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# On the Ricardian Invariable Measure of Value: The General Possibility of the Standard Commodity

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# DEPARTMENT OF ECONOMICS

## Working Paper

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On the Ricardian Invariable Measure of Value: The General Possibility of the  
Standard Commodity\*

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**Abstract**

The purpose of this paper is to examine the critical arguments made by Burmeister, Samuelson, and others, with respect to Sraffa (1960). In his arguments about the standard commodity, Sraffa assumed that a change in income distribution has no effect on the output level and choice of techniques, while those critics argue that interdependence among changes in income distribution, output level, and choice of techniques should be taken into consideration in the arguments on the invariable measure of value and the linearity of income distribution. Given this debate, the paper considers general economies with non-increasing returns to scale, where such interdependence is a universal feature, in which a generalisation of the standard commodity is defined. Moreover, it is shown that the generalised standard commodity can serve as an invariable measure of value even in those general economies. Finally, the paper also characterises the necessary and sufficient condition under which the linear functional relation of income distribution is obtained in those economies.

*Keywords:* Ricardo's invariable measure of value, Sraffa's standard commodity, Linear relation of income distribution

*JEL Classifications:* B51, D30, D51

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# 1 Introduction

As is well known, in his later years, Ricardo intensively searched for an invariable measure of value.<sup>1</sup> His struggle to find it is shown in his *Principles*, his papers entitled ‘Absolute Value and Exchangeable Value’, which were written in the last few weeks of his life (Ricardo, 1951D, pp. 361–412), and others.

An invariable measure of value can be defined as a measure that is invariable with respect to changes in both income distribution and technique (Ricardo, 1951A, chap. 1). The advantage of the invariable measure of value, if it exists, is that we can distinguish between the variations which belong to the commodity itself and those which are occasioned by a variation in the medium by which values or prices are expressed, when relative prices change (Ricardo, 1951A, p. 48). For Ricardo, the pursuit of the invariable measure of value is directly related to the completion of the embodied labour theory of value,<sup>2</sup> although the importance of the invariable measure of value was not adequately understood by his contemporaries such as Malthus.<sup>3</sup>

Although it is true that the embodied labour theory of value cannot generally hold when the rate of profit is positive, it does not mean that the invariable measure of value is no longer significant. The purpose of Ricardo’s construction of the invariable measure of value is to build a solid foundation not only to measure such important variables as national income or national wealth precisely, but also to compare those

variables intertemporally. No one can deny the importance of the invariable measure of value even today.

In the 20<sup>th</sup> century, Sraffa (1960) revived the concern about the invariable measure of value, which had fallen into oblivion since the so-called Marginal Revolution. Unlike Ricardo, he divided the problem of identifying an invariable measure of value into two parts: the first is to search for a measure of value that is invariable with respect to changes in technique, left aside the change in income distribution, and the other is to search for a measure of value that is invariable with respect to the change in income distribution, left aside the change in technique. Sraffa concentrated on the latter by constructing a special, composite commodity termed the *standard commodity*.<sup>4</sup> He also demonstrated an interesting relationship with respect to income distribution if the standard commodity is adopted as the numéraire: the linear relationship of income distribution. Although many economists have paid great attention to the results obtained by Sraffa, it seems that they have not reached a consensus on evaluating Sraffa (1960). Some economists appreciate him, whereas others do not unconditionally admit the significance of the standard commodity and the linearity of income distribution. In particular, those who are critical of Sraffa regard the assumption of a fixed technique without constant returns to scale as being too restrictive, and thus, downgrade the relevance of Sraffa. Burmeister (1968, 1975,

1977, 1980, 1984), Samuelson (2000, 2008), Samuelson and Etula (2006), and others claimed that Sraffa's analyses are irrelevant without the assumption of constant returns to scale.

We think that the views of Burmeister, Samuelson, and others are worth examining, because they point out relevant problems from the viewpoint of Ricardo (1951A) and modern economic theory, which Sraffa had not addressed. In his arguments about the standard commodity, Sraffa (1960) assumed that a change in income distribution has no effect on the output level and choice of techniques. Such an assumption is just an analytical device to construct a model. However, it is plausible that the change in income distribution is related to changes in output level or choice of techniques in actual economies. In fact, almost all modern economic theories as well as Ricardo (1951A) admit interdependence among changes in income distribution, output level, and choice of techniques, even though the logical consequences of such interdependence are different among theories. Even those who are favourable to Sraffa would not be able to deny this interdependence. Curiously enough, there is little literature on whether or not an invariable measure of value and linearity of income distribution can be obtained in models where the above-mentioned interdependence is allowed.<sup>5</sup> Therefore, we attempt to examine their critical arguments with respect to Sraffa (1960). That is, assuming quite general economies with non-increasing re-

turns to scale, where the above-mentioned interdependence is a universal feature, we define a generalisation of the standard commodity and show that it still serves as the invariable measure of value. In addition, we specify the conditions under which the linear relationship of income distribution preserves in such general economies.

This paper is organised as follows: In Section 2, we present a brief review of the concept of Ricardo's invariable measure of value and Sraffa's standard commodity. Subsequently, we briefly review the history of the debates on the standard commodity and the linear relation of income distribution that Sraffa derived. In Section 3, we discuss the generalisation of the standard commodity to a more general production economy than Sraffa's (1960), and discuss the main results in terms of the invariable measure of value and the linear relationship of income distribution. In Section 4, we present our concluding remarks.

## **2 The Invariable Measure of Value and Debates concerning Sraffa (1960)**

In this section, we briefly review the concept of Ricardo's invariable measure of value and Sraffa's standard commodity. We also review the debates concerning the significance of the standard commodity and linearity of income distribution.

## 2.1 Ricardo's invariable measure of value

Ricardo asserted that the conditions necessary to make a measure of value perfect are that it should itself have a value, and that value should itself be invariable (Ricardo, 1951D, p. 361). Concerning the first condition, he clearly argued that the labour content embodied in such a commodity represents the exchange value of the commodity. The second condition, the invariance of the value of such a commodity, perplexed him throughout his life.

Why is it difficult to obtain an invariable measure of value? First, the technique to produce it must remain unchanged. In other words, a commodity eligible to become the invariable measure of value is one 'which now and at all times required precisely the same quantity of labour to produce it.' However, Ricardo realised, 'Of such a commodity we have no knowledge, and consequently are unable to fix on any standard of value' (Ricardo, 1951A, p. 17, n. 3). In fact, Ricardo regarded money (that is, gold and silver) as the invariable measure of value, but it is just 'as near as approximation to a standard measure of value as can be theoretically conceived' (Ricardo, 1951A, p. 45). The justification is based on his recognition that the techniques of production of gold and silver are subject to fewer variations (Ricardo, 1951A, p. 87).

With respect to the second condition, even though the technique to produce



gold and silver is unchanged, these cannot be the invariable measure of value. This is because all industries have different proportions of capital and labour, different proportions of circulating and fixed capital, different degrees of durability of fixed capital, and different time-periods necessary to bring the commodity to market. In this situation, the change in the level of wage rates causes changes in relative prices. Therefore, as already mentioned, we cannot precisely know the price changes, because the prices of gold and silver themselves (the standard of value) are subject to the relative variations. The perfect invariable measure of value never existed in reality. According to Ricardo (1951A, p. 45), however, the effect of a change in income distribution on relative prices is smaller than the effect of a change in technique. Therefore, Ricardo thought of the deviation of value from the embodied quantity of labour as sufficiently slight (Ricardo, 1951B, p. 66), and he was reluctantly content to say that money can be regarded as the invariable measure of value at the first approximation.

## **2.2 Sraffa's standard commodity and income distribution**

Sraffa (1960) revived the concern about the invariable measure of value. As already mentioned, Ricardo had defined the conditions that the invariable measure of value should satisfy: the invariance of the measure of value with respect to changes in both

income distribution and technique. Ricardo was perplexed by the conditions, because he attempted to solve the two simultaneously. In contrast, Sraffa concentrated on finding a measure of value that is invariable with respect to a change in income distribution, left aside the change in techniques. Furthermore, it is Sraffa's breakthrough idea to find a special, composite commodity, termed the *standard commodity*, which plays the role of the invariable measure of value; whereas Ricardo attempted to find a single commodity that plays the role.

Let us briefly review the concept of the standard commodity in a single product system. The price system is defined as follows:

$$\mathbf{p} = (1 + \pi) \mathbf{p}\mathbf{A} + w\mathbf{L}, \tag{1}$$

where  $\mathbf{p}$ ,  $\mathbf{L}$ , and  $\mathbf{A}$  denote the price vector, the labour coefficient vector, and the physical input coefficient matrix, respectively. For the sake of simplicity,  $\mathbf{A}$  is assumed to be an indecomposable and productive matrix.  $\pi$  and  $w$  denote the rate of profit and the wage rate, respectively. In order to escape from the impasse that Ricardo faced, Sraffa attempted to find an (imaginary) industry that has a value-ratio of the net product to means of production such that the increase in profit is exactly offset by the decrease in wage when the wage rate is reduced. The value-ratio is the solution of the system that Sraffa (1960, p. 20) called the *standard system*:

$$(1 + \Pi^*) \mathbf{A} \mathbf{q}^* = \mathbf{q}^*, \quad (2)$$

$$\mathbf{L} \mathbf{q}^* = 1. \quad (3)$$

$\Pi^*$  is the value-ratio, which is now termed the *standard ratio*. It is related to the Frobenius root  $\lambda_A$  as  $\lambda_A = \frac{1}{1+\Pi^*}$ . The standard ratio is equal to the maximum rate of profit.  $\mathbf{q}^*$  is the corresponding eigenvector and is the vector denoting the output level of the industry that has the standard ratio. Now, it is termed the standard commodity. Since we assume the productiveness and indecomposability of  $\mathbf{A}$ , the above system of equations has the solution of  $\Pi^* > 0$  and  $\mathbf{q}^* > 0$  from the Perron-Frobenius theorem (Pasinetti, 1977, pp. 95-7). From formula (2), we obtain:

$$\frac{\mathbf{p}[\mathbf{I} - \mathbf{A}] \mathbf{q}^*}{\mathbf{p} \mathbf{A} \mathbf{q}^*} = \Pi^*, \quad (4)$$

where  $\mathbf{I}$  denotes the identity matrix. Formula (4) means that the ratio of the net product to means of production, measured by the standard commodity, is always constant, irrespective of price variations. Therefore,  $\Pi^*$  is a real ratio that is independent of prices. Sraffa defined the standard net product and chose it as the numéraire as follows:

$$\mathbf{p} [\mathbf{I} - \mathbf{A}] \mathbf{q}^* = 1. \quad (5)$$

Although the price of *any* numéraire is invariant by definition, the standard commodity is special in that the cause of price change as a result of the change in income distribution is absent in the industry producing it. It is only when the numéraire is the standard commodity that the absence of the price change caused by the change in income distribution in the industry producing the numéraire is ensured. Therefore, the standard commodity is eligible to become the invariable measure of value under the assumption of fixed technique.<sup>6</sup> Note that the standard commodity does not need to be actually produced; it is a “purely auxiliary construction” (Sraffa, 1960, p. 31).

In Sraffa’s model, nothing except income distribution ever changes; the technique in use, output level, and proportion of means of production to labour are all fixed. Therefore, no assumption on returns to scale needs to be made, as Sraffa (1960, p. v) said. Under such assumptions, he exclusively analysed the change in relative prices caused by the change in income distribution. Owing to formulae (4) and (5), there is no need for a variation in the price of  $\mathbf{q}^*$  to restore the surplus or deficit in the industry which produces that commodity, when the wage rate is reduced. Therefore, the variation in relative prices caused by a change in income distribution is solely

attributed to the variation in prices of measured commodities on the basis of the invariance property of the numéraire defined by the standard commodity.

Furthermore, the adoption of the standard commodity as the numéraire shows us the useful relation of income distribution. From (1) and (5), we obtain:

$$\pi = \Pi^* (1 - w). \quad (6)$$

Here,  $w$  denotes the wage rate or the wage share in terms of the standard commodity, whereas  $\pi$  is the actual rate of profit. The distributional relation is expressed by the straight line. The important implication of function (6) is that the rate of profit can be obtained without knowing prices, once we know the wage in terms of the standard commodity. In other words, the standard commodity enables us to treat income distribution independently of prices. As Pasinetti (2006, p. 154) pointed out, the relevance of function (6) does not lie in its linearity, but in the fact that it is independent of prices.

We conclude that Sraffa resolved the problem that Ricardo could not, but the resolution was *partial*, because Sraffa did not consider another problem. This is the problem of the measure of value invariable with respect to the change in technique.

### 2.3 After Sraffa (1960)

There have been many reactions to Sraffa (1960) since its publication. The debates focused not only on the invariable measure of value, but also on the usefulness of the standard commodity and function (6). Some arguments appreciate Sraffa's achievements, especially his contribution of constructing the standard commodity as the invariable measure of value (for example, Roncaglia, 2009). Other arguments are critical of Sraffa. First, some economists argued that the standard commodity does not play the role of the Ricardian invariable measure of value. Those arguments pointed out the flaw in Sraffa's analysis. Flaschel (1986) is a typical example. The second critical argument was that the standard commodity and function (6) are so restrictive that they are not too helpful for relevant analyses. Those arguments were mainly raised by neoclassical economists, who were interested in variations in output and proportions of means of production.

Let us examine Flaschel (1986) first. According to him, there is a specific and complete solution to the problem of determining the conditions for the invariable measure of value, but Sraffa's standard commodity does not fulfil those conditions. It seems to us that his definition of invariance is different from those of Ricardo and Sraffa. He defined that given  $\mathbf{e} - \mathbf{A}\mathbf{e}$  as the numéraire, where  $\mathbf{e}$  is a vector, all the elements of which are units, an arbitrary composite commodity  $\mathbf{b}$  has the invariance

property if and only if  $\mathbf{pb} = 1$  holds for any non-negative and non-zero  $\mathbf{p}$ , with  $\mathbf{p}[\mathbf{I} - \mathbf{A}]\mathbf{e} \equiv 1$  (Flaschel, 1986, pp. 597–8).<sup>7</sup> Certainly Sraffa (1960, p.11) adopted  $[\mathbf{I} - \mathbf{A}]\mathbf{e}$  as the numéraire, but the numéraire adopted in the context is irrelevant to the issue of the invariable measure of value, and his arguments on the standard commodity have nothing to do with the numéraire of  $[\mathbf{I} - \mathbf{A}]\mathbf{e}$ . Flaschel’s critique of the standard commodity, therefore, seems pointless.<sup>8</sup>

As for the second argument critical of Sraffa, the typical example is Burmeister (1968). The conclusions he derived are summarised as follows:

1) It is dubious what economic significance can be attached to the standard commodity.

2) The linearity of the distributional relation does not hold if wages are paid at the beginning of the production period rather than at the end.

3) Without the assumption of constant returns to scale and a fixed coefficients matrix, Sraffa’s analysis is meaningless if the quantity produced by an arbitrary industry changes.

After Burmeister (1968), he repeated conclusions similar to those above (Burmeister, 1975, 1977, 1980, 1984). However, he obviously misunderstood some aspects of Sraffa (1960).

The first conclusion made by him is a serious misunderstanding. Burmeister

regarded the standard commodity as the *actual* consumption basket by which the real wage rate  $w$  in function (6) is measured.<sup>9</sup> Therefore, he argued that the standard commodity has no economic significance; ‘Sraffa’s weights used to construct his basket of goods are seen to be determined completely from the technology without regard for consumption preferences’ (Burmeister, 1984, p. 509). However, the adoption of the standard commodity as the numéraire does not imply that people must actually consume each commodity in the same proportion as that given by the standard commodity. Moreover, it does not imply that each commodity is actually produced in the same proportion as that given by the standard commodity (see Kurz and Salvadori, 1987, pp. 876–7).

The second conclusion is correct. However, though the linearity no longer holds in this case, as Pasinetti (1977, p. 131) showed, the distributional relation is independent of prices.

The third conclusion is controversial. Samuelson (2000) and Samuelson and Etula (2006) also argued that constant returns to scale is an indispensable assumption in order to retain the significance of Sraffa’s analysis. Against these arguments, some proponents of Sraffa argued that the assumption on returns to scale is unnecessary in Sraffa’s analysis. The characteristic of the analysis is that it is based on the classical surplus approach. In the approach, the analysis of the distribution of physical



surplus comes first. Eatwell (1977) emphasised the difference in the analytical basis between classical and neoclassical economics. In the former, the size and composition of output, technique in use, and real wage are the *data*, on the basis of which the distribution of surplus, price formulation, and quantities of input and labour employed are obtained. In the latter, on the contrary, the preferences of individuals, initial endowment of commodities and/or factors of production, distribution of the initial endowments among individuals, and technology are the *data*, and all variables are determined by the interaction between supply and demand. It is based on the marginal method, and thus the assumption on returns to scale is necessary in neoclassical economics. Eatwell (1977) thus argued that the assumption of constant returns to scale is irrelevant in Sraffa's analysis, because it is based on the classical surplus approach.

However, Burmeister and Samuelson considered what happens to the model when the output level changes. Unless constant returns to scale are assumed, the technique generally changes as the output level changes. Since each coefficient matrix has the specific standard ratio, the standard ratio also changes when the technique in use changes. Therefore, function (6) no longer gives us any useful information on income distribution when a change in the output level causes a change in technique in economies without constant returns to scale. Burmeister (1977, pp. 69–70) thus

replied to Eatwell: ‘I conclude that constant returns to scale is irrelevant for Sraffa’s analysis only if one is content to pose irrelevant questions.’

Although it is true that Burmeister’s interpretation of Sraffa included the misunderstanding, it is also true that he raised important questions which Sraffa had not addressed. The questions are whether or not the invariable measure of value exists in economies where not only income distribution but also technical choice are available; and if it exists, what kind of relationship between the invariable measure of value and income distribution holds. We think it worthwhile to examine them. From the viewpoint of modern economic theories as well as Ricardo (1951A), these are natural questions, because nearly all economic theories allow for interdependence among changes in income distribution, output level, and techniques. In fact, Sraffa (1925, 1926) himself had considered the relationship between returns to scale and choice of techniques, although his consideration was related to the critique of Marshallian partial equilibrium analysis.

### 3 The Standard Commodity and Income Distribution under Non-increasing Returns to Scale

In this section, we investigate the conditions for the invariable measure of value and the linearity of income distribution under a non-increasing returns to scale production technology.

#### 3.1 Generalisation of the standard commodity

In order to analyse the invariable measure of value and income distribution in economies where the change in technique is allowed, let us introduce the production possibility set  $P$  with non-increasing returns to scale, which is the set of available production processes. A production process is defined as  $\alpha \equiv (-\alpha_l, -\underline{\alpha}, \bar{\alpha})$ , where  $\alpha_l$  is the non-negative *effective* labour input of the process,  $\underline{\alpha}$  is the non-negative vector of the inputs of the produced goods used in the process, and  $\bar{\alpha}$  is the non-negative vector of the outputs of the  $n$  goods. There are small mild restrictions on the properties of  $P$ : not to activate any production process is available; to produce any non-negative vector of the  $n$  goods as a net output, there is at least one production process available in  $P$ ; to produce any non-negative and non-zero vector of commodities, the inputs of labour and at least one type of capital goods are indis-

pensable; if there are two production processes available in  $P$ , it is also available that any proportion, say  $t \in (0, 1)$ , of one of the processes and the remaining proportion,  $1 - t$ , of the other process are jointly activated. Production set  $P$  satisfying these restrictions is so general that various types of technologies, such as Leontief production models with or without technical choices, joint production models, and even neoclassical differentiable production functions, are subject to the analysis here.<sup>10</sup>

Let us assume that one *economy* is represented by a production set  $P$ . We can define the standard commodity in an economy  $P$ .

**Definition 1:** For any economy  $P$ , a *standard commodity* is a positive vector  $y > \mathbf{0}$ , such that there exists a vector  $\alpha = (-1, -x, x + y)$  on  $P$  satisfying the following properties; (i)  $\frac{y_i}{x_i} = \frac{y_j}{x_j}$  for any  $i, j = 1, \dots, n$ ; (ii) there is no other  $\alpha' = (-1, -x', x' + y')$  with  $\frac{y'_i}{x'_i} = \frac{y'_j}{x'_j}$  for any  $i, j = 1, \dots, n$ ,  $\frac{y'_i}{x'_i} > \frac{y_i}{x_i}$  for any  $i = 1, \dots, n$ , and  $y' > y$ .

The standard commodity defined here is a generalisation of Sraffa's definition. Firstly, Definition 1 implies that the standard commodity is defined as the net product  $y$  that can be produced by a process  $\alpha = (-1, -x, x + y)$  with labour input  $\alpha_l = 1$ , capital inputs  $\underline{\alpha} = x$ , and gross outputs  $\bar{\alpha} = x + y$ . Moreover, condition (i) of Definition 1 implies that under this process the ratio of net product to means of production is uniform,  $\frac{y_i}{x_i} = \frac{y_j}{x_j}$  for any  $i, j = 1, \dots, n$ . Secondly, condition (ii) of Definition 1 is a generalisation of the maximality condition of the uniform ratio of

net product to means of production. The ratio corresponds to the standard ratio  $\Pi^*$  of equation (2) in Section 2.2. Therefore, Definition 1 is regarded as an extension of the Sraffian definition of the standard commodity characterised by equations (2) and (3) to a more general economy  $P$ , and production process  $\alpha = (-1, -x, x + y)$  is regarded as the *standard system* in the economy  $P$ .

This definition is well-defined, in that the standard commodity given by Definition 1 uniquely exists.<sup>11</sup>

### **3.2 The invariable measure of value and the linear relation of income distribution**

We examine whether or not the standard commodity defined above can function as the invariable measure of value in an economy  $P$ .

Consider a price system  $(p, w)$ , which is a non-negative and non-zero vector. Let there be the maximal rate of profit  $\pi \geq 0$  and a production process  $\alpha = (-\alpha_l, -\underline{\alpha}, \bar{\alpha})$  on  $P$  associated with  $(p, w)$ , in that

$$p\bar{\alpha} = (1 + \pi)p\underline{\alpha} + w\alpha_l \text{ and}$$

$$p\bar{\alpha}' \leq (1 + \pi)p\underline{\alpha}' + w\alpha_l'$$

hold for any  $\alpha' = (-\alpha'_l, -\underline{\alpha}', \bar{\alpha}')$  on  $P$ . Then, let us call such a price system an *equilibrium price*.<sup>12</sup> Consider a situation where an equilibrium price changes from  $(p, w)$  to  $(p', w')$ . Moreover, let  $\pi$  (resp.  $\pi'$ ) be the maximal rate of profit associated with the price system  $(p, w)$  (resp.  $(p', w')$ ). Then, let  $\Delta p \equiv p' - p$ ,  $\Delta w \equiv w' - w$ , and  $\Delta \pi \equiv \pi' - \pi$ . The following definitions are a generalisation of the invariable measure of value on the basis of Baldone (2006):

**Definition 2:** Given an economy  $P$ , let  $(p, w)$  and  $(p', w')$  be two different equilibrium prices, and  $\pi$  and  $\pi'$  the respectively associated maximal profit rates. Then, a commodity bundle  $y > \mathbf{0}$  serves as the *invariable measure of value with respect to change from  $(p, w)$  to  $(p', w')$* , if and only if there exist a non-negative and non-zero vector  $x$  and a positive number  $k > 0$ , such that the process  $(-k, -x, x + y)$  is feasible under the economy  $P$ ,  $(-k, -x, x + y) \in P$ , and  $\Delta p y = 0$  holds whenever this price change involves a *redistribution between profit and wage*, namely,  $\Delta \pi p x + \Delta w k = 0$ .

**Definition 3:** Given an economy  $P$ , a commodity bundle  $y > \mathbf{0}$  serves as the *invariable measure of value*, if and only if for any different vectors of the equilibrium prices  $(p, w)$  and  $(p', w')$ , it serves as the invariable measure of value with respect to change from  $(p, w)$  to  $(p', w')$ .

That is, a commodity bundle serves as the invariable measure of value, if and

only if for any change in the price system involving a redistribution of profit and wage, the price of this commodity bundle is invariable. The definitions faithfully follow Sraffa's one reviewed in Section 2. More precisely speaking, let us consider counterfactually a change in income distribution from  $(\pi, w)$  to  $(\pi', w')$ , while keeping the commodity price vector  $p$  constant, such that the increase (*resp.* decrease) in profit is exactly equal to the decrease (*resp.* increase) in wage in the production process  $(-k, -x, x + y)$  of the targeted commodity bundle  $y$ . Such a change may be derived from a purely political conflict on the income distribution between capital and labour, or it may involve a change in technique. In any case, however, it may result in a change in commodity prices from  $p$  to  $p'$ . Then, the commodity bundle  $y$  can serve as the invariable measure of value with respect to the change from  $(p, w)$  to  $(p', w')$  whenever  $py = p'y$ . Furthermore, if the commodity bundle satisfies such an invariable property for any change in price systems with its corresponding redistribution between wage and profit, it can serve as the invariable measure of value.

It is worth emphasising that in the above definitions, the invariable property must hold regardless of the causality of such a price change. For instance, even if the price change associated with the corresponding redistribution is generated *owing to technical change* so that the selected production process is changed in equilibrium,<sup>13</sup>

the value of the commodity bundle is required to be invariable.

The following Theorem 1 provides the necessary and sufficient condition for the standard commodity to serve as the invariable measure of value.

**Theorem 1:** *For any economy  $P$ , let us take any equilibrium prices  $(p, w)$  and  $(p', w')$ . Then, the standard commodity  $y^*$  associated with the standard system  $\alpha^* = (-1, -x^*, x^* + y^*)$  serves as the invariable measure of value with respect to change from  $(p, w)$  to  $(p', w')$ , if and only if there exist non-negative numbers  $\delta, \delta'$  such that  $py^* = \pi px^* + w - \delta$ ,  $p'y^* = \pi' p'x^* + w' - \delta'$ , and  $\delta = \delta'$  hold.*

**Proof:** See the Appendix.

In Theorem 1,  $\delta$  (resp.  $\delta'$ ) represents the shortfall of profits from the maximal level, when they are generated by operating the standard system  $\alpha^*$  at the equilibrium price  $(p, w)$  (resp.  $(p', w')$ ). The standard commodity can serve as the invariable measure of value with respect to a change from  $(p, w)$  to  $(p', w')$ , if and only if the shortfall of profits generated by operating the standard system is invariable with respect to such a change in prices. It then follows that the standard commodity can serve as the invariable measure of value, if and only if the shortfall of profits generated by operating the standard system is invariable with respect to any change in equilibrium prices.



Theorem 1 can be used to check for each given economy, whether the standard commodity can serve as the invariable measure of value. Such a test is particularly relevant in a general economy  $P$  where a change in prices could be associated interdependently with a change in technique and/or a change in produced outputs. For the standard system  $\alpha^*$  is not necessarily always a profit-rate maximiser in such a rather general economy, therefore a positive shortfall,  $\delta > 0$ , of profits from the maximal level is available, unlike in the case of single product system discussed in section 2. Even in the case of  $\delta > 0$ , as Theorem 1 suggests, the standard commodity can serve as the invariable measure of value whenever the amount of the shortfall is invariable.

The following corollary gives us a typical situation where the standard commodity serves as the invariable measure of value.

**Corollary 1:** *For any economy  $P$ , let us take any equilibrium prices  $(p, w)$  and  $(p', w')$  at both of which the standard system  $\alpha^* = (-1, -x^*, x^* + y^*)$  is a profit rate maximiser. Then, the standard commodity  $y^*$  serves as the invariable measure of value with respect to a change from  $(p, w)$  to  $(p', w')$ .*

**Proof:** See the Appendix.

Note that when the production set is represented by a simple Leontief technology (that is, a single product system as discussed in section 2), the standard system  $\alpha^*$

is a profit rate maximiser for any equilibrium price vectors. This is because in an economy with a simple Leontief technology, an equilibrium price system is associated with an equal rate of profit available at every industry, which implies that any efficient production process, including the standard system, is a profit rate maximiser. Thus, as Corollary 1 shows, the standard commodity  $y^*$  can be the invariable measure of value with respect to any change in equilibrium prices.

Let us now assume that the standard commodity  $y^*$  is selected as the numéraire. Then, by definition, any non-negative and non-zero price vector  $p$  is normalised as  $py^* = 1$ . Given such a situation, our next subject is to examine whether and under what condition the linear distributional relationship between profit and wage is preserved in an economy  $P$ . The following theorem is our second main result.

**Theorem 2:** *Given an economy  $P$ , the linear functional relation of income distribution,  $\pi' = \Pi(1 - w')$ , holds for any equilibrium price vector  $(p', w')$  associated with the maximal profit rate  $\pi'$  if and only if  $p'y^* = \pi'p'x^* + w'$  holds for any equilibrium price vector  $(p', w')$  associated with  $\pi'$ , where  $\alpha^* = (-1, -x^*, x^* + y^*)$  is the standard system and  $\Pi$  is the standard ratio.*

**Proof:** See the Appendix.

The crucial point for the above analysis is whether or not the standard system

$\alpha^* = (-1, -x^*, x^* + y^*)$  is a profit rate maximiser at all equilibrium prices available in  $P$ . This property is trivially satisfied in single product systems such as Leontief production economies and as in Sraffa (1960), as discussed above. In contrast, an economy  $P$  allows for the possibility of joint production as well as of technical choices, under which the standard system may not be a profit rate maximiser at some equilibrium price system. In such a case, Theorem 2 suggests that the linearity of income distribution no longer holds.

Though Theorem 2 *per se* suggests that the linear functional relation of income distribution does not hold in general, it is quite surprising that, as in the following theorem, the standard commodity serves as the invariable measure of value even in the general economy, which is derived from the joint application of Theorems 1 and 2 and Corollary 1.

**Theorem 3:** *For any economy  $P$ , the standard commodity  $y^*$  associated with the standard system  $\alpha^* = (-1, -x^*, x^* + y^*)$  serves as the invariable measure of value.*

**Proof:** See the Appendix.

In more detail, firstly, according to Corollary 1, the standard commodity  $y^*$  serves as the invariable measure of value with respect to a price change from  $(p, w)$  to  $(p', w')$  whenever the standard system  $\alpha^*$  is a profit rate maximiser at both equi-

librium prices. Secondly, it follows from a simple calculation with the application of Theorem 2 that any change of equilibrium prices from  $(p, w)$  to  $(p', w')$  involves the income redistribution,  $\Delta\pi p x^* + \Delta w = 0$ , in terms of Definition 2 if and only if the corresponding profit shortfalls are identical,  $\delta = \delta'$ . Therefore, by Definition 2 and Theorem 1, it follows that the standard commodity  $y^*$  serves as the invariable measure of value with respect to even such a general case of price change.

As discussed above, Definition 1 in this paper is a faithful generalisation of Sraffa's (1960) own definition of the standard commodity formulated with a domain of the single product system. The standard commodity given in Definition 1 also satisfies both the watershed and recurrence conditions (Schefold, 1986, 1989). Moreover, Definitions 2 and 3 are also faithful to Sraffa's (1960) own analysis and are the generalisation of Baldone (2006). Therefore, Theorem 3 implies the general possibility theorem of the standard commodity as the invariable measure of value.

## 4 Concluding Remarks

In this paper, the necessary and sufficient condition for the standard commodity à la Sraffa (1960) to serve as the invariable measure of value is identified under a general domain of economies with non-increasing returns to scale technology, where interdependence among changes in income distribution, output levels, and technical

choices is available. Based on the identified condition, the standard commodity is shown to serve as the invariable measure of value even under such general economies. Therefore, this paper significantly generalises the result of Baldone (2006) in which the change in price systems due to change in technique was not concerned.

Another main contribution is to identify the necessary and sufficient condition for the linear relationship of income distribution from applying the standard commodity as the numéraire to such a general domain of economies. According to the identified condition, the linearity is obtained if and only if the standard commodity is a profit rate maximiser regardless of whatever an equilibrium price system is. Most of economies, except economies with single product systems, within such a domain would not satisfy this condition. Ricardo (1952A, p. 194) conjectured, ‘the great questions of Rent, Wages, and Profits must be explained by the proportions in which the whole produce is divided between landlords, capitalists, and labourers, and which are not essentially connected with the doctrine of value,’ which suggests that the rate of profit can be obtained without knowing the structure of prices. However, our second result demonstrates that Ricardo’s conjecture is not generally valid.

This sharp contrast between the performances of the two basic functions of the standard commodity is quite interesting, which cannot appear in the standard single product system, but is a distinctive feature in the more general economies.

# Appendix: Mathematical formulation of Production possibility set and Proofs of Theorems

In the Appendix, we rigorously formulate our model presenting in Section 3.

Let  $\mathbb{R}_+$  be the set of all non-negative real numbers, and  $\mathbb{R}_{++}$  be the set of all positive numbers. Let  $\mathbb{R}_+^n$  (resp.  $\mathbb{R}_{++}^n$ ) be the  $n$ -fold Cartesian product of  $\mathbb{R}_+$  (resp.  $\mathbb{R}_{++}$ ). For any  $x, y \in \mathbb{R}_+^n$ , we write  $x \geq y$  to mean  $[x_i \geq y_i \text{ for all } i = 1, \dots, n]$ ,  $x \geq y$  to mean  $[x_i \geq y_i \text{ for all } i = 1, \dots, n \text{ and } x \neq y]$ , and  $x > y$  to mean  $[x_i > y_i \text{ for all } i = 1, \dots, n]$ .

Let there be  $n$  commodities which are reproducible. Let  $\mathbf{0}$  denote the null vector. Production technology is represented by a production set  $P$  which has elements of the form  $\alpha = (-\alpha_l, -\underline{\alpha}, \bar{\alpha})$ , where  $\alpha_l \in \mathbb{R}_+$  is the *effective* labour input of the process;  $\underline{\alpha} \in \mathbb{R}_+^n$  are the inputs of the produced goods used in the process; and  $\bar{\alpha} \in \mathbb{R}_+^n$  are the outputs of the  $n$  goods. Thus, elements of  $P$  are vectors in  $\mathbb{R}^{2n+1}$ . The following assumptions are imposed on a production set  $P$ .

**Assumption 0 (A0).**  $P$  is closed and convex in  $\mathbb{R}^{2n+1}$  and  $\mathbf{0} \in P$ .

**Assumption 1 (A1).** For all  $\alpha \in P$ , if  $\bar{\alpha} \geq \mathbf{0}$ , then  $\alpha_l > 0$  and  $\underline{\alpha} \geq \mathbf{0}$ .

**Assumption 2 (A2).** For all  $c \in \mathbb{R}_+^n$ , there is a  $\alpha \in P$  such that  $\hat{\alpha} \equiv \bar{\alpha} - \underline{\alpha} \geq c$ .

For each production possibility set  $P$ , let us denote  $\partial P \equiv \{\alpha \in P \mid \nexists \alpha' \in P : \alpha' > \alpha\}$ ,

which is the boundary of the production set  $P$ .

The model of production sets with **A0~A2** covers a broad class of production technologies. For instance, it contains the class of von Neumann production models as a subclass. It also contains a convex combination of multiple Leontief production models, which is an example of economy with the possibility of technical choices and without joint production.

Under **A0~A2**, Theorem 1 presenting in Section 3 can be rigourously expressed and proven.

**Theorem 1:** *For any economy  $P$  satisfying **A0~A2**, let us take any equilibrium prices  $(p, w)$  and  $(p', w')$ . Then, the standard commodity  $y^*$  associated with  $\alpha^* = (-1, -x^*, x^* + y^*) \in \partial P$  serves as the invariable measure of value with respect to change from  $(p, w)$  to  $(p', w')$ , if and only if there exist non-negative numbers  $\delta, \delta' \geq 0$  such that  $py^* = \pi px^* + w - \delta$ ,  $p'y^* = \pi' px^* + w' - \delta'$ , and  $\delta = \delta'$  hold.*

**Proof:** By Definition 1,  $y^* = \Pi x^*$  holds for some  $\Pi > 0$ . Since  $\alpha_i^* = 1$ ,  $py^* \leq \pi px^* + w$  and  $p'y^* \leq \pi' p'x^* + w'$  generally hold. Therefore, there are non-negative numbers  $\delta, \delta' \geq 0$  such that  $py^* = \pi px^* + w - \delta$  and  $p'y^* = \pi' p'x^* + w' - \delta'$  hold.

Then,

$$\Delta p(x^* + y^*) = (1 + \pi + \Delta\pi) \Delta px^* + (\Delta\pi px^* + \Delta w) - (\delta' - \delta).$$

Since  $y^* = \Pi x^*$ , the above equation can be reduced to

$$\Delta p y^* = \frac{1}{\Pi} (\pi + \Delta \pi) \Delta p y^* + (\Delta \pi p x^* + \Delta w) - (\delta' - \delta).$$

Thus, we have

$$\Delta p y^* = \left[ 1 - \frac{1}{\Pi} (\pi + \Delta \pi) \right]^{-1} ((\Delta \pi p x^* + \Delta w) - (\delta' - \delta)).$$

Suppose that  $y^*$  serves as the invariable measure of value with respect to a change from  $(p, w)$  to  $(p', w')$ . Then, by Definition 2,  $\Delta \pi p x^* + \Delta w = 0$  implies  $\Delta p y^* = 0$ . Then, by the above second equation,  $\delta' - \delta = 0$  must hold.

Conversely, let there be  $\delta, \delta' \in \mathbb{R}_+$  such that  $p y^* = \pi p x^* + w - \delta$ ,  $p' y^* = \pi' p' x^* + w' - \delta'$ , and  $\delta = \delta'$  hold. Then, the above last equation implies that  $\Delta p y^* = 0$  follows from  $\Delta \pi p x^* + \Delta w = 0$ . Thus, by Definition 2,  $y^*$  serves as the invariable measure of value with respect to a change from  $(p, w)$  to  $(p', w')$ . ■

Letting  $P(p, w) \equiv \left\{ \alpha \in P \mid \alpha = \arg \max_{\alpha'} \frac{p\bar{\alpha} - p\alpha' - w\alpha_l}{p\alpha'} \right\}$ , Corollary 1 in Section 3 can be rigorously expressed and proven.

**Corollary 1.** *Under any economy  $P$  satisfying  $\mathbf{A0} \sim \mathbf{A2}$ , take any equilibrium prices  $(p, w)$  and  $(p', w')$ , such that  $\alpha^* \in P(p, w) \cap P(p', w')$  holds. Then, the standard*



commodity  $y^*$  serves as the invariable measure of value with respect to a change from  $(p, w)$  to  $(p', w')$ .

**Proof.** Note that  $\alpha^* = (-1, -x^*, x^* + y^*) \in P(p, w) \cap P(p', w')$  implies  $py^* = \pi px^* + w - \delta$  and  $p'y^* = \pi'p'x^* + w' - \delta'$  hold for  $\delta = 0 = \delta'$ . Then, by Theorem 1, the desired result immediately follows. ■

Let us define the set of price vectors measured by the standard commodity as  $\Delta^{y^*} \equiv \{(p, w) \in \mathbb{R}_+^{n+1} \mid py^* = 1\}$ . Then, Theorem 2 can be rigorously expressed and proven under **A0~A2**.

**Theorem 2:** *Given  $P$  with **A0~A2**, the linear functional relation of income distribution,  $\pi' = \Pi(1 - w')$ , holds for any equilibrium price vector  $(p', w') \in \Delta^{y^*}$  associated with the maximal profit rate  $\pi'$  if and only if  $p'y^* = \pi'p'x^* + w'$  holds for any equilibrium price vector  $(p', w')$  associated with  $\pi'$ .*

**Proof.** By definition of an equilibrium price  $(p', w') \in \Delta^{y^*}$  associated with the maximal profit rate  $\pi'$ , it is generally true that  $p'(x^* + y^*) \leq (1 + \pi')p'x^* + w'$ . If there exists an equilibrium price  $(p', w')$  such that  $p'(x^* + y^*) < (1 + \pi')p'x^* + w'$ , then  $\pi' = \Pi(1 - w')$  does not hold. Indeed, from  $x^* + y^* = (1 + \Pi)x^*$ , it follows that  $p'y^* = \Pi p'x^* < \pi'p'x^* + w'$ . Since  $p'y^* = 1$ , then  $\pi' > \Pi(1 - w')$  holds. Conversely, let  $p'(x^* + y^*) = (1 + \pi')p'x^* + w'$  hold for any equilibrium price  $(p', w')$  associated

with the maximal profit rate  $\pi'$ . Then, since  $p'y^* = 1$ ,  $p'y^* = 1 = \pi'p'x^* + w'$ . Thus, since  $\Pi p'x^* = 1$ ,  $\Pi - \frac{w'}{p'x^*} = \pi'$  holds, which is equivalent to  $\pi' = \Pi(1 - w')$ . ■

Given Theorem 1 and Theorem 2, it can be shown that the standard commodity can serve as the invariable measure of value in any economy  $P$  with **A0**~**A2**.

**Theorem 3:** *For any economy  $P$  satisfying **A0**~**A2**, the standard commodity  $y^*$  associated with  $\alpha^* = (-1, -x^*, x^* + y^*) \in \partial P$  serves as the invariable measure of value.*

**Proof.** Since  $\alpha^* = (-1, -x^*, x^* + y^*) \in \partial P$ , there exists an equilibrium price vector  $(p^*, w^*) \in \Delta^{y^*}$  associated with the maximal profit rate  $\pi^* \geq 0$  such that  $\alpha^* \in P(p^*, w^*)$ , which is guaranteed by **A0**. Note that Theorem 2 suggest that  $\pi^* = \Pi(1 - w^*)$  holds, and for any equilibrium price vector  $(p, w) \in \Delta^{y^*}$  associated with the maximal profit rate  $\pi \geq 0$ ,  $\pi = \Pi(1 - w)$  holds if and only if  $\alpha^* \in P(p, w)$ .

In contrast, for any equilibrium price vector  $(p', w') \in \Delta^{y^*}$  associated with the maximal profit rate  $\pi' \geq 0$ , whenever  $\alpha^* \notin P(p, w)$  and  $p'y^* = \pi'p'x^* + w' - \delta'$  for some  $\delta' > 0$ , then  $\pi' = \Pi(1 - w') + \Pi\delta'$  holds. This is because, since  $(p', w') \in \Delta^{y^*}$ ,  $p'y^* = 1$  and so  $p'x^* = \frac{1}{\Pi}$  holds, which implies  $\pi' = \Pi(1 - w') + \Pi\delta'$ .

Let us take any equilibrium prices  $(p, w), (p', w') \in \Delta^{y^*}$ . If  $\alpha^* \in P(p, w) \cap P(p', w')$ , then  $y^*$  serves as the invariable measure of value with respect to a change

from  $(p, w)$  to  $(p', w')$ , as Corollary 1 shows.

Suppose that  $\alpha^* \in P(p, w) \setminus P(p', w')$ . Then, by the above argument,  $\pi = \Pi(1 - w)$  and  $\pi' = \Pi(1 - w') + \Pi\delta'$  for some  $\delta' > 0$ . Then, since  $\Delta\pi \equiv \pi' - \pi$ , it follows that:

$$\begin{aligned}\Delta\pi &= \pi' - \pi = \Pi(1 - w') - \Pi(1 - w) + \Pi\delta' \\ &= -\Pi(w' - w) + \Pi\delta' \\ &= -\Pi\Delta w + \Pi\delta',\end{aligned}$$

which implies that  $\Delta\pi\frac{1}{\Pi} + \Delta w = \Pi\delta'$ . Since  $px^* = \frac{1}{\Pi}$  by  $(p, w) \in \Delta^{y^*}$ , we have  $\Delta\pi px^* + \Delta w = \Pi\delta' > 0$ . Thus, the change of prices from  $(p, w)$  to  $(p', w')$  does not involve a redistribution between profit and wage, since  $\Delta\pi px^* + \Delta w > 0$ . Therefore, by Definition 2,  $y^*$  trivially serves as the invariable measure of value with respect to a change from  $(p, w)$  to  $(p', w')$ .

Suppose that  $\alpha^* \notin P(p, w) \cup P(p', w')$ . Then, by the above argument,  $\pi = \Pi(1 - w) + \Pi\delta$  and  $\pi' = \Pi(1 - w') + \Pi\delta'$  for some  $\delta, \delta' > 0$ . Then,

$$\begin{aligned}\Delta\pi &= \pi' - \pi = \Pi(1 - w') - \Pi(1 - w) + \Pi(\delta' - \delta) \\ &= -\Pi\Delta w + \Pi(\delta' - \delta),\end{aligned}$$

which implies that  $\Delta\pi\frac{1}{\Pi} + \Delta w = \Pi(\delta' - \delta)$ . Therefore, since  $px^* = \frac{1}{\Pi}$  by  $(p, w) \in \Delta^{y^*}$ , we have  $\Delta\pi px^* + \Delta w = \Pi(\delta' - \delta)$ . If  $\delta' - \delta = 0$ , then  $\Delta\pi px^* + \Delta w = 0$ , so that the change of prices from  $(p, w)$  to  $(p', w')$  involves a redistribution between profit and wage. Moreover, since  $\delta' = \delta$ , it follows from Theorem 1 that  $y^*$  serves as the invariable measure of value with respect to a change from  $(p, w)$  to  $(p', w')$ . If  $\delta' - \delta \neq 0$ , then  $\Delta\pi px^* + \Delta w \neq 0$ , so that the change of prices from  $(p, w)$  to  $(p', w')$  does not involve a redistribution between profit and wage. Therefore, by Definition 2,  $y^*$  trivially serves as the invariable measure of value with respect to a change from  $(p, w)$  to  $(p', w')$ .

In summary, for any equilibrium prices  $(p, w), (p', w') \in \Delta^{y^*}$ ,  $y^*$  serves as the invariable measure of value with respect to a change from  $(p, w)$  to  $(p', w')$ . Thus, by Definition 3,  $y^*$  serves as the invariable measure of value. ■

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## Notes

<sup>1</sup>Ricardo's concern about an invariable measure of value appeared as early as in his contributions to the 'bullionist' controversy. He had already pointed out the need for an invariable measure of value, which would enable an intertemporal comparison of values, and argued that such a measure did not exist in reality; however, money could be regarded as an invariable measure of value at least as the first approximation (Ricardo, 1951C, p. 65). However, his arguments at this stage were not based on the theory of value. See Kurz and Salvadori (1993) concerning the conceptual transition of Ricardo's invariable measure of value.

<sup>2</sup>Ricardo (1952C, p. 358) said, 'As soon as we are in possession of the knowledge of the circumstances which determine the value of commodities, we are enabled to say what is necessary to give us an invariable measure of value.' See also Sraffa (1951) for the transition of Ricardo's theory of value in detail.

<sup>3</sup>See Porta (1992) concerning the debates between Ricardo and Malthus.

<sup>4</sup>Pasinetti's (1981, 1993) *dynamic standard commodity* is one of the examples that pay attention to the former.

<sup>5</sup>One of the exceptions is Yagi (2012). Following Pasinetti (1981, 1993), he constructed a model in order to compare two different economic systems (called Period 1 economy and Period 2 economy) intertemporally. Moreover, he investigated the

invariable measure of value and linearity of income distribution

<sup>6</sup>Schefold (1986, 1989) emphasised that the watershed condition and the recurrence condition must hold in order for the standard commodity to serve as the invariable measure of value. The conditions imply that not only the industry producing the standard commodity, but also all the industries that produce the means of production necessary to produce the standard commodity, must adopt the ‘watershed’ proportion of means of production to labour (Sraffa, 1960, p. 16). In fact, only the watershed condition is sufficient for the existence of the standard commodity, insofar as the proof is based on the Perron-Frobenius theorem.

<sup>7</sup>In Flaschel (1986, p. 597), it is explicitly written as ‘the problem of invariance cannot be described unless a measure of value has already been assumed. This fact is implicitly taken into account by Sraffa ([1960], Ch. 3) in his assumption  $\mathbf{p}(\mathbf{e} - \mathbf{A}\mathbf{e}) \equiv 1$ . . . . the search for (conditions for) a “measure of value” relative to an already given measure of value! But what can be expected from the solution of such a problem?’

<sup>8</sup>See also Baldone (2006) and Bellino (2004) concerning Flaschel (1986).

<sup>9</sup>Samuelson (2008) also blundered into the same misinterpretation as Burmeister. Moreover, Samuelson (1990) mistakenly related the standard commodity to the amelioration of the fault of the labour theory of value.

<sup>10</sup>The rigorous formulation of production sets is given in the Appendix.

<sup>11</sup>The existence of the standard commodity is shown in Theorem A1 of the addendum. The unique existence of the standard commodity depends on free disposal of the production set  $P$ . If the production set  $P$  is more suitably specified, the unique existence of the standard commodity can be shown without the free disposal assumption by applying the non-linear Frobenius theorem (Fujimoto, 1979, 1980).

<sup>12</sup>Note that this concept is consistent with the ‘price’ in Sraffa (1960). Although it is true that Sraffa avoided using the term ‘equilibrium’, according to Roncaglia (2009, pp. 121–2), the equality of the rate of profit in Sraffa’s system implies that the mobility of capital between sectors, in the search of maximum profitability, would ultimately bring out a tendency of the rates of profit to converge towards this benchmark position. Moreover, the uniform rate of profit in Sraffa’s system does not require the equality of demand and supply, in contrast to the concept of ‘equilibrium’ used by ‘marginalists’. He also asserted that it is only in this sense that one can speak of ‘equilibrium’ price within the Sraffa’s system. Our concept of ‘equilibrium price’ also requires the only achievement of the maximum rate of profit in all activated ‘sectors’ under a production process  $\alpha$ .

<sup>13</sup>Such a situation does not have to be concerned when we assume only the single product system as in Baldone (2006).

## 5 Addendum

For each production possibility set  $P$ , let us denote  $SP \equiv \{\alpha \in P \mid \nexists \alpha' \in P : \alpha' \geq \alpha\}$ , which is the efficiency frontier of the production set  $P$ . Moreover, given  $k > 0$ , let  $P(\alpha_l = k) \equiv \{\alpha \in P \mid \alpha_l = k\}$  and

$$\partial P(\alpha_l = k) \equiv \{\alpha \in P(\alpha_l = k) \mid \nexists \alpha' \in P(\alpha_l = k) : (-\underline{\alpha}', \bar{\alpha}') > (-\underline{\alpha}, \bar{\alpha})\}.$$

### 5.1 Examples of production models satisfying **A0~A2**

The following two examples are typical types of production models in the class of production sets satisfying **A0~A2** presented in Appendix:

**Example 1:** Given a von Neumann technology  $(\mathbf{A}, \mathbf{B}, \mathbf{L})$ , where  $\mathbf{A}$  and  $\mathbf{B}$  are  $n \times m$  non-negative matrices and  $\mathbf{L}$  is a  $1 \times m$  positive vector. Suppose that for each sector  $j = 1, \dots, m$ , there exists at least one commodity  $i = 1, \dots, n$  such that  $a_{ij} > 0$ . we can define a production set  $P_{(\mathbf{A}, \mathbf{B}, \mathbf{L})}$  as

$$P_{(\mathbf{A}, \mathbf{B}, \mathbf{L})} \equiv \{\alpha \in \mathbb{R}_- \times \mathbb{R}_-^n \times \mathbb{R}_+^n \mid \exists x \in \mathbb{R}_+^m : \alpha \leq (-\mathbf{L}x, -\mathbf{A}x, \mathbf{B}x)\}.$$

Note that for each  $\alpha \in SP_{(\mathbf{A}, \mathbf{B}, \mathbf{L})}$ , there exists  $x \in \mathbb{R}_+^m$  such that  $\alpha = (-\mathbf{L}x, -\mathbf{A}x, \mathbf{B}x)$ .

The set  $P_{(\mathbf{A}, \mathbf{B}, \mathbf{L})}$  satisfies all of **A0~A2**. As a special case of the von Neumann tech-

nology, we can consider the case that  $m = n$  and  $\mathbf{B} = \mathbf{I}$ , which implies a Leontief technology  $(\mathbf{A}, \mathbf{I}, \mathbf{L})$ . Then, we can define  $P_{(\mathbf{A}, \mathbf{L})} \equiv P_{(\mathbf{A}, \mathbf{I}, \mathbf{L})}$  as in the definition of  $P_{(\mathbf{A}, \mathbf{B}, \mathbf{L})}$ . ■

**Example 2:** Let us consider a class of Leontief technology  $\{(\mathbf{A}^k, \mathbf{L}^k)\}_{k=1, \dots, m}$ , where for each  $k = 1, \dots, m$ ,  $\mathbf{A}^k$  is a  $n \times n$  non-negative, productive, and indecomposable matrix and  $\mathbf{L}^k$  is a  $1 \times n$  positive vector, such that for any  $k, k' = 1, \dots, m$ , and for any non-negative  $n \times 1$  vectors  $x^k$  and  $x^{k'}$ ,  $\mathbf{A}^k x^k = \mathbf{A}^{k'} x^{k'}$  implies  $x^k = x^{k'}$  and  $\mathbf{L}^k x^k = \mathbf{L}^{k'} x^{k'}$ . Given this, we can define a production set  $P_{(\mathbf{A}^k, \mathbf{L}^k)_{k=1, \dots, m}}$  as

$$P_{(\mathbf{A}^k, \mathbf{L}^k)_{k=1, \dots, m}} \equiv \left\{ \alpha \in \mathbb{R}_- \times \mathbb{R}_-^n \times \mathbb{R}_+^n \mid \exists S \equiv \{k^1, \dots, k^S\} \subseteq \{1, \dots, m\}, \right. \\ \left. \exists \{x^{k^s}\}_{k^s \in S} \subseteq \mathbb{R}_+^n : \alpha \leq \left( - \sum_{k^s \in S} \mathbf{L}^{k^s} x^{k^s}, - \sum_{k^s \in S} \mathbf{A}^{k^s} x^{k^s}, \sum_{k^s \in S} x^{k^s} \right) \right\}.$$

By the supposition of  $\{(\mathbf{A}^k, \mathbf{L}^k)\}_{k=1, \dots, m}$ , the production set  $P_{(\mathbf{A}^k, \mathbf{L}^k)_{k=1, \dots, m}}$  satisfies **A0~A2**. ■

## 5.2 The existence of the standard commodity

To provide a general existence of the standard commodity, let us introduce the following additional assumptions on the production set:

**Assumption 3 (A3).** For all  $\alpha \in P$ , and for all  $(-\alpha'_l, -\underline{\alpha}', \bar{\alpha}') \in \mathbb{R}_- \times \mathbb{R}_-^n \times \mathbb{R}_+^n$ ,

if  $(-\alpha'_l, -\underline{\alpha}', \bar{\alpha}') \leq \alpha$ , then  $(-\alpha'_l, -\underline{\alpha}', \bar{\alpha}') \in P$ .

**Assumption 4 (A4).** There exists  $r \in \mathbb{R}_{++}$  with  $r \leq 1$  such that for all  $\alpha \in P$ , and for any  $k > 0$ ,  $(-k\alpha_l, -k\underline{\alpha}, k^r\bar{\alpha}) \in P$ .

The model of production sets with **A0~A4** still covers a broad class of production technologies. Indeed, it still contains the class of von Neumann production models and the class of Leontief production models with the possibility of technical choices such as **Example 1** and **Example 2** in Appendix.

Given the above setup of the model and the definition of the standard commodity presenting in Section 3, the general existence of the standard commodity is proven.

**Theorem A1:** *Under **A0~A4**, there uniquely exists the standard commodity  $y^* \in \mathbb{R}_{++}^n$  associated with  $\alpha^{***} \in \partial P(\alpha_l = 1)$  and  $\hat{\alpha}^{***} = y^*$ .*

**Proof:** Given  $P(\alpha_l = 1)$  which is convex, let  $P_{\alpha_l=1}$  be the minimal closed convex cone containing  $P(\alpha_l = 1)$ . By definition,  $P_{\alpha_l=1}$  is a closed convex cone with  $P_{\alpha_l=1}(\alpha_l = 1) = P(\alpha_l = 1)$ . If  $r = 1$ ,  $P_{\alpha_l=1} = P$ . Given  $P_{\alpha_l=1}$ , let  $\bar{P}_{\alpha_l=1} \equiv \left\{ \alpha \in P_{\alpha_l=1} \mid \sum_{i=1, \dots, n} \bar{\alpha}_i = 1 \right\}$ . Let  $F : P_{\alpha_l=1} \rightarrow \mathbb{R}_+$  be such that for each  $\alpha \in P_{\alpha_l=1}$ ,

$F(\alpha) = \min_{i=1, \dots, n} \frac{\bar{\alpha}_i}{\underline{\alpha}_i}$  where

$$\frac{\bar{\alpha}_i}{\underline{\alpha}_i} \equiv \begin{cases} 0 & \text{if } \bar{\alpha}_i = 0 \\ +\infty & \text{if } \underline{\alpha}_i = 0 \text{ and } \bar{\alpha}_i > 0. \end{cases}$$

This mapping is continuous and well-defined by **A1**. Note that, by **A2** and **A4**, there exists  $\alpha' \in \partial P(\alpha_l = 1)$  such that  $\hat{\alpha}' > \mathbf{0}$ . Hence, for  $\frac{\alpha'}{\sum_{i=1, \dots, n} \bar{\alpha}'_i} \in \partial \bar{P}_{\alpha_l=1}$ ,  $F\left(\frac{\alpha'}{\sum_{i=1, \dots, n} \bar{\alpha}'_i}\right) > 0$ . This implies  $\sup_{\alpha \in \bar{P}_{\alpha_l=1}} F(\alpha) > 0$ . Suppose that  $\sup_{\alpha \in \bar{P}_{\alpha_l=1}} F(\alpha) = +\infty$ . Then, there exists a sequence  $\{\alpha^k\} \subseteq \bar{P}_{\alpha_l=1}$  such that  $\alpha^k \rightarrow \alpha^*$  with  $\lim_{k \rightarrow +\infty} F(\alpha^k) = F(\alpha^*) = \sup_{\alpha \in \bar{P}_{\alpha_l=1}} F(\alpha)$ . By definition of  $F$ ,  $F(\alpha^*) = +\infty$  implies that  $\alpha^* = (-l, \mathbf{0}, \bar{\alpha}^*)$  for some  $l \geq 0$  and some  $\bar{\alpha}^* > \mathbf{0}$ . Since  $\bar{P}_{\alpha_l=1}$  is closed,  $\alpha^* \in \bar{P}_{\alpha_l=1}$ . By construction,  $\bar{P}_{\alpha_l=1}$  satisfies **A1**, which is a contradiction of  $\alpha^* \in \bar{P}_{\alpha_l=1}$ . Thus,  $\sup_{\alpha \in \bar{P}_{\alpha_l=1}} F(\alpha) < +\infty$ . Then,  $\sup_{\alpha \in \bar{P}_{\alpha_l=1}} F(\alpha) = \max_{\alpha \in \bar{P}_{\alpha_l=1}} F(\alpha)$ . Let  $\alpha^* \in \arg \max_{\alpha \in \bar{P}_{\alpha_l=1}} F(\alpha)$ . Then, by the cone property,  $\frac{\alpha^*}{\alpha_l^*} \in P(\alpha_l = 1)$  and  $\frac{\alpha^*}{\alpha_l^*} \in \arg \max_{\alpha \in P(\alpha_l=1)} F(\alpha)$ . Hence, without loss of generality, let  $\alpha^* \in \arg \max_{\alpha \in P(\alpha_l=1)} F(\alpha)$ . Then,  $\alpha^* \in \partial P(\alpha_l = 1)$ . Since there exists  $\alpha' \in \partial P(\alpha_l = 1)$  such that  $F(\alpha') > 0$ ,  $\max_{\alpha \in P(\alpha_l=1)} F(\alpha) > 0$  holds, which implies that  $\bar{\alpha}^* > \mathbf{0}$ .

Define  $V \equiv \{\bar{\alpha} - F(\alpha^*) \underline{\alpha} \mid (-1, -\underline{\alpha}, \bar{\alpha}) \in P(\alpha_l = 1)\}$ . Then,  $V$  is a closed convex set with  $V \cap \mathbb{R}_{++}^n = \emptyset$ . Then, there exists  $p^* \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$  such that  $p^* [\bar{\alpha} - F(\alpha^*) \underline{\alpha}] \leq 0$  for all  $\alpha \in P(\alpha_l = 1)$  and  $p^* z > 0$  for all  $z \in \mathbb{R}_{++}^n$ . This implies that if there ex-



ists  $i \in \{1, \dots, n\}$  with  $\frac{\bar{\alpha}_i^*}{\underline{\alpha}_i^*} > F(\alpha^*)$ , then  $p_i^* = 0$ . By  $p^* \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ , there exists  $i \in \{1, \dots, n\}$  with  $\frac{\bar{\alpha}_i^*}{\underline{\alpha}_i^*} = F(\alpha^*)$  and  $p_i^* > 0$ . Thus,  $p^* [\bar{\alpha}^* - F(\alpha^*) \underline{\alpha}^*] = 0$ . Hence,  $p^*$  is a supporting vector of  $\alpha^* \in \partial P(\alpha_l = 1)$ . Let  $\alpha^{**} \in P(\alpha_l = 1)$  be such that for each  $i \in \{1, \dots, n\}$  with  $\frac{\bar{\alpha}_i^*}{\underline{\alpha}_i^*} > F(\alpha^*)$ ,  $(\underline{\alpha}_i^{**}, \bar{\alpha}_i^{**}) \in \mathbb{R}_{++}^2$  with  $\frac{\bar{\alpha}_i^{**}}{\underline{\alpha}_i^{**}} \equiv F(\alpha^*)$ . (Note that such a construction is possible by **A3**.) Furthermore, for each  $i \in \{1, \dots, n\}$  with  $\frac{\bar{\alpha}_i^*}{\underline{\alpha}_i^*} = F(\alpha^*)$ ,  $(\underline{\alpha}_i^{**}, \bar{\alpha}_i^{**}) \equiv (\underline{\alpha}_i^*, \bar{\alpha}_i^*)$ . Then, by construction,  $p^* [\bar{\alpha}^{**} - F(\alpha^*) \underline{\alpha}^{**}] = 0$ , which implies that  $\alpha^{**} \in \partial P(\alpha_l = 1)$ . Note that  $\underline{\alpha}^{**} > \mathbf{0}$  and  $\bar{\alpha}^{**} = F(\alpha^*) \underline{\alpha}^{**}$ .

Denote the set of such production processes as  $\alpha^{**}$  by  $P(F)$ . Then, for any  $\alpha^{**} \in P(F)$ ,  $\alpha^{**} \in \partial P(\alpha_l = 1)$  and  $F(\alpha^{**}) \geq F(\alpha')$  hold for all  $\alpha' \in P(\alpha_l = 1)$ . Since  $P(F)$  is compact, there exists  $\alpha^{***} \in P(F)$  such that for any  $\alpha^{**} \in P(F)$ ,  $\bar{\alpha}^{***} - \underline{\alpha}^{***} \geq \bar{\alpha}^{**} - \underline{\alpha}^{**}$ .

Let  $y^* \equiv \bar{\alpha}^{***} - \underline{\alpha}^{***}$ . Remember that there exists  $\alpha' \in \partial P(\alpha_l = 1)$  such that  $\hat{\alpha}' > \mathbf{0}$  and  $F(\alpha') > 0$ , which implies  $F(\alpha^*) \geq F(\alpha') > 1$ . Therefore,  $y^* > \mathbf{0}$ . Then, there exists a positive number  $\Pi > 0$  such that  $\Pi x^* = y^*$  for  $x^* \equiv \underline{\alpha}^{***} > \mathbf{0}$ . By Definition 1,  $y^* > \mathbf{0}$  is a standard commodity of the economy  $P$ . Note that  $1 + \Pi = \max_{\alpha \in P(\alpha_l = 1)} F(\alpha) = F(\alpha^{**})$  and  $y^* \geq \bar{\alpha}^{**} - \underline{\alpha}^{**}$  for any  $\alpha^{**} \in P(F)$ . This guarantees the uniqueness of the standard commodity  $y^*$  for the economy  $P$ . ■