and Krolzig (2004).

The inflationary climate does not matter for the evolution of the real wage $\omega = w/p$ or, due to our assumption of no productivity growth, for the wage share ω/x. The law of motion of the wage share is given by:

$$\hat{\omega} = \kappa[(1-\kappa_p)(\beta_{w_1}(e-\bar{e}) - \beta_{w_2}(\omega - \bar{\omega})) - (1-\kappa_w)(\beta_{p_1}(u-\bar{u}) + \beta_{p_2}(\omega - \bar{\omega})]. \quad (32)$$

where $\kappa = 1/(1-\kappa_w\kappa_p)$.[11] As a special case (if $\beta_{w_1} = 0$ holds) we get (a linearized version of) the law of motion for the profit share π used in Flaschel and Skott (2005) to capture Steindl's views on the determination of the markup. The above more general formula produces a wage price spiral in which demand pressures in the labor market also play a part in the law of motion for the real wage $\omega = w/p$.[12]

It should be noted perhaps that our specification of wage and price formation implies two cross-market or reduced-form wage and price PC's given by:

$$\hat{w} = \kappa[\beta_{w_1}(\cdot) - \beta_{w_2}(\cdot) + \kappa_w(\beta_{p_1}(\cdot) + \beta_{p_2}(\cdot))] + \hat{p}^c, \quad (35)$$

$$\hat{p} = \kappa[\beta_{p_1}(\cdot) + \beta_{p_2}(\cdot) + \kappa_p(\beta_{w_1}(\cdot) - \beta_{w_2}(\cdot))] + \hat{p}^c, \quad (36)$$

These equations generalize the conventional view of a single-market price PC with only one measure of demand pressure, the one in the labor market. The reduced-form PCs synthesize the influence of the inflationary climate and demand pressures from labor and goods markets in a symmetric and general way.

Note, finally, that the restrictions $\beta_{w_1}, \beta_{w_2}, \beta_{p_2} = 0, \kappa_w = 1$ imply the following reduced-form PC:

$$\hat{p} = \beta_{p_1}(u-\bar{u})/(1-\kappa_p) + \hat{p}^c$$

[11] This result follows from the equivalent representation of the two PCs, 29-30,

$$\hat{w} - \hat{p}^c = \beta_{w_1}(\cdot) - \beta_{w_2}(\cdot) + \kappa_w(\hat{p} - \hat{p}^c), \quad (33)$$

$$\hat{p} - \hat{p}^c = \beta_{p_1}(\cdot) + \beta_{p_2}(\cdot) + \kappa_p(\hat{w} - \hat{p}^c), \quad (34)$$

by solving for the variables $\hat{w} - \hat{p}^c$ and $\hat{p} - \hat{p}^c$.

[12] Laws of motion for the profit share π of the type $\dot{\pi} = g(u, \pi, e)$ (with partial derivative that are positive, negative and negative, respectively) can be considered – using simple mathematical substitutions – as nonlinear extensions of the dynamics for the real wage in equation (32); see also Flaschel and Skott (2005) in this regard.

This special case of the wage-price spiral removes the role of income distribution from the dynamics, since real wages are held constant in this situation. The equation differs from a monetarist PC in that it is demand pressure on the market for goods that matters and that the expectations augmented part is captured by an adaptive climate expression.

This ends the description of the extensive form of our KMGS dynamics, its behavioral rules and the budget equations within which this behavior takes place.

3 The KMGS dynamics and their feedback structures

Combining the equations of module 6 of the model, the actual output-capital ratio is determined by:

$$y = (1 + a\alpha_{n^d})y^e + \beta_n(\alpha_{n^d}y^e - \nu), \quad y^e = Y^e/K, \ \nu = N/K. \tag{37}$$

Aggregate demand per unit of capital is (using the consumption, investment and government expenditure functions of the model):

$$\begin{aligned} y^d &= (1 - s_w)(\omega y/x - t_w) + (1 - s_p)(1 - s_f)(\rho^e - t_p) + a + i(u - \bar{u}) + g \\ &= (1 - s_p)(1 - s_f)y^e + (s_p(1 - s_f) + s_f - s_w)\omega y/x + a + iu + \gamma \end{aligned} \tag{38}$$

where γ collects the given magnitudes in this aggregate demand expression. Using (1),(7),(8), the equations for the expected rate of profit ρ^e, the rate of employment e and the rate of capacity utilization u are:

$$\rho^e = y^e - \omega y/x, \quad e = y/(xl) \quad (= L^d/L = xL^d/xL), \quad u = y/y^p.$$

We use the following five state variables: sales expectations $y^e = Y^e/K$ and inventories $\nu = N/K$ per unit of capital (for the short run dynamics), trend growth a as part of investment behavior, the factor ratio l (in place of the rate of employment)[13], and the real wage ω. Unlike in the original KMG model, real balances per unit of capital $m = M/(pK)$ and the inflationary climate $\hat{p}^c$ no longer appear, and the so-called Keynes and Mundell-effects are excluded.

Taken together, the laws of motion of the KMGS dynamics read:

[13]The employment rate $e = y/(lx)$ might seem a more obvious choice of state variable. The evolution of l, however, follows a particularly simple law of motion.

$$\dot{y}^e = \beta_{y^e}(y^d - y^e) - i(u - \bar{u})y^e, \tag{39}$$

the law of motion for sales expectations,

$$\dot{\nu} = y - y^d - (i(u - \bar{u}) + a)\nu, \tag{40}$$

the law of motion for inventories,

$$\hat{a} = \alpha_1(u - \bar{u}) - \alpha_2(\omega - \bar{\omega}) - \alpha_3(e - \bar{e}), \tag{41}$$

the trend growth dynamics,

$$\hat{l} = -i(u - \bar{u}), \quad [\text{or stationary and given by} \quad l \equiv l_o, \quad \text{if} \quad n = I/K] \tag{42}$$

the evolution of the full employment labor intensity,

$$\hat{\omega} = \kappa[(1 - \kappa_p)(\beta_{w_1}(e - \bar{e}) - \beta_{w_2}(\omega - \bar{\omega})) - (1 - \kappa_w)(\beta_{p_1}(u - \bar{u}) + \beta_{p_2}(\omega - \bar{\omega})], \tag{43}$$

the law of motion for real wages.

We assume that

$$0 \leq s_w < s_p \leq 1, \quad 0 \leq s_f \leq 1, \quad \text{and} \quad 0 < \kappa_w, \kappa_p < 1.$$

Inserting the algebraic equations of this section into these laws of motion one obtains a nonlinear autonomous 5D system of differential equations. The properties of this system will be examined in the remainder of this paper. First, however, we briefly consider the important feedback chains.

The original KMG model contained four important feedback chains. The KMGS model excludes two of these chains but introduces two new feedback chains: a dynamic Harrodian accelerator mechanism and a Kalecki-Steindl reserve army mechanism. The feedback chains interact with each other in the full 5D dynamics, and different feedback mechanisms can become dominant, depending on parameter values.

1. The Keynes and Mundell effects: Neither the stabilizing Keynes effect nor the destabilizing Mundell effect is present in the KMGS model. The reason is simple: we have excluded any influence of the real rate of interest on investment and consumption and wealth effects on consumption are absent too. Thus, although price inflation appears in the real wage dynamics, it does not affect aggregate demand.

2. A Metzler type inventory accelerator mechanism: The Metzlerian inventory adjustment process defines two laws of motion; equations (39) – (40). The crucial parameters in these adjustment equations are the adjustment speeds, β_{y^e} and β_n, of sales expectations and of intended inventory, respectively.

From the static equation (37) it follows that output y depends positively on expected sales y^e and this effect is stronger the higher the speed of adjustment β_n of planned inventories. Using the static equation for y^d it then follows that the time rate of change of expected sales depends positively on the level of expected sales when the parameter β_n is chosen sufficiently large. Flexible adjustment of inventories coupled with a high speed of adjustment of sales expectations may therefore be expected to jeopardize economic stability through a refined multiplier-accelerator mechanism.

3. A Harrod type investment accelerator mechanism: This mechanism works through the parameters i and α_1 in the investment equations. Increased capacity utilization leads to higher investment (both directly and via the gradual changes in the trend of accumulation) and an in increase in aggregate demand. As a result, sales expectations increase and produce a further rise in output and capacity utilization. Thus a dynamic Harrodian multiplier-accelerator process interacts with distribution effects and the Metzlerian inventory adjustment process. Trend investment can be seen as a utilization climate – like the inflation climate – or as slowly evolving 'animal spirits', and it may be reasonable to assume that the direct effect on current investment is stronger than the indirect effect on trend investment.

4. A Goodwin/Rose type reserve army mechanism: The law of motion for the real wage, equation (32), implies that the level of the real wage exerts a directly stabilizing influence via the Blanchard and Katz (1999) error correction expressions following β_{w_2}, β_{p_2}. But there are additional Goodwin-Rose mechanisms. The specification of aggregate demand in the model implies that the short-term effect of real wages on goods demand is positive (via workers' consumption). Hence, real wages will be further stabilizing through the expressions following β_{w_1}, β_{p_1} if price flexibility with respect to demand pressure on the market for goods is sufficiently high and wage flexibility with respect to demand pressure on the market for labor is sufficiently low (the delayed negative effect of real wages on investment behavior will of course just establish the opposite conclusions). Flaschel and Krolzig (2004) suggest that a situation where wage flexibility dominates price flexibility applies for the US economy in the period following World War II and discuss the implied adverse real wage adjustments or adverse Rose effects in detail.[14]

This analysis suggests that if the rates e and u are positively correlated, income distribution may adjust in a stable manner when prices are more flexible than wages, while it may exhibit centrifugal forces in the opposite case. Investment demand will play a role in the dynamics as well, but it does so in a delayed form via the adjustment of the trend term a in the investment function. When considering the Jacobian of the dynamical system, the effects via consumption will appear in a clearly visible form in their minors of order two, while the effects via investment on income distribution will be more roundabout.

5. A Kalecki/Steindl type reserve army mechanism (the conflict about full employment): Represented by the parameter α_3 we assume that 'bosses dislike high employment'. Increases in the rate employment e thus exert downward pressure on the a-component in the investment demand function, leading to reduced economic activity and providing a check to further increases in the rate of employment.

The feedback channels 2-5 are summarized in Table 1.[15]

[14] These effects are closely related to the presence of 'Robinsonian instability' as defined by Marglin and Bhaduri (1990); see also Flaschel and Skott (2005).

[15] There is also a tendency towards unstable capital accumulation, described by the feedback chain: $l \overset{-}{\longmapsto} e \overset{+}{\longmapsto} \dot{\omega} \overset{+}{\longmapsto} \dot{u} \overset{-}{\longmapsto} l$.

1. *Metzlerian Accelerator Mechanism:*	$y^e \overset{+}{\longmapsto} y \overset{+}{\longmapsto} y^d \overset{+}{\longmapsto} y^e$
2. *Harrodian Accelerator Mechanism:*	$u \overset{+}{\longmapsto} \dot{a} \overset{+}{\longmapsto} \dot{u}$
3. *Goodwin/Rose Reserve Army Mechanism:*	$\omega \overset{+:C(-:I)}{\longmapsto} u, e \overset{+/-}{\longmapsto} \dot{\omega}$
4. *Kalecki/Steindl Reserve Army Mechanism:*	$e \overset{-}{\longmapsto} \dot{a} \overset{+}{\longmapsto} \dot{u} \overset{+}{\longmapsto} \dot{e}$

Table 1: The Feedback Structure of the KMGS model

Their interaction determines the stability of the interior steady state position of the model. Based on our partial analysis of the feedback channels, we expect that wage flexibility (as measured by β_{w_1}), fast inventory adjustment and fast investment trend adjustment will be destabilizing, while price flexibility (as measured by β_{w_1}) will be stabilizing if the opposite Rose effect is tamed by assuming a low parameter α_2. Hence, the manipulation of the parameters β_{p_1}, α_3 may help to create local stability and / or to ensure the boundedness and economic viability of the trajectories in the case of locally instability.

4 A simplified system

We are now in a position to formulate a baseline theorem on local asymptotic stability, instability and limit cycle behavior. Our approach to this theorem is based on 'feedback-guided stability analysis'. This methodology for the analysis of the high-dimensional dynamic models has been used extensively in Asada, Chiarella, Flaschel and Franke (2003) and Chiarella, Flaschel and Franke (2005). In the present case, the stability analysis again confirms expectations derived from the analysis of the constituent parts of the system.

This section gives an intuitive account of the KMGS dynamics in the special case $s_w = 0, s_f = 0$; the general case is investigated rigorously in the appendix I. In the special case, the static variables are given by:

$$\begin{aligned} y &= (1 + a\alpha_{n^d})y^e + \beta_n(\alpha_{n^d}y^e - \nu) \\ y^d &= \frac{\omega y}{x} - t_w + (1 - s_p)(\rho^e - t_p) + a + i(u - \bar{u}) + g \\ &= (1 - s_p)y^e + s_p\omega y/x + a + iu + const. \end{aligned}$$

while the laws of motion and the equations for the expected rate of profit ρ^e, the rate of employment e and the rate of capacity utilization u remain unchanged ($\rho^e = y^e - \omega y/x, \quad e = y/(xl), \quad u = y/y^p$).

Theorem I:

Assume that labor supply growth fulfills $n = a$. The following statements then hold with respect to the 5D dynamical system (39)-(43):

1. *There exists a unique interior steady state of the model with* $\omega_o = \bar{\omega}, e_o = \bar{e}, u_o = \bar{u}$. *Moreover, this steady state is characterized by* $y_o = \bar{u}y^p, y_o^d = y_o^e = y_o/(1 + a_o\alpha_{n^d})$, $\nu_o = \alpha_{n^d} y_o$, *and* $l_o = y_o/(x\bar{e})$. *Finally, the steady trend component in investment behavior* a_o *is the uniquely determined positive solution of the following quadratic equation, characterizing steady state goods market equilibrium (if the left hand side is larger than the right hand side at* $a_o = 0$*):*[16]

$$s_p y_o/(1 + a_o\alpha_{n^d}) = s_p\omega_o y_o/x - t_w - (1 - s_p)t_p + a_o + g.$$

   This equation implies that the steady state value a_o *of the term a depends positively on capitalists' savings rate, labor and capital productivity and the taxation rates, while it depends negatively on the real wage and the government expenditure ratio g (assuming in the case of* s_p, y_o *that the parameter* α_{n^d} *is chosen sufficiently close to zero).*

2. *The determinant of the Jacobian of the 5D dynamics at this steady state is always negative.*

3. *Assume that the parameters* $\beta_{w_1}, \alpha_2, \beta_n$ *are chosen sufficiently small and the parameters* α_3, β_{p_1} *sufficiently large. Then: the steady state of the 5D dynamical system is locally asymptotically stable.*

4. *On the other hand, if any one of the parameters* $\beta_{w_1}, \alpha_2, \beta_n$ *become sufficiently large (the latter for* β_{y^e} *sufficiently large), then the equilibrium is locally repelling and the system undergoes a Hopf-bifurcation at an intermediate value of these parameters. In general, stable or unstable limit cycles are then generated close to the bifurcation value.*

<u>Remark:</u> The theorem says that increasing β_{p_1}, α_3 is good for economic stability, while α_1, α_2 implement feedback chains (for quantity and real wage adjustment) that imply accelerating subdynamics and endanger the stability of the whole system. Note also that the investment functions (and $n = a$) enforce the steady state value $\bar{u}$ of u, while the steady state values of ω and e are jointly determined by the wage-price spiral and the accumulation trend function. The steady state value of a finally follows from the steady-state goods-market equilibrium condition. We conjecture that the steady state will be locally unstable under reasonable assumptions on the parameters of the model, but that an increasing parameter α_3 may bound the dynamics far off the steady state (as will be shown below by way of a numerical example).

Sketch of Proof:
1. It is easy to show that the steady state value of the variable a is determined in the assumed situation by a quadratic function of the following type:

$$a^2 + const_1\, a - const_2 = 0, \quad const_1, const_2 > 0.$$

This quadratic function has exactly one positive and one negative real solution, as is obvious from the sign of its value at $a = 0$. Note that the unique positive solution

[16] In the opposite case we will have two negative roots, the larger of which may be used to discuss the case of an economy that is contracting in the steady state.

of the equation can also be shown to exist by arguing that the left hand side in its original presentation is a set of two hyperbola in a_o and the right hand side a 45 degree line. These two curves intersect in the positive quadrant if and only if the 45 degree line starts below the intersection of the right hand hyperbola on the vertical axis. The comparative statics then easily follow from the displacements of the two curves when parameters change.

2. The 5D determinant of the Jacobian of the dynamics at the steady state is easily computed by exploiting (removing) the many linear dependencies that exist in this Jacobian (evaluated at the steady state). This procedure can be applied informally directly on the level of the laws of motion by just removing step by step the reappearing expression from them. So for example, the law of motion for the ratio l can be used to remove the effect of capacity utilization from all other laws of motion, of course only in view of the objective of our analysis, i.e., without causing a change in the sign of the determinant of the then established Jacobian (which is thereby altered in their structure considerably). Next the remaining terms in the law of motion for a and ω, i.e. the influence of e, ω on these state variables, can be adjusted in such a way, again satisfying our objective to get a new determinant that has the same sign as before, that only the influence of ω remains in the $\hat{\omega}$ equation and the influence of e in the a equation, which again restructures the newly obtained Jacobian considerably (without change in sign). The influence of ω where it is still present in the other laws of motion can now be suppressed without change in the sign of the considered determinants, and so on. In this way one finally arrives at the following system of differential equation which have been radically simplified and thus have not much in common with the original system, up to the fact that the sign of the determinant of the Jacobian evaluated at the steady state has not been changed through all the manipulations of the laws of motion we have considered:

$$\begin{aligned}
\dot{y}^e &= +const\ a \\
\dot{\nu} &= -const\ \nu \\
\hat{a} &= -const\ l \\
\hat{l} &= -const\ y^e \\
\hat{\omega} &= -const\ \omega
\end{aligned}$$

The determinant of these reduced dynamics is easily shown to be always negative.

3. Based on our partial knowledge of the working of the feedback channels of the 5D dynamics, we choose an independent 4D subsystem of the full 5D dynamics by setting the parameters β_{w_1}, β_n equal to zero:

$$\begin{aligned}
\hat{\omega} &= \kappa[-(1-\kappa_p)\beta_{w_2}(\omega-\bar{\omega})) - (1-\kappa_w)(\beta_{p_1}(u-\bar{u}) + \beta_{p_2}(\omega-\bar{\omega}))], \\
\dot{y}^e &= \beta_{y^e}(y^d - y^e) - i(u-\bar{u})y^e, \\
\hat{a} &= \alpha_1(u-\bar{u}) - \alpha_2(\omega-\bar{\omega}) - \alpha_3(y/(xl) - \bar{e}), \\
\hat{l} &= -i(u-\bar{u}).
\end{aligned}$$

Under the stated conditions on the parameters $\alpha_2, \alpha_3, \beta_{p_1}$ (which exclude Harrodian instability and Rose type adverse real wage adjustment) one can show that the steady

state is locally asymptotically stable, i.e., that the coefficients $a_i, i = 1, ..., 4$, of the Routh - Hurwitz polynomial (the characteristic polynomial of the considered Jacobian matrix) satisfies the conditions:

$$a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0, a_1a_2 - a_3 > 0, (a_1a_2 - a_3)a_3 - a_1^2a_4 > 0$$

This result is obtained by exploiting again the many linear dependencies in the submatrices of the considered Jacobian (evaluated at the steady state) and by making use of the simple form of the aggregate demand function of in theorem I.

Since the full determinant is positive, we know from the continuity properties of eigenvalues that small variations of the parameters β_{w_1}, β_n away from zero to positive values will preserve the negative real parts of the 4D subsystem, but push the remaining eigenvalue from zero (since the determinant was zero beforehand) towards a negative value, since the determinant of the Jacobian at the steady state of the enlarged dynamics has been shown to be negative and is the product of the five eigenvalues (four of which have already negative real parts) of the full dynamics.

4. Finally, since the determinant of the full Jacobian is always nonzero, loss of stability can only occur by way of (in general non-degenerate) Hopf-bifurcations (where eigenvalues cross the imaginary axis with a positive speed). To complete the proof, it remains to be shown that the system loses its stability if the parameters considered in assertion 4 of the theorem become sufficiently large. This is shown in the case of the wage adjustment speed in the appendix I to this paper. ■

5 The general 5D KMGS accumulation dynamics

In the general version of the model the static variables are given by:

$$y = (1 + n\alpha_{n^d})y^e + \beta_n(\alpha_{n^d}y^e - \nu) \quad (44)$$

$$y^d = (1 - s_w)(\omega y/x - t_w) + (1 - s_p)(1 - s_f)(\rho^e - t_p) + a + i(u - \bar{u}) + g \quad (45)$$

The laws of motion and the equations for ρ^e, e and u are again unchanged.

Theorem II:

Assume that n is given by a, that $\alpha_1/y^p < \alpha_3/(xl_o)$ holds and that the parameters $\alpha_{n^d}, i, \beta_{w_1}, \beta_n, \alpha_2$ are all chosen sufficiently small. Then:
1. There exists a unique interior steady state of the model with $\omega_o = \bar{\omega}, e_o = \bar{e}, u_o = \bar{u}$. Moreover, this steady state is characterized by $y_o = \bar{u}y^p, y_o^d = y_o^e = y_o/(1 + n\alpha_{n^d})$, $\nu_o = \alpha_{n^d}y_o^e$ and $l_o = y_o/(x\bar{e})$.

2. If $(1 - (1 - s_p)(1 - s_f))y_o/(1 + a_o\alpha_{n^d}) > (1 - s_w)(\omega_o y_o/x - t_w) - (1 - s_p)((1 - s_f)\omega_o y_o/x + t_p) + a_o + g$ for $\alpha_0 = 0$, *the steady state component in investment behavior a_o is the uniquely determined positive solution of the following equation, characterizing steady state goods market equilibrium:*

$$(1-(1-s_p)(1-s_f))y_o/(1+a_o\alpha_{n^d}) = (1-s_w)(\omega_o y_o/x-t_w)-(1-s_p)((1-s_f)\omega_o y_o/x+t_p)+a_o+g.$$

3. The steady state solution of the extended dynamics again fulfills the stability assertions 2. – 4. made in theorem I in the preceding section.

Proof: The proof of this theorem is provided in the appendix I of this paper.

Note that the last equation is again a quadratic equation to be solved for the positive value of a_o and that its solution must be used to determine y_o^e and $\rho_o^e = y_o^e - \omega_o y_o/x$. Note also that a unique positive solution for a_o under the stated condition will always exist, since the left hand side is again composed of two hyperbolas in a_o and the right hand side a 45 degree line in this variable (which starts below the intersection of the right hand hyperbola with the vertical axis under the assumed circumstances). Note finally however that a_o will not be positive otherwise, that is the stated condition is also a necessary one for positive steady growth. The dynamics are well defined from the economic perspective as long as y^e, y^d, ν stay positive.

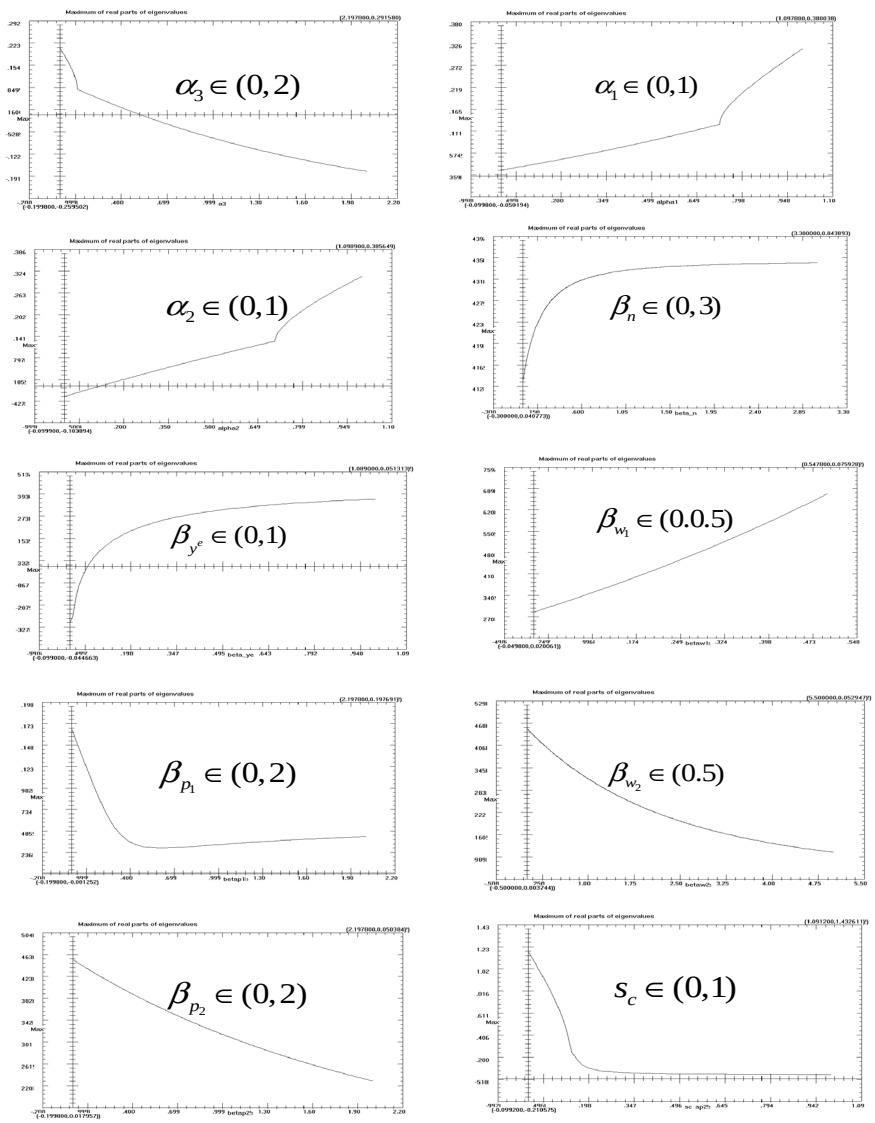

Figure 1: Eigenvalue diagrams (maximum real part) for selected parameters of the model.

Though intrinsically nonlinear, the above 5D growth dynamics are generally too weakly nonlinear to guarantee the boundedness of trajectories when the steady state is locally unstable. Indeed, the eigenvalue diagrams shown in figure 1 (and also simple simulation runs of the dynamics not shown) by and large suggest that destabilizing forces can easily become very strong.[17] When this happens, the viability of the dynamics is quickly lost as the motions then violate economic boundary conditions or even lead to a sudden breakdown of the economy. The eigenvalue diagrams also illustrate the feedback mechanisms discussed in the preceding section and the partial stabilizing or destabilizing reaction patterns they imply. Top-left in figure 1, for example, we see how the maximum real part of eigenvalues behaves when the parameter α_3, characterizing the strength of the Steindlian reserve army mechanism, is increased (all other parameters held constant). We see that the system is unstable if this influence is totally neglected, but is rapidly undergoing a Hopf-bifurcation towards stability as this parameter is increased. For the given baseline set of parameter values, moreover, a_3 is the only parameter for which a sufficient increase is capable of enforcing local asymptotic stability.

Price flexibility with respect to demand pressure and the Blanchard and Katz (1999) error correction mechanism in the wage and price Phillips curve reduce the explosiveness of the dynamics when increased, but cannot – given the other benchmark parameters – enforce the local asymptotic stability over the ranges shown in figure 1 (indeed, if increased beyond a certain point, a further increase in price flexibility may become destabilizing again, via effects on investment behavior as compared to the wage-led situation in consumption demand). As expected from our analysis of the adjustment processes, we see increasing instability if the parameters β_n (β_{y^e}), or α_1 are increased. In the numerical example we also find that an increase in α_2, characterizing the negative influence of real wages on trend accumulation, contributes to instability because an increase in this parameter can make the economy profit-led in the medium them. Although this effect is working with a delay, it may give price flexibility a destabilizing role if it becomes sufficiently pronounced. Finally, wage flexibility with respect to demand pressure on the market for goods is – as expected – destabilizing.

Overall, the example shows that most of the parameters that could be expected to enforce asymptotic stability may fail to establish this aim. The exception is the Metzlerian process as characterized by β_{y^e} and the parameter α_2 where local asymptotic stability is obtained when these adjustment speeds are chosen sufficiently small. Simple phase plot simulations (not shown) also suggest rapidly increasing amplitudes in the observed fluctuations that lead to the violation of economic nonnegativity constraints even for small deviations of the parameters from the Hopf Bifurcation point where the system loses its local asymptotic stability. Thus, extrinsic or behavioral nonlinearities may have to be added in order to ensure boundedness of the trajectories over longer time periods. In this respect increases in the value of α_3 may be the decisive mechanism to provide boundedness when the economy departs significantly from its steady state position.

[17] The parameters used in these eigenvalue calculations are $(s_w, t_w, s_f = 0)$: $\beta_{w_{1,2}} = 0.2, 0.2$, $\beta_{p_{1,2}} = 0.2, 0.2$, $\alpha_1 = 0.1$ $\alpha_2 = 0.1$, $\alpha_3 = 0.5$, $i = 0.2$, $\alpha_{n^d} = 0.1$, $\beta_n = 0.1$, $\beta_{y^e} = 0.5$, $\kappa_{w,p} = 0.5, 0.5$, $y^p = 1$, $x = 2$, $s_p = 0.8$, $t_p = 0.2$, $g = 0.2$, $\bar{e}, \bar{u} = 1$, $\bar{\omega} = 1$. At this baseline parameter set the steady state is not locally asymptotically stable (otherwise all eigenvalue calculations would exhibit sections with negative values), i.e., the eigenvalue plots in figure 1 test which parameters may enforce local asymptotic stability and which ones cannot or even make the situation worse.

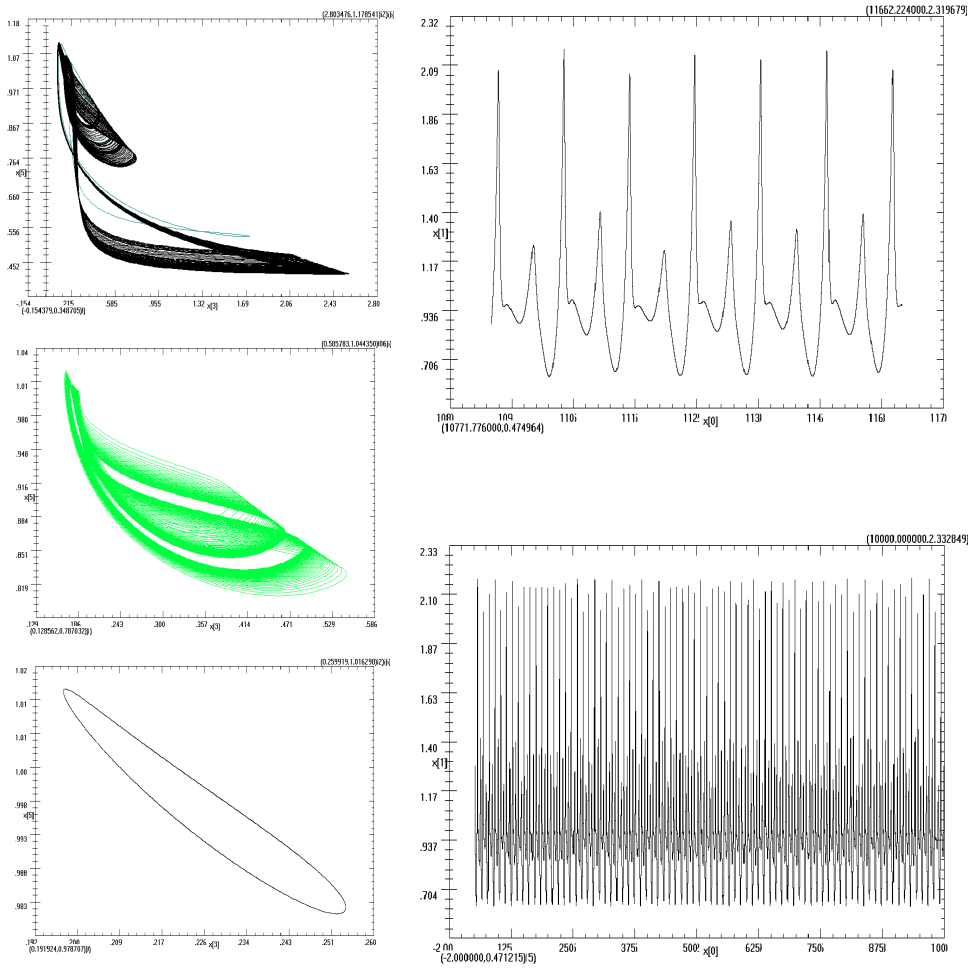

Figure 2: Bounded irregular dynamics through increasing strength of the employment effect on trend accumulation (parameters as before, except α_3 which is 0.1 now at the steady state.)

In figure 2 we have therefore added to the dynamics a nonlinearity with respect to the term $\alpha_3(e - \bar{e})$ in the accumulation function. We now assume that α_3 is an increasing function of e of the form $\alpha_3(e) = \bar{\alpha}_3 e^b, b > 1$ to the right of the steady state level $\bar{e} = 1$,[18] which is not necessarily a strong nonlinearity, depending on the choice of the parameter b. Thus, the value of α_3 now increases as the employment rate approaches the full employment ceiling (no longer explicitly specified), while to the left of the steady state $\alpha_3 = \bar{\alpha}_3$ is constant. This formulation captures the Kaleckian idea that 'bosses do not like full employment' in a simple way. We conjecture that this behavioral nonlinearity is sufficient to tame the strongly explosive nature of the 5D dynamics if the parameter b is chosen sufficiently large. Figure 2 shows that this is indeed the case. Note that the model tends to produce only long-phased cyclical fluctuations. This is to be expected since all parameters are kept time-invariant and since the stabilizing nonlinearity only affects the slow moving trend term in the accumulation function, and because the Metzlerian dynamics is weak in the considered numerical example.

[18]In our numerical example we have renormalized for expositional simplicity the steady state rate of employment such that $\bar{e} = 1$ is now the benchmark for accelerating wage inflation, see the preceding footnote.

The diagrams on the left hand side of figure 2 show the projection into the $a, \omega-$plane of the attractor generated by the simulations of the KMGS dynamics. Top-left we employ the function $0.1e^5$ to the right of the steady state rate of employment 1. In the middle we have changed the parameter b to the value 14, and the diagram at the bottom left is based on $\alpha_3 = 0.43e^7$. The diagram shows that less tension around the steady state (case 3) produces a simple limit cycle. Stronger centrifugal forces around the steady state (case 2) require a much stronger nonlinearity in order to generate this result ($b = 20$), while a weak nonlinearity leads to complicated fluctuations around the steady state as shown in case 2. In case 1 finally we have decreased the slope of the nonlinearity even more. In this case the tension between local repellers and the global stabilizer increases and leads to even more irregularity in the fluctuations, as is further exemplified for this first case on the right hand side of figure 2 (for a simulation run of 900 years and ten thousand years, respectively, with expected sales per unit of capital on the vertical axis). In this case we have a succession of smaller and larger cycles, each about 60 years long. These results are quite intuitive. Economies with a large $\bar{\alpha}_3-$value at the steady state need less nonlinearity in this function and are less volatile in amplitudes than economies with strong centrifugal forces around the steady state. These latter economies are tamed through stronger nonlinearities in the e^b function.

All phase plots share a common feature. They show a largely negative relationship between the trend growth of the capital stock and the real wage. The relation is almost linear when local instability is relatively weak but becomes nonlinear when the local tensions increase (by a decrease of the parameter in front of the e^b function.

It should be noted, finally, that we did not impose supply bottlenecks with respect to labor or capital on the shown trajectories. Thus, implicitly it is assumed that capacity ceilings are absent, despite the large fluctuations in the sales-capital ratio. The additions of such ceilings and the treatment of quantity rationing and implied nonlinearities for wage and price adjustments are not the subject of this paper; see Chiarella, Flaschel, Groh and Semmler (2000, Ch.5) on this matter.

6 Conclusions

In this paper we have combined elements of the Steindlian models of prosperity and stagnation in Flaschel and Skott with the KMG framework developed by Chiarella and Flaschel. The resulting KMGS model leads to significant modifications of the feedback structure of the KMG approach, and our analysis has focused on the real wage channel, two reserve army mechanisms and two quantity accelerator processes of Metzlerian and Harrodian type. The short-term Metzlerian mechanism, however, was not at the center of interest; it was but used for consistency reasons in the place of an incomplete, but simpler dynamic multiplier process.

We obtained local asymptotic stability of the steady state when – broadly speaking – inventory adjustments and the wage-spiral were sufficiently sluggish and the Kaleckian/Steindl reserve army mechanism operated with sufficient strength close to the steady state. These conditions, however, are unlikely to be met; the wage-price spiral may be destabilizing close to the steady state and the negative employment effect on trend in-

vestment may be weak. Thus, the more interesting case is one of local instability. We have shown by means of numerical simulations that in this case of local instability the economy may exhibit long waves of prosperity and stagnation if the Kalecki/Steindl reserve army mechanism becomes dominant close to full employment.

The KMGS model has obvious weaknesses and limitations in its present form. One problem concerns the specification of the growth of the labor supply. The endogeneity of this growth rate is justifiable, both theoretically and empirically, but our specification is less than satisfactory. It has the virtue of tractability and we have some confidence that many of the qualitative properties of the system may be robust to more satisfactory specifications. This conjecture, however, still has to be checked in future work. Another weakness of the model is the near-exclusive focus on real dynamics. The real-rate-of-interest channel, for instance, was excluded by assumption. The model should be extended to include greater interaction between real and financial variables. The financial side, moreover, should go beyond the simple Hicksian LM extension of the original KMG approach of Chiarella et al. This extension, again, is a topic for future research.

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Notation

The KMGS model of this paper is based on the following macroeconomic notation: w

A. Statically or dynamically endogenous variables:

Y	Output
Y^d	Aggregate demand $C + I + \delta K + G$
Y^e	Expected aggregate demand
N	Stock of inventories
N^d	Desired stock of inventories
$\mathcal{I}$	Desired inventory investment
L^d	Level of employment
C	Consumption
I	Fixed business investment
I^a	Actual total investment $= I + \dot{N}$ total investment)
r	Nominal rate of interest (price of bonds $p_b = 1$)
p_e	Price of Equities
$S_p = S_w + S_p$	Private savings
S_f	Savings of firms ($= Y_f$, the income of firms)
S_g	Government savings
$S = S_p + S_f + S_g$	Total savings
$T = T_w + T_p$	Real taxes
G	Government expenditure
ρ, ρ^e	Rate of profit, Expected rate of profit
$e = L^d/L$	Rate of employment
Y^p	Potential output
$u = Y/Y^p$	Rate of capacity utilization
K	Capital stock

w	Nominal wages
p	Price level
$\hat{p}^c$	Inflationary climate (medium–run expectations)
M	Money supply (index d: demand)
L	Normal labor supply
$B = B_w + B_p$	Bonds (index d: demand)
E	Equities (index d: demand)
W	Real Wealth
$\omega = w/p$	Real wage ($u = \omega/x$ the wage share)
$\nu = N/K$	Inventory-capital ratio
$y = Y/K$	Output capital ratio
a	Trend or investment climate component in investment behavior
$n = a$	Endogenous natural growth

B. Parameters

$\bar{e}$	1. NAIRU-type normal utilization rate concept (of labor)
$\bar{u}$	NAIRU-type utilization rate of the capital stock concept (of capital)
i	Investment parameter
h_1	Money demand parameter
β_{w_1}, β_{w_2}	Wage adjustment parameters
β_{p_1}, β_{p_2}	Price adjustment parameters
β_{p^c}	Inflationary expectations adjustment parameter
α_{n^d}	Desired Inventory output ratio
β_n	Inventory adjustment parameter
β_{y^e}	Demand expectations adjustment parameter
κ_w, κ_p	Weights for short– and medium–run inflation
κ	$= (1 - \kappa_w \kappa_p)^{-1}$
y^p	Potential output–capital ratio ($\neq y$, the actual ratio)
x	Output–labor ratio
g	Government expenditures per unit of capital
s_p	Savings–ratio (out of profits and interest)
s_w	Savings–ratio (out of wages and interest)
s_f	Savings–ratio of firms (out of expected profits)
b	Nonlinearity parameter in the numerical simulations
t_p	Tax to capital ratio of workers (net of interest)
t_w	Tax to capital ratio of asset holders (net of interest)
$\alpha_1, \alpha_2, \alpha_3$	Parameters of the trend investment function
$\bar{\omega}$	target real wage of firms

C. Mathematical notation

$\dot{x}$	Time derivative of a variable x
$\hat{x}$	Growth rate of x
x', x_w	Total and partial derivatives
$x_o, etc.$	Steady state values
$y = Y/K, etc.$	Real variables in intensive form
$m = M/(pK), etc.$	Nominal variables in intensive form

Appendix I: Proof of Theorem II

In this appendix we provide a rigoros and detailed proof of Theorem II considered in section 5 of the paper. In accordance with the Post-Keynesian or Kaleckian-Kaldorian macroeconomic tradition we start from the basic assumptions on differential saving habits.

Assumption 1: $0 \leqq s_w < s_p \leqq 1, \quad 0 \leqq s_f \leqq 1$

Under this assumption, we have

$$s_p(1-s_f)+s_f-s_w=(s_p-s_w)+s_f(1-s_p)>0. \tag{A1}$$

In this case, equations (44) and (45) in section 5 give rise to:

$$\begin{aligned}
y &= (1+a\alpha_{n^d})y^e+\beta_n(\alpha_{n^d}y^e-\nu)\equiv y(y^e,\nu,a); && (A2)\\
y_{y^e} &= 1+(a+\beta_n)\alpha_{n^d}>0,\\
y_\nu &= -\beta_n<0,\\
y_a &= \alpha_{n^d}y^e>0\\
y^d &= (1-s_p)(1-s_f)y^e+\{s_p(1-s_f)+s_f-s_w\}\{\omega y(y^e,\nu,a)/x\}+a\\
&+ i\{y(y^e,\nu,a)/y^p-\bar{u}\}+\gamma\equiv y^d(y^e,\nu,a,\omega); && (A3)\\
y^d_{y^e} &= (1-s_p)(1-s_f)+[\{s_p(1-s_f)+s_f-s_w\}\omega/x+i/y^p]\{1+(a+\beta_n)\alpha_{n^d}\}>0,\\
y^d_\nu &= -[\{s_p(1-s_f)+s_f-s_w\}\omega/x+i/y^p]\beta_n<0,\\
y^d_a &= 1+[\{s_p(1-s_f)+s_f-s_w\}\omega/x+i/y^p]\alpha_{n^d}y^e>0,\\
y^d_\omega &= \{s_p(1-s_f)+s_f-s_w\}y(y^e,\nu,a)/x>0
\end{aligned}$$

where $y_{y^e}=\partial y/\partial y^e, \quad y_\nu=\partial y/\partial\nu$ etc.

Substituting (A2) and (A3) into the equations (39) – (43) in the text, we obtain the following nonlinear five-dimensional system of differential equations.

(i) $\dot{y}^e=\beta_{y^e}\{y^d(y^e,\nu,a,\omega)-y^e\}-i\{y(y^e,\nu,a)/y^p-\bar{u}\}y^e\equiv F_1(y^e,\nu,a,\omega)$

(ii) $\dot{\nu}=y(y^e,\nu,a)-y^e-[i\{y(y^e,\nu,a)/y^p-\bar{u}\}+a]\nu\equiv F_2(y^e,\nu,a)$

(iii) $\dot{a}=a[\alpha_1\{y(y^e,\nu,a)/y^p-\bar{u}\}-\alpha_2(\omega-\bar{\omega})-\alpha_3\{y(y^e,\nu,a)/(xl)-\bar{e}\}], \quad \equiv F_3(y^e,\nu,a,\omega,l)$

(iv) $\dot{\omega}=\kappa\omega[(1-\kappa_p)\{\beta_{w_1}(y(y^e,\nu,a)/(xl)-\bar{e})-\beta_{w2}(\omega-\bar{\omega})\}$

$$-(1-\kappa_w)\{\beta_{p1}(y(y^e,\nu,a)/y^p-\bar{u})+\beta_{p2}(\omega-\bar{\omega})\}], \quad \equiv F_4(y^e,\nu,a,\omega,l)$$

(v) $\dot{l}=-li\{y(y^e,\nu,a)/y^p-\bar{u}\}\equiv F_5(y^e,\nu,a,l)$ (A4)

where $i\geqq 0$, and the case of $i=0$ corresponds to the case of the degenerated four-dimensional system with $l\equiv l_0$.

We can easily show that there exists a unique interior steady state which satisfies the following conditions (cf. also **Theorem II** in the text).

$$y^d(y^e,\nu,a,\omega)=y^e, \quad y(y^e,\nu,a)/y^p=\bar{u}, \quad y^e=y(y^e,\nu,a)+a\nu,\omega=\bar{\omega}, \quad y(y^e,\nu,a)/(xl)=\bar{e} \tag{A5}$$

Next, let us investigate the local stability of the equilibrium point (interior steady state) in the special case of $i=0$. We can write the Jacobian matrix of the system which is evaluated at the equilibrium point as follows.

$$J = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & 0 \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} \tag{A6}$$

where

$$\begin{aligned}
F_{11} &= \beta_{y^e}(y^d_{y^e} - 1) \\
&= \beta_{y^e}[\{s_p(1-s_f)+s_f\}\{(\bar{\omega}/x)(1+(a_0+\beta_n)\alpha_{n^d})-1\} - s_w(\bar{\omega}/x)\{1+(a_0+\beta_n)\alpha_{n^d}\}], \\
F_{12} &= \beta_{y^e} y^d_\nu = -\beta_{y^e}\beta_n\{s_p(1-s_f)+s_f-s_w\}(\bar{\omega}/x) < 0, \\
F_{13} &= \beta_{y^e} y^d_a = \beta_{y^d}[1+\{(s_p(1-s_f)+s_f-s_w)\bar{\omega}/x]\alpha_{n^d} y_0/(1+a_0\alpha_{n^d}) > 0, \\
F_{14} &= \beta_{y^e} y^d_\omega = \beta_{y^e}\{s_p(1-s_f)+s_f-s_w\}y_0/x > 0, \\
F_{21} &= y_{y^e} - 1 = (a_0+\beta_n)\alpha_{n^d} > 0, \\
F_{22} &= y_\nu - a_0 = -(\beta_n + a_0) < 0, \\
F_{23} &= y_a - \nu = \alpha_{n^d} y_0\{1/(1+a_0\alpha_{n^d})-1\} < 0, \\
F_{31} &= a_0\{\alpha_1/y^p - \alpha_3/xl_0\}\{1+(a_0+\beta_n)\alpha_{n^d}\}, \\
F_{32} &= -a_0\beta_n\{\alpha_1/y^p - \alpha_3/xl_0\}, \\
F_{33} &= a_0\{\alpha_1/y^p - \alpha_3/xl_0\}\alpha_{n^d} y_0/(1+a_0\alpha_{n^d}), \\
F_{34} &= -a_0\alpha_2 < 0, \\
F_{41} &= \kappa\bar{\omega}\{(1-\kappa_p)(\beta_{w_1}/xl_0) - (1-\kappa_w)(\beta_{p1}/y^p)\}\{1+(a_0+\beta_n)\alpha_{n^d}\}, \\
F_{42} &= -\kappa\bar{\omega}\beta_n\{(1-\kappa_p)(\beta_{w_1}/xl_0) - (1-\kappa_w)(\beta_{p1}/y^p)\}, \\
F_{43} &= \kappa\bar{\omega}\{(1-\kappa_p)(\beta_{w_1}/xl_0) - (1-\kappa_w)(\beta_{p1}/y^p)\}\alpha_{n^d} y_0/(1+a_0\alpha_{n^d}), \\
F_{44} &= -\kappa\bar{\omega}\{(1-\kappa_p)\beta_{w2} + (1-\kappa_w)\beta_{p2}\} < 0.
\end{aligned}$$

We furthermore have

$\lim_{\alpha_{n^d}\to 0} F_{11} = \beta_{y^e}[\{s_p(1-s_f)+s_f\}\{\bar{\omega}/x - 1\} - s_w(\bar{\omega}/x) < 0$ (A7) because $\bar{\omega}/x = \bar{\omega}L^d/Y$ is the equilibrium wage share, and it is less than one.

We now make the following additional assumptions.

Assumption 2. α_{n^d} is so small that we have $F_{11} < 0$.

Assumption 3. The inequality $\alpha_1/y^p < \alpha_3/xl_0$ is satisfied.

Assumption 4. β_n, α_2, β_{w2}, and β_{p2} are sufficiently small.

Assumption 3 implies that α_3 is sufficiently large relative to α_1. It is easy to see that we have the following set of inequalities under Assumption 3.

$$F_{31} < 0, \quad F_{32} > 0, \quad F_{33} < 0. \tag{A8}$$

We shall select the parameter β_{w_1} as bifurcation parameter. In this case, we have the following relationships.

$$F_{41} = F_{41}(\beta_{w_1}) \quad ; \quad F_{41}(0) < 0, \quad F'_{41}(\beta_{w_1}) = \text{constant} > 0 \tag{A9}$$

$$F_{43} = F_{43}(\beta_{w_1}) \quad ; \quad F_{43}(0) < 0, \quad F'_{43}(\beta_{w_1}) = \text{constant} > 0 \tag{A10}$$

Next, let us investigate the local stability/instability of the equilibrium point of this four-dimensional system. In the limiting case of $\beta_n = \alpha_2 = 0$, we have

$$F_{12} = F_{32} = F_{34} = F_{42} = 0. \tag{A11}$$

Under a set of conditions (A11), we can write the characteristic equation of the Jacobian matrix as follows.

$$\Delta(\lambda) \equiv |\lambda I - J| = \lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4 = 0 \tag{A12}$$

where

$$b_1 = -trace\ J = -\underset{(-)}{F_{11}} - \underset{(-)}{F_{22}} - \underset{(-)}{F_{33}} - \underset{(-)}{F_{44}} > 0 \tag{A13}$$

$b_2 =$ sum of all principal second-order minors of J

$$= \begin{vmatrix} F_{11} & 0 \\ F_{21} & F_{22} \end{vmatrix} + \begin{vmatrix} F_{11} & F_{13} \\ F_{31} & F_{33} \end{vmatrix} + \begin{vmatrix} F_{11} & F_{14} \\ F_{41}(\beta_{w_1}) & F_{44} \end{vmatrix} + \begin{vmatrix} F_{22} & F_{23} \\ 0 & F_{33} \end{vmatrix} + \begin{vmatrix} F_{22} & 0 \\ 0 & F_{44} \end{vmatrix} + \begin{vmatrix} F_{33} & 0 \\ F_{43}(\beta_{w_1}) & F_{44} \end{vmatrix}$$

$$= \underset{(-)}{F_{11}}\underset{(-)}{F_{22}} + \underset{(-)}{F_{11}}\underset{(-)}{F_{33}} - \underset{(+)}{F_{13}}\underset{(-)}{F_{31}} + \underset{(-)}{F_{11}}\underset{(-)}{F_{44}} - \underset{(+)}{F_{14}}\underset{(?)}{F_{41}(\beta_{w_1})} + \underset{(-)}{F_{22}}\underset{(-)}{F_{33}} + \underset{(-)}{F_{22}}\underset{(-)}{F_{44}} + \underset{(-)}{F_{33}}\underset{(-)}{F_{44}}$$

$$\equiv -A\beta_{w_1} + B \quad ; \quad A > 0, \quad B > 0 \tag{A14}$$

$b_3 = -($ sum of all principal third-order minors of $J)$

$$= -\begin{vmatrix} F_{22} & F_{23} & 0 \\ 0 & F_{33} & 0 \\ 0 & F_{43}(\beta_{w_1}) & F_{44} \end{vmatrix} - \begin{vmatrix} F_{11} & F_{13} & F_{14} \\ F_{31} & F_{33} & 0 \\ F_{41}(\beta_{w_1}) & F_{43}(\beta_{w_1}) & F_{44} \end{vmatrix} - \begin{vmatrix} F_{11} & 0 & F_{14} \\ F_{21} & F_{22} & 0 \\ F_{41}(\beta_{w_1}) & 0 & F_{44} \end{vmatrix} - \begin{vmatrix} F_{11} & 0 & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & 0 & F_{33} \end{vmatrix}$$

$$= -\underset{(-)}{F_{22}}\underset{(-)}{F_{33}}\underset{(-)}{F_{44}} - \underset{(-)}{F_{11}}\underset{(-)}{F_{33}}\underset{(-)}{F_{44}} + \underset{(+)}{F_{13}}\underset{(-)}{F_{31}}\underset{(-)}{F_{44}} - \underset{(-)}{F_{11}}\underset{(-)}{F_{22}}\underset{(-)}{F_{44}} + \underset{(+)}{F_{14}}\underset{(-)}{F_{22}}\underset{(?)}{F_{41}(\beta_{w_1})}$$

$$-\underset{(-)}{F_{11}}\underset{(-)}{F_{22}}\underset{(-)}{F_{33}} + \underset{(+)}{F_{13}}\underset{(-)}{F_{22}}\underset{(-)}{F_{31}} \equiv -D\beta_{w_1} + E \quad ; \quad D > 0, \quad E > 0 \tag{A15}$$

(because we have $F_{14}F_{43}(\beta_{w_1})F_{31} = F_{14}F_{33}F_{41}(\beta_{w_1})$))

$$b_4 = \det J = \begin{vmatrix} F_{11} & 0 & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & 0 \\ F_{31} & 0 & F_{33} & 0 \\ F_{41}(\beta_{w_1}) & 0 & F_{43}(\beta_{w_1}) & F_{44} \end{vmatrix} = F_{22} \begin{vmatrix} F_{11} & F_{13} & F_{14} \\ F_{31} & F_{33} & 0 \\ F_{41}(\beta_{w_1}) & F_{43}(\beta_{w_1}) & F_{44} \end{vmatrix}$$

$$= \underset{(-)}{F_{22}}\underset{(-)}{F_{44}}(\underset{(-)}{F_{11}}\underset{(-)}{F_{33}} - \underset{(+)}{F_{13}}\underset{(-)}{F_{31}}) > 0 \tag{A16}$$

We shall consider the following function in the following:

$$\Phi(\beta_{w_1}) \equiv b_1b_2b_3 - b_1^2b_4 - b_3^2 \equiv G\beta_{w_1}^2 + H\beta_{w_1} + T \tag{A17}$$

where all of A, B, D, E, G, H, T are constants.

We shall make use of the following mathematical results to prove the main proposition in this appendix.

Lemma 1. (Routh-Hurwitz conditions for a four-dimensional system)

All of the real parts of the roots of the characteristic equation (A12) are negative *if and only if* the set of inequalities

$$b_j > 0 \quad (j = 1, 2, 3, 4), \quad b_1b_2b_3 - b_1^2b_4 - b_3^2 > 0 \tag{A18}$$

is satisfied.

Lemma 2. (Asada and Yoshida(2003))

(i) The characteristic equation (A12) has a pair of purely imaginary roots and two roots with non-zero real parts *if and only if* either of the following set of conditions (A) or (B) is satisfied.

(A) $b_1 b_3 > 0, \quad b_4 \neq 0$, and $b_1 b_2 b_3 - b_1^2 b_4 - b_3^2 = 0$.

(B) $b_1 = 0, \quad b_3 = 0$, and $b_4 < 0$.

(ii) The characteristic equation (A12) has a pair of purely imaginary roots and two roots with negative real parts *if and only if* the following set of conditions (C) is satisfied.

(C) $b_1 > 0, \quad b_3 > 0, \quad b_4 > 0$, and $b_1 b_2 b_3 - b_1^2 b_4 - b_3^2 = 0$.

Now, we can prove the following proposition.

Proposition.

Under Assumptions 1 – 4, we have the following properties.

(i) There exists a parameter value $\beta_{w_1}^0 > 0$ such that the equilibrium point of the system (A4) with $i = 0$ is locally asymptotically stable for all $\beta_{w_1} \in [0, \beta_{w_1}^0)$, and it is unstable for all $\beta_{w_1} \in (\beta_{w_1}^0, +\infty)$.

(ii) The point $\beta_{w_1} = \beta_{w_1}^0$ is the unique Hopf-bifurcation point of the above system. In other words, there exist some non-constant periodic solutions at some parameter values β_{w_1} which are sufficiently close to $\beta_{w_1}^0$.

Proof of the Proposition.

Step 1.

First, we shall prove the proposition in the special case of $\beta_n = \alpha_2 = 0$. In this case, we can make use of the relationships (A13) – (A17).

(i) From (A13) – (A16), we have the following properties.

(P_1) b_1 and b_4 are always positive.

(P_2) $b_2 > 0$ for all $\beta_{w_1} \in [0, \beta_{w_1}^1)$, $\quad b_2 = 0$ for $\beta_{w_1} = \beta_{w_1}^1$, and $b_2 < 0$ for all $\beta_{w_1} \in (\beta_{w_1}^1, +\infty)$, where $\beta_{w_1}^1 \equiv B/A > 0$.

(P_3) $b_3 > 0$ for all $\beta_{w_1} \in [0, \beta_{w_1}^2)$, $\quad b_3 = 0$ for $\beta_{w_1} = \beta_{w_1}^2$, and $b_3 < 0$ for all $\beta_{w_1} \in (\beta_{w_1}^2, +\infty)$, where $\beta_{w_1}^2 \equiv E/D > 0$.

Since $\lim_{\substack{\beta_{w2} \to 0 \\ \beta_{p2} \to 0}} F_{44} = 0$, we have

$$\lim_{\substack{\beta_{w2} \to 0 \\ \beta_{p2} \to 0}} \Phi(0) = \{\underset{(+)}{F_{14}} \underset{(-)}{F_{22}} \underset{(-)}{F_{41}(0)} - \underset{(-)}{F_{11}} \underset{(-)}{F_{22}} \underset{(-)}{F_{33}}\}\{(-\underset{(-)}{F_{11}} - \underset{(-)}{F_{22}} - \underset{(-)}{F_{33}})(\underset{(-)}{F_{11}} \underset{(-)}{F_{22}} + \underset{(-)}{F_{11}} \underset{(-)}{F_{33}})$$

$$+(-\underset{(-)}{F_{22}} - \underset{(-)}{F_{33}}) \underset{(-)}{F_{22}} \underset{(-)}{F_{33}} + (\underset{(-)}{F_{11}} + \underset{(-)}{F_{33}})(\underset{(+)}{F_{13}} \underset{(-)}{F_{31}} + \underset{(+)}{F_{14}} \underset{(-)}{F_{41}(0)}\} > 0, \tag{A19}$$

which means that we have $T > 0$ if β_{w2} and β_{p2} are sufficiently small. In this case, we have

(P_4) $\Phi(\beta_{w_1})$ is a quadratic function of β_{w_1} with $\Phi(0) > 0$.

By the way, we have the following inequalities because of the properties $(P_1) - (P_3)$.

$$\Phi(\beta_{w_1}^1) = -b_1^2 b_4 - b_3^2 < 0, \quad \Phi(\beta_{w_1}^2) = -b_1^2 b_4 < 0 \tag{A20}$$

These inequalities together with (P_4) imply that there exists the unique parameter value $\beta_{w_1}^0$ such that $0 < \beta_{w_1}^0 < \min[\beta_{w_1}^1, \beta_{w_1}^2] \equiv \bar{\beta}_{w_1}$ which satisfies the following properties.

(P_5) $\Phi(\beta_{w_1}) > 0$ for all $\beta_{w_1} \in [0, \beta_{w_1}^0)$, $\quad \Phi(\beta_{w_1}) < 0$ for all $\beta_{w_1} \in (\beta_{w_1}^0, \bar{\beta}_{w_1})$, $\quad \Phi(\beta_{w_1}^0) = 0$, and $\Phi'(\beta_{w_1}^0) < 0$.

Properties $(P_1) - (P_3)$ and (P_5) imply that all of the inequalities (A18) in **Lemma 1** are satisfied for all $\beta_{w_1} \in [0, \beta^0_{w_1})$. In this case, the equilibrium point of the system is locally asymptotically stable.

On the other hand, we have $\Phi(\beta_{w_1}) < 0$ for all $\beta_{w_1} \in (\beta^0_{w_1}, \bar{\beta}_{w_1}]$, and we have $b_2 < 0$ or $b_3 < 0$ (or both) for all $\beta_{w_1} \in (\bar{\beta}_{w_1}, +\infty)$. This means that the equilibrium point of the system is unstable for all $\beta_{w_1} \in (\beta^0_{w_1}, +\infty)$.

(ii) It is apparent that a set of conditions (C) in **Lemma 2** is satisfied at $\beta_{w_1} = \beta^0_{w_1}$. This means that the characteristic equation (A12) has a pair of purely imaginary roots and two roots with negative real parts at $\beta_{w_1} = \beta^0_{w_1}$. Furthermore, we have $\Phi'(\beta^0_{w_1}) < 0$ because of the property (P_5). In other words, the real part of a pair of the complex roots is not stationary with respect to the changes of the parameter value β_{w_1} at $\beta_{w_1} = \beta^0_{w_1}$. These properties ensures that the point $\beta_{w_1} = \beta^0_{w_1}$ is in fact the Hopf-bifurcation point (cf. **Theorem A10** in the mathematical appendix of Asada, Chiarella, Flaschel and Franke (2003)).

We can prove the uniqueness of the Hopf-bifurcation point as follows. It follows from **Lemma 2** (i) that in our model a set of *necessary* conditions of the Hopf-bifurcation is given by

$$b_3 > 0, \quad \Phi(\beta_{w_1}) = 0 \tag{A21}$$

because we always have $b_1 > 0$ in our model. However, we already know that only one point (namely, the point $\beta_{w_1} = \beta^0_{w_1}$) satisfies a set of conditions (A21).

Step 2.

Next, let us extend the above results to the case in which both of β_n and α_2 are positive but sufficiently small. Suppose that β_n and α_2 slightly increased from zero. We can easily see that even in this case, all of the qualitative results which were derived in step 1 are unchanged due to the continuity of the considered relationships, except that the critical parameter value $\beta^0_{w_1}$ slightly changes according to the slight increase of the parameter values β_n and α_2. This completes the proof of the Proposition.

Appendix II: investment-saving identities and flow consistency

From the side of income generation, we get on the one hand for sectoral savings as well as aggregate savings the expressions:

$$\begin{aligned} S_w &= \omega L^d + r_o B_w/p - T_w - C_w \\ S_p &= (1-s_f)\rho^e K + r_o B_p/p - T_p - C_p \\ S_f &= Y - Y^e + s_f \rho^e K \\ S_g &= T_w + T_p - r_o(B_w + B_p)/p - G \end{aligned}$$

and hence

$$S = S_w + S_p + S_f + S_g = Y - \delta K - C - G = I + \dot{N} = I^a.$$

For allocations of savings of the four sectors we have on the other hand by way of the postulated budget equations:

$$\begin{aligned} S_w &= \dot{B}_w/p \\ S_p &= (\dot{M}_p + \dot{B}_p + p_e \dot{E}^d)/p \\ S_f &= I + \dot{N} - p_e \dot{E}/p \\ S_g &= -(\dot{M} + \dot{B})/p \end{aligned}$$

that is we in sum get

$$S = S_w + S_p + S_f + S_g = p_e \dot{E}^d/p - p_e \dot{E}/p + I + \dot{N}$$

and thus – due to $S = I + \dot{N}$ – the consistency condition:

$$p_e \dot{E}^d/p = p_e \dot{E}/p.$$

which always holds as long as government can sell the newly issued bonds (and of course the new issue of money). We thus have that the new issue of equities must be in line with the demand for them at all points in time, if this holds true with respect to money and bonds.