On the p-parts of quadratic Weyl group multiple Dirichlet series

ON THE p-PARTS OF QUADRATIC WEYL GROUP MULTIPLE DIRICHLET SERIES

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Abstract. Let Φ be a reduced root system of rank r. A *Weyl group multiple Dirichlet series* for Φ is a Dirichlet series in r complex variables $s_1, \ldots, s_r$, initially converging for $\Re(s_i)$ sufficiently large, which has meromorphic continuation to $\mathbb{C}^r$ and satisfies functional equations under the transformations of $\mathbb{C}^r$ corresponding to the Weyl group of Φ. Two constructions of such series are available, one [1–4] based on summing products of n-th order Gauss sums, the second [5] based on averaging a certain group action over the Weyl group. In each case, the essential work occurs at a generic prime p; the local factors, satisfying local functional equations, are then pieced into a global object. In this paper we study these constructions and the relationship between them. First we extend the averaging construction to obtain twisted Weyl group multiple Dirichlet series, whose p-parts are given by evaluating certain rational functions in r variables. Then we develop properties of such a rational function, giving its precise denominator, showing that the nonzero coefficients of its numerator are indexed by points that are contained in a certain convex polytope, determining the coefficients corresponding to the vertices, and showing that in the untwisted case the rational function is uniquely determined from its polar behavior and the local functional equations. We also give evidence that in the case $\Phi = A_r$, the p-part obtained here exactly matches the p-part of the twisted multiple Dirichlet series introduced in [3] when $n = 2$.

1. Introduction

Let Φ be a reduced root system of rank r. A *Weyl group multiple Dirichlet series* for Φ is a Dirichlet series in r complex variables $s_1, \cdots, s_r$, initially converging for $\Re(s_i)$ sufficiently large, which has meromorphic continuation to $\mathbb{C}^r$ and satisfies functional equations under the transformations of $\mathbb{C}^r$ corresponding to the Weyl group of Φ. For example, every Langlands L-function $L(s, \pi, r)$ is a Dirichlet series that is expected to have a functional equation of type A_1, since $s \mapsto 1 - s$ is a transformation of order 2. Let n be a fixed positive integer and let K be a fixed global field containing the

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$2n$-th roots of unity[1]. In [1–4] a broad class of Weyl group multiple Dirichlet series was exhibited, based on summing products of n-th order Gauss sums. These series are expected to be the Whittaker coefficients of minimal parabolic Eisenstein series on the n-fold cover of the simply connected algebraic group G over K whose root system is the dual of the root system Φ (i.e. Φ is the root system of the L-group ${}^LG$). In the articles cited above, the series was defined for all Φ if n is sufficiently large and for all n if $\Phi = A_r$, and its properties were established in the first of these cases, and conjectured in the second.

Let $g(c)$ denotes the n-th order Gauss sum of modulus c formed from the n-th power residue symbol $\left(\frac{a}{c}\right)_n$. Then the Chinese remainder theorem implies that for $(c_1, c_2) = 1$ the sum satisfies the twisted multiplicativity relation

$$g(c_1c_2) = \left(\frac{c_1}{c_2}\right)_n \left(\frac{c_2}{c_1}\right)_n g(c_1)\, g(c_2).$$

Similarly, the coefficients of the Weyl group multiple Dirichlet series satisfy a twisted multiplicativity (see [3], Eq. (2)). Thus, though these series are *not* in general Euler products, they may be reconstituted from a description of their p-part for a given prime p of norm q. We denote the (arithmetic part of the) coefficients at such a prime by $H(p^{k_1}, \ldots, p^{k_r})$. (See [2–4].)

The Weyl group multiple Dirichlet series in the case $n = 2$ are of particular interest; for example, the study of automorphic forms on double covers of classical groups is related to the theta correspondence. For $n = 2$, an alternative method of defining a Weyl group multiple Dirchlet series, valid for *all* simply-laced root systems, was given in [5]. (Though the arguments of n-th order Gauss sums vary for $n > 2$, for $n = 2$ they are essentially fixed, and the construction does not make direct use of Gauss sums.) In this approach, the p-part of the series is constructed as a certain average over the Weyl group under a natural but non-obvious action. By construction, this series satisfies the requisite continuation and local functional equations, and these lead to a global functional equation. However, the individual coefficients $H(p^{k_1}, \ldots, p^{k_r})$ are not identified. Nor is the size of the space of p-parts satisfying the local functional equations apparent (is the series, for example, uniquely determined by them?). By contrast, the approach of [3] provides a combinatorial description for the coefficients of a series of type A_r that is expected to have continuation and functional equations. However, except in low-rank cases, these properties are only conjectured. Moreover, it is not obvious that these two very different constructions are even related. In this paper we take a number of steps towards addressing these issues.

To describe our results further, let us review some basic features of these two constructions. We begin with the series of [1–4]. Let Φ be a simply-laced root system with Weyl group W. Fix a decomposition of $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative

[1]That K contain the n-th roots of unity is essential. The extra requirement that K contain the $2n$-th roots is made for convenience.

roots, and let $\alpha_1, \ldots, \alpha_r$ be the simple roots. Let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. If $w \in W$ and $\rho - w\rho = \sum_{i=1}^r k_i\alpha_i$, then the point $(k_1, \ldots, k_r)$ is called *stable*. The stable points form the vertices of a convex polytope called the *permutahedron* attached to Φ. The geometry of this object will be of great concern below. The series described in [1–4] have the properties that: (a) the coefficient $H(p^{k_1}, \ldots, p^{k_r})$ associated to the stable vertex attached to w is a (specific) product of $l(w)$ n-th order Gauss sums, where $l(w)$ is the length of w; (b) if n is sufficiently large (the *stable case*) then all other coefficients are zero; (c) for all $n \geq 1$, all coefficients corresponding to points outside the permutahedron are zero.

The series described in [1–4] come in two flavors: untwisted and twisted. The untwisted series should correspond to the first Whittaker coefficient for the Eisenstein series; the twisted coefficients should correspond to higher Whittaker coefficients. The adjective "twisted" is used since when the relevant indices are relatively prime, this corresponds to twisting the original Weyl group multiple Dirichlet series by characters. The description of the previous paragraph is for the untwisted series; however, it remains valid provided one does not twist by powers of p. It is precisely when one looks at the p-part of a series with twisting by powers of p that new features arise. In this case, the permutahedron expands (in a specific way; see [2]) to take into account the twisting. The vertices of this expanded permutahedron are called the *twisted stable* points. With the permutahedron replaced by this expanded one and the stable vertices by the twisted stable ones, statements (a), (b), and (c) above are still expected to hold. In [3] precise conjectures concerning the interior coefficients are formulated for all n in the case $\Phi = A_r$, while in [2] the twisted series are defined for all reduced root systems Φ and their continuations and functional equations established, provided n is sufficiently large (depending on Φ). One expects the existence of twisted Weyl group multiple Dirichlet series for all Φ and all n, and the formula of [2] for the coefficients at the twisted stable vertices (once again, a product of $l(w)$ Gauss sums; see Theorem 4.2 below) is expected to remain valid for all n. For A_r this is consistent with the conjectures in [3]; see [2].

We turn to the series described in [5]. Suppose that the root system Φ is simply-laced. Let $\Lambda = \Lambda_\Phi$ be the root lattice. For $\lambda = \sum_{i=1}^r k_i\alpha_i \in \Lambda$, let $d(\alpha) = \sum_{i=1}^r k_i$ be the usual height function, and let $\mathbf{x}^\lambda$ be the monomial $x_1^{k_1}\cdots x_r^{k_r}$. Let p be a prime of norm q. In [5], the authors construct a rational function $f(\mathbf{x})$:

$$f(\mathbf{x}) = \frac{N(\mathbf{x})}{D(\mathbf{x})}, \quad D(\mathbf{x}) := \prod_{\alpha\in\Phi^+}(1 - q^{d(\alpha)-1}\mathbf{x}^{2\alpha}),$$

which is invariant under a certain W-action. The construction will be described in detail in the next section. It is shown that $f(q^{-s_1}, \ldots, q^{-s_r})$ is the p-part of a multiple Dirichlet series (associated to Φ and $n = 2$) which has meromorphic continuation to $\mathbb{C}^r$ and a group of functional equations isomorphic to W.

The work of [5] concerns only untwisted series. However, the method can be adapted to construct twisted series as well. We carry out this extension, based on defining a *twisted action* of W on the field of rational functions in $x_1, \ldots, x_r$, in Section 2 below. Thus, our first result is a wider class of Weyl group multiple Dirichlet series with continuation to $\mathbb{C}^r$.

With this extension in hand, we pursue two goals. One is to develop properties of the construction of [5] and of this extension; the second is to study its relation to the series of [1–4]. The twisted rational function $f(\mathbf{x};\ell)$ (here $\ell \in (\mathbb{Z}_{\geq 0})^r$ is an index that specifies the twisting) is of the form $N(\mathbf{x};\ell)/D(\mathbf{x})$. Write

$$N(\mathbf{x};\ell) = \sum_{\lambda \in \Lambda} a_\lambda \mathbf{x}^\lambda.$$

Then we will show

- $N(\mathbf{x};\ell)$ is a polynomial (Theorem 2.4).
- The support of $N(\mathbf{x};\ell)$ is contained in a certain convex polytope $\Pi_{\theta(\ell)}$ (Theorem 3.2).
- The coefficients a_λ for λ a vertex of $\Pi_{\theta(\ell)}$ coincide with the stable coefficients described in [2] (Theorem 4.2).
- In the untwisted case, the series obtained in [5] is the *unique* series with denominator $D(\mathbf{x})$ satisfying the desired functional equations and having leading coefficient 1 (Corollary 5.8).

Even in the untwisted case each of these results is new. In that case $\Pi_{\theta(\ell)}$ is the convex hull in $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$ of the points

$$\rho - w\rho, \quad w \in W.$$

Based on these results, it seems natural to conjecture that if $\Phi = A_r$ then for a given prime p the series constructed here, generalizing the construction of [5], is in fact *identically the same* as the p-part of the series constructed using Gelfand-Tsetlin patterns in [3]. We note some computational evidence for this as well (see the remarks after Example 2.6).

The remainder of this paper is organized as follows. In Section 2, we construct a twisted analogue of the quadratic Weyl group multiple Dirichlet series of [5], valid for any simply laced root system Φ. The p-part is obtained from a rational function $f(\mathbf{x};\ell)$. We identify the denominator of this function, capturing the polar behavior of the series. We provide some explicit examples when $\Phi = A_2$, note that in each case this series is the same as the series of [3], and describe some computational evidence that suggests that this identification holds in general. In Section 3, we study the support of the numerator $N(\mathbf{x};\ell)$, and show that it is contained in a polytope $\Pi_{\theta(\ell)}$ as described above. This result depends on describing the polytope group-theoretically. In Section 4 we find the coefficients at the vertices of the polytope, that is, the stable coefficients (both twisted and untwisted). We show that these coefficients are

all determined from the degree 0 coefficient of $N(\mathbf{x};\ell)$. As noted, these match the coefficients of [2,3]. In Section 5, we discuss the remaining coefficients of $N(\mathbf{x};\ell)$. In the untwisted case, we show that these coefficients are all determined by the functional equations and the stable coefficients, hence the series is uniquely determined by the denominator, functional equations, and constant coefficient. Surprisingly, in the twisted case, this uniqueness no longer holds, and we explain why this is the case, due to the existence of other regular orbits for the W-action on the points in $\Pi_{\theta(\ell)}$. Finally, in Section 6 we explain how to piece together the p-parts described here into a twisted multiple Dirichlet series with analytic continuation and group of functional equations.

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2. Rationality and denominator

We first develop the definition of the rational function $f(\mathbf{x};\ell)$ which gives the p-part of a twisted quadratic Weyl group multiple Dirichlet series. We emphasize that this construction, based on averaging over a suitable group action, was introduced and carried out for the untwisted series in [5]. Here we modify the group action in [5] to accomodate twisting, and this allows us to generalize those results using similar methods. Once these p-parts satisfy the desired local functional equations, it follows that they may be combined by twisted multiplicativity to give a W-invariant function on $\mathbb{C}^r$; see Section 6 below.

Let $F = \mathbb{C}(\mathbf{x}) = \mathbb{C}(x_1, x_2, \ldots, x_r)$ be the field of rational functions in the variables $x_1, x_2, \ldots, x_r$. Let $\ell = (l_1, \ldots, l_r)$ denote an r-tuple of nonnegative integers. This is the *twisting parameter*. We define an ℓ-twisted action of W on F. This is done in stages.

Let $\sigma_1, \ldots, \sigma_r \in W$ be the simple reflections corresponding to the simple roots, and let q be a fixed prime power. First, for $\mathbf{x} = (x_1, x_2, \ldots, x_r)$ define $\sigma_i\mathbf{x} = \mathbf{x}'$, where

$$x'_j = \begin{cases} x_i x_j \sqrt{q} & \text{if } i \text{ and } j \text{ are adjacent,} \\ 1/(q x_j) & \text{if } i = j, \text{ and} \\ x_j & \text{otherwise.} \end{cases} \tag{2.1}$$

Next, define $\epsilon_i\mathbf{x} = \mathbf{x}'$, where

$$x'_j = \begin{cases} -x_j & \text{if } i \text{ and } j \text{ are adjacent,} \\ x_j & \text{otherwise.} \end{cases} \tag{2.2}$$

For $f \in F$ define

$$f^+_{i,\ell}(\mathbf{x}) = \frac{f(\mathbf{x}) + (-1)^{l_i} f(\epsilon_i \mathbf{x})}{2} \quad \text{and} \quad f^-_{i,\ell}(\mathbf{x}) = \frac{f(\mathbf{x}) - (-1)^{l_i} f(\epsilon_i \mathbf{x})}{2}. \tag{2.3}$$

Finally we can define the action of W on F for a generator $\sigma_i \in W$:

$$(f|_\ell \sigma_i)(\mathbf{x}) = -\frac{1 - qx_i}{qx_i(1-x_i)}(x_i\sqrt{q})^{l_i} f^+_{i,\ell}(\sigma_i \mathbf{x}) + \frac{1}{x_i\sqrt{q}}(x_i\sqrt{q})^{l_i} f^-_{i,\ell}(\sigma_i \mathbf{x}). \tag{2.4}$$

Note that despite the presence of the $\sqrt{q}$'s, $(f|_\ell \sigma_i)(\mathbf{x})$ will in fact be a power series in q. For example, suppose l_i is even. Then in the function f, each of the neighbors x_j of x_i gets replaced by $x_i x_j \sqrt{q}$. But the sum of degrees of the neighbors of x_i is even in $f^+_{i,\ell}$ and odd in $f^-_{i,\ell}$. Therefore both summands in (2.4) are in K. The argument is similar for l_i odd.

For $\ell = (0, \ldots, 0)$, the action (2.4) coincides with the action of σ_i on F from [5, (3.13)]. As in [5], one can prove the following lemma; the proof is essentially the same as that of [5, Lemma 3.2], and we omit the details.

Lemma 2.1. *The definition* (2.4) *extends to give an action of W on F.*

The main result of Section 3 of [5] is the construction of a W-invariant rational function with certain limiting behavior. This is summarized below.

Theorem 2.2. *Define*

$$\Delta(\mathbf{x}) = \prod_{\alpha \in \Phi^+} (1 - q^{d(\alpha)} \mathbf{x}^{2\alpha}),$$

$$j(w, \mathbf{x}) = \Delta(\mathbf{x})/\Delta(w\mathbf{x}) = \operatorname{sgn}(w) q^{d(\rho - w^{-1}\rho)} \mathbf{x}^{2(\rho - w^{-1}\rho)},$$

and

$$f(\mathbf{x}; \ell) = \Delta(\mathbf{x})^{-1} \sum_{w \in W} j(w, \mathbf{x})(1|_\ell w)(\mathbf{x}).$$

Then $f(\mathbf{x}; \ell)$ is a W-invariant rational function satisfying

(1) *for each $i = 1, 2, \ldots, r$, the function f satisfies the following limiting condition: if $x_j = 0$ for every j adjacent to i, then*

$$f(\mathbf{x}; \ell)(1 - x_i)^{m_i} \text{ is independent of } x_i, \tag{2.5}$$

where m_i is 0 if l_i is even and is 1 otherwise.

(2) $f(0, 0, \ldots, 0; \ell) = 1$.

In [5] this theorem is proven only for the untwisted action, i.e. only for $\ell = (0, \ldots, 0)$, but essentially the same proof will work here. We will not repeat the argument. We will also need below the analogues for the twisted action of Lemma 3.3(c) and Lemma 3.9 of [5]. We state these without proof.

Lemma 2.3. *Let $w \in W$ and ℓ be an r-tuple of nonnegative integers.*

(a) *Let $g, h \in F$. If g is an even function of all the x_j, then*
$$(gh|_\ell w)(\mathbf{x}) = g(w\mathbf{x}) \cdot (h|_\ell w)(\mathbf{x}).$$
(b) $\mathbf{x}^{\rho - w\rho}(1|_\ell w)(\mathbf{x})$ *is regular at the origin.*
(c) $\mathbf{x}^{\rho - \sigma_i w\rho}\left(\frac{1}{x_i}|_\ell w\right)(\mathbf{x})$ *is regular at the origin for $i = 1, 2, \ldots, r$.*

Put
$$f(\mathbf{x}; \ell) = \frac{N(\mathbf{x}; \ell)}{D(\mathbf{x})}, \quad D(\mathbf{x}) := \prod_{\alpha \in \Phi^+} (1 - q^{d(\alpha)-1}\mathbf{x}^{2\alpha}).$$

Theorem 2.4. *$N(\mathbf{x}; \ell)$ is a polynomial.*

Proof. First, we will show that $f(\mathbf{x}; \ell)D(\mathbf{x})\Delta(\mathbf{x})$ is a polynomial. In fact, we will prove, for any $w \in W$,
$$j(w, \mathbf{x})(1|_\ell w)(\mathbf{x})D_w(\mathbf{x}) \tag{2.6}$$
is a polynomial, where
$$D_w(\mathbf{x}) = \prod_{\alpha \in \Phi(w)} (1 - q^{d(\alpha)-1}\mathbf{x}^{2\alpha}).$$
Here $\Phi(w)$ denotes the subset of positive roots made negative by w. The proof that (2.6) is a polynomial will be by induction on the length $l(w)$ of w. When w is the identity, there is nothing to prove. Suppose that for $w \in W$, (2.6) is a polynomial,
$$j(w, \mathbf{x})(1|_\ell w)(\mathbf{x})D_w(\mathbf{x}) = P(\mathbf{x}),$$
say. Let σ_i be a simple reflection such that $l(w\sigma_i) = l(w) + 1$. Then
$$\begin{aligned} & j(w\sigma_i, \mathbf{x})(1|_\ell w\sigma_i)(\mathbf{x})D_{w\sigma_i}(\mathbf{x}) \\ &= j(\sigma_i, \mathbf{x})\left(\frac{P}{D_w}\bigg|_\ell \sigma_i\right)(\mathbf{x}) \cdot D_{w\sigma_i}(\mathbf{x}) \\ &= \frac{j(\sigma_i, \mathbf{x})(P|_\ell \sigma_i)(\mathbf{x})}{D_w(\sigma_i \mathbf{x})} \cdot D_{w\sigma_i}(\mathbf{x}), \quad \text{by Lemma 2.3(a).} \end{aligned} \tag{2.7}$$
By the definition (2.4) of the action of σ_i, we can write $(P|_\ell \sigma_i)(\mathbf{x})$ as $P_2(\mathbf{x})/(1 - x_i^2)$ where P_2 is a Laurent polynomial in the x_i. However, as Lemma 2.3 (b) implies that (2.6) is regular at the origin, it follows that P_2 is a polynomial. Moreover, the denominator $D_w(\sigma_i \mathbf{x})$ is equal to
$$\prod_{\alpha \in \Phi(w)} (1 - q^{d(\sigma_i\alpha)-1}\mathbf{x}^{2\sigma_i\alpha}) = D_{w\sigma_i}(\mathbf{x})/(1 - x_i^2)$$
Plugging this back into (2.7), we conclude that
$$j(w\sigma_i, \mathbf{x})(1|_\ell w\sigma_i)(\mathbf{x})D_{w\sigma_i}(\mathbf{x}) = j(\sigma_i, \mathbf{x})P_2(\mathbf{x})$$
is polynomial. Therefore $f(\mathbf{x})D(\mathbf{x})\Delta(\mathbf{x}) = N(\mathbf{x})\Delta(\mathbf{x})$ is a polynomial.

To complete the proof, it will suffice to show that

$$h(\mathbf{x}) = \sum_{w \in W} j(w, \mathbf{x})(1|_\ell w)(\mathbf{x}) \tag{2.8}$$

is divisible by $\Delta(\mathbf{x})$. This function satisfies

$$h(\mathbf{x}) = j(\sigma_i, \mathbf{x})(h|_\ell \sigma_i)(\mathbf{x}).$$

Setting $x_i = \pm 1/\sqrt{q}$ in the above equation, we deduce that

$$h(\mathbf{x})\big|_{x_i = \pm 1/\sqrt{q}} = -h(\mathbf{x})\big|_{x_i = \pm 1/\sqrt{q}},$$

and that hence $1 - qx_i^2$ divides $h(\mathbf{x})$. Similarly, we can show that $1 - q^{d(\alpha)}\mathbf{x}^{2\alpha}$ divides $h(\mathbf{x})$ for all positive roots α. This completes the proof of the theorem. □

Remark 2.5. Though we do not require this fact, we point out that that the proofs of Lemma 2.3 and Theorem 2.4 actually show that N is a polynomial in q as well.

Example 2.6. Here we give some examples of the polynomials $N(\mathbf{x}; \ell)$ for the root system A_2. We use (x, y) for the variables instead of (x_1, x_2). To get the rational function $f(\mathbf{x}; \ell)$, each of the following should be divided by $(1-x^2)(1-y^2)(1-qx^2y^2)$.

- $\ell = (0,0)$: $N = 1 + x + y - x^2y - xy^2 - x^2y^2$.
- $\ell = (1,0)$: $N = 1 - x^2 + y + (q-1)x^2y + qx^3y - qx^2y^3 - qx^3y^3$.
- $\ell = (1,1)$: $N = 1 - x^2 - y^2 + (1-q)x^2y^2 + qx^4y^2 + qx^2y^4 - qx^4y^4$.
- $\ell = (2,0)$: $N = 1 + (q-1)x^2 + qx^3 + y + (q-1)x^2y - qx^4y + (q^2-q)\,x^2y^2 + (q^2-q)\,x^3y^2 + (q^2-q)\,x^2y^3 + (q-q^2)\,x^4y^3 - q^2x^3y^4 - q^2x^4y^4$.
- $\ell = (2,1)$: $N = 1 + (q-1)x^2 + qx^3 - y^2 + (1-2q+q^2)\,x^2y^2 + (q^2-q)\,x^3y^2 + (q-q^2)\,x^4y^2 - q^2x^5y^2 + (q-q^2)\,x^2y^4 + (q^3-q^2)\,x^3y^4 + (q^2-q)\,x^4y^4 + (q^2-q^3)\,x^5y^4 + q^3x^3y^5 - q^3x^5y^5$.
- $\ell = (2,2)$: $1 + (q-1)x^2 + qx^3 + (q-1)y^2 + (1-3q+2q^2)\,x^2y^2 + (q^2-q)\,x^3y^2 + (q-2q^2+q^3)\,x^4y^2 + (q^3-q^2)\,x^5y^2 + qy^3 + (q^2-q)\,x^2y^3 + (q^3-q^2)\,x^4y^3 - q^3x^6y^3 + (q-2q^2+q^3)\,x^2y^4 + (q^3-q^2)\,x^3y^4 + (q^4-2q^3+2q^2-q)\,x^4y^4 + (q^2-2q^3+q^4)\,x^5y^4 + (q^3-q^2)\,x^2y^5 + (q^2-2q^3+q^4)\,x^4y^5 + (q^3-q^4)\,x^6y^5 - q^3x^3y^6 + (q^3-q^4)\,x^5y^6 - q^4x^6y^6$.

The invariance of $f(\mathbf{x}; \ell)$ and the limiting conditions imply that the Taylor series coefficients of f can be used to construct the p-part of a twisted multiple Dirichlet series with continuation to $\mathbb{C}^r$ and functional equation. The p-parts are combined using twisted multiplicativity, as explained in Section 6.

In fact, *we conjecture that for $\Phi = A_r$ this p-part will coincide with the p-part coming from the construction of [3] via Gelfand–Tsetlin patterns.* As evidence, we have computed $N(\mathbf{x}; \ell)$ for the root systems A_r, $r \leq 4$, for all twisting parameters ℓ with $\sum l_i \leq 6$. For each of these 2597 series the polynomial $N(\mathbf{x}; \ell)$ matches the numerator polynomial predicted by the Gelfand–Tsetlin conjecture.

3. The support of the numerator

Recall that Λ denotes the root lattice of Φ, and that for $\lambda = \sum_{i=1}^r k_i\alpha_i \in \Lambda$, $\mathbf{x}^\lambda$ denotes the monomial $\prod_{i=1}^r x_i^{k_i}$. Write

$$N(\mathbf{x};\ell) = \sum_{\lambda\in\Lambda} a_\lambda \mathbf{x}^\lambda, \tag{3.1}$$

where $N(\mathbf{x};\ell)$ is the polynomial from Theorem 2.4. Our goal in this section is to investigate the support of $N(\mathbf{x};\ell)$, that is the set

$$\operatorname{Supp} N(\mathbf{x};\ell) = \{\lambda\in\Lambda \mid a_\lambda \neq 0\}.$$

The main result (Theorem 3.2) is that $\operatorname{Supp} N(\mathbf{x};\ell)$ is contained in a certain translated weight polytope for Φ.

We begin by discussing relations that the coefficients a_λ must satisfy. Recall that $N(\mathbf{x};\ell)$ is the numerator of the rational function

$$f(\mathbf{x}) = \frac{N(\mathbf{x};\ell)}{D(\mathbf{x})}, \quad D(\mathbf{x}) := \prod_{\alpha\in\Phi^+}(1-q^{d(\alpha)-1}\mathbf{x}^{2\alpha}),$$

and that $f(\mathbf{x})$ is invariant under a certain action of the Weyl group W. The W-invariance of $f(\mathbf{x})$ and the simple form of the denominator $D(\mathbf{x})$ imply that the numerator $N(\mathbf{x};\ell)$ satisfies certain relations under the W-action. In particular, applying the reflection σ_k gives the relation

$$\begin{aligned}(q^2\mathbf{x}^{2\alpha_k}-1)N(\mathbf{x};\ell) &= q^{(1+l_k/2)}\mathbf{x}^{(l_k+1)\alpha_k}(1+\mathbf{x}^{\alpha_k})(q\mathbf{x}^{\alpha_k}-1)N^+_{k,\ell}(\sigma_k\mathbf{x}) \\ &\quad + q^{(3/2+l_k/2)}\mathbf{x}^{(l_k+1)\alpha_k}(1-\mathbf{x}^{2\alpha_k})N^-_{k,\ell}(\sigma_k\mathbf{x}).\end{aligned} \tag{3.2}$$

Inserting (3.1) in (3.2) and collecting terms, we find

$$q^2 a_{\lambda-2\alpha_k} - a_\lambda = \begin{cases} q^{d(\beta)/2}(-a_\mu + (1-1/q)a_{\mu+\alpha_k} + a_{\mu+2\alpha_k}/q) & \text{(even)}, \\ \sqrt{q}\cdot q^{d(\beta)/2}(a_\mu - a_{\mu+2\alpha_k}/q^2) & \text{(odd)}, \end{cases} \tag{3.3}$$

In (3.3) we have used the notation

- $\mu = \sigma_k\lambda + (l_k+1)\alpha_k$,
- $\beta = \lambda - \sigma_k\lambda - l_k\alpha_k = \lambda - \mu + \alpha_k$, and
- $d\colon \Lambda \to \mathbb{Z}$ is the usual height function on the root lattice;

we also are assuming that $\lambda \le \mu$, in other words that the difference $\mu - \lambda$ is a nonnegative sum of simple roots. "Even/odd" in (3.3) refers to the following. Define $\varphi_k\colon \Lambda \to \mathbb{Z}$ by

$$\varphi_k(\mu) = l_k + \sum_{j\sim k} k_j, \tag{3.4}$$

where $\mu = \sum k_j\alpha_j$, and where $j \sim k$ means that the nodes labelled j and k are adjacent in the Dynkin diagram for Φ. Then we are in the even/odd case according

to whether $\varphi_k(\mu)$ is even or odd. Note that $\varphi_k(\mu) = \varphi_k(\lambda)$, since $\mu - \lambda$ is a multiple of α_k.

The functional equation (3.3) can be simplified in the even case. Keeping the notation λ, μ, β as above, let us apply the functional equation (3.3) with λ replaced by $\mu+2\alpha_k$. Since $\sigma_k(\mu+2\alpha_k)+\alpha_k = \lambda-2\alpha_k$ and $\mu+2\alpha_k-(\lambda-2\alpha_k)+\alpha_k = -\beta+6\alpha_k$, we obtain

$$q^2 a_\mu - a_{\mu+2\alpha_k} = \begin{cases} q^{3-d(\beta)/2}(-a_{\lambda-2\alpha_k} + (1-1/q)a_{\lambda-\alpha_k} + a_\lambda/q) & \text{(even)}, \\ \sqrt{q}\cdot q^{3-d(\beta)/2}(a_{\lambda-2\alpha_k} - a_\lambda/q^2) & \text{(odd)}. \end{cases} \tag{3.5}$$

Notice that μ, $\lambda + 2\alpha_k$ are identically equal except in the k-th position, so $\varphi_k(\mu)$ is even if and only if $\varphi_k(\lambda + 2\alpha_k)$ is even (and if and only if $\varphi_k(\lambda)$ is even). In the odd case, equations (3.3), (3.5) are equivalent. However, in the even case, they are not; instead, it is is easy to combine them to obtain

$$a_\lambda + q a_{\lambda-\alpha_k} = q^{d(\beta)/2-1}(q a_\mu + a_{\mu+\alpha_k}) \qquad \text{(even)}. \tag{3.6}$$

Hence (3.6) holds for all λ in the even case. Conversely, in the even case (3.6) for all λ implies (3.3): take q times (3.6) with λ replaced by $\lambda - \alpha_k$ and subtract the original (3.6). We summarize the above discussion in the following proposition:

Proposition 3.1. *Let $\lambda, \mu \in \Lambda$ satisfy $\mu = \sigma_k\lambda + (l_k+1)\alpha_k$, and suppose $\mu \geq \lambda$. Put $\beta = \lambda - \mu + \alpha_k$, and define $\varphi_k \colon \Lambda \to \mathbb{Z}$ as in (3.4). Then the coefficients of $N(\mathbf{x}; \ell)$ satisfy*

$$q a_{\lambda-\alpha_k} + a_\lambda = q^{d(\beta)/2}(a_\mu + a_{\mu+\alpha_k}/q), \quad \textit{if } \varphi_k(\mu) \textit{ is even, and} \tag{3.7a}$$

$$q^2 a_{\lambda-2\alpha_k} - a_\lambda = \sqrt{q}\cdot q^{d(\beta)/2}(a_\mu - a_{\mu+2\alpha_k}/q^2), \quad \textit{if } \varphi_k(\mu) \textit{ is odd.} \tag{3.7b}$$

Now let $\varpi_1, \ldots, \varpi_r$ be the fundamental weights of the root system Φ; since Φ is simply-laced, these form the dual basis to the simple roots with respect to the W-invariant scalar product $\langle \cdot, \cdot \rangle$. Recall that the closure of the *dominant chamber* in $\Lambda_\mathbb{R} = \Lambda \otimes \mathbb{R}$ is defined by the inequalities

$$\langle \varpi_k, \mathbf{x} \rangle \geq 0, \quad k = 1, \ldots, r; \tag{3.8}$$

here we have slightly abused notation to let $\mathbf{x}$ denote a point in $\Lambda_\mathbb{R}$. Given a twisting parameter $\ell = (l_1, \ldots, l_r)$, let $\theta = \theta(\ell)$ be the dominant weight

$$\theta = \rho + \sum l_k \varpi_k,$$

where ρ is the sum of the fundamental weights. Note that θ is *regular*, that is the inequalities (3.8) are strict when evaluated on θ.

Let $\Pi = \Pi_\theta$ be the convex hull in $\Lambda_\mathbb{R}$ of the points

$$\theta - w\theta, \quad w \in W.$$

Our goal is to prove the following theorem:

Theorem 3.2. *The support* $\operatorname{Supp} N(\mathbf{x};\ell)$ *is contained in the polytope* Π.

Example 3.3. Figure 1 shows the support for the polynomial $N(x,y;(2,2))$ from Example 2.6. The shaded hexagon is the polygon Π. The grey dots are nonzero coefficients of $N(x,y;(2,2))$.

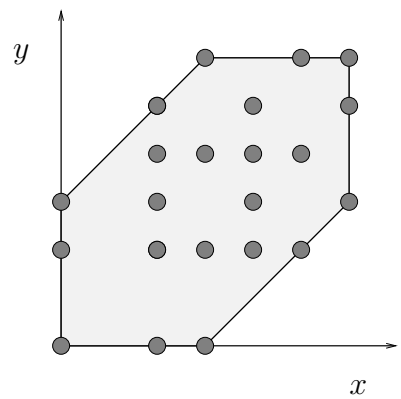

FIGURE 1. The support of $N(x,y;(2,2))$.

Before we prove the theorem we make a few remarks about the polytope Π and the geometry of the action (3.7a), (3.7b). After shifting Π by θ, we see that Π is isomorphic to the convex hull Π' of the points $\{-w\theta \mid w \in W\}$, and is thus a *Coxeterhedron* or *permutahedron* of type W. In particular, from the theory of such polytopes we know that the vertices of Π are exactly the points $\theta - w\theta$, and hence that Π has $|W|$ vertices. Shifting makes the connection between λ and μ more apparent. Suppose that λ' (respectively, μ') is the vertex of Π' obtained by translating λ (resp., μ). Then λ' and μ' are related by $\sigma_k\lambda' = \mu'$; in other words, after shifting Π to Π', the functional equations (3.7a), (3.7b) relate coefficients of monomials attached to weights that are connected by the usual reflection action of W.

It is not hard describe a set of inequalities defining Π: it is cut out by the system

$$\langle w\varpi_i, \mathbf{x} - (\theta - w\theta)\rangle \geq 0, \quad w \in W, \quad i = 1,\dots,r. \tag{3.9}$$

To prove this, one observes that the inequalities

$$\langle \varpi_i, \mathbf{x}\rangle \geq 0, \quad i = 1,\dots,r$$

define the facets containing the origin, and uses the fact that the Weyl group W acts transitively on the vertices by affine transformations. The same computation shows that the inequalities labelled by w are active at the vertex $\theta - w\theta$.

The system (3.9) is redundant. We clarify this in the following lemma, whose statement requires the notion of the *right descent set* of an element $w \in W$. By definition, this is the set $\mathscr{R}(w) = \{\sigma_i \mid l(w\sigma_i) < l(w)\}$.

Lemma 3.4. *Let $\sigma_j \in \mathscr{R}(w)$ and let $u = w\sigma_j$. Then if $j \neq k$, the inequalities*

$$\langle w\varpi_k, \mathbf{x} - (\theta - w\theta)\rangle \geq 0$$

and

$$\langle u\varpi_k, \mathbf{x} - (\theta - u\theta)\rangle \geq 0$$

are equivalent.

Proof. This follows since $\sigma_j\varpi_k = \varpi_k$ if $j \neq k$. Indeed, starting with the second inequality, we have

$$\begin{aligned}\langle u\varpi_k, \mathbf{x} - (\theta - u\theta)\rangle &= \langle u\sigma_j\varpi_k, \mathbf{x} - (\theta - u\theta)\rangle \\ &= \langle u\sigma_j\varpi_k, \mathbf{x} - (\theta - u\theta) + w\theta - w\theta\rangle \\ &= \langle w\varpi_k, \mathbf{x} - (\theta - w\theta)\rangle + \langle u\varpi_k, u\theta - w\theta)\rangle.\end{aligned}$$

(In the last line we again used $\sigma_j\varpi_k = \varpi_k$ if $j \neq k$.) By W-invariance, the second term on the last line is

$$\langle \varpi_k, \theta - \sigma_j\theta\rangle,$$

which vanishes since $\theta - \sigma_j\theta$ is a multiple of α_j. This completes the proof. □

We will also need the following geometric lemma about Π, whose statement uses the *left descent set* $\mathscr{L}(w)$ of an element $w \in W$. By definition $\mathscr{L}(w) = \{\sigma_i \mid l(\sigma_i w) < l(w)\}$. Recall that $\Phi(w)$ denotes the subset of the positive roots made negative by w. If $w = u\sigma_k$ and $l(w) = l(u) + 1$, then from the theory of Coxeter groups [7]

$$\Phi(w) = \sigma_k\Phi(u) \cup \{\alpha_k\}, \tag{3.10a}$$

$$\Phi(w^{-1}) = \Phi(u^{-1}) \cup \{u\alpha_k\}. \tag{3.10b}$$

Lemma 3.5. *Let $\mu = \theta - w\theta$ be a vertex of Π, and suppose $\sigma_k \in \mathscr{L}(w)$. Then any lattice point of the form $\mu + m\alpha_k$, where m is a positive integer, lies outside Π. Similarly, let $u = \sigma_k w$ and let $\lambda = \theta - u\theta$. Then any point of the form $\lambda - m\alpha_k$, where m is a positive integer, lies outside Π.*

Proof. For the first statement, it suffices to show that $\mu + m\alpha_k$ violates the inequalities active at μ, which are given by

$$\langle w\varpi_i, \mathbf{x} - (\theta - w\theta)\rangle \geq 0, \quad i = 1, \ldots, r.$$

Thus we want to show

$$\langle w\varpi_i, \mu + m\alpha_k - \mu\rangle = \langle w\varpi_i, m\alpha_k\rangle = -m\langle u\varpi_i, \alpha_k\rangle < 0$$

for at least one i. In fact, we will prove the stronger statement that

$$-m\langle u\varpi_i, \alpha_k\rangle < 0 \quad \text{for all } i = 1, \ldots, r.$$

Since $m > 0$, we must show

$$\langle u\varpi_i, \alpha_k\rangle = \langle \varpi_i, u^{-1}\alpha_k\rangle > 0, \quad i = 1, \ldots, r.$$

This follows if and only if $\alpha_k \notin \Phi(u^{-1})$.

So suppose on the contrary that $\alpha_k \in \Phi(u^{-1})$. By (3.10a), we have

$$\Phi(w^{-1}) = \Phi(u^{-1}\sigma_k) = \sigma_k \Phi(u^{-1}) \cup \{\alpha_k\}.$$

In particular $\sigma_k\Phi(u^{-1})$ consists of positive roots. But if $\alpha_k \in \Phi(u^{-1})$ then $\sigma_k\alpha_k$ is negative. This contradiction completes the proof of the first statement; the second statement is proved in almost exactly the same way. □

Proof of Theorem 3.2. We use induction on the length. Since $N(\mathbf{x};\ell)$ is a polynomial, we know that $a_\lambda = 0$ if λ violates the inequalities active at the origin. Indeed, otherwise $N(\mathbf{x};\ell)$ would have polar terms.

Now let $w \in W$ satisfy $l(w) > 0$, and suppose we have verified the inequalities at all vertices $\theta - u\theta$ where $l(u) < l(w)$. Let $\sigma_k \in \mathscr{R}(w)$, which is nonempty since $l(w) > 0$. Then Lemma 3.4 implies that $a_\lambda = 0$ unless λ satisfies the inequalities

$$\langle w\varpi_j, \mathbf{x} - (\theta - w\theta)\rangle \geq 0,$$

for any $j \neq k$. If $|\mathscr{R}(w)| > 1$, this shows that in fact all desired inequalities hold for the support of $N(\mathbf{x};\ell)$ at the vertex $\theta - w\theta$.

Thus we assume $\mathscr{R}(w) = \{\sigma_k\}$. We must show $a_\lambda = 0$ if λ violates

$$\langle w\varpi_k, \mathbf{x} - (\theta - w\theta)\rangle \geq 0. \tag{3.11}$$

Let $\sigma_j \in \mathscr{L}(w)$; again $\mathscr{L}(w) \neq \emptyset$ if $l(w) > 0$. Choose $\mu \in \Lambda$ such that

(1) μ violates (3.11),
(2) $a_\mu \neq 0$, and
(3) $a_{\mu'} = 0$ for all $\mu' = \mu + m\alpha_j$ with $m > 0$.

By Lemma 3.5 it is possible to find such a μ. Indeed the proof of Lemma 3.5 shows that if μ violates (3.11), so do all the points $\mu + m\alpha_j$, $m > 0$. Since $N(\mathbf{x};\ell)$ has bounded support there must be final point in the support of $N(\mathbf{x};\ell)$ on the ray $\mu + m\alpha_j$.

Now apply the relation (3.3) with σ_j to a_μ, where a_μ is the first coefficient on the right side of (3.7a), (3.7b). Note that since $a_{\mu+m\alpha_j} = 0$ for $m > 0$, as far as the right hand sides of these equations are concerned it doesn't matter whether we are in the even or the odd case. Applying σ_j produces the left hand side $qa_{\lambda-\alpha_j} + a_\lambda$ in the even case, and $q^2 a_{\lambda-2\alpha_j} - a_\lambda$ and the odd case. It is easy to see that a_λ and hence (Lemma 3.5) $a_{\lambda-\alpha_j}$ and $a_{\lambda-2\alpha_j}$ vanish by the induction hypothesis, since λ violates the inequalities active at $\theta - \sigma_j w\theta$ (cf. Figure 2). Hence a_μ vanishes. This shows that $a_\mu = 0$ unless μ satisfies (3.11), and completes the proof of the theorem. □

Remark 3.6. Theorem 3.2 shows that the support of $N(\mathbf{x};\ell)$ is contained in a translated weight polytope Π. We caution the reader that although $N(\mathbf{x};\ell)$ is constructed using the regular dominant weight θ, the polytope Π is *not* a translate of the weight

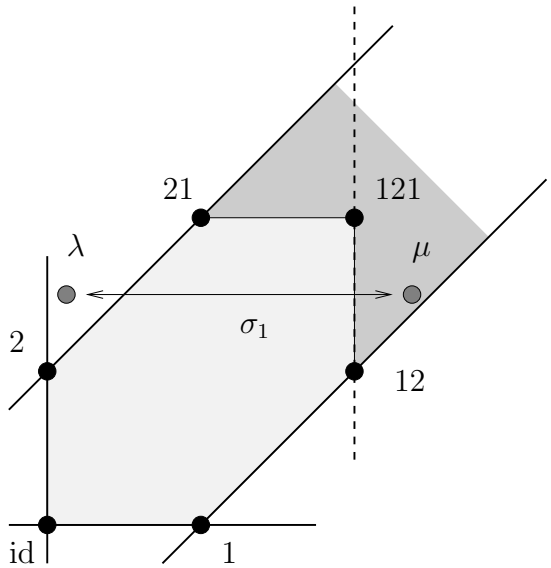

FIGURE 2. Checking the inequalities at the vertex labelled 12. By induction we assume that all desired inequalities hold at vertices labelled by w with $l(w) \leq 1$. The point μ violating the dashed inequality leads to a point λ violating the inequalities at $2 = \sigma_1 \cdot 12$.

polytope P for the representation with highest weight θ. In fact, Π is a translate of the weight polytope attached to the representation with *lowest weight* $-\theta$. This polytope differs from P in general, since $-\theta$ is not usually in the W-orbit of θ.

4. Stable coefficients

The coefficients a_λ of $N(\mathbf{x};\ell)$ attached to the vertices of Π are called the *stable coefficients*. The goal of this section is to show that the functional equations (3.3), together with the initial condition $a_0 = 1$, imply that the stable coefficients of $N(\mathbf{x};\ell)$ are given by the formulæ from [2–4].

We caution the reader on three points. The first is that in the comparison that follows, it is convenient to use a slightly different labelling convention for the vertices of Π: the element $w \in W$ now corresponds to the vertex $\theta - w^{-1}\theta$.

The second is that we are using slightly different normalizations for Gauss sums than those found in [2–4]. Hence the formula in Theorem 4.1 differs slightly from the formulæ in [2, 4], in that each factor includes a q-power denominator.

Finally, in [2], the twisting parameter $\ell = (l_1, \ldots, l_r)$ corresponds to the dominant weight $\theta' = \sum l_i \varpi_i$, whereas for us ℓ is attached to the regular dominant weight $\theta = \rho + \sum l_i \varpi_i$. Hence the results of [2, 4] are expressed in terms of the generalized height function $d'_\theta \colon \Lambda \to \mathbb{Z}$ defined by $d'_\theta(\lambda) = \langle \theta + \rho, \lambda \rangle$. Theorem 4.2, on the other

hand, uses the function $d_\theta(\lambda) = \langle \theta, \lambda \rangle$. Note that $d'_{\theta'} \equiv d_\theta$, so our statement of Theorem 4.1 is consistent with [2].

Theorem 4.1. *[2, 4] Let $\lambda = \theta - w^{-1}\theta$. Let A_λ be the stable coefficient attached to λ in [2]. Then if Φ is simply-laced, we have*

$$A_\lambda = \prod_{\alpha \in \Phi(w^{-1})} g_1(p^{d_\theta(\alpha)-1}, p^{d_\theta(\alpha)})/q^{d_\theta(\alpha)/2}, \tag{4.1}$$

where $g_1(p^a, p^b)$ is the quadratic Gauss sum, and where d_θ is the function on the root lattice defined by $d_\theta(\lambda) = \langle \theta, \lambda \rangle$.

We refer the reader to [4] for a precise definition of the Gauss sum. For our purposes all we will need to know is (4.4) below.

We now show that the stable terms of $N(\mathbf{x}; \ell)$ coincide with those given by Theorem 4.1.

Theorem 4.2. *Suppose $N(\mathbf{x}; \ell) = \sum_\lambda a_\lambda \mathbf{x}^\lambda$ where $a_0 = 1$. Then if $\lambda = \theta - w^{-1}\theta$, the coefficient a_λ is given by (4.1). In other words, $a_\lambda = A_\lambda$.*

Proof. We prove the theorem by induction on the length of w. To begin, note that $A_0 = a_0 = 1$.

Now suppose $l(\sigma_i w^{-1}) = l(w^{-1}) + 1$, and that the coefficients agree on all weights attached to $u \in W$ with $l(u) \le l(w)$. Let $\mu = \theta - \sigma_i w^{-1}\theta$. Then $\mu = \sigma_i \lambda + (l_i + 1)\alpha_i$ and $\mu > \lambda$; hence we can apply (3.7a), (3.7b), which yields

$$a_\mu = a_\lambda \times \begin{cases} q^{d(\sigma_i\lambda - \lambda + l_i\alpha_i)/2} & \text{if } \varphi_i(\mu) \text{ is even, and} \\ -q^{d(\sigma_i\lambda - \lambda + l_i\alpha_i)/2 - 1/2} & \text{if } \varphi_i(\mu) \text{ is odd.} \end{cases} \tag{4.2}$$

Note that the other coefficients of $N(\mathbf{x}; \ell)$ appearing in (3.7a), (3.7b) vanish by Theorem 3.2 and Lemma 3.5.

Applying (3.10b) in (4.1), we find

$$A_\mu = \prod_{\alpha \in \Phi(\sigma_i w^{-1})} g_1(p^{d_\theta(\alpha)-1}, p^{d_\theta(\alpha)})/q^{d_\theta(\alpha)/2} \tag{4.3a}$$

$$= g_1(p^{d_\theta(w\alpha_i)-1}, p^{d_\theta(w\alpha_i)})/q^{d_\theta(w\alpha_i)/2} \times \prod_{\alpha \in \Phi(w^{-1})} (g_1(p^{d_\theta(\alpha)-1}, p^{d_\theta(\alpha)})/q^{d_\theta(\alpha)/2}). \tag{4.3b}$$

The product on (4.3b) is just A_λ, which equals a_λ by induction. Hence we must investigate the first term, which is the Gauss sum attached to $w\alpha_i$. For the quadratic Gauss sum $g_1(p^{b-1}, p^b)$ we have

$$g_1(p^{b-1}, p^b) = \begin{cases} q^{b-1/2} & \text{if } b \text{ is odd, and} \\ -q^{b-1} & \text{if } b \text{ is even.} \end{cases} \tag{4.4}$$

Hence

$$A_\mu = A_\lambda \times \begin{cases} q^{d_\theta(w\alpha_i)/2-1/2} & \text{if } d_\theta(w\alpha_i) \text{ is odd, and} \\ -q^{d_\theta(w\alpha_i)/2-1} & \text{if } d_\theta(w\alpha_i) \text{ is even.} \end{cases} \tag{4.5}$$

It follows that to show that (4.2) and (4.5) agree, we must show that $d_\theta(w\alpha_i)$ is even if and only if $\varphi_i(\mu)$ is odd, and that $d_\theta(w\alpha_i) = d(\sigma_i\lambda - \lambda + l_i\alpha_i) + 1$.

We begin by computing $d(\sigma_i\lambda - \lambda + l_i\alpha_i)$. It is easy to see that

$$\sigma_i\lambda - \lambda + l_i\alpha_i = \mu - \lambda - \alpha_i,$$

which implies

$$d(\sigma_i\lambda - \lambda + l_i\alpha_i) = d(\mu - \lambda) - 1.$$

Now write $\mu = \sum k_j\alpha_j$. Then

$$\lambda = \sigma_i\mu + (l_i + 1)\alpha_i = \sum_{\substack{j\not\sim i \\ j\neq i}} k_j\alpha_j + \sum_{j\sim i} k_j(\alpha_i + \alpha_j) + (l_i + 1 - k_i)\alpha_i.$$

Thus

$$\mu - \lambda = \big(-\sum_{j\sim i} k_j + 2k_i - l_i - 1\big)\alpha_i,$$

and

$$d(\mu - \lambda) - 1 = 2k_i - \varphi_i(\mu) - 2 \tag{4.6}$$

is our expression for $d(\sigma_i\lambda - \lambda + l_i\alpha_i)$.

On the other hand,

$$\begin{aligned} d_\theta(w\alpha_i) &= \langle \theta, w\alpha_i\rangle \\ &= \langle \sigma_i w^{-1}\theta, -\alpha_i\rangle \\ &= \langle \theta - \sum k_j\alpha_j, -\alpha_i\rangle \\ &= -l_i - 1 + \langle \sum k_j\alpha_j, \alpha_i\rangle. \end{aligned}$$

The last equation of the above gives

$$d_\theta(w\alpha_i) = 2k_i - \varphi_i(\mu) - 1. \tag{4.7}$$

From (4.6) and (4.7), we see that $d_\theta(w\alpha_i) = d(\sigma_i\lambda - \lambda + l_i\alpha_i) + 1$. Moreover (4.7) shows that the parity of $d_\theta(w\alpha_i)$ is the opposite of that of $\varphi_i(\beta)$. This completes the proof. □

5. Unstable coefficients

In this section we investigate the implications of the relations (3.7a), (3.7b) on the coefficients a_λ attached to weights other than the vertices of Π. Such coefficients are called *unstable coefficients* [2–4]. The main result of this section, Theorem 5.7, is that in the untwisted case $\theta = \rho$, the numerator polynomial and hence $f(\mathbf{x})$ is uniquely determined by (3.7a), (3.7b) and the normalization condition $a_0 = 1$. We conclude by discussing the extent to which $N(\mathbf{x}; \ell)$ is not uniquely determined.

Recall that a weight λ is *regular* if it lies in the interior of a Weyl chamber. Equivalently, the stabilizer $P = P(\lambda)$ of λ in W is trivial. The stabilizer P of any weight is a subgroup of W generated by a subset of the simple reflections. Indeed, if λ lies on the hyperplane fixed by σ_i, then $\sigma_i \in P$, and such simple reflections generate P. Any subgroup generated by a subset of the simple reflections is called a *standard parabolic subgroup*. We recall the following basic fact about such subgroups, whose proof can be found, for example, in [7]:

Proposition 5.1. *Let $P \subset W$ be a standard parabolic subgroup. Then any coset wP contains a unique element w^P of maximal length.*

Now let $N(\mathbf{x}; \ell) = \sum a_\lambda \mathbf{x}^\lambda$ be our polynomial. By Theorem 3.2 the support of $N(\mathbf{x}; \ell)$ consists at most of the monomials $\mathbf{x}^\lambda$ where $\lambda \in \Lambda$ lies in the convex hull of the point $\theta - w\theta, w \in W$. Such λ correspond to the weights of the representation V_θ with highest weight θ, after shifting. The precise connection is as follows. Let Θ be the set of *dominant weights* of V_θ. Then the support of $N(\mathbf{x}; \ell)$ consists of all monomials $\mathbf{x}^\lambda$ where λ has the form

$$\lambda = \theta - w\xi, \quad w \in W, \, \xi \in \Theta.$$

For $\xi \in \Theta$, let O_ξ be the W-orbit $\{\theta - w\xi \mid w \in W\}$, and let $\mathscr{O} = \{O_\xi \mid \xi \in \Theta\}$. The set Θ is naturally a poset by the usual partial order on weights (note that θ is the maximal element), and we use this to turn $\mathscr{O}$ into a poset: $O_\xi \leq O_{\xi'}$ if and only if $\xi \leq \xi'$.

For any $\xi \in \Theta$, let Π_ξ be the convex hull in $\Lambda_\mathbb{R}$ of the points in O_ξ. If ξ is regular then Π_ξ is isomorphic to a permutahedron, with vertices in bijection with W. Otherwise, the orbit O_ξ has fewer than $|W|$ vertices. Indeed, if $P = P(\theta) \subset W$ is the stabilizer of ξ, then $|O_\xi|$ equals the number of cosets $|W/P|$. Accordingly, we can also write

$$O_\theta = \{\theta - wP\xi \mid w \in W\}.$$

Remark 5.2. Although we do not need it, it is known that Π_ξ is a degeneration of Π_θ obtained by contracting the edges in Π_θ labelled by the simple reflections in P. Another way to express the relationship between Π_ξ and Π_θ is to observe that Π_ξ is obtained from Π_θ by parallel translation of the facets of Π_θ until some of the faces collapse. This has the consequence that essentially the same system of inequalities

(3.9) describes the polytope Π_ξ. In particular, Π_ξ is described by

$$\langle w\varpi_k, \mathbf{x} - (\theta - wP\xi)\rangle \geq 0, \quad k = 1, \ldots, r, \quad w \in W. \tag{5.1}$$

As before this system is redundant. Also, some inequalities do not define facets, and instead are active only on higher-codimension faces. Nevertheless Π_ξ is cut out by the system (5.1).

We are now ready to prove the geometric results that allow us to analyze unstable coefficients. The main point is the following generalization of Lemma 3.5:

Lemma 5.3. *Let $\lambda = \theta - uP\xi$ be a vertex of Π_ξ, where $P = P(\xi)$, and suppose $u \in uP$ is the unique maximal element in this coset. Let $w = \sigma_k u$ and suppose $l(w) > l(u)$. Then $\mu = \theta - wP\xi$ is a different vertex of Π_ξ. Moreover, if any point of the form $\mu + m\alpha_k$, $m \geq 1$ lies in an orbit $O \in \mathscr{O}$, we have $O > O_\xi$. Similarly, if any point of the form $\lambda - m\alpha_k$, $m \geq 1$ lies in an orbit $O \in \mathscr{O}$, we have $O > O_\xi$.*

Before we prove Lemma 5.3, we need another lemma about the geometry of dominant weights:

Lemma 5.4. *Let ξ and η be weights such that $\eta > \xi$. Let η' be the unique dominant weight in the W-orbit of η. Then $\eta' > \xi$.*

Proof. Let $C(\Phi^+)$ be the cone generated by the positive roots. Then $\eta > \xi$ implies $\eta - \xi \in C(\Phi^+)$. On the other hand if η' is the dominant weight in the orbit of η, then certainly $\eta' - \eta \in C(\Phi^+)$. But then $\eta' - \xi \in C(\Phi^+)$, since $C(\Phi^+)$ is convex. This completes the proof. □

Proof of Lemma 5.3. First, it is clear that $\lambda \neq \mu$, since $l(w) > l(u)$ and u is the maximal element of the coset uP.

We now show $O > O_\xi$, where O is the orbit corresponding to $\mu + m\alpha_k$, $m > 1$. We have $\xi = w^{-1}(\theta - \mu)$. Let

$$\eta = w^{-1}(\theta - \mu - m\alpha_k) = \xi - mw^{-1}\alpha_k,$$

and let η' be the unique dominant weight in the W-orbit of η. Then $O = O_{\eta'}$. Now

$$\Phi(w^{-1}) = \Phi(u^{-1}\sigma_k) = \sigma_k\Phi(u^{-1}) \cup \{\alpha_k\}, \tag{5.2}$$

which implies

$$w^{-1}\alpha_k \in \Phi^-.$$

Thus $\eta > \xi$. By Lemma 5.4 we have $\eta' > \xi$, which proves $O > O_\xi$.

The statement about $\lambda - m\alpha_k$ is proved in almost the same way. The computation boils down to

$$\alpha_k \notin \Phi(u^{-1}).$$

This is clearly true, since by (5.2) the set $\sigma_k\Phi(u^{-1})$ consists of positive roots. □

Hence in applications of (3.7a), (3.7b), if λ and μ are weights with $\mu = \sigma_k\lambda + (l_k + 1)\alpha_k$ and $\mu > \lambda$, then we know that the weights $\lambda - 2\alpha_k$, $\lambda - \alpha_k$, $\mu + \alpha_k, \mu + 2\alpha_k$ live in bigger orbits in $\mathscr{O}$ and are attached to previously determined coefficients.

We likewise have a version of Lemma 3.5 even if σ_k fixes a vertex:

Lemma 5.5. *Let $\lambda = \theta - uP\xi$ be a vertex of Π_ξ. Let $w = \sigma_k u$ with $l(w) > l(u)$, and suppose $\mu = \theta - wP\xi$ equals λ. Then if any point of the form $\mu + m\alpha_k$, $m \geq 1$ lies in an orbit $O \in \mathscr{O}$, we have $O > O_\xi$. Similarly if any point of the form $\lambda - m\alpha_k$, $m \geq 1$ lies in an orbit O, we have $O > O_\xi$.*

Proof. The proofs of both statements are essentially the same as those of Lemma 5.3, even though the points λ, μ coincide. Again the key points are that $\alpha_k \notin \Phi(u^{-1}), \alpha_k \in \Phi(w^{-1})$. □

Remark 5.6. It is perhaps inaccurate to describe Lemmas 5.3 and 5.5 as generalizations of Lemma 3.5, since the statements are so different. However they really are the same, since it is the same geometric fact about left descent sets that is behind all of them.

Also, the inequalities (5.1) are lurking here as well. The points $\mu + m\alpha_k$, $\lambda - m\alpha_k$ lie outside Π_ξ. The fact that these points violate (5.1) again boils down to $\alpha_k \notin \Phi(u^{-1}), \alpha_k \in \Phi(w^{-1})$.

We can now prove the main result of this section.

Theorem 5.7. *Let $N(\mathbf{x}; \ell)$ be the numerator, normalized so that $a_0 = 1$. Suppose that θ is the only regular dominant weight in the representation V_θ of highest weight θ. Then all other coefficients of $N(\mathbf{x}; \ell)$ are uniquely determined by the functional equations* (3.7a), (3.7b).

Proof. The proof is by descending induction over the orbit poset $\mathscr{O}$. To begin, we know from Theorem 4.2 that all the coefficients attached to the elements of the orbit O_θ are uniquely determined once we know $a_0 = 1$.

Now fix an orbit O_ξ, where $\xi \in \Theta$ is different from θ, and assume we have determined the coefficients attached to all orbits O with $O > O_\theta$. By assumption ξ is not regular, so we can find a simple functional equation from (3.7a), (3.7b) relating the corresponding coefficient a_λ, $\lambda = \theta - \xi$, to itself. It is trivial to see that if $\varphi_k(\lambda)$ is even, then a_λ appears on both sides of (3.7a) with different coefficients; if $\varphi_k(\lambda)$ is odd then clearly a_λ appears on both sides of (3.7b) with different coefficients. By Lemma 5.5, all other $a_{\lambda'}$ in (3.7a), (3.7b) come from previously determined orbits. Thus a_λ is determined.

Now, successively applying Lemma 5.5 we can determine the remaining coefficients of the form $\theta - w\xi$, where $w \in W$. The basic strategy is the same as in the proof of Theorem 4.2; if there is more than one point in this orbit, all one needs to be able to do is move from one to another by left multiplication by a simple reflection

$$\mu := \theta - wP\xi \longrightarrow \theta - \sigma_k wP\xi =: \mu',$$

where w is maximal in the coset wP and $l(\sigma_k w) > l(w)$. Then Lemma 5.3 shows that a_μ and the coefficients from higher orbits determine $a_{\mu'}$. □

Corollary 5.8. *The only regular dominant weight for the representation V_ρ is ρ. Thus the numerator in the untwisted case, $N(\mathbf{x}) = N(\mathbf{x}; (0, \ldots, 0))$, is uniquely determined by the functional equations* (3.7a), (3.7b) *after setting $a_0 = 1$.*

Proof. Suppose $\lambda \neq \rho$ is another regular dominant weight for this representation. Write $\lambda = \sum c_i \varpi_i$. Then we must have each $c_i \neq 0$, since λ is regular. Since $\rho = \sum_i \varpi_i$, it follows that $\rho - \lambda$ must be a linear combination of the ϖ_i with nonpositive coefficients.

On the other hand, $\rho - \lambda$ is a nonnegative linear combination of the simple roots, since ρ is higher than λ in the partial order. The fundamental weights are themselves positive rational linear combinations of the simple roots, as one sees by examining the inverse Cartan matrix for any simple complex Lie algebra. Thus $\rho - \lambda$ is simultaneously a nonpositive and a nonnegative linear combination of the fundamental weights. This means $\rho = \lambda$, a contradiction. Hence ρ is the only regular dominant weight in the representation V_ρ, and by Theorem 5.7 the polynomial $N(\mathbf{x})$ is uniquely determined. □

Remark 5.9. There are other regular dominant weights $\theta \neq \rho$ such that V_θ satisfies the conditions of Theorem 5.7. For example, computations show that for $\Phi = A_r$ with $r \leq 5$ the representation $V_{\rho+\varpi_1}$ has a unique regular dominant weight; presumably this representation does for all r. We do not know another characterization of weights for any given Φ with this property.

Remark 5.10. One can also ask how big the space of possible numerators $N(\mathbf{x}; \ell)$ can be if θ does not satisfy the conditions of Theorem 5.7. Examples for A_2 show that the complex dimension of this space apparently equals the number of regular dominant weights of the representation V_θ. This should be true for all Φ, although we have not checked the details.

6. The global multiple Dirichlet series

In this final section, we describe precisely how a multiple Dirichlet series can be built up out of the p-parts. We follow the methods of [5], where the untwisted case was worked out in detail. We remark, however that the H function defined below is the analogue of the H function of [1–4] rather than that of [5]. For the relation between the two, see Remark 4.3 of [5].

For simplicity, we work over $\mathbb{Q}$, the field of rational numbers, and let $\left(\frac{d}{m}\right)$ denote the usual quadratic residue symbol for d, m odd and relatively prime. For an arbitrary global field K, one needs to work over the ring $\mathcal{O}_S$ of S-integers of K for a sufficiently large set of primes S and to replace p by $q = |\mathcal{O}_S/p\mathcal{O}_S|$. Additionally, some care

needs to be taken to define the quadratic residue symbol properly. We refer the reader to [6] and [5] for details.

Given an odd prime p and twisting parameter ℓ, we write

$$N(\mathbf{x};\ell)=\sum_{\lambda\in\Lambda}a_\lambda(p,\ell)\mathbf{x}^\lambda. \tag{6.1}$$

Fix an r-tuple of positive odd integers $\mathbf{t}=(t_1,t_2,\dots,t_r)$. The goal of this section is to define the $\mathbf{t}$-twisted multiple Dirichlet series associated to the root system Φ for $n=2$. Actually, we will need to introduce a family of series

$$Z(s_1,\dots,s_r;\Psi,\mathbf{t};\Phi)$$

where Ψ ranges over r-tuples $\Psi=(\psi_1,\psi_2,\dots,\psi_r)$ of Dirichlet characters unramified outside of 2. We abbreviate this series by $Z(\mathbf{s};\mathbf{t},\Psi)$, and it is understood that Φ remains fixed. Each series will be a sum over r-tuples of odd positive integers.

We call an r-tuple $\mathbf{m}=(m_1,\dots,m_r)$ of positive integers *odd* if each of the m_i's is odd. We denote by $\Psi(\mathbf{m})$ the product

$$\prod_i\psi_i(m_i)$$

and by $H(\mathbf{m};\mathbf{t})$ the coefficient $H(m_1,m_2,\dots,m_r;\mathbf{t})$ defined below.

Definition 6.1. The coefficient $H(m_1,m_2\dots,m_r;\mathbf{t})$ is defined by the following two conditions:

(1) Suppose $\mathbf{m}=(p^{k_1},\dots,p^{k_r})$, where p is an odd prime and $p^{l_i}||t_i$. Suppose $\lambda=\sum_{i=1}^r k_i\alpha_i\in\Lambda$. Then

$$H(p^{k_1},\dots,p^{k_r};\mathbf{t})=a_\lambda(p;\ell)$$

where $\ell=(l_1,\dots,l_r)$.

(2) Given m_j,m_j' odd with $(m_1m_2\cdots m_r,m_1'm_2'\cdots m_r')=1$ we have

$$\frac{H(m_1m_1',\dots,m_rm_r';\mathbf{t})}{H(m_1,\dots,m_r;\mathbf{t})H(m_1',\dots,m_r';\mathbf{t})}=\prod_{\substack{i,j\text{ adj.}\\ i<j}}\left(\frac{m_i}{m_j'}\right)\left(\frac{m_i'}{m_j}\right)$$

Finally, for an r-tuple $\mathbf{s}=(s_1,\dots,s_r)$ of complex numbers, define

$$Z(\mathbf{s};\mathbf{t},\Psi)=N(\mathbf{s})\sum_{\mathbf{m}=(m_1,m_2,\dots,m_r)\ odd}\frac{\Psi(\mathbf{m})H(\mathbf{m};\mathbf{t})}{\prod_j m_j^{s_j}}\prod_{i=1}^r\left(\frac{t_i^\#}{\hat{m}_i}\right) \tag{6.2}$$

where $t_i^\#$ is the squarefree part of t_i, $\hat{m}_i$ is the part of m_i relatively prime to $t_i^\#$ and $N(\mathbf{s})$ is the normalizing zeta factor

$$N(\mathbf{s})=\prod_{\alpha\in\Phi^+}\zeta(2\langle\alpha,\mathbf{s}\rangle-d(\alpha)+1),\quad \langle\alpha,\mathbf{s}\rangle=\alpha_1s_1+\cdots+\alpha_rs_r. \tag{6.3}$$

The series (6.2) converges for $\Re(s_i) \gg 1$, $1 \leq i \leq r$.

Now, given the invariance of the p-parts of $Z(\mathbf{s};\mathbf{t},\Psi)$ under the action of W defined by (2.4), we may mimic the techniques of Section 5 of [5] to show the following: for fixed twisting parameter $\mathbf{t}$, the vector of all $Z(\mathbf{s};\mathbf{t},\Psi)$ has analytic continuation and satisfies functional equations relating the values at $\mathbf{s} = (s_1, \ldots, s_r)$ to the values at $\sigma_{j_0}\mathbf{s} = (s'_1, \ldots, s'_r)$ for $j_0 = 1, 2, \ldots, r$, where

$$s'_j = \begin{cases} s_j + s_{j_0} - 1/2 & \text{if } j \text{ and } j_0 \text{ are adjacent,} \\ 1 - s_{j_0} & \text{if } j = j_0, \text{ and} \\ s_j & \text{otherwise.} \end{cases} \tag{6.4}$$

These functional equations are involutions generating a group of functional equations of $Z(\mathbf{s};\mathbf{t},\Psi)$. We then deduce

Theorem 6.2. *Fix a twisting parameter* $\mathbf{t}$. *Each function* $Z(\mathbf{s};\mathbf{t},\Psi)$ *has analytic continuation to* $\mathbb{C}^r$. *The collection of these functions as* Ψ *ranges over* r*-tuples of Dirichlet characters unramified outside of* 2 *satisfies a group of functional equations isomorphic to* W. *Finally, each* $Z(\mathbf{s};\mathbf{t},\Psi)$ *is analytic outside the hyperplanes* $(w\mathbf{s})_j = 1$ *for* $w \in W, 1 \leq j \leq r$, *where* $(w\mathbf{s})_j$ *denotes the* j^{th} *component of* $w\mathbf{s}$.

The proof of the theorem is very similar to the proof of Theorem 5.5 of [5], and so we leave the details to the reader.

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