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Household Debt and Housing Bubble: A Minskian Approach to Boom-Bust Cycles

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Soon Ryoo*

Abstract

This paper examines macroeconomic dynamics of household debt and housing prices. Drawing on Minsky's insights into financial instability and cycles, our framework combines household debt dynamics with behavioral asset price dynamics in a Keynesian macro model. We show that endogenous boom-bust cycles can emerge through the interaction between household debt and housing price dynamics. The resulting long waves are combined with a Kaldorian model of short-run business cycles.

keywords household debt, asset bubble, limit cycle, financial instability hypothesis

JEL classification E12, E32, E44

1 Introduction

Minsky's financial instability hypothesis (Minsky, 1986, 1982) has received a renewed interest since the 2008 financial crisis. The crisis appears to vindicate the hypothesis that a long period of prosperity sows a seed of its destruction by encouraging riskier financial practices. A body of the literature inspired by Minsky has tried to formalize various aspects of Minsky's financial instability hypothesis (Taylor and O'Connell, 1985, Foley, 1987, Skott, 1994, Fazzari et al., 2008, Ryoo, 2010, 2013a, Chiarella and Di Guilmi, 2011) but paid little attention to the interaction between household debt

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and housing prices, which was at the center of the recent crisis. This may not be surprising as Minsky's main analysis focuses on the interaction between the firms' liability structure and the prices of capital assets.[1]

This paper draws on Minsky's insights and examines the mechanism of instability and cycles that emerge from the interaction between household debt accumulation and housing prices. There is a strand of the literature that has investigated the implications of household debt in post-Keynesian models (Palley, 1994, 1996, 2010, Dutt, 2006, Charpe et al., 2009, 2012, Isaac and Kim, 2013). These studies stress the interaction among household debt, aggregate demand and income distribution. Our model retains this Keynesian emphasis on aggregate demand in the study of household debt but pays close attention to the implications of asset price dynamics for debt accumulation which has been by and large neglected in those models.[2]

The approach to modeling debt and asset price dynamics in this paper is close to those in Ryoo (2010). The focus of the present paper, however, is not on the interaction between corporate debt and stock prices, but on that between household debt and housing prices.

Some key elements of the model are in order. First, the macro model we adopt here is an extension of Kaldor's Keynesian model of growth and distribution (Kaldor, 1956, 1966) where endogenous adjustments in profit margins play a key role in bringing aggregate saving in line with investment. Assuming that the saving rate out of profits is higher than that of wages, fluctuations in aggregate demand have distributional implications in the Kaldorian framework: any rise in aggregate demand is reflected in an increase in the profit share.[3] The present paper extends the Kaldorian framework to incorporate the borrowing-lending relation within the household sector. In our model, borrowers are credit-constrained and the amount of their borrowing is determined by bankers' lending practices which in turn depend on borrowers' income flow as well as their balance sheet positions (net worth). As the state of household indebtedness and housing wealth affects both borrowers' income flow and net worth, it influences the trajectory of borrowing and therefore borrowers' spending. This has an implication for aggregate demand and thus income distribution between borrowers and lenders, which in turn feed back into debt accumulation and asset price dynamics.

Second, the Keynesian framework is combined with a behavioral model of asset price dynamics. The specification of asset price dynamics in our model shares key features with some behavioral literature (Beja and Goldman, 1980, Chiarella, 1992,

[1]Minsky often argues that the household sector plays only a secondary role in the mechanism of instability because 'Household debt-financing of consumption is almost always hedge-financing.'(Minsky, 1982)[p.32]

[2]The neglect of asset prices in the post-Keynesian models is somewhat curious. Many of those models are motivated by Minsky's theory of instability but Minsky's emphasis on asset prices and their role in financial instability has been largely left out of the scene.

[3]The idea dates back to Keynes (1930) and Robertson (1933). Hahn (1951) applies the same mechanism to a short-run macro model.

Sethi, 1996, Brock and Hommes, 1998). The studies in this tradition generally stress out-of-equilibrium dynamics in asset markets. The approach does not presume that agents instantaneously adjust their portfolios to optimal portfolios. The discrepancy between actual and optimal portfolios induces agents to adjust their positions which affect the actual trajectories of asset prices. The evolution of asset prices in turn feeds back to agents' portfolio decisions but the effect is filtered through their expectations and beliefs. Under uncertain environments, agents follow several rules of thumb or various learning processes in predicting the future trajectories of prices. The adaptive and evolutionary nature of learning and expectations formation is not dismissed in this approach just because it violates the rational expectations hypothesis.

Our formalization retains a key Minskian feature, the centrality of the interaction between debt and asset prices dynamics. The source of instability and cycles in our model, however, is rooted in the household sector unlike the Minsky's own benchmark framework. Instability and cycles emerge from the interaction between household debt and housing price dynamics under certain conditions. The resulting cycles are the long waves around which short-run business cycles fluctuate. Thus our analysis formalizes, in a particular framework, Minsky's general idea of long waves, which is largely based on narrative accounts.[4] In addition, income distribution is endogenously determined and interacts with debt and asset prices dynamics in our framework, whereas the role of income distribution in Minsky's own account of financial instability appears to be less clear.

The paper is structured as follows. Section 2 sets out the model. Section 3 analyzes the properties of debt and asset price dynamics and examines the conditions for endogenous financial cycles. Section 4 briefly discusses the empirical relevance of our analytic results. Section 5 studies the effects of financial cycles on the real sector by combining our model of long-run financial cycles with a Kaldorian model of business cycle. Section 6 relaxes some restrictive assumptions in the baseline model and examines some extensions. Section 7 offers some concluding remarks.

[4]Minsky argues 'The more severe depressions of history occur after a period of good economic performance, with only minor cycles disturbing a generally expanding economy'(Minsky, 1995)[p.85] and 'the stable mechanism which has generated the long swings centers around the cumulative changes in financial variables that take place over the long-swing expansions and contractions.'(Minsky, 1964)[p.324] Palley (2011) stresses the importance of Minsky's idea of long cycles. Ryoo (2010, 2013a,b,c) advocates the long-wave interpretation of Minsky's financial instability hypothesis and provides formal models of Minskian long waves.

2 Model

2.1 Firms

Firms produce a homogeneous good, using labor and capital, with fixed coefficients technology. Let us denote output, capital and labor as $Y(t)$, $K(t)$ and $L(t)$, respectively. There is no labor hoarding in this economy but production activities do not necessarily occur at full capacity utilization. Let us denote full capacity output at a given level of capital stock as $Y^f(t)$ and the ratio of full-capacity output to capital as $\sigma = Y^f(t)/K(t)$. Since σ is constant by assumption, the ratio of actual output to capital can serve as a measure of capacity utilization.[5]

$$u(t) = Y(t)/K(t) \tag{1}$$

Following Harrod (1939), we assume that firms have a well-defined desired rate of capacity utilization and takes it as exogenous for simplification. In Harrod's theory of short-run business cycles, the movement of utilization plays a central role in generating economic fluctuations. This is so because changes in $u(t)$ affect firms' investment in physical capital, which in turn influences aggregate demand and output. If the actual utilization rate is higher than the desired rate, firms undertake more investment to build up productive capacity. If the actual utilization rate is lower than the desired rate, firms slow down investment to reduce the undesired reserve of excess productive capacity.

The Harrodian approach taken in this paper assumes that although the actual rate of utilization may deviate from the desired rate in the short run due to unfulfilled demand expectations and the sluggish adjustment of capital, the actual rate cannot persistently deviate from the desired rate in the long run because the adjustment of capital stocks is more flexible over a longer time span.[6] If $u(t)$ fluctuates around u^d, the *long-run average* of $u(t)$ will be approximately equal to u^d. Denoting the long-run average rate of utilization as $\overline{u}(t)$, we then have:

$$\overline{u}(t) = u^d \tag{2}$$

The purpose of our analysis in this paper is to study the dynamic properties of

[5]The utilization rate is defined as the ratio of actual output to full capacity output and $\frac{Y(t)}{Y^f(t)} = \frac{Y(t)}{K(t)} \cdot \frac{K(t)}{Y^f(t)} = \frac{u(t)}{\sigma}$. Since σ is constant, $\frac{Y(t)}{Y^f(t)}$ can be proxied by $u(t)$.

[6]The assumptions regarding accumulation behavior have been contentious in the post Keynesian/structuralist literature. Unlike our Harrodian perspective, Kaleckian models assume either that the actual utilization rate does not have to equal the desired rate even in the long-run or that the desired rate itself adjusts to the actual rate in the face of persistent discrepancies between the two rates. The mechanisms and the properties of the models depend critically on the nature of invesment behavior. Palley (2010), Dutt (2006), Isaac and Kim (2013), for instance, study the issues of consumer debt in Kaleckian models.

household debt accumulation and asset price movements over long periods.[7] Thus we abstract from short-run fluctuations in our analysis of long-run evolution of debt and asset prices. In so doing, we focus only on the long-run average rate of utilization, assuming the deviations of actual utilization from the desired rate, by and large, cancel off each other over long periods.

(2) has implications for the long-run average rates of capital accumulation and output growth. Since the utilization rate is kept at the desired rate on average, investment will neither speed up nor slow down and therefore capital accumulation occurs at a constant rate. The approximately constant output-capital ratio also implies that the long-run average rate of output growth equals that of capital. In this paper, we assume a mature economy with no technical progress.[8] In such an economy, the availability of labor constrains output expansion and the growth rate of output cannot persistently deviate from that of labor force. Denoting the growth rate of labor force as n, we then have:

$$\overline{g}(t) = n \tag{3}$$

where $\overline{g}(t)$ is the long-run average value of the growth rate of capital stock.[9]

We will use these long-run approximations, (2) and (3), throughout our analysis of long-run financial dynamics. In other words, the utilization and the accumulation rates in our analysis of long-run financial dynamics refer to the long-run averages $\overline{u}(t)$ and $\overline{g}(t)$, not the actual rates. Two remarks are in order. First, (2) and (3) do not have any connotation that actual trajectories of utilization and accumulation always follow a steady state path. To the contrary. Harrodian investment behavior makes the steady growth path unstable and implies that the path of capital accumulation tends to be exploding. Labor constraints, however, may turn the exploding trajectory into bounded fluctuations. Skott (1989), for instance, provides a mechanism of short-run business cycles where actual $u(t)$ and $g(t)$ fluctuate around u^d and n.[10] The perpetual fluctuations of $u(t)$ and $g(t)$ around u^d and n justify our long run approximations. In section 5, we integrate such a model of short cycles with our model of long-run financial dynamics. Second, these approximations help simplify but are not necessary to our analysis. For instance, (2) and (3) can be relaxed to allow the average values to follow a moving average process. Section 6.1 examines such an extension. The exogeneity of u^d can be also dropped by allowing it to be endogenously determined.[11]

[7] Our focus on long-run financial dynamics is in line with Minsky's own interpretation of his financial instability hypothesis as a theory of long waves.

[8] The analysis can be extended to allow exogenous Harrod-neutral technical progress. The analytic results, however, will be different in a Lewis-type labor-surplus economy or in an economy where technical progress responds to the scarcity of labor supply.

[9] Let $g(t)$ the actual growth rate of capital stock $K(t)$, i.e., $g(t) = \dot{K}(t)/K(t)$. $\overline{g}(t)$ is the average value of $g(t)$ over sufficiently long periods.

[10] Also see Fazzari et al. (2013) for a recent contribution of a model with Harrodian instability.

[11] See Ryoo (2010) for details. The way of endogenizing the desired rate of utilization in Ryoo (2010)

To focus on household debt, the analysis abstracts from the firms' debt. Firms finance real investment and dividends using profits and equity issues. Firms pay out a constant fraction of profits net of depreciation to their shareholders.

$$p(t)I(t) + (1 - s_f)[\Pi(t) - \delta p(t)K(t)] = \Pi(t) + v(t)\dot{N}(t) \tag{4}$$

where $p(t)$ is the output price, $I(t)$ real investment, $1 - s_f$ the dividend pay-out ratio, $\Pi(t)$ gross profits, δ the rate of depreciation of capital stock, $v(t)$ the unit price of shares, and $N(t)$ the amount of new share issues by firms. $N(t)$ is the endogneous variable that ensures (4) holds.

By dividing both sides by $p(t)K(t)$, rearranging the terms gives us

$$\frac{I(t)}{K(t)} - \delta \equiv \overline{g}(t) = s_f[\pi(t)\overline{u}(t) - \delta] + \frac{v(t)\dot{N}(t)}{p(t)K(t)} \tag{5}$$

where $\pi(t)$ is the share of profits in total revenue, i.e. $\pi(t) \equiv \frac{\Pi(t)}{p(t)Y(t)}$. The interpretation of (5) is as follows. The profit share $\pi(t)$ is determined endogenously (see section 2.3). From (2) and (3), $\overline{u}(t) = u^d$ and $\overline{g}(t) = n$. Therefore equity finance must adjust to fill the gap between investment and retained earnings.[12]

2.2 Households

The overall structure of the economy in this paper is a modified version of a two-class economy that is standard in the structuralist/post-Keynesian literature. The household sector is divided into workers' and capitalists' households. Workers use bank loans and wage income to finance their consumption and to pay interest on loans. They do not hold firms' stocks but own housing wealth. Capitalists hold stocks and receive dividend income. In addition to dividend income, they make deposits in banks and earn interest income. In the baseline model, we leave out the complications that naturally arise from capitalists' portfolio decision problems by making a heroic assumption that capitalists do not hold housing and the composition of stocks and deposts in capitalists' portfolios is constant. Section 6.2 drops these assumptions and briefly looks at the implications of capitalists' endogenous changes in capitalists' portfolios and the distribution of housing between classes.

2.2.1 Workers

The workers' consumption and total amount of outstanding debt are denoted as $C^w(t)$ and $M(t)$, respectively. The amount of the workers' housing stock is denoted as H^w.

is different from that in various Kaleckian models.

[12] If retained earnings exceed investment, firms buy back their stocks from shareholders. Increasing stock buybacks have been a characteristic feature of the U.S. economy since the early 1980s (Skott and Ryoo, 2008).

The level of H^w is assumed to be fixed in the baseline analysis, but we relax this assumption in section 6.3.[13] The assumption of constant housing stock[14] is not necessary to our analytic results but simplifies the analysis. Section 6.3 relaxes this assumption and introduces a construction sector.

Workers have total wage income, $W(t)$:

$$W(t) = p(t)Y(t) - \Pi(t)$$

The interest rate on loans is assumed to be equal to the rate on deposits.[15] We also assume that banks' assets consist of loans only and banking entails no cost but its payments on deposits. The economy is cashless and all transactions are made via bank accounts. Under these assumptions, the equality between the rates on loans and deposits implies that the amount of workers' loans, $M(t)$, equals that of capitalists' deposits. The amount of workers' outstanding debt measured in physical capital units is denoted as $m(t)$,[16] i.e.,

$$m(t) = \frac{M(t)}{p(t)K(t)} \tag{6}$$

Workers in the aggregate have the following budget constraint:

$$\begin{aligned}\frac{C^w(t)}{K(t)} &= \frac{W(t) - i(t)M(t) + \dot{M}(t)}{p(t)K(t)} = \frac{W(t) - i(t)M(t) + \hat{M}(t)M(t)}{p(t)K(t)} \\ &= \frac{[1-\pi(t)]p(t)Y(t) - i(t)M(t) + (\hat{p}(t) + n + \hat{m})M(t)}{p(t)K(t)} \\ &= [1-\pi(t)]u^d - rm(t) + \dot{m}(t) + nm(t) \end{aligned} \tag{7}$$

[13] Even withH^w fixed, the distribution of the housing stock changes as a result of the transactions among worker households. The current framework allows housing rental *within* class, but not between classes. Any rental income on housing is netted out by the rental payment by other members within the same class. To be concrete, consider

$$\dot{C}^w + R_p^w + p^h \dot{H}_d^w + \dot{D}^w = W + R_i^w + i\dot{D}^w + p^h \dot{H}_s^d - i\dot{M}_h^w - i\dot{M}_o^w + \dot{M}_h^w + \dot{M}_o^w.$$

where $\dot{H}_d^w$ and $\dot{H}_s^w$ are the purchase and sale of houses; p^h the housing price; W wage income; i the nominal interest rate; R_p^w and R_i^w rental payment and income, respectively; M_h^w and M_o^w the stock of home mortgages and other debts; D^w workers' bank deposits. The payment and the receipt of housing rents are netted out $R^p = R^i$; the assumption of constant housing stock means $H_d^w = H_s^w$; setting $M = M_h^w + M_o^w - D^w$, we obtain the budget equation (7).

[14] The assumption of constant housing stock is also found in Iacoviello (2005)

[15] The assumption can be relaxed without affecting main results by allowing a margin between the two rates which may depend positively on the workers' indebtedness, assuming bankers' profits from the existence of the margin are fully distributed to capitalists' households.

[16] Throughout this paper, nominal (real) variables are normalized by the value of (real) capital. Due to our long-run assumption that capital grows at the natural rate on average, the quantity of a normalized variable is proportional to the quantity in per capita terms (or in efficiency units).

where $i(t)$ and r are the nominal and real interest rates on loans, and the two rates are related via the familiar Fisher equation ($i(t) = r + \hat{p}(t)$). The last two terms, $\dot{m}(t) + nm(t)$, in (7) represents the amount of real borrowing (scaled by capital stock): on a steady growth path, the amount of real borrowing equals $nm(t)$, but out of steady growth path, the amount of borrowing is greater than $nm(t)$ if $\dot{m}(t) > 0$ or less than $nm(t)$ if $\dot{m}(t) < 0$. The size of $\dot{m}(t)$, the pace of credit supply, is determined by bankers.

The real interest rate on loans, r, is assumed to be set exogenously by bankers. Workers face a credit constraint imposed by banks and thus the level of consumption is limited by the availability of consumer credit.[17] In our specification, the amount of credit depends positively on workers' income and net worth. Banks use borrowers' income (=wages net of interest paid) as an indicator of their creditworthiness. In the face of any increase in perceived risk of borrowers, bankers, we assume, respond by adjusting credit supply rather than adjusting the loan rate because they may see increasing the rate as undesirable due to typical rationing reasons. Financial innovation also tends to make credit supply more elastistic with limited variations in the interest rate.

The effect of net worth on household borrowing is one of the important features in this model. This effect can be justified by the usual collateral effect: houses may serve as collateral and relax households' credit constraints.[18] Based on these considerations, the dynamics of workers' debt is given by[19]

$$\dot{m}(t) = \mu\left(y^w(t), \omega^w(t)\right); \quad \mu_y > 0 \quad \mu_\omega > 0 \tag{8}$$

where

$$y^w(t) \equiv [1 - \pi(t)]u^d - rm(t) \tag{9}$$

and

$$\omega^w(t) \equiv h^w(t) - m(t)$$

$h^w(t)$ is the value of housing wealth scaled by the value of productive capital

$$h^w(t) = \frac{p^h(t)H^w(t)}{p(t)K(t)}$$

and thus $\omega^w(t)$ is workers' net worth. The credit supply function (8) highlights Minsky's emphasis on margins of safety in banks' lending decisions. According to Minsky,

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[17] Dutt (2006), Palley (2010), Charpe et al. (2012) and Isaac and Kim (2013) also consider credit-constrained borrowers but they do so without introducing the effect of asset prices on credit supply.

[18] A literature has studied the implications of the collateral-credit-consumption nexus for the monetary transmission mechanism, e.g., Aoki et al. (2004), Iacoviello (2005).

[19] Equation (8) can be rewritten as $\hat{M}(t) = \hat{p}(t) + n + \left[\mu\left(y^w(t), \omega^w(t)\right)/m(t)\right]$, implying bankers keep the growth of outstanding loans in line with the sum of the long-run average rates of inflation and economic growth ($\hat{p}+n$) if the workers' profile of income and net worth satisfies $\dot{m} = \mu\left(y^w(t), \omega^w(t)\right) = 0$. Higher (lower) income or net worth accelerates credit supply at a rate above (below) $\hat{p} + n$.

‘ ...bankers are not simpletons who accept all that is put forward for them to finance as being worthy of financing. In their relations with businessmen, households and governments that require financing, bankers are *designated sceptics*.’(Minsky, 1996)[p.76] Therefore, Minsky argues, banks insist on margins of safety when they consider granting loans. The ‘fundamental margin of safety’ is the excess of a unit’s expected operating income over the payment committed by debt contracts. In our formulation, the influence of $y^w(t)$ on credit supply captures the fundamental margin of safety. The collateral value is another margin of safety in bankers’ lending decisions.

(7) and (8) determine the consumption of worker households:

$$\frac{C^w(t)}{K(t)} \equiv c^w(t) = y^w(t) + \mu\left(y^w(t), \omega^w(t)\right) + nm(t) \tag{10}$$

An increase in income raises the workers’ consumption directly and indirectly via its effect on borrowing. The workers’ net worth stimulates consumption by relaxing the credit constraint.

We assume that workers has a desired ratio of housing stock to consumption and the desired ratio depends positively on expected capital gains on housing (and negatively on the constant real interest rate).

$$p^h(t)H^d(t) = \eta(\rho^e(t))p(t)C^w(t), \qquad \eta' > 0 \tag{11}$$

(11), along with (10), is part of the workers’ integrated decisions on consumption and balance sheet positions. By assumption, housing is the only asset for workers, but workers use debt and constantly make their balance sheet positions. Under our specification, workers see as desired the position of their balance sheet consistent with (10) and (11). Since workers are credit-constrained by bankers, the size of debt is not under their control and therefore their desired balance sheet position is achieved by adjusting the size of their balance sheet, i.e., the size of housing wealth. Specification (15) plausibly assumes that the workers’ desired balance sheet position is determined with reference to their level of consumption.[20] Equation (11) implies that the demand for housing stock is given by:

$$H^d(t) = \frac{\eta(\rho^e(t))p(t)C^w(t)}{p^h(t)} \tag{12}$$

Following the disequilibrium approach to asset prices, we assume that excess demand in the housing market does not vanish instantaneously[21] and causes housing price inflation.

[20]The early introduction of stock-flow specifications of consumption/portfolio behavior is found in Skott (1981).

[21]The housing market will be always in equilibrium if it instantaneously establishes the real housing price of $p^h(t)/p(t) = \eta(\rho^e(t))C^w(t)/H^w$. The assumption of instantaneous market clearing sounds

More specifically, we consider

$$\hat{p}^h(t) = \hat{p}(t) + n + \kappa\left(\frac{H^d(t)}{H^w} - 1\right), \quad \kappa > 0 \tag{13}$$

where $\hat{p}(t) + n$ amounts to the housing price inflation required to support a steady growth path with a constant housing/capital ratio (i.e., housing wealth grows in line with the size of the economy on a steady growth path), and the deviations of the rate of housing price inflation from the steady state value are driven by the excess demand in housing market.

Denoting $h^w(t) \equiv p^h(t)H^w/(p(t)K(t))$, (12) and (13) can be rewritten as[22]

$$\dot{h}^w(t) = \kappa\left[\eta(\rho^e(t))c^w(t) - h^w(t)\right] \tag{14}$$

In other words, the underlying disequilibrium dynamics in the housing market can be seen as a gradual adjustment of housing wealth to the desired level.

Households' expectations on capital gains are assumed to follow an adaptive mechanism:

$$\dot{\rho}^e(t) = \nu[\rho(t) - \rho^e(t)] \tag{15}$$

where ν is a positive constant and $\rho(t)$ is the rate of capital gains, i.e. (real) housing price inflation. Thus using the definition of $h^w(t)$, the rate of housing price inflation is given by:

$$\rho(t) \equiv \frac{\dot{p}^h(t)}{p^h(t)} - \frac{\dot{p}(t)}{p(t)} = \frac{\dot{h}^w(t)}{h^w(t)} + n \tag{16}$$

Substituting (14) and (16) in (15), the dynamics of expected housing price inflation can be written as

$$\begin{aligned}\dot{\rho}^e(t) &= \nu\left(\frac{\dot{h}^w(t)}{h^w(t)} + n - \rho^e(t)\right) \\ &= \nu\left(\frac{\kappa\left[\eta(\rho^e(t), y^b(t)) - h^w(t)\right]}{h^w(t)} + n - \rho^e(t)\right) \end{aligned} \tag{17}$$

extreme and then the question is how fast the adjustment of real housing prices is. This is utterly an empirical question. It should be noted, however, that the slow adjustment of *real* housing prices does not imply that nominal housing prices are sticky: the movement of nominal housing prices may have an order of frequencies close to that of output prices. The decades-long process of 'fixing the household balance sheets' after housing bubbles collapsed in Japan and the US may provide an anecdotal evidence for our disequilibrium dynamics approach.

[22] Equation (14) can be derived from the interaction between chartists and fundamentalists with constant wealth distribution. See Appendix A.

2.2.2 Capitalists: ultimate lenders

Capitalists' income scaled by capital stock, $y^c(t)$, is given by

$$\begin{aligned} y^c(t) &\equiv \frac{(1-s_f)(\Pi(t)-\delta p(t)K(t))+rM(t)}{p(t)K(t)} \\ &= (1-s_f)(\pi(t)u^d-\delta)+rm(t) \end{aligned} \tag{18}$$

Disregarding capitalists' housing wealth for the moment, their wealth, $\omega^c(t)$ consists of stocks and deposits:

$$\omega^c(t) \equiv \frac{v(t)N(t)+M(t)}{p(t)K(t)}$$

Denoting as $\alpha(t)$ the ratio of equities to deposits, the capitalists' wealth can be rewritten as

$$\omega^c(t) = [1+\alpha(t)]m(t) \tag{19}$$

In general, $\alpha(t)$ is affected by a number of factors including the rates of return on stocks and deposits. For instance, consider the following simple specification:

$$\alpha(t) = \alpha^*\left(\frac{(1-s_f)(\Pi(t)-\delta p(t)K(t))}{v(t)N(t)}, r\right)$$

where $\alpha(t)$ depends positively on the dividend yield (and negatively on the interest rate on deposits). One can easily show, however, that this specification makes capitalist wealth increasing in their dividend income and deposit holdings, and, under a conventional specification of consumption such as (20) below, the main analytic results in this paper are qualitatively the same as in the case where $\alpha(t)$ is constant. Therefore, we take $\alpha(t)$ as exogenous in the baseline model for the sake of simplicity.

Alternatively, $\alpha(t)$ may be endogenized along the line of the disequilibrium approach similar to our specification of housing market dynamics. In this case, expectations of capital gains on stocks may produce another source of boom and bust cycles.[23] In addition, it is natural to introduce housing into capitalists' portfolio problem in such an extension. Section 6.2 considers an extended model along the lines.

We adopt a conventional specification of consumption behavior for capitalists: their consumption depends on income and wealth.

$$\frac{C^c(t)}{K(t)} = f(y^c(t), \omega^c(t)); \quad 0 < f_y < 1,\ f_\omega > 0 \tag{20}$$

where f_y and f_ω is capitalist' marginal propensity to consume out of income and wealth, respectively.

[23]Endogenous changes in portfolios play an important role in many models of boom-bust cycles. See, for instance, (Asada et al., 2010, Ryoo, 2010, Skott, 2013, Taylor and O'Connell, 1985).

2.3 Determination of income distribution

Aggregate consumption, $C(t)$, consists of lenders' and borrowers' consumption.

$$\begin{aligned} \frac{C(t)}{K(t)} &= f(y^c(t), \omega^c(t)) + y^w(t) + \mu(y^w(t), \omega^w(t)) + m(t)n \qquad (21) \\ &\equiv \chi(\pi(t), h^w(t), m(t)) \end{aligned}$$

Let us examine how consumption demand responds to changes in income distribution and the key financial variables.

$$\begin{aligned} \chi_\pi &= -u^d[1 - (1 - s_f)f_y + \mu_y] \equiv -u^d \Delta < 0 \\ \chi_{h^w} &= \mu_\omega > 0 \\ \chi_m &= [rf_y + f_\omega(1 + \alpha)] + [n - r(1 + \mu_y) - \mu_\omega] \gtreqless 0 \end{aligned}$$

Aggregate consumption is decreasing in $\pi(t)$: a rise in $\pi(t)$ represents the distribution of income in favor of capitalists whose propensity to consume is lower than workers. Consumption is increasing in $h^w(t)$: the higher the collateral value the more amount workers can borrow and the more consumption. The effect of changes in $m(t)$ on consumption, however, is ambiguous. Capitalists' consumption will unambiguously increase as their income and wealth increase but workers' consumption may decrease as their burden of debt increases. The equilibrium condition for the goods market is given by

$$\begin{aligned} u^d &= \frac{C(t)}{K(t)} + n + \delta \\ &= \chi(\pi(t), h^w(t), m(t)) + n + \delta \qquad (22) \end{aligned}$$

Following Keynes (1930) and Kaldor (1956), the product market equilibrium is achieved through variations in the profit share. The validity of our application of the Kaldorian adjustment mechanism to longer-run financial cycles depends on two things: first, Marshallian ultra-short equilibrium is stable, and, second, the system of short cycles generate bounded fluctuations of accumulation around the natural rate. The first condition is met if aggregate saving is increasing in the profit share. This assumption is satisfied if the profit earners' propensity to save is higher than the wage earners', but note the latter condition is not necessary for the Marshallian stability in the presence of retained earning.[24] The condition certainly holds in our framework, i.e. $\chi_\pi < 0$. The second condition depends on the details of the system of short cycles, and our specification we borrowed from Skott (1989) produces the desired property (see section 5).

[24] If firms retain a fraction of profits, the saving propensity out of profits will be greater than that of wages even if there is no difference in personal saving rates.

Since $\chi_\pi \neq 0$, (22) implicitly defines $\pi(t)$ as a function of $m(t)$ and $h^w(t)$:

$$\pi^*(t) = \tilde{\pi}(h^w(t), m(t)) \tag{23}$$

such that $\chi(\pi^*(t), h^w(t), m(t)) + n + \delta = u^d$. The effects of financial variables on income distribution are given by

$$\tilde{\pi}_{h^w} = -\frac{\chi_{h^w}}{\chi_\pi} = \frac{\mu_\omega}{u^d\Delta} > 0 \tag{24}$$

$$\tilde{\pi}_m = -\frac{\chi_m}{\chi_\pi} = \frac{[rf_y + f_\omega(1+\alpha)] + [n - r(1+\mu_y) - \mu_\omega]}{u^d\Delta} \gtreqless 0 \tag{25}$$

The larger housing wealth $h^w(t)$ the higher the profit share. An increase in housing wealth allows workers to take on more loans, which stimulates their consumption demand. The increase in aggregate demand raises profit margins. The effect of consumer debt $m(t)$ on the profit share, however, is ambiguous as the effect of $m(t)$ on aggregate demand can be either way.

The workers' income is important for the behavior of the system. Substituting (23) in (9), we write

$$\tilde{y}^w(m(t), h^w(t)) \equiv [1 - \tilde{\pi}(h^w(t), m(t))]u^d - rm(t) \tag{26}$$

Using (24), (25) and (26), we have

$$\tilde{y}^w_h = -\frac{\mu_\omega}{\Delta} < 0 \tag{27}$$

$$\tilde{y}^w_m = \frac{-\left[\{f_\omega(1+\alpha) + n\} + rs^f f_y\right] + \mu_\omega}{\Delta} \tag{28}$$

The workers' net income is decreasing in housing wealth (eq. 27). An increase in housing wealth increases borrowing and consumption demand, which is reflected in higher profit margins. The increase in the profit share shift income away from workers to capitalists.

The macroeconomic effect of changes in workers' indebtedness on their net income is ambiguous. Inspecting (28), the ambiguity comes from the last term in the numerator, μ_ω. This positive term is explained by the negative effect of m on net worth and demand, which tends to reduce the profit share and to increase the workers' income. If this effect is small, then an increase in the debt ratio has a negative effect on the workers' income.

3 Dynamics

3.1 Pure debt dynamics

Let us first examine the mechanism of debt dynamics in isolation, assuming that housing prices do not respond to excess demand ($\kappa = 0$). Under our specification of housing price dynamics, this means that real housing price inflation coincides with the natural rate of growth in the economy, i.e., $h^w(t)$ remains constant.

Plugging (10) and (26) in (8), we have

$$\dot{m}(t) = \mu(\tilde{y}^w(m(t), h^w(t)), h^w(t) - m(t)) \equiv F(m(t), h^w(t)) \tag{29}$$

Let us first examine the sign of F_m, the effect of variations in $m(t)$ on $\dot{m}(t)$.

$$\frac{\partial \dot{m}(t)}{\partial m(t)} \equiv F_m = \mu_y \tilde{y}^c_m - \mu_\omega \tag{30}$$

If the workers' net income $\tilde{y}^w$ remains constant, an increase in $m(t)$ reduces credit supply because of a fall in net wealth (see the the second term, $-\mu_\omega$, in (30)). The fall in net worth also affects credit supply via its effect on income distribution. This induced effect, however, is dominated by the initial negative effect of changes in net worth on credit supply. Algebraically,

$$\begin{aligned} F_m &= \mu_y \cdot \left(\frac{-\left[\{f_\omega(1+\alpha)+n\} + rs^f f_y\right] + \mu_\omega}{\Delta} \right) - \mu_\omega \\ &= -\frac{\mu_y \left[\{f_\omega(1+\alpha)+n\} + rs^f f_y\right]}{\Delta} - \left(1 - \frac{\mu_y}{\Delta}\right) \mu_\omega < 0 \end{aligned} \tag{31}$$

The inequality in (31) comes from the fact that $\mu_y/\Delta < 1$. Thus the initial negative impact effect of $m(t)$ on $\dot{m}(t)$ always dominates, i.e. $F_m < 0$.

Next an increase in $h^w(t)$ speeds up debt accumulation, $F_{h^w} > 0$. An increase in $h^w(t)$ raises $\dot{m}(t)$ through the collateral-lending channel, holding $\tilde{y}^w(t)$ constant. The decrease in $\tilde{y}^w(t)$ caused by the rise in $h^w(t)$ partially offsets the initial positive effect of $h^w(t)$ on $\dot{m}(t)$. The overall effect will be unambiguously positive:

$$\frac{\partial \dot{m}(t)}{\partial h^w(t)} \equiv F_{h^w} = \mu_y \tilde{y}^w_h + \mu_\omega = \frac{\mu_\omega[1-(1-s_f)f_y]}{\Delta} > 0 \tag{32}$$

With $h^w(t)$ taken as exogenous, (29) is a one-dimensional differential equation of $m(t)$, which we call 'pure' debt dynamics in the sense that the debt dynamics is not disturbed by the movement of asset prices. Proposition 1 shows that the pure debt dynamics converges to a unique stationary point and the stationary debt ratio in the uni-dimensional system is increasing in the workers' housing wealth ($h^w(t)$).

Proposition 1 *For a given $h^w(t) > 0$, (29) has a unique stationary point $m^*(t)$ in $(0, m^+)$ such that $F(m^*(t), h^w(t)) = 0$ if there exists a debt ratio $m^+ > 0$ for which*

$$F(0, h^w(t)) > 0 > F(m^+, h^w(t)) \tag{33}$$

The stationary point is globally stable and increasing in $h^w(t)$.

Proof. Since $\frac{\partial \dot{m}(t)}{\partial m(t)} = F_m < 0$, F is strictly decreasing in $m(t)$ for a given $h^w(t)$. With the condition (33) given, the intermediate value theorem[25] ensures that there exists $m^*(t)$ in $(0, m^+)$ such that

$$F(m^*(t), h^w(t)) = 0 \tag{34}$$

(34) implicitly defines $m^*(t)$ as a function of $h^w(t)$ and therefore allows us to write

$$m^*(t) \equiv \tilde{m}(h^w(t))$$

$m^*(t)$ is globally stable because $\frac{\partial \dot{m}(t)}{\partial m(t)} = F_m < 0$ for all $m(t)$. Moreover

$$\tilde{m}'(h^w(t)) = -\frac{F_{h^w}}{F_m} > 0$$

since $F_m < 0$ and $F_{h^w} > 0$. ■

The interpretation of (33) is straightforward: if workers have no initial debt, then banks perceive their financial structure as robust and banks are willing to start providing loans ($F(0, h^w(t)) > 0$); if workers are highly indebted, banks restrict loans ($F(m^+, h^w(t)) < 0$). Since $\dot{m}(t)$ is decreasing in $m(t)$, there must exist a stationary debt ratio between 0 and m^+.

The pure debt dynamics is stable for a given level of assets $h^w(t)$. Increases in the borrowers' wealth, however, stimulate credit supply through the collateral-lending channel. In Minsky's terminology, a high level of borrowers' assets tends to validate high indebtedness. The stationary debt ratio, $\tilde{m}(h^w(t))$, can be seen as the debt ratio that bankers are content with for a given level of assets. Since $\tilde{m}'(h^w(t)) > 0$, the desired debt ratio depends positively on housing wealth. This aspect of the model captures the importance of the effect of asset prices on debt dynamics in Minsky's theory of financial instability. Minsky distinguishes 'loans based on the value of cash flows' from 'loans based on the value of pledged collateral' and emphasizes that the latter tends to destabilize the financial system (Minsky, 1986)[pp. 233-234].

> ...the overall fragility-robustness of the financial structure, upon which the cyclical stability of the economy depends, emerges out of loans made by bankers. A cash-flow orientation by bankers is conducive to sustaining a robust financial structure. An emphasis by bankers on the collateral value and the expected values of assets is conducive to the emergence of a fragile financial structure (Minsky, 1986)[p.234]

In our credit supply function, (8), the effect of the workers' net income captures the 'cash-flow orientation by bankers' whereas the effect of net worth represents 'the emphasis by bankers on the collateral value and the expected values of assets.' The stability

[25] We assume throughout this paper that the relevant functions are continuously differential.

result in Proposition 1 shows that, without the effect of housing prices on credit supply, an increase in indebtedness of workers has a self-stabilizing feedback because it increases the workers' burden of debt service and makes bankers more skeptic about the borrowers' ability to repay their debt. This confirms Minsky's argument that 'a cash-flow orientation by bankers is conducive to sustaining a robust financial structure.' As will be shown, the stabilizing debt dynamics is disturbed by fluctuations in housing prices.

3.2 Pure asset price dynamics

Instead of (8), let us consider a different regime of the bankers' credit supply: the amount of total loans grows at $\hat{p}(t)+n$ with no reference to the worker's income or net worth. In this case, the amount of debt per worker remains constant and the properties of housing price dynamics can be examined independently of debt dynamics.

With $\dot{m}(t)=0$ and $m(t)=m$, (14) and (17) form a two-dimensional pure asset price dynamics:

$$\dot{h}^w(t)=\kappa\left[\eta(\rho^e(t))c^w(m)-h^w(t)\right] \tag{14}$$

$$\dot{\rho}^e=\nu\left(\frac{\kappa\left[\eta(\rho^e(t))c^w(m)-h^w(t)\right]}{h^w(t)}+n-\rho^e(t)\right) \tag{17}$$

where $c^w(m)$ corresponds to the level of workers' consumption when $\dot{m}(t)=0$. Note that $c^w(m)$ is independent of $h(t)$ since the effect of $h(t)$ on the goods market works only through the collateral-lending channel which is absent under the assumption of $\dot{m}(t)=0$. One can easily show that $c^{w\prime}(m)<0$: higher debt squeezes the worker' consumption due to the strong negative effect of higher burden of debt servicing.[26]

There exists a unique stationary point of this system:

$$\tilde{h}=\eta(n)c^w(m) \text{ and } \tilde{\rho}^e=n \tag{35}$$

To see the stability property of the system (14) and (17), consider the Jacobian matrix evaluated at the stationary point

$$\tilde{J}=\begin{bmatrix} -\kappa & \kappa\eta' c^w \\ -\frac{\nu\kappa}{\tilde{h}} & \nu\left(\frac{\kappa\eta'}{\eta}-1\right)\end{bmatrix}$$

$$\begin{aligned}\mathrm{tr}(\tilde{J}) &= -\kappa+\nu\left(\frac{\kappa\eta'}{\eta}-1\right)\gtreqless 0\\ \det(\tilde{J}) &= \nu\kappa>0\end{aligned}$$

[26] Formally, we have:

$$c^{w\prime}(m)=\frac{-f_y[rs_f+n(1-s_f)]-f_\omega(1+\alpha)}{1-f_y(1-s_f)}<0$$

The determinant is always positive, and therefore the saddle path instability is excluded. The stability of the system depends on the sign of the trace. The stationary point (35) is locally stable (unstable) if and only if the trace of the Jacobian is negative (positive). Thus we have

Proposition 2 *With $m(t)$ fixed, the stationary point of the asset price dynamics (14) and (17) is locally stable if* $-\kappa+\nu\left(\frac{\kappa\eta'}{\eta}-1\right)<0$ *and locally unstable if* $-\kappa+\nu\left(\frac{\kappa\eta'}{\eta}-1\right)>0$

Proposition 2 tells us that local instability requires η', κ and ν to be sufficiently large. If the value of these parameters is sufficiently large, an increase in capital gains stimulates the demand for housing stock which fuels a further appreciation in housing prices. Thus rising $h^w(t)$ and $\rho^e(t)$ will reinforce each other. Moreover, the introduction of a plausible nonlinearity into the η-function may turn instability into undamped perpetual cycles. If $\eta(\rho^e(t))$ is bounded from above and below by positive constants, the exploding tendency of the actual housing stock is likely tamed in the neighborhood of the specified bounds, thereby creating a turning point. The underlying mechanism – a positive feedback between actual and expected outcomes under extrapolating behavior – is a characteristic feature of early models of boom and bust cycles (see the related literature in section 1). Proposition 3 corroborates the earlier result in the current context.

Proposition 3 *The trajectories of $h^w(t)$ and $\rho^e(t)$ generated by (14) and (17) converge to a closed orbit if* $-\kappa+\nu\left(\frac{\kappa\eta'}{\eta}-1\right)>0$ *and $\eta(\rho^e(t))$ is bounded such that*

$$0<\underline{\eta}\leq\eta(\rho^e(t))\leq\overline{\eta} \tag{36}$$

for all $\rho^e(t)$.

Proof See Appendix B.

(36) ensures that housing wealth $h^w(t)$ as well as the rate of capital gains $\rho(t)$ are bounded. It is easy to show that the boundedness of $\rho(t)$ implies that of $\rho^e(t)$ under our adaptive specification (15). Given the boundedness of $(h^w(t), \rho^e(t))$ and the existence of a unique and unstable equilibrium, the emergence of a stable limit cycle is a direct consequence of the Poincare-Bendixon theorem (Hirsch and Smale, 1974). The economic story behind Proposition 3 is not difficult to grasp. Suppose that $\rho(t)>\rho^e(t)$ and $h^w(t)<h^d(t)$. Then a housing market boom follows: housing wealth $h^w(t)$ and expected capital gains $\rho^e(t)$ increase. There will be a positive feedback between increasing $h^w(t)$ and $\rho^e(t)$ for a while. Such an upward movement is unsustainable because as actual housing wealth $(h^w(t))$ gets closer to $\overline{\eta}$, even a large increase in the

expected rate translates into only a small adjustment of desired housing wealth, which in turn generates only a small rise in actual capital gains. It will result in sluggish increases in capital gains and, at some point, the relation between the actual and expected capital gains will be reversed so that $\rho(t) < \rho^e(t)$. A period of optimism then will give a way to a period of pessimistic expectations. As $h^d(t)$ falls below $h^w(t)$, the level of actual housing wealth starts to fall.

Before moving onto the analysis of the interaction of debt and asset price dynamics in a three dimensional system, it would be instructive to look at the effects of changes in indebtedness ($m(t)$) on the system of asset price dynamics. First, higher indebtedness of workers is associated with a higher steady state value of housing wealth in pure asset price dynamics. This suggests that high indebtedness induced by a housing boom, if any, tends to constrain the upward instability of housing prices by shifting income distribution against workers and reducing their consumption. Second, the assumption of $\dot{m}(t) = 0$ implies that changes in $h(t)$ do not affect income distribution. Once the assumption is replaced by our credit supply function (8), increases in housing prices stimulate the workers' borrowing in the collateral-lending channel. In our framework, the rise in workers' borrowing has a paradoxical effect on income distribution and consumption: it tends to further squeeze the workers' consumption as the rise in aggregate demand driven by increasing borrowing induces a large shift in income distribution in favor of capitalists. The reduction in the workers' consumption decreases their desired housing demand.

3.3 Putting Debt and asset price dynamics together

Putting together debt and asset price dynamics, we now have a three dimensional dynamical system:

$$\begin{aligned} \dot{m}(t) &= \mu(\tilde{y}^w(m(t), h^w(t)), h^w(t) - m(t)) \equiv F(m(t), h^w(t)) && (37) \\ \dot{h}^w(t) &= \kappa\left[\eta(\rho^e(t))\tilde{c}^w(m(t), h^w(t))) - h^w(t)\right] \equiv G(m(t), h^w(t), \rho^e(t)) && (38) \\ \dot{\rho}^e(t) &= \nu\left[\frac{G(m(t), h^w(t), \rho^e(t))}{h^w(t)} + n - \rho^e(t)\right] && (39) \end{aligned}$$

Let us first examine the existence of a steady state. Proposition 4 shows that there exists a unique steady state under plausible conditions.

Proposition 4 *There exists a unique stationary point of (37)-(39), $(m(t), h^w(t), \rho^e(t)) = (m^*, h^*, n)$ such that $0 < m^* < m^+$ and $0 < h^* = \eta(n)c^w(m^*) < h^+$ if m^+ and h^+ are chosen such that*

$$F^*(0,0) > 0 > F^*(m^+, h^+) \tag{40}$$

$$0 < \eta(n)c^w(\tilde{m}^*(h^+)) < h^+ \tag{41}$$

where $F^*(m(t), h(t)) = \mu[y^w(m(t)), h^w(t) - m(t)]$, $y^w(m(t))$ stands for the workers' income under $\dot{m}(t) = 0$, and $\tilde{m}^*(h(t))$ is the solution of $m(t)$ for $F^*(m(t), h(t)) = 0$.

The meaning of condition (40) is similar to that of assumption (33) that was already discussed in section 3.1. Condition (41) implies, assuming the workers' debt is kept at what bankers want it to be ($m(t) = \tilde{m}^*(h(t))$), their desired housing wealth is positive if $h(t) = 0$,[27] but the desired housing wealth falls short of the actual holdings if the latter is sufficiently large. The system therefore permits an interior positive solution of $h(t)$ in the steady state. Since $c^w(m(t))$ is decreasing in $m(t)$ and $\tilde{m}(h(t))$ is increasing in $h(t)$, the desired housing wealth is decreasing in $h(t)$ and the steady state solution is unique.

Before we turn to the logic of instability and cycles, it may be illuminating to look at a condition under which the system exhibits stability.

Proposition 5 *If ν is sufficiently low, the stationary point of the dynamical system (37)-(39) is locally stable.*

Proof See Appendix C.

Proposition 5 tells us that if the workers' expectations of capital gains is relatively insensitive to the movement of capital gains, the steady state of the system is locally stable. In an extreme case with $\nu = 0$, $\rho^e(t)$ remains constant forever and the system (37)-(39) is reduced to a two dimensional sub-system of (37)-(38) given a fixed $\rho^e(t)$. The trace of the sub-system is always negative and its determinant is positive.[28] Thus the trajectory will converge to a stable point. Proposition 5 suggests that if ν is sufficiently small, the system retains such a stable property. This stability result highlights the importance of the dynamics of expectations and capital gains for the behavior of the system. The following proposition shows that the system loses its stability as expected capital gains change more sensitively to variations in actual capital gains.

Proposition 6 *Suppose that $\frac{G_{\rho^e}}{h^{w*}} - 1 > 0$. There exists a Hopf bifurcation value of ν for the system of(37)-(39). As ν rises passing through the bifurcation value, the system loses its stability, giving rise to a limit cycle.*

Proof See Appendix C.

Asset prices dynamics strongly shape debt dynamics. An increase in housing wealth in a booming housing market accelerates credit supply as it raises the workers' collateral, i.e. $F_{h^w} > 0$. Therefore an asset bubble (bust) is typically accompanied by increasing

[27] Since $c^w(\tilde{m}^*(h(t))$ is decreasing in $h(t)$, $\eta(n)c^w(\tilde{m}^*(h^+)) > 0$ implies $\eta(n)c^w(\tilde{m}^*(0)) > 0$.

[28] The detailed expression for the determinant $F_m G_h - F_h G_m$ is found in the proof of proposition 7 in Appendix B.

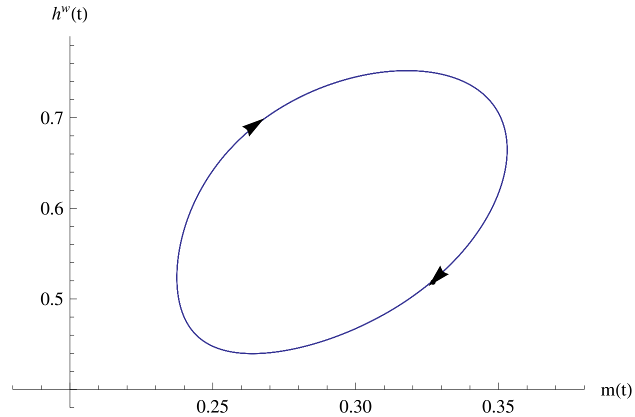

Figure 1: debt and housing wealth

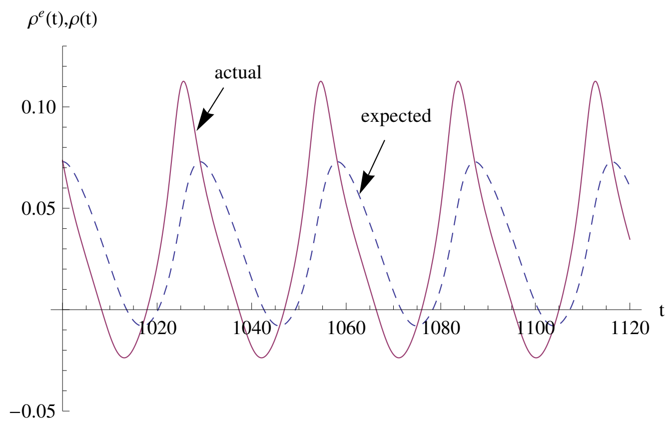

Figure 2: capital gains on housing wealth (housing price inflation)

(decreasing) indebtedness. Figure 1 illustrates such boom and bust dynamics of housing wealth and the debt ratio.[29] The movement of capital gains on housing wealth is the driving force behind the boom and bust cycles. Figure 2 depicts the movements of the expected and the actual rates of capital gains (= housing price inflation). During a boom, the rate of housing price inflation exceeds the expected rate, driving up the expected inflation rate, and during a downturn, the actual rate is lower than the expected rate, dragging down the expected rate.

Proposition 6 tells us the local property of the system. The following proposition

[29]Figures 1, 2 and 6 are based on the same parameter values and functions: $u = 0.5$, $\delta = 0.08$, $s^f = 0.5$, $r = 0.03$, $\alpha = 1$, $n = 0.03$, $\mu(y^w(t), \omega^w(t)) = 0.1y^w(t) + 0.1\omega^w(t) - 0.0606$, $f(y^l(t), \omega^l(t)) = 0.75y^l(t) + 0.048\omega^l(t)$, $\kappa = 0.2$, $\nu = 0.2$, and $\eta(\rho^e(t)) = 1.733 + \tanh[23(\rho^e - 0.03)]$. The purpose of the simulation is to demonstrate the emergence of endogenous cycles itself, and producing realistic details of long waves, including the asymmetry of boom and bust, may require the precise calibration of functional forms as well as parameter values.

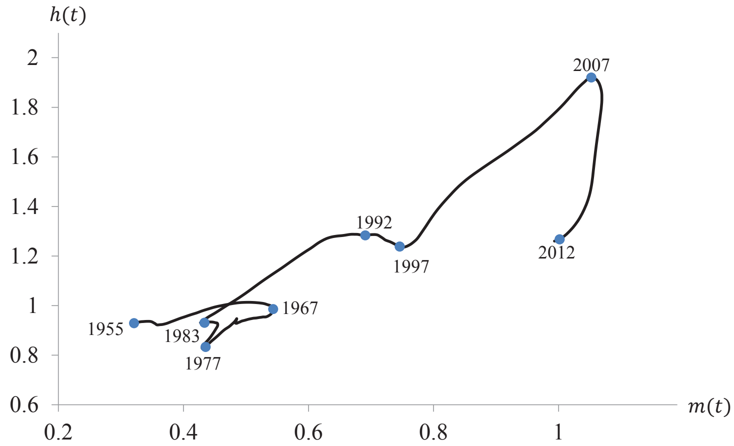

Figure 3: household debt and housing wealth
$m(t)$ and $h(t)$ are three-year moving average values of ‘credit market instruments’ (household liabilities) and household real estate, respectively, taken from Table B.100 in Flow of Funds Accounts of the United States (1952-2012)

provides a useful global property of the system behavior and complements the local analysis.

Proposition 7 *The trajectories of $(m(t), h^w(t), \rho^e(t))$ in the system of (37)-(39) are bounded if (36), (40) and (41) are satisfied.*

Proof See Appendix B.

Due to the boundedness of the trajectories, the local instability of the fixed point engenders perpetual fluctuations.

4 A cursory glance at the U.S. data

We have examined conditions under which the model generates instability ad endogenous cycles. The purpose of this section is not to provide conclusive empirical evidence for the theoretical framework but to present a preliminary look at the empirical data.

Our model can generate clockwise cycles on the $(m(t), h^w(t))$-space (see Figure 1). Using the U.S. data from 1952 to 2012, the picture is not clear-cut but we can identify two periods each of which appears to have seen a clockwise cycle of debt and housing wealth. The first period is the one between 1952 and the early 1980s and saw a relatively small cycle of debt and housing wealth (Figure 3). Household debt – scaled by the capital stock of the nonfarm nonfinanical corporation sector – had risen from 0.26 in 1952 to 0.55 in 1966. The ratio of housing wealth to capital stock had increased

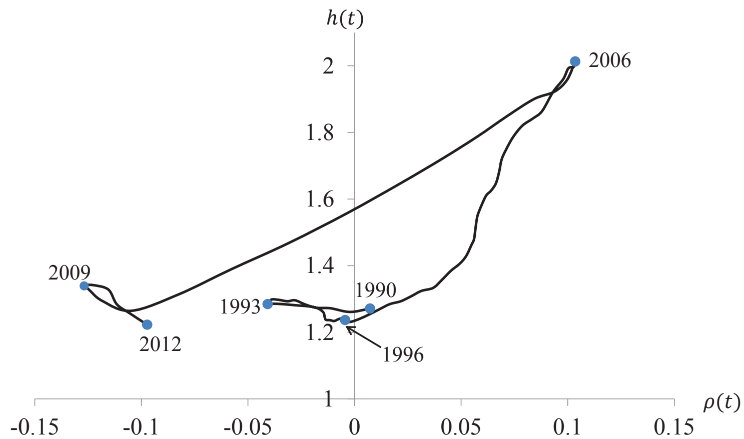

Figure 4: household wealth and housing price inflation
The definition and source of $h(t)$ are the same as in Figure 3. $\rho(t)$ is the three year moving average value of the annual percentage change in the Case-Shiller housing price index.

moderately from 0.85 in 1952 to 1.02 in 1963. The ratio of housing wealth to capital had declined from 1963 until 1977.

The next period, starting in the early 1980s, saw much more dramatic increases in both household debt and housing wealth than the previous period. The period of financial expansion, interrupted only by a relatively mild downturn in the housing market in the 1990s, lasted for more than twenty years until 2007. The housing market collapse in 2007 and the deleverging in the household sector thereafter are shown vividly in Figure 3. The boom-bust cycle in this period is also reflected in the movement of the rate of housing price inflation. Our model produces the counter-clockwise cycle in the (ρ, h)-space. Figure 4 identifies a similar pattern from the data: a sustained increase in housing price inflation was associated with the rapid accumulation of housing wealth for a period run-up to the financial crisis. Both housing wealth and housing price inflation had plummeted thereafter.

In our model, rising borrowers' net worth is a driving force behind credit expansion during a boom: $\dot{m}(t)$ depends positively on net worth $\omega^w(t)$. Figure 5 shows that increases (decreases) in indebtedness are largely associated with high (low) net worth.

5 Real effects of financial dynamics: long waves and short cycles

A prolonged period of prosperity is sustained by increasing asset prices. Such strong asset markets allow households to increase their borrowing and have a positive effect on aggregate demand. In our model, increases in aggregate demand result in increasing

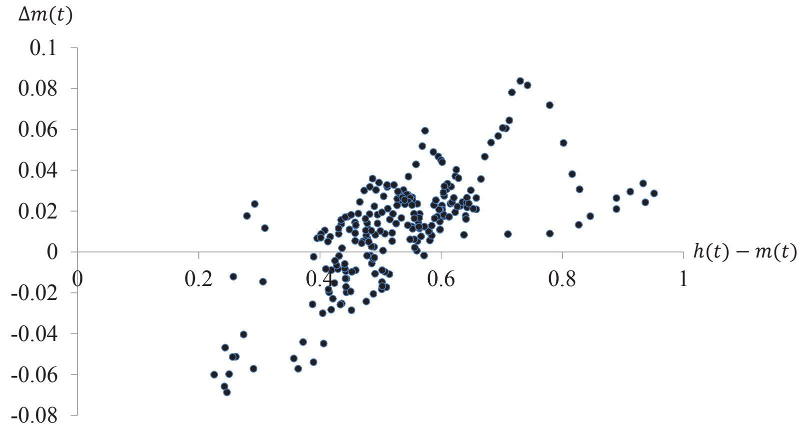

Figure 5: Change in indebtedness and net worth
Sources: The definition of $h(t)$ and $m(t)$ are the same as in Figure 3. $\Delta m(t)$ is the annual change in $m(t)$. All variables are the three year moving average values.

profitability (see Figure 6).

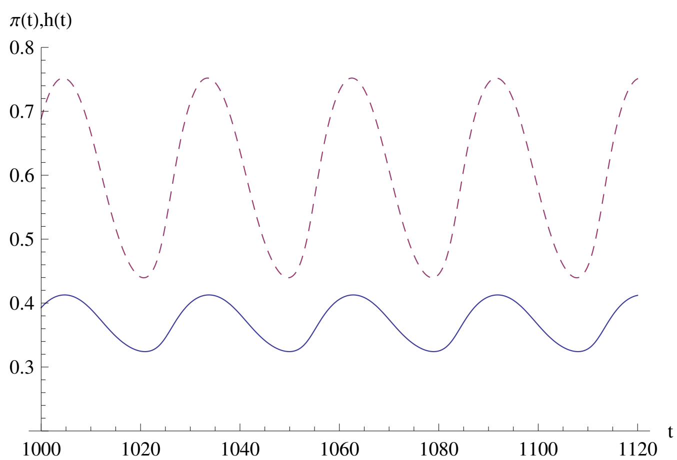

Figure 6: Profit share and housing wealth

The fluctuations of the profit share caused by changes in housing wealth and the debt ratio influence production, employment and accumulation. To examine the real effect of financial cycles, we explicitly introduce a model of short-run business cycles. The basic structure of the model of short cycles below is identical with that in Skott (1989).

The short-run profit share $\pi^s(t)$ is derived from the goods market equilibrium. In order to obtain $\pi^s(t)$,we use the actual utilization rate $u(t)$ instead of its long-run average (u^d). Changes in the actual utilization rate drive actual accumulation over short periods. The short-run investment function is given by[30]:

$$g(t) = n + \phi(u(t) - u^d), \quad \phi'(\cdot) > 0, \quad \phi(0) = 0 \tag{42}$$

Using this investment function and the consumption function evaluated at the actual rate of utilization, we have the equilibrium condition for the goods market:

$$\begin{aligned}[1 - \pi^s(t)]u(t) - rm(t) + \mu([1 - \pi^s(t)]u(t) - rm(t), h^w(t) - m(t)) + nm(t) \\ +f((1 - s^f)[\pi^s(t)u(t) - \delta] + rm(t), (1 + \alpha)m(t)) \\ +n + \phi(u(t) - u^d) = u(t)\end{aligned} \tag{43}$$

The solution of $\pi^s(t)$ can be written as a function of $u(t)$ as well as $m(t)$ and $h^w(t)$:

$$\pi^s(t) \equiv \tilde{\pi}^s(u(t), m(t), h^w(t)), \quad \tilde{\pi}^s_u > 0, \; \tilde{\pi}^s_h > 0, \; \tilde{\pi}^s_m \gtreqless 0 \tag{44}$$

The Harrodian approach assumes that investment is more sensitive to changes in utilization than saving and the condition is formally given by

$$\phi' > 1 - [1 - \pi^s(t)](1 - f_y + \mu_y) - f_y[1 - s^f \pi^s(t)] \tag{45}$$

The right-hand side of this inequality refers to the (aggregate) marginal propensity to save out of total income. The high sensitivity of investment to utilization makes the short-run profit share depend positively on the utilization rate, i.e., $\tilde{\pi}^s_u > 0$.

Changes in short-run profitability have implications for production decisions. We follow the specification of output expansion in Skott (1989):

$$\frac{\dot{Y}(t)}{Y(t)} = g^y(\pi^s(t), e(t)), \quad g^y_\pi > 0, \quad g^y_e < 0 \tag{46}$$

where $e(t) \equiv L(t)/\bar{L}(t)$, $L(t)$ is the number of employed workers and $\bar{L}(t)$ is the labor force that grows at the natural rate n. Behind (46) is the idea that output expansion is subject to the adjustment cost and responds positively to short-run profitability which reflects the condition of the goods market. The state of the labor market affects the adjustment cost and therefore the speed at which firms expand output. A tight labor market captured by a high employment rate is associated with a high cost of output adjustment[31] and thus slows down output growth.

[30](42) may be seen as a special case of the general specification where accumulation is affected by the firms' longer-run expectations of sales growth as well as the current utilization gap. (42) assumes that the firms' longer-run expectations of sales growth are anchored by the natural rate of growth. The analysis based on the general case is given in section 6.1.

[31]A higher rate of employment, for instance, tends to raise recruitment costs and shop-floor militancy, which make it difficult for firms to expand production. Skott (1989) [chapter 4] discusses the behavioral foundation of (46) in greater detail.

Since labor productivity is assumed to be constant, $L(t)$ is proportional to $Y(t)$. Therefore we have

$$\frac{\dot{e}(t)}{e(t)} = \frac{\dot{Y}(t)}{Y(t)} - n = g^y(\pi^s(t), e(t)) - n \tag{47}$$

Using the definition of $u(t)$ and $g(t)$ in (1) and (42), we can derive a differential equation that governs the trajectory of utilization.

$$\frac{\dot{u}(t)}{u(t)} = \frac{\dot{Y}(t)}{Y(t)} - g(t) = g^y(\pi^s(t), e(t)) - n - \phi(u(t) - u^d) \tag{48}$$

Plugging (44) in (47) and (48), we obtain a two dimensional system of utilization and employment if $m(t)$ and $h^w(t)$ are taken as given. In this case the system becomes recursive and is essentially the same as Skott (1989). Unless the negative effect of the employment rate on output expansion (g^y_e) is implausibly large, the dynamical system of $u(t)$ and $e(t)$ has a unique unstable steady state under certain assumptions. The source of instability here lies in the interaction between demand and production in the goods market. Given the Harrodian assumption (45) an increase in utilization raises aggregate demand and profitability which stimulates output expansion. If the induced increase in output growth is strong enough to exceed the change in the growth of capital stock, the utilization rate increases further. The increase in the employment ratio, on the other hand, tends to constrain output growth and may turn the otherwise exploding trajectories into perpetual cycles (see Skott (1989)). The cyclical trajectories of $u(t)$ and $e(t)$ are, however, contingent upon $m(t)$ and $h^w(t)$ as these financial variables affect the profit share. We have shown that the interaction between debt and housing price dynamics can produce long waves of $m(t)$ and $h^w(t)$. Long-run fluctuations of $m(t)$ and $h^w(t)$ sets the long-run trend of profitability around which the system of short cycles represented by (47) and (48) fluctuates. Figures 7 - 10 illustrate short-run business cycles fluctuate with reference to long swings.[32]

6 Extensions

6.1 Interaction between long waves and short cycles

Our long-run assumptions, (2) and (3), implies that the system of long waves is independent of that of short cycles, while the latter depends on the former. In addition, the investment function (42) is calibrated so that capital accumulation takes place at the natural rate if the actual utilization rate coincides with the desired rate. One may see these assumptions to be restrictive. The current section relaxes them.

[32]Figures 7 - 10 are based on the five dimensional system of (37), (38), (39), (47), and (48) where $g(t) = 0.03 + 1.12(u(t) - u^d)$, $g^y(t) = -0.035 + \frac{0.10}{1+\exp[-64.5\pi^s(t) - 14\ln(1.1 - e(t)) - 3.33]}$, and other parameters and functions are the same as those in footnote 29.

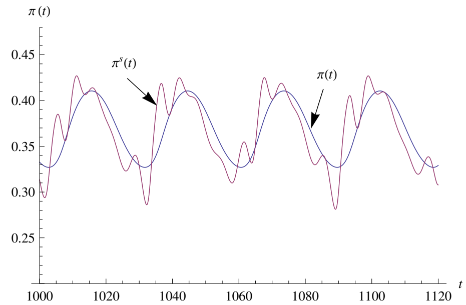

Figure 7: Profit Share

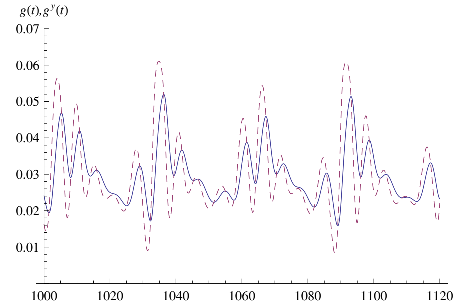

Figure 8: Output growth (dotted) and capital accumulation

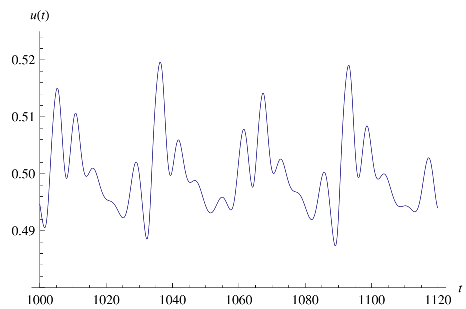

Figure 9: Utilization

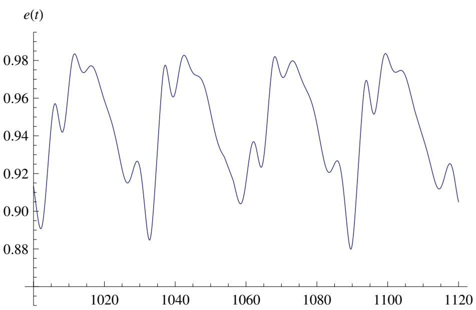

Figure 10: Employment

Let us suppose that the pace of accumulation is affected by the firms' expectations of sales growth as well as the discrepancy between the actual and the desired rates of utilization. This new assumption on accumulation behavior replaces (42) by:

$$g(t) = g^{ye}(t) + \phi(u(t) - u^d) \tag{49}$$

where g^{ye} is the expected growth rate of output (sales). Let us introduce an adaptive specification of expectations formation:

$$\dot{g}^{ye}(t) = \lambda_1 \cdot [g^y(\pi^s(t), e(t)) - g^{ye}(t)], \quad \lambda_1 > 0 \tag{50}$$

Under this adaptive specification, our original accumulation function (42) represents a special case where firms' long-term sales expectations are anchored by the natural rate of growth: $\lambda_1 = 0$ with $g^{ye}(t) = n$.

The assumptions on the long-run average values of the utilization and accumulation rates are relaxed into

$$\dot{\bar{u}}(t) = \lambda_2 \cdot [u(t) - \bar{u}(t)] \tag{51}$$

$$\bar{g}(t) = g^{ye}(t) + \phi[\bar{u}(t) - u^d] \tag{52}$$

The new specification makes the system of long waves also depend on short-run cycles since $\bar{u}(t)$ and $\bar{g}(t)$ are gradually revised based on the information on the short-run fluctuations in utilization. Our benchmark model corresponds to the case in which $\lambda_2 = \lambda_1 = 0$ with $\bar{u}(t) = u^d$ and $g^{ye}(t) = n$.

The modified assumptions lead to a seven dimensional system with $m(t)$, $h^w(t)$, $\rho^e(t)$, $u(t)$, $e(t)$, g^{ye} and $\bar{u}(t)$ being the corresponding state variables. Analytic results are difficult to obtain due to high dimensionality, but the intuition based on a perturbation argument suggests that the main feature of the baseline model, the coexistence of long waves and short cycles, survives if λ_1 and λ_2 are small. With the small adjustment parameters, large short-run fluctuations of $u(t)$ and $g(t)$ are filtered through the averaging process (51) and (52) so that their long-run averages exhibit moderate variations. Changes in the variations of $\bar{u}(t)$ and $\bar{g}(t)$ feed back into long-run financial cycles. A prolonged period of high utilization with rapid output growth, for instance, tends to push the average rate $\bar{u}(t)$ above the desired rate u^d and increases the long run average rate of accumulation, thereby reinforcing the expansionary effect of a housing market boom on the long-run profit share. The higher the values of λ_1 and λ_2, the stronger the feedback effect of short cycles on long waves. It is expected, not surprisingly, that high values of λ_1 and λ_2 – the fast adjustment of 'long-run averages' to 'actual values'– tend to obliterate the distinction between long-run trends and short-run variations.[33] Our analysis in previous sections was based on the premise that the conceptual distinction between long and short cycles is meaningful and pursued the idea in a simplest form where the long-run trends are invariant to short-run fluctuations.

6.2 Capitalists' portfolio decisions

This section drops the assumption that capitalists do not hold housing and allows endogenous changes in their portfolios. Relaxing these assumptions make the analysis more complicated. First, the purchase and sale of housing by capitalists and workers changes their respective shares of housing stock even under the assumption of the fixed total housing stock. The endogenous change in the class share of housing affects both housing price dynamics and aggregate demand (thus income distribution) under our specification of consumption/saving behavior. Second, the portfolio decisions are affected by the relative rates of returns on stocks and housing. The implications of changes in the rates of return on stocks and housing for aggregate demand and income distribution are not straightforward. Results generally depend on precise specifications and parameter values.

Let us assume that capitalists change their holdings of housing and stocks according

[33]These results are vindicated through numerical experiments. Simulation details are available upon request.

to the following adjustment rules:

$$\dot{h}^c(t) = \kappa_1 \left[\eta^c(\rho^e(t), r_s^e(t))c^c(t) - h^c(t)\right], \quad \eta^c_{\rho^e} > 0, \quad \eta^c_{r_s^e} < 0, \quad \kappa_1 > 0 \tag{53}$$

$$\dot{\epsilon}(t) = \kappa_2 \left[\epsilon^*(\rho^e(t), r_s^e(t))c^c(t) - \epsilon(t)\right], \qquad \epsilon^*_{\rho^e} < 0, \quad \epsilon^*_{r_s^e} > 0, \quad \kappa_2 > 0 \tag{54}$$

where $h^c(t) = p^h(t)H^c(t)/(p(t)K(t)$ and $\epsilon(t) = v(t)N(t)/(p(t)K(t))$. The idea behind these specifications is the same as in (14): asset holdings adjust to achieve the desired asset-consumption ratios, which depend on expected rates of return. The expected rate of return on stocks is assumed to follow an adaptive process:

$$\dot{r}_s^e(t) = j \cdot [r_s(t) - r_s^e(t)], \quad j > 0 \tag{55}$$

where the rate of return on stocks $r_s(t)$ is given by[34]

$$r_s(t) = \frac{(1 - s_f)[\Pi(t) - \delta p(t)K(t)]}{v(t)N(t)} + \hat{v}(t) - \hat{p}(t) \tag{56}$$

$$= \frac{\pi(t)u^d - \delta - n}{\epsilon(t)} + \hat{\epsilon}(t) + n. \tag{57}$$

The expression for the rate of return on housing is modified as it depends on the class share of housing:

$$\rho(t) = \gamma(t)\hat{h}^c(t) + [1 - \gamma(t)]\hat{h}^w(t) + n \tag{58}$$

where $\gamma(t)$ is the capitalists' share in total housing, i.e.

$$\gamma(t) = h^c(t)/[h^c(t) + h^w(t)] \tag{59}$$

The sum of the workers' consumption and acquisition of housing, $z^w(t)$, is given by

$$z^w(t) \equiv c^w(t) + h^w(t)\hat{H}^w(t) = y^w(t) + \mu(y^w(t), \omega^w(t)) + nm(t) \tag{60}$$

It can be shown that the capitalists' net aquisition of housing equals

$$h_c(t)\hat{H}_c(t) = \frac{[1 - \gamma(t)]\dot{h}^c(t) - \gamma(t)\kappa[\eta(\cdot)z^w(t) - h_w(t)]}{1 + \kappa\gamma(t)} \tag{61}$$

Finally, the condition for the goods market equilibrium,

$$f(y^c(t), \omega^c(t)) + h^c(t)\hat{H}^c(t) + y^w(t) + \mu(y^w(t), \omega^w(t)) + nm(t) + n + \delta = u^d, \tag{62}$$

gives us the equilibrium profit share as a function of the six state variables, $m(t)$, $h^w(t)$, $h^c(t)$, $\epsilon(t)$, $\rho^e(t)$ and $r_s^e(t)$ and closes our six dimensional system of differential equations:

$$\pi(t) = \pi^*(h^c(t), h^w(t), \epsilon(t), m(t), \rho^e(t), r_s^e(t)) \tag{63}$$

[34](57) is obtained from (56) using the firms' budget constraint (5) and the definition of $\epsilon(t)$.

The complete analysis is difficult for the six dimensional system. Given the current specifications, however, some features of the model may merit attention.

First of all, endogenous changes in capitalists' portfolios bring into the system a possibility of boom and bust cycles driven by the stock market. The interaction between income distribution and stock price dynamics is likely to be mutually reinforcing: an increase in the rate of return on stocks raises capitalists' wealth and stimulates aggregate demand; the resulting increase in profitability tends to increase the rate of return on stocks and tends to justify high expected returns; the increase in capitalists' consumption also raises the desired holdings of stocks and strengthens the tendency toward upward instability.

Second, the capitalists' substitution between stocks and housing may attenuate the instability potential in the housing market. If the effect of income distribution on the stock market is strong enough, a boom in the housing market can be short-lived as increasing profit shares shift capitalists' portfolios away from housing wealth. The forces that lead to a boom and bust cycle in the housing market, however, is also influenced by the workers' demand for housing. Given the assumption that workers do not own stocks (meaning there is no substitution effect on the workers' side), the behavior of the housing market can be dominated by the workers' decision on their balance sheet positions. If workers accumulate their housing wealth faster than capitalists, the workers' share of housing $\gamma(t)$ increases and the movement of housing price inflation will be dominated by the pace of the workers' accumulation of housing wealth[35], leading to the same kind of positive feedback as in the baseline model.

Finally, although endogenous shifts in the profit share may constrain the destabilizing potential created by the workers' strong demand for housing,[36] the very notion of bubbles suggests that the influence of economic fundamentals on asset prices dynamics is limited at times and may lag behind. Depending on the absolute and relative size of adjustment parameters, endogenous cycles of boom and bust can emerge from the stock or the housing markets. The two markets may be both subject to instability, and the resulting cycles may be asynchronous. Thus one can imagine a possibility in which the substitutability of housing and stocks generates bubbles on two assets alternately rather than completely eliminate destabilizing forces in the economy.

[35]The precise magnitude of the degree of substitution between stocks and housing is the subject of empirical studies. To the extent that the degree of substitutability is likely higher for capitalists than workers, the assumption of no stock holdings by workers may be defensible.

[36]This is true even for the mechanism of our baseline model. Recall that reductions in c^w caused by worsening income distribution contribute to the emergence of a turning point from a boom to a bust in the baseline model.

6.3 Housing supply

The amount of real housing stock available for workers H^w has been assumed to be constant. This section relaxes this assumption. The purpose of this extension is to examine the implications of induced changes in housing supply for the dynamics of debt and asset prices in a stylized manner rather than to build a full-fledged multisectoral model. The construction of houses, it is assumed, does not require any labor input but some adjustment cost in the form of the existing final good and the price of homes is determined at the level just enough to cover this adjustment cost. Given these assumptions, the housing price relative to output price, $p^h(t)/p(t)$, must equal the amount of the final good required to build $\dot{h}^w(t)$ units of houses. It appears reasonable to assume that the adjustment cost is increasing in the rate of expansion in housing stock. In particular, we assume that $p^h(t)/p(t) = \psi^{-1}(\hat{h}^w(t))$ with $\psi'(\cdot) > 0$: the new construction of homes at a rate of $\hat{h}^w(t)$ costs $\psi^{-1}(\hat{h}^w(t))$ units of the final good. The postulated relation between the adjustment cost and the growth of housing stock can be written as[37]:

$$\hat{H}^w(t) = \psi\left(\frac{p^h(t)}{p(t)}\right), \quad \psi'(\cdot) > 0 \tag{64}$$

i.e., the growth rate of housing units is increasing in the housing price relative to output price.[38] Furthermore, we assume that $\psi(\cdot)$ is bounded from above and below, and satisfies

$$\lim_{x(t)\to 0} \psi\left(x(t)\right) < n < \lim_{x(t)\to\infty} \psi\left(x(t)\right) \tag{65}$$

The variability of aggregate housing stock modifies the workers' budget constraint from equation (7) to

$$p(t)C^w(t) + p^h(t)\dot{H}^w(t) = W(t) - i(t)M(t) + \dot{M}(t) \tag{66}$$

or

$$\frac{C^w(t)}{K(t)} + h^w(t)\psi\left(\frac{p^h(t)}{p(t)}\right) = y^w(t) + \mu\left(y^w(t), \omega^b(t)\right) + nm(t) \tag{67}$$

In words, the workers' net income plus borrowing is used to finance the net acquisition of newly built homes as well as their consumption. The goods market equilibrium condition is rewritten as

$$p(t)C^w(t) + p^h(t)\dot{H}^w(t) + p(t)C^c(t) + p(t)I(t) = p(t)Y(t) \tag{68}$$

or

$$\left[\frac{C^w(t)}{K(t)} + h^w(t)\psi\left(\frac{p^h(t)}{p(t)}\right)\right] + \frac{C^c(t)}{K(t)} + n + \delta = u \tag{69}$$

[37]The specification of housing supply (64) and its interpretation are suggested by Peter Skott.

[38]The similar assumption on the relation between housing supply and housing price is also found in Poterba (1984).

Residential investment is a component of aggregate demand, but the expression for the equilibrium profit share (23) does not change. Under the current specification, the level of aggregate demand contributed by workers is determined solely by the sum of the workers' net income and borrowing (see the right-hand side of (69)). The introduction of the construction sector affects the division of workers' spending into consumption and housing investment but does not change the sum of those two components.[39]

The variability of housing stock requires the change in the expression for capital gains. Let us define $\xi(t) \equiv \frac{H^w(t)}{K(t)}$. The definition of $h^w(t)$ implies

$$\frac{p^h(t)}{p(t)} = \frac{h^w(t)}{\xi(t)} \tag{70}$$

and

$$\rho(t) \equiv \frac{R(\xi(t))}{h^w(t)} + \left[\frac{\dot{p}^h(t)}{p^h(t)} - \frac{\dot{p}(t)}{p(t)}\right] = \frac{R(\xi(t))}{h^w(t)} + \left[\frac{\dot{h}^w(t)}{h^w(t)} - \frac{\dot{\xi}(t)}{\xi(t)}\right] \tag{71}$$

where $\rho(t)$ is redefined as the rate of return on housing by including not only capital gains but also the rental rate of housing services, $R(\xi(t))$. We further assume that $R'(\xi(t)) < 0$. The negative dependence of the rental price of housing services on housing stock follows a standard argument (for instance, see Poterba (1984)).[40]

From the definition of $\xi(t)$, we have

$$\frac{\dot{\xi}(t)}{\xi(t)} = \hat{H}^w(t) - \hat{K}(t) = \psi\left(\frac{h^w(t)}{\xi(t)}\right) - n \tag{72}$$

Substituting (14) and (71) in (15), the dynamics of the expected rate of return on housing can be written as

$$\begin{aligned} \dot{\rho}^e(t) &= \nu\left(\frac{R(\xi(t)) + \dot{h}^w(t)}{h^w(t)} - \frac{\dot{\xi}(t)}{\xi(t)} - \rho^e(t)\right) \\ &= \nu\left[\frac{R(\xi(t)) + \kappa\{\eta(\rho^e(t))c^w(t) - h^w(t)\}}{h^w(t)} - \psi\left(\frac{h^w(t)}{\xi(t)}\right) + n - \rho^e(t)\right] \end{aligned} \tag{73}$$

Equations (37), (38), (72) and (73) constitute a four-dimensional system of differential equations. On a steady growth path, housing and capital stock grow at the same

[39]Note that this feature – no effect of the construction sector on the goods market equilibrium condition – results from our special assumption on the construction sector – construction requires no labor input but only a certain amount of final goods in the form of adjustment cost – as well as our specification of workers' consumption behavior. Thus it should not be taken as a general feature.

[40]Suppose the demand for the flow of housing services is decreasing in the rental rate and the flow supply is increasing in the stock of houses. The temporary equilibrium between supply and demand makes the equilibrium rental rate decreasing in the stock of houses.

rate n so as to maintain $\xi(t)$ at a constant level: $\hat{H}^w(t) = \hat{K}(t) = n$. Assumption (65) ensures the existence of $h^w(t)/\xi(t)$ that satisfies $\hat{H}^w(t) = n$. Thus we can write $\xi(t) = h^w(t)/\psi^{-1}(n)$. The steady-state value of $\rho(t)$ and $\rho^e(t)$ must equal the rental rate of housing services, $\rho(t) = \rho^e(t) = R(\xi(t)) = R(h^w(t)/\psi^{-1}(n))$, and $h^w(t)$ must be constant. The steady state requirements boil down to

$$\eta\left(R\left(\frac{h^w(t)}{\psi^{-1}(n)}\right)\right)c^w(\tilde{m}(h^w(t))) - h^w(t) = 0 \tag{74}$$

Note that the left-hand side of (74) is still decreasing in $h^w(t)$[41] and assumptions (40) and (41) ensure the existence of a unique value of $h^w(t)$, namely h^{**}, which satisfies (74). The steady-state value of $\xi(t)$ is given by $\xi^{**} = h^{**}/\psi^{-1}(n)$, the steady-state rate of return on housing $R(\xi^{**})$, and the steady-state debt ratio $\tilde{m}(h^{**})$.

The increase in the dimensionality due to the construction sector complicates the stability analysis but the main analytic results in previous sections carry over to the current extension if $R'(\cdot)$ and $\psi'(\cdot)$ are sufficiently small. In this case, the 3D subsystem of (37), (38) and (73) is semi-separable from the dynamics of $\xi(t)$, equation (72). Therefore, proposition 6, which is based on the inspection of the 3×3 sub-matrix of the full Jacobian matrix, must hold true for the dynamics of $m(t)$, $h^w(t)$ and $\rho(t)$. With the low price sensitivity of housing supply (low $\psi'(\cdot)$), the system continues to retain the destabilizing potential under the postulates in the baseline model.[42]

The high sensitivity of housing supply to the relative price (a large $\psi'(\cdot)$), however, will exert a stabilizing force for an intutive reason. The elastic supply of homes tends to undermine capital gains and therefore weaken the self-reinforcing positive feedback between housing demand and capital gains.

7 Conclusion

There exists a steady state in our model where the debt ratio and the actual level of housing wealth remain at the desired levels and the expectations are fulfilled. However, we did not presuppose that such an expectational equilibrium is instantaneously attained. Instead we asked whether plausible behavioral rules would justify the convergence process to the equilibrium. The convergence may fail and perpetual cycles of

[41]Note that the partial derivative of the left-hand side of (74) with respect to $h^w(t)$ equals $\frac{\eta' R'}{\psi^{-1}(n)} + \eta c^{w\prime}\tilde{m}' - 1 < 0$.

[42]If the housing construction is driven by the expected housing price, $p^{he}(t)$, rather than the actual price, i.e., $\hat{H}(t) = \psi(p^{he}(t)/p(t))$, and the process of expectations formation is adaptive, destabilizing forces of the system are likely to be stronger. The expected housing price that lags behind the actual price during a boom means the speed of the housing construction is more sluggish than in the case where the construction is determined by the actual price. Next, the construction may require borrowing and this aspect also affect the implication of construction activities for housing market dynamics, but our simple specification of housing construction does not allow us to pursue such an issue.

booms and busts emerge under the prevalence of extrapolating rules. A fast adjustment of expectations toward actual outcomes is in fact destabilizing. Advocates for the rational expectations hypothesis may dismiss the assumption of adaptive expectations and see the instability results built on out-of-equilibrium dynamics as irrelevant, but the approach taken in this paper may be justified by its realism.[43]

Motivated by Minsky's financial instability hypothesis, we have shown that the interaction between household debt accumulation and housing price dynamics can generate a prolonged period of expansion followed by a long downturn. Our model provides an explanation of two distinct cycles: long waves and short-run business cycles. Since the early 1980s, the U.S economy had experienced a long period of upward expansions with relatively small downturns in the 1991 and the 2001 recessions in the run-up to the Great Recession. During this period, the economics profession was increasingly dominated by an optimistic perspective about not only economic fundamentals and the power of monetary policy but also the state of the profession itself[44]. Such optimism, however, may have been a mere symptom of an upward phase of Minskian long waves.

Our analysis mainly concerns what Minsky called 'a skeletal model of a capitalist economy,' abstracting from a number of important aspects. The banking sector in our model, for instance, is highly stylized and has an obvious limitation in capturing Minsky's stress on the role of financial innovation and institutional changes in the financial sector. In addition, we have paid little attention to the analysis of stabilization policies as well as open economy complications. Furthermore, the analysis of financial instability in this paper needs to be complemented by careful empirical and historical analyses. Addressing related issues are left for future study.

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[43] The simple process of adaptive expectations can emerge from the process of econometric learning under constant-gain algorithms (Evans and Honkapohja, 2001).

[44] For instance, see Blanchard (2008) for a discussion of the state of macroeconomics.

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Appendix A: Derivation of equation (14)

Suppose that fundamentalists and chartists are different only in their expectations on capital gains: fundamentalists know the correct equilibrium value of the rate of capital gains (in our model, $\rho^e(t) = n$), whereas chartists use an adaptive rule along the lines of (14) and (15). Suppose that s is the chartists' share of housing wealth and the mass of total investors is normalized to one. Denoting the housing demand by individual chartists and fundamentalists as $h^t(t)$ and $h^f(t)$ respectively, we can write $\dot{h}^f(t) = \kappa[\overline{\eta}(n)c^w(t) - h^f(t)]$ and $\dot{h}^t(t) = \kappa[\overline{\eta}(\rho^e(t))c^w(t) - h^t(t)]$. Since $h^w(t) = (1-s)h^f(t) + sh^t(t)$, we have:

$$\begin{aligned} \dot{h}^w(t) &= (1-s)\kappa[\overline{\eta}(n)c^w(t) - h^f(t)] + s\kappa[\overline{\eta}(\rho^e(t))c^w(t) - h^t(t)] \\ &= \kappa\left[(1-s)\overline{\eta}(n)c^w(t) + s\overline{\eta}\left(\rho^e(t)\right)c^w(t)\right) - h^w(t)] \end{aligned} \tag{75}$$

Setting $\eta(\rho^e(t))c^w(t) \equiv (1-s)\overline{\eta}(n)c^w(t) + s\overline{\eta}(\rho^e(t))c^w(t)$, we obtain (14) from (75). As long as the share of chartists is non-zero, $\eta_{\rho^e} = s\overline{\eta}_{\rho^e}$ is non-zero and positive. The higher the share of chartists the larger the response of aggregate desired portfolio to changes in expected capital gains.

Appendix B: Proof of Propositions 3 and 7

Proof of Proposition 3 Let $c^w(m(t)) = c^w$. Equation (17) and assumption 36 implies

$$\kappa\underline{\eta}c^w < \dot{h}(s) + \kappa h(s) < \kappa\overline{\eta}c^w$$

Multiplity this by $\exp(\kappa s)$ and integrating it over $[0, t]$ gives us

$$[\exp(\kappa t) - 1]\underline{\eta}c^w < \exp(\kappa t)h^w(t) - h(0) < [\exp(\kappa t) - 1]\overline{\eta}c^w$$

Mutiplying by $\exp(-\kappa t)$ and rearranging the terms, we have:

$$[1 - \exp(-\kappa t)]\underline{\eta}c^w + \exp(-\kappa t)h(0) < h^w(t) < [1 - \exp(-\kappa t)]\overline{\eta}c^w + \exp(-\kappa t)h(0)$$

Since $0 < \exp(-\kappa t) \leq 1$ over $t \in [0, \infty)$, we have:

$$\underline{h} \leq h^w(t) \leq \overline{h}$$

where $\underline{h} = \min\{\underline{\eta}c^w, h(0)\}$ and $\overline{h} = \max\{\overline{\eta}c^w, h(0)\}$. There $h^w(t)$ is bounded. Since $\dot{h}^w(t)$ is continuous in $h^w(t)$ and η is bounded by assumption, $\dot{h}^w(t)$ is bounded as well. To prove the boundedness of $\rho^e(t)$, consider

$$\rho(t) = \frac{\dot{h}^w(t)}{h^w(t)} + n$$

Since $\dot{h}^w(t)$ and $h^w(t)$ are bounded, $\rho(t)$ is clearly bounded as long as $h(0) > 0$. Note that if $h^w(t) = 0$, $\rho(t)$ may explode but if $h(0) > 0$, this case is ruled out: $\underline{\eta} > 0$ by assumption and therefore $h^w(t) > \underline{h} \equiv \min\{\underline{\eta}c^w, h(0)\}$ >0 if $h(0) > 0$. Because $\rho(t)$ is bounded, the same method as in the proof of the boundedness of $h^w(t)$ can be applied to prove that of $\rho^e(t)$ by using (15). Since the trajectories are bounded and $(\eta(n)c^w, n)$ is a unique unstable fixed point, if $-\kappa + \nu\left(\frac{\kappa\eta'}{\eta} - 1\right) > 0$, the trajectories of $h^w(t)$ and $\rho^e(t)$ must converge to a closed orbit according to the Poincare-Bendixson theorem. ■

Proof of Proposition 7 To prove the boundedness of the trajectories of the 3D system, we first construct a set on the $(m(t), h^w(t))$ space from which any trajectory cannot escape once it enters *independently of* the value of $\rho^e(t)$ (i.e., the projection onto the $(m(t), h^w(t))$ space of a positively invariant set). We can confine our initial analysis to the $(m(t), h^w(t))$ space thanks to the boundedness of $\eta(\rho^e)$, i.e., assumption (36). Consider the following set:

$$\begin{aligned}\mathbf{A} \equiv &\{(m(t), h(t)) \in [m_1 - a, m_2 + a] \times [h_1 - a, h_2 + a] \mid \\ &a > 0,\ \underline{\eta}\tilde{c}^w(m_1, h_1) = h_1, \overline{\eta}c^w(m_2, h_2) = h_2, m_i = \tilde{m}(h_i), i = 1, 2\}\end{aligned}$$

where a sufficiently small positive a can be chosen for $\mathbf{A}$ to include $\{[m_1, m_2] \times [h_1, h_2]\}$ as its proper subset and to ensure $m_1 - a > 0$ and $h_1 - a > 0$. It can be easily shown that the gradient at any point on the boundaries of $\mathbf{A}$ points inward the set. It can be also shown that any trajectory from an arbitrary initial condition eventually enter $\mathbf{A}$. The intuitive explanation is as follows: for any given $\rho^e(t)$, the 2D sub-system $(m(t), h(t))$ has a unique fixed point and the fixed point is locally stable since the trace and the determinant of the subsystem are negative and positive, respectively. The fixed point depends continuously and monotonically on the value of $\eta(\rho^e(t))$. The set of all fixed points of the 2D subsystem is a finite and closed segment on the $\dot{m}(t)$-nullcline (Note that the $\dot{h}^w(t)$-nullcline depends continuously and monotonically on the value of $\rho^e(t)$, but the area it spans is limited by the boundedness of the η-function). We can choose a set that includes the set of fixed points as a proper subset. Such a set has the desired property: any trajectory cannot escape from it. $\mathbf{A}$ is an example of those sets with such a property. In our proof, we used the fact that the determinant of the 2D sub-system is positive, which can be checked:

$$\begin{aligned}F_m G_h - F_h G_m = &\frac{\kappa}{1 + \mu_y - f_y(1 - s_f)} \times [\{r f_y s_f + f_\omega(1 + \alpha)\}(\mu_y + \eta\mu_\omega) \\ &+ \mu_\omega\{1 - (1 - s_f)f_y\} + n\{\mu_y + \eta\mu_\omega f_y(1 - s_f)\}] > 0\end{aligned}$$

Since $m(t)$ and $h^w(t)$ are bounded, the boundedness of $\rho^e(t)$ follows from the argument similar to that in proposition 3. ■

Appendix C: Proof of Proposition 5 and 6

Let us consider the following Jacobian Matrix evaluated at the stationary point.

$$J = \begin{bmatrix} F_m & F_{h^w} & 0 \\ G_m & G_h & G_{\rho^e} \\ \nu\frac{G_m}{h^{w*}} & \nu\frac{G_h}{h^{w*}} & \nu\left(\frac{G_{\rho^e}}{h^{w*}} - 1\right) \end{bmatrix} \tag{76}$$

We have seen $F_m < 0$ and $F_{h^w} > 0$ in (31) and (32). We also have: $G_h = \kappa(\eta\tilde{c}_h^w - 1) < 0$ and $G_{\rho^e} = \kappa\eta'\tilde{c}^w(\cdot) > 0$.

Let us define

$$\begin{aligned} b_1 &\equiv \frac{G_{\rho^e}}{h^{w*}} - 1 \\ b_2 &\equiv F_m G_h - F_{h^w} G_m > 0 \\ b_3 &\equiv F_m + G_h < 0 \\ b_4 &\equiv \left(\frac{G_{\rho^e}}{h^{w*}} - 1\right) F_m - G_h \end{aligned}$$

Using the definition of b_i's, we have

$$\begin{aligned} \mathrm{tr}(J) &= b_3 + b_1\nu \\ \Sigma_{i=1}^3 J_i &= b_2 + b_4\nu \\ \det(J) &= -b_2\nu < 0 \\ -\mathrm{tr}(J)(\Sigma_{i=1}^3 J_i) + \det(J) &= A_0 + A_1\nu + A_2\nu^2 \end{aligned}$$

where J_i's are the first principal minors of J, and

$$\begin{aligned} A_0 &= -b_2 b_3 > 0 \\ A_1 &= -b_1 b_2 - b_3 b_4 - b_2 \\ A_2 &= -b_1 b_4 \end{aligned}$$

Proof of Proposition 5 The Routh-Hurwitz necessary and sufficient condition for the asymptotic local stability is:

$$\begin{aligned} &\mathrm{tr}(J) < 0, \quad \Sigma_{i=1}^3 J_i > 0 \\ &\det(J) < 0, \quad -\mathrm{tr}(J)(\Sigma_{i=1}^3 J_i) + \det(J) > 0 \end{aligned}$$

As $\nu \to 0$,

$$\mathrm{tr}(J) \to b_3 < 0 \tag{77}$$

$$\Sigma_{i=1}^3 J_i \to b_2 > 0 \tag{78}$$

$$-\mathrm{tr}(J)(\Sigma_{i=1}^3 J_i) + \det(J) \to A_0 > 0 \tag{79}$$

It is readily seen that for a sufficiently small positive value of ν, the signs of $\mathrm{tr}(J)$, $\Sigma_{i=1}^3 J_i$ and $-\mathrm{tr}(J)(\Sigma_{i=1}^3 J_i) + \det(J)$ should retain those of (77), (78) and (79) with $\det(J)$ being negative, thus satisfying the Routh-Hurwitz stability criterion. ■

Proof of Proposition 6 To prove the existence of a limit cycle for the system of (37)-(39) , we will show that the Jacobian matrix (76) evaluated at $(m^*(\nu),\ h^*(\nu),\ \rho^{e*}(\nu),\ \nu)$, where $(m^*(\nu),\ h^*(\nu),\ \rho^{e*}(\nu))$ is a fixed point of the system, has a negative real root and a pair of imaginary roots. If we denote the eigenvalues of the Jacobian matrix as $\lambda(\nu)$ and $\beta(\nu) \pm \theta(\nu)i$, we need to show that $\lambda(\nu^b) < 0$, $\beta(\nu^b) = 0$, and $\theta(\nu^b) \neq 0$. ν^b is called a Hopf bifurcation point. The Routh-Hurwitz criterion states that the Jacobian matrix will have a negative real root and a pair of pure imaginary roots if and only if:

$$\begin{aligned} &\mathrm{tr}(J) < 0, \qquad \Sigma_{i=1}^3 J_i > 0 \\ &\det(J) < 0, \qquad -\mathrm{tr}(J)(\Sigma_{i=1}^3 J_i) + \det(J) = 0 \end{aligned} \tag{80}$$

Let us suppose that $b_1 > 0$ and consider two cases: $b_4 > 0$ and $b_4 < 0$

Case 1. $b_4 > 0$. We then have $A_2 < 0$. Since $A_0 > 0$, the quadratic equation, $A_0 + A_1\nu + A_2\nu^2 = 0$, has one positive and one negative roots. Choose the positive root and denote it as ν^*. ν^* is given by

$$\nu^* \equiv \frac{A_1 + \sqrt{A_1^2 + 4A_0|A_2|}}{2|A_2|} > 0$$

where $-\mathrm{tr}(J)(J_1 + J_2 + J_3) + \det(J) = 0$. Because $b_2 > 0$ and $b_4 > 0$, $J_1 + J_2 + J_3 = b_2 + b_4\nu^* > 0$. $\mathrm{tr}(J) = 0$ if $\nu = \frac{b_3}{b_1} > 0$. It implies that if $\nu = \frac{b_3}{b_1}$, then $-\mathrm{tr}(J)(J_1 + J_2 + J_3) + \det(J) = \det(J) < 0$. For any $\nu > 0$, $-\mathrm{tr}(J)(J_1 + J_2 + J_3) + \det(J) = \det(J) < 0$ only if $\nu > \nu^*$. Therefore, $\nu^* < \frac{b_3}{b_1}$. Since $\mathrm{tr}(J)$ is increasing in ν ($b_1 > 0$), $\nu^* < \frac{b_3}{b_1}$ implies that $\mathrm{tr}(J) < 0$ at $\nu = \nu^*$. Therefore, we conclude that if $\nu = \nu^*$, the Routh-Hurwitz criterion (80) is satisfied and, therefore, $\lambda(\nu^*) < 0$, $\beta(\nu^{**}) = 0$, and $\theta(\nu^*) \neq 0$. For a later purpose, note that the first derivative of $-\mathrm{tr}(J)(J_1 + J_2 + J_3) + \det(J)$ with respect to ν is negative at $\nu = \nu^*$, i.e. $A_1 + 2A_2\nu^* < 0$.

Case 2. Next suppose that $b_4 < 0$. We then have $A_2 > 0$, $A_1 < 0$ and $A_0 > 0$. Furthermore, a straightforward calculation shows that:

$$A_1^2 - 4A_0A_2 = (b_1b_2 - b_3b_4)^2 + 2(b_1b_2 + b_3b_4)b_2 + b_2^2 > 0$$

The last inequality follows from the fact that $b_1 > 0$, $b_2 > 0$, $b_3 < 0$ and $b_4 < 0$. Therefore, the quadratic equation, $A_0 + A_1\nu + A_2\nu^2 = 0$, has two distinct positive roots. Denote the smaller as ν^{**}.

$$\nu^{**} \equiv \frac{|A_1| - \sqrt{A_1^2 - 4A_0A_2}}{2A_2} > 0$$

It is simple to show that $\mathrm{tr}(J) < 0$ and $J_1 + J_2 + J_3 > 0$ at $\nu = \nu^{**}$. Therefore, we conclude that if $\nu = \nu^{**}$, the Routh-Hurwitz criterion (80) is satisfied.

It remains to show that $\beta'(\nu^*) \neq 0$ and $\beta'(\nu^{**}) \neq 0$, respectively. Tedious algebra leads to:

$$\begin{aligned}\beta'(\nu^*) &= \frac{2\theta(\nu^*)[b_1b_2 + b_3b_4 + b_2 + 2b_1b_4\nu^*]}{4\lambda(\nu^*)^2\theta(\nu^*) + 4\theta(\nu^*)^3} \\ &= \frac{-2\theta(\nu^*)[A_1 + 2A_2\nu^*]}{4\lambda(\nu^*)^2\theta(\nu^*) + 4\theta(\nu^*)^3} > 0\end{aligned}$$

$$\beta'(\nu^{**}) = \frac{2\theta(\nu^{**})\left[b_4\lambda(\nu^{**}) + b_1\theta(\nu^{**})^2 + b_2\right]}{4\lambda(\nu^{**})^2\theta(\nu^{**}) + 4\theta(\nu^{**})^3} > 0$$