Cohomology of congruence subgroups of SL4(Z)

ON THE COHOMOLOGY OF CONGRUENCE SUBGROUPS OF $\mathrm{SL}_4(\mathbb{Z})$

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Abstract. We survey our joint work with Avner Ash and Mark McConnell that computationally investigates the cohomology of conguence subgroups of $\mathrm{SL}_4(\mathbb{Z})$.

1. Introduction

Let $\mathbf{G}$ be a semisimple algebraic group defined over $\mathbb{Q}$ and let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic group. The cohomology $H^*(\Gamma;\mathbb{C})$ plays an important role in number theory, in that it provides a concrete realization of certain automorphic forms (§2). For instance, if $\mathbf{G} = \mathrm{SL}_2$ and Γ is a congruence subgroup, then by the Eichler–Shimura isomorphism the cohomology space $H^1(\Gamma;\mathbb{C})$ is built from weight two modular forms of level N (§2.4). Not every automorphic form can appear in the cohomology of an arithmetic group, but those that do are widely believed to be related to arithmetic geometry, such as Galois representations and motives (§3). The cohomology and its structure as a Hecke module can be computed very explicitly in many cases. Thus the cohomology of arithmetic groups provides a tool to formulate and test conjectures about the links between automorphic forms and arithmetic.

In a series of paper with Avner Ash and Mark McConnell [8–10], we have investigated the group cohomology for $\mathbf{G} = \mathrm{SL}_4$ and $\Gamma = \Gamma_0(N)$ the congruence subgroup of $\mathrm{SL}_4(\mathbb{Z})$ with bottom row congruent to $(0,0,0,*)$ mod N. Our work has focused on a particular cohomology space attached to Γ, namely $H^5(\Gamma;\mathbb{C})$. (The choice of degree 5 is explained below.) We have computed the dimension of this cohomology space for prime levels $N \leq 211$, and in many cases have computed the action of the Hecke operators.

The purpose of this note is to give an introduction to [8–10] and to explain how we perform these computations (§4). We also discuss our computational results (§5), which indicate relationships between $H^5(\Gamma;\mathbb{C})$ and elliptic modular forms, the cohomology of subgroups of $\mathrm{SL}_3(\mathbb{Z})$, and certain Siegel modular forms, the paramodular forms.

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2. Cohomology and automorphic forms

In this section we briefly explain what the cohomology of arithmetic groups has to do with automorphic forms. For more information we refer to [13, 14, 22, 30, 36, 38, 42].

2.1. Let $\mathbf{G}$ be a semisimple connected algebraic group defined over $\mathbb{Q}$, and let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup. Let E be a finite-dimensional rational complex representation of $\mathbf{G}(\mathbb{Q})$. Then E is naturally a $\mathbb{Z}\Gamma$-module, and we are interested in the group cohomology $H^*(\Gamma; E)$.

We can compute $H^*(\Gamma; E)$ topologically as follows. Let $G = \mathbf{G}(\mathbb{R})$ be the group of real points of $\mathbf{G}$; G is a semisimple Lie group. Let $K \subset G$ be a maximal compact subgroup, and let $X = G/K$ be the associated global symmetric space. The space X is contractible with a left Γ-action. If Γ is torsion-free, then $\Gamma\backslash X$ is an Eilenberg–Mac Lane space, and we have

$$H^*(\Gamma; E) = H^*(\Gamma\backslash X; \mathscr{E}), \tag{2.1}$$

where $\mathscr{E}$ is the local coefficient system on the manifold $\Gamma\backslash X$ attached to E. In fact, since we consider only complex representations E, the isomorphism (2.1) holds even if Γ has torsion. In this case the quotient $\Gamma\backslash X$ is an orbifold, but nevertheless one can construct a local system $\mathscr{E}$ on the quotient such that (2.1) remains true.

2.2. The first step in getting automorphic forms into the picture is the de Rham theorem. Let $\Omega^p = \Omega^p(X, E)$ be the space of E-valued p-forms on X, and let $\Omega^p(X, E)^\Gamma$ be the subspace of Γ-invariant forms. We have a differential $d\colon \Omega^p \to \Omega^{p+1}$ and an isomorphism of cohomology spaces [13, Theorem 2.2.2]

$$H^*(\Gamma; E) = H^*(\Omega^*(X, E)^\Gamma).$$

Now recall that $X = G/K$. Let $\mathfrak{g}$ (respectively $\mathfrak{k}$) be the Lie algebra of G (resp. K). We can identify the tangent space to X at the basepoint with the quotient $\mathfrak{g}/\mathfrak{k}$. Since differential forms are sections of the exterior powers of the tangent bundle, one can show [42, Prop. 1.5]

$$\Omega^p(\Gamma\backslash X, \mathbb{C}) = \mathrm{Hom}_K(\wedge^p(\mathfrak{g}/\mathfrak{k}), C^\infty(\Gamma\backslash G)),$$

where on the right we take complex-valued smooth functions on $\Gamma\backslash G$, and where K acts on $\wedge^p(\mathfrak{g}/\mathfrak{k})$ by the adjoint action and on $C^\infty(\Gamma\backslash G)$ by right translations. More generally, if we want cohomology with coefficients, then we take E-valued differential forms, and we have

$$\Omega^p(\Gamma\backslash X, E) = \mathrm{Hom}_K(\wedge^p(\mathfrak{g}/\mathfrak{k}), C^\infty(\Gamma\backslash G) \otimes E). \tag{2.2}$$

The right hand side of (2.2) inherits a differential from the left. The resulting cohomology spaces are denoted

$$H^*(\mathfrak{g}, K; C^\infty(\Gamma\backslash G) \otimes E).$$

This cohomology is called the *relative Lie algebra cohomology of* $(\mathfrak{g}, K)$ *with coefficients in* E, or simply $(\mathfrak{g}, K)$*-cohomology.*

2.3. To summarize, we have identifications

$$H^*(\Gamma; E) = H^*(\Gamma\backslash X; \mathscr{E}) = H^*(\mathfrak{g}, K; C^\infty(\Gamma\backslash G) \otimes E). \tag{2.3}$$

We can use (2.3) to identify important subspaces of the cohomology.

For instance consider the space $L^2(\Gamma\backslash G)$. Let $L^2_{\mathrm{disc}}(\Gamma\backslash G) \subset L^2(\Gamma\backslash G)$ be the discrete part, and let $L^2_{\mathrm{cusp}}(\Gamma\backslash G)$ be the subspace of cusp forms [25]. We have inclusions

$$L^2_{\mathrm{cusp}}(\Gamma\backslash G)^\infty \hookrightarrow L^2_{\mathrm{disc}}(\Gamma\backslash G)^\infty \hookrightarrow C^\infty(\Gamma\backslash G),$$

where the ∞ indicates that we take the subspaces of smooth vectors. These induce a map

$$H^*(\mathfrak{g}, K; L^2_{\mathrm{cusp}}(\Gamma\backslash G)^\infty \otimes E) \to H^*(\mathfrak{g}, K; C^\infty(\Gamma\backslash G) \otimes E)$$

that is in fact injective. The image $H^*_{\mathrm{cusp}}(\Gamma; E) \subset H^*(\Gamma; E)$ is called the *cuspidal cohomology*. In a certain sense, this is the most interesting constituent of the cohomology.

Thus $H^*(\Gamma; E)$ contains a subspace corresponding to certain cuspidal automorphic forms. It is natural to ask if the rest of the cohomology can be built from automorphic forms as well. More precisely, let

$$A(\Gamma, G) \subset C^\infty(\Gamma\backslash G)$$

be the subspace of automorphic forms [25], that is, $A(\Gamma, G)$ is consists of functions of moderate growth that are right K-finite and left $Z(\mathfrak{g})$-finite, where $Z(\mathfrak{g})$ denotes the center of the universal enveloping algebra $U(\mathfrak{g})$. Then we have the following theorem of Franke [17], which verifies a conjecture of Borel:

Theorem 2.1. [17] *The inclusion $A(\Gamma, G) \to C^\infty(\Gamma\backslash G)$ induces an isomorphism*

$$H^*(\mathfrak{g}, K; A(\Gamma, G) \otimes E) \to H^*(\mathfrak{g}, K; C^\infty(\Gamma\backslash G) \otimes E).$$

Thus we can think of $H^*(\Gamma; E)$ as being a concrete realization of certain automorphic forms, namely those whose associated automorphic representations have nonvanishing $(\mathfrak{g}, K)$-cohomology. The unitary representations with this property were classified by Vogan–Zuckerman [43]. In general the automorphic forms contributing to the cohomology are far from typical, in the sense that most automorphic forms cannot appear in the cohomology. Nevertheless, those that do are very interesting, and are expected to have direct connections with arithmetic (cf. §3).

2.4. A prototypical example of the connection between cohomology and automorphic forms is that of $\mathbf{G} = \mathrm{SL}_2$. In this case we have $G = \mathrm{SL}_2(\mathbb{R})$, $K = \mathrm{SO}(2)$, and the symmetric space X is the upper halfplane. Let $\Gamma = \Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$, the subgroup of matrices upper triangular modulo N. Let E_k be the k-dimensional complex representation of G, say on the vector space of degree $k-1$ homogeneous complex polynomials in two variables. By work of Eichler and Shimura, we have

$$H^1(\Gamma; E_k) \simeq S_{k+1}(\Gamma) \oplus \overline{S}_{k+1}(\Gamma) \oplus \mathrm{Eis}_{k+1}(\Gamma),$$

where S_{k+1} is the space of holomorphic weight $k+1$ modular forms, Eis_{k+1} is the space of weight $k+1$ Eisenstein series, and the bar denotes complex conjugation. Thus holomorphic modular forms of weights ≥ 2 appear in the cohomology; Maass forms and weight 1 holomorphic forms do not.

2.5. In general the quotient $\Gamma\backslash X$ has cohomology in many degrees. Which cohomology groups are the most interesting?

Clearly $H^i(\Gamma\backslash X;\mathscr{E}) = 0$ if $i > \dim\Gamma\backslash X = \dim X$. But one actually knows that sometimes the cohomology vanishes in degrees less than this. More precisely, let $q = q(\mathbf{G})$ be the $\mathbb{Q}$-rank of $\mathbf{G}$, which by definition is the dimension of a maximal torus split over $\mathbb{Q}$. Then we have the following theorem of Borel–Serre:

Theorem 2.2. [12] *For all Γ and E as above, we have $H^i(\Gamma;E) = 0$ if $i > \dim X - q$.*

The number $\nu = \dim X - q$ is called the *virtual cohomological dimension.*

One also knows that the cuspidal cohomology does not necessarily appear in every cohomological degree. In fact, for SL_n one can show that $H^i_{\mathrm{cusp}}(\Gamma;E) = 0$ unless the degree i lies in a small interval about $(\dim X)/2$ [30]. As we shall see, this makes computations much more difficult to perform for large n. Table 1 shows the cuspidal range for subgroups of $\mathrm{SL}_n(\mathbb{Z})$ for $n \leq 9$ [35].

n	2	3	4	5	6	7	8	9
$\dim X$	2	5	9	14	20	27	35	44
$\nu(\Gamma)$	1	3	6	10	15	21	28	36
top degree of H^*_{cusp}	1	3	5	8	11	15	19	24
bottom degree of H^*_{cusp}	1	2	4	6	9	12	16	20

Table 1. The virtual cohomological dimension and the cuspidal range for subgroups of $\mathrm{SL}_n(\mathbb{Z})$.

3. Connections with arithmetic geometry

The groups $H^*(\Gamma;E)$ have an action of the *Hecke operators*, which are endomorphisms of the cohomology associated to certain finite index subgroups of Γ. The eigenclasses of these operators reveal the arithmetic information in the cohomology. We review this now.

3.1. We begin with Galois representations. Let $\mathbf{G} = \mathrm{SL}_n/\mathbb{Q}$ and let $\Gamma = \Gamma_0(N)$ be as in the introduction. Let $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of $\mathbb{Q}$. Let $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\mathbb{Q}_p)$ be a continuous semisimple Galois representation unramified outside pN. For any prime l not dividing pN let Frob_l be the Frobenius conjugacy class over l. Then we can consider the characteristic polynomial

$$\det(1 - \rho(\mathrm{Frob}_l)T) \in \mathbb{Q}_p[T].$$

3.2. On the cohomology side, for each prime l not dividing N we have Hecke operators $T(l,k)$, $k = 1,\ldots,n-1$. These operators generalize the classical operator T_l on modular forms; for an exposition of these operators and the structure of the algebra they generate, see [37, Ch. 3]. If ξ is a simultaneous eigenclass for these operators, define the *Hecke polynomial*

$$H(\xi) = \sum_k (-1)^k l^{k(k-1)/2} a(l,k) T^k \in \mathbb{C}[T].$$

where $a(l,k)$ is the eigenvalue of $T(l,k)$.[1]

3.3. Now fix an isomorphism $\varphi\colon \mathbb{C} \to \overline{\mathbb{Q}}_p$. Here is one way to express the conjectural connection between cohomology and arithmetic:

Conjecture 3.1. *For any Hecke eigenclass ξ of level N, there is a Galois representation $\rho\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{Q}_p)$ unramified outside pN such that for every prime l not dividing pN, we have*

$$\varphi(H(\xi)) = \det(1 - \rho(\mathrm{Frob}_l)T).$$

This conjecture is a translation to the setting of the cohomology of arithmetic groups of a more general conjecture of Clozel [15, Conjecture 4.5], which precisely predicts the cuspidal automorphic representations that should be attached to motives. This is the conjecture that we are ultimately testing.

We are primarily interested in Conjecture 3.1 in the case of eigenclasses that are not *essentially selfdual*. (By definition, a Hecke eigenclass is essentially selfdual if the associated automorphic representation π satisfies $\pi \simeq \widetilde{\pi} \otimes \chi$, where the tilde denotes contragredient and χ is a 1-dimensional automorphic representation.) This is because in many cases one knows how to attach motives to essentially selfdual eigenclasses by realizing the motive in the étale cohomology of a Shimura variety (see for instance [16], which treats certain selfdual classes). For eigenclasses that are not essentially selfdual it is unknown in general how to do this. In our case, for $n \geq 3$ the symmetric space $\Gamma\backslash X$ is not an algebraic variety, and it is completely unclear how to directly connect the Galois group with Hecke eigenclasses.

4. Techniques

In this section, we describe the topological techniques we use to compute the cohomology and the action of the Hecke operators.

4.1. For modular forms, i.e. the cohomology of subgroups of $\mathrm{SL}_2(\mathbb{Z})$, we can use *modular symbols* to perform computations. Recall that the symmetric space X in this case is the upper halfplane. Let $X^* = X \cup \mathbb{Q} \cup \{i\infty\}$, the standard partial compactification of X obtained by adjoining cusps; equip X^* with the Satake topology. Given two cusps $q_1, q_2 \in X^* \smallsetminus X$, we can form the ideal geodesic in X from q_1 to q_2 and can look at its image in $\Gamma\backslash X^*$. This gives a relative homology class

$$[q_1, q_2] \in H_1(\Gamma\backslash X^*, \mathrm{cusps}; \mathbb{C}) \simeq H^1(\Gamma\backslash X; \mathbb{C}). \tag{4.1}$$

Let V be the complex vector space generated by the symbols $[q_1, q_2]$. Let S be V modulo the subspace spanned by all elements of form $[q_1, q_2] + [q_2, q_1]$. The group Γ acts on V by

$$\gamma \cdot [q_1, q_2] = [\gamma q_1, \gamma q_2].$$

[1]Note that the Hecke polynomial $H(\xi)$ is normalized such that if $H(\xi)$ were used as a local factor for the L-function of the automorphic representation π_ξ attached to ξ, via the substitution $T = l^{-s}$, then the functional equation of the L-function would have the form $L(s, \pi_\xi) = L(n-s, \tilde{\pi}_\xi)$, where the tilde denotes contragredient. This should be kept in mind when studying the examples in §5.

This action descends to S, and we let S_Γ be the quotient of coinvariants. The space S_Γ is called the space of *modular symbols*. We obtain a map

$$S_\Gamma \to H^1(\Gamma;\mathbb{C}) \tag{4.2}$$

induced by (4.1) whose kernel is easily determined: it is spanned by sums of the form

$$[q_1,q_2]+[q_2,q_3]+[q_3,q_1]. \tag{4.3}$$

Let M_Γ be the quotient of S_Γ by the subspace generated by (4.3); this space is isomorphic to $H^1(\Gamma;\mathbb{C})$.

Thus modular symbols provide a combinatorial model for $H^1(\Gamma;\mathbb{C})$. One can also enrich modular symbols by adding extra data to account for the representation E_k from §2.4 to compute with modular forms of higher weights; see [38] for more details.

4.2. The Hecke action on the cohomology can be encoded in an action on the modular symbols. Suppose we are given a Hecke operator T_l. The construction of the Hecke algebra guarantees that we can find finitely many matrices $\{\gamma_i\} \subset \mathrm{GL}_2(\mathbb{Q}) \cap M_2(\mathbb{Z})$ such that, if $\xi = [q_1,q_2] \in S_\Gamma$, then

$$T_l\xi = \sum_i [\gamma_i q_1, \gamma_i q_2]. \tag{4.4}$$

It is easy to verify that this is an action of the Hecke operators on S_Γ that is compatible with the isomorphism (4.2).

4.3. Hence we have a concrete model for the cohomology space $H^1(\Gamma;\mathbb{C})$, as well as a Hecke action on our model. But unfortunately this is not good enough for practical computation of Hecke eigenvalues.

The first problem is that the space S_Γ is infinite-dimensional. Since $H^1(\Gamma;\mathbb{C})$ is finite-dimensional, one might expect to be able to identify a finite-dimensional subspace of S_Γ that maps onto M_Γ. A very convenient subspace with this property is provided by the *unimodular symbols*. These are the images in S_Γ of the $\mathrm{SL}_2(\mathbb{Z})$-translates of $[0, i\infty]$. It is easy to see that these images generate a finite-dimensional subspace that is still large enough to surject onto the cohomology. For instance, if $\Gamma = \Gamma(N)$, the principal congruence subgroup of matrices congruent to the identity mod N, then on $X(N) = \Gamma(N)\backslash X^*$ the ideal geodesics giving the unimodular symbols become the edges of a highly-symmetric triangulation. The unimodular subspace of S_Γ is just the first chain group attached to this triangulation, and the relations (4.3) correspond to the boundaries of triangles.

4.4. But now one encounters another problem. The Hecke operators act on modular symbols, and the unimodular symbols span the cohomology, but the Hecke operators do not preserve the subspace of unimodular symbols. In fact, no finite set of modular symbols modulo Γ admits a Hecke action.

This is easy to see; one can attach a integer "determinant" $n(\xi)$ to a modular symbol $\xi = [q_1,q_2]$ as follows. First write $q_i = a_i/b_i$, $i = 1,2$, where the rational numbers are in reduced terms. Then define $n(\xi) = |a_1b_2 - a_2b_1|$. The integer $n(\xi)$ is well-defined modulo the defining relations for S_Γ. We have $n(\xi) = 1$ if and only if ξ is unimodular.[2] Now if

[2]This is of course why they are called unimodular.

one applies T_l to a modular symbol ξ with $n(\xi) = d$, then in general the modular symbols on the right of (4.4) will have determinant ld. Since we can take l to arbitrarily large, no finite spanning set can be preserved by all the T_l.

Nevertheless one can circumvent this difficulty. There is an algorithm—due to Manin [31]—that enables one to write any modular symbol as a linear combination of unimodular symbols. Essentially Manin's algorithm is nothing more than the continued fraction algorithm. Conjugating by $\mathrm{SL}_2(\mathbb{Q})$ we may assume the nonunimodular symbol ξ has the form $[0, q]$. Then ξ is taken to a linear combination of the symbols $\xi_i = [a_i/b_i, a_{i+1}/b_{i+1}]$, where the a_i/b_i are the convergents in the continued fraction expansion of q. Standard facts from elementary number theory imply that the ξ_i are unimodular. This enables us to compute the effect of a Hecke operator on a basis of unimodular symbols, which allows us compute the Hecke eigenvalues and eigenclasses.

4.5. Now we consider $n > 2$. First of all, there is an analogue of the modular and unimodular symbols, and they provide a model for $H^\nu(\Gamma; \mathbb{C})$. The space of modular symbols is generated by n-tuples of cusps $[q_1, \ldots, q_n]$ modulo relations, where now cusp means a minimal boundary component in a certain Satake compactification X^* of X [34]. There is an analogue of Manin's algorithm, due to Ash–Rudolph [11], that enables us to compute the Hecke action on $H^\nu(\Gamma; \mathbb{C})$.

The problem now is that *usually* $H^\nu_{\mathrm{cusp}} = 0$. In fact, looking at Table 1, we see that ν lies in the cuspidal range only for SL_2, SL_3. Thus in these cases we can use modular symbols, together with the Ash–Rudolph algorithm, to compute Hecke eigenclasses. For SL_3 this was done originally by Ash–Grayson–Green [7], and later by van Geemen–Top [40, 41], as well as some other groups. As a result of these investigations, we have lots of convincing evidence that cuspidal cohomology is related to arithmetic geometry [1, 3, 40].

4.6. Our case of interest is $n = 4$. Here there are several tools that allow us to overcome the problems above. Most of these tools work for all n: in principle we can compute the cohomology for all n, even though this computation quickly gets impractical as n increases. On the other hand, we do not know if the Hecke operator algorithm will generalize to allow computation of the Hecke eigenvalues in the cuspidal range for $n > 4$.

We can construct an explicit homological complex S_k, $k \geq 0$, the *sharbly complex*, that computes $H^*(\Gamma; \mathbb{C})$. This complex, defined by Ash [6], generalizes the modular symbols, but now we have chain groups in all nonnegative degrees. The sharbly complex has a Γ-action, and by the Borel–Serre duality theorem [12] the homology of the complex of coinvariants $(S_*)_\Gamma$ gives the cohomology of Γ. For example, in the case of SL_2, the space S_0 coincides with the space S from §4.1, and the image of the boundary $S_1 \to S_0$ consists of sums of the form (4.3). Passing to the coinvariants we obtain our presentation of $H^1(\Gamma; \mathbb{C})$

The sharbly complex admits a Hecke action, and as before there are infinitely many sharblies modulo Γ. The analogue of the unimodular symbols is a certain subcomplex of *reduced sharblies* that is finite mod Γ. This subcomplex can be constructed using the well-rounded retract of Ash [5], or equivalently Voronoi's work on perfect quadratic forms [44].

To compute the Hecke operators, we need an analogue of the Manin trick to compute with $H^{\nu-1}(\Gamma; \mathbb{C})$. This is actually much more subtle since we work below the cohomological dimension. The technique we have developed to do this has been described in several places

[19, 21, 38], so we only give the idea. The main point is that to compute the cohomology $H^{\nu-1}(\Gamma;\mathbb{C})$ one works with sharbly cycles consisting of n-simplices glued together along "submodular symbols." (In the case of $\mathrm{SL}_2(\mathbb{Z})$, such cycles are built of geodesic triangles with usual modular symbols as faces.) To compute the Hecke action, one must move a general sharbly cycle to a sum of reduced cycles by simultaneously applying the Ash–Rudolph algorithm to each of its submodular symbols. For more details we refer to the above references.

Remark 4.1. The technique described in [19] to compute the Hecke operators on the cohomology of subgroups of $\mathrm{SL}_4(\mathbb{Z})$ is also useful in other contexts. One needs a linear group where the cuspidal range goes up to one below the cohomological dimension. We have explored this in some cases:

- $\mathbf{G} = R_{F/\mathbb{Q}}(\mathrm{GL}_2)$, where F is real quadratic [21]. This is the *Hilbert modular case*; the cohomology classes are related to weight $(2,2)$ Hilbert modular forms over F. Note that we use GL_2 instead of SL_2. We do this because the GL_2-symmetric space X_{GL_2} can be represented as a certain cone of positive-definite binary quadratic forms modulo homotheties. One can then apply the generalization of Voronoĭ's theory of perfect forms, see for example [2, 4, 28, 29] to get a cellular decomposition of X_{GL_2}. However this does not affect the results of the computation in a serious way, because the associated locally symmetric space $\Gamma\backslash X_{\mathrm{GL}_2}$ is a circle bundle over a Hilbert modular surface.
- $\mathbf{G} = R_{F/\mathbb{Q}}(\mathrm{GL}_2)$, where F is complex quartic [20]. Again we work with the GL_2-symmetric space.

5. Results

We have computed $H^5(\Gamma_0(N);\mathbb{C})$ for N prime and ≤ 211, and for composite N up to 52. Our biggest computation involved computing the Smith normal form of a matrix of size 845712×3277686 ($N = 211$). Such computations cannot be done by merely using existing numerical linear algebra software, because of the special needs of our computation. For instance, instead of working over $\mathbb{C}$ we actually work over a large finite field. We also need to compute change of basis matrices to be able to compute Hecke operators. The specific numerical techniques we developed are described in detail in [10, §2].

Our computation results can be summarized as follows:

- In the range of our computations, we found no nonselfdual cuspidal classes. We know of no reason why they should not exist for larger N, but no one has proven their existence.[3]
- We found Eisenstein classes (§5.1) attached to weight 2 and weight 4 modular forms (§5.2).
- We found Eisenstein classes attached to SL_3 cuspidal cohomology (§5.3).
- We found selfdual cuspidal classes that are apparently functorial lifts of Siegel modular forms (§5.4).

[3]At the conference, Neil Dummigan explained how some of his recent work suggests that there should exist nonselfdual classes on SL_4 at level 1 and high weight.

For N prime we believe that our description of the Eisenstein cohomology is complete. However, if N is not prime then there are other Eisenstein classes, as mentioned already in [8].

5.1. **Eisenstein cohomology.** We return for the moment to a general setting. Our goal is to explain the notion of Eisenstein cohomology, due to Harder [24]. We use the notation of §2.1, so that X is a global symmetric space and Γ is an arithmetic group.

Let $\overline{X}$ be the partial compactification of X due to Borel–Serre [12]. This is an enlargment of X that adds additional spaces at infinity. The quotient $\Gamma\backslash\overline{X}$ is called the *Borel–Serre compactification* of $\Gamma\backslash X$. If Γ is torsion-free, then $\Gamma\backslash\overline{X}$ has the structure of a manifold with corners. Otherwise, $\Gamma\backslash\overline{X}$ is an orbifold with corners. Let $\partial(\Gamma\backslash\overline{X}) = \Gamma\backslash\overline{X} \smallsetminus \Gamma\backslash X$ be the boundary.

The homotopy equivalence $\Gamma\backslash\overline{X} \to \Gamma\backslash X$ induces an isomorphism

$$H^*(\Gamma\backslash\overline{X};\mathbb{C}) \simeq H^*(\Gamma\backslash X;\mathbb{C}),$$

and the inclusion $\partial(\Gamma\backslash\overline{X}) \hookrightarrow \Gamma\backslash\overline{X}$ induces a restriction map

$$H^*(\Gamma\backslash\overline{X};\mathbb{C}) \to H^*(\partial(\Gamma\backslash\overline{X});\mathbb{C}).$$

The *Eisenstein classes* are certain classes that form a basis of the image of a splitting of this restriction map.

5.2. **Weights 2 and 4.** Now let $\Gamma = \Gamma_0(N) \subset \mathrm{SL}_4(\mathbb{Z})$, where N is prime. In $H^5(\Gamma;\mathbb{C})$ we found Eisenstein classes corresponding to weight 2 and weight 4 holomorphic modular forms of level N.

Let f be a weight 2 newform of level N. Then f contributes to $H^5(\Gamma;\mathbb{C})$ in two different ways, with the Hecke polynomials

$$(1-l^2T)(1-l^3T)(1-\alpha T+lT^2)$$

and

$$(1-T)(1-lT)(1-l^2\alpha T+l^5T^2),$$

where $T_l f = \alpha f$.

Now let g be a weight 4 newform. Then whether or not g contributes an Eisenstein class depends on the central special value of the L-function of g. Namely, if this special value vanishes, then g corresponds to an eigenclass with Hecke polynomial

$$(1-lT)(1-l^2T)(1-\beta T+l^3T^2),$$

where $T_l g = \beta g$. This dependence on special values is typical for Eisenstein cohomology. We remark also that this particular Eisenstein class is apparently a "ghost class" in the sense of [23]. That is, this class in $H^5(\Gamma;\mathbb{C})$ restricts nontrivially to the boundary of the Borel–Serre compactification, but it restricts trivially on each face of the boundary.

5.3. SL_3 **cuspidal classes.** We also found cohomology classes corresponding to cuspidal cohomology classes on subgroups of $\mathrm{SL}_3(\mathbb{Z})$. These cohomology classes were originally computed by Ash–Grayson–Green [7].

Let $\eta \in H^3_{\mathrm{cusp}}(\Gamma_0^*(N);\mathbb{C})$ be a cuspidal cohomology class on $\Gamma_0^*(N) \subset \mathrm{SL}_3(\mathbb{Z})$. Let γ, γ' be the Hecke eigenvalues of this class with respect to the Hecke operators $T(l,1), T(l,2)$ (computed of course when $n = 3$.). Then η contributes in two different ways, with the Hecke polynomials

$$(1 - l^3T)(1 - \gamma T + l\gamma' T^2 - l^3T^3)$$

and

$$(1 - T)(1 - l\gamma T + l^3\gamma' T^2 - l^6T^3).$$

5.4. **Siegel modular forms.** Finally, we describe the contributions to the cohomology coming from Siegel modular forms. For more background on Siegel modular forms we refer to [26, 39].

Let $K(p)$ be the *paramodular group* of prime level, which by defintion is the subgroup

$$K(p) = \begin{pmatrix} \mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & p^{-1}\mathbb{Z} \\ \mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} \end{pmatrix} \subset \mathrm{Sp}_4(\mathbb{Q}).$$

Let $S^3(p)$ be the space of weight three paramodular forms. Note that all such forms are cuspforms; there are no Eisenstein series. This space contains the subspace $S^3_{\mathrm{G}}(p)$ of *Gritsenko lifts*, which are lifts from certain weight 3 Jacobi forms to $S^3(p)$ [18]. Let $S^3_{\mathrm{nG}}(p)$ be the Hecke complement to $S^3_{\mathrm{G}}(p)$ in $S^3(p)$. The forms in $S^3_{\mathrm{nG}}(p)$ will be those that appear in the cohomology.

The space of cuspidal paramodular forms at prime level is known pretty explicitly. First we have a dimension formula due to Ibukiyama [26, 27].

Let $\kappa(a)$ be the Kronecker symbol $(\frac{a}{p})$. Define functions $f, g\colon \mathbb{Z} \to \mathbb{Q}$ by

$$f(p) = \begin{cases} 2/5 & \text{if } p \equiv 2, 3 \bmod 5, \\ 1/5 & \text{if } p = 5, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g(p) = \begin{cases} 1/6 & \text{if } p \equiv 5 \bmod 12, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5.1 (Ibukiyama)**.** *For p prime we have* $\dim S^3(2) = \dim S^3(3) = 0$. *For $p \geq 5$, we have*

$$\begin{aligned} \dim S^3(p) = {} & (p^2 - 1)/2880 \\ & + (p+1)(1 - \kappa(-1))/64 + 5(p-1)(1 + \kappa(-1))/192 \\ & + (p+1)(1 - \kappa(-3))/72 + (p-1)(1 + \kappa(-3))/36 \\ & + (1 - \kappa(2))/8 + f(p) + g(p) - 1. \end{aligned}$$

Using this one can easily compute the dimension of $S^3_{\mathrm{nG}}(p)$.

Next, Poor and Yuen [32, 33] have developed a technique to compute Hecke eigenvalues for weight three paramodular forms, and in particular for the forms in $S^3_{\mathrm{nG}}(p)$. Using Ibukiyama's formula and data supplied to us by Poor and Yuen, we find the following:

- For all prime levels N, the dimension of the subspace of $H^5(\Gamma_0(N);\mathbb{C})$ not accounted for by the Eisenstein classes above matches $2\dim S^3_{\mathrm{nG}}(N)$.
- In cases where we have computed the Hecke action on this subspace, we find full agreement with the data produced by Poor–Yuen. That is, the Hecke polynomials of the paramodular forms exactly match the polynomials we compute for an eigenbasis of the complement to the Eisenstein classes.

5.5. **Future problems.** We conclude by indicating some open questions and prospects for future work.

- Can one prove that the Eisenstein classes we found actually occur in the cohomology? In principle, one should be able to apply standard techniques from Eisenstein cohomology to answer this question, although it appears at the moment that current knowledge is insufficient to provide a proof.
- Can one prove the lift from paramodular forms to the cohomology? We have not tried to investigate this question at all. Perhaps it is not difficult to solve using known cases of functoriality in the literature.
- Study the Hecke action on torsion classes in the cohomology. This is currently under investigation in joint work with Ash and McConnell.

References

[1] A. Ash and M. McConnell, *Experimental indications of three-dimensional galois representations from the cohomology of $SL(3,\mathbb{Z})$*, Experiment. Math. **1** (1992), no. 3, 209–223.

[2] A. Ash, D. Mumford, M. Rapoport, and Y. Tai., *Smooth compactifications of locally symmetric varieties*, Math. Sci. Press, Brookline, Mass., 1975.

[3] A. Ash, R. Pinch, and R. Taylor, *An $\widehat{A_4}$ extension of $\mathbb{Q}$ attached to a non-selfdual automorphic form on $GL(3)$*, Math. Ann. **291** (1991), 753–766.

[4] A. Ash, *Deformation retracts with lowest possible dimension of arithmetic quotients of self-adjoint homog eneous cones*, Math. Ann. **225** (1977), 69–76.

[5] ———, *Small-dimensional classifying spaces for arithmetic subgroups of general linear groups*, Duke Math. J. **51** (1984), no. 2, 459–468.

[6] ———, *Unstable cohomology of $SL(n,\mathscr{O})$*, J. Algebra **167** (1994), no. 2, 330–342.

[7] A. Ash, D. Grayson, and P. Green, *Computations of cuspidal cohomology of congruence subgroups of* $\mathrm{SL}(3,\mathbf{Z})$, J. Number Theory **19** (1984), no. 3, 412–436.

[8] A. Ash, P. E. Gunnells, and M. McConnell, *Cohomology of congruence subgroups of* $\mathrm{SL}_4(\mathbb{Z})$, J. Number Theory **94** (2002), no. 1, 181–212.

[9] ———, *Cohomology of congruence subgroups of* $\mathrm{SL}(4,\mathbb{Z})$. *II*, J. Number Theory **128** (2008), no. 8, 2263–2274.

[10] ———, *Cohomology of congruence subgroups of* $\mathrm{SL}(4,\mathbb{Z})$. *III*, preprint, 2009.

[11] A. Ash and L. Rudolph, *The modular symbol and continued fractions in higher dimensions*, Invent. Math. **55** (1979), no. 3, 241–250.

[12] A. Borel and J.-P. Serre, *Corners and arithmetic groups*, Comm. Math. Helv. **48** (1973), 436–491.

[13] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, second ed., Mathematical Surveys and Monographs, vol. 67, American Mathematical Society, Providence, RI, 2000.

[14] A. Borel, *Introduction to the cohomology of arithmetic groups*, Lie groups and automorphic forms, AMS/IP Stud. Adv. Math., vol. 37, Amer. Math. Soc., Providence, RI, 2006, pp. 51–86.

[15] L. Clozel, *Motifs et formes automorphes: applications du principe de fonctorialité*, Automorphic forms, Shimura varieties, and L-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 77–159.

[16] ———, *Représentations galoisiennes associées aux représentations automorphes autoduales de* $\mathrm{GL}(n)$, Inst. Hautes Études Sci. Publ. Math. (1991), no. 73, 97–145.

[17] J. Franke, *Harmonic analysis in weighted L_2-spaces*, Ann. Sci. École Norm. Sup. (4) **31** (1998), no. 2, 181–279.

[18] V. Gritsenko, *Arithmetical lifting and its applications*, Number theory (Paris, 1992–1993), London Math. Soc. Lecture Note Ser., vol. 215, Cambridge Univ. Press, Cambridge, 1995, pp. 103–126.

[19] P. E. Gunnells, *Computing Hecke eigenvalues below the cohomological dimension*, Experiment. Math. **9** (2000), no. 3, 351–367.

[20] P. E. Gunnells, F. Hajir, and D. Yasaki, *Modular forms and elliptic curves over* $\mathbb{Q}(\zeta_5)$, in preparation.

[21] P. E. Gunnells and D. Yasaki, *Hecke operators and Hilbert modular forms*, Algorithmic number theory (Banff, 2008), Lecture Notes in Comput. Sci., vol. 5011, Springer, Berlin, 2008, pp. 387–401.

[22] G. Harder, *Cohomology of arithmetic groups*, book in preparation available from Harder's website.

[23] ———, *Some results on the Eisenstein cohomology of arithmetic subgroups of* GL_n, Cohomology of arithmetic groups and automorphic forms (Luminy-Marseille, 1989), Lecture Notes in Math., vol. 1447, Springer, Berlin, 1990, pp. 85–153.

[24] ———, *Eisenstein cohomology of arithmetic groups and its applications to number theory*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990) (Tokyo), Math. Soc. Japan, 1991, pp. 779–790.

[25] Harish-Chandra, *Automorphic forms on semisimple Lie groups*, Notes by J. G. M. Mars. Lecture Notes in Mathematics, No. 62, Springer-Verlag, Berlin, 1968.

[26] T. Ibukiyama, *Dimension formulas of Siegel modular forms of weight* 3 *and supersingular abelian varieties*, Proceedings of the Fourth Spring Conference on Modular Forms and Related Topics, Siegel Modular Forms and Abelian Varieties, 2007, pp. 39–60.

[27] ———, *Siegel modular forms of weight three and conjectural correspondence of Shimura type and Langlands type*, The Conference on L-Functions, World Sci. Publ., Hackensack, NJ, 2007, pp. 55–69.

[28] M. Koecher, *Beiträge zu einer Reduktionstheorie in Positivitätsbereichen. I*, Math. Ann. **141** (1960), 384–432.

[29] ———, *Beiträge zu einer Reduktionstheorie in Positivitätsbereichen. II*, Math. Ann. **144** (1961), 175–182.

[30] J.-S. Li and J. Schwermer, *Automorphic representations and cohomology of arithmetic groups*, Challenges for the 21st century (Singapore, 2000), World Sci. Publ., River Edge, NJ, 2001, pp. 102–137.

[31] Y.-I. Manin, *Parabolic points and zeta-functions of modular curves*, Math. USSR Izvestija **6** (1972), no. 1, 19–63.

[32] C. Poor and D. Yuen, *Paramodular cusp forms*, Proceedings of the Fourth Spring Conference on Modular Forms and Related Topics, Siegel Modular Forms and Abelian Varieties, 2007, pp. 198–215.

[33] ———, *Paramodular cusp forms*, preprint, 2009.

[34] I. Satake, *On compactifications of the quotient spaces for arithmetically defined discontinuous groups*, Ann. Math. **72** (1960), 555–580.

[35] J. Schwermer, *Holomorphy of Eisenstein series at special points and cohomology of arithmetic subgroups of* $\mathrm{SL}_n(\mathbf{Q})$, J. Reine Angew. Math. **364** (1986), 193–220.

[36] ———, *Cohomology of arithmetic groups, automorphic forms and L-functions*, Cohomology of arithmetic groups and automorphic forms (Luminy-Marseille, 1989), Lecture Notes in Math., vol. 1447, Springer, Berlin, 1990, pp. 1–29.

[37] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, vol. 11, Princeton University Press, Princeton, NJ, 1994, Reprint of the 1971 original, Kanô Memorial Lectures, 1.

[38] W. Stein, *Modular forms, a computational approach*, Graduate Studies in Mathematics, vol. 79, American Mathematical Society, Providence, RI, 2007, With an appendix by Paul E. Gunnells.

[39] G. van der Geer, *Siegel modular forms and their applications*, The 1-2-3 of modular forms, Universitext, Springer, Berlin, 2008, pp. 181–245.

[40] B. van Geemen and J. Top, *A non-selfdual automorphic representation of* GL_3 *and a Galois representation*, Invent. Math. **117** (1994), no. 3, 391–401.

[41] B. van Geemen, W. van der Kallen, J. Top, and A. Verberkmoes, *Hecke eigenforms in the cohomology of congruence subgroups of* $\mathrm{SL}(3, \mathbf{Z})$, Experiment. Math. **6** (1997), no. 2, 163–174.

[42] D. A. Vogan, Jr., *Cohomology and group representations*, Representation theory and automorphic forms (Edinburgh, 1996), Proc. Sympos. Pure Math., vol. 61, Amer. Math. Soc., Providence, RI, 1997, pp. 219–243.

[43] D. A. Vogan, Jr. and G. J. Zuckerman, *Unitary representations with nonzero cohomology*, Compositio Math. **53** (1984), no. 1, 51–90.

[44] G. Voronoĭ, *Sur quelques propriétés des formes quadratiques positives partaites*, Crelle's Jour. **133** (1908), 97–178.

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