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# Determination of $\langle(\pi\pi)_{I=2}|\mathcal{Q}_{7,8}|K^0\rangle$ in the Chiral Limit

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## Abstract

We reconsider the dispersive evaluation of the weak matrix elements  $\langle(\pi\pi)_{I=2}|\mathcal{Q}_{7,8}|K^0\rangle$  in the chiral limit. The perturbative matching is accomplished fully within the scheme dependence used in the two loop weak OPE calculations. The effects of dimension eight (and higher dimension) operators are fully accounted for. We perform a numerical determination of the weak matrix elements using our dispersive sum rules fortified by constraints from the classical chiral sum rules. A careful assessment of the attendant uncertainties is given.

## I. INTRODUCTION

In this paper, we present an updated and improved procedure for obtaining analytical expressions and numerical evaluations of the matrix elements  $\langle(\pi\pi)_{I=2}|\mathcal{Q}_{7,8}|K^0\rangle$  in the chiral limit. Recall [1] that the ratio  $\epsilon'/\epsilon$  can be expressed numerically in terms of operator matrix elements as evaluated at the scale  $\mu = 2$  GeV in the  $\overline{MS}$ -NDR renormalization scheme [2]

$$\frac{\epsilon'}{\epsilon} = 20 \times 10^{-4} \left( \frac{\text{Im}\lambda_t}{1.3 \cdot 10^{-3}} \right) \left[ 2.0 \text{ GeV}^{-3} \cdot \langle\mathcal{Q}_6\rangle_{2 \text{ GeV}}^{(0)} (1 - \Omega_{\text{IB}}) - 0.50 \text{ GeV}^{-3} \cdot \langle\mathcal{Q}_8\rangle_{2 \text{ GeV}}^{(2)} - 0.06 \right] \quad (1)$$

where

$$\langle\mathcal{Q}_6\rangle^{(0)} \equiv \langle(\pi\pi)_{I=0}|\mathcal{Q}_6|K^0\rangle \quad \text{and} \quad \langle\mathcal{Q}_8\rangle^{(2)} \equiv \langle(\pi\pi)_{I=2}|\mathcal{Q}_8|K^0\rangle \quad , \quad (2)$$

and all other notation is as in Ref. [2]. In a previous work [3], it was shown that in the chiral limit the matrix element  $\langle\mathcal{Q}_8\rangle_\mu^{(2)}$  (and also  $\langle\mathcal{Q}_7\rangle_\mu^{(2)}$ ) is expressible in terms of certain vacuum matrix elements<sup>1</sup>

$$\begin{aligned} \lim_{p=0} \langle(\pi\pi)_{I=2}|\mathcal{Q}_7|K^0\rangle_\mu &= -\frac{2}{F_\pi^{(0)3}} \langle\mathcal{O}_1\rangle_\mu \quad , \\ \lim_{p=0} \langle(\pi\pi)_{I=2}|\mathcal{Q}_8|K^0\rangle_\mu &= -\frac{2}{F_\pi^{(0)3}} \left[ \frac{1}{3}\langle\mathcal{O}_1\rangle_\mu + \frac{1}{2}\langle\mathcal{O}_8\rangle_\mu \right] \end{aligned} \quad (3)$$

where  $F_\pi^{(0)}$  is the pion decay constant evaluated in the chiral limit and the operators  $\mathcal{O}_{1,8}$  are defined as

$$\begin{aligned} \mathcal{O}_1 &\equiv \bar{q}\gamma_\mu \frac{\tau_3}{2} q \bar{q}\gamma^\mu \frac{\tau_3}{2} q - \bar{q}\gamma_\mu \gamma_5 \frac{\tau_3}{2} q \bar{q}\gamma^\mu \gamma_5 \frac{\tau_3}{2} q \quad , \\ \mathcal{O}_8 &\equiv \bar{q}\gamma_\mu \lambda^a \frac{\tau_3}{2} q \bar{q}\gamma^\mu \lambda^a \frac{\tau_3}{2} q - \bar{q}\gamma_\mu \gamma_5 \lambda^a \frac{\tau_3}{2} q \bar{q}\gamma^\mu \gamma_5 \lambda^a \frac{\tau_3}{2} q \quad . \end{aligned} \quad (4)$$

In the above,  $q = u, d, s$ ,  $\tau_3$  is a Pauli (flavor) matrix,  $\{\lambda^a\}$  are the Gell Mann color matrices and the subscripts on  $\mathcal{O}_1, \mathcal{O}_8$  refer to the color carried by their currents. Donoghue and Golowich showed in Ref. [3] how to obtain dispersive sum rules (which we shall refer to as DG1 and DG2 in this paper) for the vacuum matrix elements of the dimension six operators  $\mathcal{O}_1$  and  $\mathcal{O}_8$ . In a later work [4], the presence of higher dimension operators (*i.e.* those having dimension  $d > 6$ ) was identified and their impact discussed.

This paper will extend previous work in several significant respects:

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<sup>1</sup>In this paper, we work with the operator  $\mathcal{Q}_8 \equiv \bar{s}_a \Gamma_L^\mu d_b \left( \bar{u}_b \Gamma_\mu^R u_a - \frac{1}{2} \bar{d}_b \Gamma_\mu^R d_a - \frac{1}{2} \bar{s}_b \Gamma_\mu^R s_a \right)$  to be contrasted with  $\mathcal{Q}_8^{(3/2)} \equiv \bar{s}_a \Gamma_L^\mu d_b \left( \bar{u}_b \Gamma_\mu^R u_a - \bar{d}_b \Gamma_\mu^R d_a \right) + \bar{s}_a \Gamma_L^\mu u_b \bar{u}_b \Gamma_\mu^R d_a$  as used in Ref. [3]. In particular, one has  $\langle(\pi\pi)_{I=2}|\mathcal{Q}_8^{(3/2)}|K^0\rangle = 2\langle(\pi\pi)_{I=2}|\mathcal{Q}_8|K^0\rangle$ . Throughout we define  $\Gamma_{\text{L,R}}^\mu \equiv \gamma^\mu (1 \pm \gamma_5)$ .

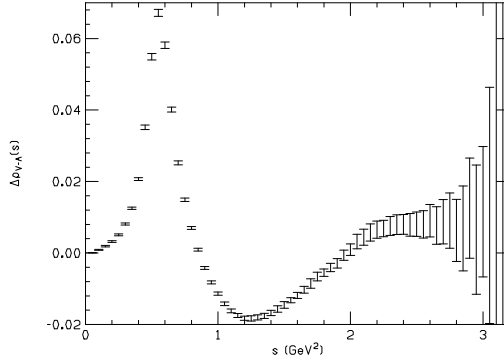


FIG. 1. ALEPH determination of  $\Delta\rho(s)$

1. We provide a two-loop determination of the dimension-six contributions to the operator product expansion (OPE) for the isospin correlator  $\Delta\Pi(q^2)$ ,

$$\begin{aligned}
 & i \int d^4x e^{iq \cdot x} \langle 0 | T (V_3^\mu(x) V_3^\nu(0) - A_3^\mu(x) A_3^\nu(0)) | 0 \rangle \\
 & = (q^\mu q^\nu - q^2 g^{\mu\nu}) \Delta\Pi(q^2) - q^\mu q^\nu \Pi_{A,3}^{(0)}(q^2) \quad .
 \end{aligned} \tag{5}$$

We give our results in  $\overline{\text{MS}}$  renormalization, using both NDR and HV schemes for  $\gamma_5$ , and adopting the same evanescent operators scheme as in Refs. [1,5,6].

2. We extend previous work on dispersive sum rules [3] with an updated derivation which takes into account the NLO renormalization scheme dependence, and incorporates contributions from so-called higher-dimensional operators [4].
3. We perform a dispersive evaluation of  $\langle \mathcal{O}_1 \rangle_\mu$  and  $\langle \mathcal{O}_8 \rangle_\mu$ . Our numerical analysis unifies input from experiment (the existing data base for the spectral function  $\Delta\rho(s)$ ) with rigorous theoretical constraints embodied by the Weinberg [7] and pion mass difference [8] chiral sum rules. An important consequence is the assignment of realistic uncertainties to our results.

## II. THEORETICAL ANALYSIS

In Ref. [3] it was shown how to relate the vacuum matrix elements of  $\mathcal{O}_1$  and  $\mathcal{O}_8$  to the correlator  $\Delta\Pi(q^2)$ . In this section we describe an improved version of the theoretical analysis of Ref. [3], beginning with some considerations on the central object of the analysis, the vacuum correlator  $\Delta\Pi(q^2)$ , defined in Eq. (5). Our work exploits a combination of the two nonperturbative representations available for the correlator, the dispersive representation and the OPE representation at large spacelike momenta.

The dispersive representation reads

$$\Delta\Pi(Q^2) = \frac{1}{Q^4} \int_0^\infty ds \frac{s^2}{s + Q^2} \Delta\rho(s) \quad , \tag{6}$$

where  $Q^2 \equiv -q^2$  is the variable for spacelike momenta and the difference of spectral functions is denoted by  $\Delta\rho(s) \equiv [\rho_{V,3} - \rho_{A,3}](s)$ . The ALEPH [9] determination of  $\Delta\rho(s)$  for  $4m_\pi^2 \leq s \leq m_\tau^2$  is displayed in Fig. 1.

For  $Q^2 \gg \Lambda_{\text{QCD}}^2$ ,  $\Delta\Pi(Q^2)$  can be represented via the OPE. Through  $\mathcal{O}(\alpha_s^2)$  one has

$$\Delta\Pi(Q^2) \sim \sum_d \frac{1}{Q^d} \left[ a_d(\mu) + b_d(\mu) \ln \frac{Q^2}{\mu^2} \right] \quad (d = 2, 4, 6, 8, 10, \dots) \quad , \quad (7)$$

where  $a_d(\mu)$  and  $b_d(\mu)$  are combinations of vacuum expectation values of local operators of dimension  $d$ . We list some properties of the OPE:

1. For  $d \leq 6$ , the coefficients  $a_d, b_d$  are known to  $\mathcal{O}(\alpha_s^2)$  for  $d = 2$  and  $\mathcal{O}(\alpha_s)$  for  $d = 4, 6$ .
2. For  $d < 6$ ,  $a_d, b_d$  are  $\mathcal{O}(m_q)$  or  $\mathcal{O}(m_q^2)$  and thus vanish in the chiral limit.
3.  $a_6$  and  $b_6$  are related to vacuum matrix elements of the operators  $\mathcal{O}_1$  and  $\mathcal{O}_8$ .
4. For  $d > 6$ ,  $a_d, b_d$  are partially known (an analysis of  $d = 8$  can be found in Ref. [10] for the vector correlator and Ref. [11] for the V-A correlator). In this work their detailed form is not needed. Hereafter we denote the collective  $d > 6$  contributions to the OPE as  $\Delta\bar{\Pi}(Q^2)$ , that is

$$\Delta\bar{\Pi}(Q^2) \sim \sum_{d>6} \frac{1}{Q^d} \left[ a_d(\mu) + b_d(\mu) \ln \frac{Q^2}{\mu^2} \right] \quad . \quad (8)$$

By virtue of item 3, the  $d = 6$  OPE coefficients  $a_6, b_6$  are of special interest. Here, we consider the  $\overline{\text{MS}}$  renormalization scheme with NDR and HV prescriptions for  $\gamma_5$  and the evanescent operator basis used in Refs. [5,6]. Including  $\mathcal{O}(\alpha_s^2)$  terms we find for  $N_c = 3$  and  $n_f = 3, 4$ ,

$$\begin{aligned} a_6(\mu) &= 2\pi \langle \alpha_s \mathcal{O}_8 \rangle_\mu + A_8 \langle \alpha_s^2 \mathcal{O}_8 \rangle_\mu + A_1 \langle \alpha_s^2 \mathcal{O}_1 \rangle_\mu \quad , \\ b_6(\mu) &= B_8 \langle \alpha_s^2 \mathcal{O}_8 \rangle_\mu + B_1 \langle \alpha_s^2 \mathcal{O}_1 \rangle_\mu \quad . \end{aligned} \quad (9)$$

The coefficients  $A_1, A_8$  and  $B_1, B_8$  are displayed in Table I, in terms of their dependence on the renormalization scheme and the active number of flavors. We shall present a derivation of the above results in Sect. II A. An additional piece of information which will be needed in our analysis is the form of  $a_6(\mu)$  at  $\mathcal{O}(\alpha_s)$  in  $d = 4 - \epsilon$  dimensions, [3]

$$a_6(\mu, \epsilon) = 2\pi \langle \alpha_s \mathcal{O}_8 \rangle_\mu \left( 1 + k_s \epsilon \right) \quad . \quad (10)$$

The scheme-dependent coefficient  $k_s$  can be found in Table I.

TABLE I. Collection of coefficients needed in the calculation at various stages.

Scheme	Three Active Flavors ( $n_f = 3$ )		Four Active Flavors ( $n_f = 4$ )	
	NDR	HV	NDR	HV
$A_1$	2	$-10/3$	2	$-10/3$
$A_8$	$25/4$	$21/4$	$205/36$	$169/36$
$B_1$	$8/3$	$8/3$	$8/3$	$8/3$
$B_8$	$-1$	$-1$	$-2/3$	$-2/3$
$C_8$	$-1/6$	$11/6$	$-1/6$	$11/6$
$k_s$	$-1/4$	$3/4$	$-1/4$	$3/4$

### A. Determination of $a_6(\mu)$ and $b_6(\mu)$

In this section we derive the expression for the dimension six terms in the OPE for  $\Delta\Pi(Q^2)$  at NLO in QCD. We begin by defining an amplitude  $\mathcal{M}$  which describes the coupling of isospin vector and axialvector currents via W-boson exchange,

$$\mathcal{M} \equiv \frac{g_2^2}{16F_\pi^2} \int d^4x \mathcal{D}(x, M_W^2) \langle 0|T (V_3^\mu(x)V_{\mu,3}(0) - A_3^\mu(x)A_{\mu,3}(0)) |0\rangle . \quad (11)$$

Our strategy is to perform two different analyses of this amplitude at NLO in QCD (one of which involves the vacuum correlator we want to study), and then match the two: consistency then determines  $a_6(\mu)$  and  $b_6(\mu)$ .

Within the first approach, we write [3] the amplitude  $\mathcal{M}$  in the language of an effective theory, involving the operators  $\mathcal{O}_1$ ,  $\mathcal{O}_8$  and their Wilson coefficients  $c_1$ ,  $c_8$ ,

$$\mathcal{M} \simeq \frac{G_F}{2\sqrt{2}F_\pi^2} [c_1(\mu)\langle\mathcal{O}_1\rangle_\mu + c_8(\mu)\langle\mathcal{O}_8\rangle_\mu] , \quad (12)$$

where  $\mu$  is the renormalization scale. We define the effective theory via dimensional regularization within  $\overline{\text{MS}}$  renormalization. The specification of how  $\gamma_5$  is treated in  $d$ -dimensions (NDR or HV scheme) and of the evanescent operator basis is needed to uniquely define the effective theory. We follow here the prescriptions given in Ref. [5]. The Wilson coefficients  $c_1(\mu)$  and  $c_8(\mu)$  are found by performing perturbative matching of the full and effective theories at scale  $\mu = M_W$  and then evolving them according to the renormalization group equations

$$\mu \frac{d}{d\mu} c_k(\mu) = (\gamma_{\ell k} - 2\gamma_J \delta_{\ell k}) c_\ell(\mu) . \quad (13)$$

Here  $\{\gamma_{\ell k}\}$  is the anomalous dimension matrix for the operators  $\mathcal{O}_{1,8}$ , while  $\gamma_J$  is the weak current anomalous dimension<sup>2</sup>. For an NLO analysis we need  $c_{1,8}(M_W)$  up to non-logarithmic

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<sup>2</sup> $\gamma_J$  is nonzero at two loops in the HV scheme, when subtracting minimally. In this work we stick to this version of  $\overline{\text{MS}}$  – HV renormalization scheme.

terms of order  $\alpha_s$ , and also the two loop anomalous dimension. The matching calculation required here has been performed in Ref. [3] and results in

$$c_1(\mu \simeq M_W) = 1 + \mathcal{O}(\alpha_s^2), \quad c_8(\mu \simeq M_W) = -\frac{3\alpha_s(M_W)}{8\pi} \left( \frac{3}{2} + 2d_s \right) \quad (14)$$

where  $d_s$  is given by

$$d_s = \begin{cases} -5/6 & \text{(NDR)} \\ 1/6 & \text{(HV)} \end{cases}. \quad (15)$$

The anomalous dimension matrix  $\{\gamma_{\ell k}\}$  is parameterized in terms of the number of active quark flavors  $n_f$  and the number of quark colors  $N_c$ . At next-to-leading order (NLO), it has the form

$$\gamma_{\text{NLO}} = \frac{\alpha_s(\mu)}{4\pi} \gamma^{(0)} + \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 \gamma^{(1)}. \quad (16)$$

The next key observation is that the anomalous dimension matrix for the operators  $\mathcal{O}_{1,8}$  can be inferred by the restriction of the full  $10 \times 10$  matrix of Ref. [5,6] to the  $\mathcal{Q}_{7,8}$  subspace. The two sets of operators  $\mathcal{O}_{1,8}$  and  $\mathcal{Q}_{7,8}$  have the same Dirac structure, but differ in the flavor structure and in the basis for the color structure. Since only current-current diagrams contribute to the anomalous dimension matrix of  $\mathcal{O}_{1,8}$ , and these contributions are insensitive to flavor, we only need to handle the different colour properties. The difference in the color-structure basis is taken care of by a simple linear transformation. Denoting by  $B^{(0,1)}$  the anomalous dimension matrices of Refs. [5,6] restricted to  $\mathcal{Q}_{7,8}$ , those we need for  $\mathcal{O}_{1,8}$  are given by

$$\gamma^{(0,1)} = M B^{(0,1)} M^{-1}, \quad (17)$$

with

$$M = \begin{pmatrix} 1 & 0 \\ -2/3 & 2 \end{pmatrix}. \quad (18)$$

Using the above ingredients and the NLO evolution operator [1,6], we have calculated the Wilson coefficients in  $\overline{\text{MS}}$  renormalization for both NDR and HV schemes, taking  $n_f = 3, 4$  and  $N_c = 3$ . Upon using the expression for the 2-loop running  $\alpha_s$ , we have then expanded  $c_{1,8}(\mu)$  in powers of  $\alpha_s(\mu)$ , finding

$$\begin{aligned} c_1(\mu) &= 1 + \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 \left[ \frac{3A_1}{16} \ln \frac{M_W^2}{\mu^2} + \frac{3B_1}{32} \ln^2 \frac{M_W^2}{\mu^2} \right] + \dots, \\ c_8(\mu) &= \frac{\alpha_s(\mu)}{\pi} \left[ \frac{3}{8} \ln \frac{M_W^2}{\mu^2} - \frac{3}{8} \left( \frac{3}{2} + 2d_s \right) \right] + \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 \left[ \frac{3A_8}{16} \ln \frac{M_W^2}{\mu^2} + \frac{3B_8}{32} \ln^2 \frac{M_W^2}{\mu^2} \right] + \dots, \end{aligned} \quad (19)$$

where  $A_1, A_8$  are the scheme-dependent coefficients of Table I.

The alternate analysis of the amplitude  $\mathcal{M}$  relies on its expression in terms of the correlator  $\Delta\Pi(Q^2)$ :

$$\mathcal{M} = \frac{3G_F M_W^2}{32\sqrt{2}\pi^2 F_\pi^2} \int_0^\infty dQ^2 \frac{Q^4}{Q^2 + M_W^2} \Delta\Pi(Q^2) . \quad (20)$$

We partition the amplitude  $\mathcal{M}$  as

$$\mathcal{M} = \mathcal{M}_<(\mu) + \mathcal{M}_>(\mu) \quad (21)$$

where the component  $\mathcal{M}_<(\mu)$  arises from contributions with  $Q < \mu$ ,

$$\mathcal{M}_<(\mu) = \frac{3G_F}{32\sqrt{2}\pi^2 F_\pi^2} \int_0^{\mu^2} dQ^2 Q^4 \Delta\Pi(Q^2) + \mathcal{O}(\mu^2/M_W^2) \quad (22)$$

and the component  $\mathcal{M}_>(\mu)$  contains the contributions with  $Q > \mu$ ,

$$\mathcal{M}_>(\mu) = \frac{3G_F M_W^2}{32\sqrt{2}\pi^2 F_\pi^2} \int_{\mu^2}^\infty dQ^2 \frac{Q^4}{Q^2 + M_W^2} \Delta\Pi(Q^2) . \quad (23)$$

Employing the OPE for  $Q^6 \Delta\Pi(Q^2)$  in  $\mathcal{M}_>(\mu)$  and performing the  $Q^2$  integration, one can compare this representation with Eqs. (12)-(19). Requiring the consistency of the two approaches yields the expressions of Eq. (9) for  $a_6(\mu)$  and  $b_6(\mu)$ , as well as the sum rule DG1 (*cf.* Eq. (25) below), to be derived in a different way in the next section.

## B. The Two DG Sum Rules

In this section we present a short derivation of the two DG sum rules [3], including effects which were neglected in the original derivation. Let us start with the first sum rule. Consider the V-A correlator in coordinate space for  $x \rightarrow 0$ , regularized dimensionally

$$\begin{aligned} \langle \mathcal{O}_1 \rangle_\mu &\equiv \langle 0 | T (V_3^\mu(0) V_{\mu,3}(0) - A_3^\mu(0) A_{\mu,3}(0)) | 0 \rangle_\mu \\ &= \frac{(d-1)\mu^{4-d}}{(4\pi)^{d/2} \Gamma(d/2)} \int_0^\infty dQ^2 Q^d \Delta\Pi(Q^2) . \end{aligned} \quad (24)$$

Let us now break up the above integral at a scale  $\mu^2$ , within the region of applicability of perturbative QCD. The integration over  $Q^2 < \mu^2$  is UV finite and can be performed in  $d = 4$ . The integration for  $Q^2 > \mu^2$  generates UV singularities. Using the OPE representation for  $\Delta\Pi(Q^2)$  in this region, one sees that the UV divergence is related to the  $Q^{-6}$  term in the expansion, while  $\Delta\bar{\Pi}(Q^2)$  leads to an UV finite term. Employing Eq. (10) for  $a_6(\mu, \epsilon)$ , performing the  $Q^2$  integration and subtracting the  $1/\epsilon$  pole according to the  $\overline{\text{MS}}$  scheme, we obtain

$$\langle \mathcal{O}_1 \rangle_\mu - \frac{3C_8}{8\pi} \langle \alpha_s \mathcal{O}_8 \rangle_\mu = \bar{I}_1(\mu) \quad (\text{DG1}) \quad (25)$$



where the scheme-dependent coefficient  $C_8$  can be found in Table I,<sup>3</sup> and  $\bar{I}_1(\mu)$  is defined in terms of the dispersive integrals  $I_1(\mu)$ ,  $H_1(\mu)$  as follows

$$\bar{I}_1(\mu) = \frac{3}{(4\pi)^2} [I_1(\mu) + H_1(\mu)] \quad , \quad (26)$$

where

$$I_1(\mu) \equiv \int_0^\infty ds \, s^2 \ln\left(\frac{s+\mu^2}{s}\right) \Delta\rho(s) \equiv \int_0^{\mu^2} dQ^2 \, Q^4 \Delta\Pi(Q^2) \quad , \quad (27)$$

$$H_1(\mu) \equiv \int_{\mu^2}^\infty dQ^2 \, Q^4 \Delta\bar{\Pi}(Q^2) \quad . \quad (28)$$

Evaluating the OPE of Eq. (7) at the point  $Q = \mu$  yields the second DG sum rule, [3]

$$2\pi\langle\alpha_s\mathcal{O}_8\rangle_\mu + A_1\langle\alpha_s^2\mathcal{O}_1\rangle_\mu + A_8\langle\alpha_s^2\mathcal{O}_8\rangle_\mu = 2\pi\alpha_s(\mu)\bar{I}_8(\mu) \quad (\text{DG2}) \quad (29)$$

where we have made use of Eq. (9) and define  $\bar{I}_8(\mu)$  as

$$\bar{I}_8(\mu) = \frac{1}{2\pi\alpha_s(\mu)} [I_8(\mu) - H_8(\mu)] \quad , \quad (30)$$

where

$$I_8(\mu) \equiv \int_0^\infty ds \, s^2 \frac{\mu^2}{s+\mu^2} \Delta\rho(s) = \mu^6 \Delta\Pi(\mu) \quad , \quad (31)$$

$$H_8(\mu) \equiv \mu^6 \Delta\bar{\Pi}(\mu) \quad . \quad (32)$$

In Ref. [3] the terms  $H_{1,8}(\mu)$ , subleading at high  $\mu$ , were neglected. They encode the effect of higher dimensional operators and lead to potentially large effects [4,12].

We can summarize the work thus far via the linear relations

$$\begin{pmatrix} 1 & -3C_8\alpha_s(\mu)/8\pi \\ A_1\alpha_s(\mu)/2\pi & (1 + A_8\alpha_s(\mu)/2\pi) \end{pmatrix} \begin{pmatrix} \langle\mathcal{O}_1\rangle_\mu \\ \langle\mathcal{O}_8\rangle_\mu \end{pmatrix} = \begin{pmatrix} \bar{I}_1(\mu) \\ \bar{I}_8(\mu) \end{pmatrix} \quad , \quad (33)$$

which allow us to keep track of the scheme dependence of the matrix elements at NLO. As a check on our calculation of the coefficients  $A_{1,8}, C_8$ , we can derive the relation between the HV and NDR matrix elements of  $\mathcal{Q}_{7,8}$ . To do so, we need to solve Eq. (33) for  $\langle\mathcal{O}_{1,8}\rangle_\mu$  in both schemes and then convert to  $\mathcal{Q}_{7,8}$  using the matrix  $M$  of Eq. (18). We find

$$\begin{pmatrix} \langle\mathcal{Q}_7\rangle_\mu \\ \langle\mathcal{Q}_8\rangle_\mu \end{pmatrix}^{HV} = \begin{pmatrix} \langle\mathcal{Q}_7\rangle_\mu \\ \langle\mathcal{Q}_8\rangle_\mu \end{pmatrix}^{NDR} + \frac{\alpha_s(\mu)}{\pi} \begin{pmatrix} -1/2 & 3/2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \langle\mathcal{Q}_7\rangle_\mu \\ \langle\mathcal{Q}_8\rangle_\mu \end{pmatrix}^{NDR} \quad , \quad (34)$$

in agreement with the result given in Refs. [5,13].

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<sup>3</sup>Although we have employed three distinct quantities  $k_s, d_s, C_8$  to indicate how scheme-dependence enters in different parts of the analysis, they are in fact related by  $C_8 = 1/3 + 2k_s = 3/2 + 2d_s$ .

### C. Additional Comments

Let us take note of the differences of the preceding analysis from that in the previous work by two of the present authors [3]. In the earlier work, the information about perturbative corrections in the NDR scheme was taken from the paper of Lanin *et al.* [14] in the literature on QCD sum rules. This is now seen to differ in the renormalization conventions and use of evanescent operators from the choices made in the two loop analysis of the weak Hamiltonian. In the present work we have adopted the conventions of the weak interaction studies. One fortunate benefit from this change is that the results are less sensitive to higher order perturbative corrections than in the previous analysis. In the earlier work we also did not present all results for the HV scheme, whereas this is accomplished in the present study. Finally, while the previous work led directly to the uncovering of the effects of higher dimension operators, these effects were not explicitly incorporated into the formulas or numerical work of that paper. In the present paper we have explicitly included these effects.

The integrals  $I_8(\mu)$  and  $I_1(\mu)$  will be central to our analysis. They are related by

$$I_8(\mu) = \mu^2 \frac{d}{d\mu^2} I_1(\mu) \quad , \quad (35)$$

and with the aid of the Weinberg sum rules [7], it is possible to express  $I_8(\mu)$  in several equivalent forms, *e.g.*

$$I_8(\mu) = -\mu^4 \int_0^\infty ds \frac{s}{s + \mu^2} \Delta\rho(s) \quad . \quad (36)$$

Moreover, there is an alternative pathway to Eqs. (27),(31). One simply evaluates two contour integrals in the complex  $s \equiv q^2$  plane,

$$\oint_{\Gamma_k} ds K_k(s, \mu) \Delta\Pi(s) = 0 \quad (k = 1, 8) \quad , \quad (37)$$

where the weights  $K_k(s, \mu)$  are

$$K_1(s, \mu) \equiv s^2 \ln \frac{s + \mu^2}{s} \quad , \quad K_8(s, \mu) \equiv \frac{\mu^2 s^2}{s + \mu^2} \quad . \quad (38)$$

The contours  $\Gamma_8, \Gamma_1$  are shown in Figs. 2(a),(b), with the understanding that the radius of each outer circle is to be taken to infinity. The unitarity cut in Figs. 2(a),(b) is denoted by a wiggly line along the positive real axis. The pole in  $K_8(s, \mu)$  at  $s = -\mu^2$  is denoted in Fig. 2(a) by the cross on the negative  $s$  axis. The cut in  $K_1(s, \mu)$  occurring for  $-\mu^2 \leq s \leq 0$  is denoted by the bold straight line.

Obtaining numerical values (along with meaningful error estimates) for  $I_8(\mu)$  and  $I_1(\mu)$  turns out to be a highly nontrivial task. It is this problem that we turn to in the next section.

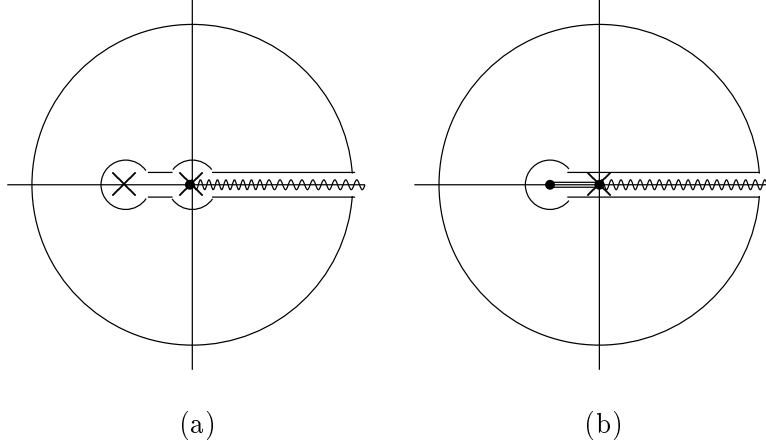


FIG. 2. Integration contours for (a)  $I_8$ , (b)  $I_1$

### III. NUMERICAL ANALYSIS

The goal of this section is to arrive at numerical estimates for the input vector on the right hand side of Eq. (33). We shall accomplish this by means of a constrained evaluation of the dispersive integrals  $I_{1,8}(\mu)$  using an approach which we call the *residual weight approximation*. This determination can be applied over a range of possible scales, *e.g.*  $2 \leq \mu(\text{GeV}) \leq 4$ . In fact, we choose a scale  $\mu = 4$  GeV sufficiently large for higher-dimension effects to be negligible and then employ a two-loop renormalization group analysis to evolve our result down to scale  $\mu = 2$  GeV. The decision to set  $H_{1,8}(4) = 0$  is supported by a preliminary study using Finite Energy Sum Rules (FESR) [15]. Work on this topic continues and will be reported on in a separate publication [16].

#### A. Residual Weight Approximation

Direct evaluation of  $I_1(\mu)$  and  $I_8(\mu)$  via Eqs. (27) and (31) is impossible since data for the spectral function  $\Delta\rho(s)$  exists only in the interval  $4m_\pi^2 \leq s \leq m_\tau^2$ . Even direct integration of the contribution from the region  $s < m_\tau^2$  is potentially problematic since strong cancellations significantly enhance the impact of experimental errors. This is compounded by the fact that the weights  $K_{1,8}(s, \mu)$  are strongly increasing with  $s$  and hence weight the high- $s$  region, where data errors are large, more strongly than the low- $s$ , low error region. As an example, defining  $I_{1,8}(s_0, \mu)$  to be versions of Eqs. (27) and (31) containing spectral contributions only up to  $s = s_0$ , the errors on  $I_{1,8}(s_0, \mu)$ , determined using the ALEPH covariance matrix, already exceed 50% of our final central values for  $s_0 \sim 2.1$  GeV<sup>2</sup>. The possibility of obtaining reliable determinations of  $I_{1,8}(\mu)$  thus depends critically on our ability to impose additional constraints on these integrals.

Although  $\Delta\rho(s)$  is not known above  $s = m_\tau^2$ , and is determined with insufficient precision for our purposes between 2 GeV<sup>2</sup> and  $m_\tau^2$ , indirect information on its behavior in these regions, in the chiral limit, is provided by the two Weinberg sum rules [7]

$$W1 \equiv \int_0^\infty ds \Delta\rho(s) = F_\pi^{(0)2}, \quad W2 \equiv \int_0^\infty ds s \Delta\rho(s) = 0, \quad (39)$$

where  $F_\pi^{(0)}$  is the pion decay constant in the chiral limit, and the sum rule for the EM pion mass splitting [8]

$$W3 \equiv \int_0^\infty ds s \ln \frac{s}{\Lambda^2} \Delta\rho(s) = -F_\pi^{(0)2} \frac{4\pi}{3\alpha} \Delta m_\pi^{(0)2} \quad , \quad (40)$$

where  $\Delta m_\pi^{(0)2}$  is the pion squared-mass splitting in the chiral limit,  $\Delta m_\pi^{(0)2} \simeq m_{\pi^\pm}^2 - m_{\pi^0}^2$ , and we will take  $\Lambda = 1 \text{ GeV}$ , for definiteness, in what follows (the LHS of Eq. (40) is independent of  $\Lambda$ ). An additional constraint is provided by the asymptotic OPE form for  $\Delta\rho(s)$ ,

$$\Delta\rho(s) \sim \frac{1}{s^3} \left[ B_1 \langle \alpha_s^2 \mathcal{O}_1 \rangle_\mu + B_8 \langle \alpha_s^2 \mathcal{O}_8 \rangle_\mu \right] + \dots \quad , \quad (41)$$

valid for sufficiently large  $s$ , say  $s > s_A$ .

Because of the asymptotic constraint of Eq. (41), uncertainties in the evaluation of any spectral integral are dominated by contributions from the region  $s \sim 2 - 2.5 \text{ GeV} \rightarrow s_A$ , for which spectral data is absent, or has large errors. Letting  $K(s, \mu)$  stand for either  $K_1(s, \mu)$  or  $K_8(s, \mu)$ , an obvious way to take advantage of the constraints provided by the chiral sum rules is to write  $K(s, \mu)$  in the form

$$K(s, \mu) = C(s, \mu) + \Delta K(s, \mu) \quad , \quad (42)$$

where  $C(s, \mu)$  is an arbitrary linear combination of the weights occurring in the chiral sum rules W1, W2 and W3,

$$C(s, \mu) \equiv x + ys + zs \ln \left( \frac{s}{\Lambda^2} \right) \quad , \quad (43)$$

and the ‘residual weight’,  $\Delta K(s, \mu)$ , is defined trivially by

$$\Delta K(s, \mu) \equiv K(s, \mu) - C(s, \mu) \quad . \quad (44)$$

The residual weight representation for  $K(s, \mu)$  generates analogous representations for the integrals  $I \equiv I_{1,8}$ ,

$$I(\mu) = I_{\text{chiral}}(\mu) + \Delta I(\mu) \quad , \quad (45)$$

where

$$I_{\text{chiral}}(\mu) \equiv \int_0^\infty ds C(s, \mu) \Delta\rho(s) = F_\pi^{(0)2} \left[ x - z \left( \frac{4\pi}{3\alpha} \right) \Delta m_\pi^{(0)2} \right] \quad , \quad (46)$$

and

$$\Delta I(\mu) \equiv \int_0^\infty ds \Delta K(s, \mu) \Delta\rho(s) \quad . \quad (47)$$

Errors on the determination of  $I_{1,8}(\mu)$  associated with uncertainties in our knowledge of  $\Delta\rho(s)$  can then be reduced by adjusting the free parameters  $x, y, z$  so as to make  $\Delta K(s, \mu)$  small in the region between  $\sim 2.5 \text{ GeV}^2$  and  $s_A$ . We will refer to this region as the ‘‘matching region’’ in what follows.

In order to choose  $x, y, z$  in such a way as to minimize the total errors on  $I_{1,8}$ , it is necessary to take into account the uncertainties in our knowledge of the chiral limit values,  $F_\pi^{(0)}$  and  $\Delta m_\pi^{(0)2}$ , which enter the W1 and W3 chiral sum rules. Taking as input the values [17]

$$F_\pi^{(0)} = (0.0871 \pm 0.0026) \text{ GeV} \quad \text{and} \quad \Delta m_\pi^{(0)2} = (0.001174 \pm 0.000055) \text{ GeV}^2 \quad , \quad (48)$$

the error,  $E_{\text{chiral}}$ , on  $I_{\text{chiral}}(\mu)$  becomes

$$E_{\text{chiral}} = \pm 0.000453 \text{ GeV}^2 |x - 0.674 \text{ GeV}^2 z| \pm 0.000240 \text{ GeV}^4 z \quad . \quad (49)$$

Based on the NNLO chiral expansion for  $F_\pi$  given in Ref. [17], we believe that the difference between the physical and NNLO chiral values represents an extremely conservative estimate of the uncertainty on  $F_\pi^{(0)}$ . We have therefore taken half this value ( $\pm 0.0026$ ) as the error cited above in an attempt to employ a ‘one-sigma’ theoretical error akin to that used in assigning experimental error.

The corresponding  $x, y, z$ -dependent error on  $\Delta I(\mu)$  is obtained as follows. We first partition the range of integration into three intervals,

$$\begin{aligned} \Delta I &= [\Delta I]_{\text{data}} + [\Delta I]_{\text{int}} + [\Delta I]_{\text{asy}} \quad , \\ &= \left[ \int_0^{m_\tau^2} + \int_{m_\tau^2}^{s_A} + \int_{s_A}^\infty \right] ds \Delta K(s, \mu) \Delta \rho(s) \quad , \end{aligned} \quad (50)$$

where, as above,  $s_A$  represents the point beyond which one can employ the OPE representation of  $\Delta \rho(s)$ . Our results are insensitive to the actual choice of  $s_A$  but for definiteness we work with  $s_A = 5 \text{ GeV}^2$ . The three regions (‘data’, ‘intermediate’ and ‘asymptotic’) are identified by the different ways in which the spectral function  $\Delta \rho(s)$  is treated. For  $[\Delta I]_{\text{data}}$ , the compilation of  $\Delta \rho(s)$  provided by ALEPH is used. In practice, we sum over those experimental bins covering the range  $s \leq 3.15 \text{ GeV}^2$ . For  $[\Delta I]_{\text{int}}$ , we assume  $|\Delta \rho(s)| < \Delta \rho_{\text{max}} = 0.005$  throughout the interval. Since the three known peaks in  $\Delta \rho(s)$  decrease in magnitude with increasing  $s$  (as expected given the  $1/s^3$  asymptotic fall-off) and since  $\Delta \rho(s) \simeq 0.005$  for  $s \simeq m_\tau^2$ , this seems to us a reasonable bound. The treatment of  $[\Delta I]_{\text{int}}$  is discussed in more detail below. For  $[\Delta I]_{\text{asy}}$ , we take the form given in Eq. (41) but with a numerical value assumed for the numerator, *viz*  $\Delta \rho(s) \sim 7.3 \cdot 10^{-5} \text{ GeV}^6/s^3$ , which is compatible with our final results. The error on  $\Delta I(\mu)$  is obtained by adding the errors associated with each of these three contributions in quadrature. The determination of the individual error contributions is discussed below.

The source of the uncertainty on  $[\Delta I]_{\text{data}}$  is  $\Delta \rho(s)$ , the most important component being that due to the experimental errors determined by ALEPH as part of their extraction of  $\Delta \rho(s)$  from hadronic  $\tau$  decay data. For each weighted spectral integral, this error is computed using the ALEPH covariance matrix. A much less significant uncertainty arises from the fact that our theoretical analysis corresponds to the chiral world, whereas our data sample is taken in the physical world. It is argued in Ref. [18] that such an effect is anticipated to be small,  $\mathcal{O}(m_\pi^2/m_\rho^2)$ . This expectation is borne out in the analysis of Ref. [19].

For the intermediate contribution  $[\Delta I]_{\text{int}}$ , we adopt the strategy of assigning the value

$$[\Delta I]_{\text{int}} = 0 \pm E_{\text{int}} \quad , \quad (51)$$

where  $E_{\text{int}}$  is determined by using the Cauchy-Schwarz inequality,

$$|\Delta I_{\text{int}}| < E_{\text{int}} \equiv |\Delta\rho_{\text{max}}| \sqrt{s_A - m_\tau^2} \cdot \sqrt{\int_{m_\tau^2}^{s_A} ds [\Delta K(s, \mu)]^2} . \quad (52)$$

We recall that the quite reasonable bound  $|\Delta\rho_{\text{max}}| = 0.005$  is employed throughout the intermediate region.

For the asymptotic contribution  $[\Delta I]_{\text{asy}}$ , we assume a 100% uncertainty. Although this assignment of error is arbitrary, the asymptotic contribution turns out to be so tiny that the 100% error does not affect the overall error-minimization in any way.

At this point, we have a residual weight representation for the integral  $I$  ( $I_1$  or  $I_8$ ) parameterized in terms of three constants  $x, y, z$ . Since  $E_{\text{chiral}}$  is independent of  $y$ , we may reduce  $E_{\text{int}}$  without adversely affecting  $E_{\text{chiral}}$ , for any  $x, z$ , by choosing  $y$  so as to appropriately minimize  $|\Delta K(s, \mu)|$  over the matching region. The lower edge of this region is chosen to be  $s_0 = 2.5 \text{ GeV}^2$ , rather than  $m_\tau^2$ , in order that contributions from that portion of the spectrum where experimental errors are large will be suppressed once one has performed this minimization. We thus fix  $y = y(x, z)$  by minimizing

$$\int_{s_0}^{s_A} ds \left[ K(s, \mu) - x - ys - zs \ln\left(\frac{s}{\Lambda^2}\right) \right]^2 . \quad (53)$$

Like  $E_{\text{chiral}}$ , the errors on the three contributions to  $\Delta I(\mu)$  now depend only on  $x$  and  $z$ , so we may combine all errors in quadrature, and minimize the total error with respect to  $x, z$ .

The results of the optimized versions of the  $I_{1,8}(\mu)$  residual weight analyses are

$\mu \text{ GeV}$	$I_1(\mu) \text{ (GeV}^6\text{)}$	$I_8(\mu) \text{ (GeV}^6\text{)}$
2	$-(39.7 \pm 3.1) \cdot 10^{-4}$	$-(26.2 \pm 3.0) \cdot 10^{-4}$
3	$-(63.1 \pm 5.9) \cdot 10^{-4}$	$-(31.4 \pm 5.9) \cdot 10^{-4}$
4	$-(82.1 \pm 9.3) \cdot 10^{-4}$	$-(34.4 \pm 9.4) \cdot 10^{-4}$

(54)

The uncertainty in each case is acceptable.

Let us study the various contributions to the  $I_{1,8}$  integrals at some scale, say,  $\mu = 4 \text{ GeV}$ . For each integral  $I$ , there will be four contributing sources:  $I_{\text{chiral}}$ ,  $\Delta I_{\text{data}}$ ,  $\Delta I_{\text{interm}}$  and  $\Delta I_{\text{asypm}}$ . Each source will have an associated uncertainty, except for the chiral contribution where we list *two* errors, the first from  $F_\pi^{(0)}$  and the second from  $\Delta m_\pi^{(0)2}$ . The numerics are as follows:

Source	$I_1(4) \text{ (GeV}^6\text{)}$	$I_8(4) \text{ (GeV}^6\text{)}$
Chiral	$-(23.9 \pm 1.4 \pm 9.1) \cdot 10^{-4}$	$(24.3 \pm 1.5 \pm 7.5) \cdot 10^{-4}$
Data	$-(58.1 \pm 0.9) \cdot 10^{-4}$	$-(58.8 \pm 3.2) \cdot 10^{-4}$
Intermed.	$(0 \pm 0.7) \cdot 10^{-4}$	$(0 \pm 4.6) \cdot 10^{-4}$
Asymp.	$-(0.1 \pm 0.1) \cdot 10^{-4}$	$(0.1 \pm 0.1) \cdot 10^{-4}$

(55)

The largest source of uncertainty in our results turns out to originate with the 4.6% error in Eq. (48) for the pion squared-mass splitting (equivalent to  $\Delta m_\pi^{\text{QCD}} = 0.32 + / - 0.20 \text{ MeV}$ ). [17] Our knowledge of this  $(m_u - m_d)^2$  effect arises mostly from studying the kaon mass difference. There, the effect of electromagnetism (which shows significant SU(3)

breaking) is subtracted and the remaining kaon mass difference fit by  $m_d - m_u$  in lowest order chiral perturbation theory. However, the next order chiral corrections could easily be sizeable at the kaon mass scale, and one must also cope with  $\pi^0$ - $(\eta, \eta')$  mixing. In light of these effects, we feel the current 4.6% error cited in Ref. [17] is realistic.

Finally, we point out that a possible modification of the procedure described above would be to include the  $F_\pi^{(0)3}$  factors of Eq. (3) *ab initio* in our error minimization analysis. In other words, we would be determining integrals  $J_{1,8} \equiv I_{1,8}/F_\pi^{(0)3}$  rather than just  $I_{1,8}$ . Such a program produces results completely compatible with those of Eq. (54) and with no significant reduction in the size of the errors.

## B. Renormalization Group Evolution and Results

In this section we outline the procedure for arriving at the physical matrix elements  $\langle(\pi\pi)_{I=2}|\mathcal{Q}_{7,8}|K^0\rangle$  in the chiral limit.

Using the values for  $I_{1,8}(\mu = 4 \text{ GeV})$  obtained in the previous section and setting  $H_{1,8}(\mu = 4 \text{ GeV}) = 0$ , we solve Eq. (33) for  $\langle\mathcal{O}_{1,8}\rangle_{\mu=4 \text{ GeV}}$ . Since we work at the scale  $\mu = 4 \text{ GeV}$  we use the values of our coefficients corresponding to four active flavors ( $n_f = 4$ ). Moreover we use  $\Lambda_{\overline{MS}}^{n_f=4} = (340 \pm 50) \text{ MeV}$ . We subsequently apply the NLO renormalization group evolution, to obtain  $\langle\mathcal{O}_{1,8}\rangle_{\mu=2 \text{ GeV}}$ . Denoting by  $U(\mu_L, \mu_H)$  the operator governing the NLO evolution of the Wilson coefficients  $(c_1(\mu), c_8(\mu)) = \vec{c}^T(\mu)$  between the scales  $\mu_L$  and  $\mu_H$ ,

$$\vec{c}(\mu_L) = U(\mu_L, \mu_H) \vec{c}(\mu_H) , \quad (56)$$

the local operators  $(\mathcal{O}_1(\mu), \mathcal{O}_8(\mu)) = \vec{\mathcal{O}}^T(\mu)$  evolve according to:

$$\langle\vec{\mathcal{O}}\rangle(\mu_L) = \left[ U^T(\mu_L, \mu_H) \right]^{-1} \langle\vec{\mathcal{O}}\rangle(\mu_H) . \quad (57)$$

The results for  $\langle\mathcal{O}_{1,8}\rangle_\mu$  and  $\langle(\pi\pi)_{I=2}|\mathcal{Q}_{7,8}|K^0\rangle_\mu$  are reported in Table II for the scales  $\mu = 2, 4 \text{ GeV}$ . The vacuum matrix elements  $\langle\mathcal{O}_{1,8}\rangle_\mu$  are converted to the  $K \rightarrow \pi\pi$  matrix elements through Eqs. (3). We have added in quadrature the uncertainties from the vacuum matrix elements and from the  $1/F_\pi^{(0)3}$  factor.

## IV. FINAL COMMENTS

Our goals in this paper have been to exactly match the dispersive description of the weak amplitudes  $\langle(\pi\pi)_{I=2}|\mathcal{Q}_{7,8}|K^0\rangle$  in the chiral limit to the specific conventions of the existing two-loop calculations of the Wilson coefficients which appear in the effective weak hamiltonian, and to provide a numerical evaluation using as input only experimental data and the rigorous theoretical constraints of QCD.

In order to accomplish the matching, we adopted the operator basis and renormalization conventions of Ref. [5]. On one hand we analyze the OPE by matching the hamiltonian at the scale  $M_W$  and evolving the operators to a lower scale  $\mu$ . By simultaneously considering the dispersive representation of the matrix element, we use the OPE to identify the  $\overline{\text{MS}}$

TABLE II. Numerical values at  $\mu = 2$  GeV and  $\mu = 4$  GeV.

$\mu$ (GeV)	$\langle \mathcal{O}_1 \rangle_\mu (\text{GeV}^6)$		$\langle \mathcal{O}_8 \rangle_\mu (\text{GeV}^6)$	
	NDR	HV	NDR	HV
2	$-(0.53 \pm 0.34) \cdot 10^{-4}$	$-(1.64 \pm 0.18) \cdot 10^{-4}$	$-(14.4 \pm 4.3) \cdot 10^{-4}$	$-(15.2 \pm 4.4) \cdot 10^{-4}$
4	$-(1.47 \pm 0.18) \cdot 10^{-4}$	$-(2.59 \pm 0.33) \cdot 10^{-4}$	$-(19.3 \pm 5.4) \cdot 10^{-4}$	$-(20.3 \pm 5.6) \cdot 10^{-4}$
$\mu$ (GeV)	$\langle (\pi\pi)_{I=2}   \mathcal{Q}_7   K^0 \rangle_\mu (\text{GeV}^3)$		$\langle (\pi\pi)_{I=2}   \mathcal{Q}_8   K^0 \rangle_\mu (\text{GeV}^3)$	
	NDR	HV	NDR	HV
2	$0.16 \pm 0.10$	$0.49 \pm 0.07$	$2.22 \pm 0.67$	$2.46 \pm 0.70$
4	$0.44 \pm 0.07$	$0.78 \pm 0.12$	$3.06 \pm 0.87$	$3.32 \pm 0.90$

operators at the scale  $\mu$  in both the NDR and HV schemes. Our results also identify the role of higher dimensional operators, which enter the dimensionally regularized matrix elements.

In the numerical analysis we have developed the ‘residual weight’ method, which provides an evaluation requiring as input only data and the chiral-limit values  $F_\pi^{(0)}$  and  $\Delta m_\pi^{(0)}$ . This method expresses the dispersive integrals  $I_1(\mu)$  and  $I_8(\mu)$  in terms of known quantities (the other known dispersive sum rules) plus a residual integral. For the residual integral, we make use of the ALEPH data and employ only a conservative bound on the spectral function in the region where  $\tau$  decay data is not available. We have chosen to evaluate  $I_1(\mu)$  and  $I_8(\mu)$  at large values of  $\mu$  where the effects of higher dimension operators are negligible. We then use the renormalization group to obtain the values of the matrix elements at other scales

Our results lead to the matrix elements given in Table II and Table III. Because we only use real experimental input, in combination with very conservative assumptions about the uncertainties on  $F_\pi^{(0)}$  and  $\Delta m_\pi^{(0)}$ , the final errors we quote are very conservative ones. These are the maximal error bars of the dispersive evaluation — our method has been designed to give as pure and conservative an evaluation from data as possible, and other methods may reduce the error bars. For example, Eq. (54) shows that the uncertainty on the residual weight analysis decreases dramatically at lower values of  $\mu$ . To directly work at  $\mu = 2$  GeV one needs to evaluate, and correct for, the effects of higher dimension operators. We have been studying this problem through the use of FESR and preliminary indications yield results consistent with the present evaluation but with reduced error bars [15]. However that analysis requires the development of further techniques and consistency checks, and we will present it in detail elsewhere [16]. The present method stands on its own as being especially simple and model independent.

In Table III, we compare our results at the scale  $\mu = 2$  GeV to other determinations: the lattice Monte Carlo simulation [20], the large  $N_c$  approach [12], the so-called X-boson approach [21] and the vacuum saturation approximation (VSA). We make the following remarks regarding this comparison.

The Monte Carlo simulation of Ref. [20] is displayed in the ‘Lattice’ row of Table III. Note that their HV results have been converted to the HV scheme used in this work. As regards comparison between the lattice results and ours, we should point out that the two approaches correspond to somewhat different limits. The lattice evaluation extrapolates the



results to the physical values of the quark masses, while our dispersive analysis applies to the amplitude in the chiral limit. Even so, the extrapolation between the chiral limit and the physical masses has been addressed by both analytic [22,23] and lattice methods [24], and the differences are not large enough to account for the discrepancy. This means that the difference between the lattice and dispersive values for  $\mathcal{Q}_8$  represents a serious disagreement. It will be important to pursue an understanding of the physics behind this disagreement, as in principle both methods are fully rigorous.

In the case of the large  $N_c$  evaluation [12], there are differences in principle with our approach because they have not performed the complete matching at two loops. In addition, instead of using data for the evaluation of the vacuum polarization functions, they use a vector meson dominance model. Nevertheless, their results are consistent with ours, especially once the NLO radiative corrections are included in their work.

A recent manuscript [21] also discusses matching and dimension-eight effects, but in the context of an ‘X-boson method’. The greatest procedural difference with our work comes in the evaluation of the  $\mathcal{Q}_8$  matrix element. They find small non-factorizable effects such that their dominant contributions are the factorized terms, refined to include the scale dependence from the matching procedure.

The VSA values are obtained from

$$\langle(\pi\pi)_{I=2}|\mathcal{Q}_8|K^0\rangle_{\text{VSA}} = 3\langle(\pi\pi)_{I=2}|\mathcal{Q}_7|K^0\rangle_{\text{VSA}} = \frac{2F_\pi M_K^4}{(m_s + m_d)_{\mu=2 \text{ GeV}}^2} \simeq 0.94 \text{ GeV}^3 \quad , \quad (58)$$

where we take  $(m_s + m_d)_{\mu=2 \text{ GeV}} = 110 \text{ MeV}$  based on the value of  $m_s$  given by the midpoint of the combined sum rule and lattice ranges in the recent review Ref. [25] in combination with quark mass ratios obtained in ChPT analyses [26].

The matrix element for  $\mathcal{Q}_7$  has little phenomenological interest, and will be useful primarily for the comparison with the lattice as both the dispersive and lattice evaluations improve. However the matrix element for  $\mathcal{Q}_8$  is one of the main contributions to  $\epsilon'/\epsilon$ , and gives a negative contribution to that quantity. Our rather large result implies that the electroweak penguin contribution is

$$\left. \frac{\epsilon'}{\epsilon} \right|_{\text{EWP}} = (-2.2 \pm 0.7) \times 10^{-3} \quad . \quad (59)$$

The gluonic penguin matrix element needs to be large enough to bring the result up to the experimental value. Given this important consequence, the resolution of the disagreement of the lattice and dispersive values will be specially interesting.

TABLE III. Comparison of Matrix Element Determinations at  $\mu = 2$  GeV.

Method of Calculation	$\langle(\pi\pi)_{I=2} \mathcal{Q}_7 K^0\rangle(\text{GeV}^3)$		$\langle(\pi\pi)_{I=2} \mathcal{Q}_8 K^0\rangle(\text{GeV}^3)$	
	NDR	HV	NDR	HV
This work	$0.16 \pm 0.10$	$0.49 \pm 0.07$	$2.22 \pm 0.67$	$2.46 \pm 0.70$
Lattice [20]	$0.11 \pm 0.04$	$0.18 \pm 0.06$	$0.51 \pm 0.10$	$0.57 \pm 0.12$
Large $N_c$ [12]	$0.11 \pm 0.03$	$0.67 \pm 0.20$	$3.5 \pm 1.1$	$3.5 \pm 1.1$
X-boson [21]	$0.26 \pm 0.03$	$0.39 \pm 0.06$	$1.2 \pm 0.5$	$1.3 \pm 0.6$
VSA	0.32	0.32	0.94	0.94

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