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Graviton Physics

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Abstract

The interactions of gravitons with matter are calculated in parallel with the familiar photon case. It is shown that graviton scattering amplitudes can be factorized into a product of familiar electromagnetic forms, and cross sections for various reactions are straightforwardly evaluated using helicity methods.

1 Introduction

The calculation of photon interactions with matter is a staple in an introductory (or advanced) quantum mechanics course. Indeed the evaluation of the Compton scattering cross section is a standard exercise in relativistic quantum mechanics, since gauge invariance together with the masslessness of the photon allow the results to be presented in terms of simple analytic forms[1].

On the surface, a similar analysis should be applicable to the interactions of gravitons. Indeed, like photons, such particles are massless and subject to a gauge invariance, so that similar analytic results for graviton cross sections can be expected. Also, just as virtual photon exchange leads to a detailed understanding of electromagnetic interactions between charged systems, a careful treatment of virtual graviton exchange allows an understanding not just of Newtonian gravity, but also of spin-dependent phenomena associated with general relativity which are to be tested in the recently launched gravity probe B[2]. However, despite this obvious parallel, examination of quantum mechanics texts reveals that (with one exception[3]) the case of graviton interactions is not discussed in any detail. There are at least three reasons for this situation:

i) the graviton is a spin-two particle, as opposed to the spin-one photon, so that the interaction forms are somewhat more complex, involving symmetric and traceless second rank tensors rather than simple Lorentz four-vectors;

ii) there exist few experimental results with which to compare the theoretical calculations;

iii) as we will see later in some processes, in order to guarantee gauge invariance one must include, in addition to the usual Born and seagull diagrams, the contribution from a graviton pole term, involving a triple-graviton coupling. This vertex is a sixth rank tensor and contains a multitude of kinematic forms.

However, recently, using powerful (string-based) techniques, which simplify conventional quantum field theory calculations, it has recently been demonstrated that the elastic scattering of gravitons from an elementary target of arbitrary spin must factorize[4], a feature that had been noted ten years previously by Choi et al. based on gauge theory arguments[5]. This factorization

permits a relatively painless evaluation of the various graviton amplitudes. Below we show how this factorization comes about and we evaluate some relevant cross sections. Such calculations can be used as an interesting auxiliary topic within an advanced quantum mechanics course

In the next section we review the simple electromagnetic case and develop the corresponding gravitational formalism. In section 3 we give the factorization results and calculate the relevant cross sections, and our results are summarized in a concluding section 4. Two appendices contain some of the formalism and calculational details.

2 Photon Interactions: a Lightning Review

Before treating the case of gravitons it is useful to review the case of photon interactions, since this familiar formalism can be used as a bridge to our understanding of the gravitational case. We begin by generating the photon interaction Lagrangian, which is accomplished by writing down the free matter Lagrangian together with the minimal substitution[6]

$$i\partial_\mu \longrightarrow iD_\mu \equiv i\partial_\mu - eA_\mu$$

where e is the particle charge and A_μ is the photon field. As examples, we discuss below the case of a scalar field and a spin 1/2 field, since these are familiar to most readers. Thus, for example, the Lagrangian for a free charged Klein-Gordon field is known to be

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + m^2 \phi^\dagger \phi \tag{1}$$

which becomes

$$\mathcal{L} = (\partial_\mu - ieA_\mu)\phi^\dagger(\partial^\mu + ieA^\mu)\phi + m^2\phi^\dagger\phi \tag{2}$$

after the minimal substitution. The corresponding interaction Lagrangian can then be identified—

$$\mathcal{L}_{int} = -ieA_\mu(\partial^\mu\phi^\dagger\phi - \phi^\dagger\partial^\mu\phi) + e^2 A^\mu A_\mu \phi^\dagger\phi \tag{3}$$

Similarly, for spin 1/2, the free Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i \not{\nabla} - m)\psi \tag{4}$$

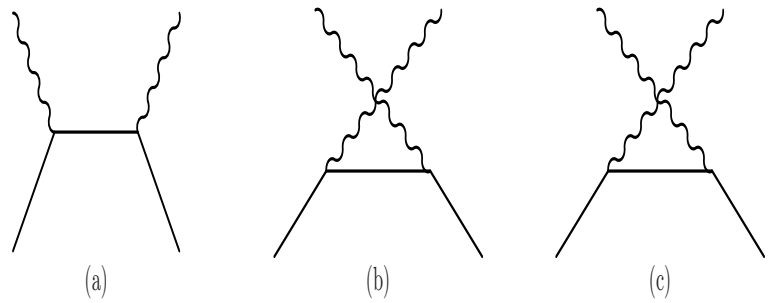

Figure 1: Diagrams relevant to Compton scattering.

becomes

$$\mathcal{L} = \bar{\psi}(i \not{\nabla} - e \not{A} - m)\psi \tag{5}$$

whereby the interaction Lagrangian is found to be

$$\mathcal{L} = -e\bar{\psi} \not{A}\psi \tag{6}$$

The single-photon vertices are then

$$< p_f|V^{\mu}_{em}|p_i >_{S=0} = e(p_f + p_i)^{\mu} \tag{7}$$

for spin zero and

$$< p_f|V^{\mu}_{em}|p_i >_{S=\frac{1}{2}} = e\bar{u}(p_f)\gamma^{\mu}u(p_i) \tag{8}$$

for spin 1/2—and the amplitudes for photon (Compton) scattering—*cf.* Figure 1—can be calculated. In the case of spin zero, *three* diagrams are involved—two Born terms and a seagull—and the total amplitude is

$$\mathrm{Amp}_{Compton}(S=0) = 2e^2 \left[\frac{2\epsilon_i \cdot p_i \epsilon^*_f \cdot p_f}{p_i \cdot k_i} - \frac{\epsilon_i \cdot p_f \epsilon^*_f \cdot p_i}{p_i \cdot k_f} - \epsilon^*_f \cdot \epsilon_i \right] \tag{9}$$

Note that all three diagrams must be included in order to satisfy the stricture of gauge invariance, which requires that the amplitude be unchanged under a gauge change

$$\epsilon_{\mu} \longrightarrow \epsilon_{\mu} + \lambda k_{\mu}$$

Indeed, we easily verify that under such a change for the incident photon

$$\begin{aligned} \delta\mathrm{Amp}_{Compton}(S=0) &= \lambda 2e^2 \left[\frac{p_i \cdot k_i \epsilon^*_f \cdot p_f}{k_i \cdot p_i} - \frac{k_i \cdot p_f \epsilon_f \cdot p_i}{p_f \cdot k_i} - k_i \cdot \epsilon^*_f \right] \\ &= \lambda 2e^2 \epsilon^*_f \cdot (p_f - p_i - k_i) = \lambda 2e^2 \epsilon^*_f \cdot k_f = 0 \end{aligned} \tag{10}$$

In the case of spin 1/2, there exists no seagull diagram and only the two Born diagrams exist, yielding[7]

$$\mathrm{Amp}_{Compton}(S=\frac{1}{2}) = e^2\bar{u}(p_f)\left[\frac{\not{\epsilon}_f^*(\not{p}_i+\not{k}_i+m)\ \not{\epsilon}_i}{2p_i\cdot k_i} - \frac{\not{\epsilon}_i(\not{p}_f-\not{k}_i+m)\ \not{\epsilon}_f^*}{2p_f\cdot k_i}\right]u(p_i) \tag{11}$$

Again, one can easily verify that this amplitude is gauge-invariant—

$$\begin{aligned}\delta\mathrm{Amp}_{Compton}(S=\frac{1}{2}) &= \lambda e^2\bar{u}(p_f)\left[\frac{\not{\epsilon}_f{}^*(\not{p}_i+\not{k}_i+m)\ \not{k}_i}{2p_i\cdot k_i} - \frac{\not{k}_i(\not{p}_f-\not{k}_i+m)\ \not{\epsilon}_f^*}{2p_f\cdot k_i}\right]u(p_i)\\ &= \bar{u}(p_f)\left[\not{\epsilon}_f^*-\not{\epsilon}_f^*\right]u(p_i)=0\end{aligned} \tag{12}$$

The corresponding cross sections can then be found via standard methods, as shown in many texts[7]. The results are usually presented in the laboratory frame—$p_i=(m,\vec{0})$—wherein the incident and final photon energies are related by

$$\omega_f = \frac{\omega_i}{1+2\frac{\omega_i}{m}\sin^2\frac{1}{2}\theta} \tag{13}$$

where θ is the scattering angle. For unpolarized scattering, we sum (average) over final (initial) spins via

$$\sum_\lambda \epsilon_\lambda^{*\mu}\epsilon_\lambda^\nu = -\eta^{\mu\nu}, \quad \sum_s u(p,s)_i\bar{u}(p,s)_j = \frac{(\not{p}+m)_{ij}}{2m} \tag{14}$$

and the resulting cross sections are well known—

$$\frac{d\sigma_{lab}(S=0)}{d\Omega} = \frac{\alpha^2}{m^2}(\frac{\omega_f}{\omega_i})^2\frac{1}{2}(1+\cos^2\theta) \tag{15}$$

and

$$\begin{aligned}\frac{d\sigma_{lab}(S=1/2)}{d\Omega} &= \frac{\alpha}{2m^2}(\frac{\omega_f}{\omega_i})^2[\frac{\omega_f}{\omega_i}+\frac{\omega_i}{\omega_f}-1+\cos^2\theta)]\\ &= \frac{\alpha^2}{m^2}\frac{\omega_f^2}{\omega_i^2}[\frac{1}{2}(1+\cos^2\theta)(1+2\frac{\omega_i}{m}\sin^2\frac{1}{2}\theta)+2\frac{\omega_i^2}{m^2}\sin^4\frac{1}{2}\theta]\end{aligned} \tag{16}$$

2.1 Helicity Methods

For use in the gravitational case it is useful to derive these results in an alternative fashion, using the so-called "helicity formalism," wherein one decomposes the amplitude in terms of components of definite helicity[8]. Here helicity is defined by the projection of the particle spin along its momentum direction. In the case of a photon moving along the z-direction, we choose states

$$\epsilon_i^{\lambda_i} = -\frac{\lambda_i}{\sqrt{2}}(\hat{x} + i\lambda_i\hat{y}), \qquad \lambda_i = \pm \tag{17}$$

while for a photon moving in the direction

$$\hat{k}_f = \sin\theta\hat{x} + \cos\theta\hat{z}$$

we use states

$$\epsilon_f^{\lambda_f} = -\frac{\lambda_f}{\sqrt{2}}(\cos\theta\hat{x} + i\lambda_f\hat{y} - \sin\theta\hat{z}), \quad \lambda_f = \pm \tag{18}$$

Working in the center of mass frame we can then calculate the amplitude for transitions between states of definite helicity. Using

$$\epsilon_i^{\pm} \cdot p_f = -\epsilon_f^{*\pm} \cdot p_i = \mp\frac{p}{\sqrt{2}}\sin\theta$$

$$\epsilon_f^{*\pm} \cdot \epsilon_i^{\pm} = -\frac{1}{2}(1 + \cos\theta), \quad \epsilon_f^{*\pm} \cdot \epsilon_i^{\mp} = -\frac{1}{2}(1 - \cos\theta)$$

we find for spin zero Compton scattering

$$\begin{aligned} A_{++} &= A_{--} = -e^2\left(1 + \cos\theta + \frac{p^2\sin^2\theta}{p_i \cdot k_f}\right) \\ A_{+-} &= A_{-+} = -e^2\left(1 - \cos\theta - \frac{p^2\sin^2\theta}{p_i \cdot k_f}\right) \end{aligned} \tag{19}$$

While these results can be found by direct calculation, the process can be simplified by realizing that under a parity transformation the momentum reverses but the spin stays the same. Thus the helicity reverses, so parity conservation assures the equality of $A_{a,b}$ and $A_{-a,-b}$ while under time reversal both spin and momentum change sign, as do initial and final states, guaranteeing that helicity amplitudes are symmetric—$A_{a,b} = A_{b,a}$.

Using the standard definitions

$$s = (p_i + k_i)^2, \quad t = (k_i - k_f)^2, \quad u = (p_i - k_f)^2$$

it is easy to see from simple kinematical considerations that

$$\begin{aligned} p^2 = \frac{(s-m^2)^2}{4s}, \quad \cos\frac{1}{2}\theta &= \frac{((s-m^2)^2+st)^{\frac{1}{2}}}{s-m^2} = \frac{(m^4-su)^{\frac{1}{2}}}{s-m^2} \\ \sin\frac{1}{2}\theta &= \frac{(-st)^{\frac{1}{2}}}{(s-m^2)} \end{aligned} \tag{20}$$

We can write then[9]

$$\begin{aligned} A_{++} &= A_{--} = 2e^2\frac{(s-m^2)^2+st}{(s-m^2)(u-m^2)} \\ A_{+-} &= A_{-+} = 2e^2\frac{-m^2t}{(s-m^2)(u-m^2)} \end{aligned} \tag{21}$$

(It is interesting that the general form of these amplitudes follows from simple kinematical constraints, as shown in ref. [10].) The cross section can now be written in terms of Lorentz invariants as

$$\begin{aligned} \frac{d\sigma}{dt} &= \frac{1}{16\pi(s-m^2)^2}\frac{1}{2}\sum_{i,j=\pm}|A_{ij}|^2 \\ &= 4e^4\frac{(m^4-su)^2+m^4t^2}{16\pi(s-m^2)^4(u-m^2)^2} \end{aligned} \tag{22}$$

and can be evaluated in any desired frame. In particular, in the laboratory frame we have

$$\begin{aligned} s-m^2 &= 2m\omega_i, \quad u-m^2 = -2m\omega_f \\ m^4-su &= 4m^2\omega_i\omega_f\cos^2\frac{1}{2}\theta, \quad m^2t = -4m^2\omega_i\omega_f\sin^2\frac{1}{2}\theta \end{aligned} \tag{23}$$

Since

$$\frac{dt}{d\Omega} = \frac{d}{2\pi d\cos\theta}\left(-\frac{2\omega_i^2(1-\cos\theta)}{1+\frac{\omega_i}{m}(1-\cos\theta)}\right) = \frac{\omega_f^2}{\pi} \tag{24}$$

the laboratory cross section is found to have the form

$$\frac{d\sigma_{lab}}{d\Omega}(S=0) = \frac{d\sigma(S=0)}{dt}\frac{dt}{d\Omega} = \frac{\alpha^2}{m^2}\frac{\omega_f^2}{\omega_i^2}(\cos^4\frac{1}{2}\theta + \sin^4\frac{1}{2}\theta) \tag{25}$$

and, using the identity

$$\cos^4\frac{1}{2}\theta + \sin^4\frac{1}{2}\theta = \frac{1}{2}(1+\cos^2\theta)$$

Eq. 25 is seen to be identical to Eq. 15 derived by conventional means.

A corresponding analysis can be performed for spin 1/2. Working again in the center of mass frame and using helicity states for both photons and spinors, one can calculate the various amplitudes. In this case it is convenient to define the photon as the "target" particle, so that the corresponding polarization vectors are

$$\begin{aligned} \epsilon_i^{\lambda_i} &= -\frac{\lambda_i}{\sqrt{2}}(-\hat{x} + i\lambda_i\hat{y}) \\ \epsilon_f^{\lambda_f *} &= -\frac{\lambda_f}{\sqrt{2}}(\sin\theta\hat{x} + \cos\theta\hat{z} - i\lambda_f\hat{y}) \end{aligned} \tag{26}$$

for initial (final) state helicity λ_i (λ_f). Working in the center of mass, the corresponding helicity amplitudes can then be evaluated via

$$B_{s_f\lambda_f;s_i\lambda_i} = \bar{u}(p_f,s_f)[\frac{2\epsilon_i\cdot p_i\ \not{\epsilon}_f^*}{p_i\cdot k_i} - \frac{2\epsilon_i\cdot p_f\ \not{\epsilon}_f^*}{p_i\cdot k_f} + \frac{\not{\epsilon}_f^*\ \not{k}_i\ \not{\epsilon}_i}{p_i\cdot k_i} - \frac{\not{\epsilon}_i\ \not{k}_i\ \not{\epsilon}_f^*}{p_i\cdot k_f}]u(p_i,s_i) \tag{27}$$

Useful identities in this evaluation are

$$\begin{aligned} \mathcal{O}_1 &\equiv \not{\epsilon}_f^*\ \not{k}_i\ \not{\epsilon}_i = \frac{p\lambda_i\lambda_f}{2}\begin{pmatrix} A+\Sigma & \lambda_i(A+\Sigma) \\ -\lambda_i(A+\Sigma) & -(A+\Sigma)\end{pmatrix} \\ \mathcal{O}_2 &\equiv \not{\epsilon}_i\ \not{k}_i\ \not{\epsilon}_f^* = \frac{p\lambda_i\lambda_f}{2}\begin{pmatrix} A-\Sigma & -\lambda_i(A-\Sigma) \\ \lambda_i(A-\Sigma) & -(A-\Sigma)\end{pmatrix} \\ \mathcal{O}_3 &\equiv \epsilon_i\cdot p_f\ \not{\epsilon}_f^* = \frac{p\sin\theta\lambda_i\lambda_f}{2}\begin{pmatrix} 0 & \Upsilon \\ -\Upsilon & 0\end{pmatrix} \end{aligned} \tag{28}$$

where

$$\begin{aligned} A &= \lambda_i\lambda_f + \cos\theta \\ \Sigma &= (\cos\theta\lambda_i + \lambda_f)\sigma_z + \lambda_i\sin\theta\sigma_x - i\sin\theta\sigma_y \\ \Upsilon &= -\cos\theta\sigma_x + \sin\theta\sigma_z - i\lambda_f\sigma_y \end{aligned} \tag{29}$$

We have then

$$\begin{aligned} B_{s_f\lambda_f;s_i\lambda_i} &= \frac{E+m}{2m}\begin{pmatrix} \chi_f^\dagger & \frac{-2s_f p}{E+m}\chi_f^\dagger \end{pmatrix} \\ &\times \left[\frac{1}{s-m^2}\mathcal{O}_1 - \frac{1}{u-m^2}(\mathcal{O}_2+\mathcal{O}_3)\right]\begin{pmatrix} \chi_i \\ \frac{2s_i p}{E+m}\chi_i \end{pmatrix} \end{aligned} \tag{30}$$

and, after a straightforward (but tedious) exercise, one finds the amplitudes[11]

$$
\begin{aligned}
B_{\frac{1}{2}1;\frac{1}{2}1} &= B_{-\frac{1}{2}-1;-\frac{1}{2}-1} = \frac{2\sqrt{s}e^2 p\cos\frac{1}{2}\theta}{m^2-u}(-\frac{1}{m}+\frac{m}{s}\sin^2\frac{1}{2}\theta) \\
B_{\frac{1}{2}1;\frac{1}{2}-1} &= B_{-\frac{1}{2}1;-\frac{1}{2}-1} = B_{\frac{1}{2}-1;\frac{1}{2}1} = B_{-\frac{1}{2}-1;-\frac{1}{2}1} = -\frac{2e^2 mp}{(m^2-u)\sqrt{s}}\sin^2\frac{1}{2}\theta\cos\frac{1}{2}\theta \\
B_{\frac{1}{2}-1;\frac{1}{2}-1} &= B_{-\frac{1}{2}1;-\frac{1}{2}1} = \frac{-2\sqrt{s}e^2 p}{m(m^2-u)}\cos^3\frac{1}{2}\theta \\
B_{\frac{1}{2}1;-\frac{1}{2}1} &= -B_{-\frac{1}{2}-1;\frac{1}{2}-1} = B_{-\frac{1}{2}1;\frac{1}{2}1} = -B_{\frac{1}{2}-1;-\frac{1}{2}-1} = -\frac{2e^2 p}{(m^2-u)}\sin\frac{1}{2}\theta\cos^2\frac{1}{2}\theta \\
B_{\frac{1}{2}-1;-\frac{1}{2}1} &= -B_{-\frac{1}{2}1;\frac{1}{2}-1} = \frac{-2e^2 p}{m^2-u}\sin^3\frac{1}{2}\theta \\
B_{\frac{1}{2}1;-\frac{1}{2}-1} &= -B_{-\frac{1}{2}-1;\frac{1}{2}1} = -\frac{2e^2 m^2 p}{s(m^2-u)}\sin^3\frac{1}{2}\theta
\end{aligned} \tag{31}
$$

Summing (averaging) over final (initial) spin 1/2 states we define spin-averaged photon helicity quantities

$$
\begin{aligned}
|B_{++}|^2_{av} &= |B_{--}|^2_{av} = \frac{2p^2e^4 s\sin^2\frac{1}{2}\theta}{m^2(m^2-u)^2}\left[1+\cos^4\frac{1}{2}\theta+\sin^4\frac{1}{2}\theta\frac{m^4}{s^2}(1-2\frac{s}{m^2})\right] \\
|B_{+-}|^2_{av} &= |B_{-+}|^2_{av} = \frac{2p^2e^4\sin^4\frac{1}{2}\theta}{(m^2-u)^2}\left[2\frac{m^2}{s}\cos^2\frac{1}{2}\theta+(1+\frac{m^4}{s^2})\sin^2\frac{1}{2}\theta\right]
\end{aligned} \tag{32}
$$

which, in terms of invariants, have the form

$$
\begin{aligned}
|B_{++}|^2_{av} &= |B_{--}|^2_{av} = \frac{e^4}{2m^2(u-m^2)^2(s-m^2)^2}(m^4-su)(2(m^4-su)+t^2) \\
|B_{+-}|^2_{av} &= |B_{-+}|^2_{av} = \frac{e^4t^2(2m^2-t)}{2(s-m^2)^2(u-m^2)^2}
\end{aligned} \tag{33}
$$

The laboratory frame cross section can be determined as before (note that this expression differs from Eq. 25 by the factor $4m^2$, which is due to the different normalizations for fermion and boson states—m/E for fermions and $1/2E$ for bosons)

$$
\begin{aligned}
\frac{d\sigma(S=1/2)}{d\Omega} &= \frac{4m^2\omega_f^2}{16\pi^2(s-m^2)^2}[|B_{++}|^2_{av}+|B_{+-}|^2_{av}] \\
&= \frac{\alpha^2}{m^2}\frac{\omega_f^2}{\omega_i^2}[\frac{1}{2}(1+\cos^2\theta)(1+2\frac{\omega_i}{m}\sin^2\frac{1}{2}\theta)+2\frac{\omega_i^2}{m^2}\sin^4\frac{1}{2}\theta]
\end{aligned} \tag{34}
$$

which is seen to be identical to the previously form—Eq. 16.

So far, all we have done is to derive the usual forms for Compton cross sections by non-traditional means. However, in the next section we shall see how the use of helicity methods allows the derivation of the corresponding graviton cross sections in an equally straightforward fashion.

3 Gravitation

The theory of graviton interactions can be developed in direct analogy to that of electromagnetism. Some of the details are given in Appendix A. Here we shall be content with a brief outline.

Just as the electromagnetic interaction can be written in terms of the coupling of a vector current j_μ to the vector potential A^μ with a coupling constant given by the charge e

$$\mathcal{L}_{int} = -ej_\mu a^\mu \tag{35}$$

the gravitational interaction can be described in terms of the coupling of the energy-momentum tensor $T_{\mu\nu}$ to the gravitational field $h^{\mu\nu}$ with a coupling constant κ

$$\mathcal{L}_{int} = -\frac{1}{2}\kappa T_{\mu\nu}h^{\mu\nu} \tag{36}$$

Here the field tensor is defined in terms of the metric via

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{37}$$

while κ is defined in terms of Newton's constant via $\kappa^2 = 32\pi G$. The energy-momentum tensor is defined in terms of the free matter Lagrangian via

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\sqrt{-g}\mathcal{L}_{int}}{\delta g^{\mu\nu}} \tag{38}$$

where

$$\sqrt{-g} = \sqrt{-\mathrm{det}g} = \exp\frac{1}{2}\mathrm{trlog}g \tag{39}$$

is the square root of the determinant of the metric. This prescription yields the forms

$$T_{\mu\nu} = \partial_\mu\phi^\dagger\partial_\nu\phi + \partial_\nu\phi^\dagger\partial_\mu\phi - g_{\mu\nu}(\partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi) \tag{40}$$

for a scalar field and

$$T_{\mu\nu} = \bar{\psi}[\frac{1}{4}\gamma_\mu i \overleftrightarrow{\nabla}_\nu + \frac{1}{4}\gamma_\nu i \overleftrightarrow{\nabla}_\mu - g_{\mu\nu}(\frac{i}{2} \overleftrightarrow{\not{\nabla}} - m)]\psi \tag{41}$$

for spin 1/2, where we have defined

$$\bar{\psi} i \overleftrightarrow{\nabla}_\mu \psi \equiv \bar{\psi} i \nabla_\mu \psi - (i\nabla_\mu \bar{\psi})\psi \tag{42}$$

The matrix elements of $T_{\mu\nu}$ can now be read off as

$$< p_f|T_{\mu\nu}|p_i >_{S=0} = p_{f\mu}p_{i\nu} + p_{f\nu}p_{i\mu} - \eta_{\mu\nu}(p_f \cdot p_i - m^2) \tag{43}$$

and

$$< p_f|T_{\mu\nu}|p_i >_{S=\frac{1}{2}} = \bar{u}(p_f)[\frac{1}{4}\gamma_\mu(p_f + p_i)_\mu + \frac{1}{4}\gamma_\nu(p_f + p_i)_\mu]u(p_i) \tag{44}$$

We shall work in harmonic (deDonder) gauge which satisfies, in lowest order,

$$\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h \tag{45}$$

where

$$h = \mathrm{tr} h_{\mu\nu} \tag{46}$$

and yields a graviton propagator

$$D_{\alpha\beta;\gamma\delta}(q) = \frac{i}{2q^2}(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma} - \eta_{\alpha\beta}\eta_{\gamma\delta}) \tag{47}$$

Then just as the (massless) photon is described in terms of a spin-one polarization vector ϵ_μ which can have projection (helicity) either plus or minus one along the momentum direction, the (massless) graviton is a spin two particle which can have the projection (helicity) either plus or minus two along the momentum direction. Since $h_{\mu\nu}$ is a symmetric tensor, it can be described in terms of a simple product of unit spin polarization vectors—

$$\begin{aligned} \text{helicity} &= +2: \quad h^{(2)}_{\mu\nu} = \epsilon^+_\mu \epsilon^+_\nu \\ \text{helicity} &= -2: \quad h^{(-2)}_{\mu\nu} = \epsilon^-_\mu \epsilon^-_\nu \end{aligned} \tag{48}$$

and just as in electromagnetism, there is a gauge condition—in this case Eq. 45—which must be satisfied. Note that the helicity states given in Eq. 48 are consistent with the gauge requirement, since

$$\eta^{\mu\nu}\epsilon_\mu\epsilon_\nu = 0, \quad \text{and} \quad k^\mu \epsilon_\mu = 0 \tag{49}$$

With this background we can now examine various interesting reactions involving gravitons, as we detail below.

3.1 Graviton Photoproduction

Before dealing with our ultimate goal, which is the treatment of graviton Compton scattering, we first warm up with a simpler process—that of graviton photoproduction

$$\gamma + s \to g + s, \quad \gamma + f \to g + f$$

The relevant diagrams are shown in Figure 2 and include the Born diagrams accompanied by a seagull and by the photon pole. The existence of a seagull is required by the feature that the energy-momentum tensor is momentum-dependent and therefore yields a contact interactions when the minimal substitution is made, yielding the amplitudes

$$< p_f; k_f, \epsilon_f \epsilon_f | T | p_i; k_i, \epsilon_i >_{seagull} = \kappa e \begin{cases} -2\epsilon_f^* \cdot \epsilon_i \epsilon_f^* \cdot (p_f + p_i) & S = 0 \\ \bar{u}(p_f) \not{\epsilon}_f^* \epsilon_f^* \cdot \epsilon_i u(p_i) & S = \frac{1}{2} \end{cases} \tag{50}$$

For the photon pole diagram we require a new ingredient, the graviton-photon coupling, which can be found from the expression for the photon energy-momentum tensor[6]

$$T_{\mu\nu} = -F_{\mu\alpha} F_\nu^\alpha + \frac{1}{2} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \tag{51}$$

This yields the photon pole term

$$< p_f; k_f, \epsilon_f \epsilon_f | T | | p_i; k_i \epsilon_i >_{\gamma - pole} = e < p_f | j_\alpha | p_i > \frac{1}{(p_f - p_i)^2}$$
$$\times \frac{\kappa}{2} [2\epsilon_f^{*\alpha} (k_f \cdot k_i \epsilon_f^* \cdot \epsilon_i - \epsilon_f^* \cdot k_i \epsilon_i \cdot k_f) + 2\epsilon_i \cdot k_f (\epsilon_f^* \cdot \epsilon_i k_f^\alpha - \epsilon_i \cdot k_f \epsilon_f^{*\alpha})] \tag{52}$$

Adding the four diagrams together, we find (after considerable but simple algebra—*cf.* Appendix B) a remarkably simple result

$$< p_f; k_f, \epsilon_f \epsilon_f | T | p_i; k_i, \epsilon_i >= H \times \left(\epsilon_{f\alpha}^* \epsilon_{i\beta} T_{Compton}^{\alpha\beta}(S) \right) \tag{53}$$

where H is the factor

$$H = \frac{\kappa}{4e} \frac{\epsilon_f^* \cdot p_f k_f \cdot p_i - \epsilon_f^* \cdot p_i k_f \cdot p_f}{k_i \cdot k_f} \tag{54}$$

and $\epsilon_{f\alpha}^* \epsilon_{i\beta} T_{Compton}^{\alpha\beta}(S)$ is the Compton scattering amplitude for particles of spin S calculated in the previous section. The gauge invariance of Eq. 53

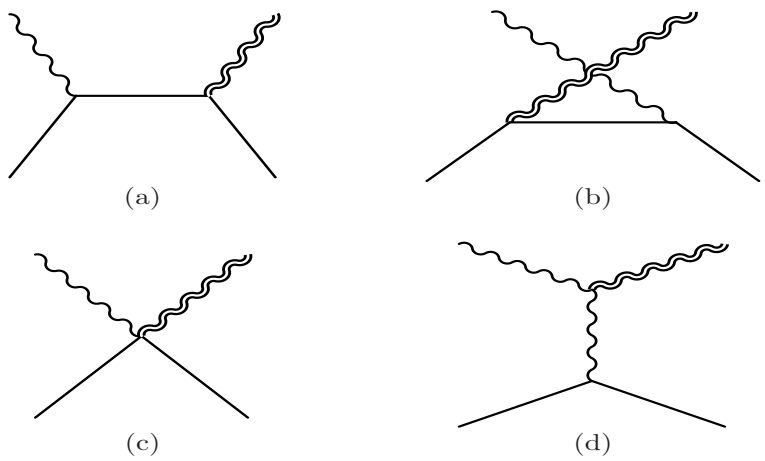

Figure 2: Diagrams relevant to graviton photoproduction.

is obvious, since it follows directly from the gauge invariance already shown for the corresponding photon amplitudes together with that of the factor H under

$$\epsilon_f \to \epsilon_f + \lambda k_f.$$

Also we note that in Eq. 53 the factorization condition mentioned in the introduction is made manifest, and consequently the corresponding cross sections can be obtained trivially. In principle, one can use conventional techniques, but this is somewhat challenging in view of the tensor structure of the graviton polarization vector. However, factorization means that helicity amplitudes for graviton photoproduction are simple products of the corresponding photon amplitudes times the universal factor H, and the cross sections are then given by the simple photon forms times the universal factor H^2. In the CM frame we have $\epsilon_f^* \cdot p_i = -\epsilon_f^* \cdot k_i$ and the factor H assumes the form

$$|H| = \frac{\kappa}{4e}|\frac{\epsilon_f^* \cdot k_i k_f \cdot p_f}{k_i \cdot k_f}| = \frac{\kappa}{4e}\frac{p\sin\theta}{\sqrt{2}}\frac{s-m^2}{-t} = \frac{\kappa}{2e}\left[\frac{m^4 - st}{-2t}\right]^{\frac{1}{2}} \tag{55}$$

In the lab frame this takes the form

$$|H_{lab}|^2 = \frac{\kappa^2 m^2}{8e^2}\frac{\cos^2\frac{1}{2}\theta}{\sin^2\frac{1}{2}\theta} \tag{56}$$

and the graviton photoproduction cross sections are found to be

$$\frac{d\sigma}{d\Omega} = G\alpha\cos^2\frac{1}{2}\theta$$
$$\times \begin{cases} (\frac{\omega_f}{\omega_i})^2[\mathrm{ctn}^2\frac{1}{2}\theta\cos^2\frac{1}{2}\theta + \sin^2\frac{1}{2}\theta] & S=0 \\ (\frac{\omega_f}{\omega_i})^3[(\mathrm{ctn}^2\frac{1}{2}\theta\cos^2\frac{1}{2}\theta + \sin^2\frac{1}{2}\theta) + \frac{2\omega_i}{m}(\cos^4\frac{1}{2}\theta + \sin^4\frac{1}{2}\theta) + 2\frac{\omega_i^2}{m^2}\sin^2\frac{1}{2}\theta] & S=1/2 \end{cases} \tag{57}$$

The form of the latter has previously been given by Voronov.[12]

3.2 Graviton Compton Scattering

Finally, we can proceed to our primary goal, which is the calculation of graviton Compton scattering. In order to produce a gauge invariant scattering amplitude in this case we require *four* separate contributions, as shown in Figure 3. Two of these diagrams are Born terms and can be written down straightforwardly. However, there are also seagull terms both for spin 0 and for spin 1/2, whose forms can be found in Appendix A. For the scalar case we have

$$\begin{aligned} < p_f; k_f, \epsilon_f\epsilon_f|T|p_i; k_i\epsilon_i\epsilon_i >_{seagull} &= \left(\frac{\kappa}{2}\right)^2 \Big[-2\epsilon_f^*\cdot\epsilon_i(\epsilon_i\cdot p_i\epsilon_f^*\cdot p_f + \epsilon_i\cdot p_f\epsilon_f^*\cdot p_i) \\ &- \frac{1}{2}(\epsilon_f^*\cdot\epsilon_i)^2 k_i\cdot k_f\Big] \end{aligned} \tag{58}$$

while in the case of spin 1/2

$$\begin{aligned} < p_f; k_f, \epsilon_f\epsilon_f|T|p_i; k_i\epsilon_i\epsilon_i >_{seagull} &= \left(\frac{\kappa}{2}\right)^2 \bar{u}(p_f) \\ &\times \Big[\frac{3}{16}\epsilon_f^*\cdot\epsilon_i(\not{\epsilon}_i\epsilon_f^*\cdot(p_i+p_f) + \not{\epsilon}_f^*\epsilon_i\cdot(p_i+p_f)) \\ &+ \frac{i}{16}\epsilon_f^*\cdot\epsilon_i\epsilon^{\rho\sigma\eta\lambda}\gamma_\lambda\gamma_5(\epsilon_{i\eta}\epsilon_{f\sigma}^* k_{f\rho} - \epsilon_{f\eta}\epsilon_{i\sigma}k_{i\rho})\Big] u(p_i) \end{aligned} \tag{59}$$

Despite the complex form of the various contributions, the final form, which results (after considerable algebra-*cf.* Appendix B) upon summation of the various components, is remarkably simple:

$$\epsilon_{f\alpha}\epsilon_{f\beta}M_{grav}^{\alpha\beta;\gamma\delta}\epsilon_{i\gamma}\epsilon_{i\delta} = F\times\left(\epsilon_{f\mu}^*\epsilon_{i\nu}T_{Compton}^{\mu\nu}(S=0)\right)\times\left(\epsilon_{f\alpha}^*\epsilon_{i\beta}T_{Compton}^{\alpha\beta}(S)\right) \tag{60}$$

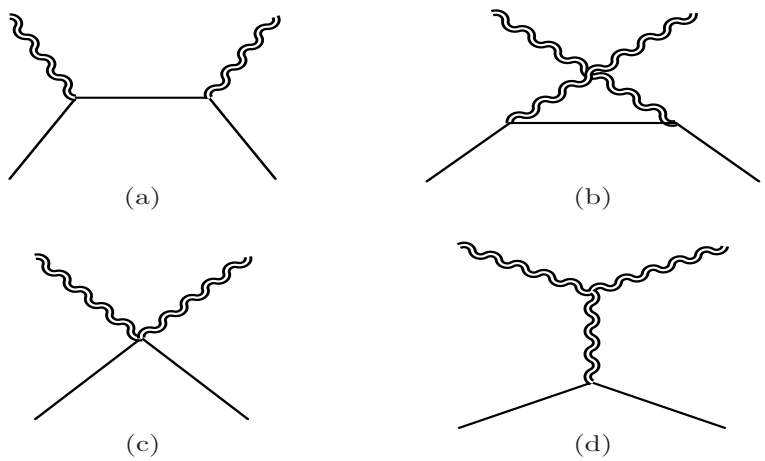

Figure 3: Diagrams relevant for gravitational Compton scattering.

where F is the universal factor

$$F = \frac{\kappa^2}{8e^4} \frac{p_i \cdot k_i p_i \cdot k_f}{k_i \cdot k_f} \tag{61}$$

and the Compton amplitudes are those calculated in section 2. The gauge invariance of this form is again obvious from the already-demonstrated gauge invariance of the photon amplitudes and this is the factorized form guaranteed by general arguments[5]. Again, it is in principle possible but very challenging to evaluate the cross section by standard means, but the result follows directly by the use of helicity methods. From the form of Eq. 60 it is clear that the helicity amplitudes for graviton scattering have the simple form of a product of corresponding helicity amplitudes for spinless and spin S Compton scattering. That is, for graviton scattering from a spinless target we have

$$\begin{aligned} |C_{++}|^2 &= |C_{--}|^2 = F^2 |A_{++}|^4 \\ |C_{+-}|^2 &= |C_{-+}|^2 = F^2 |A_{+-}|^4 \end{aligned} \tag{62}$$

while for scattering from a spin 1/2 target, we find for the target-spin averaged helicity amplitudes

$$\begin{aligned} |D_{++}|^2_{av} &= |C_{--}|^2_{av} = F^2 |A_{++}|^2 |B_{++}|^2_{av} \\ |C_{+-}|^2_{av} &= |C_{-+}|^2_{av} = F^2 |A_{+-}|^2 |B_{+-}|^2_{av} \end{aligned} \tag{63}$$

Here the factor F has the form

$$F = \frac{\kappa^2}{8e^4}\frac{(s-m^2)(u-m^2)}{t} \tag{64}$$

whose laboratory frame value is

$$F_{lab} = \frac{\kappa^2 m^2}{8e^4}\frac{1}{\sin^2\frac{1}{2}\theta} \tag{65}$$

The corresponding laboratory cross sections are found then to be

$$\begin{aligned}\frac{d\sigma_{lab}}{d\Omega}(S=0) &= \frac{\omega_f^2}{\pi}F^2\frac{1}{16\pi(s-m^2)^2}(|C_{++}|^2+|C_{+-}|^2) \\ &= G^2m^2(\frac{\omega_f}{\omega_i})^2[\mathrm{ctn}^4\frac{1}{2}\theta\cos^4\frac{1}{2}\theta+\sin^4\frac{1}{2}\theta]\end{aligned} \tag{66}$$

for a spinless target and

$$\begin{aligned}\frac{d\sigma_{lab}}{d\Omega}(S=\frac{1}{2}) &= \frac{\omega_f^2}{\pi}F^2\frac{1}{16\pi(s-m^2)^2}(|D_{++}|^2_{av}+|D_{+-}|^2_{av}) \\ &= G^2m^2(\frac{\omega_f}{\omega_i})^3[\mathrm{ctn}^4\frac{1}{2}\theta\cos^4\frac{1}{2}\theta+\sin^4\frac{1}{2}\theta) \\ &+ 2\frac{\omega_i}{m}(\mathrm{ctn}^2\frac{1}{2}\theta\cos^6\frac{1}{2}\theta+\sin^6\frac{1}{2}\theta)+2\frac{\omega_i^2}{m^2}(\cos^6\frac{1}{2}\theta+\sin^6\frac{1}{2}\theta)]\end{aligned} \tag{67}$$

for a spin 1/2 target. The latter form agrees with that given by Voronov[12].

We have given the results for unpolarized scattering from an unpolarized target, but having the form of the helicity amplitudes means that we can also produce cross sections involving polarized photons or gravitons. That is, however, a subject for a different time and a different paper.

4 Summary

While the subject of photon interactions with charged particles is a standard one in any quantum mechanics course, the same is not true for that of graviton interactions with masses despite the obvious parallels between these two topics. The origin of this disparity lies with the complications associated with

the tensor structure of gravity and the inherent nonlinearity of gravitational theory. We have argued above that this need not be the case. Indeed in an earlier work we showed how the parallel between the exchange of *virtual* gravitons and photons could be used in order to understand the phenomena of geodetic and Lense-Thirring precession in terms of the the related spin-orbit and spin-spin interactions in quantum electrodynamics[2]. In the present paper, we have shown how the treatment of graviton scattering processes can benefit from use of this analogy. Of course, such amplitudes are inherently more complex, in that they must involve tensor polarization vectors and the addition of somewhat complex photon or graviton pole diagrams. However, it is remarkable that when all effects are added together, the resulting amplitudes factorize into simple products of photon amplitudes times kinematic factors. Using helicity methods, this factorization property then allows the relatively elementary calculation of cross sections since they involve simple products of the already known photon amplitudes times kinematical factors. It is hoped that this remarkable result will allow introduction of graviton reactions into the quantum mechanics cirriculum in at least perhaps a special topics presentation. In any case, the simplicity associated with this result means that graviton interactions can be considered a topic which is no longer only associated with advanced research papers.

Appendix A: Gravitational Formalism

Here we present some of the basics of gravitational field theory. Details can be found in various references[13, 14]. The full gravitational action is given by

$$S_g = \int d^4x\sqrt{-g}\left(\frac{1}{16\pi G}R + \mathcal{L}_m\right) \tag{68}$$

where $\mathcal{L}_m$ is the Lagrange density for matter and R is the scalar curvature. Variation of Eq. 68 via

$$g_{\mu\nu} \to \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

yields the Einstein equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu} \tag{69}$$

where the energy-momentum tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\partial}{\partial g^{\mu\nu}}(\sqrt{-g}\mathcal{L}_m) \tag{70}$$

We work in the weak field limit, with an expansion in powers of the gravitational coupling G

$$\begin{aligned} g_{\mu\nu} &\equiv \eta_{\mu\nu} + \kappa h^{(1)}_{\mu\nu} + \dots \\ g^{\mu\nu} &= \eta^{\mu\nu} - \kappa h^{(1)\mu\nu} + \kappa^2 h^{(1)\mu\lambda} h^{(1)}{}_{\lambda}{}^{\nu} + \dots \end{aligned} \tag{71}$$

where here the superscript indicates the number of powers of G which appear and indices are understood to be raised or lowered by $\eta_{\mu\nu}$. We shall also need the determinant which is given by

$$\sqrt{-g} = \exp \frac{1}{2} \mathrm{tr} \log\, g = 1 + \frac{1}{2}\kappa h^{(1)} + \dots \tag{72}$$

The corresponding curvatures are given by

$$\begin{aligned} R^{(1)}_{\mu\nu} &= \frac{\kappa}{2}\left[\partial_\mu\partial_\nu h^{(1)} + \partial_\lambda\partial^\lambda h^{(1)}_{\mu\nu} - \partial_\mu\partial_\lambda h^{(1)}{}^{\lambda}{}_{\nu} - \partial_\nu\partial_\lambda h^{(1)}{}^{\lambda}{}_{\mu}\right] \\ R^{(1)} &= \eta^{\mu\nu} R^{(1)}_{\mu\nu} = \kappa\left[\Box h^{(1)} - \partial_\mu\partial_\nu h^{(1)\mu\nu}\right] \end{aligned} \tag{73}$$

In order to define the graviton propagator, we must make a gauge choice and we shall work in harmonic (or deDonder) gauge—$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0$—which requires, to first order in the field expansion,

$$0 = \partial^\beta h^{(1)}_{\beta\alpha} - \frac{1}{2}\partial_\alpha h^{(1)} \tag{74}$$

Using these results, the Einstein equation reads, in lowest order,

$$\Box h^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h^{(1)} - \partial_\mu\left(\partial^\beta h^{(1)}_{\beta\nu} - \frac{1}{2}\partial_\nu h^{(1)}\right) - \partial_\nu\left(\partial^\beta h^{(1)}_{\beta\mu} - \frac{1}{2}\partial_\mu h^{(1)}\right) = -16\pi G T^{\mathrm{matt}}_{\mu\nu} \tag{75}$$

which, using the gauge condition Eq. 74, can be written as

$$\Box\left(h^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^{(1)}\right) = -16\pi G T^{\mathrm{matt}}_{\mu\nu} \tag{76}$$

or in the equivalent form

$$\Box h^{(1)}_{\mu\nu} = -16\pi G\left(T^{\mathrm{matt}}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^{\mathrm{matt}}\right) \tag{77}$$

Gravitational Interactions: Spin 0

The coupling to matter via one-graviton and two-graviton vertices can be found by expanding the spin zero matter Lagrangian

$$\sqrt{-g}\mathcal{L}_m = \sqrt{-g}\left(\frac{1}{2}D_\mu\phi g^{\mu\nu}D_\nu\phi - \frac{1}{2}m^2\phi^2\right) \tag{78}$$

via

$$\begin{aligned}
\sqrt{-g}\mathcal{L}_m^{(0)} &= \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) \\
\sqrt{-g}\mathcal{L}_m^{(1)} &= -\frac{\kappa}{2}h^{(1)\mu\nu}\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}(\partial_\alpha\phi\partial^\alpha\phi - m^2\phi^2)\right) \\
\sqrt{-g}\mathcal{L}_m^{(2)} &= \frac{\kappa^2}{2}\left(h^{(1)\mu\lambda}h^{(1)\nu}{}_\lambda - \frac{1}{2}h^{(1)}h^{(1)\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi \\
&- \frac{\kappa^2}{8}\left(h^{(1)\alpha\beta}h^{(1)}_{\alpha\beta} - \frac{1}{2}h^{(1)2}\right)(\partial^\alpha\phi\partial_\alpha\phi - m^2\phi^2)
\end{aligned} \tag{79}$$

The one- and two-graviton vertices are then respectively

$$\begin{aligned}
\tau_{\alpha\beta}(p,p') &= \frac{-i\kappa}{2}\left(p_\alpha p'_\beta + p'_\alpha p_\beta - \eta_{\alpha\beta}(p\cdot p' - m^2)\right) \\
\tau_{\alpha\beta,\gamma\delta}(p,p') &= i\kappa^2\left[I_{\alpha\beta,\rho\xi}I^\xi{}_{\sigma,\gamma\delta}\left(p^\rho p'^\sigma + p'^\rho p^\sigma\right)\right. \\
&- \frac{1}{2}\left(\eta_{\alpha\beta}I_{\rho\sigma,\gamma\delta} + \eta_{\gamma\delta}I_{\rho\sigma,\alpha\beta}\right)p'^\rho p^\sigma \\
&- \left.\frac{1}{2}\left(I_{\alpha\beta,\gamma\delta} - \frac{1}{2}\eta_{\alpha\beta}\eta_{\gamma\delta}\right)\left(p\cdot p' - m^2\right)\right]
\end{aligned} \tag{80}$$

where we have defined

$$I_{\alpha\beta;\gamma\delta} = \frac{1}{2}(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma})$$

We also require the triple graviton vertex $\tau^{\mu\nu}_{\alpha\beta,\gamma\delta}(k,q)$ whose form is

$$\begin{aligned}
\tau^{\mu\nu}_{\alpha\beta,\gamma\delta}(k,q) &= \frac{i\kappa}{2}\Big\{(I_{\alpha\beta,\gamma\delta}-\frac{1}{2}\eta_{\alpha\beta}\eta_{\gamma\delta})\left[k^\mu k^\nu+(k-q)^\mu(k-q)^\nu+q^\mu q^\nu-\frac{3}{2}\eta^{\mu\nu}q^2\right] \\
&+ 2q_\lambda q_\sigma\left[I^{\lambda\sigma,}{}_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta}+I^{\lambda\sigma,}{}_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta}-I^{\lambda\mu,}{}_{\alpha\beta}I^{\sigma\nu,}{}_{\gamma\delta}-I^{\sigma\nu,}{}_{\alpha\beta}I^{\lambda\mu,}{}_{\gamma\delta}\right] \\
&+ [q_\lambda q^\mu(\eta_{\alpha\beta}I^{\lambda\nu,}{}_{\gamma\delta}+\eta_{\gamma\delta}I^{\lambda\nu,}{}_{\alpha\beta})+q_\lambda q^\nu(\eta_{\alpha\beta}I^{\lambda\mu,}{}_{\gamma\delta}+\eta_{\gamma\delta}I^{\lambda\mu,}{}_{\alpha\beta}) \\
&- q^2(\eta_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta}+\eta_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta})-\eta^{\mu\nu}q^\lambda q^\sigma(\eta_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}+\eta_{\gamma\delta}I_{\alpha\beta,\lambda\sigma})] \\
&+ [2q^\lambda(I^{\sigma\nu,}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k-q)^\mu+I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k-q)^\nu \\
&- I^{\sigma\nu,}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\mu-I^{\sigma\mu,}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\nu) \\
&+ q^2(I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\nu+I_{\alpha\beta,\sigma}{}^\nu I^{\sigma\mu,}{}_{\gamma\delta})+\eta^{\mu\nu}q^\lambda q_\sigma(I_{\alpha\beta,\lambda\rho}I^{\rho\sigma,}{}_{\gamma\delta}+I_{\gamma\delta,\lambda\rho}I^{\rho\sigma,}{}_{\alpha\beta})] \\
&+ [(k^2+(k-q)^2)\left(I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\nu+I^{\sigma\nu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\mu-\frac{1}{2}\eta^{\mu\nu}P_{\alpha\beta,\gamma\delta}\right) \\
&- (k^2\eta_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta}+(k-q)^2\eta_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta})]\Big\} \qquad (81)
\end{aligned}$$

Gravitational Interactions: Spin 1/2

For the case of spin 1/2 we require some additional formalism in order to extract the gravitational couplings. In this case the matter Lagrangian reads

$$\sqrt{e}\mathcal{L}_m=\sqrt{e}\bar{\psi}(i\gamma^a e_a{}^\mu D_\mu-m)\psi \qquad (82)$$

and involves the vierbein $e_a{}^\mu$ which links global coordinates with those in a locally flat space[15, 16]. The vierbein is in some sense the "square root" of the metric tensor $g_{\mu\nu}$ and satisfies the relations

$$\begin{aligned}
e^a{}_\mu e^b{}_\nu\eta_{ab} &= g_{\mu\nu}, \qquad e^a{}_\mu e_{a\nu}=g_{\mu\nu} \\
e^{a\mu}e_{b\mu} &= \delta^a_b, \qquad e^{a\mu}e_a{}^\nu=g^{\mu\nu} \qquad (83)
\end{aligned}$$

The covariant derivative is defined via

$$D_\mu\psi=\partial_\mu\psi+\frac{i}{4}\sigma^{ab}\omega_{\mu ab} \qquad (84)$$

where

$$\begin{aligned}
\omega_{\mu ab} &= \frac{1}{2}e_a{}^\nu(\partial_\mu e_{b\nu}-\partial_\nu e_{b\mu})-\frac{1}{2}e_b{}^\nu(\partial_\mu e_{a\nu}-\partial_\nu e_{a\mu}) \\
&+ \frac{1}{2}e_a{}^\rho e_b{}^\sigma(\partial_\sigma e_{c\rho}-\partial_\rho e_{c\sigma})e_\mu{}^c \qquad (85)
\end{aligned}$$

The connection with the metric tensor can be made via the expansion

$$e^a{}_\mu = \delta^a_\mu + \kappa c^{(1)a}_\mu + \dots \tag{86}$$

The inverse of this matrix is

$$e_a{}^\mu = \delta^\mu_a - \kappa c^{(1)\mu}_a + \kappa^2 c^{(1)\mu}_b c^{(1)b}_a + \dots \tag{87}$$

and we find

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa c^{(1)}_{\mu\nu} + \kappa c^{(1)}_{\nu\mu} + \dots \tag{88}$$

For our purposes we shall use only the symmetric component of the c-matrices, since these are physical and can be connected to the metric tensor. We find then

$$c^{(1)}_{\mu\nu} \to \frac{1}{2}(c^{(1)}_{\mu\nu} + c^{(1)}_{\nu\mu}) = \frac{1}{2}h^{(1)}_{\mu\nu}$$

We have

$$\det e = 1 + \kappa c + \dots = 1 + \frac{\kappa}{2}h + \dots$$

and, using these forms, the matter Lagrangian has the expansion

$$\begin{aligned}
\sqrt{e}\mathcal{L}^{(0)}_m &= \bar\psi(\frac{i}{2}\gamma^\alpha\delta^\mu_\alpha\overleftrightarrow{\nabla}_\mu - m)\psi \\
\sqrt{e}\mathcal{L}^{(1)}_m &= -\frac{\kappa}{2}h^{(1)\alpha\beta}\bar\psi i\gamma_\alpha\overleftrightarrow{\nabla}_\beta\psi - \frac{\kappa}{2}h^{(1)}\bar\psi(\frac{i}{2}\overleftrightarrow{\not\nabla} - m)\psi \\
\sqrt{e}\mathcal{L}^{(2)}_m &= \frac{\kappa^2}{8}h^{(1)}_{\alpha\beta}h^{(1)\alpha\beta}\bar\psi i\gamma^\gamma\overleftrightarrow{\nabla}_\lambda\psi + \frac{\kappa^2}{16}(h^{(1)})^2\bar\psi i\gamma^\gamma\overleftrightarrow{\nabla}_\gamma\psi \\
&- \frac{\kappa^2}{8}h^{(1)}\bar\psi i\gamma^\alpha h_\alpha{}^\lambda\overleftrightarrow{\nabla}_\lambda\psi + \frac{3\kappa^2}{16}h^{(1)}_{\delta\alpha}h^{(1)\alpha\mu}\bar\psi i\gamma^\delta\overleftrightarrow{\nabla}_\mu\psi \\
&+ \frac{\kappa^2}{4}h^{(1)}_{\alpha\beta}h^{(1)\alpha\beta}\bar\psi m\psi - \frac{\kappa^2}{8}(h^{(1)})^2\bar\psi m\psi \\
&+ \frac{i\kappa^2}{16}h^{(1)}_{\delta\nu}(\partial_\beta h^{(1)\nu}_\alpha - \partial_\alpha h^{(1)\nu}_\beta)\epsilon^{\alpha\beta\delta\epsilon}\bar\psi\gamma_\epsilon\gamma_5\psi
\end{aligned} \tag{89}$$

The corresponding one- and two-graviton vertices are found then to be

$$
\begin{aligned}
\tau_{\alpha\beta}(p,p') &= \frac{-i\kappa}{2}\left[\frac{1}{4}(\gamma_\alpha(p+p')_\beta+\gamma_\beta(p+p')_\alpha)-\frac{1}{2}\eta_{\alpha\beta}(\frac{1}{2}(\not{p}+\not{p}')-m)\right] \\
\tau_{\alpha\beta,\gamma\delta}(p,p') &= i\kappa^2\left\{-\frac{1}{2}(\frac{1}{2}(\not{p}+\not{p}')-m)P_{\alpha\beta,\gamma\delta}\right. \\
&- \frac{1}{16}[\eta_{\alpha\beta}(\gamma_\gamma(p+p')_\delta+\gamma_\delta(p+p')_\gamma) \\
&+ \eta_{\gamma\delta}(\gamma_\alpha(p+p')_\beta+\gamma_\beta(p+p')_\alpha)] \\
&+ \frac{3}{16}(p+p')^\epsilon\gamma^\xi(I_{\xi\phi,\alpha\beta}I^\phi{}_{\epsilon,\gamma\delta}+I_{\xi\phi,\gamma\delta}I^\phi{}_{\epsilon,\alpha\beta}) \\
&+ \left.\frac{i}{16}\epsilon^{\rho\sigma\eta\lambda}\gamma_\lambda\gamma_5(I_{\alpha\beta,\eta}{}^\nu I_{\gamma\delta,\sigma\nu}k'_\rho-I_{\gamma\delta,\eta}{}^\nu I_{\alpha\beta,\sigma\nu}k_\rho)\right\} \qquad (90)
\end{aligned}
$$

Appendix B: Graviton Scattering Amplitudes

In this section we summarize the independent contributions to the various graviton scattering amplitudes which must be added in order to produce the complete amplitudes quoted in the text. We leave it to the (perspicacious) reader to perform the appropriate additions and to verify the factorized forms shown earlier.

Graviton Photoproduction: Spin 0

$$
\begin{aligned}
\mathrm{Born-a}: &\quad \mathrm{Amp}_a = 4e\kappa\frac{(\epsilon_f^*\cdot p_f)^2\epsilon_i\cdot p_i}{2p_i\cdot k_i} \\
\mathrm{Born-b}: &\quad \mathrm{Amp}_b = -4e\kappa\frac{(\epsilon_f^*\cdot p_i)^2\epsilon_i\cdot p_f}{2p_i\cdot k_f} \\
\mathrm{Seagull}: &\quad \mathrm{Amp}_c = -2e\kappa\epsilon_f^*\cdot\epsilon_i\epsilon_f^*\cdot(p_i+p_f) \\
\gamma-\mathrm{pole}: &\quad \mathrm{Amp}_d = \frac{e\kappa}{k_i\cdot k_f}[\epsilon_f^*\cdot(p_i+p_f)(k_i\cdot k_f\epsilon_f^*\cdot\epsilon_i-\epsilon_f^*\cdot k_i\epsilon_i\cdot k_f) \\
&+ \epsilon_f^*\cdot k_i(\epsilon_f^*\cdot\epsilon_i k_i\cdot(p_i+p_f)-\epsilon_f^*\cdot k_i\epsilon_i\cdot(p_i+p_f)] \qquad (91)
\end{aligned}
$$

Graviton Photoproduction: Spin 1/2

$$
\begin{aligned}
\text{Born}-\text{a}: \quad & \text{Amp}_a = e\kappa \frac{\epsilon_f^* \cdot p_f}{2p_i \cdot k_i} \bar{u}(p_f)[\not{\epsilon}_f{}^*(\not{p}_i + \not{k}_i + m)\ \not{\epsilon}_i]u(p_i) \\
\text{Born}-\text{b}: \quad & \text{Amp}_b = -e\kappa \frac{\epsilon_f^* \cdot p_i}{2p_i \cdot k_f} \bar{u}(p_f)[\not{\epsilon}_i(\not{p}_i - \not{k}_f + m)\ \not{\epsilon}_f^*]u(p_i) \\
\text{Seagull}: \quad & \text{Amp}_c = -e\kappa \bar{u}(p_f)\ \not{\epsilon}_f^* u(p_i) \\
\gamma-\text{pole}: \quad & \text{Amp}_d = e\kappa \frac{1}{k_i \cdot k_f} \bar{u}(p_f)[\not{\epsilon}_f^*(k_i \cdot k_f \epsilon_f^* \cdot \epsilon_i - \epsilon_f^* \cdot k_i \epsilon_i \cdot k_f) \\
+ \quad & \not{k}_f \epsilon_f^* \cdot \epsilon_i \epsilon_f^* \cdot k_i - \not{\epsilon}_i(\epsilon_f^* \cdot k_i)^2]u(p_i)
\end{aligned} \tag{92}
$$

Graviton Scattering: Spin 0

$$
\begin{aligned}
\text{Born}-\text{a}: \quad & \text{Amp}_a = 2\kappa^2 \frac{(\epsilon_i \cdot p_i)^2 (\epsilon_f^* \cdot p_f)^2}{p_i \cdot k_i} \\
\text{Born}-\text{b}: \quad & \text{Amp}_b = -2\kappa^2 \frac{(\epsilon_f^* \cdot p_i)^2 (\epsilon_i \cdot p_f)^2}{p_i \cdot k_f} \\
\text{Seagull}: \quad & \text{Amp}_c = \kappa^2 \left[\epsilon_f^* \cdot \epsilon_i (\epsilon_i \cdot p_i \epsilon_f^* \cdot p_f + \epsilon_i \cdot p_f \epsilon_f^* \cdot p_i) - \frac{1}{2} k_i \cdot k_f (\epsilon_f^* \cdot \epsilon_i)^2 \right] \\
\text{g}-\text{pole}: \quad & \text{Amp}_d = \frac{4\kappa^2}{k_i \cdot k_f} \left[\epsilon_f^* \cdot p_f \epsilon_f^* \cdot p_i (\epsilon_i \cdot (p_i - p_f))^2 + \epsilon_i \cdot p_i \epsilon_i \cdot p_f (\epsilon_f^* \cdot (p_i + p_f))^2 \right. \\
+ \quad & \epsilon_i \cdot (p_i - p_f) \epsilon_f^* \cdot (p_f - p_i)(\epsilon_f^* \cdot p_f \epsilon_i \cdot p_i + \epsilon_f^* \cdot p_i \epsilon_i \cdot p_f) \\
- \quad & \epsilon_f^* \cdot \epsilon_i \big(\epsilon_i \cdot (p_i - p_f) \epsilon_f^* \cdot (p_f - p_i)(p_i \cdot p_f - m^2) \\
+ \quad & k_i \cdot k_f (\epsilon_f^* \cdot p_f \epsilon_i \cdot p_i + \epsilon_f^* \cdot p_i \epsilon_i \cdot p_f) + \epsilon_i \cdot (p_i - p_f)(\epsilon_f^* \cdot p_f p_i \cdot k_f + \epsilon_f^* \cdot p_i p_f \cdot k_f) \\
+ \quad & \epsilon_f^* \cdot (p_f - p_i)(\epsilon_i \cdot p_i p_f \cdot k_i + \epsilon_i \cdot p_f p_i \cdot k_i) \big) \\
+ \quad & (\epsilon_f^* \cdot \epsilon_i)^2 \left(p_i \cdot k_i p_f \cdot k_i + p_i \cdot k_f p_f \cdot k_f - \frac{1}{2}(p_i \cdot k_i p_f \cdot k_f + p_i \cdot k_f p_f \cdot k_i) \right. \\
+ \quad & \left. \left. \frac{3}{2} k_i \cdot k_f (p_i \cdot p_f - m^2)^2 \right) \right]
\end{aligned} \tag{93}
$$

Graviton Scattering: Spin 1/2

$$
\begin{aligned}
\text{Born}-\text{a}: \quad & \text{Amp}_a = \kappa^2 \frac{\epsilon_f^* \cdot p_f \epsilon_i \cdot p_i}{8 p_i \cdot k_i} \bar{u}(p_f)[\not{\epsilon}_f{}^*(\not{p}_i + \not{k}_i + m)\ \not{\epsilon}_i] u(p_i) \\
\text{Born}-\text{b}: \quad & \text{Amp}_b = -\kappa^2 \frac{\epsilon_f^* \cdot p_i \epsilon_i \cdot p_f}{8 p_i \cdot k_f} \bar{u}(p_f)[\not{\epsilon}_i(\not{p}_i - \not{k}_f + m)\ \not{\epsilon}_f^*] u(p_i) \\
\text{Seagull}: \quad & \text{Amp}_c = \kappa^2 \bar{u}(p_f) \left[\frac{3}{16} \epsilon_f^* \cdot \epsilon_i (\not{\epsilon}_i \epsilon_f^* \cdot (p_i + p_f) + \not{\epsilon}_f^* \epsilon_i \cdot (p_i + p_f)) \right. \\
+ \quad & \left. \frac{i}{8} \epsilon_f^* \cdot \epsilon_i \epsilon^{\rho\sigma\eta\lambda} \gamma_\lambda \gamma_5 (\epsilon_{i\eta} \epsilon_{f\sigma}^* k_{f\rho} - \epsilon_{f\eta}^* \epsilon_{i\sigma} k_{i\rho}) \right] u(p_i) \\
\text{g}-\text{pole}: \quad & \text{Amp}_d = \frac{\kappa^2}{k_i \cdot k_f} \bar{u}(p_f) \left[(\not{\epsilon}_i \epsilon_f^* \cdot k_i + \not{\epsilon}_f^* \epsilon_i \cdot k_f)(\epsilon_i \cdot p_i \epsilon_f^* \cdot p_f - \epsilon_i \cdot p_f \epsilon_f^* \cdot p_i) \right. \\
- \quad & (\epsilon_f^* \cdot \epsilon_i) \big(k_i \cdot k_f (\not{\epsilon}_f^* \epsilon_i \cdot k_f + \not{\epsilon}_i \epsilon_f^* \cdot p_i) \\
+ \quad & \not{k}_i (\epsilon_f^* \cdot p_f \epsilon_i \cdot p_i - \epsilon_f^* \cdot p_i \epsilon_i \cdot p_f) + p_i \cdot k_i (\not{\epsilon}_i \epsilon_f^* \cdot k_i + \not{\epsilon}_f^* \epsilon_i \cdot k_f) \big) \\
+ \quad & \left. (\epsilon_f^* \cdot \epsilon_i)^2\ \not{k}_i (p_i \cdot k_i - \frac{1}{2} k_i \cdot k_f) \right] u(p_i)
\end{aligned}
\tag{94}
$$

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$$H = \sqrt{m^2 + \vec{p}^2}$$

is replaced by

$$H - e\phi = \sqrt{m^2 + (\vec{p} - e\vec{A})^2}$$

Making the quantum mechanical substituations

$$H \to i\frac{\partial}{\partial t} \qquad \vec{p} \to -i\vec{\nabla}$$

we find the minimal substitution given in the text.

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