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Knot Contact Homology and Open Strings

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KNOT CONTACT HOMOLOGY AND OPEN STRINGS

A Dissertation Presented

by

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ABSTRACT

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In this thesis, we give a topological interpretation of knot contact homology, by considering intersections of a particular class of chains of open strings with the knot itself. In doing so, we provide evidence toward a differential graded algebra structure on the algebra generated by chains of open strings.

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INTRODUCTION

A knot is a smooth oriented submanifold of $\mathbb{R}^3$ which is diffeomorphic to S^1.

In section 1, we give a definition of combinatorial knot contact homology, which is an invariant of smooth knots introduced by Ng in [4]. Combinatorial knot contact homology is the homology of a differential graded algebra, the knot contact algebra, which is completely described in terms of a knot diagram of K, see [6]. In [5], Ng shows that a part of the invariant admits a purely topological description in terms of cords. A cord is a continuous path in the complement of the knot which starts and ends on the knot. The cord algebra is the quotient of the algebra generated by homotopy classes of cords by an ideal generated by certain skein relations.

In chapter 2, we extend this result in two ways. An *open string* is a path in $\mathbb{R}^3$ which starts and ends on the knot. Unlike cords, an open string may intersect the knot.

By considering homotopy classes of open strings, we obtain a graded algebra which admits two anti-commuting differentials: one defined by cutting an open string along its intersections with the knot, the other defined by resolving intersections with the knot. This gives a natural generalization of the cord algebra, which we call the homotopy string algebra. In fact, the cord algebra is the degree 0 part of the homology of the homotopy string algebra, where the differential is given by the difference of the resolution and the cutting differentials.

Let K be a knot in $\mathbb{R}^3$ and denote by $\mathcal{K}$ the space of knots which are ambient isotopic to K.

Theorem 2.1.15. The natural action of $\pi_1(\mathcal{K}, K)$ on the cord algebra of K extends to an action on the homology of the homotopy string algebra.

As the action on the cord algebra is nontrivial, the action on the homology of the homotopy string algebra is also nontrivial. Unfortunately, the homotopy string algebra is not isomorphic to knot contact homology; the invariants differ for the unknot.

We describe a second approach. Taking inspiration from string topology, we consider chains of open strings instead of homotopy classes of open strings. The cutting and resolution operators are then defined for a subset of admissible chains. As this section is primarily motivational, we do not aim to make rigorous statements, but instead try to give a heuristic interpretation of the features of such a theory.

In section 3.1, we describe a specific subcomplex of admissible chains for which the cutting and resolution operators are well-defined. Given a knot diagram of a knot, we construct a 2-parameter family of open strings which is a homotopy retract of the space of open strings. The wicket algebra is a graded algebra generated by a cubical decomposition of this family. We show that the cutting and resolution operators on the wicket algebra are well-defined, and prove

Theorem 3.1.6. The wicket algebra is a differential graded algebra, with differential given in terms of the boundary, cutting and resolution operators.

After constructing the wicket algebra, the confirmation that $d^2 = 0$ is a direct calculation; see appendix A.

We prove in section 3.2 the following

Theorem 3.2.1. The homology of the wicket algebra is isomorphic to knot contact homology.

CHAPTER 1

KNOT CONTACT HOMOLOGY

We give two descriptions of knot contact homology, both due to Ng.

1.1 Combinatorial knot contact homology

For the purposes of this paper, a knot is the image of a smooth embedding of S^1 in $\mathbb{R}^3$.

Let D be an oriented knot diagram with n crossings. Label the arcs of the diagram cyclically; each crossing is then uniquely described as a triple (r, l, o). Here o is the over-crossing arc, r is the arc to the right of the crossing and l is the arc to the left of the crossing, where left and right are determined by the orientation of o. See figure 1.1.

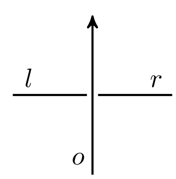


Figure 1.1. The crossing α is uniquely determined by the arcs r, l and o.

Definition 1.1.1. For a labeled, oriented knot diagram D, the graded unital tensor $\mathbb{Z}$-algebra $\mathcal{A}(D)$ is generated in degree 0 by a_{ij}, where i and j are distinct arcs; in degree 1 by $b_{\alpha i}$ and $c_{i\alpha}$, where i is an arc and α is a crossing; and in degree 2 by $d_{\alpha\beta}$ and e_α, where α and β are crossings.

Let $d_{\mathcal{A}} : \mathcal{A}(D) \to \mathcal{A}(D)$ be the map defined on generators as follows:

$$a_{ij} \mapsto 0,$$

$$b_{\alpha i} \mapsto a_{ri} + a_{si} - a_{st}a_{ti},$$

$$c_{i\alpha} \mapsto a_{ir} + a_{is} - a_{it}a_{ts},$$

$$d_{\alpha\beta} \mapsto c_{r\beta} + c_{s\beta} - a_{st}c_{t\beta} - b_{\alpha x} - b_{\alpha y} + b_{\alpha z}a_{zy} \text{ and}$$

$$e_\alpha \mapsto c_{r\alpha} - b_{\alpha s} + b_{\alpha t}a_{ts}.$$

Here (r, s, t) is the crossing α and (x, y, z) is the crossing β. We set a_{ii} equal to 2, and we extended $d_{\mathcal{A}}$ over products by Leibniz.

The *knot contact algebra* of the diagram D is the pair $(\mathcal{A}(D), d_{\mathcal{A}})$.

Theorem 1.1.2 (Ng [6])**.** *The pair $(\mathcal{A}(D), d_{\mathcal{A}})$ is a differential graded algebra, and the stable tame isomorphism class of $\mathcal{A}(D)$ is an invariant of smooth knots.*

Example 1.1.3. Let D be a figure eight diagram of unknot. Then the diagram has one arc and one crossing. The algebra $\mathcal{A}(D)$ is the unital tensor algebra generated by $b_{\alpha 1}$, $c_{1\alpha}$, $d_{\alpha\alpha}$ and e_{α}; the differential is given by the map

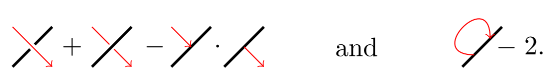
$$
\begin{aligned}
b_{\alpha 1} &\mapsto 0 \\
c_{1\alpha} &\mapsto 0 \\
d_{\alpha\alpha} &\mapsto 0 \\
e_{\alpha} &\mapsto c_{1\alpha} + b_{\alpha 1}.
\end{aligned}
$$

Thus $H(\mathcal{A}(D), d_{\mathcal{A}})$ is the unital tensor algebra $T(\mathbb{Z} \oplus \mathbb{Z}[1] \oplus \mathbb{Z}[2])$. Here we use the convention that for a graded module $C = \oplus C_i$, the graded module $C[n]$ is defined so that $C[n]_i = C_{i-n}$.

1.2 The cord algebra

We now describe a topological interpretation of a part of knot contact homology.

A *cord* is a continuous map $\gamma : [0, 1] \to \mathbb{R}^3$ with $\gamma^{-1}(K) = \{0, 1\}$. Two cords γ_0 and γ_1 are homotopic if there is a homotopy Γ from γ_0 to γ_1 such that Γ_t is a cord for each t. Let $\mathcal{A}_{\text{cord}}(K)$ be the free unital tensor algebra generated by homotopy classes of cords. Then the *cord algebra* is algebra $\mathcal{A}_{\text{cord}}(K)/\mathcal{I}$ where $\mathcal{I}$ is the ideal generated by skein relations



We understand these relations to depicting a small ball in $\mathbb{R}^3$, outside of which the cords are identical.

Theorem 1.2.1 (Ng [5])**.** *The cord algebra is isomorphic to the degree zero part of knot contact homology.*

CHAPTER 2
OPEN STRINGS

In this chapter, we give two generalization of the cord algebra. The first, the homotopy string algebra, comes from considering homotopy classes of strings which non-tangentially intersect the knot. The algebra generated by these homotopy classes admits two anti-commuting differentials called resolution and cutting. The skein relations in the cord algebra are then reinterpreted as being given by the sum of the resolution and cutting differentials.

The second comes from considering opens strings, which are paths which start and end on the knot. Drawing inspiration from the work of Chas and Sullivan [1, 7], we consider cubical chains of open strings. Unlike in the case of the homotopy string algebra, the resolution and cutting operations are defined only for cubes which are suitably generic. Furthermore, they fail to be differentials – they are not even chain maps.

A few words of caution. The homotopy string algebra is a rigorous generalization of the cord algebra, but it does not extend to give a topological interpretation of knot contact homology. On the other hand, the open string algebra is not rigorous; since the resolution and cutting operations are only partially defined, we present the construction only as a heuristic. However, we apply heuristic in chapter 3 to give a rigorously defined topological interpretation of knot contact homology.

2.1 Homotopy string algebra

Let M be a smooth oriented manifold, and suppose K is a codimension 2 cooriented submanifold.

Definition 2.1.1. The space of *parameterized open strings* Γ in M is the set of C^1 maps from the unit interval $I = [0, 1]$ to M which map ∂I to K. We topologize Γ as a subspace of $C^1(I, M)$.

The submanifold K of M defines a discriminant set in Γ. We make this explicit by introducing a notion of homotopic parameterized open strings, where the homotopy preserves the number of intersections of the parameterized open string with K.

Definition 2.1.2. Let n be an integer. Denote by Γ_n the subset of parameterized strings which intersects K at least $n+2$ times, that is, $\Gamma_n = \{\gamma \in \Gamma : \#\gamma^{-1}(K) \geq n+2\}$. A parameterized string γ in Γ_n *cleanly intersects at n points* if γ is not in the closure of Γ_{n+1}.

The following lemmata are immediate.

Lemma 2.1.3. *A parameterized string γ cleanly intersects at n points if and only if γ is nowhere tangential to K and $\#\gamma^{-1}(K) = n+2$.*

Lemma 2.1.4. *The space $\Gamma_n \setminus \overline{\Gamma_{n+1}}$ is locally path connected.*

Definition 2.1.5. Two parameterized strings which cleanly intersect at n points are *homotopic* if both are in the same connected component of $\Gamma_n \setminus \overline{\Gamma_{n+1}}$. An *$n$-string* is a connected component of $\Gamma_n \setminus \overline{\Gamma_{n+1}}$.

Definition 2.1.6. Let R be a ring with unity. The *unreduced homotopy string algebra* $\mathcal{S}_K(M; R)$ is the graded unital tensor algebra over R generated in degree n by the set of n-strings.

The unreduced homotopy string algebra admits the additional structure of a resolution map ρ and a cutting map δ, both of which lower grading by 1.

Let γ be a parameterized string which cleanly intersects at n points $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$ and denote by $[0, n+1]$ the homotopy class of γ. Since γ cleanly intersects, any restriction of γ to a subinterval also cleanly intersects. Denote by $[i, j]$ the homotopy class of the restriction of γ to the interval $[t_i, t_j]$, reparameterized linearly. We say $[i, j]$ is a *substring* contained in $[0, n+1]$. As each component of $\Gamma_n \setminus \overline{\Gamma_{n+1}}$ is path connected, the substring $[i, j]$ of $[0, n+1]$ does not depend on the chosen representative γ.

Definition 2.1.7. The *cutting map* is the map $\delta : \mathcal{S}_K(M; R) \to \mathcal{S}_K(M; R)$ given on generators by

$$\delta[0, n+1] = \sum_{i=1}^{n} (-1)^i \, [0, i][i, n+1].$$

We extend δ over products by Leibnitz.

Note that the cutting map reduces degree by one.

Another natural way to reduce the degree of an n-string is as follows. For each interior intersection $t_1, \ldots, t_n$ of a representative γ, choose a generic vector field V_i supported on the interval $[(t_{i-1} + t_i)/2, (t_i + t_{i+1})/2]$ such that $\dot{\gamma}(t_i) \wedge V_i(t_i)$ is oriented. Denote by $[0, \hat{i}_r, n+1]$ the homotopy class of a small perturbation by V_i and by $[0, \hat{i}_l, n+1]$ the homotopy class of a small perturbation by $-V_i$.

Definition 2.1.8. Let μ be a unit in R. The *resolution map* $\rho : \mathcal{S}_K(M;R) \to \mathcal{S}_K(M;R)$ is given by

$$\rho[0,n+1] = \sum_{i=1}^{n} (-1)^i \left([0, \hat{i}_l, n+1] + \mu[0, \hat{i}_r, n+1] \right),$$

and extends over products by Leibnitz.

Proposition 2.1.9. *The cutting map δ and the resolution map ρ are anticommuting differentials on $\mathcal{S}_K(M;R)$.*

Proof. Direct calculation shows that

$$\begin{aligned}
\delta^2[0,n+1] &= \sum_{i=1}^{n} (-1)^i \, (\delta[0,i]) \, [i,n+1] - \sum_{i=1}^{n} [0,i] \, (\delta[i,n+1]) \\
&= \sum_{j<i} (-1)^{i+j} \, [0,j][j,i][i,n+1] - \sum_{i<j} (-1)^{j-i} \, [0,i][i,j][j,n+1] \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\rho^2[0,n+1] &= \sum_{i=1}^{n} (-1)^i \, \rho[0, \hat{i}_l, n+1] + \sum_{i=1}^{n} (-1)^i \, \mu\rho[0, \hat{i}_r, n+1] \\
&= \sum_{j<i} (-1)^{i+j} \, ([0, \hat{j}_l, \hat{i}_l, n+1] + \mu[0, \hat{j}_r, \hat{i}_l, n+1]) \\
&\quad + \sum_{j<i} (-1)^{i+j} \, \mu([0, \hat{j}_l, \hat{i}_r, n+1] + \mu[0, \hat{j}_r, \hat{i}_r, n+1]) \\
&\quad + \sum_{j>i} (-1)^{i+j-1} \, ([0, \hat{i}_l, \hat{j}_l, n+1] + \mu[0, \hat{i}_l, \hat{j}_r, n+1]) \\
&\quad + \sum_{j>i} (-1)^{i+j-1} \, \mu([0, \hat{i}_r, \hat{j}_l, n+1] + \mu[0, \hat{i}_r, \hat{j}_r, n+1]) \\
&= 0.
\end{aligned}$$

To prove the second statement, we compute

$$
\begin{aligned}
\delta\rho[0,n+1] &= \sum_{i=1}^{n}(-1)^i \left(\delta[0,\hat{i}_l,n+1]+\mu\delta[0,\hat{i}_r,n+1]\right) \\
&= \sum_{j<i}(-1)^{i+j} \left([0,j][j,\hat{i}_l,n+1]+\mu[0,j][j,\hat{i}_r,n+1]\right) \\
&\quad + \sum_{j>i}(-1)^{i+j-1} \left([0,\hat{i}_l,j][j,n+1]+\mu[0,\hat{i}_r,j][j,n+1]\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\rho\delta[0,n+1] &= \sum_{j=1}^{n}(-1)^j \left(\rho[0,j]\cdot[j,n+1]+(-1)^{j-1}[0,j]\cdot\rho[j,n+1]\right) \\
&= \sum_{j>i}(-1)^{j+i} \left([0,\hat{i}_l,j][j,n+1]+\mu[0,\hat{i}_r,j][j,n+1]\right) \\
&\quad + \sum_{j<i}(-1)^{i+j-1} \left([0,j][j,\hat{i}_l,n+1]+\mu[0,j][j,\hat{i}_r,n+1]\right).
\end{aligned}
$$

Thus $\delta\rho+\rho\delta=0$ as claimed. □

Definition 2.1.10. Let M be a oriented manifold and K a cooriented codimension two submanifold. The *unreduced homotopy string algebra* of (M,K) is the differential graded algebra $\mathcal{S}_K(M;R)$ with differential $d=\delta-\rho$.

Definition 2.1.11. A 0-string s is *contractible* if there is a representative γ of s which is arbitrary close to the constant string. An n-string s is *negligible* if it contains a contractible substring.

Let $\mathcal{I}$ denote the ideal in $\mathcal{S}_K(M;R)$ generated by the following:

- if s a contractible 0-string, then $s-2\in\mathcal{I}$ and
- if s a negligible n-string, then $s\in\mathcal{I}$.

Proposition 2.1.12. *The ideal $\mathcal{I}$ in $\mathcal{S}_K(M;R)$ is a differential ideal, i.e. $d(\mathcal{I})\subset\mathcal{I}$, and so the differential on $\mathcal{S}_K(M;R)$ induces a differential on the quotient $\mathcal{S}_K(M;R)/\mathcal{I}$.*

The proof is trivial.

We call the algebra $\mathcal{S}_K(M;R)/\mathcal{I}$ with induced differential the *homotopy string algebra.*

Theorem 2.1.13. *The cord algebra is isomorphic to $H_0(\mathcal{S}_K(M;\mathbb{Z})/\mathcal{I},d)$, the zeroth homology of the homotopy string algebra, where $\mu=1$.*

Proof. The cord algebra and $(\mathcal{S}_K(M;\mathbb{Z}))_0$ have the same generators, since there is a bijective correspondence between cords and 0-strings. The relations on the cord algebra are precisely the relations given by the the ideal $\mathcal{I}$ and by the differential of 1-strings. □

The following example shows that the homology of the homotopy string algebra is not identical to knot contact homology.

Example 2.1.14. If K is the unknot, then every n-string contains a contractible substring. Thus the homology of the homotopy string algebra for the unknot is $\mathbb{Z}$.

We now consider a monodromy result.

Theorem 2.1.15. *There is a natural action of* $\pi(\mathcal{K}, K)$ *on the homotopy string algebra of* K*, and the induced action on the homology of the open string algebra extends the natural action on the cord algebra.*

Proof. A loop in $\pi(\mathcal{K}, K)$ gives an isotopy ϕ_t of $\mathbb{R}^3$, such that ϕ_0 is the identity map and $\phi_1(K) = K$. The diffeomorphism ϕ_1 induces an algebra automorphism $\mathcal{S}_K(M; R)/\mathcal{I}$; this automorphism is a chain equivalence. □

Example 2.1.16. Consider the knot $K = 3_1 \# 3_1$, illustrated in figure 2.1. The cord algebra of K is the $\mathbb{Z}$-algebra generated by generators x, y, z and w, modulo the ideal generated by $x^2 - x - 2$, $y^2 - y - 2$, $(1 + y)(x - w)$, $(x - z)(1 + y)$, $(1 + x)(y - z)$ and $(y - w)(1 + x)$.

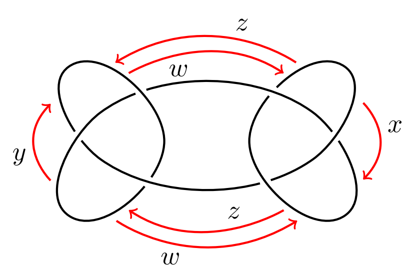


Figure 2.1. The knot $3_1\#3_1$. The red lines are cords which correspond to the generators x, y, z and w of the cord algebra.

Suppose γ is a cord in the homotopy class a. Write $\bar{a}$ for the homotopy class of the cord $t \mapsto \gamma(1 - t)$. In this example, we have $\bar{x} = x$, $\bar{y} = y$ and $\bar{w} = z$.

If we rotate K in the plane, swapping the prime components of the knot, the induced map on the cord algebra sends x to y and w to z and vice versa. If instead we rotate K around the horizontal axis, the induced map on the cord algebra is simply the identity map.

2.2 Open string algebra

Let M be a smooth oriented manifold. Let K be a codimension 2 submanifold of M which is closed, connected and oriented. We assume that the normal bundle $\nu(K)$ of K in M is trivial.

An *open string* is a smooth map $\gamma : [0,1] \to M$ which starts and ends on K, that is, $\gamma(0)$ and $\gamma(1)$ are in K. We topologize $\Gamma(M,K)$, the set of all open strings, as a subspace of $C^\infty([0,1],M)$.

Before proceeding, we give a quick review of the definition of the singular cubical chain complex of a topological space. Denote by I the closed interval $[-1,1]$.

The n-cube is the space I^n. For $i = 1,\ldots,n$, the face maps $a_i, b_i : I^{n-1} \to I^n$ are given by

$$a_i(x_1,\ldots,x_{n-1}) = (x_1,\ldots,x_{i-1},0,x_i,\ldots,x_{i-1})$$

and

$$b_i(x_1,\ldots,x_{n-1}) = (x_1,\ldots,x_{i-1},1,x_i,\ldots,x_{i-1})$$

and the degeneracy maps $s_i : I^{n+1} \to I^n$ are given by

$$s_i(x_1,\ldots,x_{n+1}) = (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+1}).$$

Fix a topological space X. A singular n-cube in X is a continuous map $I^n \to X$. If x is a singular n-cube in X, the i-th front face of x is the $(n-1)$-cube $A_i(x)$ given by $x \circ a_i$. Similarly, the i-th back face of x is the $(n-1)$-cube $B_i(x) = x \circ b_i$. A k-face of x is a k-cube of the form $F_{i_1} \cdots F_{i_{n-k}}(x)$, where F_{i_j} is a face map. We say x is degenerate if there is an $(n-1)$-cube y such that $x = y \circ s_i$, otherwise x is nondegenerate.

We write $C_n(X)$ for the singular cubical n-chains of X: this is the quotient $Q_n(X)/D_n(X)$, where $Q_n(X)$ is the abelian group generated by singular n-cubes and $D_n(X)$ is the abelian group generated by degenerate singular n-cubes. The groups $C_n(X)$ form a chain complex with differential

$$\partial_n(x) = \sum_{i=1}^{n} (-1)^n \left(A_i(x) - B_i(x)\right).$$

2.2.1 Admissible cubes

Let x be an singular n-cube in the space of open strings $\Gamma(M,K)$. For p a point in I^n, write γ_p for the open string $x(p)$. As $\Gamma(M,K)$ is a function space, we say that x is *smooth* if the evaluation map

$$\hat{x} : I^n \times [0,1] \to M$$
$$(p,t) \mapsto \gamma_p(t)$$

is smooth. For x a smooth singular n-cube of open strings, write $\hat{x}_{int}$ for the restriction of $\hat{x}$ to $I^n \times (0,1)$. Let $T\hat{x} : I^n \times [0,1] \to TM$ be the map which sends (p,t) to $(\gamma_p(t), \dot{\gamma}_p(t))$; write $T\hat{x}_{int}$ and $T\hat{x}_{bdy}$ for the restriction of $T\hat{x}$ to $I^n \times (0,1)$ and $I^n \times \{0,1\}$, respectively.

Definition 2.2.1. Let x be a smooth singular n-cube in $\Gamma(M,K)$. Denote by $x_{\mathcal{H}}$ the union ${\hat{x}_{int}}^{-1}(K) \cup {T\hat{x}_{bdy}}^{-1}(TK)$. Then x is *pre-admissible* if

- $\hat{x}_{int}$ is transverse to K;
- $T\hat{x}_{int}$ and $T\hat{x}_{bdy}$ are transverse to TK; and
- $x_{\mathcal{H}}$ is a neat submanifold of $I^n \times [0,1]$.

We say x is *admissible* if every k-face of x is pre-admissible, for $0 \le k \le n$.

For x a pre-admissible n-cube, $x_{\mathcal{H}}$ is a smooth oriented codimension 2 submanifold of $I^n \times (0,1)$, and so we may interpret it as a smooth $(n-1)$-chain in $I^n \times [0,1]$. Likewise we may interpret $x_{\partial\mathcal{H}}$, the preimage ${T\hat{x}_{bdy}}^{-1}(TK)$, as a $(n-2)$-chain in $I^n \times [0,1]$ oriented as a boundary of $x_{\mathcal{H}}$. We may assume that faces of the chains $x_{\mathcal{H}}$ and $x_{\partial\mathcal{H}}$ are transverse to $T\hat{x}^{-1}(TK)$.

Proposition 2.2.2. *If x is an admissible n-cube, the boundary of $x_{\mathcal{H}}$ is $(\partial x)_{\mathcal{H}} + x_{\partial\mathcal{H}}$.*

Proof. Since $x_{\mathcal{H}}$ is a neat submanifold, the boundary of $x_{\mathcal{H}}$ is the intersection of $x_{\mathcal{H}}$ with the boundary of $I^n \times [0,1]$. Thus

$$\partial x_{\mathcal{H}} = x_{\mathcal{H}} \cap \partial(I^n \times [0,1]) = (x_{\mathcal{H}} \cap \partial I^n \times [0,1]) \cup (x_{\mathcal{H}} \cap I^n \times \partial[0,1])$$

as sets, and it is straightforward to show that $x_{\mathcal{H}} \cap \partial I^n \times [0,1] = (\partial x)_{\mathcal{H}}$ and $x_{\mathcal{H}} \cap I^n \times \partial[0,1] = x_{\partial\mathcal{H}}$. □

2.2.2 Cutting

For a pre-admissible n-cube x, a point in $x_{\mathcal{H}}$ is a pair (p,t) such that the open string $\gamma_p = x(p)$ intersects K at $\gamma_p(t)$. If we consider the restrictions of γ_p to the intervals $[0,t]$ and $[t,1]$, these are paths which start and end on K. Thus, after reparameterization, we obtain open strings $r_1(p)$ and $r_2(p)$.

We may choose reparameterizations such that the maps $r_1, r_2 : x_{\mathcal{H}} \to \Gamma(M, K)$ are smooth. Let y be a smooth singular cubical chain in $x_{\mathcal{H}}$. Composition with r_i then gives a smooth chain $r_i(y)$ of open strings.

Definition 2.2.3. Let x be a pre-admissible n-cube, and let Θ be a diagonal approximation of the singular cubical complex $C_*(I^n \times [0,1])$. Let y be a $(n-1)$-chain which represents $x_{\mathcal{H}}$ in $C_*(I^n \times [0,1])$. A *cutting* of x is a chain of the form

$$\delta(x) = r_1 \otimes r_2 \circ \Theta(y)$$

in $C_*(\Gamma(M,K)) \otimes C_*(\Gamma(M,K))$.

Remark 2.2.4. It is important to note that taking a cutting of x does **not** necessarily commute with taking the boundary, since the map $x \mapsto y$ is not a chain map.

2.2.3 Resolution

We now consider the definition of a resolution operation ρ. For an admissible n-cube, the commutative diagram

$$\begin{array}{ccc} I^n \times [0,1] & \xrightarrow{\hat{x}} & M \\ \uparrow & & \uparrow \\ x_{\mathcal{H}} & \longrightarrow & K \end{array}$$

induces a diagram of bundles

$$\begin{array}{ccccc} \nu(x_{\mathcal{H}}) & \xrightarrow{f} & \hat{x}^*\nu(K) & \longrightarrow & \nu(K) \\ \downarrow & & \downarrow & & \downarrow \\ x_{\mathcal{H}} & \longrightarrow & x_{\mathcal{H}} & \xrightarrow{\hat{x}} & K. \end{array}$$

Let ζ be the section $\frac{\partial}{\partial t}|_{(p,t)} + T_{(p,t)}(x_{\mathcal{H}})$ of $\nu(x_{\mathcal{H}})$. Then $f(\zeta)$ is the section $(p,t) \mapsto \gamma_p'(t) + T_{\gamma_p(t)}K$. For x generic, the section $f(\zeta)$ also vanishes on a codimension 2 subvariety of $x_{\mathcal{H}}$.

As K is cooriented, the bundle $\hat{x}^*\nu(K)$ is oriented as well. Since $\nu(K)$ is trivial, and $f : \nu(x_{\mathcal{H}}) \to \hat{x}^*\nu(K)$ is a fiber isomorphisms except along $x_{\partial\,\mathcal{H}}$, we have

Lemma 2.2.5. *There exists a section η of $\nu(x_{\mathcal{H}})$ such that*

- *$\eta = 0$ whenever $\zeta = 0$ and*

- *if ζ is nonzero, $f(\eta)$ and $f(\zeta)$ give an oriented basis for $\hat{x}^*\nu(K)_{(p,t)}$.*

Such a section η is called a *positive thickening* of $x_{\mathcal{H}}$.

Denote by $\mathrm{proj} : I^n \times [0,1] \to \Gamma(M,K)$ the projection $(p,t) \mapsto \gamma_p$. Since $x_{\mathcal{H}}$ is a neat submanifold of $I^n \times [0,1]$, it admits a tubular neighborhood, and so a positive thickening η of $x_{\mathcal{H}}$ may be considered as a chain in $I^n \times [0,1]$. Let p_η be the corresponding chain.

Definition 2.2.6. A *resolution* of a cube x is the chain

$$\rho(x) = p_\eta(x_{\mathcal{H}}).$$

Note that a resolution of a cube depends on the choice of positive thickening.

2.2.4 A conjecture

Consider the subcomplex $\Delta_*(\Gamma)$ of admissible chains in $C_*(\Gamma)$. The cutting map, resolution map and boundary map extend to derivations on the tensor algebra $T(\Delta_*(\Gamma))$. The *algebra of open strings* $\mathcal{A}(\Delta_*(\Gamma))$ is then the quotient of $T(\Delta_*(\Gamma))$ setting constant open strings equal to 2 and chains of constant open strings equal to 0.

Now consider the operation $d = \partial + (-1)^k(2\rho - \delta)$ on $\mathcal{A}(\Delta_*(\Gamma))$. Since $\partial^2 = 0$,

$$d^2 = (-1)^k(2[\partial,\rho] - [\partial,\delta]) - 4\rho^2 + 2\rho\delta + 2\delta\rho - \delta^2,$$

where $[a,b]$ is the commutator $ab - ba$ of a and b.

Proposition 2.2.7. *Let x be an n-cube of open strings. Then for any choice of cutting δ and a generic choice of resolution ρ, there is a choice of cuttings and resolutions on the chains $\partial(x)$, $\delta(x)$ and $\rho(x)$ such that*

$$\rho^2(x) = 0 \text{ and}$$

$$2[\partial,\rho](x) - [\partial,\delta](x) = 0.$$

Proof. Let η be the positive thickening of $x_{\mathcal{H}}$ which corresponds to $\rho(x)$. If $p_\eta(x_{\mathcal{H}})$ is self-transverse, we may pull back η to obtain a positive thickening for the iterated intersection $(p_\eta(x_{\mathcal{H}}))_{\mathcal{H}}$. The fact that $\rho^2(x) = 0$ then immediately follows from our sign conventions.

For the second equation, note that the commutator of ∂ and ρ is

$$\begin{aligned}[\partial,\rho](x) &= p_\eta\left(\partial(x_{\mathcal{H}})\right) - p_\eta\left((\partial x)_{\mathcal{H}}\right) \\ &= p_\eta\left(x_{\partial\,\mathcal{H}}\right)\end{aligned}$$

while the commutator of ∂ and δ is

$$\begin{aligned}[\partial,\delta](x) &= r_1 \otimes r_2 \circ \theta\left(\partial(x_{\mathcal{H}})\right) - r_1 \otimes r_2 \circ \theta\left((\partial x)_{\mathcal{H}}\right) \\ &= r_1 \otimes r_2 \circ \theta\left(x_{\partial\,\mathcal{H}}\right).\end{aligned}$$

Since any point (p,t) of $x_{\partial\,\mathcal{H}}$ corresponds to an open string which lies tangent to K at an endpoint, the section η is equal to zero at (p,t), and so the first term is

$$p_\eta(x_{\partial\,\mathcal{H}}) = \operatorname{proj}(x_{\partial\,\mathcal{H}}).$$

Furthermore, either r_1 or r_2 applied to $x_{\partial\,\mathcal{H}}$ is then a chain of constant strings, depending on which endpoints of the family of strings lay tangent to K. Since n-chains of constant strings are equal to zero, when $n > 0$, we have

$$r_1 \otimes r_2 \circ \theta\left(x_{\partial\,\mathcal{H}}\right) = 2\operatorname{proj}(x_{\partial\,\mathcal{H}}).$$

□

We conjecture that the map d satisfies "$d^2 = 0$ up to homotopy". More precisely, we state the following

Conjecture 2.2.8. *The algebra $\mathcal{A}(\Delta_*(\Gamma))$ is a differential graded algebra with differential*

$$d = \partial + (-1)^k(2\rho - \delta) + d_2 + d_3 + \cdots$$

where $d_n : \Delta_*(\Gamma) \to \Delta_*(\Gamma)^{\otimes n}$ are degree -1 maps.

CHAPTER 3

THE WICKET ALGEBRA

3.1 Wickets and the wicket algebra

Despite the difficulties in establishing a rigorous definition of open string topology, we can establish a finite dimensional model for which Conjecture 2.2.8 holds in the case of smooth knots in $\mathbb{R}^3$.

3.1.1 Wickets

Denote by $pr : \mathbb{R}^3 \to \mathbb{R}^2$ the projection $(x, y, z) \mapsto (x, y)$. Let ε_1,ε_2,ε_3 and ε be positive constants such that $\varepsilon_1 < \varepsilon_2 < \varepsilon_3 < 1$ and $3\varepsilon_3 < \varepsilon$.

We consider a knot K and diagram D such that

- K lies between the planes $z = 0$ and $z = \varepsilon_1$;
- away from a crossing, the knot lies in the plane $z = 0$;
- near a crossing of D, the arcs are straight lines;
- near a crossing of D, the upper branch lies in $z = \varepsilon_1$ and the lower branch lies in $z = 0$;
- the distance between any two crossings of D is bigger then ε; and
- the curvature of K is bounded above by a constant.

Such a knot will be called an *admissible lift* of the corresponding diagram. Any knot diagram admits an admissible lift after planar isotopy.

Given an admissible knot we construct a smooth 2-parameter family of open strings. Let R be the rectangle $[-\varepsilon_3, \varepsilon_3] \times [0, 1]$. We consider a family of smooth paths $\gamma_s : [0, 1] \to R$ such that

- for $0 \leq s \leq \varepsilon_2$, the path γ_s is a half circle $\gamma_s = (s \cos \pi t, s \sin \pi t)$,
- for $s = \varepsilon_3$, the path γ_{ε_3} is a parameterization of line segments $(\{-\varepsilon_3\} \times [0, 1]) \cup ([-\varepsilon_3, \varepsilon_3] \times \{1\}) \cup (\{\varepsilon_3\} \times [0, 1])$ oriented counterclockwise and

- for $\varepsilon_2 < s < \varepsilon_3$, the path γ_s is given by linear interpolation between γ_{ε_2} and γ_{ε_3}, i.e. if $s' = (s - \varepsilon_2)/(\varepsilon_3 - \varepsilon_2)$, then

$$\gamma_s(t) = (1 - s')\,\gamma_{\varepsilon_2}(t) + s'\gamma_{\varepsilon_3}(t).$$

We assume that γ_{ε_3} is chosen so that the images of the paths γ_s are leaves of a singular foliation of the rectangle R.

Lemma 3.1.1. *Let p and q be distinct points in $\mathbb{R}^3$ which lie between the planes $z = 0$ and $z = \varepsilon_1$. If the planar distance between p and q is less then $2\varepsilon_3$, there is a unique isometric embedding $\iota : R \to \mathbb{R}^3$ such that*

- *R lies on the vertical plane containing p and q,*
- *the intersection of R and the xy-plane is the line $[-\varepsilon_3, \varepsilon_3] \times \{0\}$ in R,*
- *there is a curve γ_s on R such that $\iota \circ \gamma_s$ contains p and q and*
- *$\tau_1 = (\iota \circ \gamma_s)^{-1}(p) < \tau_2 = (\iota \circ \gamma_s)^{-1}(q)$.*

Furthermore, $\iota \circ \gamma_s$ induces an open string

$$\gamma_{p,q}(t) = \iota \circ \gamma_s\left((\tau_2 - \tau_1)t + \tau_1\right)$$

which starts on p and ends on q.

Proof. Note that for any two points p and q in $\mathbb{R}^3$ there is a unique point r on the $z = 0$ plane such that the distance $d(r,p) = d(r,q)$ is minimal. Let P be the vertical plane containing p, q and r, and embed R in P such that $(0,0)$ maps to r and the preimage of p is to the right of the preimage of q. Then the preimages of p and q lie on leaves of the foliation of R. If both lie on the same leaf, we are done. If not, we translate R along P until they do. □

Definition 3.1.2. Let K be an admissible knot. Suppose p and q are points on K. If p and q satisfy the conditions of Lemma 3.1.1, denote by (p,q) the open string $\gamma_{p,q}$. Otherwise we set (p,q) to be a rectilinear path from p to $p + (0,0,1)$ to $q + (0,0,1)$ to q such that the evaluation map $ev : \mathcal{W} \times [0,1] \to \mathbb{R}$ is smooth. Such open strings are called *wickets*, and we write $\mathcal{W}$ for a family of wickets.

The *discriminant* $\mathcal{H}$ is the set of wickets in $\mathcal{W}$ which either intersect the knot or are tangent to the knot.

See figure 3.1 for an example.

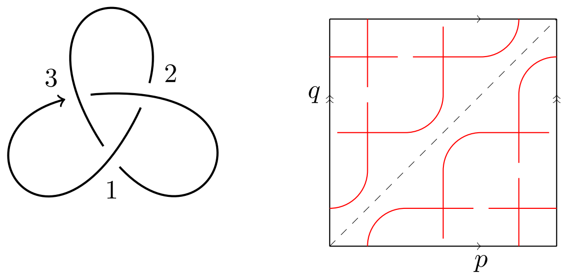


Figure 3.1. The fundamental domain (right) for a family $\mathcal{W}$ of wickets for a diagram of the trefoil (left). Wickets along the red curves are in $\mathcal{H}$, while wickets along the dashed line are constant.

The following is an immediate consequence of the construction of $\mathcal{W}$.

Proposition 3.1.3. *The discriminant $\mathcal{H}$ is the image of a 1 dimensional immersed submanifold with boundary. In particular, the boundary of $\mathcal{H}$ consists of wickets which are tangent at one endpoint to the knot K.*

Away from the boundary of $\mathcal{H}$, the submanifold consists of wickets which cleanly intersect K. This allows us to give a coorientation on $\mathcal{H}$, as follows. Let γ be a wicket in $\mathcal{H}$ which intersects K at $\gamma(t_0)$ and v be an oriented tangent vector of $T_{\gamma(t_0)}K$. Given a vector $\eta \in T_\gamma \mathcal{W}$, the vector η is *oriented with respect to* t_0 if the triple product

$$\left(\gamma_*(t_0)\left(\frac{\partial}{\partial t}\right) \times v\right) \cdot ev_*(\gamma, t_0)\,(\eta, 0)$$

is positive. Here $\times$ is the cross product and $\cdot$ is the dot product in $\mathbb{R}^3$. Note that the double points of $\mathcal{H}$ correspond to wickets which cleanly intersect the knot K twice. Thus a vector η may be oriented with respect to t_0 but not with respect to t_1.

Given this, it is natural to instead consider a line bundle $\mathcal{L}$ over $\mathcal{H}$, and a map $f : \mathcal{L} \to \mathcal{H}$. This may be thought of as the image of positive thickening of the pullback of K by the evaluation map for the family of wickets. Let σ be a section of $\mathcal{L}$. Denote by $ev_\sigma : \mathcal{H} \times \mathbb{R} \to \mathcal{W}$ the map $ev_\sigma((\gamma, t), s) = f(s\sigma(\gamma, t))$ We say that $f \circ \sigma$ is a *resolution* of $\mathcal{H}$ if $ev_{\sigma *}\left(\frac{\partial}{\partial s}\right)$ is oriented with respect to t for every (γ, t) in $\mathcal{H}$ which cleanly intersects K at t, and $\sigma = 0$ otherwise.

3.1.2 The complex $\Delta_*(\mathcal{W})$

Let $\iota : S^1 \to K$ be an oriented parameterization of K. Since D is a generic knot diagram, the composition $pr \circ \iota$ is an immersion. Let m be the length of D in $\mathbb{R}^2$. We identify K with $\mathbb{R}/m\mathbb{Z}$ by choosing a basepoint on S^1 and pulling back the metric on $\mathbb{R}^2$ via $pr \circ \phi$.

Definition 3.1.4. Suppose $u_1, \ldots, u_n$ are the undercrossing points of K and that $o_1, \ldots, o_n$ are the corresponding overcrossing points. Let $\Delta_*(K)$ be the cubical complex given by dividing K into 1-cubes at the points o_i, $o_i \pm \varepsilon$, and $u_i \pm \varepsilon$.

Before proceeding further, we fix our notation. Suppose v_0, v_1 and v_2 are 0-cells in a cubical complex C_*. If there is a unique singular 2-cube σ with vertices v_0, v_1 and v_2, we write $\sigma = [v_0, v_1, v_2]$. If C_* is a product of two cubical complexes A_* and B_*, then we write $c = a \times b$ to denote the k-cube in C_* which is the product of the i-cube a in A_* and the j-cube b in B_*, where $i + j = k$.

We are interested in building a finite cubical complex $\Delta_*(\mathcal{W})$ for the family of wickets $\mathcal{W}$. Let $\Delta_*(K \times K)$ be the product complex induced by the decomposition $\Delta_*(K)$ of K. Let a_i, b_i^- and b_i^+ be the 1-cubes $[u_i - \varepsilon, u_i + \varepsilon]$, $[o_i - \varepsilon, o_i]$ and $[o_i, o_i + \varepsilon]$ in $\Delta_*(K)$ respectively.

Definition 3.1.5. The cubical complex $\Delta_*(\mathcal{W})$ is the refinement of $\Delta_*(K \times K)$ given by subdivision along the diagonal of $K \times K$, plus the additional cells arising from attaching additional 2-cubes as follows

- for each wicket (u_i, o_i), two cubical 2-cubes which share an edge, one shares an edge with $a_i \times b_i^-$ and the other with $a_i \times b_i^+$, and all four 2-cubes share a vertex;
- for each wicket (o_i, u_i), two cubical 2-cubes which share an edge, one shares an edge with $b_i^- \times a_i$ and the other with $b_i^+ \times a_i$, and all four 2-cubes share a vertex; and
- for each overcrossing o_i of the knot, four singular 2-cubes:
  - $[(o_i - \varepsilon_1, o_i - \varepsilon_1), (o_i, o_i), (o_i - \varepsilon, o_i)]$ along $[o_i - \varepsilon, o_i] \times o_i$,
  - $[(o_i - \varepsilon_1, o_i - \varepsilon_1), (o_i, o_i - \varepsilon), (o_i, o_i)]$ along $o_i \times [o_i - \varepsilon, o_i]$,
  - $[(o_i + \varepsilon_1, o_i + \varepsilon_1), (o_i, o_i), (o_i + \varepsilon, o_i)]$ along $[o_i, o_i + \varepsilon] \times o_i$, and
  - $[(o_i + \varepsilon_1, o_i + \varepsilon_1), (o_i, o_i + \varepsilon), (o_i, o_i)]$ along $o_i \times [o_i, o_i + \varepsilon]$.

We give more details regarding the additional cells and their attaching maps in section 3.1.4.4.

We note that the decomposition of $\mathcal{W}$ into a cubical complex $\Delta_*(\mathcal{W})$ also induces the structure of a cubical complex on the discriminant $\mathcal{H}$, since the 1-skeleta of $\Delta_*(\mathcal{W})$ is transverse to the interior of $\mathcal{H}$.

3.1.3 The wicket algebra

Let $T(\mathcal{W})$ be the unital graded tensor algebra over $\mathbb{Z}$ generated by the cubical complex $\Delta_*(\mathcal{W})$, where we think of $\Delta_*(\mathcal{W})$ as a graded set. The wicket algebra $\mathcal{A}(\mathcal{W})$ is the quotient of $T(\mathcal{W})$ by the following relation: 0-cells on the diagonal are set equal to 2 and 1-cells on the diagonal are set equal to zero.

In the following subsection, we define two degree -1 maps $\Delta_*(\mathcal{W}) \to \mathcal{A}(\mathcal{W})$, a resolution map ρ and a cutting map δ. We extend ρ and δ as derivations over $\mathcal{A}(\mathcal{W})$.

Recall that the coorientation of the discriminant $\mathcal{H}$ may be interpreted as a "thickening" of $\mathcal{H}$, given by given by a section of a line bundle $\mathcal{L}$ over $\mathcal{H}$ and smooth map $\iota : \mathcal{L} \to \mathcal{W}$ of the total space to the family of wickets which extends the immersion of the discriminant.

We are interested in an intersection map $\cdot \cap \mathcal{H} : \mathbb{Z}[\Delta_*(\mathcal{W})] \to \mathbb{Z}[\Delta_{*-1}(\mathcal{H})]$ of graded $\mathbb{Z}$-modules. For an oriented chain x in $\Delta_*(\mathcal{W})$, the sign of the intersection $x \cap \mathcal{H}$ is determined by the following convention:

$$(\text{the orientation of } x \cap \mathcal{H})(\text{the coorientation of } \mathcal{H}) = (\text{the orientation of } x).$$

Suppose that the section $\eta : \mathcal{H} \to \mathcal{L}$ is a resolution of $\mathcal{H}$. A *resolution map* ρ is the composition

$$\rho(a) = \iota\eta(a \cap \mathcal{H}).$$

As both $\Delta_*(\mathcal{H})$ and $\Delta_*(\mathcal{W})$ are finite complexes, this map is a priori not well-defined, as it depends on the choice of section η and thickening ι of $\mathcal{H}$.

The cutting map δ is slightly more complicated. Let γ be a wicket in $\mathcal{H}$ which intersects K at $\gamma(t)$. By construction of the family of wickets $\mathcal{W}$, there are maps $r_0, r_1 : \mathcal{H} \to \mathcal{W}$ such that $r_0(\gamma)$ is equal to the wicket $(\gamma(0), \gamma(t))$, the wicket which starts at $\gamma(0)$ and ends at $\gamma(1)$; similarly $r_1(\gamma)$ is the wicket $(\gamma(t), \gamma(1))$. Then the cutting map is

$$\delta(a) = (r_0 \otimes r_1) \circ \Theta(a \cap \mathcal{H}),$$

where Θ is a diagonal approximation $\Delta_*(\mathcal{H}) \to \Delta_*(\mathcal{H}) \otimes \Delta_*(\mathcal{H})$. Again, this map is not a priori well defined, as it depends on the choice of diagonal approximation.

If $(\partial a) \cap \mathcal{H} = \partial(a \cap \mathcal{H})$, then both ρ and δ are chain maps (of singular chains). However, due to the presence of the boundary of $\mathcal{H}$, the intersection map $\cdot \cap \mathcal{H}$ is not a chain map. Despite this, we have

Theorem 3.1.6. *Let K be an admissible lift of a knot diagram D, and $\mathcal{W}$ the corresponding family of wickets. Then there exist a resolution map ρ and cutting map δ such that the wicket algebra $(\mathcal{A}(\mathcal{W}), d)$ is a differential graded algebra with differential*

$$d(a) = \partial(a) + (-1)^k(2\rho(a) - \delta(a)),$$

for a generator a of $\mathcal{A}(\mathcal{W})$ of degree k.

We explain how the maps ρ and δ are constructed in section 3.1.4. We then compute $d^2 = 0$ in appendix A.

3.1.4 Cubes in $\Delta_*(\mathcal{W})$

We say that a cube *intersects* $\mathcal{H}$ if the cube is not in the kernel of $\cdot \cap \mathcal{H}$. Note that for degree reasons, $dx = 0$ for any 0-cube x in $\Delta_*(\mathcal{W})$. Likewise, if x does not intersect $\mathcal{H}$, then the cutting $\delta(x)$ and the resolution $\rho(x)$ are both zero.

In this section, we do not indicate the appropriate signs and orientations of the cutting and resolution of a cube. We instead focus on the geometry of cubes which intersect $\mathcal{H}$. See appendix A for a summary of the proper signs.

For an n-cube x, write $S(x)$ for the image of the degeneracy map $x \circ s_1$.

3.1.4.1 1-cubes in $\Delta_*(\mathcal{W})$

Let u be an undercrossing and o the corresponding overcrossing. Let a be the 1-cube $[u-\varepsilon, u+\varepsilon]$ in $\Delta_*(K)$. Then for any 0-cube b in $\Delta_*(K)$ distinct from o, the 1-cubes $a \times b$ and $b \times a$ in $\Delta_*(\mathcal{W})$ intersect $\mathcal{H}$. Namely, $a \times b \cap \mathcal{H} = (\gamma, t)$, where γ is the wicket which starts at u and ends at b and $\gamma(t) = o$ as a point on K.

Consider $x = a \times b$, where b is not the overcrossing o. The cutting map applied to x gives $\delta(x) = \pm(u, o) \otimes (o, b)$, which is a product of wickets. Here the sign is determined by the coorientation of $\mathcal{H}$ and the orientation of x. The sign also determines the choice of resolution; the resolution of x is either $\rho(x) = -(u + \varepsilon, b)$ or $\rho(x) = +(u - \varepsilon, b)$. The case for $x = b \times a$ is analogous and is treated in appendix A.

If $b = o$ then x does not intersect $\mathcal{H}$.

3.1.4.2 2-cubes which intersect $\mathcal{H}$ in a 1-parameter family

There are two types of 2-cubes in $\Delta_*(\mathcal{W})$ which intersect $\mathcal{H}$ in a 1-parameter family.

We consider the first case: x is a 2-cubes which is not along the diagonal. Let $x = a \times b$ where only one of the intervals $a = [a_0, a_1]$ and $b = [b_0, b_1]$ contains an undercrossing. See figure 3.2.

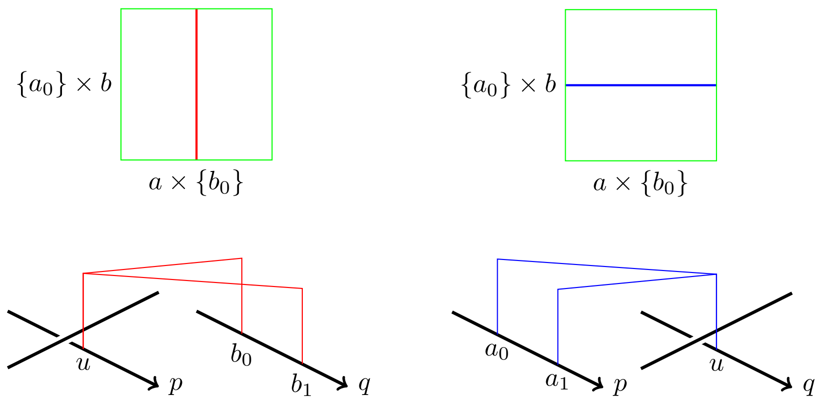


Figure 3.2. Above: Two 2-cells which intersect $\mathcal{H}$ in a one parameter family. Below: wickets corresponding to the boundary of the intersection of a and $\mathcal{H}$.

Let u be an undercrossing and suppose that the 1-cell $b = [b_0, b_1]$ in $\Delta_*(K)$ does not contain an undercrossing. We consider the 2-cell $a = a \times b$, where $a = [a_0, a_1] = [u - \varepsilon, u + \varepsilon]$. The boundary of a and the intersection of a with $\mathcal{H}$ are given by

$$\partial x = -(\partial a) \times b + a \times (\partial b)$$

and

$$x \cap \mathcal{H} = \pm\{u\} \times b$$

respectively. Denote by o the overcrossing associated to u. The cutting is

$$\begin{aligned}\delta(x) &= \pm(S(u, o) \otimes (o, b_1) + (u, o) \otimes \{o\} \times b) \\ &= \pm(u, o) \otimes \{o\} \times b.\end{aligned}$$

We choose the resolution to be one of the boundaries parallel to $\mathcal{H}$: the resolution is $\rho(x) = \pm\{u \mp \varepsilon\} \times b$.

We now consider the second case: x is a 2-cube which lies along the diagonal. Let $a = u - \varepsilon$ and $b = u + \varepsilon$, and consider the cells along the diagonal given by the singular cubes $[(a, a), (b, a), (b, b)]$

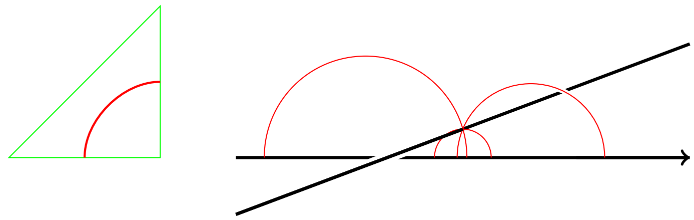


Figure 3.3. A δ-small 2-cell: as a simplex and its geometric realization.

and $[(a, a), (b, b), (a, b)]$. In either case, the cell near the diagonal intersects $\mathcal{H}$ in a 1-parameter family.

We consider $x = [(a, a), (b, a), (b, b)]$. The boundary and intersection are

$$\partial a = [a, b] \times \{a\} + \{b\} \times [a, b] + [(a, a) \to (b, b)]$$

$$a \cap \mathcal{H} = \pm[(u, u - \varepsilon) \to_{\mathcal{H}} (u + \varepsilon, u)].$$

Here by $[p \to_{\mathcal{H}} q]$ we mean the 1-cell in $\mathcal{H}$ with boundary $q - p$. The cutting is

$$\delta(x) = \pm[u, u + \varepsilon] \times \{o\} \otimes (o, u) \pm (u, o) \otimes \{o\} \times [u - \varepsilon, u]$$

and the resolution is either

$$\rho(x) = -S(b, a) \quad \text{or} \quad r(x \cap \mathcal{H}) = +[(a, a) \to (b, b)]$$

depending on the orientation of $x \cap \mathcal{H}$. However, note that $\rho(x) = 0$ in $\mathcal{A}(\mathcal{W})$.

3.1.4.3 2-cubes which intersect $\mathcal{H}$ in two transverse 1-parameter families

A 2-cell $a = x \times y$ will intersect $\mathcal{H}$ in two transverse families when $a = [u_a - \varepsilon, u_a + \varepsilon]$ and $b = [u_b - \varepsilon, u_b + \varepsilon]$, where u_a and u_b are distinct undercrossings. See figure 3.4.

The topological boundary and intersection with $\mathcal{H}$ of x are given by

$$\partial x = -(\partial a) \times b + a \times (\partial b),$$

$$x \cap \mathcal{H} = \mathfrak{o}_1 \{u_a\} \times b + \mathfrak{o}_2\, a \times \{u_b\},$$

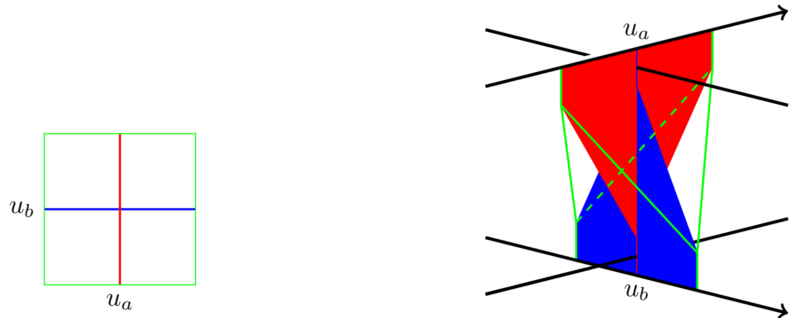


Figure 3.4. A ε-large 2-cell which intersects $\mathcal{H}$ in two transverse families: a simplex and a model of its geometric realization. Note that the model does not actually depict wickets.

where $\mathfrak{o}_1$ and $\mathfrak{o}_2$ are either $+1$ or -1, depending on orientation. The cutting of the intersection is a sum of 1-cubes

$$\delta(x) = \mathfrak{o}_1\, r_1 \otimes r_2 \circ \Theta(\{u_a\} \times b) + \mathfrak{o}_2\, r_1 \otimes r_2 \circ \Theta(a \times \{u_b\})$$

where

$$\begin{aligned} r_1 \otimes r_2 \circ \Theta(\{u_a\} \times b) &= S(u_a, o_a) \otimes (o_a, b_1) + (u_a, o_a) \otimes \{o_a\} \times b \\ &= (u_a, o_a) \otimes \{o_a\} \times b, \\ r_1 \otimes r_2 \circ \Theta(a \times \{u_b\}) &= a \times \{o_b\} \otimes (o_b, u_b) + (a_0, o_b) \otimes S(o_b, u_b) \\ &= a \times \{o_b\} \otimes (o_b, u_b). \end{aligned}$$

The resolution is also a sum of 1-cubes

$$\rho(x) = \mathfrak{o}_1\, \iota\eta(\{u_a\} \times b) + \mathfrak{o}_2\, \iota\eta(a \times \{u_b\})$$

where

$$\begin{aligned} \iota\eta(\{u_a\} \times b) &= \{u_a - \mathfrak{o}_1\, \varepsilon\} \times b \\ \iota\eta(a \times \{u_b\}) &= a \times \{u_b + \mathfrak{o}_2\, \varepsilon\}. \end{aligned}$$

3.1.4.4 2-cubes which intersect $\mathcal{H}$ in a 1-parameter family with boundary

Let $a = [u - \varepsilon, u + \varepsilon]$ and $b = [o, o + \varepsilon]$ or $b = [o - \varepsilon, o]$, where u is an undercrossing and o is the corresponding overcrossing. Then $x = a \times b$ and $y = b \times a$ are 2-cells which intersect $\mathcal{H}$ in a one parameter family with boundary. See figure 3.5.

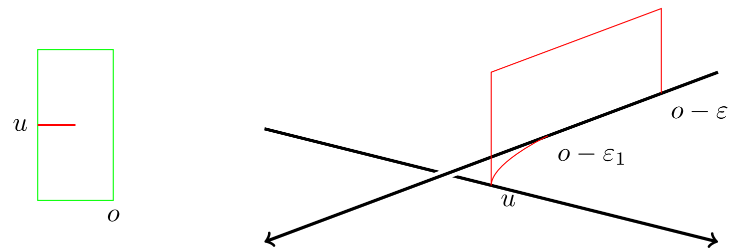


Figure 3.5. A 2-cell which contains an wicket which is tangent to K.

We consider the 2-cell $x = a \times b$, where $a = [o - \varepsilon, o]$ and $b = [u - \varepsilon, u + \varepsilon]$. The boundary of x and the intersection of x with $\mathcal{H}$ are

$$\partial a = -(\partial a) \times b + a \times (\partial b),$$

$$a \cap \mathcal{H} = \pm[o - \varepsilon, o - \varepsilon_1] \times \{u\}.$$

The cutting is

$$\delta(x) = \pm[(o - \varepsilon, o) \to_{\mathcal{H}} (o - \varepsilon_1, o - \varepsilon_1)] \otimes (o - \varepsilon_1, u) \pm (o - \varepsilon, o) \otimes [o, o - \varepsilon_1] \times \{u\}.$$

additional 1-cell $c = [(o - \varepsilon, o) \to_{\mathcal{H}} (o - \varepsilon_1, o - \varepsilon_1)]$

As the resolution may be thought of a section of the normal bundle, we may choose a straight line path to represent it. Note that the resolution coincides with $a \cap \partial \mathcal{H}$ along the boundary of $\mathcal{H}$.

$$\rho(x) = [(o - \varepsilon, \{u + \mathfrak{o}\,\varepsilon\}) \to (o - \varepsilon_1) \times \{u\})].$$

In order to compensate for the addition of these 1-cells, we add two additional 2-cells to our picture, a cell m to deal with the resolution and the $[o, o - \varepsilon_1] \times \{u\}$ term of the cutting, and another cell n to deal with the c term of the cutting.

[New figure needed]

3.2 Relation to knot contact homology

In this section we prove the following

Theorem 3.2.1. *Let D be a generic diagram of a smooth knot in $\mathbb{R}^3$ with at least one crossing. Fix an admissible lift K of the diagram. Then the wicket algebra $\mathcal{A}(\mathcal{W})$ of K is stable tame isomorphic to the combinatorial knot contact algebra $\mathcal{A}(D)$ of D.*

Corollary 3.2.2. *The cord algebra is the degree zero part of knot contact homology.*

Proof of corollary. Every element of the cord algebra can be represented in terms of a product of 0-cubes in $\Delta_*(\mathcal{W})$. The relations given by the differential of 1-cubes either relate two 0-cubes by a homotopy as cords, or relate four 0-cubes by the skein relation. □

We outline the proof of the theorem.

1. Identify the degree zero generators a_{ij} of $\mathcal{A}(D)$ with the 0-cells of wickets.

2. Stabilize the algebra by adding the faces of 2-cells which do not intersect $\mathcal{H}$.

3. Identify the degree one generators $b_{\alpha k}$ and $c_{k\alpha}$ with geometric 1-chains of wickets.

4. Stabilize the algebra by adding 2-cells which intersect $\mathcal{H}$ in a one-parameter family, except those immediately along the diagonal.

5. Identify the degree two generators $d_{\alpha\beta}$ and e_α with geometric 2-chains of wickets.

The resulting algebra will be $\mathcal{A}(\mathcal{W})$.

Our principle tool is the following lemma.

Lemma 3.2.3. *Let $\mathcal{A} = (T(V), d)$ be a differential graded R-algebra. Suppose $\mathcal{A}' = (T(V \oplus R\langle x, y\rangle), d')$, where $d'v = dv$ for v in V and $d'x = y - w$ for w a word in $T(V)$. Then the map $\mathcal{A} \to \mathcal{A}'$ induced by the inclusion $V \to V \oplus R\langle x, y\rangle$ is a stable tame isomorphism.*

Remark 3.2.4 (Notation). A sequence $a_0, \ldots, a_{n+1}$ of 0-cells in $\Delta_*(K)$ is *consecutive* if for each i the oriented interval $[a_i, a_{i+1}]$ is an 1-cell in $\Delta_*(K)$. Suppose that $a_0, \ldots, a_{n+1}$ and $b_0, \ldots, b_{m+1}$ are both consecutive sequences. We write $a \times b_0$, $a_0 \times b$ and $a \times b$ to denote the chains

$$a \times b_0 = \sum_i [a_i, a_{i+1}] \times b_0,$$

$$a_0 \times b = \sum_j a_0 \times [b_j, b_{j+1}] \text{ and}$$

$$a \times b = \sum_{i,j} [a_i, a_{i+1}] \times [b_j, b_{j+1}]$$

in $\Delta_*(\mathcal{W})$. Here we assume that none of the 2-cells $[a_i, a_{i+1}] \times [b_j, b_{j+1}]$ intersect the discriminant $\mathcal{H}$ in a one parameter family with boundary. In that case, we define $a \times b$ to be a different chain of 2-cells, see remark 3.2.5 and remark B.3.1.

We adopt a similar convention for triangular 2-chains: for $a_0, \ldots, a_{n+1}$ consecutive 0-cells,

$$\begin{aligned}
[(a_0, a_0), (a_{n+1}, a_{n+1}), (a_0, a_{n+1})] &= \sum_{i=j}[(a_i, a_i), (a_{i+1}, a_{i+1}), (a_i, a_{i+1})] \\
&\quad + \sum_{i<j}[a_i, a_{i+1}] \times [a_j, a_{j+1}] \text{ and} \\
[(a_0, a_0), (a_{n+1}, a_0), (a_{n+1}, a_{n+1})] &= \sum_{i=j}[(a_i, a_i), (a_{i+1}, a_i), (a_{i+1}, a_{i+1})] \\
&\quad + \sum_{i>j}[a_i, a_{i+1}] \times [a_j, a_{j+1}].
\end{aligned}$$

Now let $\mathcal{A}$ and $\mathcal{A}'$ be graded algebras generated by the graded R-modules $V \oplus Ra$ and $V \oplus Ra'$, respectively. Write $a \mapsto \boxed{a'} + w$ for the tame automorphism $\mathcal{A} \to \mathcal{A}'$ which sends the generator a to $a' + w$ and restricts to an isomorpism on $T(V)$. If $\mathcal{A}$ is a differential graded algebras, a tame automorphism induces a differential on $\mathcal{A}'$. In particular, this implies that $d_{\mathcal{A}'}(a') = d_{\mathcal{A}}(a) - dw$ as elements of $T(V)$. We will abuse notation and write a for the chain $a' + w$ in $\mathcal{A}'$.

An expression of the form

$$d\boxed{x} = \boxed{y} - w$$

indicates a stabilization as in lemma 3.2.3.

3.2.1 Replacing a_{ij}

For a knot diagram with at least one crossing, the arcs of the knot diagram are in one-to-one correspondence with the undercrossing points. Since the diagram is oriented, we assume that the arcs are cyclically ordered. Then the tame isomorphisms

$$a_{ij} \mapsto \boxed{(u_{i+1} - \varepsilon, u_{j+1} - \varepsilon)}$$

identify the degree zero generators of $\mathcal{A}(D)$ with 0-cells in $\Delta_*(\mathcal{W})$; here the arc i on the diagram is the image of the open interval (u_i, u_{i+1}) of the knot. We will abuse notation and write (i, j) for the geometric 0-cell $(u_{i+1}-\varepsilon, u_{j+1}-\varepsilon)$. More generally, we identify the arc i with the 0-cell $u_{i+1}-\varepsilon$ in $\Delta_*(K)$.

3.2.2 Add faces of 2-cells which do not intersect $\mathcal{H}$

Recall that for a cubical complex a cell x is a *face* of the cell y if there is a sequence of face maps $y \to \cdots \to x$. In particular, y is a face of y. Let X be the set of faces of 2-cells in $\Delta_*(\mathcal{W})$ which do not intersect $\mathcal{H}$, and let $(T(X), d)$ be the differential graded algebra generated by X, with differential induced by $\mathcal{A}(\mathcal{W})$. Since X consists of cells which do not intersect $\mathcal{H}$, the differential d is simply the boundary ∂, modulo the quotient which sends chains of constant strings to 2 or 0.

Denote by $T(Y)$ the algebra generated by the 0-cells (i, j), where i and j are arcs in the diagram. Then $\mathcal{A}(D) \coprod_{T(Y)} T(X)$ is stable tame isomorphic to $\mathcal{A}(D)$.

3.2.3 Replacing $b_{\alpha k}$ and $c_{k\alpha}$

We now consider the algebraic generators of degree 1. Let $\alpha = (r, l, o)$ be a crossing of the knot diagram, where o is the overcrossing arc and l and r are the arcs to the left and the right, respectively, of the crossing. We will identify a degree 1 algebraic generator with a 1-chain of wickets. See figure 3.6.

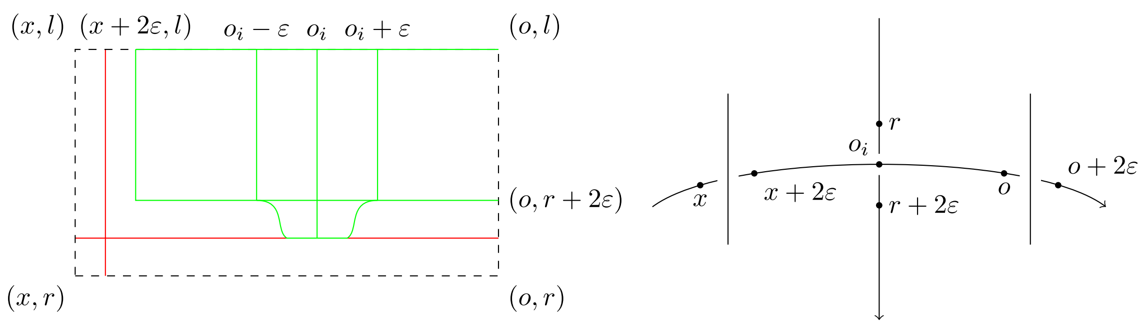


Figure 3.6. In black: the algebraic generator $d_{\beta\alpha}$. The crossing $\alpha = (r, l, o)$ has $r = u_i - \varepsilon$, $l = u_{i+1} - \varepsilon$ and $o = u_{j+1} - \varepsilon$, while $\beta = (x, o, z)$ with $x = u_j - \varepsilon$ and $z \neq l$. In green: 0-,1- and 2-cells from $T(V)$. In red: the intersection of $\mathcal{H}$ with $\mathcal{W}$.

Since the orientation of the lower branch of K at the crossing α determines an ordering on the arcs r and l, we must consider two separate cases when identifying $b_{\alpha k}$ and $c_{k\alpha}$ with chains of wickets. We consider one of the cases in detail here. The second in analogous and is treated in section B.

Let $a = [a_0, a_1] = [u_i - \varepsilon, u_i + \varepsilon]$, and suppose that $r = u_i - \varepsilon$ and $l = u_{i+1} - \varepsilon$. For the generators $b_{\alpha k}$ and $c_{k\alpha}$, if $k = o$, the upper branch of the crossing, we apply the tame isomorphisms

$$c_{o\alpha} \mapsto [o_i, o] \times r - \boxed{o_i \times a} - [o_i, o] \times a_1 - o \times [a_1, l]$$

and

$$b_{\alpha o} \mapsto r \times [o_i, o] - \boxed{a \times o_i} - a_1 \times [o_i, o] - [a_1, l] \times o.$$

Recall that by remark 3.2.4, an expressions such as $[o_i, o] \times r$ refers to a chain of 1-cubes, if it is not a generator of the algebra.

We must check that the differential induced by the above tame maps gives the "correct" differential on the new generator, that is, the induced differential is the same as the differential in the wicket algebra. Since $d(c_{o\alpha}) = a_{or} - a_{ol} = (o, r)$, we compute

$$\begin{aligned} d(o_i \times a) &= d\left(c_{o\alpha} - [o_i, o] \times r + [o_i, o] \times a_1 + o \times [a_1, l]\right) \\ &= (o, r) - (o, l) - (o, r) + (o_i, r) + (o, a_1) - (o_i, a_1) + (o, l) - (o, a_1) \\ &= (o_i, r) - (o_i, a_1), \end{aligned}$$

since each of the 1-chains do not intersect the discriminant $\mathcal{H}$. A similar computation shows that $d(a \times o_i) = (a_1, o_i) - (r, o_i)$.

If $k \neq o$, we use the tame automorphisms

$$c_{k\alpha} \mapsto \boxed{k \times a} + k \times [a_1, l] - k \times [o_i, o] \otimes (o_i, u_i) - (k, o) \otimes o_i \times [u_i, l] - (k, o) \otimes [o_i, o] \times l$$

and

$$b_{\alpha k} \mapsto -\boxed{a \times k} + [a_1, l] \times k - (u_i, o_i) \otimes [o_i, o] \times k - [u_i, l] \times o_i \otimes (o, k) - l \times [o_i, o] \otimes (o, k)$$

to replace the algebraic generators with geometric generators. Again we must check that the differential is correct. Since $d(c_{k\alpha}) = a_{kr} + a_{kl} - a_{ko}a_{ol} = (k, r) + (k, l) - (k, o) \otimes (o, l)$, we compute

$$\begin{aligned} d(k \times a) &= d\big(c_{k\alpha} - k \times [a_1, l] + k \times [o_i, o] \otimes (o_i, u_i) + (k, o) \otimes o_i \times [u_i, l] + (k, o) \otimes [o_i, o] \times l\big) \\ &= (k, r) + (k, l) - (k, o) \otimes (o, l) - (k, l) + (k, a_1) \\ &\quad + (k, o) \otimes (o_i, u_i) - (k, o_i) \otimes (o_i, u_i) + (k, o) \otimes (o_i, l) \\ &\quad - (k, o) \otimes (o_i, u_i) + (k, o) \otimes (o, l) - (k, o) \otimes (o_i, l) \\ &= (k, r) + (k, a_1) - (k, o_i) \otimes (o_i, u_i), \end{aligned}$$

which is precisely $\partial(k \times a) - 2\rho(k \times a) + \delta(k \times a)$. A similar calculation shows that $d(a \times k) = \partial(a \times k) - 2\rho(a \times k) + \delta(a \times k)$, as desired.

3.2.4 Add 2-cells which intersect $\mathcal{H}$ in a 1-parameter family and which are not along the diagonal

The above tame automorphisms replace all the algebraic generators of degree 1. After replacing these algebraic generators, we stabilize the algebra to add in the 1-cells and 2-cells which intersect $\mathcal{H}$ in a 1-parameter family.

Let $b_1 = u_j + \varepsilon, \ldots, b_{n+1} = u_{j+1} - \varepsilon$ be consecutive 0-cells in $\Delta_*(K)$, and let $\alpha = (r, s, t)$ be a crossing with undercrossing u_i and overcrossing o_i, where i and j are distinct. Let $a_0 = u_i - \varepsilon$, $a_1 = u_i + \varepsilon$ and $a = [a_0, a_1]$. There are two cases to consider.

3.2.4.1 Case 1: None of the b_k are equal to o_i

By the previous section, the 1-cells $a \times b_{n+1}$ and $b_{n+1} \times a$ are in the algebra. We apply the stable tame automorphisms which add the boxed generators

$$\begin{aligned}
\partial \boxed{a \times [b_k, b_{k+1}]} &= a_1 \times [b_k, b_{k+1}] - a_0 \times [b_k, b_{k+1}] - a \times b_{k+1} + \boxed{a \times b_k} \\
&\quad + 2\rho(a \times [b_k, b_{k+1}]) - \delta(a \times [b_k, b_{k+1}]) \\
\partial \boxed{[b_k, b_{k+1}] \times a} &= b_{k+1} \times a - \boxed{b_k \times a} - [b_k, b_{k+1}] \times a_1 + [b_k, b_{k+1}] \times a_0 \\
&\quad + 2\rho([b_k, b_{k+1}] \times a) - \delta([b_k, b_{k+1}] \times a),
\end{aligned}$$

working down from $k = n$ to $k = 1$.

3.2.4.2 Case 2: One of the b_k is equal to o_i

Note that $b_{k-1} = o_i - \varepsilon$ and $b_{k+1} = o_i + \varepsilon$, so the 2-cells $a \times [b_l, b_{l+1}]$ and $[b_l, b_{l+1}] \times a$ intersect $\mathcal{H}$ in a 1-parameter family with boundary if $l = k - 1$ or $l = k$. We add cells of the form

$$\begin{aligned}
\partial \boxed{a \times [b_{k-1}, b_k]} &= a_1 \times [b_{k-1}, b_k] - a_0 \times [b_{k-1}, b_k] - a \times b_k + \boxed{a \times b_{k-1}} \\
&\quad - 2\rho(a \times [b_{k-1}, b_k]) + \delta(a \times [b_{k-1}, b_k])
\end{aligned}$$

and

$$\begin{aligned}
\partial \boxed{[b_{k-1}, b_k] \times a} &= b_k \times a - \boxed{b_{k-1} \times a} - [b_{k-1}, b_k] \times a_1 + [b_{k-1}, b_k] \times a_0 \\
&\quad - 2\rho([b_{k-1}, b_k] \times a) + \delta([b_{k-1}, b_k] \times a)
\end{aligned}$$

starting with $l = k-1$, and work down to $l = 1$ as in the previous case.

We then add cells of the form

$$\partial \boxed{a \times [b_k, b_{k+1}]} = a_1 \times [b_k, b_{k+1}] - a_0 \times [b_k, b_{k+1}] - \boxed{a \times b_{k+1}} + a \times b_k$$
$$- 2\rho(a \times [b_k, b_{k+1}]) + \delta(a \times [b_k, b_{k+1}])$$

and

$$\partial \boxed{[b_k, b_{k+1}] \times a} = \boxed{b_{k+1} \times a} - b_k \times a - [b_k, b_{k+1}] \times a_1 + [b_k, b_{k+1}] \times a_0$$
$$- 2\rho([b_k, b_{k+1}] \times a) + \delta([b_k, b_{k+1}] \times a)$$

and work up from $l = k+1$ to $l = n$ by adding generators

$$\partial \boxed{a \times [b_l, b_{l+1}]} = a_1 \times [b_l, b_{l+1}] - a_0 \times [b_l, b_{l+1}] - \boxed{a \times b_{l+1}} + a \times b_l$$
$$+ 2\rho(a \times [b_l, b_{l+1}]) - \delta(a \times [b_l, b_{l+1}])$$
$$\partial \boxed{[b_l, b_{l+1}] \times a} = \boxed{b_{l+1} \times a} - b_l \times a - [b_l, b_{l+1}] \times a_1 + [b_l, b_{l+1}] \times a_0$$
$$+ 2\rho([b_l, b_{l+1}] \times a) - \delta([b_l, b_{l+1}] \times a).$$

3.2.5 Replacing $d_{\alpha\alpha}$ and e_α

Let $\alpha = (r, s, t)$ be a crossing. Write $a = [u_i - \varepsilon, u_i + \varepsilon]$, where u_i is the undercrossing at α. As before, there are two cases which must be considered; the second is postponed to appendix B.

Suppose $r = u_i - \varepsilon$ and $s = u_{i+1} - \varepsilon$. We first replace $d_{\alpha\alpha}$ with the geometric 2-cell $x = [(a_0, a_0), (a_1, a_0), (a_1, a_1)]$. Since the differential of $d_{\alpha\alpha}$ is

$$d(d_{\alpha\alpha}) = c_{s\alpha} - a_{st}c_{t\alpha} - b_{\alpha r} + d(e_\alpha)$$

while the differential of x is

$$d(x) = a_1 \times a + a \times a_0 + (u_i, o_i) \otimes o_i \times [a_0, u_i] + [u_i, a_1] \times o_i \otimes (o_i, u_i),$$

a direct calculation shows that the difference is

$$d\left(d_{\alpha\alpha} - x\right) = dw$$

where

$$w = e_\alpha + [a_1, s] \times a + [(a_1, a_1), (s, a_1), (s, s)]$$
$$+ [u_i, s] \times o_i \otimes o_i \times [a_0, u_i] - [u_i, s] \times o_i \otimes [o_i, t] \times a_0$$
$$+ s \times [o_i, t] \otimes o_i \times [a_0, u_i] - s \times [o_i, t] \otimes [o_i, t] \times a_0 + (s, t) \otimes [o_i, t] \times [a_1, s].$$

Thus $d_{\alpha\alpha} \mapsto \boxed{x} + w$ is a tame automorphism.

We now consider e_α. Since

$$d(e_\alpha) = c_{r\alpha} - b_{\alpha s} + b_{\alpha t} a_{ts},$$

we will replace e_α with $x = [(a_0, a_0), (a_1, a_1), (a_0, a_1)]$. The differential of x is

$$d(x) = -a \times a_1 - a_0 \times a + (a_0, o_i) \otimes o_i \times [u_i, a_1] + [a_0, u_i] \times o_i \otimes (o_i, a_1)$$

and so

$$d(e_\alpha + x) = dw$$

where

$$w = -a \times [a_1, s] + [(a_1, a_1), (s, s), (a_1, s)]$$
$$+ [a_0, u_i] \times o_i \otimes o_i \times [a_1, s] + [a_0, u_i] \times o_i \otimes [o_i, t] \times s$$
$$- a_0 \times [o_i, t] \otimes o_i \times [u_i, s] - a_0 \times [o_i, t] \otimes [o_i, t] \times s + [a_1, s] \times [o_i, t] \otimes (t, s)$$

and so $d_{\alpha\alpha} \mapsto -\boxed{x} + w$ is a tame automorphism.

3.2.6 Replacing $d_{\alpha\beta}$ for α and β distinct

We finally consider algebraic generators of the form $d_{\alpha\beta}$ where $\alpha = (r, s, t)$ and $\beta = (x, y, z)$ are distinct crossings.

Let $a = [u_i - \varepsilon, u_i + \varepsilon]$ and $b = [u_j - \varepsilon, u_j + \varepsilon]$ be neighborhoods of the lower branches of α and β, respectively. Since the orientations of the lower branches of K determine the ordering of the lower arcs, there are four separate cases. Suppose that $r = u_i - \varepsilon$, $s = u_{i+1} - \varepsilon$, $x = u_j - \varepsilon$ and $y = u_{j+1} - \varepsilon$. The other three configurations are analogous and are treated in appendix B.

Given these orientations, there are two primary cases to consider.

3.2.6.1 Case 1:

Suppose that t is distinct from x, y and z and that z is distinct from r and s. Then after the stable tame isomorphisms in the previous steps, the image of $d_{\alpha\beta}$ has differential

$$d(d_{\alpha\beta}) = c_{r\beta} + c_{s\beta} - a_{st}c_{t\beta} - b_{\alpha x} - b_{\alpha y} + b_{\alpha z}a_{zy}$$

where

$$\begin{aligned}
c_{r\beta} &= r \times b + r \times [b_1, y] - r \times [o_j, z] \otimes (o_j, u_j) - (r, z) \otimes o_j \times [u_j, y] - (r, z) \otimes [o_j, z] \times y \\
c_{s\beta} &= s \times b + s \times [b_1, y] - s \times [o_j, z] \otimes (o_j, u_j) - (s, z) \otimes o_j \times [u_j, y] - (s, z) \otimes [o_j, z] \times y \\
a_{st}c_{t\beta} &= (s, t) \otimes \big\{ + t \times b + t \times [b_1, y] - t \times [o_j, z] \otimes (o_j, u_j) \\
&\qquad - (t, z) \otimes o_j \times [u_j, y] - (t, z) \otimes [o_j, z] \times y \big\} \\
b_{\alpha x} &= -a \times x + [a_1, s] \times x - (u_i, o_i) \otimes [o_i, t] \times x - [u_i, s] \times o_i \otimes (t, x) - s \times [o_i, t] \otimes (t, x) \\
b_{\alpha y} &= -a \times y + [a_1, s] \times y - (u_i, o_i) \otimes [o_i, t] \times y - [u_i, s] \times o_i \otimes (t, y) - s \times [o_i, t] \otimes (t, y) \\
b_{\alpha z}a_{zy} &= \big\{ - a \times z + [a_1, s] \times z - (u_i, o_i) \otimes [o_i, t] \times z \\
&\qquad - [u_i, s] \times o_i \otimes (t, z) - s \times [o_i, t] \otimes (t, z) \big\} \otimes (z, y).
\end{aligned}$$

We wish to replace $d_{\alpha\beta}$ with the 2-cell $a \times b$. This geometric 2-cell has differential

$$\begin{aligned}
d(a \times b) &= \partial(a \times b) + 2\rho(a \times b) - \delta(a \times b) \\
&= a_1 \times b - a_0 \times b - a \times b_1 + a \times b_0 - 2a \times b_0 - 2a_1 \times b \\
&\quad + a \times o_j \otimes (o_j, u_j) + (u_i, o_i) \otimes o_i \times b.
\end{aligned}$$

Note that the differential of $a \times b$ consists of geometric cells already in the stabilization of the algebra $\mathcal{A}(D)$. Thus we may compute the sum of the differential of the algebraic generator $d_{\alpha\beta}$ and geometric cell $a \times b$; a lengthy but direct computation shows that

$$d(d_{\alpha\beta} + a \times b) = dw_{\alpha\beta}$$

where $w_{\alpha\beta}$ is the 2-chain

$$
\begin{aligned}
w_{\alpha\beta} = [a_1, s] \times [b_1, y] & \\
& - a \times [b_1, y] + a \times [o_j, z] \otimes (o_j, u_j) + a \times z \otimes o_j \times [u_j, y] + a \times z \otimes [o_j, z] \times y \\
& + [a_1, s] \times b - (u_i, o_i) \otimes [o_i, t] \times b - [u_i, s] \times o_i \otimes t \times b - s \times [o_i, t] \otimes t \times b \\
& - [a_1, s] \times [o_j, z] \otimes (o_j, u_j) - [a_1, s] \times z \otimes o_j \times [u_j, y] - [a_1, s] \times z \otimes [o_j, z] \times y \\
& - (u_i, o_i) \otimes [o_i, t] \times [b_1, y] - [u_i, s] \times o_i \otimes t \times [b_1, y] - s \times [o_i, t] \otimes t \times [b_1, y] \\
& + (u_i, o_i) \otimes [o_i, t] \times [o_j, z] \otimes (o_j, u_j) + (u_i, o_i) \otimes [o_i, t] \times z \otimes o_j \times [u_j, y] \\
& + (u_i, o_i) \otimes [o_i, t] \times z \otimes [o_j, z] \times y + [u_i, s] \times o_i \otimes (t, z) \otimes o_j \times [u_j, y] \\
& + [u_i, s] \times o_i \otimes (t, z) \otimes [o_j, z] \times y + [u_i, s] \times o_i \otimes t \times [o_j, z] \otimes (o_j, u_j) \\
& + s \times [o_i, t] \otimes t \times [o_j, z] \otimes (o_j, u_j) + s \times [o_i, t] \otimes (t, z) \otimes o_j \times [u_j, y] \\
& + s \times [o_i, t] \otimes (t, z) \otimes [o_j, z] \times y.
\end{aligned}
$$

Thus the map $d_{\alpha\beta} \mapsto -a \times b + w_{\alpha\beta}$ is an tame isomorphism.

3.2.6.2 Case 2:

We now consider the cases where the arcs r, s, t and x, y, z are not all distinct. Before proceeding, we introduce a collection of words which will greatly simplify matters.

Remark 3.2.5. Suppose $\alpha = (r, s, t)$ and that $r = u_i - \varepsilon$ and $s = u_{i+1} - \varepsilon$. We consider the other case in remark B.3.1. Let $a = [u_i - \varepsilon, u_i + \varepsilon]$.

We write

$$
f_{t,\alpha} = -[o_i, t] \times a + 2m_{t,\alpha} - 2[o_i, t] \times [a_1, s] + n_{t,\alpha} \otimes (o_i, u_i)
$$

where $m_{t,\alpha}$ and $n_{t,\alpha}$ are the 2-chains satisfying

$$
\begin{aligned}
d(m_{t,\alpha}) &= \rho([o_i, t] \times a) - [o_i, o_i + \varepsilon_1] \times u_i - o_i \times [a_0, u_i] + [o_i, t] \times a_0 \\
d(n_{t,\alpha}) &= [(t, o_i), (t, t)] + [(o_i + \varepsilon_1, o_i + \varepsilon_1), (t, o_i)].
\end{aligned}
$$

The differential of $f_{t,\alpha}$ is

$$
\begin{aligned}
d(f_{t,\alpha}) = [o_i, t] \times a_0 - o_i \times a - [o_i, t] \times a_1 - t \times [a_1, s] & \\
& - (t \times a + t \times [a_1, s] - t \times [o_i, t] \otimes (o_i, u_i) - 2o_i \times [u_i, s] - 2[o_i, t] \times s).
\end{aligned}
$$

Similarly we write

$$f_{\alpha,t} = -a \times [o_i, t] + 2m_{\alpha,t} + 2[a_1, s] \times [o_i, t]$$
$$+ u_i \times [o_i, o_i + \varepsilon_1] \otimes [(o_i + \varepsilon_1, o_i + \varepsilon_1), (o_i, t)]) + (u_i, o_i) \otimes n_{\alpha,t}$$

where

$$d(m_{\alpha,t}) = \rho(a \times [o_i, t]) - u_i \times [o_i, o_i + \varepsilon_1] + [u_i, a_1] \times o_i + a_1 \times [o_i, t]$$
$$d(n_{\alpha,t}) = [o_i, t] \times t + [(o_i + \varepsilon_1, o_i + \varepsilon_1), (o_i, t)].$$

Then

$$d(f_{\alpha,t}) = a_0 \times [o_i, t] - a \times o_i - a_1 \times [o_i, t] - [a_1, s] \times t$$
$$-(-a \times t + [a_1, s] \times t - (u_i, o_i) \otimes [o_i, t] \times t - 2[u_i, s] \times o_i - 2s \times [o_i, t]).$$

Note that the differential of $f_{t,\alpha}$ is literally the difference between the two versions of $c_{t,\alpha}$ given in section 3.2.3.

Now suppose that $b_0 = o_i - \varepsilon$ and $b_1 = o_i + \varepsilon$. Consider the expression $a \times [b_0, b_1]$. Since o_i lies between b_0 and b_1, this is not a geometric chain in our complex, nor is it a 2-chain by our notation convention.

We define $a \times [b_0, b_1]$ to be the 2-chain

$$a \times [b_0, b_1] = a \times [b_0, o_i] - 2m_0 + (u_i, o_i) \otimes n_0 + a \times [o_i, b_1] - 2m_1 - (u_i, o_i) \otimes n_1$$
$$- u_i \times [o_i, o_i + \varepsilon_1] \otimes [(o_i - \varepsilon_1, o_i - \varepsilon_1), (o_i, b_1)]$$

where m_i and n_i are the 2-chains which satisfy

$$d(m_0) = \rho(a \times [b_0, o_i]) + a_1 \times [b_0, o_i] - [u_i, a_1] \times o_i - u_i \times [o_i - \varepsilon_1, o_i],$$
$$d(n_0) = [(o_i, b_0), (o_i, o_i)] - [(o_i, b_0), (o_i - \varepsilon_1, o_i - \varepsilon_1)],$$
$$d(m_1) = \rho(a \times [o_i, b_1]) - u_i \times [o_i, o_i + \varepsilon_1] + [u_i, a_1] \times o_i + a_1 \times [o_i, b_1],$$
$$d(n_1) = -[(o_i, o_i), (o_i, b_1)] + [(o_i - \varepsilon_1, o_i - \varepsilon_1), (o_i, b_1)].$$

The differential of $a \times [b_0, b_1]$ is then

$$d(a \times [b_0, b_1]) = -a_1 \times [b_0, o_i] - a_1 \times [o_i, b_1] - a_0 \times [b_0, o_i] - a_0 \times [o_i, b_1] - a \times b_1 + a \times b_0$$
$$+ (u_i, o_i) \otimes o_i \times [b_0, o_i] + (u_i, o_i) \otimes o_i \times [o_i, b_1]$$

Similarly, we define $[b_0, b_1] \times a$ to be the 2-chain

$$[b_0, b_1] \times a = [b_0, o_i] \times a - 2m_0 - n_0 \otimes (o_i, u_i) - [(b_0, o_i), (o_i - \varepsilon_1, o_i - \varepsilon_1)] \otimes [o_i - \varepsilon_1, o_i] \times u_i$$
$$+ [o_i, b_1] \times a - 2m_1 - n_1 \otimes (o_i, u_i)$$

where

$$d(m_0) = \rho([b_0, o_i] \times a) + [b_0, o_i] \times a_0 + o_i \times [a_0, u_i] - [o_i - \varepsilon_1, o_i] \times u_i,$$
$$d(n_0) = -[b_0, o_i] \times o_i + [(b_0, o_i), (o_i - \varepsilon_1, o_i - \varepsilon_1)],$$
$$d(m_1) = \rho([o_i, b_1] \times a) - [o_i, o_i + \varepsilon_1] \times u_i - o_i \times [a_0, u_i] + [o_i, b_1] \times a_0,$$
$$d(n_1) = -[o_i, b_1] \times o_i + [(o_i - \varepsilon_1, o_i - \varepsilon_1), (b_1, o_i)].$$

Then the differential of $[b_0, b_1] \times a$ is

$$d[b_0, b_1] \times a = b_1 \times a - b_0 \times a - [b_0, o_i] \times a_1 - [o_i, b_1] \times a_1 - [o_i, b_1] \times a_0 - [b_0, o_i] \times a_0$$
$$+ [b_0, o_i] \times o_i \otimes (o_i, u_i) + o_i \times [o_i, b_1] \otimes (o_i, u_i).$$

We now consider the cases where the arcs in α and β are not all distinct.

Suppose that $t = x$. Then $d(d_{\alpha\beta}) = c_{r\beta} + c_{s\beta} - a_{st}c_{t\beta} - b_{\alpha t} - b_{\alpha y} + b_{\alpha z}a_{zy}$, where

$$d(b_{\alpha t}) = r \times [o_i, t] - a \times o_i - a_1 \times [o_i, t] - [a_1, s] \times t.$$

Since the chain $b_{\alpha t}$ does not intersect $\mathcal{H}$, we consider the differential of the 2-chain $d_{\alpha\beta} + a \times b + f_{\alpha,t}$. Since the differential of $f_{\alpha,t}$ is the difference between the word $b_{\alpha t}$ which does not intersect $\mathcal{H}$ and the word which does, the differential of the 2-chain is then literally

$$d(d_{\alpha\beta} + a \times b + f_{\alpha,t}) = d(w_{\alpha\beta})$$

where $w_{\alpha\beta}$ is defined as above. Thus $d_{\alpha\beta} \mapsto -\boxed{a \times b} + w_{\alpha\beta} - f_{\alpha,t}$ is the appropriate stable tame isomorphism.

If instead $t = y$, we have $d(d_{\alpha\beta}) = c_{r\beta} + c_{s\beta} - a_{st}c_{t\beta} - b_{\alpha x} - b_{\alpha t} + b_{\alpha z}a_{zt}$, where

$$d(b_{\alpha t}) = r \times [o_i, t] - a \times o_i - a_1 \times [o_i, t] - [a_1, s] \times t.$$

Again, we consider the differential of the 2-chain $d_{\alpha\beta} + a \times b + f_{\alpha,t}$, which is as before the differential of $w_{\alpha\beta}$. Note however that the term $a \times [b_1, y]$ in the 2-chain $w_{\alpha\beta}$ must be interpreted in terms of remark 3.2.5: since $t = y$ is the overcrossing arc for the crossing α, the overcrossing occurs between b_1 and y. As in the previous case, we find that $d_{\alpha\beta} \mapsto -\boxed{a \times b} + w_{\alpha\beta} - f_{\alpha,t}$ is the appropriate stable tame isomorphism.

Now suppose that $t = z$. Then $d(d_{\alpha\beta}) = c_{r\beta} + c_{s\beta} - a_{st}c_{t\beta} - b_{\alpha x} - b_{\alpha y} + b_{\alpha t}a_{ty}$ where

$$c_{t\beta} = [o_j, t] \times x - o_j \times b - [o_j, t] \times b_1 - t \times [b, y]$$
$$b_{\alpha t} = x \times [o_i, t] - a \times o_i - a_1 \times [o_i, t] - [a_1, s] \times t.$$

Since $c_{t\beta}$ and $b_{\alpha t}$ are 1-chains which do not intersect $\mathcal{H}$, we compute

$$d\left(d_{\alpha\beta} + a \times b + (s, t) \otimes f_{t,\beta} - f_{\alpha,t} \otimes (t, y)\right) = d(w_{\alpha\beta}).$$

If $[o_i, o_j]$ is oriented, then $[o_i, t] \times b$ intersects $\mathcal{H}$ and $a \times [o_j, t]$ does not. Otherwise, $a \times [o_j, t]$ intersects $\mathcal{H}$ and $[o_i, t] \times b$ does not. Thus we use the stable tame isomorphism

$$d_{\alpha\beta} \mapsto -\boxed{a \times b} - (s, t) \otimes f_{t,\beta} + f_{\alpha,t} \otimes (t, y) + w_{\alpha\beta}.$$

As the other cases are completely analogous, this completes the proof of the theorem.

APPENDIX A

GENERATORS FOR THE WICKET ALGEBRA AND $d^2 = 0$

In this section we explicitly define the resolution map ρ and cutting map δ on $\mathcal{A}(\mathcal{W})$. We compute $d = \partial + (-1)^k(2\rho - \delta)$, and a straightforward computation confirms that $d^2 = 0$ for all the generators of $\mathcal{A}(\mathcal{W})$.

A.1 1-cubes which intersect $\mathcal{H}$

Let u be an undercrossing and o the corresponding overcrossing. Let x and y be 0-cubes in $\Delta_*(K)$.

A.1.1

Let $a = [u - \varepsilon, u + \varepsilon] \times y$. If $a \cap \mathcal{H} = (u, y)$, then

$$\rho(a) = (u + \varepsilon, y)$$

$$\delta(a) = (u, o) \otimes (o, y)$$

$$d(a) = -(u + \varepsilon, y) - (u - \varepsilon, y) + (u, o) \otimes (o, y).$$

If $a \cap \mathcal{H} = -(u, y)$ then

$$\rho(a) = -(u - \varepsilon, y)$$

$$\delta(a) = -(u, o) \otimes (o, y)$$

$$d(a) = (u + \varepsilon, y) + (u - \varepsilon, y) - (u, o) \otimes (o, y).$$

A.1.2

Let $b = x \times [u - \varepsilon, u + \varepsilon]$. If $b \cap \mathcal{H} = (x, u)$, then

$$\rho(b) = (x, u + \varepsilon)$$

$$\delta(b) = (x, o) \otimes (o, u)$$

$$d(b) = -(x, u + \varepsilon) - (x, u - \varepsilon) + (x, o) \otimes (o, u).$$

If $b \cap \mathcal{H} = -(x, u)$, then

$$\rho(b) = -(x, u - \varepsilon)$$

$$\delta(b) = (x, o) \otimes (o, u)$$

$$d(b) = (x, u + \varepsilon) + (x, u - \varepsilon) - (x, o) \otimes (o, u).$$

A.2 2-cubes which intersect $\mathcal{H}$ in a 1-parameter family

Let u be an undercrossing and o the corresponding overcrossing. Let x and y be consecutive 0-cubes in $\Delta_*(K)$ and suppose there is no undercrossing in the arc $[x, y]$

A.2.1 Singular 2-cubes away from the diagonal

A.2.1.1

Let $a = [u - \varepsilon, u + \varepsilon] \times [x, y]$. If $a \cap \mathcal{H} = u \times [x, y]$, then

$$\rho(a) = \{u - \varepsilon\} \times [x, y]$$

$$\delta(a) = (u, o) \otimes o \times [x, y]$$

$$\begin{aligned} d(a) &= \{u + \varepsilon\} \times [x, y] + \{u - \varepsilon\} \times [x, y] \\ &\quad - [u - \varepsilon, u + \varepsilon] \times y + [u - \varepsilon, u + \varepsilon] \times x - (u, o) \otimes o \times [x, y]. \end{aligned}$$

If $a \cap \mathcal{H} = -u \times [x, y]$, then

$$\rho(a) = -\{u + \varepsilon\} \times [x, y]$$

$$\delta(a) = -(u, o) \otimes o \times [x, y]$$

$$\begin{aligned} d(a) &= -\{u + \varepsilon\} \times [x, y] - \{u - \varepsilon\} \times [x, y] \\ &\quad - [u - \varepsilon, u + \varepsilon] \times y + [u - \varepsilon, u + \varepsilon] \times x + (u, o) \otimes o \times [x, y]. \end{aligned}$$

A.2.1.2

Let $b = [x, y] \times [u - \varepsilon, u + \varepsilon]$. If $b \cap \mathcal{H} = [x, y] \times u$, then

$$\rho(b) = [x, y] \times \{u + \varepsilon\}$$

$$\delta(b) = [x, y] \times o \otimes (o, u)$$

$$\begin{aligned} d(b) &= y \times [u - \varepsilon, u + \varepsilon] - x \times [u - \varepsilon, u + \varepsilon] \\ &\quad + [x, y] \times \{u + \varepsilon\} + [x, y] \times \{u - \varepsilon\} - [x, y] \times o \otimes (o, u). \end{aligned}$$

If $b \cap \mathcal{H} = -[x, y] \times u$, then

$$\rho(b) = -[x, y] \times \{u - \varepsilon\}$$

$$\delta(b) = -[x, y] \times o \otimes (o, u)$$

$$\begin{aligned} d(b) &= y \times [u - \varepsilon, u + \varepsilon] - x \times [u - \varepsilon, u + \varepsilon] \\ &\quad - [x, y] \times \{u + \varepsilon\} - [x, y] \times \{u - \varepsilon\} + [x, y] \times o \otimes (o, u). \end{aligned}$$

A.2.2 Singular 2-cubes along the diagonal

Let $v_0 = (u - \varepsilon, u - \varepsilon)$, $v_1 = (u + \varepsilon, u - \varepsilon)$, $v_2 = (u + \varepsilon, u + \varepsilon)$ and $v_3 = (u - \varepsilon, u + \varepsilon)$.

A.2.2.1

Let $a = [v_0, v_1, v_2]$. For the chain $x = [(u, u - \varepsilon), (u + \varepsilon, u)]$ in $\Delta_*(\mathcal{H})$, we choose the diagonal approximation

$$\Theta(x) = (u + \varepsilon, u) \otimes [(u, u - \varepsilon), (u + \varepsilon, u)] + [(u, u - \varepsilon), (u + \varepsilon, u)] \otimes (u + \varepsilon, u).$$

If $a \cap \mathcal{H} = x$, then

$$\rho(a) = 0$$

$$\begin{aligned} \delta(a) &= (u + \varepsilon, o) \otimes [(o, u - \varepsilon), (o, u)] + [(u, o), (u + \varepsilon, o)] \otimes (o, u) \\ &= (u + \varepsilon, o) \otimes o \times [u - \varepsilon, u] + [u, u + \varepsilon] \times o \otimes (o, u) \end{aligned}$$

$$\begin{aligned} d(a) &= [v_1, v_2] + [v_0, v_1] + 2\rho(a) - \delta(a) \\ &= \{u + \varepsilon\} \times [u - \varepsilon, u + \varepsilon] + [u - \varepsilon, u + \varepsilon] \times \{u - \varepsilon\} \\ &\quad - (u + \varepsilon, o) \otimes o \times [u - \varepsilon, u] - [u, u + \varepsilon] \times o \otimes (o, u) \end{aligned}$$

If $a \cap \mathcal{H} = -x$, then

$$\rho(a) = 0$$

$$\delta(a) = -(u + \varepsilon, o) \otimes o \times [u - \varepsilon, u] - [u, u + \varepsilon] \times o \otimes (o, u)$$

$$\begin{aligned} d(a) &= \{u + \varepsilon\} \times [u - \varepsilon, u + \varepsilon] + [u - \varepsilon, u + \varepsilon] \times \{u - \varepsilon\} \\ &\quad + (u + \varepsilon, o) \otimes o \times [u - \varepsilon, u] + [u, u + \varepsilon] \times o \otimes (o, u) \end{aligned}$$

A.2.2.2

Let $b = [v_0, v_2, v_3]$. For the chain $y = [(u-\varepsilon, u), (u, u+\varepsilon)]$ in $\Delta_*(\mathcal{H})$, we choose the diagonal approximation

$$\Theta(y) = (u-\varepsilon, u) \otimes [(u-\varepsilon, u), (u, u+\varepsilon)] + [(u-\varepsilon, u), (u, u+\varepsilon)] \otimes (u, u+\varepsilon).$$

If $b \cap \mathcal{H} = y$, then

$$\begin{aligned}
\rho(b) &= 0 \\
\delta(b) &= (u-\varepsilon, o) \otimes [(o, u), (o, u+\varepsilon)] + [(u-\varepsilon, o), (u, o)] \otimes (o, u+\varepsilon) \\
d(b) &= -[v_3, v_2] - [v_0, v_3] + 2\rho(b) - \delta(b) \\
&= -[u-\varepsilon, u+\varepsilon] \times \{u+\varepsilon\} - \{u-\varepsilon\} \times [u-\varepsilon, u+\varepsilon] \\
&\quad - (u-\varepsilon, o) \otimes [(o, u), (o, u+\varepsilon)] - [(u-\varepsilon, o), (u, o)] \otimes (o, u+\varepsilon).
\end{aligned}$$

If $b \cap \mathcal{H} = -y$, then

$$\begin{aligned}
\rho(b) &= 0 \\
\delta(b) &= -(u-\varepsilon, o) \otimes [(o, u), (o, u+\varepsilon)] - [(u-\varepsilon, o), (u, o)] \otimes (o, u+\varepsilon) \\
d(b) &= -[v_3, v_2] - [v_0, v_3] + 2\rho(b) - \delta(b) \\
&= -[u-\varepsilon, u+\varepsilon] \times \{u+\varepsilon\} - \{u-\varepsilon\} \times [u-\varepsilon, u+\varepsilon] \\
&\quad + (u-\varepsilon, o) \otimes [(o, u), (o, u+\varepsilon)] + [(u-\varepsilon, o), (u, o)] \otimes (o, u+\varepsilon).
\end{aligned}$$

A.3 2-cubes which intersect $\mathcal{H}$ in a 1-parameter family with boundary

Let $x = [u-\varepsilon, u+\varepsilon]$, $y = [o-\varepsilon, o]$ and $z = [o, o+\varepsilon]$, where o is the overcrossing of u.

A.3.1

Consider the 2-cube $a = x \times y$. Let b be the 1-cell $u \times [o-\varepsilon, o-\varepsilon_1]$ in $\Delta_*(\mathcal{H})$. We use the diagonal approximation

$$\Theta(b) = b_0 \otimes b + b \otimes b_1.$$

If $a \cap \mathcal{H} = b$, then

$$\rho(a) = [(u, o - \varepsilon_1), (x_0, y_0)]$$

$$\begin{aligned}\delta(a) &= (u, o) \otimes [(o, o - \varepsilon), (o - \varepsilon_1, o - \varepsilon_1)] + [(u, o), (u, o - \varepsilon_1)] \otimes (o - \varepsilon_1, o - \varepsilon_1) \\ &= (u, o) \otimes [(o, o - \varepsilon), (o - \varepsilon_1, o - \varepsilon_1)] - 2u \times [o - \varepsilon_1, o]\end{aligned}$$

and so

$$d(a) = x_1 \times y - x_0 \times y - x \times y_1 + x \times y_0 + 2\rho(a) - \delta(a).$$

If $a \cap \mathcal{H} = -b$, then

$$\rho(a) = [(x_1, y_0), (u, o - \varepsilon_1)]$$

$$\delta(a) = -(u, o) \otimes [(o, o - \varepsilon), (o - \varepsilon_1, o - \varepsilon_1)] + 2u \times [o - \varepsilon_1, o]$$

and so

$$d(a) = x_1 \times y - x_0 \times y - x \times y_1 + x \times y_0 + 2\rho(a) - \delta(a).$$

A.3.2

Consider the 2-cube $a = x \times z$. Let b be the 1-cell $u \times [o + \varepsilon_1, o + \varepsilon]$ in $\Delta_*(\mathcal{H})$. We use the diagonal approximation

$$\Theta(b) = b_0 \otimes b + b \otimes b_1.$$

If $a \cap \mathcal{H} = b$, then

$$\rho(a) = [(u, o + \varepsilon_1), (x_0, z_1)]$$

$$\delta(a) = (u, o + \varepsilon_1) \otimes [(o + \varepsilon_1, o + \varepsilon_1), (o, o + \varepsilon)] + [(u, o + \varepsilon_1), (u, o)] \otimes (o, o + \varepsilon)$$

and so

$$d(a) = x_1 \times z - x_0 \times z - x \times z_1 + x \times z_0 + 2\rho(a) - \delta(a).$$

If $a \cap \mathcal{H} = -b$, then

$$\rho(a) = [(x_1, z_1), (u, o + \varepsilon_1)]$$

$$\delta(a) = -(u, o + \varepsilon_1) \otimes [(o + \varepsilon_1, o + \varepsilon_1), (o, o + \varepsilon)] - [(u, o + \varepsilon_1), (u, o)] \otimes (o, o + \varepsilon)$$

and so

$$d(a) = x_1 \times z - x_0 \times z - x \times z_1 + x \times z_0 + 2\rho(a) - \delta(a).$$

A.3.3

Consider the 2-cube $a = y \times x$. Let b be the 1-cell $[o - \varepsilon, o - \varepsilon_1] \times u$ in $\Delta_*(\mathcal{H})$. We use the diagonal approximation

$$\Theta(b) = b_0 \otimes b + b \otimes b_1.$$

If $a \cap \mathcal{H} = b$, then

$$\rho(a) = [(y_0, x_1), (o - \varepsilon_1, u)]$$

$$\begin{aligned}\delta(a) &= (o - \varepsilon, o) \otimes [(o, u), (o - \varepsilon_1, u)] + [(o - \varepsilon, o), (o - \varepsilon_1, o - \varepsilon_1)] \otimes (o - \varepsilon_1, u)] \\ &= (o - \varepsilon, o) \otimes [(o, u), (o - \varepsilon_1, u)] + [(o - \varepsilon, o), (o - \varepsilon_1, o - \varepsilon_1)] \otimes (o - \varepsilon_1, u)]\end{aligned}$$

and so

$$d(a) = y_1 \times x - y_0 \times x - y \times x_1 + y \times x_0 + 2\rho(a) - \delta(a).$$

If $a \cap \mathcal{H} = -b$, then

$$\rho(a) = [(o - \varepsilon_1, u), (y_0, x_0)]$$

$$\delta(a) = -(o - \varepsilon, o) \otimes [(o, u), (o - \varepsilon_1, u)] - [(o - \varepsilon, o), (o - \varepsilon_1, o - \varepsilon_1)] \otimes (o - \varepsilon_1, u)]$$

and so

$$d(a) = y_1 \times x - y_0 \times x - y \times x_1 + y \times x_0 + 2\rho(a) - \delta(a).$$

A.3.4

Consider the 2-cube $a = z \times x$. Let b be the 1-cell $[o + \varepsilon_1, o + \varepsilon] \times u$ in $\Delta_*(\mathcal{H})$. We use the diagonal approximation

$$\Theta(b) = b_0 \otimes b + b \otimes b_1.$$

If $a \cap \mathcal{H} = b$, then

$$\rho(a) = [(o + \varepsilon_1, u), (z_1, x_1)]$$
$$\begin{aligned}\delta(a) &= (o + \varepsilon_1, o + \varepsilon_1) \otimes [(o + \varepsilon_1, u), (o + \varepsilon, u)] + [(o + \varepsilon_1, o + \varepsilon_1), (o + \varepsilon, o)] \otimes (o, u) \\ &= -2[o + \varepsilon, o + \varepsilon_1) \times u + [(o + \varepsilon_1, o + \varepsilon_1), (o + \varepsilon, o)] \otimes (o, u)\end{aligned}$$

and so

$$d(a) = z_1 \times x - z_0 \times x - z \times x_1 + z \times x_0 + 2\rho(a) - \delta(a).$$

If $a \cap \mathcal{H} = -b$, then

$$\rho(a) = [(z_1, x_0), (o + \varepsilon_1, u)]$$
$$\delta(a) = +2[o + \varepsilon, o + \varepsilon_1) \times u - [(o + \varepsilon_1, o + \varepsilon_1), (o + \varepsilon, o)] \otimes (o, u)$$

and so

$$d(a) = z_1 \times x - z_0 \times x - z \times x_1 + z \times x_0 + 2\rho(a) - \delta(a).$$

A.4 2-cubes which intersect $\mathcal{H}$ in two 1-parameter families

Let $a = [u_i - \varepsilon, u_i + \varepsilon]$ and $b = [u_j - \varepsilon, u_j + \varepsilon]$. Let $x = u_i \times b$ and $y = a \times u_j$ in $\Delta_*(\mathcal{H})$. Let o_i be the overcrossing of u_i and o_j the overcrossing of u_j.

A.4.1

If $a \cap \mathcal{H} = x + y$, we choose the diagonal approximations

$$\Theta(x) = x_1 \otimes x + x \times x_0$$
$$\Theta(y) = y_1 \otimes y + y \times y_0$$

we have

$$\rho(a) = a_0 \times b + a \times b_1$$
$$\delta(a) = (u_i, o_i) \otimes o_i \times b + a \times o_j \otimes (o_j, u_j)$$

and so

$$\begin{aligned} d(a) &= a_1 \times b + a_0 \times b + a \times b_1 + a \times b_0 \\ &\quad - (u_i, o_i) \otimes o_i \times b - a \times o_j \otimes (o_j, u_j). \end{aligned}$$

A.4.2

If $a \cap \mathcal{H} = x - y$, we choose the diagonal approximations

$$\Theta(x) = x_0 \otimes x + x \times x_1$$
$$\Theta(y) = y_1 \otimes y + y \times y_0$$

we have

$$\rho(a) = a_0 \times b - a \times b_0$$
$$\delta(a) = (u_i, o_i) \otimes o_i \times b - a \times o_j \otimes (o_j, u_j)$$

and so

$$\begin{aligned} d(a) &= a_1 \times b + a_0 \times b - a \times b_1 - a \times b_0 \\ &\quad - (u_i, o_i) \otimes o_i \times b + a \times o_j \otimes (o_j, u_j). \end{aligned}$$

A.4.3

If $a \cap \mathcal{H} = -x + y$, we choose the diagonal approximations

$$\Theta(x) = x_1 \otimes x + x \otimes x_0$$
$$\Theta(y) = y_0 \otimes y + y \otimes y_1$$

we have

$$\rho(a) = -a_1 \times b + a \times b_1$$
$$\delta(a) = -(u_i, o_i) \otimes o_i \times b + a \times o_j \otimes (o_j, u_j)$$

and so

$$\begin{aligned} d(a) = &-a_1 \times b - a_0 \times b + a \times b_1 + a \times b_0 \\ &+ (u_i, o_i) \otimes o_i \times b - a \times o_j \otimes (o_j, u_j). \end{aligned}$$

A.4.4

If $a \cap \mathcal{H} = -x - y$, we choose the diagonal approximations

$$\Theta(x) = x_0 \otimes x + x \otimes x_1$$
$$\Theta(y) = y_0 \otimes y + y \otimes y_1$$

we have

$$\rho(a) = -a_1 \times b - a \times b_0$$
$$\delta(a) = -(u_i, o_i) \otimes o_i \times b - a \times o_j \otimes (o_j, u_j)$$

and so

$$\begin{aligned} d(a) = &-a_1 \times b - a_0 \times b - a \times b_1 - a \times b_0 \\ &+ (u_i, o_i) \otimes o_i \times b + a \times o_j \otimes (o_j, u_j). \end{aligned}$$

APPENDIX B

ADDITIONAL CALCULATIONS FOR $KCH_*(K) = H_*(\mathcal{A}(\mathcal{W}))$

In this appendix we tersely deal with the cases not considered in section 3.2.

B.1 Replacing $b_{\alpha k}$ and $c_{k\alpha}$

Suppose $\alpha = (r, l, o)$ and that $l = u_i - \varepsilon$ and $r = u_{i+1} - \varepsilon$. Let $a = [a_0, a_1] = [u_i - \varepsilon, u_i + \varepsilon]$. We apply the stable tame isomorphisms

$$c_{o\alpha} \mapsto -[o_i, o] \times l + \boxed{o_i \times a} + [o_i, o] \times a_1 + o \times [a_1, r]$$
$$b_{\alpha o} \mapsto -l \times [o_i, o] + \boxed{a \times o_i} + a_1 \times [o_i, o] + [a_1, r] \times o$$

to give 1-chains which do not intersect $\mathcal{H}$. If $k \neq o$, the stable tame isomorphisms

$$c_{k\alpha} \mapsto -\boxed{k \times a} + k \times [a_1, r] - k \times [o_i, o] \otimes (o_i, u_i) + (k, o) \otimes o_i \times [l, u_i] - (k, o) \otimes [o_i, o] \times l$$
$$b_{\alpha k} \mapsto \boxed{a \times k} + [a_1, r] \times k - (u_i, o_i) \otimes [o_i, o] \times k + [l, u_i] \times o_i \otimes (o, k) - l \times [o_i, o] \otimes (o, k)$$

give 1-chains which intersect $\mathcal{H}$.

B.2 Replacing $d_{\alpha\alpha}$ and e_α

Suppose $\alpha = (r, s, t)$ is a crossing where $s = u_i - \varepsilon$ and $r = u_{i+1} - \varepsilon$. Write $a = [u_i - \varepsilon, u_i + \varepsilon]$, where u_i is the undercrossing at α. We denote by o_i the overcrossing at α.

In order to replace the algebraic generator $d_{\alpha\alpha}$, which has differential

$$d(d_{\alpha\alpha}) = c_{s\alpha} - a_{st}c_{t\alpha} - b_{\alpha r} + d(e_\alpha),$$

we compare it to $x = [(a_0, a_0), (a_1, a_1), (a_0, a_1)]$, which has differential

$$d(x) = -a \times a_1 - a_0 \times a - (a_0, o_i) \otimes o_i \times [u_i, a_1] - [a_0, u_i] \times o_i \otimes (o_i, a_1).$$

A direct calculation shows

$$d(d_{\alpha\beta} + x) = dw$$

where

$$w = a \times [a_1, r] - [(a_1, a_1), (r, r), (a_1, r)] + e_\alpha$$
$$+ [a_0, u_i] \times o_i \otimes [o_i, t] \times r + [a_0, u_i] \times o_i \otimes o_i \times [a_1, r]$$
$$- a_0 \times [o_i, t] \otimes o_i \times [u_i, r] - a_0 \times [o_i, t] \otimes [o_i, t] \times r - (a_0, t) \otimes [o_i, t] \times [a_1, r];$$

hence we use the stable tame map $d_{\alpha\alpha} \mapsto -\boxed{x} + w$a.

For e_α we have

$$d(e_\alpha) = c_{r\alpha} - b_{\alpha s} + b_{\alpha t} a_{ts},$$

and here we replace with $x = [(a_0, a_0), (a_1, a_0), (a_1, a_1)]$ since

$$d(x) = a_1 \times a + a \times a_0 - (u_i, o_i) \otimes o_i \times [a_0, u_i] - [u_i, , a_1] \times o_i \otimes (o_i, u_i)$$

Computing the differential of $e_\alpha + x$ gives

$$d(e_\alpha + x) = dw$$

where

$$w = -[a_1, r] \times a + [(a_1, a_1), (r, a_1), (r, r)]$$
$$+ r \times [o_i, t] \otimes o_i \times [a_0, u_i] - r \times [o_i, t] \otimes [o_i, t] \times a_0$$
$$- [u_i, r] \times o_i \otimes [o_i, t] \times a_0 + [u_i, r] \times o_i \otimes o_i \times [a_0, u_i] - [a_1, r] \times [o_i, t] \otimes (t, a_0),$$

and so we use $e_\alpha \to -\boxed{x} + w$ to replace the algebraic generator e_α with a singular 2-cube.

B.3 *

Replacing $d_{\alpha\beta}$ for α and β distinct We treat the cases which were not considered previously. Before doing so, we define the 2-chains corresponding to the other case of remark 3.2.5.

Remark B.3.1 (Notation). Suppose $\alpha = (r, s, t)$ is a crossing where $s = u_i - \varepsilon$ and $r = u_{i+1} - \varepsilon$. Let $a = [u_i - \varepsilon, u_i + \varepsilon]$.

We write

$$f_{t,\alpha} = [o_i, t] \times a - 2m_{t,\alpha} + n_{t,\alpha} \otimes (o_i, u_i)$$

where $m_{t,\alpha}$ and $n_{t,\alpha}$ are the 2-chains satisfying

$$d(m_{t,\alpha}) = \rho([o_i, t] \times a) - [o_i, t] \times a_1 - o_i \times [u_i, a_1] + [o_i, o_i + \varepsilon_1] \times u_i$$
$$d(n_{t,\alpha}) = [(t, o_i), (t, t)] + [(o_i + \varepsilon_1, o_i + \varepsilon_1), (t, o_i)].$$

Then

$$d(f_{t,\alpha}) = -[o_i, t] \times a_0 + o_i \times a + [o_i, t] \times a_1 + t \times [a_1, r]$$
$$- (-t \times a + t \times [a_1, r] - t \times [o_i, t] \otimes (o_i, u_i) + 2o_i \times [a_0, u_i] - 2[o_i, o] \times a_0)$$

Similarly we write

$$f_{\alpha,t} = a \times [o_i, t] - 2m_{\alpha,t} + u_i \times [o_i, o_i + \varepsilon_1] \otimes [(o_i + \varepsilon_1, o_i + \varepsilon_1), (o_i, t)] + (u_i, o_i) \otimes n_{\alpha,t}$$

where

$$d(m_{\alpha,t}) = \rho(a \times [o_i, t]) - a_0 \times [o_i, t] + [a_0, u_i] \times o_i + u_i \times [o_i, o_i + \varepsilon_1]$$
$$d(n_{\alpha,t}) = [(o_i, t), (t, t)] + [(o_i + \varepsilon_1, o_i + \varepsilon_1), (o_i, t)]$$

Then

$$d(f_{\alpha,t}) = -a_0 \times [o_i, t] + a \times o_i + a_1 \times [o_i, t] + [a_1, r] \times t$$
$$- (a \times t + [a_1, r] \times t - (u_i, o_i) \otimes [o_i, t] \times t + 2[a_0, u_i] \times o_i - 2a_0 \times [o_i, t]).$$

Now suppose that $b_0 = o_i - \varepsilon$ and $b_1 = o_i + \varepsilon$. By $a \times [b_0, b_1]$ we mean the 2-chain

$$a \times [b_0, b_1] = a \times [b_0, o_i] - 2m_0 - (u_i, o_i) \otimes n_0 + a \times [o_i, b_1] - 2m_1 + (u_i, o_i) \otimes n_1$$
$$+ u_i \times [o_i, o_i + \varepsilon_1] \otimes [(o_i + \varepsilon_1, o_i + \varepsilon_1), (o_i, b_1)].$$

Here m_i and n_i are 2-chains which satisfy

$$d(m_0) = \rho(a \times [b_0, o_i]) + u_i \times [o_i - \varepsilon_1, o_i] - [a_0, u_i] \times o_i - a_0 \times [b_0, o_i]$$

$$d(n_0) = -[(o_i, b_0), (o_i - \varepsilon_1, o_i - \varepsilon_1)] + [(o_i, b_0), (o_i, o_i)]$$

$$d(m_1) = \rho(a \times [o_i, b_1]) - a_0 \times [o_i, b_1] + [a_0, u_i] \times o_i + u_i \times [o_i, o_i + \varepsilon_1]$$

$$d(n_1) = +[(o_i, b_1), (o_i, o_i)] + [(o_i + \varepsilon_1, o_i + \varepsilon_1), (o_i, b_1)],$$

and so

$$d(a \times [b_0, b_1]) = a_1 \times [b_0, o_i] + a_1 \times [o_i, b_1] + a_0 \times [b_0, o_i] + a_0 \times [o_i, b_1] - a \times b_1 + a \times b_0$$
$$- (u_i, o_i) \otimes [(o_i, b_0), (o_i, o_i)] - (u_i, o_i) \otimes o_i \times [o_i, b_1].$$

Likewise we write $[b_0, b_1] \times a$ for the 2-chain

$$[b_0, b_1] \times a = [b_0, o_i] \times a - 2m_0 + n_0 \otimes (o_i, u_i) + [(b_0, o_i), (o_i - \varepsilon_1, o_i - \varepsilon_1)] \otimes [o_i - \varepsilon_1, o_i] \times u_i$$
$$+ [o_i, b_1] \times a - 2m_1 + n_1 \otimes (o_i, u_i),$$

where

$$m_0 = \rho([b_0, o_i] \times a) + [o_i - \varepsilon_1, o_i] \times u_i + o_i \times [u_i, a_1] - [b_0, o_i] \times a_1$$

$$n_0 = -[b_0, o_i] \times o_i + [(b_0, o_i), (o_i - \varepsilon_1, o_i - \varepsilon_1)]$$

$$m_1 = \rho([o_i, b_1] \times a) - [o_i, b_1] \times a_1 - o_i \times [u_i, a_1] + [o_i, o_i + \varepsilon_1] \times u_i$$

$$n_1 = -[o_i, b_1] \times o_i + [(o_i + \varepsilon_1, o_i + \varepsilon_1), (b_1, o_i)].$$

The differential of the 2-chain is then

$$d([b_0, b_1] \times a) = b_1 \times a - b_0 \times a + [b_0, o_i] \times a_1 + [o_i, b_1] \times a_1 + [b_0, o_i] \times a_0 + [o_i, b_1] \times a_0$$
$$- [b_0, o_i] \times o_i \otimes (o_i, u_i) - [o_i, b_1] \times o_i \otimes (o_i, u_i).$$

We consider the orientations not considered in section 3.2.6. Suppose that $\alpha = (r, s, t)$ and $\beta = (x, y, z)$ are distinct crossings. We assume in all the following cases that t is distinct from x, y and z and that z is distinct from r and s. If the arcs are not distinct, then we modify the stable tame maps using the 2-chains in remarks 3.2.5 and B.3.1. The modification is completely analogous to that in section 3.2.6.2.

B.3.1

Suppose that $r = u_i - \varepsilon$, $s = u_{i+1} - \varepsilon$, and that $y = u_j - \varepsilon$, $x = u_{j+1} - \varepsilon$. Let $a = [u_i - \varepsilon, u_i + \varepsilon]$ and $b = [u_j - \varepsilon, u_j + \varepsilon]$. Then $d(d_{\alpha\beta}) = c_{r\beta} + c_{s\beta} - a_{st}c_{t\beta} - b_{\alpha x} - b_{\alpha y} + b_{\alpha z}a_{zy}$, where

$$
\begin{aligned}
c_{r\beta} &= -r \times b + r \times [b_1, x] - r \times [o_j, z] \otimes (o_j, u_j) + (r, z) \otimes o_j \times [y, u_j] - (r, z) \otimes [o_j, z] \times y \\
c_{s\beta} &= -s \times b + s \times [b_1, x] - s \times [o_j, z] \otimes (o_j, u_j) + (s, z) \otimes o_j \times [y, u_j] - (s, z) \otimes [o_j, z] \times y \\
a_{st}c_{t\beta} &= (s, t) \otimes \big\{ - t \times b + t \times [b_1, x] - t \times [o_j, z] \otimes (o_j, u_j) \\
&\qquad + (t, z) \otimes o_j \times [y, u_j] - (t, z) \otimes [o_j, z] \times y \big\} \\
b_{\alpha x} &= -a \times x + [a_1, s] \times x - (u_i, o_i) \otimes [o_i, t] \times x - [u_i, s] \times o_i \otimes (t, x) - s \times [o_i, t] \otimes (t, x) \\
b_{\alpha y} &= -a \times y + [a_1, s] \times y - (u_i, o_i) \otimes [o_i, t] \times y - [u_i, s] \times o_i \otimes (t, y) - s \times [o_i, t] \otimes (t, y) \\
b_{\alpha z}a_{zy} &= \big\{ - a \times z + [a_1, s] \times z - (u_i, o_i) \otimes [o_i, t] \times z \\
&\qquad - [u_i, s] \times o_i \otimes (t, z) - s \times [o_i, t] \otimes (t, z) \big\} \otimes (z, y).
\end{aligned}
$$

We wish to replace $d_{\alpha\beta}$ with the 2-cube $a \times b$, the differential of this 2-cube is

$$
\begin{aligned}
d(a \times b) &= \partial(a \times b) + 2r(a \times b) - \delta(a \times b) \\
&= a_1 \times b - a_0 \times b - a \times b_1 + a \times b_0 + 2a \times b_1 - 2a_1 \times b \\
&\quad - a \times o_j \otimes (o_j, u_j) + (u_i, o_i) \otimes o_i \times b.
\end{aligned}
$$

The difference of $d_{\alpha\beta}$ and $a \times b$ has differential

$$
d(d_{\alpha\beta} - a \times b) = d(w_{\alpha\beta}),
$$

where $w_{\alpha\beta}$ is the 2-chain

$$
\begin{aligned}
w_{\alpha\beta} = {} & [a_1, s] \times [b_1, x] \\
& - a \times [b_1, x] + a \times [o_j, z] \otimes (o_j, u_j) - a \times z \otimes o_j \times [b_0, u_j] + a \times z \otimes [o_j, z] \times b_0 \\
& - [a_1, s] \times b - (u_i, o_i) \otimes [o_i, t] \times b + [u_i, s] \times o_i \otimes t \times b + s \times [o_i, t] \otimes t \times b \\
& - [a_1, s] \times [o_j, z] \otimes (o_j, u_j) + [a_1, s] \times z \otimes o_j \times [b_0, u_j] - [a_1, s] \times z \otimes [o_j, z] \times b_0 \\
& - (u_i, o_i) \otimes [o_i, t] \times [b_1, x] - [u_i, s] \times o_i \otimes t \times [b_1, x] - s \times [o_i, t] \otimes t \times [b_1, x] \\
& + (u_i, o_i) \otimes [o_i, t] \times [o_j, z] \otimes (o_j, u_j) - (u_i, o_i) \otimes [o_i, t] \times z \otimes o_j \times [b_0, u_j] \\
& + (u_i, o_i) \otimes [o_i, t] \times z \otimes [o_j, z] \times b_0 - [u_i, s] \times o_i \otimes (t, z) \otimes o_j \times [b_0, u_j] \\
& + [u_i, s] \times o_i \otimes (t, z) \otimes [o_j, z] \times b_0 + [u_i, s] \times o_i \otimes t \times [o_j, z] \otimes (o_j, u_j) \\
& + s \times [o_i, t] \otimes t \times [o_j, z] \otimes (o_j, u_j) - s \times [o_i, t] \otimes (t, z) \otimes o_j \times [b_0, u_j] \\
& + s \times [o_i, t] \otimes (t, z) \otimes [o_j, z] \times b_0.
\end{aligned}
$$

This shows that the map $d_{\alpha\beta} \mapsto a \times b + w_{\alpha\beta}$ is the required stable tame isomorphism.

B.3.2

Suppose that $s = u_i - \varepsilon$, $r = u_{i+1} - \varepsilon$, and that $x = u_j - \varepsilon$, $y = u_{j+1} - \varepsilon$. Let $a = [u_i - \varepsilon, u_i + \varepsilon]$ and $b = [u_j - \varepsilon, u_j + \varepsilon]$. Then after the stable tame isomorphisms in the previous steps, we have that $d(d_{\alpha\beta}) = c_{r\beta} + c_{s\beta} - a_{st}c_{t\beta} - b_{\alpha x} - b_{\alpha y} + b_{\alpha z}a_{zy}$, where

$$
\begin{aligned}
c_{r\beta} &= r \times b + r \times [b_1, y] - r \times [o_j, z] \otimes (o_j, u_j) - (r, z) \otimes o_j \times [u_j, y] - (r, z) \otimes [o_j, z] \times y \\
c_{r\beta} &= s \times b + s \times [b_1, y] - s \times [o_j, z] \otimes (o_j, u_j) - (s, z) \otimes o_j \times [u_j, y] - (s, z) \otimes [o_j, z] \times y \\
a_{st}c_{t\beta} &= (s, t) \otimes \big\{ + t \times b + t \times [b_1, y] - t \times [o_j, z] \otimes (o_j, u_j) \\
&\qquad - (t, z) \otimes o_j \times [u_j, y] - (t, z) \otimes [o_j, z] \times y \big\} \\
b_{\alpha x} &= a \times x + [a_1, r] \times x - (u_i, o_i) \otimes [o_i, t] \times x + [s, u_i] \times o_i \otimes (t, x) - s \times [o_i, t] \otimes (t, x) \\
b_{\alpha y} &= a \times y + [a_1, r] \times y - (u_i, o_i) \otimes [o_i, t] \times y + [s, u_i] \times o_i \otimes (t, y) - s \times [o_i, t] \otimes (t, y) \\
b_{\alpha z}a_{zy} &= \big\{ a \times z + [a_1, r] \times z - (u_i, o_i) \otimes [o_i, t] \times z \\
&\qquad + [s, u_i] \times o_i \otimes (t, z) - s \times [o_i, t] \otimes (t, z) \big\} \otimes (z, y).
\end{aligned}
$$

We wish to replace $d_{\alpha\beta}$ with the 2-cube $a \times b$. The 2-cube has differential

$$
\begin{aligned}
d(a \times b) = {} & a_1 \times b - a_0 \times b - a \times b_1 + a \times b_0 - 2a \times b_0 + 2a_0 \times b \\
& + a \times o_j \otimes (o_j, u_j) - (u_i, o_i) \otimes o_i \times b.
\end{aligned}
$$

The sum of the differential of $d_{\alpha\beta}$ and $a \times b$ is

$$d(d_{\alpha\beta} - a \times b) = d(w_{\alpha\beta}),$$

where $w_{\alpha\beta}$ is the 2-chain

$$\begin{aligned} w_{\alpha\beta} = {} & [a_1, r] \times [b_1, y] \\ & + a \times [b_1, y] + [a_1, r] \times b - a \times [o_j, z] \otimes (o_j, u_j) - a \times z \otimes o_j \times [u_j, y] \\ & - a \times z \otimes [o_j, z] \times y - (u_i, o_i) \otimes [o_i, t] \times b + [a_0, u_i] \times o_i \otimes t \times b - a_0 \times [o_i, t] \otimes t \times b \\ & - [a_1, r] \times [o_j, z] \otimes (o_j, u_j) - [a_1, r] \times z \otimes o_j \times [u_j, y] - [a_1, r] \times z \otimes [o_j, z] \times y \\ & - (u_i, o_i) \otimes [o_i, t] \times [b_1, y] + [a_0, u_i] \times o_i \otimes t \times [b_1, y] - a_0 \times [o_i, t] \otimes t \times [b_1, y] \\ & + (u_i, o_i) \otimes [o_i, t] \times [o_j, z] \otimes (o_j, u_j) + (u_i, o_i) \otimes [o_i, t] \times z \otimes o_j \times [u_j, y] \\ & + (u_i, o_i) \otimes [o_i, t] \times z \otimes [o_j, z] \times y - [a_0, u_i] \times o_i \otimes (t, z) \otimes o_j \times [u_j, y] \\ & - [a_0, u_i] \times o_i \otimes (t, z) \otimes [o_j, z] \times y - [a_0, u_i] \times o_i \otimes t \times [o_j, z] \otimes (o_j, u_j) \\ & + a_0 \times [o_i, t] \otimes t \times [o_j, z] \otimes (o_j, u_j) + a_0 \times [o_i, t] \otimes (t, z) \otimes o_j \times [u_j, y] \\ & + a_0 \times [o_i, t] \otimes (t, z) \otimes [o_j, z] \times y, \end{aligned}$$

and so the map $d_{\alpha\beta} \mapsto \boxed{a \times b} + w_{\alpha\beta}$ is the required stable tame isomorphism.

B.3.3

Suppose that $s = u_i - \varepsilon$, $r = u_{i+1} - \varepsilon$, and that $y = u_j - \varepsilon$, $x = u_{j+1} - \varepsilon$. Let $a = [u_i - \varepsilon, u_i + \varepsilon]$ and $b = [u_j - \varepsilon, u_j + \varepsilon]$. The generator $d_{\alpha\beta}$ has differential $d(d_{\alpha\beta}) = c_{r\beta} + c_{s\beta} - a_{st}c_{t\beta} - b_{\alpha x} - b_{\alpha y} + b_{\alpha z}a_{zy}$; here

$$\begin{aligned} c_{r\beta} &= -r \times b + r \times [b_1, x] - r \times [o_j, z] \otimes (o_j, u_j) \\ &\quad + (r, z) \otimes \{o_j\} \times [y, u_j] - (r, z) \otimes [o_j, z] \times y \\ c_{s\beta} &= -s \times b + s \times [b_1, x] - s \times [o_j, z] \otimes (o_j, u_j) \\ &\quad + (s, z) \otimes \{o_j\} \times [y, u_j] - (s, z) \otimes [o_j, z] \times y \\ a_{st}c_{t\beta} &= (s, t) \otimes \big\{ - t \times b + t \times [b_1, x] - t \times [o_j, z] \otimes (o_j, u_j) \\ &\qquad + (t, z) \otimes \{o_j\} \times [y, u_j] - (t, z) \otimes [o_j, z] \times y \big\} \end{aligned}$$

and

$$b_{\alpha x} = a \times x + [a_1, r] \times x - (u_i, o_i) \otimes [o_i, t] \times x + [s, u_i] \times o_i \otimes (t, x) - s \times [o_i, t] \otimes (t, x)$$

$$b_{\alpha y} = a \times y + [a_1, r] \times y - (u_i, o_i) \otimes [o_i, t] \times y + [s, u_i] \times o_i \otimes (t, y) - s \times [o_i, t] \otimes (t, y)$$

$$b_{\alpha z} a_{zy} = \big\{a \times z + [a_1, r] \times z - (u_i, o_i) \otimes [o_i, t] \times z + [s, u_i] \times o_i \otimes (t, z) - s \times [o_i, t] \otimes (t, z)\big\} \otimes (z, y).$$

We wish to replace $d_{\alpha\beta}$ with the 2-cell $a \times b$ which has differential

$$\begin{aligned} d(a \times b) &= \partial(a \times b) + 2r(a \times b) - \delta(a \times b) \\ &= +a_1 \times b - a_0 \times b - a \times b_1 + a \times b_0 + 2a \times b_1 + 2a_0 \times b \\ &\quad - a \times \{o_j\} \otimes (o_j, u_j) - (u_i, o_i) \otimes \{o_i\} \times b. \end{aligned}$$

The sum of the differential of the algebraic generators $d_{\alpha\beta}$ and the geometric cell $a \times b$ is

$$d(d_{\alpha\beta} + a \times b) = d(w_{\alpha\beta},$$

where $w_{\alpha\beta}$ is the 2-chain

$$\begin{aligned} w_{\alpha\beta} = {} & [a_1, r] \times [b_1, x] \\ & + a \times [b_1, x] - a \times [o_j, z] \otimes (o_j, u_j) + a \times z \otimes o_j \times [b_0, u_j] - a \times z \otimes [o_j, z] \times b_0 \\ & - [a_1, r] \times b + (u_i, o_i) \otimes [o_i, t] \times b - [a_0, u_i] \times o_i \otimes t \times b + a_0 \times [o_i, t] \otimes t \times b \\ & - [a_1, r] \times [o_j, z] \otimes (o_j, u_j) + [a_1, r] \times z \otimes o_j \times [b_0, u_j] - [a_1, r] \times z \otimes [o_j, z] \times b_0 \\ & - (u_i, o_i) \otimes [o_i, t] \times [b_1, x] + [a_0, u_i] \times o_i \otimes t \times [b_1, x] - a_0 \times [o_i, t] \otimes t \times [b_1, x] \\ & + (u_i, o_i) \otimes [o_i, t] \times [o_j, z] \otimes (o_j, u_j) - (u_i, o_i) \otimes [o_i, t] \times z \otimes o_j \times [b_0, u_j] \\ & + (u_i, o_i) \otimes [o_i, t] \times z \otimes [o_j, z] \times b_0 + [a_0, u_i] \times o_i \otimes (t, z) \otimes o_j \times [b_0, u_j] \\ & - [a_0, u_i] \times o_i \otimes (t, z) \otimes [o_j, z] \times b_0 - [a_0, u_i] \times o_i \otimes t \times [o_j, z] \otimes (o_j, u_j) \\ & + a_0 \times [o_i, t] \otimes t \times [o_j, z] \otimes (o_j, u_j) - a_0 \times [o_i, t] \otimes (t, z) \otimes o_j \times [b_0, u_j] \\ & + a_0 \times [o_i, t] \otimes (t, z) \otimes [o_j, z] \times b_0. \end{aligned}$$

The map $d_{\alpha\beta} \mapsto -\boxed{a \times b} + w_{\alpha\beta}$ is the required stable tame isomorphism.

APPENDIX C

DIFFERENTIAL GRADED ALGEBRAS

In this appendix we review some basic facts of differential graded algebras. For a more comprehensive exposition, see [3].

Fix a ring k. Recall that a graded algebra $\mathcal{A}$ is an k-module $\mathcal{A} = \oplus_n \mathcal{A}_n$ with an associative product $\cdot : \mathcal{A}_m \times \mathcal{A}_n \to \mathcal{A}_{m+n}$ which is k-linear. An algebra morphism between two graded algebras is a k-linear map which intertwines the product structure.

Let $f, g : \mathcal{A} \to \mathcal{B}$ be algebra maps. A k-linear map $S : \mathcal{A} \to \mathcal{B}$ is an *(f, g)-derivation S of degree n* if $S(\mathcal{A}_m) \subset \mathcal{B}_{m+n}$ and

$$S(x \cdot y) = S(x) \cdot f(y) + (-1)^{n|x|} g(x) \cdot S(y),$$

for x of degree $|x|$. If $\mathcal{A} = \mathcal{B}$ and f and g are the identity morphisms on $\mathcal{A}$, then S is a *derivation* of $\mathcal{A}$. In particular, a degree -1 derivation d on $\mathcal{A}$ which satisfies $d^2 = 0$ is a *differential* on $\mathcal{A}$.

We briefly describe the category $\mathbf{DGA}_k$. The objects of $\mathbf{DGA}_k$ consist of differential graded algebras over k, and the morphisms are chain maps. A *differential graded algebra* is a graded algebra $\mathcal{A}$ and a differential d on $\mathcal{A}$. A *chain map* is an algebra morphism $f : \mathcal{A} \to \mathcal{B}$ which satisfies $f d_{\mathcal{A}} = d_{\mathcal{B}} f$.

Two chain maps $f, g : \mathcal{A} \to \mathcal{B}$ are *chain homotopic* if there is an (f, g)-derivation s of degree 1 which satisfies

$$s d_{\mathcal{A}} + d_{\mathcal{B}} s = f - g.$$

Here s is a *chain homotopy* between f and g; we write $f \simeq g$. Two differential graded algebras $\mathcal{A}$ and $\mathcal{B}$ are *chain homotopy equivalent* if there exist chain maps $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{A}$ such that $gf \simeq \mathrm{Id}_{\mathcal{A}}$ and $fg \simeq \mathrm{Id}_{\mathcal{B}}$.

A differential graded algebra is *semifree* if it is isomorphic as a graded algebra to a finitely generated unital tensor algebra, and $d(1) = 0$ for $1 \in A_0$.

Lemma C.0.2. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are semifree differential graded algebras.*

- *If $f : \mathcal{A} \to \mathcal{B}$ is a graded algebra morphism and $f d_{\mathcal{A}} = d_{\mathcal{B}} f$ for all generators of $\mathcal{A}$, then f is a chain map.*

- *If $f, g : \mathcal{A} \to \mathcal{B}$ are chain maps and there is a (f, g)-derivation K such that $K d_{\mathcal{A}} + d_{\mathcal{B}} K = f - g$ for all generators of $\mathcal{A}$, then K is a chain homotopy.*

For semifree differential graded algebras, there is an equivalence relation introduced by Chekanov [2]. Let $\mathcal{A}$ be the unital tensor algebra $T(a_1, \ldots, a_n)$ over k. An algebra automorphism of $\mathcal{A}$ is *elementary* if for some $i \in \{1, \ldots, n\}$, $a_i \mapsto u a_i + w$ and $a_j \mapsto a_j$ if $j \neq i$, where u is a unit in k and w a word in $T(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$. Two unital tensor algebras are *tame isomorphic* if there is an algebra isomorphism which is a composition of elementary automorphisms and a map which sends generators of $\mathcal{A}$ to $\mathcal{B}$.

The differential graded algebra $\mathcal{S}^i$ is the unital tensor algebra over k generated by e_i and de_i, with $|e_i| = |de_i| + 1 = i$, and $d(e_i) = de_i$ and $d(de_i) = 0$. The stabilization of a differential graded algebra is the map with the algebra $\mathcal{S}^i$. The stabilization $S^i(\mathcal{A}, d_{\mathcal{A}}) = (S^i, \partial) \coprod (\mathcal{A}, d_{\mathcal{A}})$ is chain homotopy equivalent with $(\mathcal{A}, d_{\mathcal{A}})$. Concretely, $S^i(\mathcal{A}, d_{\mathcal{A}})$ is the unital tensor algebra $T(a_1, \ldots, a_n, e_i, de_i)$

Two semifree differential graded algebras $\mathcal{A}$ and $\mathcal{B}$ are stable tame isomorphic if there are stabilizations of $\mathcal{A}$ and of $\mathcal{B}$ which are tame isomorphic.

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