University of Massachusetts - Amherst
ScholarWorks@UMass Amherst

Physics Department Faculty Publication Series | Physics

2008

Spin Effects in Long Range Electromagnetic Scattering

BR Holstein
University of Massachusetts Amherst, holstein@physics.umass.edu

Andreas Ross
University of Massachusetts - Amherst

Follow this and additional works at: http://scholarworks.umass.edu/physics_faculty_pubs

Part of the Physical Sciences and Mathematics Commons

Holstein, BR and Ross, Andreas, "Spin Effects in Long Range Electromagnetic Scattering" (2008). *Physics Department Faculty Publication Series.* Paper 522.
http://scholarworks.umass.edu/physics_faculty_pubs/522

This Article is brought to you for free and open access by the Physics at ScholarWorks@UMass Amherst. It has been accepted for inclusion in Physics Department Faculty Publication Series by an authorized administrator of ScholarWorks@UMass Amherst. For more information, please contact scholarworks@library.umass.edu.

arXiv:0802.0715v1 [hep-ph] 5 Feb 2008

Spin Effects in Long Range Electromagnetic Scattering

Barry R. Holstein[a*] and Andreas Ross[a,b†]

[a] Department of Physics – LGRT
University of Massachusetts
Amherst, MA 01003, USA

[b] Department of Physics
Yale University
New Haven, CT 06520, USA

February 8, 2008

Abstract

We analyze the electromagnetic scattering of massive particles with and without spin and, using the techniques of effective field theory, we isolate the leading long distance effects beyond one photon exchange, both classical and quantum mechanical. Spin-independent and spin-dependent effects are isolated and shown to have a universal structure.

*holstein@physics.umass.edu
†andreas.ross@yale.edu

1 Introduction

There has been a good deal of recent interest in higher order corrections to Coulomb scattering. In particular the one-photon exchange approximation, which has traditionally been used to analyze electron scattering has been shown to be inadequate when applied to the problem of isolating nucleon form factors via a Rosenbluth separation. Inclusion of two-photon exchange contributions have been found to be essential in resolving small discrepancies with the values of these same form factors as obtained from spin correlation measurements [1]. A second arena where two-photon exchange effects are needed is in the analysis of transverse polarization asymmetry measurements in electron scattering. Such quantities vanish in the one-photon exchange approximation meaning that the sizable effects found experimentally must arise from two-photon effects [2].

Much has been written about such higher order photon processes and a number of groups have undertaken precision calculation of such effects [3]. It is not our purpose here to attempt such detailed calculations or to confront experimental data. Rather our goal is to use the methods of effective field theory in order to analyze the very longest range (smallest momentum transfer) contributions to the scattering process. These long range components are associated with pieces of the scattering amplitude which are nonanalytic in the momentum transfer, and most of them are also singular in the limit of a vanishing momentum transfer (the exception being part of the the correction to the spin-spin coupling component where an extra factor of q^2 arises). Some of these corrections are classical ($\hbar$-independent) and behave as $1/\sqrt{-q^2}$ while others are quantum mechanical ($\hbar$-dependent) and behave as $\log -q^2$, where q^2 is the invariant momentum transfer squared [4]. Below we shall examine both types of structures in the context of the electromagnetic scattering of two distinguishable massive particles of unit charge e with and without spin. In this case the lowest order interaction, which arises from one-photon exchange, is the simple Coulomb interaction

$$V(r) = \frac{\alpha}{r} \tag{1}$$

where $\alpha = e^2/4\pi$ is the fine structure constant. We find that two-photon exchange processes at threshold ($v \to 0$) yield corrections of the form

$$V(r) = \frac{\alpha}{r}\left(1 + A_C \frac{\alpha}{mr} + A_Q\, \hbar \frac{\alpha}{(mr)^2}\right) \tag{2}$$

where A_C and A_Q are the coefficients of the classical and quantum corrections respectively and are evaluated below.

To see how such terms arise, in the next section we sketch our calculational techniques in the context of spin-independent scattering. This is a problem addressed nearly two decades ago by Feinberg and Sucher using dispersive methods [5]. Even earlier Iwasaki had studied the classical piece of this problem using standard noncovariant perturbation theory [6]. Our quantum corrections are found to agree completely with those found by Feinberg and Sucher. However, our classical potential is at variance with that found both in [5] and [6, 7]. The origin of these differences is found in terms of differing contributions from the iterated piece of the lowest order potential, which must be subtracted from the scattering amplitude in order to produce a properly defined higher order potential [8]. Our work has also been motivated by more recent calculations in gravitational scattering where corrections to Newton's law are obtained [9].

In the following sections we extend these effective field theoretic methods to the problem of spin-dependent scattering and demonstrate that the results are *universal*, in that they can be written in terms of forms which are independent of spin. The calculation of spin-0 – spin-1/2 scattering reveals new structures of spin-orbit character whose universal form is also obtained when we consider spin-0 – spin-1 scattering. The next extension consists of spin-1/2 – spin-1/2 scattering where again we find the universal spin-independent and spin-orbit pieces as well as new (presumably universal) spin-spin coupling interactions. Our results are summarized in a short concluding chapter and the calculational details are found in the appendices. In Appendix D we give generalized results for arbitrary charges and g-factors of the scattered particles and argue for a multipole expansion like scheme which explains the universalities found.

2 Spin-Independent Scattering

We first set the generic framework for our study. We examine the electromagnetic scattering of two charged particles—particle a with mass m_a, charge e and incoming four-momentum p_1 and particle b with mass m_b, charge e and incoming four-momentum p_3. After undergoing scattering the final four-momenta of particle a is $p_2 = p_1 - q$ and that of particle b is $p_4 = p_3 + q$—*cf.* Fig. 1. Now we need to be more specific.

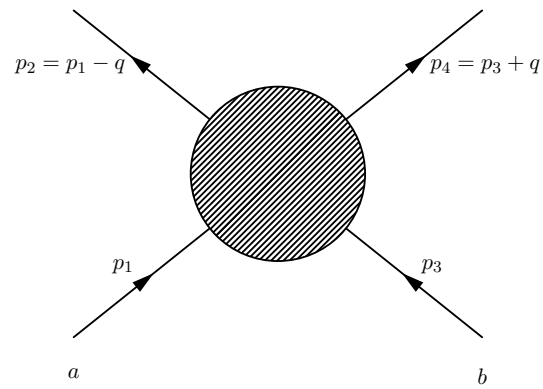

Figure 1: Basic kinematics of electromagnetic scattering.

2.1 Spin-0 – Spin-0 Scattering

We begin by examining the electromagnetic scattering of two spinless particles. The electromagnetic interaction follows from making the minimal substitution in the Klein-Gordon Lagrangian density, yielding

$$\mathcal{L} = (iD_\mu\phi)^\dagger iD^\mu\phi - m^2\phi^\dagger\phi \tag{3}$$

where $iD_\mu = i\partial_\mu - eA_\mu$ is the covariant derivative, and leads to the one- and two-photon vertices

$$\begin{aligned} \tau^{(1)}_\mu(p_2, p_1) &= -ie(p_2 + p_1)_\mu \\ \tau^{(2)}_{\mu\nu}(p_2, p_1) &= 2ie^2\eta_{\mu\nu} \end{aligned} \tag{4}$$

Single photon exchange then leads to the familiar amplitude (in Feynman gauge)

$$\begin{aligned} {}^0\mathcal{M}^{(1)}(q) &= \frac{-i}{\sqrt{2E_1 2E_2 2E_3 2E_4}}\,\tau^{(1)}_\mu(p_1, p_2)\,\frac{-i\eta^{\mu\nu}}{q^2}\,\tau^{(1)}_\nu(p_3, p_4) \\ &= \frac{8\pi\alpha}{\sqrt{2E_1 2E_2 2E_3 2E_4}}\,\frac{s - m_a^2 - m_b^2 + \frac{1}{2}q^2}{q^2} \end{aligned} \tag{5}$$

with $s = (p_1 + p_3)^2$ the square of the center of mass energy.

One way to define the nonrelativistic potential is as the Fourier transform of the nonrelativistic amplitude evaluated in the center of mass frame. We

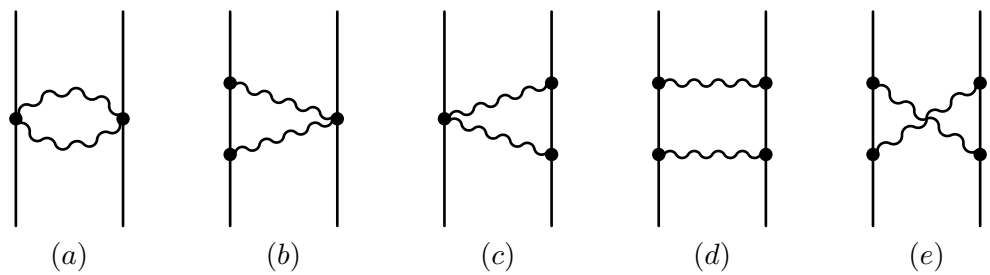

Figure 2: One loop diagrams in electromagnetic scattering.

will use a symmetric center of mass frame[1] with incoming momenta $\vec{p}_1 = \vec{p} + \vec{q}/2$ and $\vec{p}_3 = -\vec{p}_1 = -\vec{p} - \vec{q}/2$ and with outgoing momenta $\vec{p}_2 = \vec{p} - \vec{q}/2$ and $\vec{p}_4 = -\vec{p} + \vec{q}/2$. Conservation of energy then requires $\vec{p} \cdot \vec{q} = 0$ so that $\vec{p}_i^{\,2} = \vec{p}^{\,2} + \vec{q}^{\,2}/4$ for $i = 1, 2, 3, 4$ and $q^2 = -\vec{q}^{\,2}$. In the nonrelativistic limit—$\vec{q}^{\,2}, \vec{p}^{\,2} \ll m^2$ —the amplitude reads

$$ {}^0\mathcal{M}^{(1)}(\vec{q}) \simeq -\frac{4\pi\alpha}{\vec{q}^{\,2}} \left(1 + \frac{\vec{p}^{\,2}}{m_a m_b} + \ldots\right) + \frac{\pi\alpha}{m_a m_b} \left(0 + \frac{(m_a^2 + m_b^2)\vec{p}^{\,2}}{2m_a^2 m_b^2} + \ldots\right) + \ldots \tag{6} $$

yielding the potential

$$ \begin{aligned} {}^0V_C^{(1)}(\vec{r}) &= -\int \frac{d^3q}{(2\pi)^3}\, {}^0\mathcal{M}^{(1)}(\vec{q})\, e^{-i\vec{q}\cdot\vec{r}} \\ &= \frac{\alpha}{r}\left(1 + \frac{\vec{p}^{\,2}}{m_a m_b} + \ldots\right) - \frac{\pi\alpha}{m_a m_b}\,\delta^3(\vec{r})\left(0 + \frac{(m_a^2 + m_b^2)\vec{p}^{\,2}}{2m_a^2 m_b^2} + \ldots\right) \end{aligned} \tag{7} $$

The first component of Eq. (7) is recognized as the usual Coulomb potential (accompanied by small kinematic effects) while the second piece is a short range correction.

Our purpose in this paper is to study the long distance corrections to this form which arise from the two-photon exchange diagrams shown in Fig. 2. This problem has been previously studied by Iwasaki using nonrelativistic perturbation theory [6] and by Feinberg and Sucher using dispersive methods

[1] These symmetric momentum labels of the center of mass frame are chosen so that the leading order coordinate space potential is real in the calculation of spin-0 – spin-1 scattering presented below.

[5]. Our approach will be to use the methods of effective field theory, wherein we evaluate these diagrams by keeping only the leading nonanalytic structure in q^2, since it is these pieces which lead to the long range corrections to the potential. This nonanalytic behavior is of two forms—

i) terms in $1/\sqrt{-q^2}$ which are $\hbar$-independent and therefore classical

ii) terms in $\log -q^2$ which are $\hbar$-dependent and therefore quantum mechanical

The former terms, when Fourier transformed lead to corrections to the nonrelativistic potential of the form $V_{classical}(r) \sim 1/r^2$ while the latter lead to $V_{quantum}(r) \sim \hbar/mr^3$ corrections. For typical masses and separations the quantum mechanical forms are themselves numerically insignificant. However, they are intriguing in that their origin appears to be associated with zitterbewegung. That is, classically we can define the potential by measuring the energy when two objects are separated by distance r. However, in the quantum mechanical case the distance between two objects is uncertain by an amount of order the Compton wavelength due to zero point motion—$\delta r \sim \hbar/m$. This leads to the replacement

$$V(r) \sim \frac{1}{r^2} \longrightarrow \frac{1}{(r \pm \delta r)^2} \sim \frac{1}{r^2} \mp 2\frac{\hbar}{mr^3}$$

which is the form found in our calculations.

The calculational details are described in Appendix A. Here we present only the results. Defining

$$S = \frac{\pi^2}{\sqrt{-q^2}} \quad \text{and} \quad L = \log -q^2$$

we have, from diagrams (a)-(e) of Fig. 2 respectively

$$\begin{aligned}
{}^0\mathcal{M}^{(2)}_{2_a}(q) &= \frac{\alpha^2}{m_a m_b}(-2L) \\
{}^0\mathcal{M}^{(2)}_{2_b}(q) &= \frac{\alpha^2}{m_a m_b}(2L + m_a S) \\
{}^0\mathcal{M}^{(2)}_{2_c}(q) &= \frac{\alpha^2}{m_a m_b}(2L + m_b S)
\end{aligned}$$

$$
\begin{aligned}
{}^0\mathcal{M}^{(2)}_{2_d}(q) &= \frac{\alpha^2}{m_a m_b}\left[L\left(\frac{4m_a m_b}{q^2}+\frac{5(m_a^2+m_b^2)}{4m_a m_b}-\frac{1}{2}\right)+S(m_a+m_b)\right] \\
&- i4\pi\alpha^2\frac{L}{q^2}\sqrt{\frac{m_a m_b}{s-s_0}} \\
{}^0\mathcal{M}^{(2)}_{2_e}(q) &= \frac{\alpha^2}{m_a m_b}\left[L\left(-\frac{4m_a m_b}{q^2}-\frac{5(m_a^2+m_b^2)}{4m_a m_b}-\frac{23}{6}\right)-S(m_a+m_b)\right]
\end{aligned}
\tag{8}
$$

where $s=(p_1+p_3)^2$ is the square of the center of mass energy and $s_0=(m_a+m_b)^2$ is its threshold value. Summing, we find the final result

$$
{}^0\mathcal{M}^{(2)}_{tot}(q)=\frac{\alpha^2}{m_a m_b}\left[(m_a+m_b)S-\frac{7L}{3}\right]-i4\pi\alpha^2\frac{L}{q^2}\sqrt{\frac{m_a m_b}{s-s_0}} \tag{9}
$$

We observe that in addition to the expected terms involving L and S there arises a piece of the second order amplitude which is imaginary. The origin of this imaginary piece is, of course, from the second Born approximation to the Coulomb potential, and reminds us that in order to define a proper correction to the first order Coulomb potential we must subtract off such terms. Before performing the necessary subtraction, we also point out that the imaginary part arising at this order of the amplitude is very singular in the nonrelativistic limit, even more than the leading order amplitude. It stems from an overall phase of the amplitude [19] and thus cannot contribute to any observable since observables are proportional to $|\mathcal{M}|^2$. At our order we have calculated $\mathcal{M}={}^0\mathcal{M}^{(1)}+{}^0\mathcal{M}^{(2)}$ where an observable such as a differential cross section has a leading piece $\mathcal{O}(\alpha^2)$ and the corrections we have calculated contribute to order $\mathcal{O}(\alpha^3)$, but the imaginary part does not contribute to $\mathcal{O}(\alpha^3)$.

In order to subtract the second Born piece, we will work in the nonrelativistic limit and the center of mass frame—$\vec{p}_1+\vec{p}_3=0$. We have then

$$
s-s_0=2\sqrt{m_a^2+\vec{p}_1^{\,2}}\sqrt{m_b^2+\vec{p}_1^{\,2}}+2\vec{p}_1^{\,2}-2m_a m_b \tag{10}
$$

and

$$
\sqrt{\frac{m_a m_b}{s-s_0}}\simeq\frac{m_r}{p_0} \tag{11}
$$

where $m_r=m_a m_b/(m_a+m_b)$ is the reduced mass and $p_0\equiv|\vec{p}_i|,\ i=1,2,3,4$. The transition amplitude then assumes the form

$$
{}^0\mathcal{M}^{(2)}_{tot}(\vec{q})\simeq\frac{\alpha^2}{m_a m_b}\left[(m_a+m_b)S-\frac{7L}{3}\right]-i4\pi\alpha^2\frac{L}{q^2}\frac{m_r}{p_0} \tag{12}
$$

For the iteration we shall use the simple potential

$$ {}^0V_C^{(1)}(\vec{r}) = \frac{\alpha}{r} \tag{13} $$

which reproduces the lowest order amplitude for spin-0 – spin-0 Coulomb scattering—Eq. (7)—in the nonrelativistic limit and which reads in momentum space

$$ {}^0V_C^{(1)}(\vec{q}) \equiv \left\langle \vec{p}_f \left| {}^0\hat{V}_C^{(1)} \right| \vec{p}_i \right\rangle = \frac{e^2}{\vec{q}^2} = \frac{e^2}{(\vec{p}_i - \vec{p}_f)^2} \tag{14} $$

where we identify $\vec{p}_i = \vec{p}_1$ and $\vec{p}_f = \vec{p}_2$. The second Born term is then

$$ \begin{aligned} {}^0\mathrm{Amp}_C^{(2)}(\vec{q}) &= -\int \frac{d^3\ell}{(2\pi)^3} \frac{\left\langle \vec{p}_f \left| {}^0\hat{V}_C^{(1)} \right| \vec{\ell} \right\rangle \left\langle \vec{\ell} \left| {}^0\hat{V}_C^{(1)} \right| \vec{p}_i \right\rangle}{E(p_0) - E(\ell) + i\epsilon} \\ &= i \int \frac{d^3\ell}{(2\pi)^3} {}^0V_C^{(1)}(\vec{\ell} - \vec{p}_f)\, G^{(0)}(\vec{\ell})\, {}^0V_C^{(1)}(\vec{p}_i - \vec{\ell}) \end{aligned} \tag{15} $$

where

$$ G^{(0)}(\ell) = \frac{i}{\frac{p_0^2}{2m_r} - \frac{\ell^2}{2m_r} + i\epsilon} \tag{16} $$

is the free propagator. Note that in Eq. (15) we take both the leading order potential as well as the total energies $E(p_0)$ and $E(\ell)$ in the nonrelativistic limit. The remaining integration can be performed exactly, as discussed in Appendix C, by including a "photon mass" term λ^2 as a regulator, yielding

$$ \begin{aligned} {}^0\mathrm{Amp}_C^{(2)}(\vec{q}) &= i \int \frac{d^3\ell}{(2\pi)^3} \frac{e^2}{|\vec{p}_2 - \vec{\ell}|^2 + \lambda^2} \frac{i}{\frac{p_0^2}{2m_r} - \frac{\ell^2}{2m_r} + i\epsilon} \frac{e^2}{|\vec{\ell} - \vec{p}_1|^2 + \lambda^2} \\ &\xrightarrow{\lambda \to 0} H = -i4\pi\alpha^2 \frac{L}{q^2} \frac{m_r}{p_0} \end{aligned} \tag{17} $$

which reproduces the imaginary component of ${}^0\mathcal{M}_{tot}^{(2)}(\vec{q})$, as expected.[2]

In order to produce a properly defined second order potential ${}^0V_C^{(2)}(\vec{r})$, we must then subtract this second order Born term from the second order scattering amplitude Eq. (12), yielding the result

$$ {}^0V_C^{(2)}(\vec{r}) = -\int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \left[{}^0\mathcal{M}_{tot}^{(2)}(\vec{q}) - {}^0\mathrm{Amp}_C^{(2)}(\vec{q}) \right] $$

[2]We have omitted the IR singularity in the limit $\lambda \to 0$ since it does not contain nonanalytic dependence on q^2. However, we note that it is present in the iteration as well as in the amplitude of the box diagram, and it is easily shown to cancel for the potential.

$$
\begin{aligned}
&= \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \frac{\alpha^2}{m_a m_b} \left[-S(m_a+m_b) + \frac{7}{3}L \right] \\
&= -\frac{\alpha^2(m_a+m_b)}{2m_a m_b r^2} - \frac{7\alpha^2\hbar}{6\pi m_a m_b r^3} \qquad (18)
\end{aligned}
$$

The result given in Eq. (18) agrees with that previously given by Feinberg and Sucher for the quantum mechanical—$\sim 1/r^3$—piece but disagrees for the classical—$1/r^2$—term. The classical contribution has also been calculated by Iwasaki [6], who determined zero for this second order potential. The resolution of this issue was given by Sucher, who pointed out that the classical term depends upon the precise definition of the first order potential used in the iteration [8] and on whether one uses relativistic expressions in the iteration. Use of the simple lowest order form Eq. (13) within a nonrelativistic iteration yields our result for the iteration amplitude given in Eq. (17) and is sufficient to remove the offending imaginary piece of the scattering amplitude. However, if one uses relativistic expressions for the potential and the relativistic form of the energy in the iteration then alternate forms result with a different classical piece of the potential. Moreover, there are ambiguities in the leading order potential used for the iteration so that a unique definition of the second order potential does not exist [8].

For example, Feinberg and Sucher [5] calculate the iteration amplitude

$$
{}^0\mathrm{Amp}^{(2)}_{FS}(\vec{q}) = -\int \frac{d^3\ell}{(2\pi)^3} \frac{\left\langle \vec{p}_f \left| {}^0\hat{V}^{(1)}_{FS} \right| \vec{\ell} \right\rangle \left\langle \vec{\ell} \left| {}^0\hat{V}^{(1)}_{FS} \right| \vec{p}_i \right\rangle}{E(p_0) - E(\ell) + i\epsilon} \qquad (19)
$$

using fully relativistic expressions for the total energies in the denominator $E(p_0) = E_a(p_0) + E_b(p_0) = \sqrt{m_a^2 + p_0^2} + \sqrt{m_b^2 + p_0^2}$ and $E(\ell) = E_a(\ell) + E_b(\ell)$ and a potential including relativistic corrections

$$
{}^0\hat{V}^{(1)}_{FS} = \sqrt{1 + \frac{\hat{p}^2}{\hat{E}_a(\hat{p})\hat{E}_b(\hat{p})}} \; \frac{e^2}{\hat{q}^2} \; \sqrt{1 + \frac{\hat{p}^2}{\hat{E}_a(\hat{p})\hat{E}_b(\hat{p})}}. \qquad (20)
$$

where we do not display the short distance part of the potential and with hats denoting operators whose ordering matters. Sucher calls this a "Feynman gauge inspired" potential whose operator ordering would be a natural choice when working in Feynman gauge [8]. One can evaluate the iteration integral by keeping the leading relativistic modifications of our previous results. Thus the Feinberg-Sucher potential—Eq. (20)—becomes

$$
\left\langle \vec{p}_f \left| {}^0\hat{V}^{(1)}_{FS} \right| \vec{p}_i \right\rangle \simeq \frac{e^2}{\vec{q}^2} \left(1 + \frac{\vec{p}_i^{\,2} + \vec{p}_f^{\,2}}{2m_a m_b} \right) \qquad (21)
$$

Similarly, the energy difference appearing in the propagator becomes

$$
\begin{aligned}
E(p_0) - E(\ell) &\simeq \left(m_a + m_b + \frac{p_0^2}{2m_a} + \frac{p_0^2}{2m_b} - \frac{p_0^2}{8m_a^3} - \frac{p_0^2}{8m_b^3} \right) \\
&- \left(m_a + m_b + \frac{\ell^2}{2m_a} + \frac{\ell^2}{2m_b} - \frac{\ell^2}{8m_a^3} - \frac{\ell^2}{8m_b^3} \right) \\
&= \left(\frac{p_0^2}{2m_r} - \frac{\ell^2}{2m_r} \right) \left[1 - \left(\frac{p_0^2}{4m_r^2} + \frac{\ell^2}{4m_r^2} \right) \left(1 - 3\frac{m_r^2}{m_a m_b} \right) \right]
\end{aligned} \tag{22}
$$

We can now perform the Feinberg-Sucher iteration integral given in Eq. (19)

$$
\begin{aligned}
{}^0\mathrm{Amp}^{(2)}_{FS}(\vec{q}) &\simeq -\int \frac{d^3\ell}{(2\pi)^3} \frac{e^2}{|\vec{p}_f - \vec{\ell}|^2} \frac{1}{\frac{p_0^2}{2m_r} - \frac{\ell^2}{2m_r} + i\epsilon} \frac{e^2}{|\vec{\ell} - \vec{p}_i|^2} \\
&\qquad \times \left\{ 1 + \frac{(p_0^2 + \ell^2)}{m_a m_b} \left[1 + \frac{m_a m_b}{4m_r^2} \left(1 - 3\frac{m_r^2}{m_a m_b} \right) \right] \right\} \\
&\simeq H + \frac{1}{m_a m_b} \left(p_0^2 H + \delta_{rs} H_{rs} \right) \left[1 + \frac{m_a m_b}{4m_r^2} \left(1 - 3\frac{m_r^2}{m_a m_b} \right) \right] \\
&\simeq -i4\pi\alpha^2 \frac{m_r}{p_0} \frac{L}{q^2} + \frac{4\alpha^2}{m_a + m_b} S \left[1 + \frac{(m_a + m_b)^2}{4m_a m_b} - \frac{3}{4} \right] \\
&\simeq -i4\pi\alpha^2 \frac{m_r}{p_0} \frac{L}{q^2} + \frac{\alpha^2}{m_a m_b} \left(m_a + m_b + \frac{m_a m_b}{m_a + m_b} \right) S
\end{aligned} \tag{23}
$$

which agrees precisely to this order with the exact relativistic evaluation given in Ref. [5].

Subtracting from ${}^0\mathcal{M}^{(2)}_{tot}(\vec{q})$ we find then the second order potential

$$
\begin{aligned}
{}^0V^{(2)}_{FS}(\vec{r}) &= -\int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \left[{}^0\mathcal{M}^{(2)}_{tot}(\vec{q}) - {}^0\mathrm{Amp}^{(2)}_{FS}(\vec{q}) \right] \\
&= \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \frac{\alpha^2}{m_a m_b} \left[\frac{m_a m_b}{m_a + m_b} S + \frac{7}{3} L \right] \\
&= \frac{\alpha^2}{2(m_a + m_b) r^2} - \frac{7\alpha^2 \hbar}{6\pi m_a m_b r^3}
\end{aligned} \tag{24}
$$

which is the form given by Feinberg and Sucher [5].

On the other hand Sucher [8] also discusses an alternative version of the one-photon exchange potential which is a natural choice when working in

Coulomb gauge. It includes relativistic expressions, and we refer the reader to [8] for the details of this "Coulomb gauge inspired" order α potential ${}^0\hat{V}^{(1)}_{SP}$. We note that *on-shell* the two potentials coincide

$$\left\langle \vec{p}_f \left| {}^0\hat{V}^{(1)}_{FS} \right| \vec{p}_i \right\rangle = \left\langle \vec{p}_f \left| {}^0\hat{V}^{(1)}_{SP} \right| \vec{p}_i \right\rangle \tag{25}$$

but when used *off-shell* in the iteration then the second order amplitude becomes

$$\begin{aligned} {}^0\mathrm{Amp}^{(2)}_{SP}(\vec{q}) &= -\int \frac{d^3\ell}{(2\pi)^3} \frac{\left\langle \vec{p}_f \left| {}^0\hat{V}^{(1)}_{SP} \right| \vec{\ell} \right\rangle \left\langle \vec{\ell} \left| {}^0\hat{V}^{(1)}_{SP} \right| \vec{p}_i \right\rangle}{E(p_0) - E(\ell) + i\epsilon} \\ &= \frac{\alpha^2}{m_a m_b}(m_a + m_b)S - i4\pi\alpha^2 \frac{m_r}{p_0}\frac{L}{q^2} + \ldots \end{aligned} \tag{26}$$

where fully relativistic expressions for the total energies in the denominator were used. Subtracting from ${}^0\mathcal{M}^{(2)}_{tot}(\vec{q})$ we find then the second order potential

$$\begin{aligned} {}^0V^{(2)}_{SP}(\vec{r}) &= -\int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \left[{}^0\mathcal{M}^{(2)}(\vec{q}) - {}^0\mathrm{Amp}^{(2)}_{SP}(\vec{q}) \right] \\ &= \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \frac{7}{3}\frac{\alpha^2 L}{m_a m_b} = -\frac{7\alpha^2\hbar}{6\pi m_a m_b r^3} \end{aligned} \tag{27}$$

This vanishing result for the classical piece of the $\mathcal{O}(\alpha^2)$ potential was also obtained by Iwasaki [6] and by Spruch [7]. Comparing Eqs. (18), (24) and (27) we note that the quantum mechanical contribution is *invariant*—only the classical term changes. As we have seen, even two fully relativistic iteration treatments do not yield the same answer for the classical piece, and the reason is that ambiguities arise when defining an $\mathcal{O}(\alpha)$ potential operator ${}^0\hat{V}^{(1)}$. We have used an on-shell matrix element to find ${}^0\hat{V}^{(1)}$ whereas the "Coulomb gauge inspired" leading order potential of [8] is defined from an off-shell matrix element. When we use these leading order potentials in an iteration where we integrate over all possible intermediate states, terms that vanish on-shell can contribute and yield differing results for the iteration amplitude and thus for the $\mathcal{O}(\alpha^2)$ potential. However, since the potential itself is not an observable this is not an issue. What *is* an observable is the transition amplitude. In each case we find the same result

$${}^0\mathcal{M}_{tot}(\vec{q}) = -\int d^3r e^{i\vec{q}\cdot\vec{r}} \left[{}^0V^{(1)}_i(\vec{r}) + {}^0V^{(2)}_i(\vec{r}) \right] + {}^0\mathrm{Amp}_i(\vec{q}) \quad i = C, FS, SP \tag{28}$$

In this paper, since we are interested only in the threshold behavior of the transition amplitude, we shall utilize the simple Coulomb form for the potential and a nonrelativistic iteration, since this is sufficient to remove any pieces of the amplitude that would prevent us from writing down a well defined second order potential.

Before moving on to details, it is useful also to point out the parallels between our calculational methods and those of the effective field theory NRQED [11] which has been set up to analyze non-relativistic bound states. The latter involves a systematic expansion in powers of the relative velocity v of the two particles. In this picture the one loop corrections to the amplitude involve terms of order $1/v^3$ and higher. For example, the calculation of Manohar and Stewart[3] evaluates corrections for the scattering of a particle-antiparticle pair up to order v^0 [12]. In coordinate space such terms include a combination of both short and long distance corrections. Our calculation involves a different sort of expansion looking only at the *longest range* terms in coordinate space, and it is not optimized for bound states since we do not use a power counting based on the virial theorem. Correspondingly, in momentum space we look for the nonanalytic components of the scattering amplitude.

In their work in NRQED, Manohar and Stewart have performed a one loop matching calculation to $\mathcal{O}(v^0)$ [12]. They calculate the full QED amplitude as well as the amplitude in their formulation of NRQED—vNRQED—which describes interactions of nonrelativistic fermions, ultra-soft photons and soft photons. For our discussion, the essential difference between vNRQED and full QED is that potential photons have been integrated out and their effects are described by effective four-fermion operators in vNRQED. The coefficients in front of these four-fermion operators are the potential of Manohar and Stewart.

In this paper we do not match onto a well-defined theory such as NRQED. Instead, our matching corresponds solely to the subtraction of the second Born iteration amplitude from the QED amplitude before Fourier transforming to a second order potential in coordinate space. Thus we regard our

[3]Manohar and Stewart (and collaborators) have performed a number of NRQED and NRQCD calculations of great phenomenological importance [13], such as the Lamb shift and hyperfine splitting or the $t\bar{t}$ production cross section near threshold. Much of their work includes impressive two loop results and involved applications of renormalization group methods. When we refer to Manohar and Stewart in our comparison here, we refer specifically to Ref. [12] which we found most suitable when comparing with our work.

potential as a nice way to display our resulting scattering amplitudes in coordinate space, but we emphasize that our main results are the long distance components of the scattering amplitude. Since we do not match onto a theory containing photons whereas the Manohar-Stewart potential arises in the matching to vNRQED which has soft and ultrasoft photons as degrees of freedom, we expect that the potentials differ. It is seen that our quantum corrections proportional to $1/r^3$ stemming from $\log q^2$ pieces of the amplitude are absent from Manohar and Stewart's potential because the exchange of soft photons in vNRQED yield the complete QED contribution of $\log q^2$ terms so that their matching does not yield any quantum pieces of the potential[4].

Despite these differences, Manohar and Stewart's calculation [12] of spin-1/2 – spin-1/2 scattering gives us the opportunity to compare our results for the scattering amplitudes and iterations with theirs. Note that one further difference is that our calculation deals with non-identical particle scattering, while that of Manohar and Stewart involves quark-quark pairs or quark-antiquark pairs of equal mass. Thus we do not have the exchange piece of the amplitude or the annihilation channel contributions given in [12]. We verify that the classical spin-independent iteration amplitude in vNRQED found by these authors using on-shell matching—given by the sum of iterated terms involving $\mathcal{V}_c \times \mathcal{V}_c$ (including relativistic corrections to the free propagator) and $\mathcal{V}_c \times \mathcal{V}_r$ in their notation—yields for the classical piece

$$\mathrm{Amp}^{(2)}_{MS}(\vec{q}) = \frac{5}{2m}\alpha^2 S \tag{29}$$

and agrees with the iterated Feinberg-Sucher amplitude Eq. (23) when we set $m_a = m_b = m$. Note that Manohar and Stewart also emphasize the nonuniqueness of the classical potential, which they associate in part with the existence of off-shell terms of the lowest order potentials such as

$$\mathcal{V}_{\Delta 2}(\vec{q}) \propto \frac{(\vec{p}_f^{\,2} - \vec{p}_i^{\,2})^2}{4m^2\vec{q}^{\,4}}$$

[4]The alternative formulation of potential NRQED (pNRQED) of Pineda and Soto [14], [15] differs from vNRQED in that it only contains ultra-soft photons as degrees of freedom whereas soft photons are integrated out. (Further differences between vNRQED and pNRQED are not important for our discussion here.) Therefore, the potential in pNRQED does exhibit the same quantum pieces of the potential as ours—cf. Eq. (2.17) in Ref. [15] for example. However, we note that Pineda and Soto use off-shell matching in Coulomb gauge yielding a vanishing classical piece of the potential.

which vanish on-shell. Since we perform the iteration using on-shell potentials these terms do not contribute, but they can contribute in some forms such as those used by Iwasaki [6] and in the Coulomb inspired form of Sucher [7].

However, despite the agreement of many forms, the corrections which we examine are often higher order in the relative velocity than included in the Manohar-Stewart vNRQED exposition of [12]. Thus in the case of the spin-orbit term, the one loop corrections which we consider are order v in the Manohar-Stewart expansion and are therefore outside the quoted pieces of their potential. Likewise, the spin-spin correlation corrections which we retain are order v in vNRQED and again are not found in the Manohar-Stewart expressions of [12].

3 Spin-Dependent Scattering: Spin-Orbit Interaction

3.1 Spin-0 – Spin-1/2

Having determined the form of the scattering amplitude and the resulting potential for the spinless scattering case we move on to the case of scattering of particles carrying spin. We begin with the scattering of a spinless particle a from a spin-1/2 particle b. From the Dirac Lagrangian density

$$\mathcal{L} = \bar{\psi}(x)(i \not{D} - m)\psi(x) \tag{30}$$

we determine the one- and two-photon vertices for a spin-1/2 particle to be

$$\begin{aligned} \tau^{(1)}_{\mu}(p_4, p_3) &= -ie\gamma_\mu \\ \tau^{(2)}_{\mu\nu}(p_4, p_3) &= 0 \end{aligned} \tag{31}$$

and the resulting transition amplitude at tree level is found to be

$$\tfrac{1}{2}\mathcal{M}^{(1)}(q) = \frac{4\pi\alpha}{q^2}\frac{1}{m_a}\,\bar{u}(p_4)\,\not{p}_1 u(p_3)\sqrt{\frac{m_a^2 m_b^2}{E_1 E_2 E_3 E_4}} \tag{32}$$

where our spinors are normalized as $\bar{u}(p)\,u(p) = 1$. Defining the spin vector as

$$S_b^\mu = \frac{1}{2}\bar{u}(p_4)\gamma_5\gamma^\mu u(p_3) \tag{33}$$

where $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$, we find the identity[5]

$$\bar{u}(p_4)\gamma_\mu u(p_3) = \left(\frac{1}{1-\frac{q^2}{4m_b^2}}\right)\left[\frac{(p_3+p_4)_\mu}{2m_b}\bar{u}(p_4)u(p_3) - \frac{i}{m_b^2}\epsilon_{\mu\beta\gamma\delta}q^\beta p_3^\gamma S_b^\delta\right] \quad (34)$$

whereupon the nonanalytic part of the transition amplitude can be written in the form

$$\tfrac{1}{2}\mathcal{M}^{(1)}(q) = \frac{4\pi\alpha}{q^2}\left[\bar{u}(p_4)u(p_3) + \frac{i}{m_a m_b^2}\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta\right]. \quad (35)$$

Now we again give the nonrelativistic amplitude in the symmetric center of mass frame ($\vec{p}_1 = -\vec{p}_3 = \vec{p} + \vec{q}/2$) where

$$S_b^\alpha \xrightarrow{NR} (0, \vec{S}_b) \qquad \text{with} \qquad \vec{S}_b = \frac{1}{2}\chi_f^{b\dagger}\vec{\sigma}\chi_i^b, \quad (36)$$

$$\bar{u}(p_4)u(p_3) \xrightarrow{NR} \chi_f^{b\dagger}\chi_i^b - \frac{i}{2m_b^2}\vec{S}_b\cdot\vec{p}\times\vec{q} \quad (37)$$

and

$$\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta \xrightarrow{NR} (m_a+m_b)\left(1+\frac{\vec{p}^{\,2}}{2m_a m_b}\right)\vec{S}_b\cdot\vec{p}\times\vec{q}, \quad (38)$$

so that we find

$$\tfrac{1}{2}\mathcal{M}^{(1)}(\vec{q}) \simeq -\frac{4\pi\alpha}{\vec{q}^{\,2}}\left[\chi_f^{b\dagger}\chi_i^b + \frac{i(m_a+2m_b)}{2m_a m_b^2}\vec{S}_b\cdot\vec{p}\times\vec{q}\right] \quad (39)$$

whereby the lowest order potential becomes

$$\begin{aligned}\tfrac{1}{2}V^{(1)}(\vec{r}) &= -\int\frac{d^3q}{(2\pi)^3}\,\tfrac{1}{2}\mathcal{M}^{(1)}(\vec{q})\,e^{-i\vec{q}\cdot\vec{r}} \\ &\simeq \frac{\alpha}{r}\chi_f^{b\dagger}\chi_i^b - \frac{m_a+2m_b}{2m_a m_b^2}\,\vec{S}_b\cdot\vec{p}\times\vec{\nabla}\frac{\alpha}{r} \\ &\simeq \frac{\alpha}{r}\chi_f^{b\dagger}\chi_i^b - \frac{\alpha}{r^3}\frac{m_a+2m_b}{2m_a m_b^2}\,\vec{L}\cdot\vec{S}_b \end{aligned} \quad (40)$$

where $\vec{L} = \vec{r}\times\vec{p}$ is the angular momentum and $\vec{r} \equiv \vec{r}_a - \vec{r}_b$—the modification of the leading spin-independent potential has a spin-orbit character.

[5]Here ϵ^{0123} is taken to be $+1$.

When evaluating the one loop corrections we encounter an additional complication: The calculation contains two independent kinematic variables, the momentum transfer q^2 and $s - s_0$ which is to leading order proportional to p_0^2 (where $p_0 \equiv |\vec{p}_i|,\ i = 1, 2, 3, 4$) in the center of mass frame. We find that our results differ if we perform an expansion first in $s - s_0$ and then in q^2 or vice versa. This ordering issue only occurs for the box diagram, diagram (d) of Fig. 2, where it stems from the reduction of vector and tensor box integrals. Their reduction in terms of scalar integrals involves the inversion of a matrix whose Gram determinant vanishes in the nonrelativistic threshold limit $q^2, s - s_0 \to 0$. More precisely, the denominators or the vector and tensor box integrals (see Appendix A) involve a factor of $(4p_0^2 - \vec{q}^{\,2})$ when expanded in the nonrelativistic limit. Since $q^2 = 4p_0^2 \sin^2 \frac{\theta}{2}$ with θ the scattering angle, we notice that $4p_0^2 > \vec{q}^{\,2}$ unless we consider backward scattering where $\theta = \pi$ and where the scattering amplitude diverges. And since p_0^2 originates from the relativistic structure $s - s_0$, we therefore have to first expand our vector and tensor box integrals in q^2 and then in $s - s_0$. Evaluating the diagrams (a)-(e) of Fig. 2 and we find the results

$$
\begin{aligned}
\tfrac{1}{2}\mathcal{M}^{(2)}_{2_a}(q) &= 0 \\
\tfrac{1}{2}\mathcal{M}^{(2)}_{2_b}(q) &= 0 \\
\tfrac{1}{2}\mathcal{M}^{(2)}_{2_c}(q) &= \frac{\alpha^2}{m_a m_b}\left[\bar{u}(p_4)u(p_3)\, S m_b\right] \\
\tfrac{1}{2}\mathcal{M}^{(2)}_{2_d}(q) &= \frac{\alpha^2}{m_a m_b}\Bigg[\bar{u}(p_4)u(p_3)\Bigg(-S\frac{m_a m_b(m_a+2m_b)}{s-s_0} \\
&\qquad\qquad - S m_b - L\,\frac{2m_a^2+3m_a m_b-2m_b^2}{6m_a m_b}\Bigg) \\
&\qquad + \frac{1}{m_a}\bar{u}(p_4)\,\not{p}_1 u(p_3)\Bigg(4L\frac{m_a m_b}{q^2} + S\frac{m_a m_b(m_a+2m_b)}{s-s_0} \\
&\qquad\qquad + S(m_a+m_b) + L\,\frac{10m_a^2+11m_b^2}{12m_a m_b}\Bigg)\Bigg] \\
&\quad - i4\pi\alpha^2\frac{L}{q^2}\sqrt{\frac{m_a m_b}{s-s_0}}\,\frac{1}{m_a}\bar{u}(p_4)\,\not{p}_1 u(p_3) \\
\tfrac{1}{2}\mathcal{M}^{(2)}_{2_e}(q) &= \frac{\alpha^2}{m_a m_b}\Bigg[\bar{u}(p_4)u(p_3)\Bigg(S\frac{m_a-2m_b}{4} + L\frac{2m_a^2-3m_a m_b-2m_b^2}{6m_a m_b}\Bigg)
\end{aligned}
$$

$$+\frac{1}{m_a}\bar{u}(p_4)\,\not{p}_1 u(p_3)\left(-4L\frac{m_a m_b}{q^2}-S\frac{3m_a+2m_b}{4}\right.$$
$$\left.-L\,\frac{10m_a^2+16m_a m_b+11m_b^2}{12m_a m_b}\right)\Bigg] \quad (41)$$

Summing, we determine

$$\begin{aligned}
\tfrac{1}{2}\mathcal{M}^{(2)}_{tot}(q) \;=\; & \frac{\alpha^2}{m_a m_b}\Bigg[L\left(-\bar{u}(p_4)u(p_3)-\frac{4}{3}\frac{1}{m_a}\bar{u}(p_4)\,\not{p}_1 u(p_3)\right)\\
& +S\left(\frac{m_a-2m_b}{4}\,\bar{u}(p_4)u(p_3)+\frac{m_a+2m_b}{4}\frac{1}{m_a}\bar{u}(p_4)\not{p}_1 u(p_3)\right)\\
& -\frac{(m_a+2m_b)m_a m_b S}{s-s_0}\left(\bar{u}(p_4)u(p_3)-\frac{1}{m_a}\bar{u}(p_4)\,\not{p}_1 u(p_3)\right)\Bigg]\\
& -\,i4\pi\alpha^2\frac{L}{q^2}\sqrt{\frac{m_a m_b}{s-s_0}}\;\frac{1}{m_a}\bar{u}(p_4)\,\not{p}_1 u(p_3). \qquad (42)
\end{aligned}$$

Using the identity Eq. (34) and

$$p_1\cdot(p_3+p_4)=2m_a m_b+s-s_0+\frac{q^2}{2}$$

Eq. (42) becomes

$$\begin{aligned}
\tfrac{1}{2}\mathcal{M}^{(2)}_{tot}(q) \;=\; & \frac{\alpha^2}{m_a m_b}\Bigg[L\left(-\frac{7}{3}\bar{u}(p_4)u(p_3)-\frac{4i}{3m_a m_b^2}\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta\right)\\
& +S\left((m_a+m_b)\bar{u}(p_4)u(p_3)+\frac{i(m_a+2m_b)}{4m_a m_b^2}\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta\right)\\
& +\frac{iS(m_a+2m_b)}{m_b(s-s_0)}\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta\Bigg]\\
& -i4\pi\alpha^2\frac{L}{q^2}\sqrt{\frac{m_a m_b}{s-s_0}}\left(\bar{u}(p_4)u(p_3)+\frac{i}{m_a m_b^2}\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta\right). \quad (43)
\end{aligned}$$

Finally, working in the symmetric center of mass frame and taking the non-relativistic limit using Eqs. (37), (38) and

$$\frac{1}{s-s_0}\xrightarrow{NR}\frac{m_a m_b}{(m_a+m_b)^2}\frac{1}{p_0^2}+\frac{(m_a-m_b)^2}{4m_a m_b(m_a+m_b)^2} \qquad (44)$$

we find

$$
\begin{aligned}
{}^{\frac{1}{2}}\mathcal{M}^{(2)}_{tot}(\vec{q}) \;\simeq\; & \left[\frac{\alpha^2}{m_a m_b}\left((m_a+m_b)S-\frac{7}{3}L\right)-i4\pi\alpha^2\frac{L}{q^2}\frac{m_r}{p_0}\right]\chi_f^{b\dagger}\chi_i^b \\
+\; & \left[\frac{\alpha^2}{m_a m_b}\left(\frac{m_a^2+2m_a m_b+2m_b^2}{2m_a(m_a+m_b)}\,S-\frac{m_a+8m_b}{6m_a m_b}\,L\right)\right. \\
& \left.+\frac{\alpha^2(m_a+2m_b)}{(m_a+m_b)}\left(-i\frac{2\pi L}{p_0 q^2}+\frac{S}{p_0^2}\right)\right]\frac{i}{m_b}\vec{S}_b\cdot\vec{p}\times\vec{q} \qquad (45)
\end{aligned}
$$

We note from Eq. (45) that the scattering amplitude consists of two pieces—a spin-independent component proportional to $\chi_f^{b\dagger}\chi_i^b$ whose functional form

$$
\frac{\alpha^2}{m_a m_b}\left[(m_a+m_b)S-\frac{7}{3}L\right]-i4\pi\alpha^2\frac{L}{q^2}\frac{m_r}{p_0} \qquad (46)
$$

is *identical* to that of spinless scattering—together with a spin-orbit component proportional to

$$
\frac{i}{m_b}\vec{S}_b\cdot\vec{p}\times\vec{q}
$$

whose functional form is

$$
\begin{aligned}
& \frac{\alpha^2}{m_a m_b}\left(\frac{m_a^2+2m_a m_b+2m_b^2}{2m_a(m_a+m_b)}\,S-\frac{m_a+8m_b}{6m_a m_b}\,L\right) \\
+\; & \frac{\alpha^2(m_a+2m_b)}{(m_a+m_b)}\left(-i\frac{2\pi L}{p_0 q^2}+\frac{S}{p_0^2}\right) \qquad (47)
\end{aligned}
$$

We note in Eq. (47) the presence of an imaginary final state rescattering term proportional to i/p_0 as before together with a completely new type of kinematic form, proportional to $1/p_0^2$ which diverges at threshold. Unlike the term proportional to i/p_0, the spin-dependent piece proportional to $1/p_0^2$ or proportional to $1/s-s_0$ is intriguing in that it is not imaginary but real and therefore *does* contribute to observables. (However, for bound state systems where the virial theorem $\frac{1}{2}mv^2\sim\frac{\alpha}{r}$ holds, this piece seems to be of the same order as the leading order $\mathcal{O}(\alpha)$ spin-orbit piece.) Such a kinematic form has been seen before by other researchers when looking at spin-dependent scattering. It appears in the form

$$
\frac{1}{p_0^2\cos^2\frac{\theta}{2}}=\frac{1}{p_0^2-\frac{1}{4}\vec{q}^{\,2}}
$$

and has been previously identified by Feinberg and Sucher in their evaluation of spin-0 – spin-1/2 scattering [10] and by Manohar and Stewart [12]. (Note that in our situation, since kinematics guarantees that $\vec{q}^2 \leq 4p_0^2$ we can expand via

$$\frac{1}{p_0^2 - \frac{1}{4}\vec{q}^2} = \frac{1}{p_0^2}\left(1 + \frac{\vec{q}^2}{4p_0^2} + \frac{\vec{q}^4}{16p_0^4} + \dots\right)$$

and drop the terms higher order in $\vec{q}^2/4p_0^2$ since after Fourier-transforming, such terms are higher order in $1/r^2$ and are therefore shorter distance than the terms which we retain.) In any case, the presence of *either* of these two forms proportional to i/p_0 and $1/p_0^2$ would prevent us from writing down a well defined second order potential.

The solution to this problem is, as before, to properly subtract the iterated first order potential

$$\tfrac{1}{2}\mathrm{Amp}_C^{(2)}(\vec{q}) \;=\; -\int \frac{d^3\ell}{(2\pi)^3} \frac{\left\langle \vec{p}_f \left| \tfrac{1}{2}\hat{V}_C^{(1)} \right| \vec{\ell} \right\rangle \left\langle \vec{\ell} \left| \tfrac{1}{2}\hat{V}_C^{(1)} \right| \vec{p}_i \right\rangle}{\frac{p_0^2}{2m_r} - \frac{\ell^2}{2m_r} + i\epsilon} \tag{48}$$

where we now use the one-photon exchange potential $\frac{1}{2}V_C^{(1)}(\vec{r})$ given in Eq. (40). Splitting this lowest order potential into spin-independent and spin-dependent components—

$$\left\langle \vec{p}_f \left| \tfrac{1}{2}\hat{V}_C^{(1)} \right| \vec{p}_i \right\rangle = \left\langle \vec{p}_f \left| \tfrac{1}{2}\hat{V}_{S-I}^{(1)} \right| \vec{p}_i \right\rangle + \left\langle \vec{p}_f \left| \tfrac{1}{2}\hat{V}_{S-O}^{(1)} \right| \vec{p}_i \right\rangle \tag{49}$$

where

$$\begin{aligned}
\left\langle \vec{p}_f \left| \tfrac{1}{2}\hat{V}_{S-I}^{(1)} \right| \vec{p}_i \right\rangle &= \frac{e^2}{\vec{q}^2}\,\chi_f^{b\dagger}\chi_i^b = \frac{e^2}{(\vec{p}_i - \vec{p}_f)^2}\,\chi_f^{b\dagger}\chi_i^b \\
\left\langle \vec{p}_f \left| \tfrac{1}{2}\hat{V}_{S-O}^{(1)} \right| \vec{p}_i \right\rangle &= \frac{e^2}{\vec{q}^2}\frac{m_a + 2m_b}{2m_a m_b}\,\frac{i}{m_b}\vec{S}_b \cdot \vec{p} \times \vec{q} \\
&= \frac{e^2}{(\vec{p}_i - \vec{p}_f)^2}\frac{m_a + 2m_b}{2m_a m_b}\,\frac{i}{m_b}\vec{S}_b \cdot \frac{1}{2}(\vec{p}_i + \vec{p}_f) \times (\vec{p}_i - \vec{p}_f)
\end{aligned} \tag{50}$$

we find that the iterated amplitude splits also into spin-independent and spin-dependent pieces. The leading spin-independent amplitude is

$$\tfrac{1}{2}\mathrm{Amp}_{S-I}^{(2)}(\vec{q}) \;=\; -\int \frac{d^3\ell}{(2\pi)^3} \frac{\left\langle \vec{p}_f \left| \tfrac{1}{2}\hat{V}_{S-I}^{(1)} \right| \vec{\ell} \right\rangle \left\langle \vec{\ell} \left| \tfrac{1}{2}\hat{V}_{S-I}^{(1)} \right| \vec{p}_i \right\rangle}{\frac{p_0^2}{2m_r} - \frac{\ell^2}{2m_r} + i\epsilon}$$

$$
\begin{aligned}
&= i\sum_{s_\ell}\int\frac{d^3\ell}{(2\pi)^3}\frac{e^2\chi_f^{b\dagger}\chi_{s_\ell}^b}{|\vec{p}_f-\vec{\ell}|^2+\lambda^2}\frac{i}{\frac{p_0^2}{2m_r}-\frac{\ell^2}{2m_r}+i\epsilon}\frac{e^2\chi_{s_\ell}^{b\dagger}\chi_i^b}{|\vec{\ell}-\vec{p}_i|^2+\lambda^2}\\
&\xrightarrow{\lambda\to 0} \chi_f^{b\dagger}\chi_i^b H = -i4\pi\alpha^2\frac{L}{q^2}\frac{m_r}{p_0}\chi_f^{b\dagger}\chi_i^b \qquad (51)
\end{aligned}
$$

and the leading spin-dependent term is

$$
\begin{aligned}
{}^{\frac{1}{2}}\mathrm{Amp}^{(2)}_{S-O}(\vec{q}) &= -\int\frac{d^3\ell}{(2\pi)^3}\frac{\left\langle\vec{p}_f\left|\frac{1}{2}\hat{V}^{(1)}_{S-I}\right|\vec{\ell}\right\rangle\left\langle\vec{\ell}\left|\frac{1}{2}\hat{V}^{(1)}_{S-O}\right|\vec{p}_i\right\rangle}{\frac{p_0^2}{2m_r}-\frac{\ell^2}{2m_r}+i\epsilon}\\
&\quad -\int\frac{d^3\ell}{(2\pi)^3}\frac{\left\langle\vec{p}_f\left|\frac{1}{2}\hat{V}^{(1)}_{S-O}\right|\vec{\ell}\right\rangle\left\langle\vec{\ell}\left|\frac{1}{2}\hat{V}^{(1)}_{S-I}\right|\vec{p}_i\right\rangle}{\frac{p_0^2}{2m_r}-\frac{\ell^2}{2m_r}+i\epsilon}\\
&= \frac{i(m_a+2m_b)}{2m_am_b^2}\vec{S}_b\cdot\\
&\quad\left(i\int\frac{d^3\ell}{(2\pi)^3}\frac{e^2}{|\vec{p}_f-\vec{\ell}|^2+\lambda^2}\frac{i}{\frac{p_0^2}{2m_r}-\frac{\ell^2}{2m_r}+i\epsilon}\frac{e^2\,\frac{1}{2}(\vec{p}_i+\vec{\ell})\times(\vec{p}_i-\vec{\ell})}{|\vec{\ell}-\vec{p}_i|^2+\lambda^2}\right.\\
&\quad\left.+i\int\frac{d^3\ell}{(2\pi)^3}\frac{e^2\,\frac{1}{2}(\vec{\ell}+\vec{p}_f)\times(\vec{\ell}-\vec{p}_f)}{|\vec{p}_f-\vec{\ell}|^2+\lambda^2}\frac{i}{\frac{p_0^2}{2m_r}-\frac{\ell^2}{2m_r}+i\epsilon}\frac{e^2}{|\vec{\ell}-\vec{p}_i|^2+\lambda^2}\right)\\
&\xrightarrow{\lambda\to 0}\frac{i(m_a+2m_b)}{2m_am_b^2}\,\vec{S}_b\cdot\vec{H}\times\vec{q}\\
&= \frac{\alpha^2(m_a+2m_b)}{(m_a+m_b)}\left(-i\frac{2\pi L}{p_0q^2}+\frac{S}{p_0^2}\right)\frac{i}{m_b}\vec{S}_b\cdot\vec{p}\times\vec{q} \qquad (52)
\end{aligned}
$$

which we have checked against Manohar and Stewart's expression for the iteration amplitude to this order [12] setting $m_a = m_b = m$. In principle we would also have to iterate the leading order spin-orbit piece twice. However this procedure yields only terms higher order in p^2. We observe that when the amplitudes Eqs. (52) and (51) are subtracted from the full one loop scattering amplitude Eq. (47) both the terms involving $1/p_0^2$ and those proportional to i/p_0 disappear leaving behind a well-defined second order potential

$$
\begin{aligned}
{}^{\frac{1}{2}}V_C^{(2)}(\vec{r}) &= -\int\frac{d^3q}{(2\pi)^3}e^{-i\vec{q}\cdot\vec{r}}\left[\tfrac{1}{2}\mathcal{M}^{(2)}_{tot}(\vec{q})-{}^{\frac{1}{2}}\mathrm{Amp}^{(2)}_C(\vec{q})\right]\\
&= -\frac{\alpha^2}{m_am_b}\int\frac{d^3q}{(2\pi)^3}e^{-i\vec{q}\cdot\vec{r}}\left[\left((m_a+m_b)S-\frac{7}{3}L\right)\chi_f^{b\dagger}\chi_i^b\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{m_a^2+2m_am_b+2m_b^2}{2m_a(m_a+m_b)}\,S-\frac{m_a+8m_b}{6m_am_b}\,L\right)\frac{i}{m_b}\vec{S}_b\cdot\vec{p}\times\vec{q}\Bigg]\\
=&\left[-\frac{\alpha^2(m_a+m_b)}{2m_am_br^2}-\frac{7\alpha^2\hbar}{6\pi m_am_br^3}\right]\chi_f^{b\dagger}\chi_i^b\\
+&\frac{1}{m_b}\vec{S}_b\cdot\vec{p}\times\vec{\nabla}\left[\frac{\alpha^2(m_a^2+2m_am_b+2m_b^2)}{4m_a^2m_b(m_a+m_b)r^2}+\frac{\alpha^2(m_a+8m_b)\hbar}{12\pi m_a^2m_b^2r^3}\right]\\
=&\left[-\frac{\alpha^2(m_a+m_b)}{2m_am_br^2}-\frac{7\alpha^2\hbar}{6\pi m_am_br^3}\right]\chi_f^{b\dagger}\chi_i^b\\
+&\left[\frac{\alpha^2(m_a^2+2m_am_b+2m_b^2)}{2m_a^2m_b^2(m_a+m_b)r^4}+\frac{\alpha^2(m_a+8m_b)\hbar}{4\pi m_a^2m_b^3r^5}\right]\vec{L}\cdot\vec{S}_b \qquad (53)
\end{aligned}
$$

We observe then that the second order potential for long range Coulomb scattering of a spinless and a spin-1/2 particle has one component which is independent of the spin of particle b and which is identical to the potential found for the case of spinless scattering, accompanied by a spin-orbit interaction with a new form for its classical and quantum potentials.

Finally we would like to note that vertex corrections on the side of the spin-1/2 particle, particle b, give corrections to the g-factor of the spin-1/2 particle altering the tree level value $g_b^{(0)}=2$ to its $\mathcal{O}(\alpha)$ corrected value $g_b^{(1)}=2+\frac{\alpha}{\pi}$. Since the g-factor is implicitly a parameter of the spin-orbit coupling piece of our leading order amplitude Eq. (39) and potential Eq. (40) the vertex correction which we have neglected will yield a long range contribution of $\mathcal{O}(\alpha^2)$ with the same distance dependence as the leading order contributions proportional to $\vec{L}\cdot\vec{S}_b/r^3$. We will neglect these contributions for now and include them later by writing our results for particles with arbitrary charges and g-factors in Appendix D. Therefore by using the physical values of the mass, charge and g-factor of the scattered particles it is sufficient to only consider the two-photon exchange diagrams displayed in Fig. 2 when considering the leading long distance corrections.

3.2 Spin-0 – Spin-1

It is tempting to speculate that if we extend our considerations to higher spin then this pattern continues—a spin-independent component identical to the spin-0 – spin-0 potential, accompanied by a spin-orbit interaction which is the same for all spins, plus additional terms which have no spin-0 or spin-1/2

analog. In order to test this hypothesis, we move to spin-0 – spin-1 scattering, and we take the spin-1 particle to be a W^+ boson. In order to determine the correct interaction vertices we must recall that the electroweak interaction is a non-abelian gauge theory [16]. This means that the spin one Lagrangian which contains the charged-W has the Proca form—

$$\mathcal{L} = -\frac{1}{4}(\vec{U}_{\mu\nu})^2 + \frac{m^2}{2}\vec{U}_\mu^2 \tag{54}$$

but the SU(2) field tensor $\vec{U}_{\mu\nu}$ contains an additional term on account of the required gauge invariance [16]

$$\vec{U}_{\mu\nu} = D_\mu \vec{U}_\nu - D_\nu \vec{U}_\mu - ic_{SU(2)}\vec{U}_\mu \times \vec{U}_\nu \tag{55}$$

where $c_{SU(2)}$ is the SU(2) electroweak coupling constant. This additional term in the field tensor is responsible for the interactions involving three and four W-bosons and for an "extra" interaction term which has the form of an anomalous magnetic moment and, when added to the simple Proca moment, increases the predicted gyromagnetic ratio from its naive value—$g_{W^\pm}^{\text{naive}} = 1$—to its standard model value—$g_{W^\pm}^{\text{sm}} = 2$. As discussed in [17] there are various theoretical reasons why elementary particles have a g-factor $g = 2$. The resulting one- and two-photon vertices are then found to be

$$\begin{aligned} \tau^{(1)}_{\mu,\beta\alpha}(p_4,p_3) &= ie\left[\eta_{\alpha\beta}(p_4+p_3)_\mu - \eta_{\mu\beta}(2p_4-p_3)_\alpha - \eta_{\mu\alpha}(2p_3-p_4)_\beta\right] \\ \tau^{(2)}_{\mu\nu,\beta\alpha}(p_4,p_3) &= -ie^2\left[2\eta_{\mu\nu}\eta_{\alpha\beta} - \eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\beta}\eta_{\nu\alpha}\right] \end{aligned} \tag{56}$$

for an incoming massive spin-1 particle with momentum p_3 and Lorentz index α and the outgoing one with momentum p_4 and Lorentz index β.

We assign particle b to be the massive spin-1 particle with the incoming polarization vector ϵ_i^b satisfying $\epsilon_i^b \cdot p_3 = 0$ and the polarization vector ϵ_f^b satisfying $\epsilon_f^b \cdot p_4 = 0$. The lowest order scattering amplitude then has the form

$$\begin{aligned} {}^1\mathcal{M}^{(1)}(q) = \frac{8\pi\alpha}{\sqrt{2E_1 2E_2 2E_3 2E_4}} \Bigg[& \frac{s - m_a^2 - m_b^2 + \frac{1}{2}q^2}{q^2}\left(-\epsilon_f^{b*}\cdot\epsilon_i^b\right) \\ & - \frac{2}{q^2}\left(\epsilon_f^{b*}\cdot q\,\epsilon_i^b\cdot p_1 - \epsilon_f^{b*}\cdot p_1\,\epsilon_i^b\cdot q\right)\Bigg] \end{aligned} \tag{57}$$

Now we rewrite this expression using the identity

$$\epsilon_{f\mu}^{b*}\,\epsilon_i^b\cdot q - \epsilon_{i\mu}^b\,\epsilon_f^{b*}\cdot q = \frac{1}{1-\frac{q^2}{4m_b^2}}\left[\frac{i}{m_b}\epsilon_{\mu\beta\gamma\delta}p_3^\beta q^\gamma S_b^\delta - \frac{(p_3+p_4)_\mu}{2m_b^2}\epsilon_f^{b*}\cdot q\,\epsilon_i^b\cdot q\right] \tag{58}$$

where we have defined the spin vector

$$S_{b\mu} = \frac{i}{2m_b}\epsilon_{\mu\beta\gamma\delta}\epsilon_f^{b*\beta}\epsilon_i^{b\gamma}(p_3+p_4)^\delta \tag{59}$$

The leading one-photon exchange amplitude can then be written as

$$^1\mathcal{M}^{(1)}(q) = \frac{4\pi\alpha}{q^2}\left[-\epsilon_f^{b*}\cdot\epsilon_i^b + \frac{i}{m_a m_b^2}\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta - \frac{1}{m_b^2}\epsilon_f^{b*}\cdot q\,\epsilon_i^b\cdot q\right] \tag{60}$$

Now in the nonrelativistic limit we have

$$\epsilon_i^{b0} \simeq \frac{1}{m_b}\hat{\epsilon}_i^b\cdot\vec{p_3}, \quad \epsilon_f^{b0} \simeq \frac{1}{m_b}\hat{\epsilon}_f^b\cdot\vec{p_4} \tag{61}$$

so that

$$\begin{aligned}\epsilon_f^{b*}\cdot\epsilon_i^b &\simeq -\hat{\epsilon}_f^{b*}\cdot\hat{\epsilon}_i^b + \frac{1}{m_b^2}\hat{\epsilon}_f^{b*}\cdot\vec{p}_4\hat{\epsilon}_i^b\cdot\vec{p}_3 \\ &\simeq -\hat{\epsilon}_f^{b*}\cdot\hat{\epsilon}_i^b + \frac{1}{2m_b^2}\hat{\epsilon}_f^{b*}\times\hat{\epsilon}_i^b\cdot\vec{p}_4\times\vec{p}_3 \\ &\quad + \frac{1}{2m_b^2}\left(\hat{\epsilon}_f^{b*}\cdot\vec{p}_4\hat{\epsilon}_i^b\cdot\vec{p}_3 + \hat{\epsilon}_f^{b*}\cdot\vec{p}_3\hat{\epsilon}_i^b\cdot\vec{p}_4\right)\end{aligned} \tag{62}$$

Since

$$-i\hat{\epsilon}_f^{b*}\times\hat{\epsilon}_i^b = \left\langle 1, m_f\left|\vec{S}_b\right|1, m_i\right\rangle, \tag{63}$$

Eq. (62) becomes

$$\begin{aligned}\epsilon_f^{b*}\cdot\epsilon_i^b &\simeq -\hat{\epsilon}_f^{b*}\cdot\hat{\epsilon}_i^b - \frac{i}{2m_b^2}\vec{S}_b\cdot\vec{p}_3\times\vec{p}_4 + \frac{1}{2m_b^2}\left(\hat{\epsilon}_f^{b*}\cdot\vec{p}_4\hat{\epsilon}_i^b\cdot\vec{p}_3 + \hat{\epsilon}_f^{b*}\cdot\vec{p}_3\hat{\epsilon}_i^b\cdot\vec{p}_4\right) \\ &\simeq -\hat{\epsilon}_f^{b*}\cdot\hat{\epsilon}_i^b + \frac{1}{m_b^2}\hat{\epsilon}_f^{b*}\cdot\vec{p}\,\hat{\epsilon}_i^b\cdot\vec{p} + \frac{i}{2m_b^2}\vec{S}_b\cdot\vec{p}\times\vec{q} - \frac{1}{4m_b^2}\hat{\epsilon}_f^{b*}\cdot\vec{q}\,\hat{\epsilon}_i^b\cdot\vec{q}\end{aligned} \tag{64}$$

in the symmetric center of mass frame and the transition amplitude assumes the form

$$\begin{aligned}^1\mathcal{M}^{(1)}(\vec{q}) \simeq -\frac{4\pi\alpha}{\vec{q}^2}\Bigg[&\hat{\epsilon}_f^{b*}\cdot\hat{\epsilon}_i^b - \frac{1}{m_b^2}\hat{\epsilon}_f^{b*}\cdot\vec{p}\,\hat{\epsilon}_i^b\cdot\vec{p} + \frac{i(m_a+2m_b)}{2m_a m_b^2}\vec{S}_b\cdot\vec{p}\times\vec{q} \\ &- \frac{3}{4m_b^2}\hat{\epsilon}_f^{b*}\cdot\vec{q}\,\hat{\epsilon}_i^b\cdot\vec{q}\Bigg]\end{aligned} \tag{65}$$

The spin-independent and spin-orbit terms here are identical in form to those found in the spin-0 – spin-1/2 case but now are accompanied by new terms which are quadrupole in nature, as can be seen from the identity

$$\begin{aligned} T^b_{cd} &\equiv \frac{1}{2}\left(\hat{\epsilon}^{b*}_{fc}\hat{\epsilon}^b_{id} + \hat{\epsilon}^b_{ic}\hat{\epsilon}^{b*}_{fd}\right) - \frac{1}{3}\delta_{cd}\hat{\epsilon}^{b*}_f \cdot \hat{\epsilon}^b_i \\ &= -\left\langle 1, m_f \left| \frac{1}{2}(S_c S_d + S_d S_c) - \frac{2}{3}\delta_{cd} \right| 1, m_i \right\rangle \end{aligned} \tag{66}$$

The corresponding lowest order potential is then

$$\begin{aligned} {}^1V^{(1)}_C(\vec{r}) &= -\int \frac{d^3q}{(2\pi)^3}\, {}^1\mathcal{M}^{(1)}(\vec{q})\, e^{-i\vec{q}\cdot\vec{r}} \\ &\simeq \frac{\alpha}{r}\left(\hat{\epsilon}^{b*}_f \cdot \hat{\epsilon}^b_i - \frac{1}{m_b^2}\hat{\epsilon}^{b*}_f \cdot \vec{p}\, \hat{\epsilon}^b_i \cdot \vec{p}\right) - \frac{m_a + 2m_b}{2m_a m_b^2}\, \vec{S}_b \cdot \vec{p} \times \vec{\nabla}\frac{\alpha}{r} \\ &\quad + \frac{3}{4m_b^2}\hat{\epsilon}^{b*}_f \cdot \vec{\nabla}\, \hat{\epsilon}^b_i \cdot \vec{\nabla}\, \frac{\alpha}{r} \\ &\simeq \frac{\alpha}{r}\left(\hat{\epsilon}^{b*}_f \cdot \hat{\epsilon}^b_i - \frac{1}{m_b^2}\, \vec{p} : T^b : \vec{p}\right) - \frac{\alpha}{r^3}\frac{m_a + 2m_b}{2m_a m_b^2}\vec{L} \cdot \vec{S}_b \\ &\quad + \frac{\alpha}{r^5}\frac{9}{4m_b^2}\, \vec{r} : T^b : \vec{r} \end{aligned} \tag{67}$$

where we have defined

$$\vec{w} : T^b : \vec{s} \equiv w_c T^b_{cd} s_d$$

and which agrees precisely with its spin-1/2 analog—Eq. (40)—up to tensor and quadrupole corrections.

The calculation of the one loop corrections proceeds as before, but with increased complexity due to the unit spin. We find

$$\begin{aligned} {}^1\mathcal{M}^{(2)}_{2a}(q) &= \frac{\alpha^2}{m_a m_b}\frac{3}{2}L\, \epsilon^{b*}_f \cdot \epsilon^b_i \\ {}^1\mathcal{M}^{(2)}_{2b}(q) &= \frac{\alpha^2}{m_a m_b}\Bigg[(2L + m_a S)\Bigg(-\epsilon^{b*}_f \cdot \epsilon^b_i + \frac{1}{m_a^2}\epsilon^{b*}_f \cdot p_1 \epsilon^b_i \cdot p_1 \\ &\qquad - \frac{1}{2m_a^2}(\epsilon^{b*}_f \cdot q\epsilon^b_i \cdot p_1 + \epsilon^{b*}_f \cdot p_1 \epsilon^b_i \cdot q)\Bigg) \\ &\quad + (16L + 7m_a S)\frac{1}{32m_a^2}\epsilon^{b*}_f \cdot q\epsilon^b_i \cdot q\Bigg] \end{aligned}$$

$$
\begin{aligned}
{}^1\mathcal{M}^{(2)}_{2_c}(q) \;=\; & \frac{\alpha^2}{m_a m_b}\left[-\frac{3L+2m_bS}{2}\epsilon_f^{b*}\cdot\epsilon_i^b - m_bS\frac{3}{4m_b^2}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\right] \\
{}^1\mathcal{M}^{(2)}_{2_d}(q) \;=\; & \frac{\alpha^2}{m_a m_b}\left[4\frac{m_a m_b}{q^2}L\left(-\epsilon_f^{b*}\cdot\epsilon_i^b + \frac{1}{m_a m_b}\left(\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q - \epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1\right)\right)\right. \\
& +\frac{S}{s-s_0}\left(\left(\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q - \epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1\right)(m_a+2m_b)\right. \\
& \qquad\left. + \epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\,\frac{m_a(m_a+3m_b)}{2m_b}\right) \\
& +S\left(-\epsilon_f^{b*}\cdot\epsilon_i^b(m_a+m_b) - \epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot p_1\,\frac{1}{2m_a}\right. \\
& \quad -\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1\left(\frac{1}{m_b}+\frac{9}{8m_a}\right) + \epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q\left(\frac{1}{m_b}+\frac{13}{8m_a}\right) \\
& \quad\left. -\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\left(-\frac{1}{2m_b}+\frac{7}{64m_a}\right)\right) \\
& +L\left(-\epsilon_f^{b*}\cdot\epsilon_i^b\left(-\frac{1}{2}+\frac{5m_b}{4m_a}+\frac{m_a}{4m_b}\right)\right. \\
& \quad -\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot p_1\left(\frac{1}{m_a^2}-\frac{4}{3m_am_b}\right) \\
& \quad -\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q\left(-\frac{95}{48m_a^2}+\frac{2}{3m_am_b}-\frac{7}{6m_b^2}\right) \\
& \quad -\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1\left(\frac{47}{48m_a^2}+\frac{2}{3m_am_b}+\frac{7}{6m_b^2}\right) \\
& \quad\left.\left. -\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\left(\frac{1}{4m_a^2}-\frac{1}{3m_am_b}-\frac{1}{6m_b^2}-\frac{7m_a}{24m_b^3}\right)\right)\right] \\
& - i4\pi\alpha^2\frac{L}{q^2}\sqrt{\frac{m_am_b}{s-s_0}}\left(-\epsilon_f^{b*}\cdot\epsilon_i^b - \frac{1}{m_am_b}(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1 - \epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q)\right) \\
{}^1\mathcal{M}^{(2)}_{2_e}(q) \;=\; & \frac{\alpha^2}{m_am_b}\left[4\frac{m_am_b}{q^2}L\left(\epsilon_f^{b*}\cdot\epsilon_i^b - \frac{1}{m_am_b}\left(\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q - \epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1\right)\right)\right. \\
& +S\left(\epsilon_f^{b*}\cdot\epsilon_i^b(m_a+m_b) - \epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot p_1\frac{1}{2m_a}\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q\left(\frac{5}{8m_a}+\frac{3}{4m_b}\right)+\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1\left(\frac{3}{4m_b}+\frac{9}{8m_a}\right)\\
&-\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\left(\frac{7}{64m_a}-\frac{1}{8m_b}+\frac{m_a}{8m_b^2}\right)\Bigg)\\
-L\Bigg(&-\epsilon_f^{b*}\cdot\epsilon_i^b\left(\frac{23}{6}+\frac{5m_b}{4m_a}+\frac{m_a}{4m_b}\right)\\
&+\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot p_1\left(\frac{1}{m_a^2}+\frac{4}{3m_am_b}\right)\\
&-\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1\left(\frac{95}{48m_a^2}+\frac{2}{3m_am_b}+\frac{7}{6m_b^2}\right)\\
&+\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q\left(\frac{47}{48m_a^2}+\frac{2}{3m_am_b}+\frac{7}{6m_b^2}\right)\\
&+\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\left(\frac{1}{4m_a^2}+\frac{1}{3m_am_b}-\frac{1}{6m_b^2}+\frac{7m_a}{24m_b^3}\right)\Bigg)\Bigg]
\end{aligned}
\tag{68}
$$

Combining, we find the complete one loop amplitude

$$
\begin{aligned}
{}^1\mathcal{M}^{(2)}_{tot}(q) \;=\; \frac{\alpha^2}{m_am_b}\Bigg[&\frac{S}{s-s_0}\Bigg(-\left(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1-\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q\right)(m_a+2m_b)\\
&\qquad+\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\frac{m_a(m_a+3m_b)}{2m_b}\Bigg)\\
&+S\Bigg(-\epsilon_f^{b*}\cdot\epsilon_i^b(m_a+m_b)-\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\frac{m_a+m_b}{8m_b^2}\\
&\qquad-\left(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1-\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q\right)\frac{m_a+2m_b}{4m_am_b}\Bigg)\\
&+L\Bigg(-\epsilon_f^{b*}\cdot\epsilon_i^b\left(-\frac{7}{3}\right)+\frac{1}{3m_b^2}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\\
&\qquad+\frac{4}{3m_am_b}\left(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1-\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q\right)\Bigg)\Bigg]\\
-\;i4\pi\alpha^2\frac{L}{q^2}&\sqrt{\frac{m_am_b}{s-s_0}}\left(-\epsilon_f^{b*}\cdot\epsilon_i^b-\frac{1}{m_am_b}(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1-\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q)\right)
\end{aligned}
$$

(69)

Using the identity Eq. (58) this becomes

$$
\begin{aligned}
{}^1\mathcal{M}^{(2)}_{tot}(q) \;=\; & \frac{\alpha^2}{m_a m_b}\Bigg[-\epsilon_f^{b*}\cdot\epsilon_i^b\left(-\frac{7}{3}L + S(m_a+m_b)\right) \\
& + \frac{i}{m_a m_b^2}\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta\left(-\frac{4}{3}L + \frac{m_a+2m_b}{4}S\right) \\
& + \frac{1}{m_b^2}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\,\left(\frac{5}{3}L - S\left(\frac{7}{8}m_a + \frac{13}{8}m_b\right)\right) \\
& + \frac{iS(m_a+2m_b)}{m_b(s-s_0)}\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta \\
& - \frac{Sm_a(m_a+m_b)}{2m_b(s-s_0)}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\Bigg] \\
- \; & i4\pi\alpha^2\frac{L}{q^2}\sqrt{\frac{m_a m_b}{s-s_0}}\Bigg(-\epsilon_f^{b*}\cdot\epsilon_i^b + \frac{i}{m_a m_b^2}\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta \\
& - \frac{1}{m_b^2}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\Bigg)
\end{aligned}
\tag{70}
$$

Notice here that without the $\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q$ terms, Eq. (70) has an identical structure to that of the case of spin-0 – spin-1/2 scattering—Eq. (42)—provided we substitute $\bar{u}(p_4)u(p_3) \longrightarrow -\epsilon_f^{b*}\cdot\epsilon_i^b$.

Finally, taking the nonrelativistic limit we find

$$
\begin{aligned}
{}^1\mathcal{M}^{(2)}_{tot}(\vec{q}) \simeq & \left[\frac{\alpha^2}{m_a m_b}\left((m_a+m_b)S - \frac{7}{3}L\right) - i4\pi\alpha^2\frac{L}{q^2}\frac{m_r}{p_0}\right]\left(\hat{\epsilon}_f^{b*}\cdot\hat{\epsilon}_i^b - \frac{1}{m_b^2}\,\vec{p}:T^b:\vec{p}\right) \\
& + \left[\frac{\alpha^2}{m_a m_b}\left(\frac{m_a^2+2m_a m_b+2m_b^2}{2m_a(m_a+m_b)}\,S - \frac{m_a+8m_b}{6m_a m_b}\,L\right)\right. \\
& \left. + \frac{\alpha^2(m_a+2m_b)}{(m_a+m_b)}\left(-i\frac{2\pi L}{p_0 q^2} + \frac{S}{p_0^2}\right)\right]\frac{i}{m_b}\,\vec{S}_b\cdot\vec{p}\times\vec{q} \\
& + \left[\frac{\alpha^2}{m_a m_b}\left(-\frac{3m_a^2+7m_a m_b+6m_b^2}{4(m_a+m_b)}\,S + \frac{13}{12}\,L\right)\right. \\
& \left. + \frac{\alpha^2 m_a m_b}{2(m_a+m_b)}\left(i\frac{6\pi L}{p_0 q^2} - \frac{S}{p_0^2}\right)\right]\frac{1}{m_b^2}\,\vec{q}:T^b:\vec{q}
\end{aligned}
\tag{71}
$$

As found in the earlier calculations, there exist terms involving both i/p_0 and $1/p_0^2$ which prevent the defining of a simple second order potential. The solution now is well known—subtraction of the iterated first order potential. Since the form of the spin-independent—$\hat{\epsilon}_B^* \cdot \hat{\epsilon}_A$—and spin-orbit—$\vec{S}_b \cdot \vec{p} \times \vec{q}$—terms is identical to that found for the case of spin-1/2, it is clear that their subtraction goes through as before and that the corresponding pieces of the second order potential have the same form as found for spin-1/2. However, there are now two new pieces of the amplitude, the quadrupole structure $\vec{q} : T^b : \vec{q}$ which multiplies terms involving both i/p_0 and $1/p_0^2$ and the tensor structure $\vec{p} : T^b : \vec{p}$ multiplying only i/p_0. In order to remove these we must iterate the full first order potential including these quadrupole and tensor components. However, we find that our simple nonrelativistic iteration fails to remove them! We suspect the reason to be the presence of the tensor structure $\vec{p} : T^b : \vec{p}$ in the lowest order potential which is in some sense a relativistic correction but which when iterated yields also quadrupole pieces $\vec{q} : T^b : \vec{q}$. A fully relativistic iteration should thus be performed which we will not include in this paper. It would be interesting to investigate if the requirement to cancel all i/p_0 and $1/p_0^2$ forms in the quadrupole and tensor pieces could clarify the ambiguity in the iteration of the leading order potential as discussed for the spinless case.

Since we did not perform the proper iteration of the quadrupole and tensor pieces we merely include the spin-independent and spin-orbit pieces in the resulting second order potential

$$\begin{aligned}
{}^1V_C^{(2)}(\vec{r}) &= -\int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \left[{}^1\mathcal{M}_{tot}^{(2)}(\vec{q}) - {}^1\mathrm{Amp}_C^{(2)}(\vec{q}) \right] \\
&= \left[-\frac{\alpha^2(m_a+m_b)}{2m_a m_b r^2} - \frac{7\alpha^2\hbar}{6\pi m_a m_b r^3} \right] \hat{\epsilon}_f^{b*} \cdot \hat{\epsilon}_i^b \\
&+ \frac{1}{m_b} \vec{S}_b \cdot \vec{p} \times \vec{\nabla} \left[\frac{\alpha^2(m_a^2+2m_a m_b+2m_b^2)}{4m_a^2 m_b(m_a+m_b)r^2} + \frac{\alpha^2(m_a+8m_b)\hbar}{12\pi m_a^2 m_b^2 r^3} \right] + {}^1V_T^{(2)}(\vec{r}) \\
&= \left[-\frac{\alpha^2(m_a+m_b)}{2m_a m_b r^2} - \frac{7\alpha^2\hbar}{6\pi m_a m_b r^3} \right] \hat{\epsilon}_f^{b*} \cdot \hat{\epsilon}_i^b \\
&+ \left[\frac{\alpha^2(m_a^2+2m_a m_b+2m_b^2)}{2m_a^2 m_b^2(m_a+m_b)r^4} + \frac{\alpha^2(m_a+8m_b)\hbar}{4\pi m_a^2 m_b^3 r^5} \right] \vec{L} \cdot \vec{S}_b + {}^1V_T^{(2)}(\vec{r})
\end{aligned} \tag{72}$$

where ${}^1V_T^{(2)}(\vec{r})$ denotes the tensor pieces not explicitly shown. Comparison

with the corresponding form of $\frac{1}{2}V_C^{(2)}(\vec{r})$ given in Eq. (53) confirms the universality which we have suggested—the spin-independent and spin-orbit terms have identical forms. The next task is to see whether this universality applies when both scattered particles carry spin. For this purpose we consider the case of spin-1/2 – spin-1/2 scattering.

4 Spin-Dependent Scattering: Spin-Spin Interaction

In order to check universality when both scattering particles carry spin and to study possible spin-spin interactions, we now consider the case where both particles a and b carry spin-1/2.

4.1 Spin-1/2 – Spin-1/2

For this calculation the vertices have been given previously and the calculation proceeds as before. The one-photon exchange amplitude is given by

$$^{\frac{1}{2}\frac{1}{2}}\mathcal{M}^{(1)}(q) = \frac{4\pi\alpha}{q^2}\,\bar{u}(p_2)\gamma_\alpha u(p_1)\,\bar{u}(p_4)\gamma^\alpha u(p_3)\sqrt{\frac{m_a^2 m_b^2}{E_1E_2E_3E_4}} \tag{73}$$

Using the spin-1/2 identity Eq. (34) and its pendant for particle a

$$\bar{u}(p_2)\gamma_\mu u(p_1) = \left(\frac{1}{1-\frac{q^2}{4m_a^2}}\right)\left[\frac{(p_1+p_2)_\mu}{2m_a}\bar{u}(p_2)u(p_1) + \frac{i}{m_a^2}\epsilon_{\mu\beta\gamma\delta}q^\beta p_1^\gamma S_a^\delta\right] \tag{74}$$

where we have defined the spin vector

$$S_a^\mu = \frac{1}{2}\bar{u}(p_2)\gamma_5\gamma^\mu u(p_1)$$

for particle a, the one-photon exchange amplitude becomes

$$\begin{aligned} ^{\frac{1}{2}\frac{1}{2}}\mathcal{M}^{(1)}(q) \;=\; & \frac{4\pi\alpha}{q^2}\Bigg[\bar{u}(p_2)u(p_1)\bar{u}(p_4)u(p_3) + \frac{i}{m_a m_b^2}\bar{u}(p_2)u(p_1)\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta \\ & + \frac{i}{m_b m_a^2}\bar{u}(p_4)u(p_3)\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_a^\delta \end{aligned}$$

$$+\frac{1}{m_a m_b}(S_a\cdot q S_b\cdot q - q^2 S_b\cdot S_a)\Bigg] \tag{75}$$

We observe that in addition to the spin-orbit pieces found previously, a spin-spin interaction is also present. Taking the nonrelativistic limit and working in the center of mass frame, we find

$$\begin{aligned}
{}^{\frac{1}{2}\frac{1}{2}}\mathcal{M}^{(1)}(\vec{q}) \;\simeq\; & -\frac{4\pi\alpha}{\vec{q}^{\,2}}\Bigg[\chi_f^{a\dagger}\chi_i^a\,\chi_f^{b\dagger}\chi_i^b + \frac{i(m_a+2m_b)}{2m_a m_b^2}\vec{S}_b\cdot\vec{p}\times\vec{q}\,\chi_f^{a\dagger}\chi_i^a \\
& + \frac{i(2m_a+m_b)}{2m_a^2 m_b}\vec{S}_a\cdot\vec{p}\times\vec{q}\,\chi_f^{b\dagger}\chi_i^b \\
& + \frac{1}{m_a m_b}\left(\vec{S}_a\cdot\vec{q}\,\vec{S}_b\cdot\vec{q} - \vec{q}^{\,2}\vec{S}_a\cdot\vec{S}_b\right)\Bigg]
\end{aligned} \tag{76}$$

whereby the lowest order potential for spin-1/2 – spin-1/2 scattering is of the form

$$\begin{aligned}
{}^{\frac{1}{2}\frac{1}{2}}V_C^{(1)}(\vec{r}) \;=\; & -\int\frac{d^3q}{(2\pi)^3}\,{}^{\frac{1}{2}\frac{1}{2}}\mathcal{M}^{(1)}(\vec{q})\,e^{-i\vec{q}\cdot\vec{r}} \\
\simeq\; & \frac{\alpha}{r}\chi_f^{a\dagger}\chi_i^a\chi_f^{b\dagger}\chi_i^b - \frac{(m_a+2m_b)}{2m_a m_b^2}\vec{S}_b\cdot\vec{p}\times\vec{\nabla}\frac{\alpha}{r}\,\chi_f^{a\dagger}\chi_i^a \\
& -\frac{(2m_a+m_b)}{2m_a^2 m_b}\vec{S}_a\cdot\vec{p}\times\vec{\nabla}\frac{\alpha}{r}\,\chi_f^{b\dagger}\chi_i^b - \frac{1}{m_a m_b}\vec{S}_a\cdot\vec{\nabla}\vec{S}_b\cdot\vec{\nabla}\frac{\alpha}{r} \\
\simeq\; & \frac{\alpha}{r}\chi_f^{a\dagger}\chi_i^a\chi_f^{b\dagger}\chi_i^b - \frac{\alpha}{r^3}\frac{(m_a+2m_b)}{2m_a m_b^2}\vec{L}\cdot\vec{S}_b\,\chi_f^{a\dagger}\chi_i^a \\
& -\frac{\alpha}{r^3}\frac{(2m_a+m_b)}{2m_a^2 m_b}\vec{L}\cdot\vec{S}_a\,\chi_f^{b\dagger}\chi_i^b \\
& -\frac{\alpha}{r^5}\frac{1}{m_a m_b}\left(3\vec{S}_a\cdot\vec{r}\,\vec{S}_b\cdot\vec{r} - r^2\vec{S}_a\cdot\vec{S}_b\right)
\end{aligned} \tag{77}$$

Note that since the piece proportional to $\vec{S}_a\cdot\vec{S}_b$ in Eq. (76) is analytic in $\vec{q}^{\,2}$ it only gives a short distance contribution which is omitted in the potential (77).

In this case when we evaluate the loop diagrams (a)-(e) of Fig. 2, we find that part of the spin-spin structure piece contains the form $q^2 S_a\cdot S_b$ multiplying the nonanalytic structures L and S. Due to this extra factor of

q^2 in this form, we must expand all loop integrals to one order higher in q^2 than before in order to be consistent. This has been done and does make a difference in our results, but the very lengthy expressions for the vector and tensor integrals prevent us from explicitly listing them to this order in Appendix A. The results for the diagrams (a)-(e) in Fig. 2 are then[6]

$$
\begin{aligned}
{}^{\frac{1}{2}\frac{1}{2}}\mathcal{M}^{(2)}_{2a}(q) &= 0 \\
{}^{\frac{1}{2}\frac{1}{2}}\mathcal{M}^{(2)}_{2b}(q) &= 0 \\
{}^{\frac{1}{2}\frac{1}{2}}\mathcal{M}^{(2)}_{2c}(q) &= 0 \\
{}^{\frac{1}{2}\frac{1}{2}}\mathcal{M}^{(2)}_{2d}(q) &= \frac{\alpha^2}{m_a m_b}\Bigg[\mathcal{U}_a\mathcal{U}_b\left(L\left(\frac{4m_am_b}{q^2}+\frac{3m_a^2+m_am_b+3m_b^2}{2m_am_b}\right)+S\frac{3}{2}(m_a+m_b)\right) \\
&\quad +i\frac{\mathcal{E}_a\mathcal{U}_b}{m_a^2m_b}\left(L\left(\frac{4m_am_b}{q^2}+\frac{10m_a^2+11m_b^2}{6m_am_b}\right)\right. \\
&\qquad \left.+S\left(\frac{m_am_b(2m_a+m_b)}{s-s_0}+(m_a+m_b)\right)\right) \\
&\quad +i\frac{\mathcal{U}_a\mathcal{E}_b}{m_am_b^2}\left(L\left(\frac{4m_am_b}{q^2}+\frac{11m_a^2+10m_b^2}{6m_am_b}\right)\right. \\
&\qquad \left.+S\left(\frac{m_am_b(m_a+2m_b)}{s-s_0}+(m_a+m_b)\right)\right) \\
&\quad +\frac{S_b\cdot qS_a\cdot q}{m_am_b}L\left(\frac{4m_am_b}{q^2}+\frac{4m_a^2+3m_am_b+4m_b^2}{3m_am_b}\right) \\
&\quad -\frac{q^2S_a\cdot S_b}{m_am_b}L\left(\frac{2m_am_b}{q^2}+\frac{8m_a^2+13m_am_b+8m_b^2}{6m_am_b}\right) \\
&\quad +\frac{S_a\cdot qS_b\cdot q-q^2S_a\cdot S_b}{m_am_b}S(m_a+m_b)\left(\frac{m_am_b}{s-s_0}+2\right) \\
&\quad +\Big(2S_a\cdot p_3S_b\cdot p_1+S_a\cdot qS_b\cdot p_1-S_a\cdot p_3S_b\cdot q\Big)\frac{7L}{3m_am_b}\Bigg] \\
&\quad - i4\pi\alpha^2\frac{L}{q^2}\sqrt{\frac{m_am_b}{s-s_0}}\Bigg(\mathcal{U}_a\mathcal{U}_b+i\frac{\mathcal{E}_a\mathcal{U}_b}{m_a^2m_b}+i\frac{\mathcal{U}_a\mathcal{E}_b}{m_am_b^2}
\end{aligned}
$$

[6]The results found for the spin-1/2 – spin-1/2 case are rather lengthy, and so we quote the results *after* the identities Eqs. (34) and (74) have been used.

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad + \frac{S_a\cdot q S_b\cdot q - \frac{1}{2}q^2 S_a\cdot S_b}{m_a m_b}\Bigg) \\
{}^{\frac{1}{2}\frac{1}{2}}\mathcal{M}^{(2)}_{2e}(q) \;=\; \frac{\alpha^2}{m_a m_b}\Bigg[&\mathcal{U}_a\mathcal{U}_b\Bigg(L\left(-\frac{4m_a m_b}{q^2} - \frac{9m_a^2+17m_a m_b+9m_b^2}{6m_a m_b}\right) \\
&\qquad -S\,\frac{m_a+m_b}{2}\Bigg) \\
&+i\frac{\mathcal{E}_a\mathcal{U}_b}{m_a^2 m_b}\Bigg(L\left(-\frac{4m_a m_b}{q^2} - \frac{10m_a^2+8m_a m_b+11m_b^2}{6m_a m_b}\right) \\
&\qquad -S\,\frac{2m_a+3m_b}{4}\Bigg) \\
&+i\frac{\mathcal{U}_a\mathcal{E}_b}{m_a m_b^2}\Bigg(L\left(-\frac{4m_a m_b}{q^2} - \frac{11m_a^2+8m_a m_b+10m_b^2}{6m_a m_b}\right) \\
&\qquad -S\,\frac{3m_a+2m_b}{4}\Bigg) \\
&+\frac{S_b\cdot q S_a\cdot q}{m_a m_b}\,L\left(-\frac{4m_a m_b}{q^2} - \frac{4m_a^2+4m_b^2}{3m_a m_b}\right) \\
&-\frac{q^2 S_a\cdot S_b}{m_a m_b}\,L\left(-\frac{2m_a m_b}{q^2} - \frac{8m_a^2+9m_a m_b+8m_b^2}{6m_a m_b}\right) \\
&+\frac{S_a\cdot q S_b\cdot q - q^2 S_a\cdot S_b}{m_a m_b}\,S(m_a+m_b)\left(-\frac{1}{4}\right) \\
&-\Big(2S_a\cdot p_3 S_b\cdot p_1 + S_a\cdot q S_b\cdot p_1 - S_a\cdot p_3 S_b\cdot q\Big)\frac{7L}{3m_a m_b}\Bigg]
\end{aligned}
\tag{78}
$$

where we have defined

$$
\mathcal{U}_a = \bar{u}(p_2)u(p_1) \qquad\qquad \mathcal{U}_b = \bar{u}(p_4)u(p_3) \tag{79}
$$

and

$$
\mathcal{E}_i = \epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_i^\delta \tag{80}
$$

with $i = a, b$. The sum is found then to be

$$
\begin{aligned}
{}^{\frac{1}{2}\frac{1}{2}}\mathcal{M}^{(2)}_{tot}(q) \;=\; & \frac{\alpha^2}{m_a m_b}\Bigg[\mathcal{U}_a\mathcal{U}_b\left((m_a+m_b)S-\frac{7}{3}L\right) \\
& +i\frac{\mathcal{E}_a\mathcal{U}_b}{m_a^2 m_b}\left(\frac{2m_a+m_b}{4}S-\frac{4}{3}L+\frac{m_a m_b(2m_a+m_b)}{s-s_0}S\right) \\
& +i\frac{\mathcal{U}_a\mathcal{E}_b}{m_a^2 m_b}\left(\frac{m_a+2m_b}{4}S-\frac{4}{3}L+\frac{m_a m_b(m_a+2m_b)}{s-s_0}S\right) \\
& +S(m_a+m_b)\,\frac{S_a\cdot qS_b\cdot q-q^2S_a\cdot S_b}{m_a m_b}\left(\frac{7}{4}+\frac{m_a m_b}{s-s_0}\right) \\
& +L\,\frac{S_a\cdot qS_b\cdot q-\frac{2}{3}q^2S_a\cdot S_b}{m_a m_b} \\
- \;\; & i4\pi\alpha^2\frac{L}{q^2}\sqrt{\frac{m_a m_b}{s-s_0}}\Bigg(\mathcal{U}_a\mathcal{U}_b+i\frac{\mathcal{E}_a\mathcal{U}_b}{m_a^2 m_b}+i\frac{\mathcal{U}_a\mathcal{E}_b}{m_a m_b^2} \\
& +\frac{S_a\cdot qS_b\cdot q-\frac{1}{2}q^2S_a\cdot S_b}{m_a m_b}\Bigg)
\end{aligned}
\tag{81}
$$

Comparison with the result Eq. (43) reveals again the universality which has been found in other cases—the forms for the scalar density and antisymmetric tensor components is identical and symmetric between particles a and b. However, there is also a new component—a spin-spin interaction. Performing the nonrelativistic reduction yields for the amplitude

$$
\begin{aligned}
{}^{\frac{1}{2}\frac{1}{2}}\mathcal{M}^{(2)}_{tot}(\vec{q}) \;\simeq\; & \left[\frac{\alpha^2}{m_a m_b}\left((m_a+m_b)S-\frac{7}{3}L\right)-i4\pi\alpha^2\frac{L}{q^2}\frac{m_r}{p_0}\right]\chi_f^{a\dagger}\chi_i^a\,\chi_f^{b\dagger}\chi_i^b \\
+ \;\; & \Bigg[\frac{\alpha^2}{m_a m_b}\left(\frac{2m_a^2+2m_a m_b+m_b^2}{2m_b(m_a+m_b)}\,S-\frac{8m_a+m_b}{6m_a m_b}\,L\right) \\
& +\frac{\alpha^2(2m_a+m_b)}{(m_a+m_b)}\left(-i\frac{2\pi L}{p_0 q^2}+\frac{S}{p_0^2}\right)\Bigg]\frac{i}{m_a}\vec{S}_a\cdot\vec{p}\times\vec{q}\,\chi_f^{b\dagger}\chi_i^b \\
+ \;\; & \Bigg[\frac{\alpha^2}{m_a m_b}\left(\frac{m_a^2+2m_a m_b+2m_b^2}{2m_a(m_a+m_b)}\,S-\frac{m_a+8m_b}{6m_a m_b}\,L\right) \\
& +\frac{\alpha^2(m_a+2m_b)}{(m_a+m_b)}\left(-i\frac{2\pi L}{p_0 q^2}+\frac{S}{p_0^2}\right)\Bigg]\chi_f^{a\dagger}\chi_i^a\;\frac{i}{m_b}\vec{S}_b\cdot\vec{p}\times\vec{q}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{\alpha^2}{m_a m_b} \frac{2m_a^2 + 3m_a m_b + 2m_b^2}{m_a + m_b} S \frac{\vec{S}_a \cdot \vec{q}\, \vec{S}_b \cdot \vec{q} - \vec{q}^{\,2} \vec{S}_a \cdot \vec{S}_b}{m_a m_b} \\
&+ \quad \frac{\alpha^2}{m_a m_b} L \frac{\vec{S}_a \cdot \vec{q}\, \vec{S}_b \cdot \vec{q} - \frac{2}{3}\vec{q}^{\,2} \vec{S}_a \cdot \vec{S}_b}{m_a m_b} \\
&+ \quad \frac{\alpha^2 m_a m_b}{m_a + m_b} \frac{S}{p_0^2} \frac{\vec{S}_a \cdot \vec{q}\, \vec{S}_b \cdot \vec{q} - \vec{q}^{\,2} \vec{S}_a \cdot \vec{S}_b}{m_a m_b} \\
&+ \quad \frac{\alpha^2 m_a m_b}{m_a + m_b} \left(-i \frac{4\pi L}{p_0 q^2} \right) \frac{\vec{S}_a \cdot \vec{q}\, \vec{S}_b \cdot \vec{q} - \frac{1}{2}\vec{q}^{\,2} \vec{S}_a \cdot \vec{S}_b}{m_a m_b}
\end{aligned} \tag{82}
$$

Again we verify universality: the same spin-independent piece as in the previous calculations and now two spin-orbit coupling pieces, one for the spin of each particle, with again the same form as found earlier. The novel spin-spin coupling piece consists of the last four lines of Eq. (82).

Note that unlike in the case of spin-0 – spin-1 scattering where there were relativistic forms $\hat{\epsilon}_f^{b*} \cdot \vec{p}\, \hat{\epsilon}_i^b \cdot \vec{p}$ along with the quadrupole forms $\hat{\epsilon}_f^{b*} \cdot \vec{q}\, \hat{\epsilon}_i^b \cdot \vec{q}$ in the leading order potential, the leading order spin-1/2 – spin-1/2 potential contains no analog relativistic term $\vec{S}_a \cdot \vec{p}\, \vec{S}_b \cdot \vec{p}$ along with the spin-spin terms $\vec{S}_a \cdot \vec{q}\, \vec{S}_b \cdot \vec{q}$. We will see that now our nonrelativistic second Born iteration succeeds in removing all terms involving $1/p_0^2$ and i/p_0 and we find the spin-spin coupling piece of the $\mathcal{O}(\alpha^2)$ potential.

Due to the universalities we obtained, it is clear that the iteration for the spin-independent piece and the spin-orbit pieces proceeds as shown before in the spin-0 – spin-1/2 case. As before, the second Born amplitude is

$$
{}^{\frac{1}{2}\frac{1}{2}}\mathrm{Amp}_C^{(2)}(\vec{q}) \;=\; -\int \frac{d^3\ell}{(2\pi)^3} \frac{\left\langle \vec{p}_f \left| {}^{\frac{1}{2}\frac{1}{2}}\hat{V}_C^{(1)} \right| \vec{\ell} \right\rangle \left\langle \vec{\ell} \left| {}^{\frac{1}{2}\frac{1}{2}}\hat{V}_C^{(1)} \right| \vec{p}_i \right\rangle}{\frac{p_0^2}{2m_r} - \frac{\ell^2}{2m_r} + i\epsilon} \tag{83}
$$

where we now use the one-photon exchange potential ${}^{\frac{1}{2}\frac{1}{2}}V_C^{(1)}(\vec{r})$ given in Eq. (77). Splitting this lowest order potential into spin-independent, spin-orbit and spin-spin components—

$$
\left\langle \vec{p}_f \left| {}^{\frac{1}{2}\frac{1}{2}}\hat{V}_C^{(1)} \right| \vec{p}_i \right\rangle = \left\langle \vec{p}_f \left| {}^{\frac{1}{2}\frac{1}{2}}\hat{V}_{S-I}^{(1)} \right| \vec{p}_i \right\rangle + \left\langle \vec{p}_f \left| {}^{\frac{1}{2}\frac{1}{2}}\hat{V}_{S-O}^{(1)} \right| \vec{p}_i \right\rangle + \left\langle \vec{p}_f \left| {}^{\frac{1}{2}\frac{1}{2}}\hat{V}_{S-S}^{(1)} \right| \vec{p}_i \right\rangle \tag{84}
$$

where

$$
\left\langle \vec{p}_f \left| {}^{\frac{1}{2}\frac{1}{2}}\hat{V}_{S-I}^{(1)} \right| \vec{p}_i \right\rangle \;=\; \frac{e^2}{\vec{q}^{\,2}} \chi_f^{a\dagger} \chi_i^a \chi_f^{b\dagger} \chi_i^b
$$

$$
\begin{aligned}
\left\langle \vec{p}_f \left| {}^{\frac{1}{2}\frac{1}{2}}\hat{V}^{(1)}_{S-O} \right| \vec{p}_i \right\rangle &= \frac{e^2}{\vec{q}^2}\frac{2m_a+m_b}{2m_a m_b}\,\frac{i}{m_a}\vec{S}_a\cdot\vec{p}\times\vec{q}\,\chi_f^{b\dagger}\chi_i^b \\
&+ \frac{e^2}{\vec{q}^2}\frac{m_a+2m_b}{2m_a m_b}\,\chi_f^{a\dagger}\chi_i^a\,\frac{i}{m_b}\vec{S}_b\cdot\vec{p}\times\vec{q} \\
\left\langle \vec{p}_f \left| {}^{\frac{1}{2}\frac{1}{2}}\hat{V}^{(1)}_{S-S} \right| \vec{p}_i \right\rangle &= \frac{e^2}{\vec{q}^2}\frac{1}{m_a m_b}\,\vec{S}_a\cdot\vec{q}\,\vec{S}_b\cdot\vec{q} \qquad (85)
\end{aligned}
$$

we again find that the iterated amplitude splits also into spin-independent, spin-orbit and spin-spin pieces. As mentioned above the calculation of the leading spin-independent amplitude ${}^{\frac{1}{2}\frac{1}{2}}\mathrm{Amp}^{(2)}_{S-I}(\vec{q})$ and the leading spin-orbit amplitude ${}^{\frac{1}{2}\frac{1}{2}}\mathrm{Amp}^{(2)}_{S-O}(\vec{q})$ works out exactly as in the $0-1/2$ case, cf. Eqs. (51) and (52), and we will not repeat it here again.

The leading spin-spin term of the second Born iteration amplitude is new and we compute

$$
\begin{aligned}
{}^{\frac{1}{2}\frac{1}{2}}\mathrm{Amp}^{(2)}_{S-S}(\vec{q}) &= -\int\frac{d^3\ell}{(2\pi)^3}\,\frac{\left\langle \vec{p}_f \left| {}^{\frac{1}{2}\frac{1}{2}}\hat{V}^{(1)}_{S-I} \right| \vec{\ell} \right\rangle \left\langle \vec{\ell} \left| {}^{\frac{1}{2}\frac{1}{2}}\hat{V}^{(1)}_{S-S} \right| \vec{p}_i \right\rangle}{\frac{p_0^2}{2m_r}-\frac{\ell^2}{2m_r}+i\epsilon} \\
&\quad -\int\frac{d^3\ell}{(2\pi)^3}\,\frac{\left\langle \vec{p}_f \left| {}^{\frac{1}{2}\frac{1}{2}}\hat{V}^{(1)}_{S-S} \right| \vec{\ell} \right\rangle \left\langle \vec{\ell} \left| {}^{\frac{1}{2}\frac{1}{2}}\hat{V}^{(1)}_{S-I} \right| \vec{p}_i \right\rangle}{\frac{p_0^2}{2m_r}-\frac{\ell^2}{2m_r}+i\epsilon} \\
&= \frac{1}{m_a m_b}\,S_a^r\,S_b^s \\
&\quad \left(i\int\frac{d^3\ell}{(2\pi)^3}\,\frac{e^2}{|\vec{p}_f-\vec{\ell}|^2+\lambda^2}\,\frac{i}{\frac{p_0^2}{2m_r}-\frac{\ell^2}{2m_r}+i\epsilon}\,\frac{e^2(p_i-\ell)^r(p_i-\ell)^s}{|\vec{\ell}-\vec{p}_i|^2+\lambda^2}\right. \\
&\quad \left. +i\int\frac{d^3\ell}{(2\pi)^3}\,\frac{e^2(\ell-p_f)^r(\ell-p_f)^s}{|\vec{p}_f-\vec{\ell}|^2+\lambda^2}\,\frac{i}{\frac{p_0^2}{2m_r}-\frac{\ell^2}{2m_r}+i\epsilon}\,\frac{e^2}{|\vec{\ell}-\vec{p}_i|^2+\lambda^2}\right) \\
&\xrightarrow{\lambda\to 0} \frac{1}{m_a m_b}\left[\left(\vec{S}_a\cdot\vec{p}_i\,\vec{S}_b\cdot\vec{p}_i+\vec{S}_a\cdot\vec{p}_f\,\vec{S}_b\cdot\vec{p}_f\right)H\right. \\
&\qquad -\vec{S}_a\cdot(\vec{p}_i+p_f)\vec{S}_b\cdot\vec{H}-\vec{S}_a\cdot\vec{H}\,\vec{S}_b\cdot(\vec{p}_i+p_f) \\
&\qquad \left.+2\,S_a^r S_b^s\,H^{rs}\right] \\
&= \frac{\alpha^2 m_a m_b}{m_a+m_b}\,\frac{S}{p_0^2}\,\frac{\vec{S}_a\cdot\vec{q}\,\vec{S}_b\cdot\vec{q}-\vec{q}^2\vec{S}_a\cdot\vec{S}_b}{m_a m_b}
\end{aligned}
$$

$$+ \frac{\alpha^2 m_a m_b}{m_a+m_b}\left(-i\frac{4\pi L}{p_0 q^2}\right)\frac{\vec{S}_a\cdot\vec{q}\,\vec{S}_b\cdot\vec{q}-\frac{1}{2}\vec{q}^{\,2}\vec{S}_a\cdot\vec{S}_b}{m_a m_b} \tag{86}$$

in agreement with the corresponding terms in [12] in the equal mass limit $m_a = m_b = m$. With that, the total second Born iteration amplitude becomes

$$\begin{aligned}
{}^{\frac{1}{2}\frac{1}{2}}\mathrm{Amp}_C^{(2)}(\vec{q}) &= {}^{\frac{1}{2}\frac{1}{2}}\mathrm{Amp}_{S-I}^{(2)}(\vec{q}) + {}^{\frac{1}{2}\frac{1}{2}}\mathrm{Amp}_{S-O}^{(2)}(\vec{q}) + {}^{\frac{1}{2}\frac{1}{2}}\mathrm{Amp}_{S-S}^{(2)}(\vec{q}) \\
&= -i4\pi\alpha^2\frac{L}{q^2}\frac{m_r}{p_0}\,\chi_f^{a\dagger}\chi_i^a\,\chi_f^{b\dagger}\chi_i^b \\
&+ \frac{\alpha^2(2m_a+m_b)}{m_a+m_b}\left(-i\frac{2\pi L}{p_0 q^2}+\frac{S}{p_0^2}\right)\frac{i}{m_a}\vec{S}_a\cdot\vec{p}\times\vec{q}\,\chi_f^{b\dagger}\chi_i^b \\
&+ \frac{\alpha^2(m_a+2m_b)}{m_a+m_b}\left(-i\frac{2\pi L}{p_0 q^2}+\frac{S}{p_0^2}\right)\chi_f^{a\dagger}\chi_i^a\,\frac{i}{m_b}\vec{S}_b\cdot\vec{p}\times\vec{q} \\
&+ \frac{\alpha^2 m_a m_b}{m_a+m_b}\frac{S}{p_0^2}\frac{\vec{S}_a\cdot\vec{q}\,\vec{S}_b\cdot\vec{q}-\vec{q}^{\,2}\vec{S}_a\cdot\vec{S}_b}{m_a m_b} \\
&+ \frac{\alpha^2 m_a m_b}{m_a+m_b}\left(-i\frac{4\pi L}{p_0 q^2}\right)\frac{\vec{S}_a\cdot\vec{q}\,\vec{S}_b\cdot\vec{q}-\frac{1}{2}\vec{q}^{\,2}\vec{S}_a\cdot\vec{S}_b}{m_a m_b}
\end{aligned} \tag{87}$$

and we observe that when this amplitude is subtracted from the full one loop scattering amplitude Eq. (82), all terms involving $1/p_0^2$ and i/p_0 disappear leaving behind a well-defined second order potential

$$\begin{aligned}
{}^{\frac{1}{2}\frac{1}{2}}V_C^{(2)}(\vec{r}) &= -\int\frac{d^3q}{(2\pi)^3}e^{-i\vec{q}\cdot\vec{r}}\left[{}^{\frac{1}{2}\frac{1}{2}}\mathcal{M}_{tot}^{(2)}(\vec{q}) - {}^{\frac{1}{2}\frac{1}{2}}\mathrm{Amp}_C^{(2)}(\vec{q})\right] \\
&= \left[-\frac{\alpha^2(m_a+m_b)}{2m_a m_b r^2}-\frac{7\alpha^2\hbar}{6\pi m_a m_b r^3}\right]\chi_f^{a\dagger}\chi_i^a\,\chi_f^{b\dagger}\chi_i^b \\
&+ \frac{1}{m_a}\vec{S}_a\cdot\vec{p}\times\vec{\nabla}\left[\frac{\alpha^2(2m_a^2+2m_a m_b+m_b^2)}{4m_a m_b^2(m_a+m_b)r^2}+\frac{\alpha^2(8m_a+m_b)\hbar}{12\pi m_a^2 m_b^2 r^3}\right]\chi_f^{b\dagger}\chi_i^b \\
&+ \chi_f^{a\dagger}\chi_i^a\frac{1}{m_b}\vec{S}_b\cdot\vec{p}\times\vec{\nabla}\left[\frac{\alpha^2(m_a^2+2m_a m_b+2m_b^2)}{4m_a^2 m_b(m_a+m_b)r^2}+\frac{\alpha^2(m_a+8m_b)\hbar}{12\pi m_a^2 m_b^2 r^3}\right] \\
&+ \frac{\vec{S}_a\cdot\vec{\nabla}\vec{S}_b\cdot\vec{\nabla}-\vec{\nabla}^2\vec{S}_a\cdot\vec{S}_b}{m_a m_b}\left[\frac{\alpha^2(2m_a^2+3m_a m_b+2m_b^2)}{2m_a m_b(m_a+m_b)r^2}\right] \\
&+ \frac{\vec{S}_a\cdot\vec{\nabla}\vec{S}_b\cdot\vec{\nabla}-\frac{2}{3}\vec{\nabla}^2\vec{S}_a\cdot\vec{S}_b}{m_a m_b}\left[-\frac{\alpha^2\hbar}{2\pi m_a m_b r^3}\right] \\
&= \left[-\frac{\alpha^2(m_a+m_b)}{2m_a m_b r^2}-\frac{7\alpha^2\hbar}{6\pi m_a m_b r^3}\right]\chi_f^{a\dagger}\chi_i^a\,\chi_f^{b\dagger}\chi_i^b
\end{aligned}$$

$$
\begin{aligned}
&+ \left[\frac{\alpha^2(2m_a^2+2m_am_b+m_b^2)}{2m_a^2m_b^2(m_a+m_b)r^4}+\frac{\alpha^2(8m_a+m_b)\hbar}{4\pi m_a^3m_b^2r^5}\right]\vec{L}\cdot\vec{S}_a\,\chi_f^{b\dagger}\chi_i^b \\
&+ \left[\frac{\alpha^2(m_a^2+2m_am_b+2m_b^2)}{2m_a^2m_b^2(m_a+m_b)r^4}+\frac{\alpha^2(m_a+8m_b)\hbar}{4\pi m_a^2m_b^3r^5}\right]\chi_f^{a\dagger}\chi_i^a\,\vec{L}\cdot\vec{S}_b \\
&+ \left[-\frac{2\alpha^2(2m_a^2+3m_am_b+2m_b^2)}{m_a^2m_b^2(m_a+m_b)r^4}\right]\left(\vec{S}_a\cdot\vec{S}_b-2\vec{S}_a\cdot\vec{r}\,\vec{S}_b\cdot\vec{r}/r^2\right) \\
&+ \left[\frac{\alpha^2\hbar}{2\pi m_a^2m_b^2r^5}\right]\left(7\vec{S}_a\cdot\vec{S}_b-15\vec{S}_a\cdot\vec{r}\,\vec{S}_b\cdot\vec{r}/r^2\right) \qquad (88)
\end{aligned}
$$

5 Conclusions

Above we have analyzed the electromagnetic scattering of two charged particles having nonzero mass. In lowest order the interaction arises from one-photon exchange and leads at threshold to the well known Coulomb interaction $V(r)=\alpha/r$. Inclusion of two-photon exchange effects means adding the contribution from box, cross-box, triangle, and bubble diagrams, which have a rather complex form. The calculation can be simplified, however, by using ideas from effective field theory. The point is that if one is interested only in the leading long-range behavior of the interaction, then one need retain only the leading nonanalytic small momentum-transfer piece of the scattering amplitude. Specifically, the terms which one retains are those which are nonanalytic and behave as either $\alpha^2/\sqrt{-q^2}$ or $\alpha^2\log -q^2$. When Fourier transformed, the former leads to classical ($\hbar$-independent) terms in the potential of order α^2/mr^2 while the latter generates quantum mechanical ($\hbar$-dependent) corrections of order $\alpha^2\hbar/m^2r^3$. (Of course, there are also shorter range nonanalytic contributions than these that are generated by scattering terms of order $\alpha^2q^{2n}\sqrt{-q^2}$ or $\alpha^2q^{2n}\log -q^2$. However, these pieces are higher order in momentum transfer and therefore lead to shorter distance effects than those considered above and are therefore neglected in our discussion.)

Specific calculations were done for particles with spin $0-0$, $0-1/2$, $0-1$, and $1/2-1/2$ and various universalities were found. In particular, we found that in each case there was a spin-independent contribution of the form

$$
\begin{aligned}
{}^{S_aS_b}\mathcal{M}^{(2)}_{tot}(q) &= \left[\frac{\alpha^2}{m_am_b}\left((m_a+m_b)S-\frac{7L}{3}\right)-i4\pi\alpha\frac{L}{q^2}\sqrt{\frac{m_am_b}{s-s_0}}\right] \\
&\quad\times\langle S_a,m_{af}|S_a,m_{ai}\rangle\,\langle S_b,m_{bf}|S_b,m_{bi}\rangle \qquad (89)
\end{aligned}
$$

where $L = \log -q^2$ and $S = \pi^2/\sqrt{-q^2}$ and with S_a the spin of particle a and S_b the spin of particle b with projections m_a and m_b on the quantization axis. The imaginary component of the amplitude, which would not, when Fourier-transformed lead to a real potential, is eliminated when the iterated lowest order potential contribution is subtracted, leading to a well defined spin-independent second order potential of universal form

$$\begin{aligned} {}^{S_aS_b}V^{(2)}_{S-I}(\vec{r}) &= \left[-\frac{\alpha^2(m_a+m_b)}{2m_am_br^2} - \frac{7\alpha^2\hbar}{6\pi m_am_br^3}\right] \\ &\quad \times \langle S_a, m_{af}|S_a, m_{ai}\rangle \langle S_b, m_{bf}|S_b, m_{bi}\rangle \end{aligned} \tag{90}$$

If either scattering particle carries spin then there is an additional spin-orbit contribution, whose form is also universal

$$\begin{aligned} {}^{S_aS_b}V^{(2)}_{S-O}(\vec{r}) &= \left[\frac{\alpha^2(2m_a^2+2m_am_b+m_b^2)}{2m_a^2m_b^2(m_a+m_b)r^4} + \frac{\alpha^2(8m_a+m_b)\hbar}{4\pi m_a^3m_b^2r^5}\right] \\ &\quad \times \vec{L}\cdot\vec{S}_a \langle S_b, m_{bf}|S_b, m_{bi}\rangle \\ &+ \left[\frac{\alpha^2(m_a^2+2m_am_b+2m_b^2)}{2m_a^2m_b^2(m_a+m_b)r^4} + \frac{\alpha^2(m_a+8m_b)\hbar}{4\pi m_a^2m_b^3r^5}\right] \\ &\quad \times \langle S_a, m_{af}|S_a, m_{ai}\rangle \vec{L}\cdot\vec{S}_b \end{aligned} \tag{91}$$

where we have defined

$$\vec{S}_a = \left\langle S_a, m_{af} \middle| \vec{S} \middle| S_a, m_{ai}\right\rangle \quad \text{and} \quad \vec{S}_b = \left\langle S_b, m_{bf} \middle| \vec{S} \middle| S_b, m_{bi}\right\rangle$$

In this case a well defined second order potential required the subtraction of infrared singular terms behaving as both i/p_0 *and* $1/p_0^2$ which arise from the iterated lowest order potential.

In the calculation of spin-0 – spin-1 scattering we encountered new tensor structures including a quadrupole interaction. Unfortunately, the subtraction of the i/p_0 and $1/p_0^2$ tensor pieces in the two-photon exchange amplitude was not successful with our simple nonrelativistic iteration of the leading order potential so that we cannot at this time give the form of the quadrupole component of the potential. Further work is needed to clarify this issue. The corrections to the spin-spin interaction have only been calculated in spin-1/2 – spin-1/2 scattering where we found their contributions to the scattering amplitude and to the potential. Since we verified these forms only for a

single spin configuration we have not confirmed its universality which we, however, strongly suspect. Of course, for higher spin configurations, there also exist quadrupole-quadrupole interactions, spin-quadrupole interactions, etc. However, the calculation of such forms becomes increasingly cumbersome as the spin increases, and the phenomenological importance becomes smaller. Thus we end our calculations here.

One point of view to interpret the universalities of the long distance components of the scattering amplitudes and the resulting potentials is that if we increase the spins of our scattered particles, all we do is to add additional multipole moments. The spin-independent component can then be viewed as a monopole-monopole interaction, the spin-orbit piece as a dipole-monopole interaction etc. As long as we do not change the quantum numbers that characterize the lower multipoles (such as the charge for the monopole-monopole interaction or the g-factor for the monopole-dipole interaction), an increase in spin of the scattered particles merely adds new interactions that are less important at long distances. In Appendix D we show explicitly how this multipole expansion structure arises. While it is familiar from classical electrodynamics—i.e. at the one-photon exchange level—we are not aware that this has been proven for two-photon exchange processes.

It is interesting that the same kinds of universalities of the long range components of the scattering amplitudes are also found in gravitational scattering [20] and in mixed electromagnetic-gravitational scattering [21].

Acknowledgements

We would like to thank John Donoghue for many clarifying discussions and Walter Goldberger for important comments. This work was supported in part by the National Science Foundation under award PHY05-53304 (BRH and AR) and by the the US Department of Energy under grant DE-FG-02-92ER40704 (AR).

A One loop integration in EFT

In this appendix we sketch how our results were obtained. The basic idea is to calculate the Feynman diagrams shown in Fig. 2. Since our calculations focus on long distance effects that stem from nonanalytic contributions in the squared momentum transfer q^2, we only evaluate these nonanalytic pieces of

the one loop integrals neglecting all short distance contributions which include the UV divergences. In practice, that means that all one loop diagrams where q does not run through any part of the loop can be neglected. Furthermore, only diagrams with at least two massless propagators yield nonanalytic contributions, reducing the number of contributing diagrams and thus integrals further. In the end, we need four different types of integrals for our calculations: Bubble integrals with two massless propagators and no massive propagator, triangle integrals with two massless propagators and one massive propagator and box and cross-box integrals, each with two massless propagators and two massive propagators.

For simplicity we shall assume spinless scattering. Thus for diagram (a) of Fig. 2—the bubble diagram—we find

$$\mathrm{Amp}[2a] = \frac{1}{2!}\int \frac{d^4k}{(2\pi)^4}\,\frac{\tau^{(2)}_{\mu\nu}(p_2,p_1)\eta^{\mu\alpha}\eta^{\nu\beta}\tau^{(2)}_{\alpha\beta}(p_4,p_3)}{k^2(k+q)^2}. \tag{92}$$

All vertex functions are listed in the main body of the paper, while for the integrals, all that is needed is their nonanalytic behavior. The exact expressions for the nonanalytic components of the bubble integrals read

$$\begin{aligned}
I &= \int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2(k+q)^2} = \frac{i}{16\pi^2}\,(-L) && (93)\\
I_\mu &= \int \frac{d^4k}{(2\pi)^4}\,\frac{k_\mu}{k^2(k+q)^2} = -\frac{1}{2}\,I\,q_\mu = \frac{i}{16\pi^2}\left(\frac{1}{2}\,L\right)q_\mu && (94)\\
I_{\mu\nu} &= \int \frac{d^4k}{(2\pi)^4}\,\frac{k_\mu k_\nu}{k^2(k+q)^2} = -\frac{1}{12}\,q^2\,I\,\eta_{\mu\nu} + \frac{1}{3}\,I\,q_\mu q_\nu \\
&= \frac{i}{16\pi^2}\left(\frac{1}{12}\,q^2\,L\;\eta_{\mu\nu} - \frac{1}{3}\,L\;q_\mu q_\nu\right) && (95)\\
I_{\mu\nu\rho} &= \int \frac{d^4k}{(2\pi)^4}\,\frac{k_\mu k_\nu k_\rho}{k^2(k+q)^2} = \frac{1}{24}\,q^2\,I\;3\,\eta_{(\mu\nu}q_{\rho)} - \frac{1}{4}\,I\,q_\mu q_\nu q_\rho \\
&= \frac{i}{16\pi^2}\left(-\frac{1}{24}\,q^2\,L\;3\,\eta_{(\mu\nu}q_{\rho)} + \frac{1}{4}\,L\;q_\mu q_\nu q_\rho\right) \\
I_{\mu\nu\rho\sigma} &= \int \frac{d^4k}{(2\pi)^4}\,\frac{k_\mu k_\nu k_\rho k_\sigma}{k^2(k+q)^2} \\
&= \frac{1}{240}\,q^4\,I\;3\,\eta_{(\mu\nu}\eta_{\rho\sigma)} - \frac{1}{40}\,q^2\,I\;6\,\eta_{(\mu\nu}q_\rho q_{\sigma)} + \frac{1}{5}\,I\,q_\mu q_\nu q_\rho q_\sigma \\
&= \frac{i}{16\pi^2}\left(-\frac{1}{240}\,q^4\,L\;3\,\eta_{(\mu\nu}\eta_{\rho\sigma)} + \frac{1}{40}\,q^2\,L\;6\,\eta_{(\mu\nu}q_\rho q_{\sigma)} - \frac{1}{5}\,L\,q_\mu q_\nu q_\rho q_\sigma\right)
\end{aligned}$$

Figure 3: Momentum labels for loops in triangle diagrams. On the left, the massive particle a with mass m_a runs through the loop, whereas on the right, particle b with mass m_b propagates in the loop.

(96)

where our symmetrization convention is

$$A_{(\mu_1\mu_2\mu_3\ldots\mu_n)} = \frac{1}{n!}\left(A_{\mu_1\mu_2\mu_3\ldots\mu_n} + A_{\mu_2\mu_1\mu_3\ldots\mu_n} + \ldots\right)$$

so that for example $3\,\eta_{(\mu\nu}q_{\rho)} = \eta_{\mu\nu}q_\rho + \eta_{\mu\rho}q_\nu + \eta_{\nu\rho}q_\mu$.

There are two distinct triangle diagrams, (b) and (c) in Fig. 2, with two different masses that propagate inside the loop. The momentum label conventions used in all triangle diagrams are shown in Fig. 3 for the two cases so that the expression for the amplitude for diagram (b) for example reads

$$\mathrm{Amp}[2b] = \int \frac{d^4k}{(2\pi)^4}\,\frac{\tau^{(2)}_{\mu\nu}(p_4,p_3)\eta^{\mu\alpha}\eta^{\nu\beta}\tau^{(1)}_{\beta}(p_2,p_2-k)\tau^{(1)}_{\alpha}(p_2-k,p_1)}{k^2(k+q)^2((k-p_2)^2-m_a^2)}. \quad (97)$$

In the evaluation of the integrals we use the on-shell relations

$$p_2\cdot q = -\frac{q^2}{2} \quad \text{and} \quad p_4\cdot q = +\frac{q^2}{2} \quad (98)$$

and the expressions for the triangle integrals read

$$\begin{aligned} J &= \int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2(k+q)^2((k+p)^2-m^2)} \\ &= \frac{i}{16\pi^2}\,\frac{1}{m^2}\left[-\frac{1}{2}L\left(1+\frac{q^2}{6m^2}+\mathcal{O}\left[\left(\frac{q^2}{m^2}\right)^2\right]\right)\right. \end{aligned}$$

$$-\frac{m}{2}\,S\left(1+\frac{q^2}{8m^2}+\mathcal{O}\left[\left(\frac{q^2}{m^2}\right)^2\right]\right)\Bigg] \tag{99}$$

$$\begin{aligned}
J_\mu &= \int \frac{d^4k}{(2\pi)^4}\;\frac{k_\mu}{k^2(k+q)^2((k+p)^2-m^2)} \\
&= \mathcal{F}\left[\left(-\frac{1}{4m^2}\,I-\frac{1}{2}\,J\right)q_\mu+\left(\frac{1}{2m^2}\,I+\frac{1}{4}\frac{q^2}{m^2}\,J\right)p_\mu\right] \qquad (100)\\
J_{\mu\nu} &= \int \frac{d^4k}{(2\pi)^4}\;\frac{k_\mu k_\nu}{k^2(k+q)^2((k+p)^2-m^2)} \\
&= \mathcal{F}\left(-\frac{q^2}{16m^2}\,I-\frac{q^2}{8}\,J\right)\eta_{\mu\nu} \\
&+ \mathcal{F}^2\left[\frac{5}{16m^2}\left(1-\frac{1}{10}\,\frac{q^2}{m^2}\right)I+\frac{3}{8}\,J\right]q_\mu q_\nu \\
&+ \mathcal{F}^2\left[\frac{3q^2}{16m^4}\,I+\frac{q^2}{8m^2}\left(1+\frac{1}{2}\,\frac{q^2}{m^2}\right)J\right]p_\mu p_\nu \\
&+ \mathcal{F}^2\left[-\frac{1}{4m^2}\left(1+\frac{1}{8}\,\frac{q^2}{m^2}\right)I-\frac{3q^2}{16m^2}\,J\right]2\,p_{(\mu}q_{\nu)} \qquad (101)\\
J_{\mu\nu\rho} &= \int \frac{d^4k}{(2\pi)^4}\;\frac{k_\mu k_\nu k_\rho}{k^2(k+q)^2((k+p)^2-m^2)} \\
&= \mathcal{F}^2\left[\frac{5q^2}{96m^2}\left(1-\frac{1}{10}\,\frac{q^2}{m^2}\right)I+\frac{q^2}{16}\,J\right]3\,\eta_{(\mu\nu}q_{\rho)} \\
&+ \mathcal{F}^2\left[-\frac{q^2}{24m^2}\left(1+\frac{1}{8}\,\frac{q^2}{m^2}\right)I-\frac{q^4}{32m^2}\,J\right]3\,\eta_{(\mu\nu}p_{\rho)} \\
&+ \mathcal{F}^3\left[-\frac{11}{32m^2}\left(1-\frac{13}{66}\,\frac{q^2}{m^2}+\frac{1}{66}\,\frac{q^4}{m^4}\right)I-\frac{5}{16}\,J\right]q_\mu q_\nu q_\rho \\
&+ \mathcal{F}^3\left[\frac{q^2}{12m^4}\left(1+\frac{11}{16}\,\frac{q^2}{m^2}\right)I+\frac{3q^4}{32m^4}\left(1+\frac{1}{6}\,\frac{q^2}{m^2}\right)J\right]p_\mu p_\nu p_\rho \\
&+ \mathcal{F}^3\left[\frac{1}{6m^2}\left(1+\frac{9}{32}\,\frac{q^2}{m^2}-\frac{1}{64}\frac{q^4}{m^4}\right)I+\frac{5q^2}{32m^2}\,J\right]3\,q_{(\mu}q_\nu p_{\rho)} \\
&+ \mathcal{F}^3\left[-\frac{13q^2}{96m^4}\left(1+\frac{1}{26}\,\frac{q^2}{m^2}\right)I-\frac{q^2}{16m^2}\left(1+\frac{q^2}{m^2}\right)J\right]3\,q_{(\mu}p_\nu p_{\rho)} \qquad (102)\\
J_{\mu\nu\rho\sigma} &= \int \frac{d^4k}{(2\pi)^4}\;\frac{k_\mu k_\nu k_\rho k_\sigma}{k^2(k+q)^2((k+p)^2-m^2)}
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{F}^2 \left[\frac{5q^4}{768m^2} \left(1 - \frac{1}{10} \frac{q^2}{m^2} \right) I + \frac{q^4}{128} J \right] 3\, \eta_{(\mu\nu}\eta_{\rho\sigma)} \\
&+ \mathcal{F}^3 \left[-\frac{11q^2}{256m^2} \left(1 - \frac{13}{66} \frac{q^2}{m^2} + \frac{1}{66} \frac{q^4}{m^4} \right) I - \frac{5q^2}{128} J \right] 6\, \eta_{(\mu\nu} q_\rho q_{\sigma)} \\
&+ \mathcal{F}^3 \left[\frac{q^2}{48m^2} \left(1 + \frac{9}{32} \frac{q^2}{m^2} - \frac{1}{64} \frac{q^4}{m^4} \right) I + \frac{5q^4}{256m^2} J \right] 12\, \eta_{(\mu\nu} q_\rho p_{\sigma)} \\
&+ \mathcal{F}^3 \left[-\frac{13q^4}{768m^4} \left(1 + \frac{1}{26} \frac{q^2}{m^2} \right) I - \frac{q^4}{128m^2} \left(1 + \frac{q^2}{m^2} \right) J \right] 6\, \eta_{(\mu\nu} p_\rho p_{\sigma)} \\
&+ \mathcal{F}^4 \left[\frac{93}{256m^2} \left(1 - \frac{163}{558} \frac{q^2}{m^2} + \frac{25}{558} \frac{q^4}{m^4} - \frac{1}{372} \frac{q^6}{m^6} \right) I + \frac{35}{128} J \right] q_\mu q_\nu q_\rho q_\sigma \\
&+ \mathcal{F}^4 \left[-\frac{1}{8m^2} \left(1 + \frac{29}{64} \frac{q^2}{m^2} - \frac{19}{384} \frac{q^4}{m^4} + \frac{1}{384} \frac{q^6}{m^6} \right) I - \frac{35q^2}{256m^2} J \right] 4\, q_{(\mu} q_\nu q_\rho p_{\sigma)} \\
&+ \mathcal{F}^4 \left[\frac{27q^2}{256m^4} \left(1 + \frac{7}{81} \frac{q^2}{m^2} - \frac{1}{324} \frac{q^4}{m^4} \right) I + \frac{5q^2}{128m^2} \left(1 + \frac{3}{2} \frac{q^2}{m^2} \right) J \right] 6\, q_{(\mu} q_\nu p_\rho p_{\sigma)} \\
&+ \mathcal{F}^4 \left[-\frac{q^2}{24m^4} \left(1 + \frac{83}{64} \frac{q^2}{m^2} + \frac{3}{128} \frac{q^4}{m^4} \right) I - \frac{15q^4}{256m^4} \left(1 + \frac{1}{3} \frac{q^2}{m^2} \right) J \right] 4\, q_{(\mu} p_\nu p_\rho p_{\sigma)} \\
&+ \mathcal{F}^4 \left[\frac{55q^4}{768m^6} \left(1 + \frac{5}{22} \frac{q^2}{m^2} \right) I + \frac{3q^4}{128m^4} \left(1 + 2 \frac{q^2}{m^2} + \frac{1}{6} \frac{q^4}{m^4} \right) J \right] p_\mu p_\nu p_\rho p_\sigma
\end{aligned}
\tag{103}
$$

where we have defined

$$
\mathcal{F} \equiv \frac{1}{1 - \frac{1}{4} \frac{q^2}{m^2}} \tag{104}
$$

in order to keep our notation more compact. Note that the scalar integral J has been expanded in the limit $q^2 \ll m^2$, however the expressions for the nonanalytic parts of the vector and tensor integrals are exact to all orders in q^2 when expressed in terms of the scalar integrals I and J. The triangle integrals listed in Eqs. (99-103) must be used as

$$
\left[J, J_\mu, J_{\mu\nu}, J_{\mu\nu\rho}, J_{\mu\nu\rho\sigma} \right] \Big|_{p=-p_2,\, m=m_a} \tag{105}
$$

in diagrams where particle a (incoming momentum p_1 and outgoing momentum p_2) propagates in the loop as sketched on the left side of Fig. 3, and

as

$$[J, J_\mu, J_{\mu\nu}, J_{\mu\nu\rho}, J_{\mu\nu\rho\sigma}]\Big|_{p=+p_4,\, m=m_b} \tag{106}$$

when particle b (incoming momentum p_3 and outgoing momentum p_4) propagates though the loop with momentum labels as seen on the right hand side of Fig. 3.

More challenging is the calculation of the box and cross-box diagrams—diagrams (d) and (e) in Fig. 2. For the box diagram (d) we have

$$\begin{aligned}\mathrm{Amp}[2d] = \int \frac{d^4k}{(2\pi)^4}\, & \frac{1}{k^2(k+q)^2((k-p_2)^2-m_a^2)((k+p_4)^2-m_b^2)} \\ & \times \tau_\nu^{(1)}(p_4, p_4+k)\tau_\mu^{(1)}(p_4+k, p_3)\, \eta^{\mu\alpha}\eta^{\nu\beta}\tau_\beta^{(1)}(p_2, p_2-k)\tau_\alpha^{(1)}(p_2-k, p_1).\end{aligned} \tag{107}$$

The evaluation of the box integrals has been performed earlier by others with Ref. [22] giving a nice treatment with some of the calculational details. Unfortunately, the exact expressions for the tensor integrals become extremely long so that we only give the form of the vector box integral. The Passarino-Veltman reduction of the higher tensor integrals was performed with the help of computer algebra, which is highly recommended. The expression for the scalar box integral is [22]

$$\begin{aligned}K &= \int \frac{d^4k}{(2\pi)^4}\, \frac{1}{k^2(k+q)^2((k-p_2)^2-m_1^2)((k+p_4)^2-m_2^2)} \\ &= \frac{i}{16\pi^2}\Bigg[-2\frac{L}{q^2}\,\frac{1}{\sqrt{\Lambda}}\log\left|\frac{\sqrt{\Lambda}-(s-s_0)}{-\sqrt{\Lambda}-(s-s_0)}\right| \\ &\qquad - i2\pi\frac{L}{q^2}\,\frac{1}{\sqrt{\Lambda}}\,\theta(s-s_0)\Bigg] \\ &= \frac{i}{16\pi^2}\Bigg[-2\frac{L}{q^2}\left(-\frac{1}{2m_am_b}\right)\left(1-\frac{s-s_0}{6m_am_b}+\mathcal{O}\left((s-s_0)^2\right)\right) \\ &\qquad - i2\pi\frac{L}{q^2}\,\frac{1}{2\sqrt{m_am_b}\sqrt{s-s_0}}\left(1-\frac{s-s_0}{8m_am_b}+\mathcal{O}\big((s-s_0)^2\big)\right)\theta(s-s_0)\Bigg]\end{aligned} \tag{108}$$

where

$$\Lambda \equiv (s-s_0)(4m_a m_b + s - s_0). \tag{109}$$

Note that the expression is *exact* in q^2 and we only expand it in $s-s_0$ in our calculations. The vector box integral is found to be

$$\begin{aligned}
K^\mu &= \int \frac{d^4k}{(2\pi)^4}\,\frac{k^\mu}{k^2(k+q)^2((k-p_2)^2-m_1^2)((k+p_4)^2-m_2^2)} \\
&= \frac{(2m_a^2+2m_am_b+s-s_0)J_a+(2m_b^2+2m_am_b+s-s_0)J_b-\Lambda\,K}{2\left[((m_a+m_b)^2+s-s_0)q^2+\Lambda\right]}\,q^\mu \\
&+ \frac{(4m_am_b+2(s-s_0)+q^2)J_a+(4m_b^2-q^2)J_b+q^2(2m_b^2+2m_am_b+s-s_0)K}{2\left[((m_a+m_b)^2+s-s_0)q^2+\Lambda\right]}\,p_2^\mu \\
&- \frac{(4m_a^2-q^2)J_a+(4m_am_b+2(s-s_0)+q^2)J_b+q^2(2m_a^2+2m_am_b+s-s_0)K}{2\left[((m_a+m_b)^2+s-s_0)q^2+\Lambda\right]}\,p_4^\mu
\end{aligned} \tag{110}$$

with $J_i \equiv J\big|_{m=m_i}$, and we notice that its denominator vanishes in the limit q^2, $s-s_0 \to 0$. More specifically, the denominator can be written as

$$\begin{aligned}
D_{K^\mu} &= 2\left[((m_a+m_b)^2+s-s_0)q^2+\Lambda\right] \\
&= 2\left[m_a^2+m_b^2+2\left(p_0^2+\sqrt{m_a^2+p_0^2}\sqrt{m_b^2+p_0^2}\right)\right]\left(4p_0^2-\vec{q}^{\,2}\right) \\
&= 8p_0^2\left[m_a^2+m_b^2+2\left(p_0^2+\sqrt{m_a^2+p_0^2}\sqrt{m_b^2+p_0^2}\right)\right]\left(1-\sin^2\frac{\theta}{2}\right) \\
&= 8p_0^2\left[m_a^2+m_b^2+2\left(p_0^2+\sqrt{m_a^2+p_0^2}\sqrt{m_b^2+p_0^2}\right)\right]\cos^2\frac{\theta}{2}
\end{aligned} \tag{111}$$

where we have used Eq. (10) and $q^2 = -\vec{q}^{\,2} = -4p_0^2\sin^2\frac{\theta}{2}$. We see that the denominator vanishes for $p_0 \to 0$ and for backward scattering at $\theta = \pi$. Unless we consider backward scattering where the denominator vanishes and thus the amplitude diverges, we have $4p_0^2 > \vec{q}^{\,2}$, and since p_0^2 originates from the relativistic structure $s-s_0$, we therefore expand our vector and tensor box integrals first in q^2 and then in $s-s_0$. Denominators that vanish in the limit q^2, $s-s_0 \to 0$ are a common feature for all box vector and tensor integrals, and they are the source of the $1/(s-s_0) \sim 1/p_0^2$ components in our results for the scattering amplitude.

In the case of the cross-box diagram (e) the amplitude reads

$$\mathrm{Amp}[2e] = \int \frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2(k+q)^2((k-p_2)^2-m_a^2)((k-p_3)^2-m_b^2)}$$

$$\times\, \tau^{(1)}_{\nu}(p_4, p_3-k)\tau^{(1)}_{\mu}(p_3-k, p_3)\, \eta^{\mu\alpha}\eta^{\nu\beta}\tau^{(1)}_{\beta}(p_2, p_2-k)\tau^{(1)}_{\alpha}(p_2-k, p_1). \tag{112}$$

Now we need the cross-box scalar integral which can be deduced from the result for the box scalar integral by replacing the set of Mandelstam variables (s,t) by (u,t) where $t=q^2$ and $s+t+u=2m_a^2+2m_b^2$. Again, we only give the scalar and vector integrals because the exact expressions for the higher tensor integrals become very long. The resulting expressions are

$$\begin{aligned}
K' &= \int \frac{d^4k}{(2\pi)^4}\, \frac{1}{k^2(k+q)^2((k-p_2)^2-m_1^2)((k-p_3)^2-m_2^2)} \\
&= \frac{i}{16\pi^2}\Bigg[-2\frac{L}{q^2}\left(+\frac{1}{2m_am_b}\right)\Bigg\{\left(1-\frac{s-s_0}{6m_am_b}+\mathcal{O}\left((s-s_0)^2\right)\right) \\
&\qquad -\frac{q^2}{6m_am_b}\left(1-\frac{2(s-s_0)}{5m_am_b}+\mathcal{O}\left((s-s_0)^2\right)\right) \\
&\qquad +\frac{q^4}{30m_a^2m_b^2}\left(1-\frac{9(s-s_0)}{14m_am_b}+\mathcal{O}\left((s-s_0)^2\right)\right) \\
&\qquad +\mathcal{O}\left(q^6\right)\left(1+O\left(s-s_0\right)\right)\Bigg\}\Bigg] \\
K'^{\mu} &= \int \frac{d^4k}{(2\pi)^4}\, \frac{k^{\mu}}{k^2(k+q)^2((k-p_2)^2-m_1^2)((k-p_3)^2-m_2^2)} \\
&= \frac{(2m_a^2-2m_am_b-(s-s_0)-q^2)J_a+(2m_b^2-2m_am_b-(s-s_0)-q^2)J_b-\tilde{\Lambda}K'}{2\left[((m_a+m_b)^2+s-s_0)q^2+\Lambda\right]}\, q^{\mu} \\
&\quad -\frac{(4m_am_b+2(s-s_0)+q^2)J_a-(4m_b^2-q^2)J_b-q^2(2m_b^2-2m_am_b-(s-s_0)-q^2)K'}{2\left[((m_a+m_b)^2+s-s_0)q^2+\Lambda\right]}\, p_2^{\mu} \\
&\quad +\frac{(4m_a^2-q^2)J_a-(4m_am_b+2(s-s_0)+q^2)J_b+q^2(2m_a^2-2m_am_b-(s-s_0)-q^2)K'}{2\left[((m_a+m_b)^2+s-s_0)q^2+\Lambda\right]}\, p_3^{\mu}
\end{aligned} \tag{113}$$

where

$$\tilde{\Lambda} \equiv (s-s_0+q^2)(4m_am_b+s-s_0+q^2). \tag{114}$$

We point out that we did not include an imaginary component in the case of the cross-box scalar integral whereas for the box integral in Eq. (108) we included both an imaginary and a real part. The reason for that is that the θ-function multiplying the imaginary part in Eq. (108) for the cross-box

integral becomes $\theta(u - s_0)$ and it vanishes in the kinematic region we are considering since

$$u = (m_a - m_b)^2 - (s - s_0) - q^2 < s_0 = (m_a + m_b)^2.$$

In this way all amplitudes quoted in the text can be generated.

B Fourier Transformations

In this appendix we collect all Fourier transformation integrals needed to evaluate the potentials in coordinate space.

$$\begin{aligned}
\int \frac{d^3q}{(2\pi)^3}\, e^{-i\vec{q}\cdot\vec{r}}\, \frac{1}{|\vec{q}|^2} &= \frac{1}{4\pi r} \\
\int \frac{d^3q}{(2\pi)^3}\, e^{-i\vec{q}\cdot\vec{r}}\, \frac{q_i}{|\vec{q}|^2} &= -\frac{i\, r_i}{4\pi r^3} \\
\int \frac{d^3q}{(2\pi)^3}\, e^{-i\vec{q}\cdot\vec{r}}\, \frac{q_i q_j}{|\vec{q}|^2} &= -\frac{1}{4\pi}\left(3\frac{r_i r_j}{r^5} - \frac{\delta_{ij}}{r^3}\right) \\
\int \frac{d^3q}{(2\pi)^3}\, e^{-i\vec{q}\cdot\vec{r}}\, \frac{1}{|\vec{q}|} &= \frac{1}{2\pi^2 r^2} \\
\int \frac{d^3q}{(2\pi)^3}\, e^{-i\vec{q}\cdot\vec{r}}\, \frac{q_i}{|\vec{q}|} &= -\frac{i\, r_i}{\pi^2 r^4} \\
\int \frac{d^3q}{(2\pi)^3}\, e^{-i\vec{q}\cdot\vec{r}}\, \frac{q_i q_j}{|\vec{q}|} &= -\frac{1}{\pi^2}\left(4\frac{r_i r_j}{r^6} - \frac{\delta_{ij}}{r^4}\right) \\
\int \frac{d^3q}{(2\pi)^3}\, e^{-i\vec{q}\cdot\vec{r}}\, \log|\vec{q}|^2 &= -\frac{1}{2\pi r^3} \\
\int \frac{d^3q}{(2\pi)^3}\, e^{-i\vec{q}\cdot\vec{r}}\, q_i \log|\vec{q}|^2 &= \frac{3i\, r_i}{2\pi r^5} \\
\int \frac{d^3q}{(2\pi)^3}\, e^{-i\vec{q}\cdot\vec{r}}\, q_i q_j \log|\vec{q}|^2 &= \frac{3}{2\pi}\left(5\frac{r_i r_j}{r^7} - \frac{\delta_{ij}}{r^5}\right) \\
\int \frac{d^3q}{(2\pi)^3}\, e^{-i\vec{q}\cdot\vec{r}}\, |\vec{q}|^2 \log|\vec{q}|^2 &= \frac{3}{\pi r^5}
\end{aligned} \tag{115}$$

C Iteration Integrals

In this appendix we evaluate the integrals

$$[H; H_r; H_{rs}] = i \int \frac{d^3\ell}{(2\pi)^3} \frac{e^2}{|\vec{p}_f - \vec{\ell}|^2 + \lambda^2} \frac{i[1; \ell_r; \ell_r \ell_s]}{\frac{p_0^2}{2m_r} - \frac{\ell^2}{2m_r} + i\epsilon} \frac{e^2}{|\vec{\ell} - \vec{p}_i|^2 + \lambda^2} \tag{116}$$

which are needed in order to perform the iteration of the lowest order Coulomb potentials. Note that we have introduced a small photon mass $\lambda^2 \ll p_0^2$ in order to avoid singularities, but since we are only interested in the long distance effects we do not show the singularities in λ in our expressions. The evaluation of the integral H has been given by Dalitz as [23]

$$H = i4\pi\alpha^2 \frac{m_r}{p_0} \frac{\log \vec{q}^2}{\vec{q}^2} = -i4\pi\alpha^2 \frac{m_r}{p_0} \frac{L}{q^2}. \tag{117}$$

In order to determine the vector integral H_r we define

$$H_r = A\,(p_i + p_f)_r \tag{118}$$

and contracting with $(p_i + p_f)_r$, we find

$$2A(\vec{p}_i + \vec{p}_f)^2 = (2\lambda^2 + 4p_0^2)H - 4m_r Y - X(p_i) - X(p_f) \tag{119}$$

where

$$Y = -\int \frac{d^3\ell}{(2\pi)^3} \frac{e^2}{|\vec{p}_f - \vec{\ell}|^2 + \lambda^2} \frac{e^2}{|\vec{\ell} - \vec{p}_i|^2 + \lambda^2} = -\frac{\pi^2\alpha^2}{p_0 \sin\frac{\theta}{2}} = -2\alpha^2 S \tag{120}$$

and

$$\begin{aligned} X(p_i) = X(p_f) &= ie^2 \int \frac{d^3\ell}{(2\pi)^3} \frac{e^2}{|\vec{\ell} - \vec{p}_i|^2 + \lambda^2} \frac{i}{\frac{p_0^2}{2m_r} - \frac{\ell^2}{2m_r} + i\epsilon} \\ &= -i4\pi\alpha^2 \frac{m_r}{p_0} \log \frac{i\lambda}{2p_0 + i\lambda}. \end{aligned} \tag{121}$$

Note that the integrals X only depend on p_i or p_f and therefore do not yield any terms nonanalytic in q^2. Thus we drop the contributions of the X's and we have

$$A = \frac{1}{8p_0^2(1 - \sin^2\frac{\theta}{2})} \left[4p_0^2 H - 4m_r Y\right] \simeq \alpha^2 \left(\frac{m_r}{p_0^2} S - i2\pi \frac{m_r}{p_0} \frac{L}{q^2}\right) \tag{122}$$

In the case of the tensor integral we define

$$H_{rs} = B\,\delta_{rs} + C\,(p_i + p_f)_r (p_i + p_f)_s + D\,(p_i - p_f)_r (p_i - p_f)_s \tag{123}$$

and we require three conditions in order to evaluate the coefficients B, C and D. Neglecting again the integrals X, these are

i)

$$\delta_{rs} H_{rs} : \; 3B + (4p_0^2 - \vec{q}^{\,2})C + \vec{q}^{\,2} D \simeq p_0^2 H - 2m_r Y$$

ii)

$$(p_i + p_f)^r H_{rs} : \; B + (4p_0^2 - \vec{q}^{\,2})C \simeq \frac{1}{1 - \frac{\vec{q}^{\,2}}{4p_0^2}} \left[p_0^2 H - m_r Y \right] - m_r Y$$

iii)

$$(p_i - p_f)^r H_{rs} : \; B + \vec{q}^{\,2} D \simeq 0$$

Solving, we find

$$\begin{aligned} B &\simeq -\frac{\vec{q}^{\,2}}{4}\left(H - \frac{m_r Y}{p_0^2}\right) \\ C &\simeq \frac{1}{4}\left(H - \frac{2m_r Y}{p_0^2}\right) \\ D &\simeq \frac{1}{4}\left(H - \frac{m_r Y}{p_0^2}\right) \end{aligned} \tag{124}$$

Keeping only the leading terms in $\vec{q}^{\,2}$ we have then

$$\begin{aligned} H &\simeq -i4\pi\alpha^2 \frac{m_r}{p_0}\frac{L}{q^2} \\ H_r &\simeq (p_i + p_f)_r \left(-i2\pi\alpha^2 \frac{m_r}{p_0}\frac{L}{q^2} + \alpha^2 \frac{m_r}{p_0^2} S + \ldots \right) \\ H_{rs} &\simeq \delta_{rs}\, \vec{q}^{\,2} \left(i\pi\alpha^2 \frac{m_r}{p_0}\frac{L}{q^2} - \frac{1}{2}\alpha^2 \frac{m_r}{p_0^2} S + \ldots \right) \\ &+ (p_i + p_f)_r (p_i + p_f)_s \left(-i\pi\alpha^2 \frac{m_r}{p_0}\frac{L}{q^2} + \alpha^2 \frac{m_r}{p_0^2} S + \ldots \right) \\ &+ (p_i - p_f)_r (p_i - p_f)_s \left(-i\pi\alpha^2 \frac{m_r}{p_0}\frac{L}{q^2} + \frac{1}{2}\alpha^2 \frac{m_r}{p_0^2} S + \ldots \right) \end{aligned} \tag{125}$$

D Generalized Results and Interpretation

How can we interpret the universalities we have found? As an example, let us first consider two spinless charged particles of charge e where the leading order Coulomb interaction between these two charges is

$$V(\vec{r}) = \frac{\alpha}{r}. \tag{126}$$

Now if we replace one of the two charges by a spin-1/2 particle of charge $-2e$ the spin-independent leading order Coulomb interaction becomes

$$V(\vec{r}) = \frac{-2\alpha}{r}. \tag{127}$$

We see that the universality of even the leading order Coulomb potential depends on having equal charges. In this section, we extend our calculations to arbitrary charges and g-factors, and our results lead us to the interpretation that the universalities originate from a multipole expansion of the long range scattering amplitudes and potentials.

D.1 One-photon Exchange Potential

It has long been known that a particle with spin S has $2S+1$ multipole moments [24]. The one-photon exchange potential thus exhibits a multipole expansion as we know it from classical electrodynamics:

Spin-0 – Spin-0

The Lagrangian for a spin-0 particle with arbitrary charge $q = Ze$ reads

$$\mathcal{L} = (iD_\mu\phi)^\dagger iD^\mu\phi - m^2\phi^\dagger\phi \tag{128}$$

with $D_\mu = \partial_\mu + ieZA_\mu$ and the Feynman rules for the vertices become

$$\begin{aligned} {}^0\tau^{(1)}_\mu(p_2,p_1) &= -iZe(p_2+p_1)_\mu \\ {}^0\tau^{(2)}_{\mu\nu}(p_2,p_1) &= 2i(Ze)^2\eta_{\mu\nu}. \end{aligned} \tag{129}$$

The one-photon exchange potential for a spin-0 Particle a with mass m_a, charge $q_a = Z_a e$ and a spin-0 particle b with mass m_b, charge $q_b = Z_b e$ is then

$${}^0V^{(1)}_C(\vec{r}) \simeq \frac{Z_a Z_b \alpha}{r} \tag{130}$$

and exhibits merely a monopole-monopole interaction, Coulomb's law, proportional to $1/r$.

Spin-0 – Spin-1/2

Now we introduce the Lagrangian for a spin-1/2 particle of arbitrary charge $q = Ze$ and arbitrary g-factor g,

$$\mathcal{L} = \bar{\psi}(i \not{D} - m)\psi - \frac{Ze(g-2)}{8m} F^{\mu\nu} \bar{\psi} \sigma_{\mu\nu} \psi \tag{131}$$

with again $D_\mu = \partial_\mu + ieZA_\mu$, which yields the Feynman vertex rules

$$\begin{aligned} \tfrac{1}{2}\tau_\mu^{(1)}(p_2, p_1) &= -iZe\gamma_\mu + \frac{Ze(g-2)}{4m}\sigma_{\mu\rho}(p_2 - p_1)^\rho \\ \tfrac{1}{2}\tau_{\mu\nu}^{(2)}(p_p, p_1) &= 0. \end{aligned} \tag{132}$$

The one-photon exchange potential for a spin-0 particle a with mass m_a, charge $q_a = Z_a e$, g-factor g_a and a spin-1/2 particle b with mass m_b, charge $q_b = Z_b e$, g-factor g_b is

$$\tfrac{1}{2}V_C^{(1)}(\vec{r}) \simeq \frac{Z_a Z_b \alpha}{r} \chi_f^{b\dagger} \chi_i^b - \frac{Z_a Z_b \alpha}{r^3} \frac{(g_b - 1)m_a + g_b m_b}{2 m_a m_b^2} \vec{L} \cdot \vec{S}_b \tag{133}$$

where $\vec{L} \cdot S_b = (\vec{r} \times \vec{p}) \cdot \vec{S}_b = \vec{r} \cdot (\vec{p} \times \vec{S}_b)$. Thus, besides the leading monopole-monopole interaction – Coulomb's law – proportional to $1/r$, we observe an additional monopole-dipole interaction, the spin-orbit coupling. The coefficient of the monopole-monopole interaction does not depend on the g-factor whereas the spin-orbit piece does.

Using the physical values for the masses, charges and g-factors then includes all one-photon exchange effects to all orders in α, i.e. if we take $g = 2 + \frac{\alpha}{\pi}$ (for a particle of charge $\pm e$), we take into account the $\mathcal{O}(\alpha^2)$ long-distance contribution from the one loop vertex correction diagram.

Spin-0 – Spin-1

For the spin-1 case we consider a particle of charge Ze with g-factor g in the Lagrangian

$$\mathcal{L} = -\frac{1}{2} U_{\mu\nu}^\dagger U^{\mu\nu} - m^2 \phi_\mu^\dagger \phi^\mu + iZe(g-1)\phi_\mu^\dagger \phi_\nu F^{\mu\nu} \tag{134}$$

where $U_{\mu\nu} = D_\mu \phi_\nu - D_\nu \phi_\mu$ with $D_\mu = \partial_\mu + ieZA_\mu$. Once the charge and the g-factor are determined, the quadrupole moment from this Lagrangian is fixed. If one wants to include an arbitrary quadrupole moment one has to add a dimension 6 operator [25] which we will not do here since it complicates the following two-photon exchange calculations considerably. The resulting Feynman rules read

$$\begin{aligned}
{}^1\tau^{(1)}_{\mu,\beta\alpha}(p_2,p_1) &= iZe\Big[\eta_{\alpha\beta}(p_2+p_1)_\mu \\
&\qquad -\eta_{\mu\beta}(gp_2-(g-1)p_1)_\alpha - \eta_{\mu\alpha}(gp_1-(g-1)p_2)_\beta\Big] \\
{}^1\tau^{(2)}_{\mu\nu,\beta\alpha}(p_2,p_1) &= -i(Ze)^2\left[2\eta_{\mu\nu}\eta_{\alpha\beta} - \eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\mu\beta}\eta_{\nu\alpha}\right]
\end{aligned} \tag{135}$$

and the one-photon exchange potential for a spin-0 particle a with mass m_a, charge $q_a = Z_a e$ and a spin-1 particle b with mass m_b, charge $q_b = Z_b e$, g-factor g_b is

$$\begin{aligned}
{}^1V^{(1)}_C(\vec{r}) &\simeq \frac{Z_a Z_b \alpha}{r}\,\hat{\epsilon}^{b*}_f \cdot \hat{\epsilon}^b_i - \frac{Z_a Z_b \alpha}{r^3}\frac{(g_b-1)m_a + g_b m_b}{2m_a m_b^2}\vec{L}\cdot\vec{S}_b \\
&+ \frac{Z_a Z_b \alpha}{r^5}\frac{3(2g_b-1)}{4m_b^2}\;\vec{r} : T^b : \vec{r}
\end{aligned} \tag{136}$$

where we neglected relativistic terms involving $\hat{\epsilon}^{b*}_f \cdot \vec{p}\;\hat{\epsilon}^b_i \cdot \vec{p}$. Now besides the monopole-monopole and monopole-dipole pieces seen before, a new piece of monopole-quadrupole structure constitutes the highest multipole in the expansion for the spin-1 particle.

Spin-1/2 – Spin-1/2

The one-photon exchange potential for a spin-1/2 particle a with mass m_a, charge $q_a = Z_a e$, g-factor g_a and a spin-1/2 particle b with mass m_b, charge $q_b = Z_b e$, g-factor g_b is

$$\begin{aligned}
{}^{\frac{1}{2}\frac{1}{2}}V^{(1)}_C(\vec{r}) &\simeq \frac{Z_a Z_b \alpha}{r}\chi^{a\dagger}_f \chi^a_i \chi^{b\dagger}_f \chi^b_i \\
&- \frac{Z_a Z_b \alpha}{r^3}\frac{(g_b-1)m_a + g_b m_b}{2m_a m_b^2}\vec{L}\cdot\vec{S}_b\,\chi^{a\dagger}_f \chi^a_i \\
&- \frac{Z_a Z_b \alpha}{r^3}\frac{g_a m_a + (g_a-1)m_b}{2m_a^2 m_b}\vec{L}\cdot\vec{S}_a\,\chi^{b\dagger}_f \chi^b_i
\end{aligned}$$

$$- \quad \frac{Z_a Z_b \alpha}{r^5} \frac{g_a g_b}{4 m_a m_b} \left(3 \vec{S}_a \cdot \vec{r}\, \vec{S}_b \cdot \vec{r} - r^2 \vec{S}_a \cdot \vec{S}_b \right) \tag{137}$$

In this case, we observe a monopole-monopole piece, two monopole-dipole pieces aka spin-orbit pieces and a dipole-dipole piece, the spin-spin interaction.

Clearly, the one-photon exchange potentials (and scattering amplitudes) for two charged particles of various spins exhibit a multipole expansion, and as one would expect for a multipole expansion, higher spins only add higher multipole interactions while all lower multipole interactions are universal, i.e. of identical form as for lower spins. That then implies that the numerical coefficients in the multipole expansion do not depend on structures as for example $\vec{S}^2$ or on coefficients that characterize higher multipoles, for example Coulomb's law cannot depend on the g-factors but only on the charges and the spin-orbit interaction does not depend on the quadrupole moment.

D.2 Two-photon Exchange Potential

At the two-photon exchange level the amplitudes and potentials we calculated exhibit the same universalities as found in the one-photon exchange case where they are explained in terms of a multipole expansion. For particles with arbitrary charges and g-factors the results for the second order potentials read

$$\begin{aligned}
{}^0V_C^{(2)}(\vec{r}) \;&\simeq\; -\frac{(Z_a Z_b \alpha)^2 (m_a + m_b)}{2 m_a m_b r^2} - \frac{7 (Z_a Z_b \alpha)^2 \hbar}{6 \pi m_a m_b r^3} \\
{}^{\frac{1}{2}}V_C^{(2)}(\vec{r}) \;&\simeq\; \left[-\frac{(Z_a Z_b \alpha)^2 (m_a + m_b)}{2 m_a m_b r^2} - \frac{7 (Z_a Z_b \alpha)^2 \hbar}{6 \pi m_a m_b r^3} \right] \chi_f^{b\dagger} \chi_i^b \\
&\quad + \left[\frac{(Z_a Z_b \alpha)^2 \Big((g_b - 2) m_a^3 + (2 g_b - 3) m_a^2 m_b + 2 (g_b - 1) m_a m_b^2 + g_b m_b^3 \Big)}{2 m_a^2 m_b^3 (m_a + m_b) r^4} \right. \\
&\quad \left. + \frac{(Z_a Z_b \alpha)^2 \hbar \Big((-3 g_b^2 + 16 g_b - 18) m_a + (-3 g_b^2 + 16 g_b - 4) m_b \Big)}{8 \pi m_a^2 m_b^3 r^5} \right] \vec{L} \cdot \vec{S}_b \\
{}^1V_C^{(2)}(\vec{r}) \;&\simeq\; \left[-\frac{(Z_a Z_b \alpha)^2 (m_a + m_b)}{2 m_a m_b r^2} - \frac{7 (Z_a Z_b \alpha)^2 \hbar}{6 \pi m_a m_b r^3} \right] \hat{\epsilon}_f^{b*} \cdot \hat{\epsilon}_i^b
\end{aligned}$$

$$
\begin{aligned}
&+ \left[\frac{(Z_aZ_b\alpha)^2\Big((g_b-2)m_a^3+(2g_b-3)m_a^2m_b+2(g_b-1)m_am_b^2+g_bm_b^3\Big)}{2m_a^2m_b^3(m_a+m_b)r^4}\right.\\
&\quad\left.+\frac{(Z_aZ_b\alpha)^2\hbar\Big((-3g_b^2+16g_b-18)m_a+(-3g_b^2+16g_b-4)m_b\Big)}{8\pi m_a^2m_b^3r^5}\right]\vec{L}\cdot\vec{S}_b\\
&+ {}^1V_T^{(2)}(\vec{r})\\
{}^{\frac{1}{2}\frac{1}{2}}V_C^{(2)}(\vec{r}) &\simeq \left[-\frac{(Z_aZ_b\alpha)^2(m_a+m_b)}{2m_am_br^2}-\frac{7(Z_aZ_b\alpha)^2\hbar}{6\pi m_am_br^3}\right]\chi_f^{a\dagger}\chi_i^a\,\chi_f^{b\dagger}\chi_i^b\\
&+ \left[\frac{(Z_aZ_b\alpha)^2\Big(g_am_a^3+2(g_a-1)m_a^2m_b+(2g_a-3)m_am_b^2+(g_a-2)m_b^3\Big)}{2m_a^3m_b^2(m_a+m_b)r^4}\right.\\
&\quad\left.+\frac{(Z_aZ_b\alpha)^2\hbar\Big((-3g_b^2+16g_b-4)m_a+(-3g_b^2+16g_b-18)m_b\Big)}{8\pi m_a^3m_b^2r^5}\right]SO_a\\
&+ \left[\frac{(Z_aZ_b\alpha)^2\Big((g_b-2)m_a^3+(2g_b-3)m_a^2m_b+2(g_b-1)m_am_b^2+g_bm_b^3\Big)}{2m_a^2m_b^3(m_a+m_b)r^4}\right.\\
&\quad\left.+\frac{(Z_aZ_b\alpha)^2\hbar\Big((-3g_b^2+16g_b-18)m_a+(-3g_b^2+16g_b-4)m_b\Big)}{8\pi m_a^2m_b^3r^5}\right]SO_b\\
&+ \Big[-8g_ag_bm_am_b-5g_ag_b(m_a^2+m_b^2)+2(g_am_a+g_bm_b)(m_a+m_b)\Big]\\
&\qquad\times\frac{(Z_aZ_b\alpha)^2\vec{S}_a\cdot\vec{S}_b}{4m_a^2m_b^2(m_a+m_b)r^4}\\
&+ \Big[(g_b^2+20g_b-12)g_am_a^2+(g_ag_b(g_a+g_b+32)-12(g_a+g_b))m_am_b\\
&\quad+(g_a^2+20g_a-12)g_bm_b^2\Big]\times\frac{(Z_aZ_b\alpha)^2\vec{S}_a\cdot\vec{r}\,\vec{S}_b\cdot\vec{r}\,/r^2}{8m_a^2m_b^2(m_a+m_b)r^4}\\
&+ \Big[3g_a^2g_b^2+15g_ag_b(g_a+g_b)-92g_ag_b+36(g_a+g_b)+48\Big]\\
&\qquad\times\frac{(Z_aZ_b\alpha)^2\hbar\,\vec{S}_a\cdot\vec{S}_b}{32\pi m_a^2m_b^2r^5}
\end{aligned}
$$

$$+ \left[g_a^2 g_b^2 + 14 g_a g_b (g_a + g_b) - 56 g_a g_b + 4(g_a^2 + g_b^2) + 8(g_a + g_b) + 16 \right]$$
$$\times \frac{5(Z_a Z_b \alpha)^2 \hbar \, \vec{S}_a \cdot \vec{r} \, \vec{S}_b \cdot \vec{r} \, / r^2}{64 \pi m_a^2 m_b^2 r^5} \tag{138}$$

where we have introduced the short notations $SO_a \equiv \vec{L} \cdot \vec{S}_a \chi_f^{b\dagger} \chi_i^b$ and $SO_b \equiv \chi_f^{a\dagger} \chi_i^a \, \vec{L} \cdot \vec{S}_b$.

The two-photon exchange potential exhibits a similar structure as the one-photon exchange potential, a "generalized multipole expansion". Its *classical* part of $\mathcal{O}(\hbar^0)$ starts off with a monopole-monopole piece proportional to $1/r^2$, then there is a monopole-dipole piece proportional to $\vec{L} \cdot \vec{S}/r^4$ etc. Thus the "generalized multipole expansion" of the *classical* part is similar to the multipole expansion of the one-photon exchange potential, but it has one additional power of r in the denominator. The "generalized multipole expansion" of the *quantum* $\mathcal{O}(\hbar)$ part of the two-photon exchange potential however is seen to start with a monopole-monopole piece that falls off as $1/r^3$ followed by monopole-dipole pieces that go as $\vec{L} \cdot \vec{S}/r^5$ etc.

That then suggests the interpretation of the universalities we have found for the long-distance two-photon scattering potentials and amplitudes as following from a "generalized multipole expansion" where "generalized" means that the multipole expansion of the potential does not start with a monopole-monopole term proportional to $1/r$ as for a usual multipole expansion but proportional to $1/r^n$ with $n > 1$.

It would be interesting to see if one could prove this multipole expansion scheme and thus the universalities found using low-energy theorems for Compton scattering amplitudes and combining two Compton scattering amplitudes to a two-photon exchange scattering amplitude using dispersion relations. Moreover, one could speculate that a three-photon exchange potential would exhibit a similar structure with a "generalized multipole expansion" and universalities.

References

[1] M. K. Jones *et al.* [Jefferson Lab Hall A Collaboration], Phys. Rev. Lett. **84**, 1398 (2000) [arXiv:nucl-ex/9910005].
O. Gayou *et al.*, Phys. Rev. C **64**, 038202 (2001) and Phys. Rev. Lett. **88**, 092301 (2002) [arXiv:nucl-ex/0111010].

M. E. Christy *et al.* [E94110 Collaboration], Phys. Rev. C **70**, 015206 (2004) [arXiv:nucl-ex/0401030].

[2] A. deRujula, J.M. Kaplan, and E. deRafael, Nucl. Phys. **B35**, 365 (1971).
M. Gorchtein, P. A. M. Guichon and M. Vanderhaeghen, Nucl. Phys. A **741**, 234 (2004) [arXiv:hep-ph/0404206].

[3] P. A. M. Guichon and M. Vanderhaeghen, Phys. Rev. Lett. **91**, 142303 (2003) [arXiv:hep-ph/0306007].
P. G. Blunden, W. Melnitchouk and J. A. Tjon, Phys. Rev. Lett. **91**, 142304 (2003) [arXiv:nucl-th/0306076].
A. V. Afanasev, S. J. Brodsky, C. E. Carlson, Y. C. Chen and M. Vanderhaeghen, Phys. Rev. D **72**, 013008 (2005) [arXiv:hep-ph/0502013].

[4] B. R. Holstein and J. F. Donoghue, Phys. Rev. Lett. **93**, 201602 (2004) [arXiv:hep-th/0405239].

[5] G. Feinberg and J. Sucher, Phys. Rev. D **38**, 3763 (1988) [Erratum-ibid. D **44**, 3997 (1991)].

[6] Y. Iwasaki, Prog. Theor. Phys. **46**, 1587 (1971).

[7] L. Spruch, in *Long-Range Casimir Forces: Theory and Recent Experiments in Atomic Systems*, edited by F.S. Levin and D.A. Micha (Plenum, New York, 1993).

[8] J. Sucher, Phys. Rev. D **49**, 4284 (1994).

[9] N. E. J. Bjerrum-Bohr, J. F. Donoghue and B. R. Holstein, Phys. Rev. D **67**, 084033 (2003) [Erratum-ibid. D **71**, 069903 (2005)] [arXiv:hep-th/0211072].
I. B. Khriplovich and G. G. Kirilin, J. Exp. Theor. Phys. **95**, 981 (2002) [Zh. Eksp. Teor. Fiz. **95**, 1139 (2002)] [arXiv:gr-qc/0207118].

[10] G. Feinberg and J. Sucher, Phys. Rev. D **45**, 2493 (1992).

[11] W. E. Caswell and G. P. Lepage, Phys. Lett. B **167**, 437 (1986).
G. T. Bodwin, E. Braaten and G. P. Lepage, Phys. Rev. D **51**, 1125 (1995) [Erratum-ibid. D **55**, 5853 (1997)] [arXiv:hep-ph/9407339].
P. Labelle, Phys. Rev. D **58**, 093013 (1998) [arXiv:hep-ph/9608491].

M. E. Luke and A. V. Manohar, Phys. Rev. D **55**, 4129 (1997) [arXiv:hep-ph/9610534].
A. V. Manohar, Phys. Rev. D **56**, 230 (1997) [arXiv:hep-ph/9701294].
B. Grinstein and I. Z. Rothstein, Phys. Rev. D **57**, 78 (1998) [arXiv:hep-ph/9703298].
M. E. Luke and M. J. Savage, Phys. Rev. D **57**, 413 (1998) [arXiv:hep-ph/9707313].
A. Pineda and J. Soto, Nucl. Phys. Proc. Suppl. **64**, 428 (1998) [arXiv:hep-ph/9707481].
M. Beneke and V. A. Smirnov, Nucl. Phys. B **522**, 321 (1998) [arXiv:hep-ph/9711391].

[12] A. V. Manohar and I. W. Stewart, Phys. Rev. D **62**, 074015 (2000) [arXiv:hep-ph/0003032].

[13] A. V. Manohar and I. W. Stewart, Phys. Rev. D **62**, 014033 (2000) [arXiv:hep-ph/9912226].
A. V. Manohar and I. W. Stewart, Phys. Rev. D **63**, 054004 (2001) [arXiv:hep-ph/0003107].
A. V. Manohar and I. W. Stewart, Phys. Rev. Lett. **85**, 2248 (2000) [arXiv:hep-ph/0004018].
A. V. Manohar, J. Soto and I. W. Stewart, Phys. Lett. B **486**, 400 (2000) [arXiv:hep-ph/0006096].
A. H. Hoang, A. V. Manohar, I. W. Stewart and T. Teubner, Phys. Rev. Lett. **86**, 1951 (2001) [arXiv:hep-ph/0011254].
A. H. Hoang, A. V. Manohar and I. W. Stewart, Phys. Rev. D **64**, 014033 (2001) [arXiv:hep-ph/0102257].
A. H. Hoang, A. V. Manohar, I. W. Stewart and T. Teubner, Phys. Rev. D **65**, 014014 (2002) [arXiv:hep-ph/0107144].
A. V. Manohar and I. W. Stewart, Nucl. Phys. Proc. Suppl. **94**, 130 (2001) [AIP Conf. Proc. **618**, 381 (2002)] [arXiv:hep-lat/0012002].

[14] A. Pineda and J. Soto, Nucl. Phys. Proc. Suppl. **64**, 428 (1998) [arXiv:hep-ph/9707481].

[15] A. Pineda and J. Soto, Phys. Rev. D **59**, 016005 (1999) [arXiv:hep-ph/9805424].

[16] I.J.R. Aitchison and A.J.G. Hey, **Gauge Theories in Particle Physics**, Adam Hilger, Philadelphia (1989) Ch. 8.

[17] B. R. Holstein, Am. J. Phys. **74**, 1104 (2006) [arXiv:hep-ph/0607187].

[18] See, *e.g.*, B.R. Holstein, **Weak Interactions in Nuclei**, Princeton Univ. Press, Princeton (1988), Ch. 3.

[19] S. Weinberg, Phys. Rev. **140**, B516 (1965).

[20] B.R. Holstein, A. Ross, arXiv:0802.0716 [hep-ph].

[21] B.R. Holstein, A. Ross, arXiv:0802.0717 [hep-ph].

[22] P. Van Nieuwenhuizen, Nucl. Phys. B **28**, 429 (1971).

[23] R.H. Dalitz, Proc. Roy. Soc. London A206, 509 (1951).

[24] L. D. Landau and E. M. Lifshitz, Quantum Mechanics, 2. Ed., Pergamon Press, 1965.

[25] H. Aronson, Phys. Rev. **186**, 1434 (1969).
K. Hagiwara, R. D. Peccei, D. Zeppenfeld and K. Hikasa, Nucl. Phys. B **282**, 253 (1987).
U. Baur and D. Zeppenfeld, Nucl. Phys. B **308**, 127 (1988).