Optimal Acquisition and Sorting Policies for Remanufacturing over single and Multiple Periods

OPTIMAL ACQUISITION AND SORTING POLICIES FOR REMANUFACTURING OVER SINGLE AND MULTIPLE PERIODS

A Thesis Presented

by

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To my parents, Yuya and Xiaodi.

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TABLE OF CONTENTS

LIST OF TABLES

Table **Page**

LIST OF FIGURES

Figure **Page**

CHAPTER 1

INTRODUCTION

The drastic development in industry and increase in population during the past centuries has caused the natural resources depletion, wide spread pollution and increase on land for food production. As a result, many countries and environmental protection agencies enforce stricter regulations (such as the European Union's WEEE directive 2003) for the companies to assume responsibilities for the disposal of their products as well as reduce their waste.

Those strict government regulations, along with the increasing number of environmentally conscious consumers and the progress in ecological design have been bringing the handling of used products to the forefront of business priorities ([7, 8, 26, 28, 37, 41]). Consequently, manufacturers are encouraged to take interest in product recovery and the after-sale market.

Although the remanufacturing industry emerged under these environmental concerns, it is becoming a profitable business model ([14, 16, 21, 29, 30, 31, 32]). Moreover, in contrast to the common belief that the remanufactured products may cannibalize the market of new products, researchers have found that remanufacturing may increase market share under the right circumstances ([20, 31, 36]). Therefore, remanufacturing industry, along with its high labor-intensity ([14, 30]) as well as the contribution to energy and material conservation, provides economic, environmental and social benefits.

According to Hauser and Lund [33], the size of the remanufacturing sector in the United States is $53 billion, with over 70,000 firms and 480,000 employees. The

average profit margin is estimated to exceed 20% [43]. Throughout the world, the total size of remanufacturing has reached more than $100 billion. Many independent businesses have emerged to exploit the potential profitable remanufacturing opportunities [52] such that today the Original Equipment Manufacturers (OEM) only account for five percent of the remanufacturing industry's total sales. Today, there are three sectors in this industry: the OEM and their first tier suppliers, the OEMs' subcontractors, and independent remanufacturers [46].

In this thesis, we study one major problem in remanufacturing, namely, sorting policies that specify which returned items should be remanufactured and which should be scrapped. We examine a remanufacturer who acquires used products from third party brokers or directly from the market in multiple periods. Due to the high variability of those acquired items' condition, the remanufacturer has to make three related decisions: (1) how many used items to acquire; (2) how selective to be during the sorting process and (3) how many remanufactured products to be kept in inventory for future demands. While the deterministic demands are always satisfied without backlogging, postponing acquiring may lead to a lower per-unit acquiring cost and a higher per-unit remanufacturing cost, as we take the condition deterioration into account. Moreover, as more items are acquired, and hence a higher per-unit acquiring cost, for a given demand, the remanufacturer can be more selective when sorting, and thus a lower per-unit remanufacturing cost. We study the trade-off among these factors and derive optimal acquisition, remanufacturing and inventory quantity in the presence of product condition variability for a remanufacturer facing deterministic demand. In the following sections, we first review the definition and applications of remanufacturing. Two cases, Fuji Xerox and Kodak, are studied. In Section 1.2, we review the challenges in remanufacturing research given by Guide [22], followed by an introduction on product acquisition management and sorting policies, which is the

focus of our work. At the end of this chapter, we state our research objectives and introduce the outline of this thesis.

1.1. Remanufacturing

According to The Remanufacturing Institute (http://www.reman.org/faq.htm), a product is defined as remanufactured if:

- *Its primary components come from a used product.*
- *The used product is dismantled to the extent necessary to determine the condition of its components.*
- *The used product's components are thoroughly cleaned and made free from rust and corrosion, if needed.*
- *All missing, defective, broken or substantially worn parts are either restored to sound, functionally good condition, or they are replaced with new, remanufactured, or sound, functionally good used parts.*
- *To put the product in sound working condition, such matching, rewinding, refinishing or other operations are performed as necessary.*
- *The product is reassembled and a determination is made that it will operate like a similar new product.*

In this thesis, we apply this definition and use the term "remanufacturing" interchangeably with "refurbishing" and "reconditioning".

However, not all products are suitable for remanufacturing. It depends on the availability of returned cores, the product's life span and the technical progress regarding new versus remanufactured products [48]. Usually, before the remanufacturing decision is made, a firm needs to determine whether such remanufacturing can be realized with current technologies in mass production scale. Currently, auto-parts, tires, furniture, laser toner cartridges, and computers and electrical equipment are among the appropriate products for remanufacturing. Other typical remanufactured products include mattresses and carpets.

Fuji Xerox and Kodak are good examples of firms that are successfully supplementing their business with remanufacturing capability.

Fuji Xerox

Fuji Xerox, a joint venture between Japan's Fuji Photo Film Co Ltd and the US Rank Xerox Limited from 1962, is one of the world's leading manufacturers of office equipment. By the end of 2006, this joint venture had 40,295 employees and generated 1,163.3 billion Japanese Yen revenue around the world (www.fujixerox.com).

Fuji Xerox started its business by renting its products to customers and then moved on to selling and leasing the products. Through its early practice in serving contracts with repairing or replacing consumables, the firm developed its remanufacturing capability and its closed loop supply chain. In 2000, Fuji Xerox became the first to achieve zero landfill of used products in Japan. It also built its remanufacturing facilities in Sydney in 1993, where valuable components had been considered waste even with minor defects. It is believed the new operation even improves the performance of the products [5].

In 2004 alone, Fuji Xerox diverted 128 million pounds of material from entering landfills through part recycling and reuse. From reuse alone, it saved approximately 11 million therms of energy and around 70,000 machines were remanufactured. At the same time, Fuji Xerox remains one of the most profitable firms in its industry.

Kodak

Kodak introduced its single-use cameras in 1987 to capture the market of occasional users. After taking the pictures, the cameras had to be returned to a photofinisher for printing. This is the reason such cameras are called "disposables" or "throwaways".

The product had an outstanding picture quality and received an immediate success among customers who were looking for budget picture taking solutions. However, some environmental groups raised concerns about the potential for a significant increase in the amount of solid waste generated.

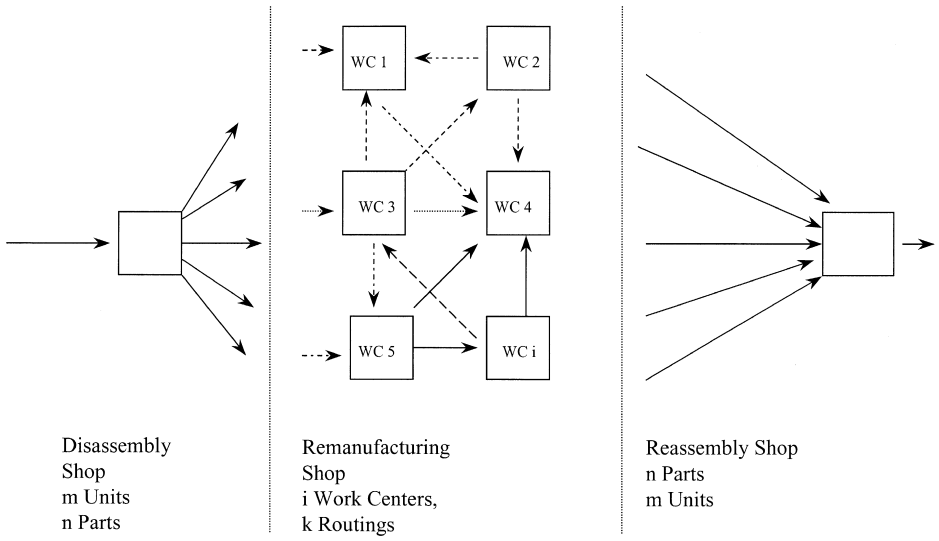

Figure 1.1. Structure of a remanufacturing facility. Adapted from Guide 2000 [22]

Therefore, Kodak began to redesign its single-use cameras in 1990 [24], which led to a remarkable global recycling and re-use program. The redesign facilitated recycling and re-use of parts. Over the years, the Kodak cameras have been so well designed that up to 90% of the product is remanufacturable. The remaining parts can be recycled such that virtually no part is sent to a landfill.

Today, the recycling rate for one-time use cameras in US is greater than 75%, the highest among all consumer products in this country. More than 90% of the parts of a new camera sold orginates from remanufacturing. The number of Kodak one-time-use cameras recycled is now more than 800 million. When considering competitors cameras that are also collected through Kodak, the total number of recycled exceeds one billion cameras.

While each firm may face its specific problem in remanufacturing, several common challenges have been identified by prior research.

1.2. Challenges in Remanufacturing

Guide divides the remanufacturing facilities into three sub-systems: disassembly, processing, and reassembly (see Figure 1.1). Given its high dependence on the stream of returned products, remanufacturing is quite different from traditional manufacturing. In his well cited article regarding potential research topics, Guide lists seven characteristics which complicate the production planning and control activities in remanufacturing [22]. They are:

1. The uncertain timing and quantity of returns. The uncertainty of a product's life cycle and the change of technology and market preference cause the variability of timing and amount of available cores. This issue is made worse as, in the early life phase when the remanufactured product may have more market share, there are fewer cores on the market, compared to the late phases. Firms are trying to attack this problem by some form of core deposit system, i.e., trying to generate a core when a remanufactured product is sold. Trade-in, charging a premium if no core is returned, and leasing instead of selling are among the examples. However, such practices have not solved the built-in uncertainty and remanufacturing firms still report that core inventories account for one-third of the inventory carried [43].

2. The need to balance returns with demands. Meeting demands with supply is important to profit-maximizing firms and has been studied extensively in the traditional production planning literature. It is well known that both lost sales and excessive inventory cut profit. However, in remanufacturing, supply and demand have a more complicated relationship; that is, they are coupled together. The excess or scarcity of cores depends on the demand met previously.

3. The disassembly of returned products. Disassembly, as the first step in the operations, involves inspecting the product and retrieving the components for

further processing. Products are disassembled to the part level, assessed as to their remanufacturability and acceptable parts are then routed to further processing. Parts that can not meet minimum remanufacturing standards may be used for spares or sold for scrap value. Moreover, products designed for easy assembly may not be well designed for disassembly. These factors in disassembly cause high variability in disassembly time and thus the lead time in remanufactured products. In fact, Guide reports the coefficients of variance for disassembly time can be as high as 5.0. This hurts the competitiveness of remanufacturers.

4. The uncertainty in materials recovered from returned items. Material Recoverability Rate, or MRR, is used to measure the recovery uncertainty [25]. MRR is applied to determine batch size for purchasing and manufacturing. The majority of firms report to estimate the MRR by simple average, but some sophisticated regressive models have also been used. Most common inputs to the MRR estimation model is historical data, although procurer's subjective estimation is also used sometimes. To decide the purchase batch size, dynamic lot sizing techniques are most commonly used. The concerns with the purchasing process includes long purchase lead time, sole suppliers for some part, uncertainty in demand and small purchase orders.

5. The requirement for a reverse logistics network. A reverse network is needed to collect the returns from the end user to the remanufacturer. For high value cores, trade-in systems, which connect the customers directly with the firms, is a common practice. However, for low value items, alternatives are required to motivate the returns. The three most used methods are core brokers, third party agencies and seed stock.

6. The complication of material matching restrictions. Some remanufacturing practices are complicated by special material matching restrictions. For example, in after-market aviation maintenance industry, remanufacturing orders are mostly driven by customers who retain the ownership of the product and require the same unit returned. This obliges the coordination between disassembly, reprocessing and assembly operations. Concerns raised here include short planning horizons and poor visibility for replacement parts, which are familiar to make-to-order production systems. The material matching restrictions may also pose a high burden on scheduling and information systems. For products with complicated structures, tens of thousands of parts may need to be numbered, tagged and tracked.

7. The problems of stochastic routings for materials for remanufacturing operations and highly variable processing times. Given the variable condition of the returns, each item may have its unique requirement of processing steps. For example, while all the returns may have to go through the cleaning process, other routings may be probabilistic and highly dependent on the age and condition of the part. This causes difficulties in estimating flow time and planning both machine and labor resources. Determining machine structure and setup times also become complicated. This characteristic is claimed to be the single most complicating factor of lot sizing decisions and scheduling.

There are several activities remanufacturers are practicing to handle the issues listed above. In the next section, we review these activities and our focus is on Production Acquisition Management, the major interest of this thesis.

1.3. Product Acquisition Management and Sorting Policies

There are three primary groups of activities in remanufacturing [31], namely, Production Returns Management (PRM), Remanufacturing Operational Issues and

Remanufactured Products Market Development, and each of them has seen extensive work during the years. PRM, which includes Product Acquisition Management (PrAM), studies the issues related to the timing, quantity, and quality of returned products ([3, 49]). Remanufacturing Operational Issues focus on reverse logistics, testing, sorting and disposing, product disassembly and remanufacturing processes ([2, 19, 44]). The Remanufactured Products Market Development includes remanufactured products marketing strategies and channels, market competition as well as cannibalization ([6, 9, 38, 51]). In this paper, our major work focuses on PRM, or to be more specific, PrAM.

Among the challenges listed in the last section, PrAM mainly helps to handle (1) uncertainty in timing and quantity of returns; (2) balancing returns with demands; (4) material recovery uncertainty; (5) reverse logistics and (6) customer-required returns. In Guide et al. 2000 [23], the authors report six responsibilities for PrAM activities:

1. Core acquisition. While core acquisition is mostly a purchasing function, inputs from operations are necessary to develop criteria in evaluating the condition of the cores acquired from various sources. These criteria may include quality, cost, and quantity.

2. Forecasting core availability. Core availability is critical to production planning in remanufacturing. Factors affecting availability include the product life-cycle position, rate of technological change and economic conditions. Given the coupling nature of core supply and previous demand, forecasting involves the cooperation of purchasing, operations and marketing. A better forecast will save inventory holding or lost sales.

3. Synchronizing return rates with demand rates. As shown in the last section, in remanufacturing, both demand and core supply need to be forecast. This calls for the cooperation between purchasing and marketing to match the return

rate with demand rate for remanufactured products. This problem is further complicated by the uncertainty in the material recovery rate, which in turn calls the inputs from operations.

4. Coordinating replacement materials. New parts and components are needed to replace materials that are technologically or economically unrecoverable. Due to the variability in quality of the returns, coordination between purchasing and operations on an on-going basis is often required.

5. Resource planning. Uncertainty in quality, quantity and time of the returns make it difficult to allocate and schedule machine and labor. Therefore, PrAM is also responsible for the inputs to capacity planning for remanufacturers.

6. Reducing the uncertainties in returns. The responsibilities listed above help to reduce the associated costs at an operational level. However, firms need to develop long term strategies to reduce the inherited uncertainty rooted in the return process. Core deposits, leasing and trade-ins are potential alternatives currently being studied.

These six responsibilities all involve firm-wise cooperation, because of the coupled (looping) nature of remanufacturing. Guide et al. 2000 [23] summarizes the involvement with a table (Table 1.1).

Table 1.1. Responsibilities of PrAM. Adapted from Guide 2000 [23]

Responsibility	Operations	Purchasing	Marketing
(1) Core acquisition		√	√
(2) Forecasting core availability		√	√
(3) Synchronizing returns with demands	√	√	√
(4) Coordinating materials replacement	√	√	√
(5) Resource planning	√	√	
(6) Reducing uncertainty in returns		√	√

The acquisition/sorting policy refers to the operational decision on which returned products should be remanufactured and which should be scrapped [18]. Not all returns may be remanufactured due to the technological or economical constraints. Items that cannot be recovered have to be scrapped; for the sake of simplicity, we do not consider other alternatives, such as recycling. Therefore, to satisfy certain demand, firms need to acquire more cores than the projected demand. In general, a less stringent sorting will generate more available cores and thus less returns are needed to be acquired. This lowers the total acquisition cost. However, this lower selectivity may require sophisticated recovery technologies and command a higher per unit recovery cost. The interaction between these two factors will drive the optimal acquisition/sorting policy in any single period.

1.4. Research Objectives and Outline

In this thesis, we extend the earlier work of Galbreth and Blackburn [18] to a multi-period model. The work of [18] will be covered in detail in Chapter 3. As in their paper, we assume the per unit remanufacturing costs are convex increasing functions in quality, to capture the fact that the remanufacturing cost increases as quality decreases.

For many consumer goods, the longer the product has been used, the worse its condition and thus, the lower its buy-back cost. We capture this effect by assuming the acquisition cost decreases in time while the remanufacturing cost increases in time.

The remanufacturer follows a "make from stock" model; i.e., products are acquired and available as needed to meet remanufacturing needs. A typical example is cellular phone industry where used products are purchased from brokers as needed to fulfill specific demands.

The remainder of this thesis is organized as follows. In Chapter 2, we will review the literature most closely related to our work. In Chapter 3, after we introduce the model developed by Galbreth and Blackburn, we extend their work to a more general case. The study and results of multi-period problem are presented in Chapter 4, where we first introduce the multi-period model, show the convexity of the general objective function and then study a special case – linear acquisition cost. We also extend our work to relax the no-core-holding constraint at the end of Chapter 4. We conclude with the outline and prospectus of future study in Chapter 5.

CHAPTER 2

RELATED LITERATURE

It is well acknowledged that variability in supply quality, quantity and timing poses a huge challenge to remanufacturing ([1, 22, 23, 47]) and extensive literature can be found in the study of such uncertainty. For example, Toktay et al. [50] use a queuing network model to study the uncertainty in processing time. Simulation models are also developed. Humphrey et al. [34] simulate the variable repair requirements in a reverse logistics network. Guide et al. [27] also incorporate stochastic processing time into their simulation model in an evaluation of various order release strategies.

Many aspects of the variability in quality of the returns have been studied. Ferguson and Toktay [13] assume the variable cost of collection increases in the quantity of the products collected/processed in a single period and use this model to study the cannibalism between the new and remanufactured products.

In his Remanufacturing Aggregate Production Planning (RAPP) model, Jayaraman [35] proposes to capture the variability of returned products' condition through a discrete distribution of nominal quality, which is expressed as an n-dimensional vector. RAPP is a unit cost minimizing linear programming model, in which the yield is assumed to be a constant within each quality category. Given the linearity nature of his model, costs are all assumed proportional to the total number. Bakal and Akcali [4] extend this work, still in a single-period setting, to a problem where both the quantity and quality of the returned products depends on the acquisition price offered. Moreover, the remanufacturer has control over the demand for the final products by pricing.

In Ferrer [15], it is found the information of yield is generally quite valuable under infinite horizon using Markov Decision Processes. Ketzenberg et al. [39] further extend it to consider uncertainty in demand, returns along with yield.

Guide [22] identifies the need to estimate used product condition to determine the appropriate disposition, which is an important step in determining the optimal recovery action. Therefore, an inspection decision is a necessity. However, the sorting policies in remanufacturing have received limited scrutiny in the literature. Guide and Wassenhove [30] study the optimal acquisition policies and pricing, where they assume products are acquired from third-party brokers with quality categories and known remanufacturing cost.

The work most closely related to ours is in Ferguson et al.[11, 12], and Galbreth and Blackburn [18]. In [11], the authors study a multi-period production planning problem and assume the number of returns in each period is known. Returns are tested and categorized into a finite number of nominal quality categories, such as, good, better and best. For each category, the remanufacturing cost is assumed to be a constant such that the cost structure is a piecewise linear convex curve in the total quantity of products remanufactured. This work is further extended to include uncertainty in quality levels with stochastic programming in [11]. The model is solved numerically and, given its nature of exponential increase in the number of outcomes, its computational complexity of this model grows very fast.

In Galbreth and Blackburn [18], the authors examine the single period case in which a remanufacturer acquires unsorted products from third party brokers in the presence of product condition variability. As more used items are acquired for a given demand, the remanufacturer can be more selective when sorting. Two related decisions are made: how many used items to acquire, and how selective to be during the sorting process while the fixed demand is always satisfied. The existence of a single optimal acquisition and sorting policy with a simple structure is proved. The authors

also show that the policy of selectivity is independent of the production amount when acquisition costs are linear. In [17], those authors extend it to study stochastic yield. They find that the deterministic yield is often a reasonable approximation to its stochastic counterpart.

Our major objective in this thesis is to extend the single period model in Galbreth and Blackburn [18]. While these authors have studied the single period problem when acquisition cost is linear, in the next chapter, we will reproduce their work and then continue to study when the acquisition cost is piecewise linear convex instead.

CHAPTER 3

SINGLE-PERIOD OPTIMAL ACQUISITION AND SORTING POLICIES

In this chapter, we formulate the single-period model and then reproduce the work by Galbreth and Blackburn [18] when the acquisition cost is linear. We will then study the case when the acquisition cost is a piecewise linear convex function.

3.1. Model Assumptions

Many electronic products have relatively short life cycles. For example, fast growing communication techniques have been making the cell phone devices' life much shorter than their functional life. For these products, a single-period model is reasonable given that future demand is not guaranteed.

As mentioned in Chapter 1, to simplify our model, we assume the returned cores are either remanufactured or scrapped. No other alternative such as raw material recycling is considered. The extension to include those alternatives should not change the sorting and acquisition decision if scrapping cost is introduced into the model, as they may be absorbed into the remanufacturing yield curve. Figure 3.1 shows the flowchart of the remanufacturing processes. At the beginning of the planning period, the firm needs to determine the total amount (p) to acquire from the market or dealers. Given that it is not economically feasible to remanufacture low quality products, the firm needs to decide, among the acquired, r items to be remanufactured and the rest $p-r$ to be scrapped ($p \geq r$). Perfect testing, which means that at the time of sorting, the corresponding remanufacturing cost is known based on the observed condition,

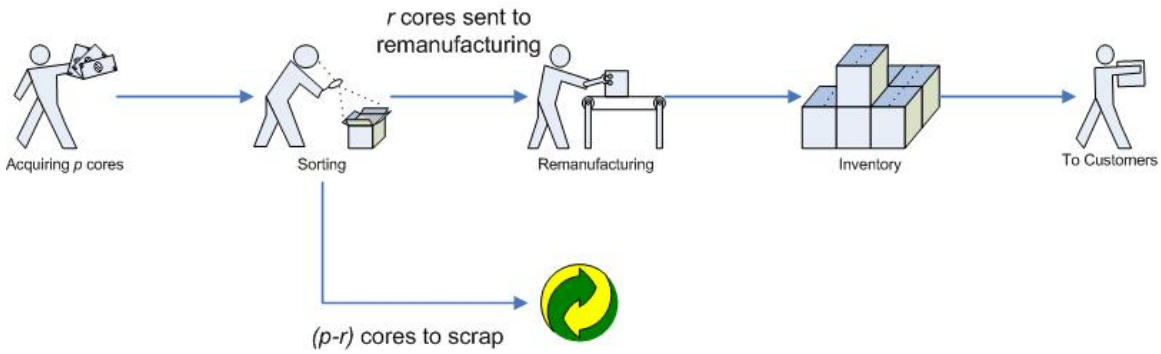

Figure 3.1. Remanufacturing Flowchart

is another assumption here. This process is defined as sorting and we assume it follows immediately after the acquisition. That is, as each product is processed, it is sorted into one of two categories, remanufacture or scrap. Initially, we assume the demand is deterministic and no backlogging is allowed. Moreover, we assume the remanufacturing cost is composed totally of the variable cost with negligible fixed cost. Hence, for single period, the only decision variable is the total amount of acquisition p. As we will see in the subsequent sections, a lot of assumptions made here will be relaxed for the linear acquisition cost case.

In Galbreth and Blackburn [18], the authors define sorting policy as the value of p when the distribution of condition of used products and the demand are fixed and known. Our model follows the same definition. While the condition may be represented in many different ways, for example, the processing time, we use the remanufacturing cost as the proxy for product condition in the study.

As the acquisition quantity is increased, sorting can be made more stringent– only products with lower remanufacturing costs are actually remanufactured. Of course, the cost of acquiring more used products offsets, to some extent, the remanufacturing cost savings achieved by increased selectivity. The purpose of the study is to model the variable returned product condition via remanufacturing cost and try to find the optimal acquisition and sorting decision in the sense of lowest total cost.

Following the same approach as Galbreth, we define "remanufacturing yield" as the percentage of the returned cores that are actually remanufactured; that is, r/p. Higher selectivity is equivalent to lower yield, higher pre-remanufacturing quality, and thus, lower remanufacturing cost.

With these assumptions in mind, we introduce the notation to be used and the two cost components, namely, the buy-back cost and the remanufacturing cost in the next section.

3.2. Model Description

3.2.1 Notation

Before we develop the model, we list the notations below. The details of the cost structure, the acquisition cost and the remanufacturing cost, will be given in the next section.

D**:** the demand for remanufactured products;

p**:** the total number of items bought back, which is the only decision variable in this model;

c_y : the remanufacturing cost threshold; we use the subscript y, since this threshold is uniquely related to the yield, with the relationship ($G(c_y) = \frac{D}{p}$) explained in the next section;

$Z(p)$**:** the total buy-back cost given p items to buy back. To capture the fact of increasing marginal cost, we assume this function to be linear or convex;

$G(x)$**:** the cumulative distribution function, or yield function, of a used product's remanufacturing cost. In other words, $G(x)$ represents the fraction of the returns that can be remanufactured at cost less than or equal to x.

$g(x)$**:** the corresponding density function of $G(x)$;

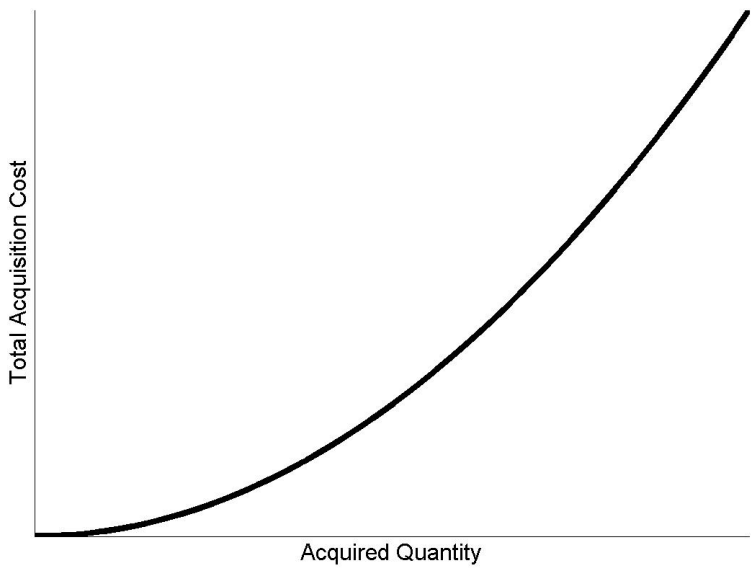

Figure 3.2. Acquisition cost vs. acquired amount

BC**:** the total buy-back cost;

RC**:** the total remanufacturing cost;

3.2.2 Buy-back Cost

We assume the marginal acquisition cost is a non-decreasing function in acquired quantity, because of scarcity and we denote it as, shown in Figure (3.2) as

$$BC = Z(p) \tag{3.1}$$

Here, $Z(p)$ is linear or convex increasing function of their arguments p, or $Z''(p) \geq 0$.

3.2.3 Remanufacturing Cost

Figure 3.3 illustrates the total remanufacturing cost versus remanufactured yield, when the total acquired amount is fixed; that is, it represents cost as a function of yield.

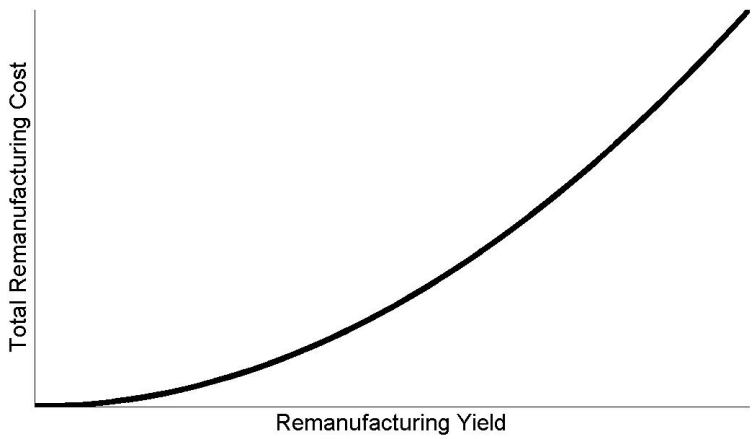

Figure 3.3. Remanufacturing cost vs. remanufacturing yield

For a given acquired quantity, the remanufacturing quantity is equivalent to the remanufacturing yield (D/p) and we re-plot its relationship to remanufacturing cost in Figure (3.4). Because we may use the remanufacturing cost to represent the condition of the returned core, the yield function can be viewed as a cumulative distribution function for the quality and we denote it as $G(c_y)$. We also denote its corresponding probability density function as $g(c_y)$. When the remanufacturer decides to remanufacture products at cost not higher than c_y (we term it remanufacturing cost threshold, or the "cut-off" cost), the yield it can realize is given by $y = G(c_y)$, as shown in Figure (3.4).

Hence, the total remanufacturing cost is

$$\begin{aligned} RC &= D\frac{\int_0^{c_y} xg(x)dx}{\int_0^{c_y} g(x)dx} \\ &= p\int_0^{c_y} xg(x)dx \end{aligned} \tag{3.2}$$

3.3. Optimization Problem

For the single period model, we have the total cost minimization problem as

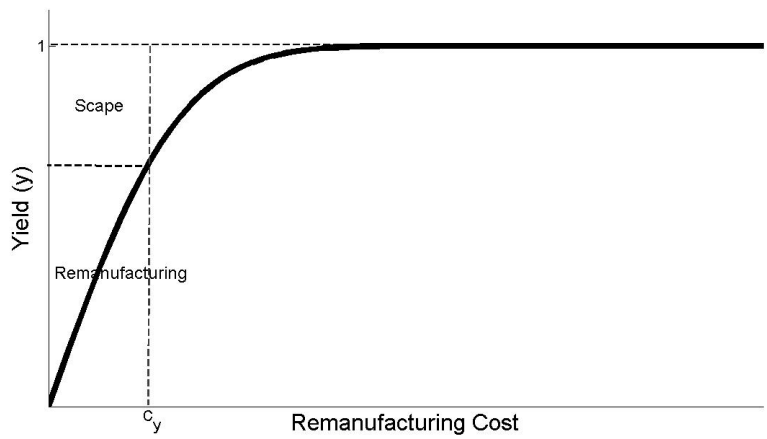

Figure 3.4. Remanufacturing cost vs. yield

$$\begin{aligned} \text{Problem} S & \\ \min C_S &= BC + RC \\ &= Z(p) + p\int_0^{c_y} xg(x)dx \end{aligned}$$

It can be proved that this problem has the following properties, as shown in Galbreth and Blackburn [18]:

Proposition 1 *For any given remanufacturing amount D, remanufacturing cost is a convex monotonically increasing function of the acquisition amount p.*

Proof

$$By definition, we have G(c_y) = \frac{D}{p}$$

We take the first order derivative of this equation over p and have

$$G'(c_y)\frac{\partial c_y}{\partial p} = \frac{-D}{p^2} \tag{3.3}$$

After reorganization, we have

$$\frac{\partial c_y}{\partial p} = \frac{-D}{p^2 G'(c_y)} \tag{3.4}$$

If we take partial derivative of Eqn. (3.4) over p and substitute (3.3) into the result, we have

$$\frac{\partial^2 c_y}{\partial p^2} = \frac{2D}{p^3 G'(c_y)} + G''(c_y)\frac{D^2}{p^4[G'(c_y)]^3} \tag{3.5}$$

Therefore, if we take the first order derivative of RC over p, we have

$$\begin{aligned}
\frac{\partial RC(p)}{\partial p} &= \frac{\partial p \int_0^{c_y} xg(x)dx}{\partial p} \\
&= \frac{\partial c_y D - p\int_0^{c_y} G(x)dx}{\partial p} \\
&= D\frac{\partial c_y}{\partial p} - \int_0^{c_y} G(x)dx - pG(c_y)\frac{\partial c_y}{\partial p} \\
&= -\int_0^{c_y} G(x)dx \\
&\geq 0
\end{aligned} \tag{3.6}$$

If we take derivative of (3.6) over p, we have

$$\frac{\partial^2 RC(p)}{\partial p^2} = \frac{D^2}{p^3 G'(c_y)} \geq 0 \tag{3.7}$$

Combine Eqn. (3.6) and 3.7, we know $RC(p)$ is monotonously increasing convex function. ∎

Proposition 2 *Given any convex acquisition cost, the total cost $C_S(p)$ is convex. Therefore, there is an optimal acquisition amount p^* and corresponding optimal remanufacturing cost threshold c_y^* which minimizes total average costs to meet a fixed demand.*

Proof From Proposition 1, we know $RC(p)$ is convex. Therefore, as long as the acquisition cost $BC(p)$ is convex, their summation, $C_S(p)$ is also convex. ∎

3.4. Model Analysis

3.4.1 Linear Acquisition Cost

As we have proved the convexity of the objective function and hence the existence of the minimal total cost, we show how to obtain this optimum when the acquisition cost is linear. That is, we assume

$$Z(p) = bp$$

where b is the per unit acquisition cost.

If we take the first order condition of the objective function over p and substitute the Eqn. (3.6) into the result, we will have

$$b = \int_0^{c_y} G(x)dx \tag{3.8}$$

From Eqn. (3.8), we have the following proposition:

Proposition 3 *In a single period, if the acquisition cost is linear, the minimal cost can be achieved at an optimal yield, which is independent of the actual acquired amount.*

Proof We define

$$F(x) = \int_0^x G(t)dt$$

As $G(t)$ is an positive increasing function, the inverse of $F(x)$ exists. Therefore, we can solve Eqn. (3.8) and find the optimal cost threshold

$$c_y^* = F^{-1}(b)$$

and the corresponding optimal yield

$$y^* = G(c_y^*)$$

∎

Since the optimal yield is independent of the total acquisition quantity, the deterministic demand restriction can actually be relaxed. Therefore, as long as the acquisition cost is linear in the quantity, the optimal acquisition and sorting problem may be solved as general inventory problems which may include setup costs, backlogging and uncertain demand.

3.4.2 Piecewise Linear Cost

While Galbreth and Blackburn [18] have shown the convexity of the objective function when the acquisition cost is convex increasing, they stop there and continue to work on linear acquisition cost for their remaining study. In this section, we extend this work and analyze the case when acquisition cost is a piecewise linear convex function. Assume the acquisition cost has the form

$$Z(p) = \begin{cases} b_1 p + \eta_1, & \texttt{if } \ p \in [0, q_1) \\ b_2 p + \eta_2, & \texttt{if } \ p \in [q_1, q_2) \\ \quad \cdots & \\ b_m p + \eta_m, & \texttt{if } \ p \in [q_{m-1}, q_m) \end{cases} \tag{3.9}$$

We prove this case has the following properties.

Proposition 4 *The optimal cost threshold c_y^* of the piecewise linear acquisition cost problem is*

$$c_y^* = \begin{cases} c_{y^i}^*, & \textit{if } \ q_{i-1} G(c_{y^i}^*) \leq D < q_i G(c_{y^i}^*) \\ G^{-1}(\frac{D}{q_i}), & \textit{if } \ q_i G(c_{y^i}^*) \leq D < q_i G(c_{y^{i+1}}^*) \end{cases}$$

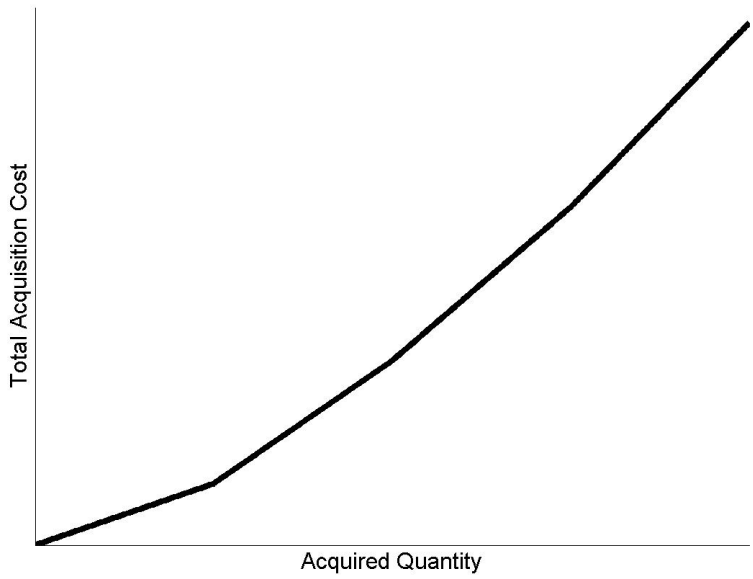

Figure 3.5. Piecewise acquisition cost

where $c^*_{y^i}$ stands for the optimal cost threshold when the acquisition cost is purely linear with the same coefficient as the i-th segment. That is, $c^*_{y^i}$ is the optimal cost threshold when the acquisition cost is purely linear and has the form of

$$Z(p) = b_i p$$

Proof We prove this by induction.

1. For $i = 1$, or the first linear segment, we have $q_{i-1} = 0$.

   - Assume $D \in [0, q_1 G(c^*_{y^1}))$. This is equivalent to the purely linear problem P^1_L with the total acquisition cost as

     $$Z_1(p) = b_1 p$$

     As this is the lowest acquisition cost segment we may get, it is optimal to remain in this interval if the demand allows us to. Therefore, the opti-

mal yield we should follow is the optimal yield for problem P_L^1, which we denoted as $G(c_{y^1}^*)$. We can see this by contradiction: if we have another optimal yield $G(c_y^*) \neq G(c_{y^1}^*)$, which gives a lower total cost, then $G(c_y^*)$ should also be the optimal yield for problem P_L^1, instead of $G(c_{y^1}^*)$. This contradicts with our conclusion in the last section.

- Assume $D \in \Big[q_1 G(c_{y^1}^*), q_1 G(c_{y^2}^*)\Big)$. The optimal acquisition amount will be fixed at $p = q_1$ and the optimal yield will increase linearly as the demand D increases. This can also be proved by contradiction: if it does not hold, the "optimal acquisition quantity" will lie in $[0, q_1)$ or in $(q_1, q_2]$. We may define two corresponding linear problem P_L^1 and P_L^2, so that their respective acquisition costs are

$$\begin{aligned} Z_1 &= b_1 p \\ Z_2 &= b_2 p \end{aligned}$$

  Suppose the quantity is in $[0, q_1)$. Follow the similar argument as the proof above, we know problem P_L^1 will have a new optimal yield $G(c_y^*) \neq G(c_{y^1}^*)$, which contradicts with the fact that $c_{y^1}^*$ is its optimal.

  Similarly, we can argue the acquisition quantity can not be in $(q_1, q_2]$. Therefore, it is bounded at $p = q_1$. While the demand increases, only the yield $(\frac{D}{p})$ increases proportionally, until the cut-off cost reaches $c_{y^2}^*$. When the demand further increase after this, the acquisition quantity will increase and enter the second linear segment.

2. Assume our proposition holds for all $i < k$.

3. Now we prove our proposition holds for $i = k$.

- Assume $D \in [q_{k-1}G(c^*_{y^k}), q_kG(c^*_{y^k}))$. When the demand increases from below into this region, the decision maker may choose to keep the previous yield or apply a new (and higher) yield. We prove it is optimal to apply the yield $G(c^*_{y^k})$ in this interval by contradiction. Assume some other yield $G(c^*_y) \neq G(c^*_{y^k})$ leads to a lower total cost. Following similar arguments in $i = 1$ case, we know this contradicts with the fact that $G(c^*_{y^k})$ is the optimal yield for the corresponding purely linear problem P^k_L. Hence, we know $G(c^*_y) = G(c^*_{y^k})$
- Assume $D \in [q_kG(c^*_{y^k}), q_kG(c^*_{y^{k+1}}),)$.As in $i = 1$, when the demand further increase to this region, the optimal acquisition quantity will be fixed at $p = q_k$ so that the optimal yield increases linearly as the demand increases. Otherwise, it will contradict with the optimal yield we arrived for the purely linear problems.

Therefore, combining the proof above, our proposition holds for all i's.

In the next section, we use a numerical example to illustrate this proposition.

3.5. Numerical Study

We proceed our study on the piecewise linear cost problem with a numerical example to generate further insight.

In this example, we assume the acquisition cost takes the functional form:

$$Z(p) = \begin{cases} p, & p \leq 2500 \\ 2p - 2500, & p > 2500 \end{cases}$$

as shown in Figure (3.6) and the remanufacturing yield curve follow a $\Gamma(5, 2)$ distribution with a mean cost of 10. The cumulative distribution and probability density functions of $\Gamma(5, 2)$ are plotted in Figure (3.7). We sweep the demand D from 1 unit

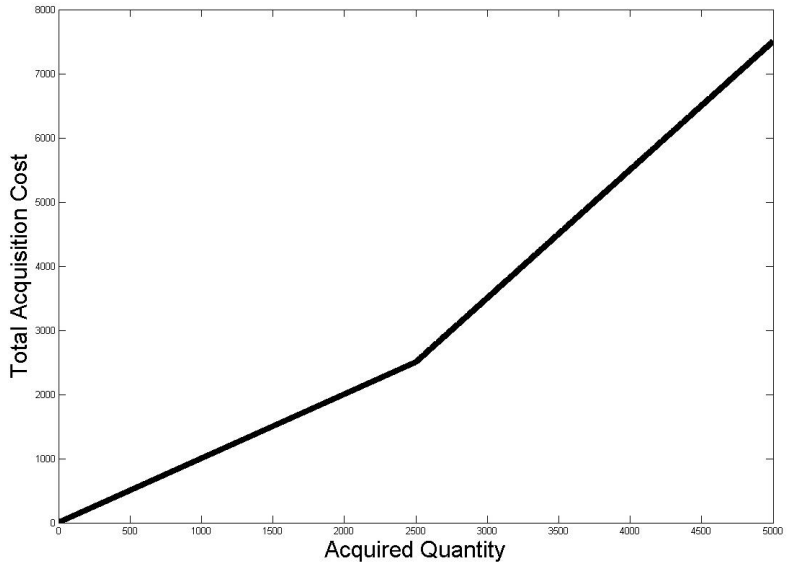

Figure 3.6. Acquisition cost vs. acquired quantity, piece-wise linear

to 4999 units and search the acquisition units from $D+1$ to $D+6000$ for the minimal total cost. Figure (3.8) illustrates the relationship between the optimal yield D/p and the demand D. From the last section, we know the optimal yield should have the form as

$$c_y^* = \begin{cases} c_{y^1}^* = F^{-1}(1) = 0.4156, & \texttt{if } \ 0 \leq \frac{D}{p} < 2500 \times 0.4156 = 1039 \\ G^{-1}(\frac{D}{2500}), & \texttt{if } \ 1039 \leq D < 2500 \times 0.5959 = 1490 \\ c_{y^2}^* = F^{-1}(2) = 0.5959, & \texttt{if } \ D \geq 1490 \end{cases}$$

which agrees well with the numerical results in the figure.

Figure (3.9) shows the relationship between the total cost and the demand. Since the optimal yield is independent on the actual acquisition quantity, we know the total cost increases linearly when the acquisition cost is purely linear. It will also hold in the piecewise linear problem when the yield is still independent on the total amount. That is, when $D \in [q_{i-1}G(c_{y^i}^*, q_iG(c_{y^i}^*))$. However, this does not hold when $D \in [q_iG(c_{y^i}^*), q_iG(c_{y^{i+1}}^*))$. In our numerical example, that is to say, the total cost

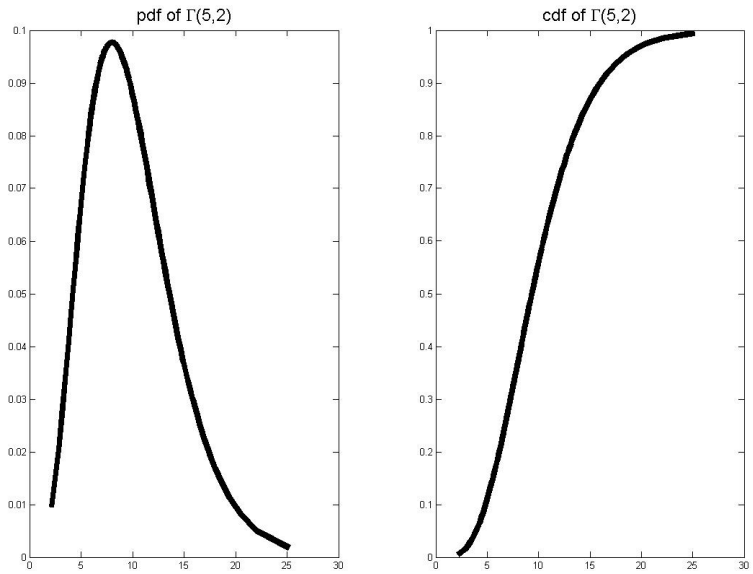

Figure 3.7. Distribution of $\Gamma(5, 2)$

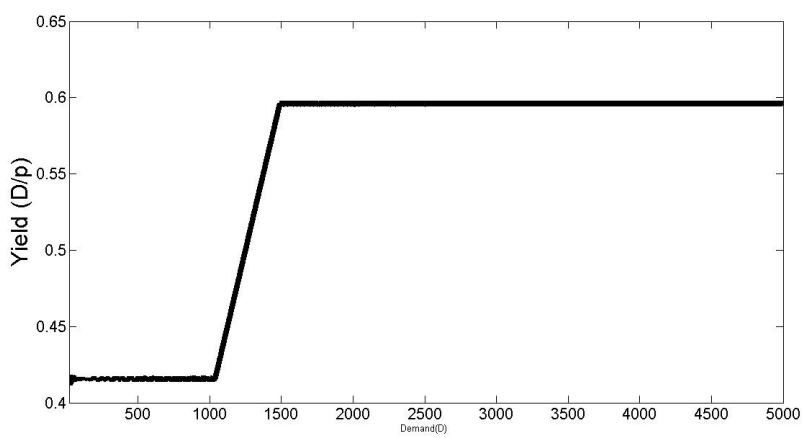

Figure 3.8. Optimal yield vs. demand

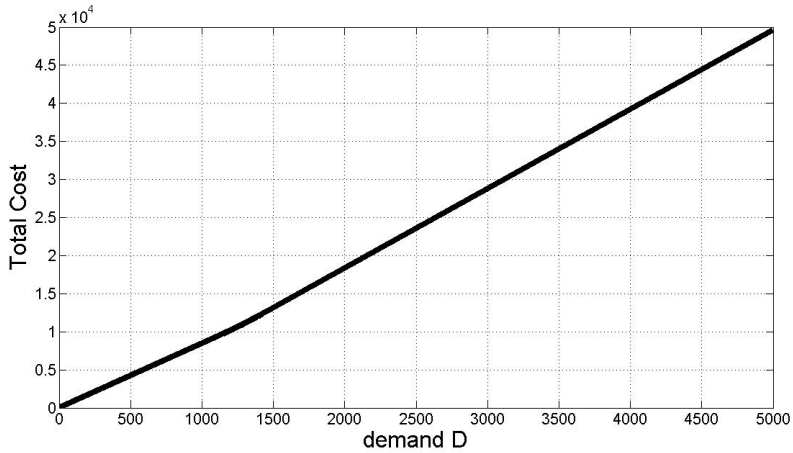

Figure 3.9. Total cost vs. demand

is not proportional to the total amount, when $D \in [1039, 1490)$. When we further increase the demand so that $D \in [q_i G(c^*_{y^{i+1}}), q_i)$, the total cost will increase linearly again. We can actually show the relationship when $D \in [q_i G(c^*_{y^i}), q_i G(c^*_{y^{i+1}}))$ by taking first order derivative of the total cost over the demand. As we know, in that region,

$$G(c_y) = \frac{D}{q_i}$$

Since the total acquisition is fixed at $p = q_i$, when we take the first order derivative over D, we have

$$\frac{\partial c_y}{\partial D} = \frac{1}{q_i g(c_y)} \tag{3.10}$$

Therefore, since

$$Z_i(p) = b_i p + \int_0^{c_y} x g(x) dx \tag{3.11}$$

we may take the first order derivative of Eqn. (3.11) and substitute with Eqn. (3.10). Then we have

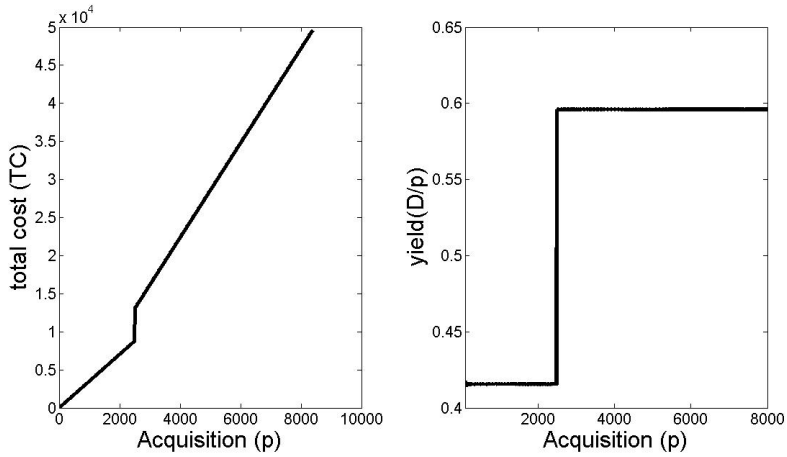

Figure 3.10. Total cost, optimal yield vs. optimal acquisition

$$\begin{aligned}\frac{\partial Z_i}{\partial D} &= c_y g(c_y)\frac{\partial c_y}{\partial D} \\ &= \frac{c_y}{q_i}\end{aligned}$$

Hence, the marginal total cost increases as the demand increases. Figure (3.10) illustrates that p is fixed in the transition region, which also agrees with our arguments in the last section.

CHAPTER 4

MULTI-PERIOD OPTIMAL ACQUISITION AND SORTING POLICIES

In this chapter, we extend the single period model to multiple periods. In section 1, we list the assumptions used. Then we explain the notation, and model components in section 2. We give the full optimization problem formulation in section 3.

After the general descriptions of the model, we prove the convexity of the total cost. At the end of this chapter, we focus on a special case: linear acquisition cost. While we have assumed only remanufactured products are kept in inventory, at the end of this chapter, we relax this constraint and show an example when the unsorted cores are kept in inventory as well. The motivation for this relaxation comes from the idea that, in some situations, it is worthwhile or cheaper to keep the unprocessed cores instead of the final products.

4.1. Model Assumptions

Here, we model how acquisition costs and remanufacturing costs vary over a planning horizon and find the optimal acquisition and sorting decision, in the sense of lowest total cost to satisfy demand over the planning horizon.

In our analysis, we assume the demand for remanufactured products at each period is deterministic. It is denoted as D_i in period i and always to be satisfied from the stock of refurbished products. That is, backlogging is not allowed here. This assumption separates the channel for remanufacturing products from the channel for new products. However, in the subsequent sections, we will see the deterministic demand restriction can actually be relaxed, thanks to the properties of this problem.

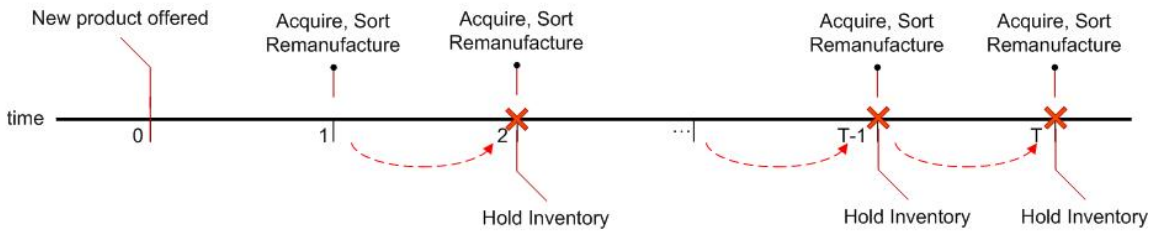

Figure 4.1. Model Time Line

Again, to simplify our model, we assume the returned cores are either remanufactured or scrapped. No other alternative such as raw material recycling is considered. Those alternatives can always be handled in our model if scrapping cost is introduced.

The flowchart of the remanufacturing processes is similar to what we have shown in Figure (3.1). At the beginning of period i, where $i = 1, 2, \cdots, T$, the firm needs to determine the total acquisition of p_i products from the market or dealers. The sorting process follows immediately after the acquisition and decides among them r_i items to be remanufactured and the rest $p_i - r_i$ to be scrapped ($p_i \geq r_i$). At this moment, we will first assume no unprocessed core will be kept in the inventory. This makes economic sense, since, otherwise, the remanufacturer should wait instead of buy back, as the acquisition cost decreases as quality decays in time. However, there are cases when the remanufacturers decide to acquire earlier so that they may obtain cores with high quality. They may decide to store unprocessed cores, for the sake of easy storage, easy transportation, and etc. At the end of this chapter, we will study the cases where unprocessed cores are allowed to be kept in inventory.

We assume demands are met from the output or the inventory at hand. Remaining items will be kept in inventory to meet future demands, at some holding cost. The time line is shown in Figure 4.1.

We keep the perfect testing assumption as before. That is, at the time of sorting, the corresponding remanufacturing cost is known based on the observed condition.

Over the T periods, the decision maker faces the following problem. As the product condition is variable, some of the units acquired may be too costly to recover and are, hence, scrapped. As the acquisition amount is increased, sorting can be made more stringent– only products with lower remanufacturing costs are actually remanufactured. Of course, the cost of acquiring more used products offsets, to some extent, the remanufacturing cost savings achieved by increased selectivity. Moreover, over the planning horizon, the acquisition cost is assumed to be a decreasing function of time, which simply means the longer the customers use the products, the less they can sell back to the remanufacturer for.

To capture the time decaying effect of product condition, the remanufacturing cost is an increasing function of time, which implies that the longer the products have been used, the more efforts the remanufacturer may have to make to recover them. The interaction of these effects will drive the acquisition amount and corresponding sorting policy.

With these assumptions at hand, we are now ready to describe the model in the next section. Firstly, we introduce the notation to be used and then the three cost components, namely, the acquisition cost, the remanufacturing cost and the inventory holding cost. After introducing the cost structure, we derive the constraints that link multiple periods.

4.2. Model Description

4.2.1 Notation

i**:** 1, 2, $\cdots$, T, subscript of periods

D_i**:** the demand for period i;

T**:** the length of the total planning horizon;

h**:** the per-unit-item and per-unit-time inventory holding cost for the remanufactured product;

p_i**:** the number of items bought back at the beginning of period i;

c_{y_i} : the remanufacturing cost threshold in period i; we use the subscript y, as this threshold is directly related to the yield;

$Z_i(p_i)$**:** the total acquisition cost at the beginning of period i, given p_i items to buy back. This should be a decreasing function of i (time);

$G_i(x)$**:** the cumulative distribution function, or yield function, of a used product's remanufacturing cost at the beginning of period i. In other words, $G_i(x)$ represents the fraction of the returns that can be remanufactured at cost less than or equal to x. This is a decreasing function of i (time);

$g_i(x)$**:** the corresponding density function of $G_i(x)$;

r_i**:** the number of items remanufactured at the beginning of period i;

I_i**:** inventory at the end of period i;

BC_i**:** the acquisition cost for period i;

BC**:** the total acquisition cost over the horizon;

RC_i**:** the remanufacturing cost for period i;

RC**:** the total remanufacturing cost over the horizon;

IC_i**:** the inventory cost for period i;

IC**:** the total inventory cost over horizon;

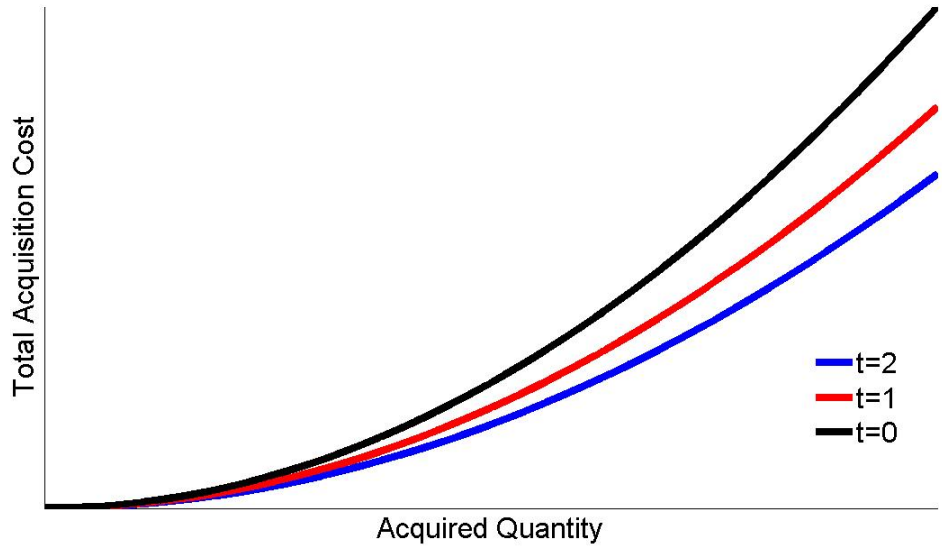

Figure 4.2. Acquisition cost for different periods.

4.2.2 acquisition Cost

As reasoned in the first section, given any fixed number of acquired units, the acquisition costs are assumed to decrease in time, which simply means the longer the customers use the product, the less they can sell back to the remanufacturer for. Figure 4.2 illustrates the total acquisition cost function family for the first three periods.

We assume the marginal acquisition cost is a non-decreasing function in acquired quantity, because of scarcity. Therefore, we denote the total acquisition cost for each period as

$$BC_i = Z_i(p_i) \tag{4.1}$$

While we do not assume specific function form here, Z_i's are linear or convex increasing function of their arguments p_i, or $Z_i''(p_i) \geq 0$ to account for the non-decreasing property of non-decreasing marginal cost.

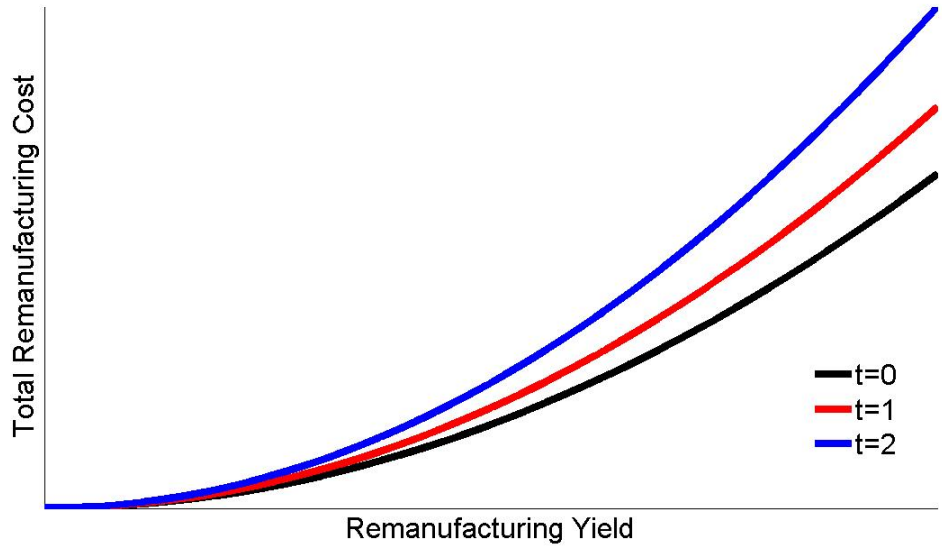

Figure 4.3. Remanufacturing cost for different periods.

The total acquisition cost is

$$\begin{aligned} BC &= \sum_{i=1}^{T} BC_i \\ &= \sum_{i=1}^{T} Z_i(p_i) \end{aligned} \tag{4.2}$$

Note here, we do not introduce extra discount factor, since it can always be introduced into the decaying acquisition cost.

4.2.3 Remanufacturing Cost

Figure 4.3 illustrates the total remanufacturing cost for the first three periods. Again, we are incorporating the idea of non-decreasing marginal remanufacturing cost into our model. However, unlike the acquisition cost, the remanufacturing cost increases in time, which implies that the longer the products have been used, the more efforts the remanufacturer may have to make to recover them.

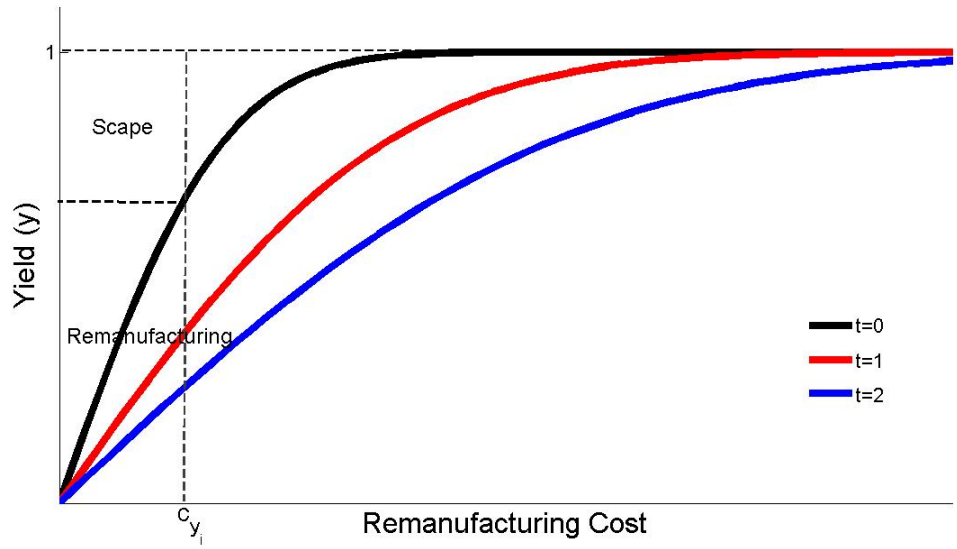

Figure 4.4. Remanufacturing cost - Yield for different periods.

For given acquisition quantity at any period, the remanufacturing quantity is equivalent to remanufacturing yield (denoted as y) and we re-plot its relationship to remanufacturing cost in Figure 4.4. Because the remanufacturing cost can represent the condition of the returned core, these yield functions can be viewed as cumulative distribution functions for the quality and we denote them as $G_i(c_{y_i})$, and the corresponding probability density function $g_i(c_{y_i})$. When the remanufacturer decides to remanufacture products at cost no higher than c_{y_i} (we term it remanufacturing cost threshold), the yield it can realize is given by $y = G_i(c_{y_i})$.

Hence, for each period, the remanufacturing cost is

$$\begin{aligned} RC_i &= r_i \frac{\int_0^{c_{y_i}} x g_i(x) dx}{\int_0^{c_{y_i}} g_i(x) dx} \\ &= p_i \int_0^{c_{y_i}} x g_i(x) dx \end{aligned} \tag{4.3}$$

and the total remanufacturing cost is

$$
\begin{aligned}
RC &= \sum_{i=1}^{T} RC_i \\
&= \sum_{i=1}^{T} p_i \int_0^{c_{y_i}} x g_i(x) dx
\end{aligned} \tag{4.4}
$$

4.2.4 Inventory Cost

While all the remanufacturing or scrapping decisions are made immediately after the acquisition, it is possible to hold inventory for the refurbished products to the successive periods at some cost.

The inventory cost for each period is given as

$$
IC_i = hI_i + \frac{1}{2} hD_i \tag{4.5}
$$

where the second term is average holding cost for the products sold in this period. The total inventory cost is

$$
\begin{aligned}
IC &= \sum_{i=1}^{T} IC_i \\
&= \sum_{i=1}^{T} (hI_i + \frac{1}{2} hD_i)
\end{aligned} \tag{4.6}
$$

4.2.5 Inventory Constraints

The inventory cost should satisfy the following constraints

$$
\begin{aligned}
I_0 &= 0 \\
I_i &= I_{i-1} - D_i + r_i, \ where \ i = 1, 2, ..., T-1 \\
I_i &\geq 0 \\
I_T &= 0
\end{aligned}
$$

4.3. Multi-Period Optimization Problem

The total optimization problem now becomes

$$
\begin{aligned}
\text{Problem } O & \\
min\ C_O &= BC + RC + HC \\
&= \sum_{i=1}^{T} [Z_i(p_i) + p_i \int_0^{c_{y_i}} x g_i(x) dx + (hI_i + \frac{1}{2} hD_i)] \qquad (4.7) \\
s.t.\ I_0 &= 0 \\
I_i &= I_{i-1} - D_i + r_i,\ where\ i = 1, 2, ..., T-1 \\
I_i &\geq 0 \\
I_T &= 0 \\
where\ c_{y_i} &= G_i^{-1}(\frac{r_i}{p_i})
\end{aligned}
$$

If we write the variables I_i's in terms of other variables, we have:

$$
\begin{aligned}
\text{Problem } O & \\
min\ C_O &= \sum_{i=1}^{T} [Z_i(p_i) + p_i \int_0^{c_{y_i}} x g_i(x) dx] \qquad (4.8) \\
& -\frac{h}{2}[(2T-1)D_1 + (2T-3)D_2 + \cdots + D_T] \\
& +h[Tr_1 + (T-1)r_2 + \cdots + r_T] \\
s.t. & \sum_{k=1}^{i} r_k \geq \sum_{k=1}^{i} D_k,\ where\ i = 1, 2, ..., T-1 \\
& \sum_{k=1}^{T} r_k = \sum_{k=1}^{T} D_k \\
where & c_{y_i} = G_i^{-1}(\frac{r_i}{p_i})
\end{aligned}
$$

Moreover, we can reformulate the problem in terms of the cumulative demand and production over the T periods. Let

$$
s_i = \sum_{k=1}^{i} r_k
$$

$$
\begin{aligned}
c_{y_i} &= G_i^{-1}(\frac{r_i}{p_i}) = G_i^{-1}(\frac{s_i - s_{i-1}}{p_i}) \\
E_i &= \sum_{k=1}^{i} D_k \\
const_1 &= \frac{1}{2} h E_T
\end{aligned}
$$

then we can write the problem as

Problem S

$$
\begin{aligned}
min\ C_S &= \sum_{i=1}^{T} [Z_i(p_i) + p_i \int_0^{c_{y_i}} x g_i(x) dx] + h \sum_{i=1}^{T} (s_i - E_i) + const_1 \\
s.t. \quad & s_i \geq E_i,\ where\ i = 1, 2, ..., T-1 \\
& s_T = E_T
\end{aligned}
$$

Note that we may also remove the equality constraint in Problem O and Problem S by embedding it into the objective function. Therefore, another simplified version is

Problem O'

$$
\begin{aligned}
min\ C_{O'} &= \sum_{i=1}^{T} [Z_i(p_i) + p_i \int_0^{c_{y_i}} x g_i(x) dx] \\
& -\frac{h}{2}[(2T-3)D_1 + (2T-5)D_2 + \cdots + D_{T-1} - D_T] \\
& +h[(T-1)r_1 + (T-2)r_2 + \cdots + r_{T-1}] \\
s.t. \quad & \sum_{k=1}^{i} r_k \geq \sum_{k=1}^{i} D_k,\ where\ i = 1, 2, ..., T-1 \\
where \quad & c_{y_i} = G_i^{-1}(\frac{r_i}{p_i})
\end{aligned}
$$

and

Problem S'

$$
\begin{aligned}
min\ C_{S'} &= \sum_{i=1}^{T}[Z_i(p_i) + p_i \int_0^{c_{y_i}} x g_i(x)dx] + h\sum_{i=1}^{T-1}(s_i - E_i) + const_1 \\
s.t. \quad & s_i \geq E_i,\ where\ i = 1, 2, ..., T-1
\end{aligned}
$$

As the problems above are equivalent, we may apply different forms when that one is easier for derivation.

In the next section, we prove the convexity of the objective function.

4.4. Convexity of the Cost Function

We first prove the following two propositions.

Proposition 5 *The function $C_O(r_1, r_2, \cdots, r_T, p_1, p_2, \cdots, p_T)$ is convex, if the acquisition cost functions Z_i's are strictly convex.*

Proof Before we continue, we need put down some important relationships here:

$$
\begin{aligned}
\frac{\partial c_{y_i}}{\partial r_i} &= \frac{1}{p_i g_i(c_{y_i})} \\
\frac{\partial^2 c_{y_i}}{\partial r_i^2} &= \frac{-g_i'(c_{y_i})}{p_i^2 g_i^3(c_{y_i})} \\
\frac{\partial c_{y_i}}{\partial p_i} &= \frac{-r_i}{p_i^2 g_i(c_{y_i})} \\
\frac{\partial^2 c_{y_i}}{\partial p_i^2} &= \frac{2p_i r_i g_i^2(c_{y_i}) - r_i^2 g_i'(c_{y_i})}{p_i^4 g_i^3(c_{y_i})} \\
\frac{\partial^2 c_{y_i}}{\partial r_i \partial p_i} &= \frac{r_i g_i'(c_{y_i}) - p_i g_i^2(c_{y_i})}{p_i^3 g_i^3(c_{y_i})}
\end{aligned}
$$

and hence

$$
\begin{aligned}
\frac{\partial C_O}{\partial r_i} &= c_{y_i} + h(T + 1 - i) \\
\frac{\partial^2 C_O}{\partial r_i^2} &= \frac{1}{p_i g_i(c_{y_i})} \\
\frac{\partial^2 C_O}{\partial r_i \partial p_i} &= \frac{-r_i}{p_i^2 g_i(c_{y_i})}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial C_O}{\partial p_i} &= Z_i'(p_i) - \int_0^{c_{y_i}} G_i(x)dx \\
\frac{\partial^2 C_O}{\partial p_i^2} &= Z_i''(p_i) + G_i(c_{y_i})\frac{r_i}{p_i^2 g_i(c_{y_i})} \\
&= Z_i''(p_i) + \frac{r_i^2}{p_i^3 g_i(c_{y_i})}
\end{aligned}
$$

$$
\frac{\partial^2 C_O}{\partial p_i \partial p_j} = \frac{\partial^2 C_O}{\partial p_i \partial r_j} = \frac{\partial^2 C_O}{\partial r_i \partial r_j} = 0 \ (i \neq j)
$$

The Hessian matrix for C_O has the following form

$$
H(C_O) = \begin{bmatrix}
\frac{\partial^2 C_O}{\partial r_1^2}, & \frac{\partial^2 C_O}{\partial r_1 \partial r_2}, & \cdots, & \frac{\partial^2 C_O}{\partial r_1 \partial r_T}, & \frac{\partial^2 C_O}{\partial r_1 \partial p_1}, & \frac{\partial^2 C_O}{\partial r_1 \partial p_2}, & \cdots, & \frac{\partial^2 C_O}{\partial r_1 \partial p_T} \\
\frac{\partial^2 C_O}{\partial r_2 \partial r_1}, & \frac{\partial^2 C_O}{\partial r_2^2}, & \cdots, & \frac{\partial^2 C_O}{\partial r_2 \partial r_T}, & \frac{\partial^2 C_O}{\partial r_2 \partial p_1}, & \frac{\partial^2 C_O}{\partial r_2 \partial p_2}, & \cdots, & \frac{\partial^2 C_O}{\partial r_2 \partial p_T} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial^2 C_O}{\partial r_T \partial r_1}, & \frac{\partial^2 C_O}{\partial r_T \partial r_2}, & \cdots, & \frac{\partial^2 C_O}{\partial r_T^2}, & \frac{\partial^2 C_O}{\partial r_T \partial p_1}, & \frac{\partial^2 C_O}{\partial r_T \partial p_2}, & \cdots, & \frac{\partial^2 C_O}{\partial r_T \partial p_T} \\
\frac{\partial^2 C_O}{\partial p_1 \partial r_1}, & \frac{\partial^2 C_O}{\partial p_1 \partial r_2}, & \cdots, & \frac{\partial^2 C_O}{\partial p_1 \partial r_T}, & \frac{\partial^2 C_O}{\partial p_1^2}, & \frac{\partial^2 C_O}{\partial p_1 \partial p_2}, & \cdots, & \frac{\partial^2 C_O}{\partial p_1 \partial p_T} \\
\frac{\partial^2 C_O}{\partial p_2 \partial r_1}, & \frac{\partial^2 C_O}{\partial p_2 \partial r_2}, & \cdots, & \frac{\partial^2 C_O}{\partial p_2 \partial r_T}, & \frac{\partial^2 C_O}{\partial p_2 \partial p_1}, & \frac{\partial^2 C_O}{\partial p_2^2}, & \cdots, & \frac{\partial^2 C_O}{\partial p_2 \partial p_T} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial^2 C_O}{\partial p_T \partial r_1}, & \frac{\partial^2 C_O}{\partial p_T \partial r_2}, & \cdots, & \frac{\partial^2 C_O}{\partial p_T \partial r_T}, & \frac{\partial^2 C_O}{\partial p_T \partial p_1}, & \frac{\partial^2 C_O}{\partial p_T \partial p_2}, & \cdots, & \frac{\partial^2 C_O}{\partial p_T^2}
\end{bmatrix}.
$$

This proof is divided into two parts, namely, when $T = 2$ and when $T > 2$. As we only consider multi-period problem, we assume $T \geq 2$. $T = 1$ has been shown in by Galbreth and Blackburn [18].

1. $T = 2$

   The Hessian matrix becomes

$$
H(C_O) = \begin{bmatrix}
\frac{1}{p_1 g_1(c_{y_1})}, & 0, & \frac{-r_1}{p_1^2 g_1(c_{y_1})}, & 0 \\
0, & \frac{1}{p_2 g_2(c_{y_2})}, & 0, & \frac{-r_2}{p_2^2 g_2(c_{y_2})} \\
\frac{-r_1}{p_1^2 g_1(c_{y_1})}, & 0, & Z_1''(p_1) + \frac{r_1^2}{p_1^3 g_1(c_{y_1})}, & 0 \\
0, & \frac{-r_2}{p_2^2 g_2(c_{y_2})}, & 0, & Z_2''(p_2) + \frac{r_2^2}{p_2^3 g_2(c_{y_2})}
\end{bmatrix}.
$$

We study the leading principal minors, represented as H_k for the k degree leading principal minors, as following:

- H_4

$$
\begin{aligned}
H_4 &= \begin{vmatrix} \frac{1}{p_1 g_1(c_{y_1})}, & 0, & \frac{-r_1}{p_1^2 g_1(c_{y_1})}, & 0 \\ 0, & \frac{1}{p_2 g_2(c_{y_2})}, & 0, & \frac{-r_2}{p_2^2 g_2(c_{y_2})} \\ \frac{-r_1}{p_1^2 g_1(c_{y_1})}, & 0, & Z_1''(p_1) + \frac{r_1^2}{p_1^3 g_1(c_{y_1})}, & 0 \\ 0, & \frac{-r_2}{p_2^2 g_2(c_{y_2})}, & 0, & Z_2''(p_2) + \frac{r_2^2}{p_2^3 g_2(c_{y_2})} \end{vmatrix} \\
&= \frac{1}{p_1 g_1(c_{y_1})} \frac{1}{p_2 g_2(c_{y_2})} [Z_1''(p_1) + \frac{r_1^2}{p_1^3 g_1(c_{y_1})}][Z_2''(p_2) + \frac{r_2^2}{p_2^3 g_2(c_{y_2})}] \\
&\quad + \frac{r_1^2}{p_1^4 g_1^2(c_{y_1})} \frac{r_2^2}{p_2^4 g_2^2(c_{y_2})} \\
&\quad - \frac{r_1^2}{p_1^4 g_1^2(c_{y_1})} \frac{1}{p_2 g_2(c_{y_2})} [Z_2''(p_2) + \frac{r_2^2}{p_2^3 g_2(c_{y_2})}] \\
&\quad - \frac{r_2^2}{p_2^4 g_2^2(c_{y_2})} \frac{1}{p_1 g_1(c_{y_1})} [Z_1''(p_1) + \frac{r_1^2}{p_1^3 g_1(c_{y_1})}] \\
&= \frac{Z_1''(p_1) Z_2''(p_2)}{p_1 g_1(c_{y_1}) p_2 g_2(c_{y_2})} > 0
\end{aligned}
$$

- H_3

$$
\begin{aligned}
H_3 &= \begin{vmatrix} \frac{1}{p_1 g_1(c_{y_1})}, & 0, & \frac{-r_1}{p_1^2 g_1(c_{y_1})} \\ 0, & \frac{1}{p_2 g_2(c_{y_2})}, & 0 \\ \frac{-r_1}{p_1^2 g_1(c_{y_1})}, & 0, & Z_1''(p_1) + \frac{r_1^2}{p_1^3 g_1(c_{y_1})} \end{vmatrix} \\
&= \frac{1}{p_1 g_1(c_{y_1})} \frac{1}{p_2 g_2(c_{y_2})} [Z_1''(p_1) + \frac{r_1^2}{p_1^3 g_1(c_{y_1})}] - \frac{r_1^2}{p_1^4 g_1^2(c_{y_1})} \frac{1}{p_2 g_2(c_{y_2})} \\
&= \frac{Z_1''(p_1)}{p_1 g_1(c_{y_1}) p_2 g_2(c_{y_2})} > 0
\end{aligned}
$$

- H_2

$$
H_2 = \begin{vmatrix} \frac{1}{p_1 g_1(c_{y_1})}, & 0 \\ 0, & \frac{1}{p_2 g_2(c_{y_2})} \end{vmatrix}
$$

$$= \frac{1}{p_1 g_1(c_{y_1})} \frac{1}{p_2 g_2(c_{y_2})} > 0 \tag{4.9}$$

- H_1

$$H_1 = \frac{1}{p_1 g_1(c_{y_1})} > 0 \tag{4.10}$$

Therefore, because all the leading principal minors are non-negative, from the Sylvester criterion, we know $H(C_O)$ is positive definite when $T = 2$.

2. $T > 2$

The Hessian matrix becomes

$$H(C_O) = \begin{bmatrix} \frac{1}{p_1 g_1(c_{y_1})}, & 0, & \cdots, & 0, & \frac{-r_1}{p_1^2 g_1(c_{y_1})}, & 0, & \cdots, & 0 \\ 0, & \frac{1}{p_2 g_2(c_{y_2})},, & \cdots, & 0, & 0, & \frac{-r_2}{p_2^2 g_2(c_{y_2})}, & \cdots, & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0, & 0, & \cdots, & \frac{1}{p_T g_T(c_{y_T})}, & 0, & 0, & \cdots, & \frac{-r_T}{p_T^2 g_T(c_{y_T})}, \\ \frac{-r_1}{p_1^2 g_1(c_{y_1})}, & 0, & \cdots, & 0, & Z_1'' + \frac{r_1^2}{p_1^3 g_1(c_{y_1})}, & 0, & \cdots, & 0 \\ 0, & \frac{-r_2}{p_2^2 g_2(c_{y_2})},, & \cdots, & 0, & 0, & Z_2'' + \frac{r_1^2}{p_1^3 g_1(c_{y_1})}, & \cdots, & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0, & 0, & \cdots, & \frac{-r_T}{p_T^2 g_T(c_{y_T})}, & 0, & 0, & \cdots, & Z_T'' + \frac{r_T^2}{p_T^3 g_T(c_{y_T})} \end{bmatrix}$$

Note here Z_i'' still represents $Z_i''(p_i)$ and the arguments are omitted to save space.

Similar to the $T = 2$ case, we study the leading principal minors. Observing the matrix above, it is easy to see all the leading principal minors with degree less than $2T$ are positive, because their only non-zero component is the product of their diagonal elements, which are all positive. Therefore, we only need to study the leading principal minor H_{2T}.

$$\begin{aligned} H_{2T} &= \prod_{i=1}^{T} \frac{1}{p_i g_i(c_{y_i})} [Z_i''(p_i) + \frac{r_i^2}{p_i^3 g_i(c_{y_i})}] + (-1)^{T^2} \prod_{i=1}^{T} \frac{r_i^2}{p_i^4 g_i^2(c_{y_i})} \\ &> \prod_{i=1}^{T} \frac{1}{p_i g_i(c_{y_i})} \frac{r_i^2}{p_i^3 g_i(c_{y_i})} - \prod_{i=1}^{T} \frac{r_i^2}{p_i^4 g_i^2(c_{y_i})} = 0 \end{aligned}$$

Therefore, from the Sylvester criterion, we know $H(C_O)$ is positive definite when $T > 2$.

In conclusion, we proved that the Hessian matrix for $C_O(r_1, r_2, \cdots, r_T, p_1, p_2, \cdots, p_T)$ is positive definite, and therefore, C_O is a convex function. While we proved this for $T \geq 2$, this can be easily extended to the trivial case when $T = 1$, though this is not of interest here. ∎

Proposition 6 *$C_S(s_1, s_2, \cdots, s_T, p_1, p_2, \cdots, p_T)$ is convex, if the acquisition cost functions Z_i's are strictly convex.*

Proof As we know,

$$\begin{aligned} r_1 &= s_1 \\ r_i &= s_i - s_{i-1} \quad (i = 2,\ 3,\ \cdots,\ T) \end{aligned}$$

or we may write it in matrix form (here, prime represents matrix transpose)

$$[r_1, \cdots, r_T, p_1, \cdots, p_T]' = A \times [s_1, \cdots, s_T, p_1, \cdots, p_T]'$$

where matrix A is defined as

$$A = \begin{bmatrix} 1 & & & & & & & & \\ -1 & 1 & & & & & & & \\ & -1 & 1 & & & & & & \\ & & \ddots & \ddots & & & & & \\ & & & -1 & 1 & & & & \\ & & & & 0 & 1 & & & \\ & & & & & \ddots & \ddots & & \\ & & & & & & 0 & 1 \end{bmatrix}$$

Therefore, we have

$$C_S(x) = C_O(Ax)$$

Because composition with an affine mapping preserves the convexity, C_S is convex, given Proposition 1. ∎

As we have proved the convexity of the function, we may use Karush-Kuhn-Tucker (KKT) condition to find the global minimum, if it is feasible.

We define Problem L:

Problem L

$$\begin{aligned}
min\ C_L &= \sum_{i=1}^{T}[Z_i(p_i) + p_i\int_0^{c_{y_i}} xg_i(x)dx] \\
&\quad -\frac{h}{2}[(2T-3)D_1 + (2T-5)D_2 + \cdots + D_{T-1} - D_T] \\
&\quad +h[(T-1)r_1 + (T-2)r_2 + \cdots + r_{T-1}] \\
&\quad +\sum_{i=1}^{T-1} u_i(\sum_{k=1}^{i} D_k - \sum_{k=1}^{i} r_k) \\
where &\quad u_i \geq 0 \\
&\quad u_i(\sum_{k=1}^{i} D_k - \sum_{k=1}^{i} r_k) = 0,\ (i = 1, 2, ..., T-1)
\end{aligned}$$

and take its first order derivative conditions

$$\begin{cases}
\frac{\partial C_L}{\partial p_i} = Z_i'(p_i) - \int_0^{c_{y_i}} G_i(x)dx = 0 \\
\frac{\partial C_L}{\partial r_i} = c_{y_i} - c_{y_T} + h(T-i) - \sum_{k=i}^{T-1} u_k = 0 \\
\quad u_i(\sum_{k=1}^{i} D_k - \sum_{k=1}^{i} r_k) = 0 \\
i = 1, 2, \cdots, T-1
\end{cases} \tag{4.11}$$

From the propositions listed above, we have the following theorem:

Theorem 1 *If the solution to Eqn. (4.11) is feasible to Problem O, it is also the optimal solution to Problem O.*

Proof Let $P = \left\{(r_1, r_2, \cdots, r_{T-1}, p_1, p_2, \cdots, p_T) | \sum_{k=1}^{i} r_i, i = 1, 2, \cdots, T-1 \geq \sum_{k=1}^{i} D_i\right\}$. That is, P is the feasible set for Problem O. It is easy to see that P is a convex set.

If we have solution $X = (r_1^*, r_2^*, \cdots, r_{T-1}^*, p_1^*, p_2^*, \cdots, p_T^*)$ to Eqn. (4.11); i.e. the solution to Problem L, and $X \in P$, combining the proposition above that objective function C_O is a convex function, we know that we reach the global minimum of Problem O, because of the Karush-Kuhn-Tucker (KKT) condition. ∎

4.5. Linear Acquisition Cost Without Core Holding

Prior research ([40, 42, 45]) justifies the linear acquisition as a valid assumption, particularly when the market is large and well-defined. In this section, we examine this special case and assume

$$Z_i(p_i) \quad = \quad b_i p_i$$

where b_i's are the per unit acquisition cost for period i and

$$b_i \geq b_j, (\ for\ 1 \leq i \leq j \leq T).$$

Hence, the optimization problem reduces to

Problem LR

$$\begin{aligned} min\ C_{LR} \quad &= \quad \sum_{i=1}^{T} [b_i p_i + p_i \int_0^{c_{y_i}} x g_i(x) dx] + h \sum_{i=1}^{T-1} (s_i - E_i) + const \\ s.t. \quad & \quad s_i \geq E_i,\ where\ i = 1, 2, ..., T-1 \\ where \quad & \quad c_{y_i} = G_i^{-1}(\frac{r_i}{p_i}) = G_i^{-1}(\frac{s_i - s_{i-1}}{p_i}) \end{aligned}$$

Given this reduced optimization problem, we have the following proposition:

Proposition 7 *For linear acquisition cost, the optimal remanufacturing yield in each period depends on the per-unit acquisition cost and the remanufacturing cost function, but it does not depend on the actual demand or acquisition quantity.*

Proof Define $f(p_1, p_2, \cdots, p_T) = C_{LR}(p_1, p_2, \cdots, p_T; s_1 = S_1, s_2 = S_2, \cdots, s_T = S_T)$, i.e., we define f as a function of arguments p_1, p_2, ... and p_T, and with the parameter set $s_1 = S_1$, $s_2 = S_2$, ... and $s_T = S_T$. The partial derivatives are as following:

$$\begin{cases} \frac{\partial f}{\partial p_i} & = & b_i - \int_0^{c_{y_i}} G_i(x)dx \\ \frac{\partial^2 f}{\partial p_i^2} & = & \frac{(s_i - s_{i-1})^2}{p_i^3 g_i(c_{y_i})} \\ \frac{\partial^2 f}{\partial p_i \partial p_j} & = & 0 \quad (i \neq j) \end{cases} \tag{4.12}$$

Therefore, the Hessian matrix becomes

$$H(f) = \begin{bmatrix} \frac{s_1^2}{p_1^3 g_1(c_{y_1})} & & \\ & \ddots & \\ & & \frac{(s_T - s_{T-1})^2}{p_T^3 g_T(c_{y_T})} \end{bmatrix}$$

and it is straightforward this Hessian matrix is positive definite. Thus, function f is convex.

We define $F_i(x) = \int_0^x G_i(t)dt$. Since $G_i(t) > 0$ for $t \in (0, 1]$, we know the inverse of $F_i(x)$ exists.

Therefore, from the first order condition in Eqn. (4.12), f takes its minimum at

$$c_{y_i}^* = F_i^{-1}(b_i)$$

which does not depend on the actual demand and acquisition quantity. ∎

Then the optimal yields are given by

$$\begin{aligned} c_{y_i}^* &= F_i^{-1}(b_i) \\ \textit{where} \qquad & F_i(x) = \int_0^x G_i(t)dt \end{aligned} \tag{4.13}$$

We define $C_{LR}^*(s_1, s_2, \cdots, s_T)$ as the optimal solution when $c_{y_i} = c_{y_i}^*$. That is, the minimum cost depends on the parameter set $(s_1, s_2, \cdots, s_T)$.

Take the first order derivatives of these parameters, we have

$$\frac{\partial C^*_{LR}}{\partial s_i} = c^*_{y_i} - c^*_{y_{i+1}} + h$$

4.5.1 Special Cases

Now, we first study the solution properties for three special cases.

1. $c^*_{y_i} - c^*_{y_{i+1}} + h > 0$ for all $i = 1, 2, \cdots, T-1$

   The minimal cost increases in s_i. Given the constraints that $s_i \geq E_i$ ($i = 1, 2, \cdots, T-1$), we have

   $$s_i = E_i$$

   This implies that the remanufacturer should not keep inventory at the end of period i, given $c^*_{y_i} - c^*_{y_{i+1}} + h > 0$. Therefore, the original multiple period problem is simplified into two sub-problems at any point i.

2. $c^*_{y_i} - c^*_{y_{i+1}} + h = 0$ for all $i = 1, 2, \cdots, T-1$

   When this happens, the remanufacturer should be indifferent to the inventory decision, as long as demand in any period is satisfied.

3. $c^*_{y_i} - c^*_{y_{i+1}} + h < 0$ for all $i = 1, 2, \cdots, T-1$

   This implies the minimal cost decreases as s_i increases. Since we have the $s_1 \leq s_2 \leq \cdots \leq s_T$, we should have $s_1 = s_2 = \cdots = s_T = E_T$. That is, it is optimal to acquire and remanufacture at the beginning of period 1 and hold inventory for the following periods.

4.5.2 General Cases

We assume a general linear acquisition cost case as shown in Figure 4.5.

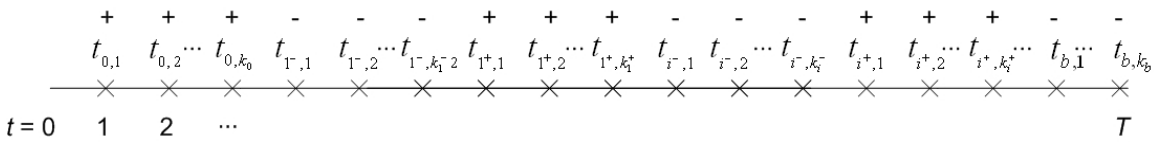

Figure 4.5. General Problem with Linear Acquisition Cost

In this figure, positive sign ("+") represents that $c^*_{y_i} - c^*_{y_{i+1}} + h > 0$ at this period and negative sign ("-") means that $c^*_{y_i} - c^*_{y_{i+1}} + h < 0$. The entire time horizon is assumed to start with a series of "+" periods $(t_{0,1}, t_{0,2}, \cdots, t_{0,k_0})$ and we denote them as set Θ_0; that is,

$$\Theta_0 = \left\{ t_{0,i} : (c^*_{y_i} - c^*_{y_{i+1}} + h) > 0 \right\}$$

Similarly, we assume the horizon ends with a series of "-" periods $(t_{b,1}, t_{b,2}, \cdots, t_{b,k_b})$ and denote them as set Θ_b, or

$$\Theta_b = \left\{ t_{b,i} : (c^*_{y_i} - c^*_{y_{i+1}} + h) < 0 \right\}$$

However, these assumptions do not sacrifice the generosity here, since we allow Θ_0 and Θ_b to be null sets. We will include the $c^*_{y_i} - c^*_{y_{i+1}} + h = 0$ at the end of this section.

s_i's within Θ_0 and Θ_b can be determined easily:

1. Θ_0

   Since the total cost increases in the s_i's within this set, the optimum will be achieved when they are at their minimum. Given the no backlogging constraints, we have

$$s_{0,i} = E_{0,i}, \quad \texttt{where} \;\; i = 1, \cdots, k_0$$

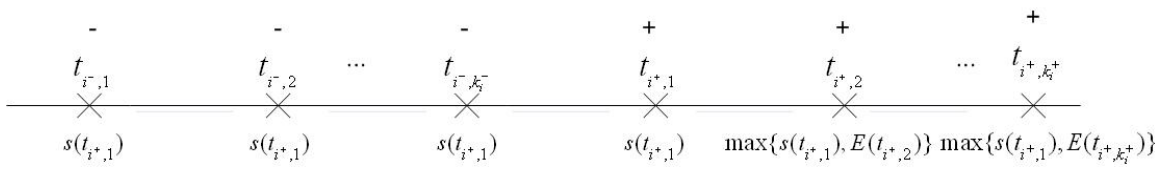

Figure 4.6. General Solution with Linear Acquisition Cost

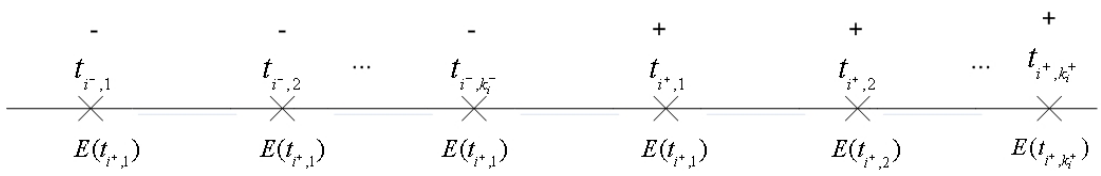

Figure 4.7. Base of General Solution for one Set

2. Θ_b

   As the cost decreases in s_i's within this set, to obtain optimum, s_i's should be all equal to E_T here. That is,

$$s_{b,i} = E_T, \quad \texttt{where} \ i = 1, \cdots, k_b$$

We separate the rest portion of the time horizon (except Θ_0 and Θ_b) into $(b-1)$ sets, such that each set i starts with a series of "-" periods (t_{i^-,j_i^-}'s) and ends with "+" ones (t_{i^+,j_i^+}'s). Here, $j_i^- \in \left\{1, 2, \cdots, k_i^-\right\}$ and $j_i^+ \in \left\{1, 2, \cdots, k_i^+\right\}$, where k_i^- and k_i^+ is the total number of "-" and "+" periods in set i, respectively.

4.5.2.1 Solution for a Single Set

Figure 4.6 illustrates one such set i. The symbols above the time line represent the beginning point of the periods, and the symbols below are the corresponding values of s_i's. Note while $s(t_{i+/-,j})$ actually stands for $s_{t_{i+/-,j}}$, we use the former to make the figure discernible. This is also the reason why we use $E(t_{i+/-,j})$ instead of $E_{t_{i+/-,j}}$ in Figure 4.8.

The general solution for the set in Figure 4.6 can be obtained as

1. "-" periods:

   As discussed above, when $c^*_{y_i} - c^*_{y_{i+1}} + h < 0$, the total cost decreases in s_i and it is optimal to maximize those s_i's while still satisfying the constraints: $s_i \leq s_j$ (for $i < j$) and $s_i \geq E_i$. Therefore, we have

$$s(t_{i^-,1}) = s(t_{i^-,2}) = \cdots = s(t_{i^-,k_i^-}) = s(t_{i^+,1})$$

2. "+" periods:

   Following the similar reasoning, when $c^*_{y_i} - c^*_{y_{i+1}} + h > 0$, s_i should be as small as possible and satisfies the constraints above. Hence, we have

$$s(t_{i^+,c}) \quad = \quad \max\left\{s(t_{i^+,1}), E(t_{i^+,j})\right\} \ \texttt{where} \ j = 1, 2, \cdots, k_i^+$$

We define a base solution for this problem, shown in Figure 4.7. From the reasoning above, we know the optimal solution has the property that, at any period, $s_{t^{+/-},j}$ should be no less than the value shown in this base solution. Therefore, to solve the problem, we may start from the base and then sum up the equation $c^*_{y_i} - c^*_{y_{i+1}} + h$ for all "-" periods. That is, let

$$S = \sum_{j=1}^{k_i^-} c^*_{y(t_{i^-,j})} - c^*_{y(t_{i^-,j+1})} + h = c^*_{y(t_{i^-,1})} - c^*_{y(t_{i^+,1})} + k_i^- h$$

From this point on, we adapt the following steps:

- Step 1: let $j = 1$.
- Step 2: let $S = S + c^*_{y(t_{i^+,j})} - c^*_{y(t_{i^+,j+1})} + h$
- Step 3: if $S \leq 0$, for $t = t_(i^+, 1), t_(i^+, 2), \cdots, t_(i^+, j-1), t_(i^+, j)$, raise the value for all $s(t)$'s to $E(t_{i^+,j})$, set $j = j + 1$ and return to step 2; otherwise, continue to Step 4

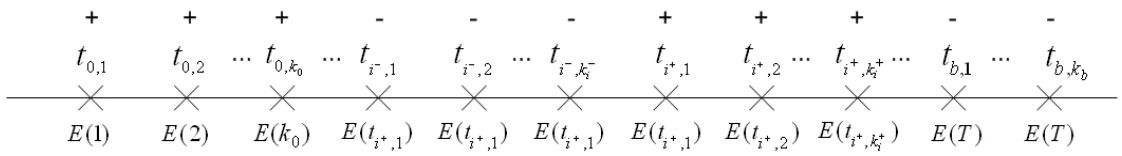

Figure 4.8. Base of General Solution

- Step 4: for the remaining periods $(t = t_(i^+, j+1), \cdots, t_(i^+, k_i^+))$ let $s(t) = E(t)$.

Proof The proof of this algorithm follows directly from the idea that, as long as the cumulative sum S is negative, we may further reduce the total cost by raising all the s_i's before the time $t = j$ ∎

4.5.2.2 Solution for General Cases

With the conclusions above, we can solve a general linear acquisition cost problem following the procedures below:

- Step 1: Obtain all $c^*_{y_i}$'s from Equation (4.13)
- Step 2: Start with the base solution as shown in Figure 4.8.
- Step 3: For periods $t_{0,i}$'s, fix $s_{0,i} = E(t_{0,i})$. Similarly, for periods t_{b,i_b}'s, let $s_{b,i_b} = E_T$.
- Step 4: Starting from period $t_{1^-,1}$, keep calculating cumulative sum S as

$$S = S + c^*_{y(t_{i+,j})} - c^*_{y(t_{i+,j+1})} + h$$

until S becomes positive at some period $t = t_{i+,j'}$ (Note, this can only happen at some positive periods). Raise all the $s_{t_{i+,j}}$'s to $E(t_{i+,j'})$, where $j \leq j'$.

- Set all the remaining $s_{t_{i^+,j}}$'s (where $j = j'+1, j'+2, \cdots, k_i^+$) to their allowed minimum. That is, $s_{t_{i^+,j}} = E(t_{i^+,j})$. In another words, keep them as in the base solution.

- Step 5: If $i+1 \neq b$, starting from period $t = t_{(i+1)^-,1}$, treating it as $t_{1^-,1}$, reset $S = 0$ and go back to step 4. Otherwise, end the process and we have reached the solution.

4.5.2.3 Quantity-Independent Optimal Yield

In this section, we will study the linear acquisition cost problem again in terms of quantity-independent optimal yield.

The equation (4.13) gives the optimal remanufacturing cost threshold, which does not depend on the actual amount desired to remanufacture. Therefore, for products acquired, remanufactured and then sold in period i, the total cost is given by

$$TC_i = b_i p_i + p_i \int_0^{c^*_{y_i}} x g_i(x) dx$$

and hence, the average per unit is

$$ATC_i = b_i / y_i^* + A_i^*$$

where we define

$$\begin{aligned} y_i^* &= G_i(c^*_{y_i}) \\ A_i^* &= \frac{\int_0^{c^*_{y_i}} x g_i(x) dx}{\int_0^{c^*_{y_i}} g_i(x) dx} \end{aligned}$$

Similarly, if these items are held in inventory and sold in period j ($j \geq i$), their average cost is

$$ATC_i = b_i / y_i^* + A_i^* + (j-i)h$$

The last component is introduced to count for the inventory holding cost. Therefore, we can prove the following proposition.

Proposition 8 *For the linear acquisition cost case above, the optimal average cost per unit in period i is given by*

$$\min_{j=1,\cdots,i}(b_j/y_j^* + A_j^* + (j-i)h) \tag{4.14}$$

Proof First of all, it is obvious that the optimal average cost in period i is no greater than in (4.14). Assume that $k = arg\{\min_{j=1,\cdots,i}(b_j/y_j^* + A_j^* + (j-i)h)\}$. the remanufacturer may acquire, remanufacture and hold inventory in period k alone to satisfy the demand in period i. Given the average cost does not change with the amount actually required, the average cost can always be realized with such policy.

Now we show that the optimal average cost cannot be lower than (4.14). Assume there exists another policy to satisfy the demand in period i, with lower average cost than in this proposition, and assume that policy requires the remanufacturer to acquire, remanufacture and hold k_j units in inventory from period j ($j = 1, \cdots, i$) to satisfy the demand of period i. That is, $D_i = \sum_{j=1}^{i} k_j$. Therefore, there must exist at least one period k^* ($k^* \in 1, \cdots, i$) where the average cost is lower than $ATC_{k^*} = b_{k^*}/y_{k^*}^* + A_{k^*}^*$, which contradicts with the conclusion we have reached that the optimal cost is given by (4.13). ∎

From this proposition, we may reach the optimal remanufacturing cost for all periods and thus the original problem reduce to a linear programming problem.

4.6. Linear Acquisition Cost With Core Holding

Till this point, we have made the assumption that the cores, once acquired, are inspected and remanufactured immediately. In practice, however, this may not hold for various reasons. For example, the inventory holding cost for raw cores may be

lower than for final products; or it may be harder to transport raw cores. Therefore, in this section, we study the cases where both inventory for uninspected raw cores and final products can be kept in inventory. However, we still assume inspected cores are remanufactured or discarded immediately after the inspection.

Unlike when raw cores inventory is not held, in each period, now we have two different sources for final products. We may acquire cores in previous period, hold them uninspected and remanufacture when we need them; or we may acquire, remanufacture and hold them as final products in inventory. However, it is easy to rule out other options. That is, it can never be optimal to acquire the raw cores and hold them for portion of the time, remanufacture then and hold the final products for the rest portion of time.

The reason is as following:

- If the average holding cost for raw cores is lower than for the final products, they should be kept in the form of raw cores until when they are needed, to save the inventory cost, since the remanufacturing yield-cost relationship does not change over time.

- Similarly, if the holding cost for raw cores is higher than for the final products, they should be kept in the form of final products ever since they are acquired.

4.6.1 A Simple Case

Before we continue with the study, we first study a simple case where inventory is only kept for uninspected raw cores, at the per unit cost of h_r per time period. That is, after acquisition, the remanufacturer decides the portion to inspect and then remanufacture or discard, and the rest is kept as raw cores. Final products are either used to satisfy demand or discarded. No inventory will be kept for them.

For period i, this is equivalent to say there are i different sources of raw cores, each of which may reduce to a similar problem as in the last section. That is, for raw

cores acquired at period j and held to period i, we may consider them as acquired at period i with acquisition cost $b_j + (i-j)h_r$ and the remanufacturing cost-yield relationship $G_j(x)$. Therefore, following the same derivation as before, we may get the optimal average cost per unit as

$$ATC_{j,i} = [b_j + (i-j)h_r]/y^*_{j,i} + A^*_{j,i} \tag{4.15}$$

where

$$\begin{aligned} y^*_{j,i} &= G_j(c^*_{y_{j,i}}) \\ c^*_{y_{j,i}} &= F_j^{-1}(b_j + (i-j)h_r) \\ A^*_{j,i} &= \frac{\int_0^{c^*_{y_{j,i}}} x g_j(x)dx}{\int_0^{c^*_{y_{j,i}}} g_j(x)dx} \end{aligned} \tag{4.16}$$

The minimum average cost for any period i will be achieved by solving the linear problem:

$$ATC^*_i = \min_{j=1,\cdots,i} [b_j + (i-j)h_r]/y^*_{j,i} + A^*_{j,i} \tag{4.17}$$

The algorithm to solve this problem is

- Step 1: Obtain $c^*_{y_{j,i}}$'s from Equation (4.16) for $i = 1, \cdots, T$ and $j = 1, \cdots, i$;
- Step 2: Solve the minimal average cost problem in (4.17) for period i ($i = 1, \cdots, T$) and obtain j^*_i, so that

$$j^*_i = \arg\min_{j=1,\cdots,i} \{[b_j + (i-j)h_r]/y^*_{j,i} + A^*_{j,i}\} \tag{4.18}$$

  Also, let $r_{j^*,i} = D_i$ and $p_{j^*,i} = \frac{r_{j^*,i}}{G_j(c^*_{y_{j,i}})}$

- Step 3: For each period j^*, let $p_{j^*} = \sum_{i=j^*}^{T} p_{j^*,i}$, which is the total units should be acquired. Moreover, $p_{j^*,i}$'s are the cores acquired at period j^* and kept in inventory to satisfy the demand in period i.

4.6.2 Linear Acquisition Cost With Core Holding

Now we are ready to study the original problem. That is, both raw cores and remanufactured products may be kept in inventory. At any period, the remanufacturer has two options:

- use the final products remanufactured before;
- inspect and remanufacture the raw cores acquired before.

The first option is equivalent to the problem we solved before, or the linear acquisition cost case without core holding; while the second option is equivalent to what we derived above. As both case give the optimal yield independent on the actual amount desired, the decision maker should always use the one which gives the lower per unit cost, instead of any combination of the two. If we let h_r represent per unit holding cost for raw cores and h_f for final products, we will have the optimal average cost for period i from the minimum of Eqn. (4.14) and Eqn. (4.17):

$$ATC_i^* = \min\left\{ \min_{j=1,\cdots,i} \left\{[b_j + (i-j)h_r]/y_{j,i}^* + A_{j,i}^*\right\}, \min_{j=1,\cdots,i} \left\{(b_j/y_j^* + A_j^* + (j-i)h_f)\right\} \right\} \quad (4.19)$$

where

$$\begin{aligned} c_{y_i}^* &= F_i^{-1}(b_i) \\ c_{y_{j,i}}^* &= F_j^{-1}(b_j + (i-j)h_r) \\ y_i^* &= G_i(c_{y_i}^*) \\ y_{j,i}^* &= G_j(c_{y_{j,i}}^*) \end{aligned}$$

$$
\begin{aligned}
A_i^* &= \frac{\int_0^{c^*_{y_i}} x g_i(x) dx}{\int_0^{c^*_{y_i}} g_i(x) dx} \\
A_{j,i}^* &= \frac{\int_0^{c^*_{y_{j,i}}} x g_j(x) dx}{\int_0^{c^*_{y_{j,i}}} g_j(x) dx}
\end{aligned}
$$

Therefore, the linear acquisition cost problem with core holding can be solved in the following steps:

- Step 1: Obtain all $c^*_{y_i}$ and $c^*_{y_{j,i}}$'s from above.
- Step 2: Solve the minimal average cost problem in (4.19) for period i ($i = 1, \cdots, T$) and obtain j_i^*, so that

  $$j_i^* = \arg\min \left\{ \min_{j=1,\cdots,i} \left\{ [b_j + (i-j)h_r]/y^*_{j,i} + A^*_{j,i} \right\}, \min_{j=1,\cdots,i} \left\{ (b_j/y^*_j + A^*_j + (j-i)h_f) \right\} \right\}$$

  If the left component is the minimum, the demand in period i should be satisfied by the raw cores acquired at period j_i^*, and remanufactured at period i. Otherwise, or if the right component is the minimum, the demand in period i should be satisfied by the final products acquired, remanufactured and carried from period j_i^*
- Step 3: From step 2, the total amount of acquiring and remanufacturing at each period is known.

Similar to the single period model, since the optimal yield is independent of the total acquisition quantity, the deterministic demand restriction can actually be relaxed. Therefore, as long as the acquisition cost is linear in the quantity, the optimal acquisition and sorting problem may be solved as general inventory problems which may include setup costs, backlogging and uncertain demand.

CHAPTER 5

CONCLUSIONS

The fast development of new technologies has brought an amazing array of high-tech consumer products. However, they have also brought an equally astounding profusion of electronic waste, among which lead, mercury and other heavy metals pose serious issues to the environment. In its January issue, *National Geographic* reported that *much of the world's electronic waste ends up in Ghana and Nigeria, where children are paid pennies a day to salvage the choice commodities inside–exposing themselves to dioxins, lead and other poisons. The rest of the waste, spanning everything from dot-matrix printers to surge protectors, ends up scattered in open dumps or washed out to sea.*

On the other hand, much of this electronic waste is in functionally good condition. In a recent issue of *Business Week*[10], the author reported a Greenwish CT based private equity firm, which sensed opportunity in discarded computers and invested $50 million in TechTurn, an e-recycler. It refurbishes the waste from companies and sells it to schools, nonprofits and poor countries. In 2007, the company booked $40 million in sales.

As more and more business comes to realize the profitability in this industry, our study provides a detailed analysis of optimal acquisition and sorting for remanufacturers facing variable condition of the returned cores in both single and multiple periods. As an extension of Galbreth and Blackburn [18], we first reproduce the results from those authors. It has been shown that, in single period model, when the acquisition cost is linear in the acquisition quantity, the optimal yield is independent

on the total acquisition amount. In that circumstance, whatever demand it has, the firm will always follow a simple optimal yield.

We extend this result to the case of piecewise linear convex acquisition costs. It has been shown that the optimal yield can be obtained by solving linear cost problems corresponding to its linear segments. When the cost is piecewise linear, the remanufacturer should still follow the optimal yield as if the cost is purely linear. However, when the optimal acquisition quantity reaches the intersect point of two segments, the quantity will be fixed and the yield will increase instead, until the yield reaches the optimal yield of the next segment. This gives the insight that, as the acquisition cost increases in some convex function form, it will be optimal to increase the yield instead of the quantity, which never happens in linear cost case.

The model is then extended to multiple periods.

We have proved that, in multiple periods, the minimal total cost exists, by showing the convexity of the objective function. This means that the problem will be relatively easy to solve using off-the-shelf non-linear optimization software. Nonetheless, we focus on identifying the optimal solution in special cases. When the acquisition cost is linear, the optimal yield in each period will be independent of the total acquisition, similar to what we have found in single period. Therefore, in multiple periods, the decision maker may still follow the single optimal policy without worrying about the actual demand. In this way, the policy will work even when the demand is not deterministic.

In most of our work, we assume remanufacturing or scrapping happen immediately after acquisition, and only remanufactured final products are kept in inventory. While this does happen in reality, sometimes it is wise to keep raw cores in inventory as well. For example, the remanufacturer may choose to remanufacture only when demand is observed. At the end of this thesis, we relax the final-product-only assumption and

allow raw cores to be kept in inventory. Still, we restrict ourselves to linear acquisition cost and derive the optimal solution to this problem.

While our study provides some insights into this emerging area of remanufacturing research, we expect more study in related problems. For example, uncertainty in the demand and the yield itself may be an intriguing problem worth more scrutiny. Algorithms to solve the general convex acquisition problem under different settings (setup costs, uncertainty in demand, backlogging, etc.) would also be of much interest. Even the piecewise linear convex acquisition cost case becomes difficult in multiple periods.

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