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Mobility of Discrete Solitons in Quadratically Nonlinear Media

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We study the mobility of solitons in second-harmonic-generating lattices. Contrary to what is known for their cubic counterparts, discrete quadratic solitons are mobile not only in the one-dimensional (1D) setting, but also in two dimensions (2D). We identify parametric regions where an initial kick applied to a soliton leads to three possible outcomes, namely, staying put, persistent motion, or destruction. For the 2D lattice, it is found that, for the solitary waves, the direction along which they can sustain the largest kick and can attain the largest speed is the diagonal. Basic dynamical properties of the discrete solitons are also discussed in the context of an analytical approximation, in terms of an effective Peierls-Nabarro potential in the lattice setting.

Introduction. In the past several years, tremendous progress has been observed in studies of nonlinear dynamical systems on lattices [1]. In a considerable part, this development is fueled by the continuous expansion of relevant physical applications, including spatial dynamics of optical beams in waveguide arrays [2], temporal evolution of Bose-Einstein condensates in deep optical lattices [3], transformations of the DNA double strand [4], and so on.

A ubiquitous dynamical-lattice system is represented by the discrete nonlinear Schrödinger equation [1, 2, 5] with cubic ($\chi^{(3)}$) nonlinearity. It has been used to model a variety of experiments featuring, among others, formation of discrete solitons, lattice modulational instability, buildup of the Peierls-Nabarro (PN) barrier impeding the soliton motion, diffraction management, and soliton interactions [6, 7, 8, 9, 10, 11].

Substantial activity has also been aimed at lattices with quadratic ($\chi^{(2)}$) nonlinearity, boosted, in particular, by the recent experimental realization of discrete $\chi^{(2)}$ solitons in nonlinear optics [12]. A variety of topics have been studied for such media both theoretically and experimentally, including the formation of 1D and 2D solitons [13, 14, 15] (see also reviews [16, 17]), observation of the modulational instability in periodically poled lithium niobate waveguide arrays, finite few-site lattices [18], $\chi^{(2)}$ photonic crystals [19], cavity solitons [20], and multi-color localized modes [21]. In addition, the same lattice models with the quadratic nonlinearity may be used to describe the dynamics of Fermi-resonance interface modes in multilayered systems based on organic crystals [22]. A variety of solutions have been obtained in the latter context [23].

A fundamental difference of $\chi^{(2)}$ continua from their $\chi^{(3)}$ counterparts [24] is that they feature no collapse in 2D and 3D cases [25], which paves the way to create stable 2D [26] and 3D [27] quadratic solitons. On the other hand, due to the presence of collapse-type phenomena in 2D and 3D $\chi^{(3)}$ continua, lattice solitons in the corresponding discrete setting are subject to *quasi-collapse*. As a consequence, they only exist with a norm ("mass") exceeding a certain threshold value [28], and they are strongly localized (on few lattice sites), hence 2D and 3D $\chi^{(3)}$ solitons are strongly pinned to the lattice and cannot be motile [29].

The absence of the trend to the catastrophic self-compression in the 2D $\chi^{(2)}$ medium suggests that the corresponding lattice solitons may be broad and therefore *mobile*, being loosely bound to the lattice. The aim of this work is to investigate the mobility of 1D and, especially, 2D solitons in $\chi^{(2)}$ lattices. Besides its importance for the use of such waves in photonic applications, the topic is of fundamental interest by itself, as it will reveal a family of *mobile solitons* in 2D lattices. Thus far, the only example of mobility was provided by solitons in a 2D lattice with saturable nonlinearity [21] (the *Vinetskii-Kukhtarev* model [30], in which the mobility of 1D solitons was examined in Ref. [31]). In this work, we identify parametric regions of stable motion of $\chi^{(2)}$ solitons on 1D and 2D lattices and, for the first time in the 2D case, we study *anisotropy* of the mobility of 2D lattice solitons (for propagation off of the principal directions of the lattice). First, we will introduce the model and develop an analytical approach to solitary waves and the respective PN barrier in $\chi^{(2)}$ lattices. Then, systematic numerical results for the soliton mobility in 1D and 2D lattices will be reported.

The model and analytical results. Following Ref. [15], we introduce a system including equations for the fundamental-frequency (FF) and second-harmonic (SH) waves, $\psi_{m,n}(t)$ and $\phi_{m,n}(t)$. In the 2D setting the model has the following form (its 1D counterpart will also be used below):

$$i\frac{d}{dt}\psi_{m,n} = -\left(C_1\Delta_2\psi_{m,n} + \psi^{\star}_{m,n}\phi_{m,n}\right), \quad (1)$$

$$i\frac{d}{dt}\phi_{m,n} = -\frac{1}{2}\left(C_2\Delta_2\phi_{m,n} + \psi^2_{m,n} + k\phi_{m,n}\right), \quad (2)$$

where $\Delta_2 u_{m,n} \equiv u_{m+1,n} + u_{m-1,n} + u_{m,n+1} + u_{m,n-1} - 4u_{mn}$ is the discrete Laplacian, C_1 and C_2 are the FF and

SH lattice-coupling constants, and k is the mismatch parameter. Equations (1) and (2) conserve the corresponding Hamiltonian and the Manley-Rowe (MR) invariant, $I = \sum_{m,n} \left(|\psi_{m,n}|^2 + 2\,|\phi_{m,n}|^2 \right)$.

Stationary solutions are looked for as $\{\psi_{m,n}(t), \phi_{mn}(t)\} = \{e^{-i\omega t}\Psi_{m,n}, e^{-2i\omega t}\Phi_{m,n}\}$, where distributions $\Psi_{m,n}, \Phi_{m,n}$ are real for fundamental solitons, and may be complex for more elaborate patterns, such as vortices [15]. To set discrete solitons in motion, one must overcome the above-mentioned PN barrier, i.e., the energy difference between static solitons, centered, respectively, on a lattice site and between sites. To obtain an approximate analytical expression for the barrier, we consider the continuum limit, in which stationary functions Ψ and Φ depend only on the radial variable which is the continuum limit of $r \equiv \sqrt{(m^2+n^2)/C_1}$ and obey equations

$$\begin{aligned} \omega\Psi + (\Psi'' + \Psi'/r) + \Psi\Phi &= 0\,, \\ (4\omega + k)\,\Phi + C\,(\Phi'' + \Phi'/r) + \Psi^2 &= 0, \end{aligned} \tag{3}$$

where $C \equiv C_2/C_1$, and the prime stands for d/dr. In this limit, the soliton is approximated by the following ansatz with amplitudes A and B:

$$\begin{aligned} \left\{ \begin{array}{c} \Psi \\ \Phi \end{array} \right\} &= \left\{ \begin{array}{c} A \\ B/\sqrt{2} \end{array} \right\} \sqrt{\frac{\sinh\left(2\sqrt{\{|\omega|,|\chi|\}}r\right)}{\sqrt{\{|\omega|,|\chi|\}}r}} \\ & \operatorname{sech}\left(2\sqrt{\{|\omega|,|\chi|\}}r\right), \end{aligned} \tag{4}$$

where ω and $\chi \equiv (4\omega + k)/C$ must be negative. These expressions have the correct 2D asymptotic forms at $r \to \infty$, $\{\Psi, \Phi\} \sim r^{-1/2} \exp\left(-\sqrt{|\{\omega,\chi\}}r\right)$. Substituting the ansatz in Eqs. (3) and demanding its validity at $r \to 0$, we obtain $B = (23/3)|\omega|$, $A = (23/3)\sqrt{\omega\,(4\omega + k)}$.

The Hamiltonian corresponding to the axially symmetric real solutions of the continuum equations is

$$\begin{aligned} H &= \pi \int_0^\infty r dr \left[2\,(\Psi_r')^2 + C\,(\Phi_r')^2 - 2\Phi\Psi^2 \right. \\ &\left. -k\Phi^2 \right] = \pi \int_0^\infty r dr \left(2\omega\Psi^2 + 4\omega\Phi^2 + \Phi\Psi^2 \right), \end{aligned} \tag{5}$$

where the derivatives were eliminated using integration by parts and Eqs. (3). To derive the PN potential, we apply the lattice discretization to final expression (5) by defining $H_{\rm latt} = \frac{1}{2}\int\int \left(2\omega\Psi^2 + 4\omega\Phi^2 + \Phi\Psi^2\right) \mathrm{gr}(x,y)\ \ dxdy$, where the grid function is

$$\mathrm{gr}(x,y) \equiv \sum_{m,n=-\infty}^{+\infty} \delta\,(x-m)\,\delta\,(y-n) = \sum_{p,q=-\infty}^{+\infty} e^{2\pi i(px+qy)}. \tag{6}$$

In the quasi-continuum approximation (which implies small $|\omega|$ and $|\chi|$), the leading terms in $H_{\rm latt}$ correspond to $(p,q) = (\pm 1, 0)$ and $(0, \pm 1)$ and yield the PN potential with an exponentially small amplitude, $U = U_0\,[\cos(2\pi\xi) + \cos(2\pi\eta)]$, where (ξ, η) are the coordinates of the soliton's center. An expression for the amplitude U_0 is simplest in the case of $|\chi| > 2|\omega|$, which corresponds to numerical results presented below (with $\omega = -0.25$, $\chi = -0.75$); in this case, the second term in $H_{\rm latt}$ dominates. Fitting the slowly varying part of the integrand to a Gaussian, we thus obtain $U_0 = -\alpha\left(|\omega|^3/|\chi|\right)\exp\left(-3\pi^2/(10|\chi|)\right)$, with $\alpha \equiv (2\pi/15)23^2 \approx 222$.

Numerical Results. In the 1D and 2D cases alike, we used lattices with periodic boundary conditions, in order to allow indefinitely long progressive motion of solitons. First, we found standing lattice-soliton solutions $\left\{\Psi^{(0)}_{m,n}, \Phi^{(0)}_{m,n}\right\}$, by means of fixed-point iterations. Next, dynamical simulations were initialized by applying a *shove* (kick) to those solutions, which corresponds to initial conditions

$$\left\{ \begin{array}{c} \psi_{m,n} \\ \phi_{m,n} \end{array} \right\} = e^{i(S/C_{1,2})(m\cos\theta + n\sin\theta)} \left\{ \begin{array}{c} \Psi^{(0)}_{m,n} \\ \Phi^{(0)}_{m,n} \end{array} \right\}, \tag{7}$$

where S and θ determine the size and orientation of the shove vector.

Examples of stable motion and destruction of the 1D lattice soliton subjected to the shove are displayed in Fig. 1, and systematic results, obtained with variation of S and $C_1 = C_2$, are summarized in Fig. 2. The destruction was registered as the outcome if the kicked soliton would eventually lose more than 30% of the initial value of its MR invariant. For the coupling strength such as that corresponding to Fig. 1, there are, practically, only two outcomes, *mobile waves* and *wave destruction,* observed with different initial kicks. However, for weaker couplings (i.e., stronger discreteness), "localization" is also possible: if the shove's strength, S, is below a lower critical value, $S^{(0)}_{\rm cr}$, the soliton survives without acquiring any velocity. The latter outcome is explained by noting that the kinetic energy, $E_{\rm kin} \sim S^2$, imparted to the soliton by the shove, may be insufficient to overcome the PN barrier ($2U_0$ as defined above). Because U_0 decays exponentially with the increase of the intersite coupling, the "localization" region in Fig. 2 is very small. General features of the 1D situation are: (i) for $S < S^{(0)}_{\rm cr}$, the soliton remains quiescent; (ii) for $S^{(0)}_{\rm cr} < S < S_{\rm cr}$, the soliton sets in the state of persistent propagation; (iii) for $S > S_{\rm cr}$, the soliton is destroyed.

We now turn to the 2D setting, which is more interesting for two reasons. First, as noted above, in the 2D case the mobility of lattice solitons is a highly nontrivial feature, practically impossible in the case of the $\chi^{(3)}$ nonlinearity; second, it is interesting to study anisotropy of the mobility, i.e., its dependence on the orientation of the initial kick relative to the (principal directions of the) lattice. Figure 3 shows two examples of stable moving regimes: one along the lattice diagonal, and, to our knowledge, the first ever example of the motion on the

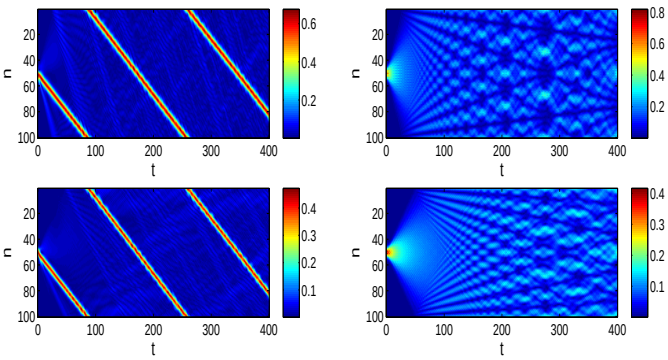

FIG. 1: (Color online) Space-time contour plots of $|\psi_{m,n}|^2$ and $|\phi_{m,n}|^2$ for the FF and SH fields (top and bottom panels) in the 1D lattice with periodic boundary conditions, for $C_1 = C_2 = 1$, $\omega = -0.25$, $k = 0.25$, and the shove's strength $S = 0.4$ and 3.0 (left and right panels respectively). The boosted soliton sets in stable motion in the former case, and is destroyed in the latter case.

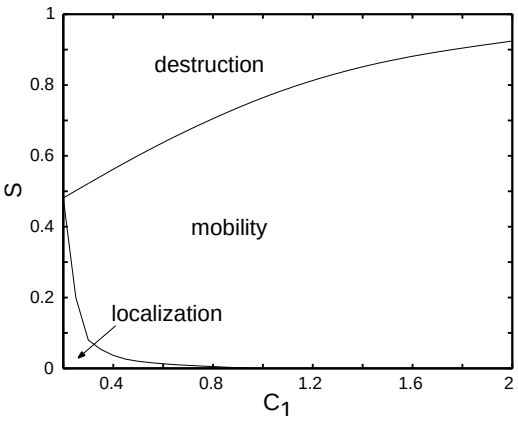

FIG. 2: A diagram in the plane of the coupling strength, $C_1 = C_2$, and shove's strength, S, showing different outcomes of kicking the quiescent soliton in the 1D lattice, for $\omega = -0.25$ and $k = 0.25$. ("localization" means that the soliton remains quiescent).

lattice in an arbitrary direction (neither diagonal, nor along the bonds).

In Fig. 4, we summarize the dependence of the mobility properties on the strength, S, and direction, θ, of the initial kick. The kicking of the 2D soliton may result in "localization" (no motion at all), in some interval $S < S_{\rm cr}^{(0)}$ ($S < S_{\rm cr}^{(0)} \approx 0.02$ in the top left panel of Fig. 4). Other generic outcomes again amount to propagation at a finite velocity, which depends on S, and destruction for a large supercritical kick $S > S_{\rm cr}$.

Particularly noteworthy features, specific to the 2D setting, are presented in the top right and bottom left panels of Fig. 4, viz., dependences of $S_{\rm cr}$ and velocity in the moving regime on θ. These dependences demonstrate that the propagation direction along which it is easiest to sustain motion (i.e., with larger speeds/range of initial kicks) on the square lattice is along the diagonal, as the motion in this direction can be sustained up to larger values of $S_{\rm cr}$, and is fastest for given S. Both facts may be qualitatively explained by the analytically predicted PN potential in the following way. Given the nearest-neighbor nature of the interactions, in order for the cen-

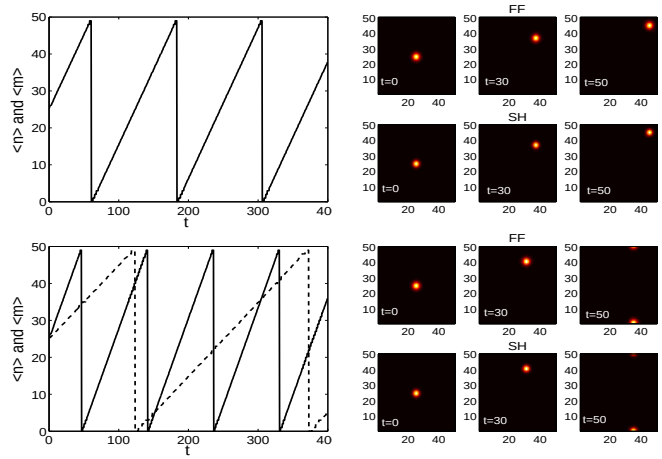

FIG. 3: (Color online) Same as Fig. 1 and with the same parameters, but in the 2D periodic lattice, for the propagation in the diagonal (45 degrees, top panels) and off-diagonal (20 degrees relative to the lattice bonds, bottom panels) directions. The left panels show trajectories of the soliton's center, while the right ones display snapshots of the moving solitons in the FF (top) and SH (bottom) fields at $t = 0, 30, 50$.

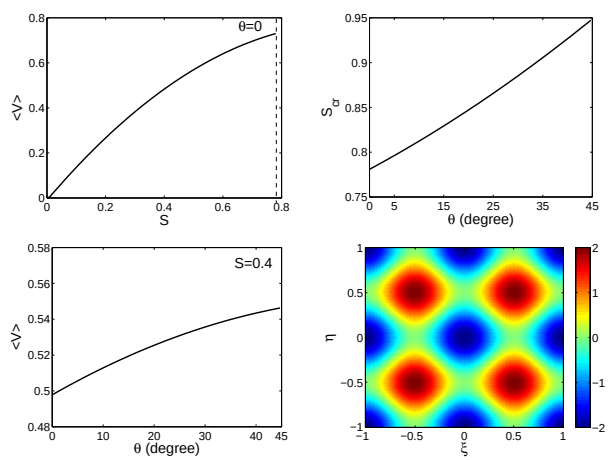

FIG. 4: (Color online) Features of the soliton motion in the 2D periodic lattice, for $C_1 = C_2 = 1$, $k = 0.25$, $\omega = -0.25$. The top left panel shows the velocity versus the shove's strength S in the propagation along the lattice bonds (at angle $\theta = 0$); the vertical dashed line indicates the value of $S_{\rm cr}$, beyond which the soliton retains less than 70% of its initial MR invariant, and is therefore categorized as destroyed. The top right panel depicts $S_{\rm cr}$ as a function of the orientation of the initial kick, θ. For a given S ($S = 0.4$), the ensuing velocity of the motion is shown versus θ in the bottom left corner. In addition, the bottom right panel shows the analytically predicted PN potential.

ter of the wave to move along the diagonal, it has to split along the two lattice directions and then recombine at the site located diagonally across from the initial position. The recurrent small-scale symmetric breakups and recombinations (observed in the numerical data) provide for an effective propagation along the diagonal direction with a minimum PN barrier (see bottom right panel of Fig. 4) and thus result in higher propagation speeds.

We have also examined the situation with $C_2 < C_1$, and obtained similar results, but with larger $S_{\rm cr}^{(0)}$. In the special case of $C_2 = 0$ (no lattice coupling in the SH field), we were not able to generate moving solitons, which can be easily explained: with $C_2 = 0$, Eq. (2) yields $\Phi_{m,n} = -\left(4\omega + k\right)^{-1}\left(\Psi_{m,n}\right)^2$, and the substitution of this in Eq. (1) makes the model equivalent to that with the cubic nonlinearity, where moving 2D discrete solitons do not exist.

Conclusions. In this work, we have examined the mobility of solitons in 1D and 2D lattices with the quadratic nonlinearity. We have shown that the solitons feature stable motion much more easily than their counterparts in 1D lattices with the cubic nonlinearity, and they may also be mobile on the 2D lattice, where the cubic solitons cannot move at all. In the 2D lattice, we have for the first time reported a possibility of motion of the soliton in an arbitrary direction (neither axial nor diagonal), while we have illustrated the interesting differences between propagating on and off of lattice directions. A qualitative explanation for some of these features was provided by an analytical approximation for the 2D Peierls-Nabarro potential.

It may be interesting to extend this type of examination to other 1D and, especially, 2D models, where mobile solitons may be expected, such as systems with competing nonlinearities (the cubic-quintic model [32], or the Salerno model with competing on-site and inter-site cubic terms [33]) and saturable nonlinearity [29], where one may expect significant potential for genuine traveling [31, 34]. While herein the mobility of solutions for realistic physical purposes in finite lattices was examined, the existence of genuinely traveling such solutions is also an interesting computational [34] and mathematical [35] problem.

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