University of Massachusetts - Amherst
ScholarWorks@UMass Amherst

Physics Department Faculty Publication Series | Physics

2008

Spin Effects in Long Range Gravitational Scattering

BR Holstein
University of Massachusetts Amherst, holstein@physics.umass.edu

Andreas Ross
University of Massachusetts - Amherst

Follow this and additional works at: http://scholarworks.umass.edu/physics_faculty_pubs
Part of the Physical Sciences and Mathematics Commons

Recommended Citation
Holstein, BR and Ross, Andreas, "Spin Effects in Long Range Gravitational Scattering" (2008). *Physics Department Faculty Publication Series*. 521.
http://scholarworks.umass.edu/physics_faculty_pubs/521

This Article is brought to you for free and open access by the Physics at ScholarWorks@UMass Amherst. It has been accepted for inclusion in Physics Department Faculty Publication Series by an authorized administrator of ScholarWorks@UMass Amherst. For more information, please contact scholarworks@library.umass.edu.

arXiv:0802.0716v1 [hep-ph] 5 Feb 2008

Spin Effects in Long Range Gravitational Scattering

Barry R. Holstein[a*] and Andreas Ross[a,b†]

[a] Department of Physics – LGRT
University of Massachusetts
Amherst, MA 01003, USA

[b] Department of Physics
Yale University
New Haven, CT 06520, USA

February 8, 2008

Abstract

We study the gravitational scattering of massive particles with and without spin in the effective theory of gravity at one loop level. Our focus is on long distance effects arising from nonanalytic components of the scattering amplitude and we show that the spin-independent and the spin-dependent long range components exhibit a universal form. Both classical and quantum corrections are obtained, and the definition of a proper second order potential is discussed.

*holstein@physics.umass.edu
†andreas.ross@yale.edu

1 Introduction

The gravitational interaction of two massive particles is described by Newton's law

$$V(r) = -\frac{Gm_a m_b}{r} \tag{1}$$

which is a good approximation for the motion of nonrelativistic particles at large separations. Einstein's theory of general relativity, however, predicts important corrections that have been verified experimentally, *e.g.*, in measurements of the precession of the perihelion of Mercury [1]. This effect can be calculated from the famous Einstein-Infeld-Hoffmann Lagrangian [2] where the terms beyond Newtonian physics stem from the relativistic $\mathcal{O}(v^2)$ corrections to the kinetic energy, from relativistic $\mathcal{O}(v^2)$ corrections to the Newtonian potential and from a new piece of the potential proportional to G^2M^3/r^2 which may be regarded as an $\mathcal{O}(GM/r)$ correction to the leading Newtonian potential. The two small expansion parameters here are v^2 and GM/r which for bound states are related due to the virial theorem. We, however, will examine scattering processes wherein the classical expansion parameters v^2 and GM/r are independent, and our focus will be on the components that vanish in the nonrelativistic limit $v \to 0$.

In the weak field limit gravitational dynamics can be described in terms of a quantum field theory which is based on expanding around flat Minkowski spacetime and quantizing the metric fluctuation in terms of a massless spin-2 field—the graviton. The resulting theory is an effective field theory whose predictions are trustworthy only for energies much smaller than the Planck scale. Nevertheless, one can in this picture calculate well-defined quantum predictions as well as classical corrections to Newtonian physics. Donoghue has pioneered the use of the effective field theory of gravity to extract the leading long distance effects in the nonrelativistic limit [3, 4] (see [5] and [6] for reviews) which yield corrections of the form

$$V(r) = -\frac{Gm_a m_b}{r}\left(1 + A_C\,\frac{GM}{r} + A_Q\,\frac{G\hbar}{r^2}\right) \tag{2}$$

where A_C and A_Q are the coefficients of the classical and quantum corrections respectively and are evaluated below. Note that while the classical expansion parameter $GM/r \sim M/M_{Pl} \times \ell_{Pl}/r$ can give measurable effects for macroscopic objects (where a large factor multiplies a tiny factor), the quantum corrections which scale as $G\hbar/r^2 \sim (\ell_{Pl}/r)^2$ are clearly tiny and phenomenologically irrelevant for macroscopic systems. The calculations of these leading long distance corrections are performed by evaluating the leading nonanalytic components of the scattering amplitude at one loop level, where $\mathcal{O}(G^2)$ effects first arise, and by defining a potential in terms of the Fourier transform of this amplitude.

Once spin is introduced into the calculation, additional interaction structures arise, such as a spin-orbit coupling and a spin-spin coupling. The leading effects of order G stemming from the tree level one-graviton exchange process give rise to the familiar geodetic precession and to the Lense-Thirring effect [1]. At the two-graviton exchange level, classical $\mathcal{O}(GM/r)$ and quantum $\mathcal{O}(G\hbar/r^2)$ corrections to these leading spin-dependent interactions arise and are calculated as part of our work.

In earlier work on loop corrections to the form factors of graviton couplings it was found that the long distance corrections to the spin-independent interactions have the same form for scalars, spin-1/2 fermions and for spin-1 bosons [7, 8]. These NLO form factor interactions constitute one component of the calculation of the full scattering amplitude at the two-graviton exchange level, so the question arises as to whether the *full* scattering amplitude also exhibits such universalities, whereby the form of the corrections to the spin-independent Newtonian potential is independent of the spins of the scattered particles — in other words, are the coefficients A_C and A_Q in Eq. (2) independent of spin?

This then is the goal of the present work — to evaluate the gravitational scattering of two massive particles having various spins, in order to check previous work and to verify the universality hypothesis. In the next section then we review our calculational methods and reproduce the long range gravitational scattering amplitude in the case of a pair of spinless particles. In the following sections, we generalize these methods to the cases of spin-0 – spin-1/2, spin-0 – spin-1 and spin-1/2 – spin-1/2 scattering and demonstrate both the expected universality as well as novel spin-dependent interactions. We summarize our work and draw general implications in a brief concluding section. Appendix A encapsulates parts of our calculational methods while we refer to our companion paper on long distance effects in electromagnetic scattering [9] for many of the details such as the required Feynman integrals, the Fourier transformations etc. In Appendix B we demonstrate how the classical equations of motion in the form of the Einstein-Infeld-Hoffmann Lagrangian can be extracted from our results.

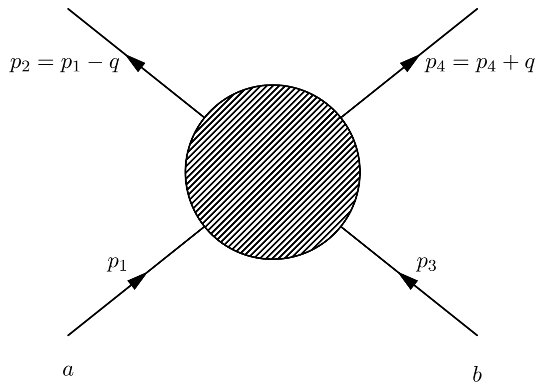


Figure 1: Basic kinematics of gravitational scattering.

2 Spin-Independent Scattering

We first set the kinematic framework for our study. We consider the gravitational scattering of two non-identical particles—particle a with mass m_a, and incoming four-momentum p_1 and particle b with mass m_b, and incoming four-momentum p_3. After interacting the final four-momentum of particle a is $p_2 = p_1 - q$ and that of particle b is $p_4 = p_3 + q$—*cf.* Fig. 1. Now we need to be more specific.

2.1 Spin-0 – Spin-0 Scattering

We begin by examining the gravitational scattering of two spinless particles. The gravitational coupling of a spin-0 particle is found by expanding the minimally coupled scalar field matter Lagrangian

$$\sqrt{-g}\mathcal{L}_m = \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right) \tag{3}$$

in terms of the gravitational field $h_{\mu\nu}$ which is a small fluctuation of the metric about flat Minkowski space defined as

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{4}$$

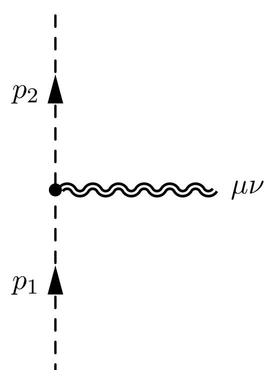
with $\kappa = \sqrt{32\pi G} \propto 1/M_P$. The inclusion of this factor κ in the definition of the graviton field $h_{\mu\nu}$ gives this field a mass-dimension of unity and thus yields a kinetic term of standard normalization without a dimensionful parameter. For matter interactions, this choice is convenient since the order of κ keeps track of the number of gravitons involved in an interaction. Once the action is written in terms of the expansion of the graviton field, all indices are understood to be lowered or raised using the Minkowski metric $\eta_{\mu\nu}$. We also require the expansion of the inverse metric and square root of the determinant of the metric tensor—

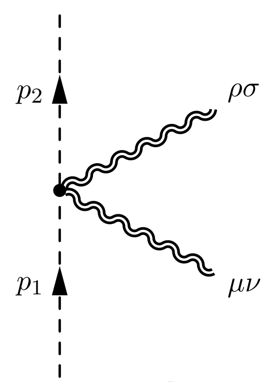
$$\begin{aligned} g^{\mu\nu} &= \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\alpha} h^{\nu}_{\alpha} + \mathcal{O}(\kappa^3) \\ \sqrt{-g} &= 1 + \frac{\kappa}{2} h + \frac{\kappa^2}{8}\left(h^2 - 2h_{\mu\nu}h^{\mu\nu}\right) + \mathcal{O}(\kappa^3). \end{aligned} \tag{5}$$

Then, expanding in powers of κ, we find

$$\begin{aligned} \sqrt{-g}\mathcal{L}_m^{(0)} &= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \\ \sqrt{-g}\mathcal{L}_m^{(1)} &= \frac{\kappa}{2} h^{\mu\nu}\left[\eta_{\mu\nu}\left(\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - \frac{1}{2}m^2\phi^2\right) - \partial_\mu\phi\partial_\nu\phi\right] \\ \sqrt{-g}\mathcal{L}_m^{(2)} &= \frac{\kappa^2}{2}\left[\frac{1}{4}\Big(h^2 - 2h_{\mu\nu}h^{\mu\nu}\Big)\Big(\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - \frac{1}{2}m^2\phi^2\Big) \right. \\ &\qquad \left. + \Big(h^{\mu\alpha}h^{\nu}_{\alpha} - \frac{1}{2}hh^{\mu\nu}\Big)\partial_\mu\phi\partial_\nu\phi\right] \end{aligned} \tag{6}$$

where $h \equiv \eta^{\alpha\beta}h_{\alpha\beta}$ represents the trace and the one- and two-graviton vertices are identified as





$$\begin{aligned} {}^0\tau^{(1)}_{\mu\nu}(p_2, p_1, m) &= \frac{-i\kappa}{2}\left[p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu} - \eta_{\mu\nu}(p_1\cdot p_2 - m^2)\right] \\ {}^0\tau^{(2)}_{\mu\nu,\rho\sigma}(p_2, p_1, m) &= \frac{i\kappa^2}{2}\Big[2I_{\mu\nu,\kappa\zeta}I^{\zeta}{}_{\lambda,\rho\sigma}(p_1^\kappa p_2^\lambda + p_1^\lambda p_2^\kappa) \\ &\qquad - (\eta_{\mu\nu}I_{\kappa\lambda,\rho\sigma} + \eta_{\rho\sigma}I_{\kappa\lambda,\mu\nu})p_1^\kappa p_2^\lambda - P_{\mu\nu,\rho\sigma}(p_1\cdot p_2 - m^2)\Big] \end{aligned} \tag{7}$$

where we have defined

$$\begin{aligned}
I_{\alpha\beta,\gamma\delta} &\equiv \frac{1}{2}(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma}) \\
P_{\alpha\beta,\gamma\delta} &\equiv \frac{1}{2}(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma} - \eta_{\alpha\beta}\eta_{\gamma\delta}).
\end{aligned} \tag{8}$$

The purely gravitational dynamics are derived from the Einstein-Hilbert action

$$S_{GR} = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\, R \tag{9}$$

where higher dimensional operators such as R^2, which are needed when performing renormalization, are not relevant here [3, 4]. Since general relativity is invariant under *local* coordinate transformations, it is a gauge theory and one must deal with the subtleties that arise in the quantization of gauge theories—we have to perform gauge fixing. The procedure works analogously to gauge fixing in Yang-Mills theories. Furthermore, we will make use of the background field method, which provides a powerful organizational scheme for the quantization of the effective field theory of gravity: Keeping the background metric general instead of restricting to the flat Minkowski metric, we use the expansion

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}. \tag{10}$$

where $\bar{g}_{\mu\nu}$ is the classical background metric (or field) and $h_{\mu\nu}$ is the quantum field. A gauge fixing condition is only imposed on the quantum field $h_{\mu\nu}$, leaving the general covariance of the background unaffected. This procedure has the advantage of ensuring that the resulting theory can be renormalized, since the loop expansion then has the same symmetry properties as the action.

Moreover, the background field method greatly simplifies calculations involving graviton loops. Gravitons running in loops must be derived from from an expansion involving the quantum field $h_{\mu\nu}$ whereas gravitons that are *not* within a loop may be derived from expanding the background field

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + \kappa H_{\mu\nu} \tag{11}$$

where $H_{\mu\nu}$ denotes an "external" graviton, *i.e.*, a graviton that is not inside a loop. At the one loop level, at most two gravitons involved in a vertex are propagating within a loop, which allows us to use a gauge fixing condition *linear* in the quantum field $h_{\mu\nu}$ and greatly simplifies the derivation of both the triple graviton vertex and the vertex that couples one graviton to the ghost fields.

In order to fix the gauge, we will use the harmonic gauge—

$$\bar{D}^{\nu} h_{\mu\nu} - \frac{1}{2}\bar{D}_{\mu} h = 0 \tag{12}$$

where $\bar{D}_{\mu}$ denotes the covariant derivative on the background metric. This condition leads to an additional gauge fixing piece of the action

$$S_{GF} = \int d^4x \sqrt{-\bar{g}} \left(\bar{D}^{\nu} h_{\mu\nu} - \frac{1}{2}\bar{D}_{\mu} h\right)\left(\bar{D}_{\rho} h^{\mu\rho} - \frac{1}{2}\bar{D}^{\mu} h\right) \tag{13}$$

as well as the ghost action

$$S_{Ghost} = \int d^4x \sqrt{-\bar{g}}\, \bar{\eta}^{\mu} \left(\bar{D}_{\mu}\bar{D}_{\nu} - R_{\mu\nu}\right) \eta^{\nu} \tag{14}$$

where η^{μ} is the ghost field that annihilates a ghost particle while $\bar{\eta}^{\mu}$ creates a ghost particle.[1]

Now we are in the position to derive the Feynman rules for the effective field theory of gravity. The complete quantum gravitational action consists of three components

$$S_{Grav} = S_{GR} + S_{GF} + S_{Ghost}, \tag{15}$$

and we will illustrate the use of the background field method during the derivations of the Feynman rules. When deriving the graviton propagator we expand the action to second order in the quantum fields $h_{\mu\nu}$ and to zeroth order in the external gravitons $H_{\mu\nu}$ and the ghosts, yielding

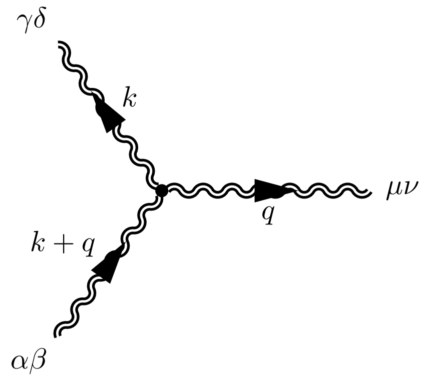
$$D_{\alpha\beta,\gamma\delta}(q) = \frac{iP_{\alpha\beta,\gamma\delta}}{q^2}. \tag{16}$$

We obtain the ghost propagator expanding the action to second order in the ghost fields and to zeroth order in the gravitons, whereby

$$D_{\mu\nu}(q) = \frac{i\eta_{\mu\nu}}{q^2}. \tag{17}$$

Our triple graviton vertex is obtained by expanding to second order in the quantum fields $h_{\mu\nu}$, to first order in the background gravitons $H_{\mu\nu}$, and to zeroth order in the ghost fields and reads

[1]Note that the ghost fields anticommute since they obey Fermi-Dirac statistics and we have to include a factor of (-1) for each closed ghost loop.

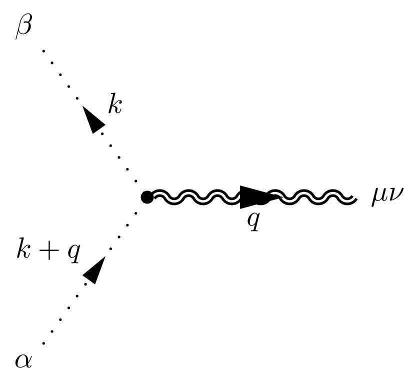


$$\tau^{\mu\nu}_{\alpha\beta,\gamma\delta}(k,q) = -\frac{i\kappa}{2}\Big\{P_{\alpha\beta,\gamma\delta}\Big[k^\mu k^\nu + (k+q)^\mu(k+q)^\nu + q^\mu q^\nu - \frac{3}{2}\eta^{\mu\nu}q^2\Big]$$
$$+2q_\lambda q_\sigma\Big[I^{\lambda\sigma,}{}_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta} + I^{\lambda\sigma,}{}_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta}$$
$$-I^{\lambda\mu,}{}_{\alpha\beta}I^{\sigma\nu,}{}_{\gamma\delta} - I^{\sigma\nu,}{}_{\alpha\beta}I^{\lambda\mu,}{}_{\gamma\delta}\Big]$$
$$+\Big[q_\lambda q^\mu(\eta_{\alpha\beta}I^{\lambda\nu,}{}_{\gamma\delta} + \eta_{\gamma\delta}I^{\lambda\nu,}{}_{\alpha\beta}) + q_\lambda q^\nu(\eta_{\alpha\beta}I^{\lambda\mu,}{}_{\gamma\delta} + \eta_{\gamma\delta}I^{\lambda\mu,}{}_{\alpha\beta})$$
$$-q^2(\eta_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta} + \eta_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta}) - \eta^{\mu\nu}q^\lambda q^\sigma(\eta_{\alpha\beta}I_{\gamma\delta,\lambda\sigma} + \eta_{\gamma\delta}I_{\alpha\beta,\lambda\sigma})\Big]$$
$$+\Big[2q^\lambda\Big(I^{\sigma\nu,}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\mu + I^{\sigma\mu,}{}_{\gamma\delta}I_{\alpha\beta,\lambda\sigma}k^\nu$$
$$-I^{\sigma\nu,}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k+q)^\mu - I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\lambda\sigma}(k+q)^\nu\Big)$$
$$+q^2(I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\nu + I_{\alpha\beta,\sigma}{}^\nu I^{\sigma\mu,}{}_{\gamma\delta})$$
$$+\eta^{\mu\nu}q^\lambda q_\sigma(I_{\alpha\beta,\lambda\rho}I^{\rho\sigma,}{}_{\gamma\delta} + I_{\gamma\delta,\lambda\rho}I^{\rho\sigma,}{}_{\alpha\beta})\Big]$$
$$+\Big[(k^2 + (k+q)^2)\Big(I^{\sigma\mu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\nu + I^{\sigma\nu,}{}_{\alpha\beta}I_{\gamma\delta,\sigma}{}^\mu - \frac{1}{2}\eta^{\mu\nu}P_{\alpha\beta,\gamma\delta}\Big)$$
$$-((k+q)^2\eta_{\alpha\beta}I^{\mu\nu,}{}_{\gamma\delta} + k^2\eta_{\gamma\delta}I^{\mu\nu,}{}_{\alpha\beta})\Big]\Big\} \qquad (18)$$

where the graviton with Lorentz indices $\mu\nu$ represents a background graviton, and therefore, it is *not* to be used within any loop![2] The final missing Feynman rule from the purely gravitational action is the vertex coupling one

[2]For each closed graviton bubble loop, we also have to include a symmetry factor of 1/2! in the amplitude.

graviton to two ghost fields which we derive by expanding to second order in the ghost fields and to first order in $H_{\mu\nu}$—



$$\tau^{\mu\nu}_{\alpha,\beta}(k,q) = \frac{i\kappa}{2}\Big[(k^2+(k+q)^2+q^2)I_{\alpha\beta,}{}^{\mu\nu} + 2\eta_{\alpha\beta}k^\lambda(k+q)^\sigma P_{\lambda\sigma,}{}^{\mu\nu} + 2q_\alpha k^\lambda I_{\beta\lambda,}{}^{\mu\nu} - 2q_\beta(k+q)^\lambda I_{\alpha\lambda,}{}^{\mu\nu} + q_\alpha q_\beta \eta^{\mu\nu}\Big] \tag{19}$$

where the graviton with Lorentz indices $\mu\nu$ is again a background graviton and not to be used as part of any loop.

We begin our calculation with the simplest tree level single-graviton exchange which leads to the amplitude[3]

$$\begin{aligned} {}^0\mathcal{M}^{(1)}(q) &= \frac{-i}{\sqrt{2E_1 2E_2 2E_3 2E_4}}\,\tau^{(1a)}_{\mu\nu}(p_2,p_1)\,\frac{iP^{\mu\nu,\alpha\beta}}{q^2}\,\tau^{(1b)}_{\alpha\beta}(p_4,p_3) \\ &= \frac{-8\pi G}{\sqrt{2E_1 2E_2 2E_3 2E_4}}\left[\frac{(s-m_a^2-m_b^2+\frac{1}{2}q^2)^2 - 2m_a^2 m_b^2 - \frac{1}{4}q^4}{q^2}\right]. \end{aligned} \tag{20}$$

A convenient way to define the nonrelativistic potential is as the Fourier transform of the nonrelativistic center of mass scattering amplitude. We utilize a symmetric center of mass frame[4] with incoming momenta $\vec{p}_1 = \vec{p}+\vec{q}/2$ and $\vec{p}_3 = -\vec{p}_1 = -\vec{p}-\vec{q}/2$ and with outgoing momenta $\vec{p}_2 = \vec{p}-\vec{q}/2$

[3]Our normalization of the amplitude is a nonrelativistic one such that after applying all Feynman rules we divide the amplitude by a factor of $\sqrt{2E_1 2E_2 2E_3 2E_4}$.

[4]These symmetric momentum labels of the center of mass frame are chosen so that the leading order coordinate space potential is real in the calculation of spin-0 – spin-1 scattering presented below.

and $\vec{p}_4 = -\vec{p} + \vec{q}/2$. Conservation of energy then requires $\vec{p}\cdot\vec{q} = 0$ so that $\vec{p}_i^{\,2} = \vec{p}^{\,2} + \vec{q}^{\,2}/4$ for $i = 1,2,3,4$ and $q^2 = -\vec{q}^{\,2}$. In the nonrelativistic limit—$\vec{q}^{\,2}, \vec{p}^{\,2} \ll m^2$ —the lowest order amplitude reads

$$\begin{aligned} {}^0\mathcal{M}^{(1)}(\vec{q}) &\simeq \frac{4\pi G m_a m_b}{\vec{q}^{\,2}}\left[1 + \frac{\vec{p}^{\,2}}{m_a m_b}\left(1 + \frac{3(m_a+m_b)^2}{2m_a m_b}\right) + \ldots\right] \\ &+ G\pi\left[\frac{3(m_a^2+m_b^2)}{2m_a m_b} + \frac{\vec{p}^{\,2}}{m_a m_b}\left(3 - \frac{5(m_a^2+m_b^2)^2}{4m_a^2 m_b^2}\right) + \ldots\right] + \ldots \end{aligned} \tag{21}$$

yielding the potential

$$\begin{aligned} {}^0V_G^{(1)}(\vec{r}) &= -\int \frac{d^3q}{(2\pi)^3}\, {}^0\mathcal{M}^{(1)}(\vec{q})\, e^{-i\vec{q}\cdot\vec{r}} \\ &= -\frac{G m_a m_b}{r}\left[1 + \frac{\vec{p}^{\,2}}{m_a m_b}\left(1 + \frac{3(m_a+m_b)^2}{2m_a m_b}\right) + \ldots\right] \\ &\quad + G\pi\delta^3(\vec{r})\left[\frac{3(m_a^2+m_b^2)}{2m_a m_b} + \frac{\vec{p}^{\,2}}{m_a m_b}\left(3 - \frac{5(m_a^2+m_b^2)^2}{4m_a^2 m_b^2}\right) + \ldots\right] \end{aligned} \tag{22}$$

The leading component of Eq. (22) is recognized as the usual Newtonian potential (accompanied by a small kinematic correction) while the second piece is a short range modification.

Our purpose in this paper is to study the long distance corrections to this lowest order potential which arise from the two-graviton exchange diagrams shown in Fig. 2. This problem has been previously studied by Iwasaki using noncovariant perturbation theory [10], and by Khriplovich and Kirilin [11, 12] and by Bjerrum-Bohr, Donoghue, and Holstein [13] using conventional Feynman diagrams. Our approach will be similar to that used in [11, 12] and [13]. That is, using the ideas of effective field theory, we evaluate these Feynman diagrams by keeping only the leading nonanalytic structure in q^2, since it is these pieces that lead to the long range corrections to the potential while components analytic in q^2 only yield short range contributions, *i.e.*, delta functions or derivatives of delta functions. The leading nonanalytic behavior is of two basic forms

i) terms in $1/\sqrt{-q^2}$ which are $\hbar$-independent and therefore classical

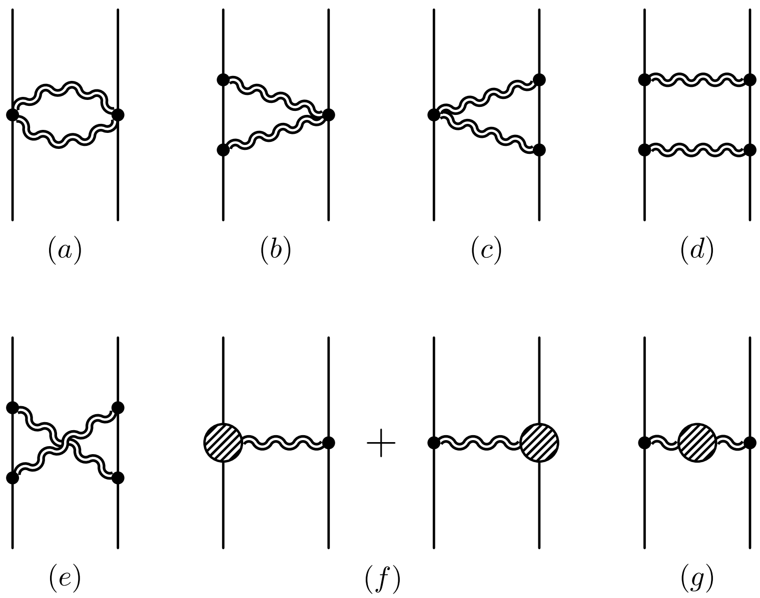


Figure 2: One loop diagrams of gravitational scattering.

ii) terms in $\log -q^2$ which are $\hbar$-dependent and therefore quantum mechanical.

The former terms, when Fourier transformed lead to corrections to the non-relativistic potential of the form $V_{classical}(r) \sim 1/r^2$ while the latter lead to $V_{quantum}(r) \sim \hbar/mr^3$ corrections. For typical masses and separations the quantum mechanical forms are themselves numerically insignificant. However, they are intriguing in that their origin appears to be associated with zitterbewegung. That is, classically we can define the potential by measuring the energy when two objects are separated by distance r. However, in the quantum mechanical case the distance between two objects is uncertain by an amount of order the Compton wavelength due to zero point motion—$\delta r \sim \hbar/m$. This leads to the replacement

$$V(r) \sim \frac{1}{r^2} \longrightarrow \frac{1}{(r \pm \delta r)^2} \sim \frac{1}{r^2} \mp 2\frac{\hbar}{mr^3}$$

which is the form found in our calculations.

The diagrams to be evaluated are shown in Fig. 2 where the "blobs" shown in diagrams (f) and (g) are explained in Fig. 3. The calculational

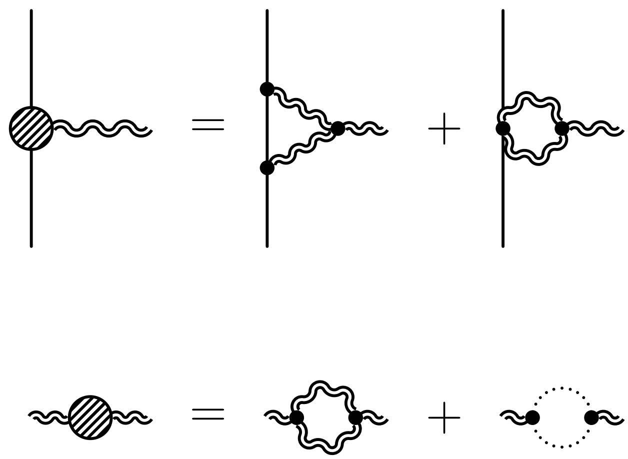


Figure 3: Loop corrections subsumed in vertex and in vacuum polarization functions.

details including the Feynman integrals are described in an appendix of our companion paper on electromagnetic scattering [9]. Here we present only the results. Defining[5]

$$S = \frac{\pi^2}{\sqrt{-q^2}} \qquad \text{and} \qquad L = \log -q^2$$

we have, from diagrams 2(a)-(g) respectively

$$\begin{aligned}
{}^0\mathcal{M}^{(2)}_{2a}(q) &= G^2 m_a m_b \left[-44L\right] \\
{}^0\mathcal{M}^{(2)}_{2b}(q) &= G^2 m_a m_b \left[28L + 8m_a S\right] \\
{}^0\mathcal{M}^{(2)}_{2c}(q) &= G^2 m_a m_b \left[28L + 8m_b S\right] \\
{}^0\mathcal{M}^{(2)}_{2d}(q) &= G^2 m_a m_b \left[\left(\frac{4m_a m_b}{q^2} + \frac{8(m_a^2 + m_b^2)}{m_a m_b} - 8\right)L + 4(m_a + m_b)S\right] \\
&- i4\pi G^2 m_a^2 m_b^2 \frac{L}{q^2}\sqrt{\frac{m_a m_b}{s - s_0}} \\
{}^0\mathcal{M}^{(2)}_{2e}(q) &= G^2 m_a m_b \left[\left(-\frac{4m_a m_b}{q^2} - \frac{8(m_a^2 + m_b^2)}{m_a m_b} - \frac{70}{3}\right)L - 4(m_a + m_b)S\right]
\end{aligned}$$

[5]Note that our definition of S differs slightly from previous publications [3, 4, 13, 7] in that it does not include a factor of mass in the numerator.

$$
\begin{aligned}
{}^0\mathcal{M}_{2_f}(q) &= G^2 m_a m_b \left[14L - 2(m_a+m_b)S\right] \\
{}^0\mathcal{M}_{2_g}(q) &= G^2 m_a m_b \left[-\frac{43}{15}L\right]
\end{aligned}
\tag{23}
$$

where $s = (p_1+p_3)^2$ is the square of the center of mass energy and $s_0 = (m_a + m_b)^2$ is its threshold value. The contributions from the form factor diagrams in Fig. 2(f) have been calculated both directly and using the results from [7, 8], and the contribution from vacuum polarization diagrams in Fig. 2(g) have been obtained both by direct evaluation and by using previous results of 't Hooft and Veltman [14] for the divergences of the graviton self-energy that allow us to infer its nonanalytic contributions[6]. Here we should note that *any* massless particle would contribute to the nonanalytic terms of the graviton vacuum energy since gravitons couple to anything. We choose here not to include other massless particles besides the graviton—in particular we do not include the contribution of the photon in the quantum piece of our potential.

Summing, we find the total result

$$
{}^0\mathcal{M}^{(2)}_{tot}(q) = G^2 m_a m_b \left[6(m_a+m_b)S - \frac{41}{5}L\right] - i4\pi G^2 m_a^2 m_b^2 \frac{L}{q^2}\sqrt{\frac{m_a m_b}{s-s_0}} \tag{24}
$$

and we observe that, in addition to the expected terms involving L and S, there arises a piece of the second order amplitude which is *imaginary*. The origin of this imaginary piece is, of course, from the second Born approximation to the Newtonian potential, and reminds us that in order to define a proper correction to the first order Newtonian potential we must subtract off such terms. For this purpose we will work in the nonrelativistic limit and the center of mass frame—$\vec{p}_1 + \vec{p}_3 = 0$ as defined above. We have then

$$
s - s_0 = 2\sqrt{m_a^2 + \vec{p}_1^{\,2}}\sqrt{m_b^2 + \vec{p}_1^{\,2}} + 2\vec{p}_1^{\,2} - 2m_a m_b \tag{25}
$$

and

$$
\sqrt{\frac{m_a m_b}{s-s_0}} \simeq \frac{m_r}{p_0} \tag{26}
$$

where $m_r = m_a m_b/(m_a+m_b)$ is the reduced mass and $p_0 \equiv |\vec{p}_i|,\ i = 1,2,3,4$. The transition amplitude Eq. (24) then assumes the form

$$
{}^0\mathcal{M}^{(2)}_{tot}(\vec{q}) \simeq G^2 m_a m_b \left[6(m_a+m_b)S - \frac{41}{5}L\right] - i4\pi G^2 m_a^2 m_b^2 \frac{L}{q^2}\frac{m_r}{p_0}. \tag{27}
$$

[6]In a massless theory the divergences of dimensional regularization are accompanied by logarithms of the momentum transfer.

For the iteration we shall use the simple potential

$$ {}^0V_G^{(1)}(\vec{r}) = -\frac{Gm_am_b}{r} \tag{28} $$

which reproduces the long distance behavior of the lowest order amplitude for spin-0 – spin-0 gravitational scattering—Eq. (22)—in the nonrelativistic limit. We will employ the momentum space representation

$$ {}^0V_G^{(1)}(\vec{q}) \equiv \left\langle \vec{p}_f \left| {}^0\hat{V}_G^{(1)} \right| \vec{p}_i \right\rangle = -\frac{4\pi Gm_am_b}{\vec{q}^2} = -\frac{4\pi Gm_am_b}{(\vec{p}_i - \vec{p}_f)^2} \tag{29} $$

where we identify $\vec{p}_i = \vec{p}_1$ and $\vec{p}_f = \vec{p}_2$, and the second Born term is then

$$ \begin{aligned} {}^0\mathrm{Amp}_G^{(2)}(\vec{q}) &= -\int \frac{d^3\ell}{(2\pi)^3} \frac{\left\langle \vec{p}_f \left| {}^0\hat{V}_G^{(1)} \right| \vec{\ell} \right\rangle \left\langle \vec{\ell} \left| {}^0\hat{V}_G^{(1)} \right| \vec{p}_i \right\rangle}{E(p_0) - E(\ell) + i\epsilon} \\ &= i \int \frac{d^3\ell}{(2\pi)^3} {}^0V_G^{(1)}(\vec{\ell} - \vec{p}_f)\, G^{(0)}(\vec{\ell})\, {}^0V_G^{(1)}(\vec{p}_i - \vec{\ell}) \end{aligned} \tag{30} $$

where

$$ G^{(0)}(\ell) = \frac{i}{\frac{p_0^2}{2m_r} - \frac{\ell^2}{2m_r} + i\epsilon} \tag{31} $$

is the nonrelativistic propagator. Note that in Eq. (30) we take both the leading order potential as well as the total energies $E(p_0)$ and $E(\ell)$ in the nonrelativistic limit. The remaining integration can be performed exactly, yielding[7]

$$ \begin{aligned} {}^0\mathrm{Amp}_G^{(2)}(\vec{q}) &= i \int \frac{d^3\ell}{(2\pi)^3} \frac{-4\pi Gm_am_b}{|\vec{\ell} - \vec{p}_2|^2 + \lambda^2} \frac{i}{\frac{p_0^2}{2m_r} - \frac{\ell^2}{2m_r} + i\epsilon} \frac{-4\pi Gm_am_b}{|\vec{p}_1 - \vec{\ell}|^2 + \lambda^2} \\ &= H = -i4\pi G^2 m_a^2 m_b^2 \frac{L}{q^2} \frac{m_r}{p_0} \end{aligned} \tag{32} $$

which precisely reproduces the imaginary component of ${}^0\mathcal{M}_{tot}^{(2)}(\vec{q})$, as expected. In order to produce a properly defined second order potential ${}^0V_G^{(2)}(\vec{r})$ we must then subtract this second order Born term from the second order

[7]Note that the iteration integrals are listed in Appendix A

transition amplitude, yielding the result

$$\begin{aligned}
{}^0V_G^{(2)}(\vec{r}) &= -\int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \left[{}^0\mathcal{M}_{tot}^{(2)}(q) - {}^0\mathrm{Amp}_G^{(2)}(q)\right] \\
&= \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} G^2 m_a m_b \left[-6S(m_a+m_b) + \frac{41}{5}L\right] \\
&= -\frac{3G^2 m_a m_b (m_a+m_b)}{r^2} - \frac{41 G^2 m_a m_b \hbar}{10\pi r^3}
\end{aligned} \tag{33}$$

The quantum mechanical—$\sim \hbar/r^3$—component of the second order potential given in Eq. (33) agrees with that previously given by Bjerrum-Bohr, Donoghue, and Holstein [13] and by Kirilin and Khriplovich [12]. However, the classical—$\sim 1/r^2$—contribution quoted by Iwasaki

$${}^0V_{IW}^{(2)}(\vec{r}) = \frac{G^2 m_a m_b (m_a+m_b)}{2r^2}. \tag{34}$$

differs from that quoted above in Eq. (33) and by Bjerrum-Bohr et al. in [13]. The resolution of this issue was given by Sucher, who pointed out that the classical term depends upon the precise definition of the first order potential used in the iteration [15]. Moreover, it depends and on whether one uses relativistic forms of the leading order potentials and the propagator $G^{(0)}(\ell)$ in the iteration. In modern terms, the potential depends on how one performs the matching—*e.g.*, Iwasaki [10] performs an off-shell matching while we match on-shell. In our companion paper on electromagnetic scattering [9] we have provided a more detailed discussion of these ambiguities[8]. Use of the simple lowest order form Eq. (29) within a nonrelativistic iteration yields our result for the iteration amplitude given in Eq. (32) and is sufficient to remove the offending imaginary piece of the scattering amplitude. In Appendix B we derive an alternative form of the $\mathcal{O}(G^2)$ classical potential which results from an iteration that includes the leading relativistic corrections and which reproduces the classical equations of motion.

Therefore, a unique definition of the potential does not exist. Of course, the ambiguities in the form of the second order classical potential should not

[8]Besides the dependence on the forms used in the iteration, the classical piece also depends on the coordinates used. The quantum piece however depends neither on the choice of coordinates [13] nor on the iteration forms [9].

be a concern, since the potential is *not* an observable. What *is* an observable is the on-shell transition amplitude, which is uniquely defined in each case as

$$ {}^{0}\mathcal{M}_{tot}(\vec{q}) = -\int d^3r e^{i\vec{q}\cdot\vec{r}} \left[{}^{0}V_i^{(1)}(\vec{r}) + {}^{0}V_i^{(2)}(\vec{r})\right] + {}^{0}\mathrm{Amp}_i(\vec{q}) \tag{35} $$

where the index i denotes differing possible definitions of the potentials and the iteration. Thus we regard our potential as a nice way to display our resulting scattering amplitudes in coordinate space, but we emphasize that our main results are the long distance components of the scattering amplitude.

3 Spin-Dependent Scattering: Spin-Orbit Interaction

3.1 Spin-0 – Spin-1/2

Having determined the form of the potential for the spinless scattering case we move on to the case of scattering of particles carrying spin. We begin with the scattering of a spinless particle a from a spin-1/2 particle b.

For the case of spin-1/2 we require some additional formalism in order to extract the gravitational couplings, which is necessary because the Dirac algebra $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ is defined with respect to the Minkoswki flat space metric. In this case the Dirac matter Lagrangian coupled to gravity reads

$$ \sqrt{-g}\mathcal{L}_m = \sqrt{-g}\,\bar{\psi}\left[\frac{i}{2}e^{\mu}{}_{a}\{\gamma^a, D_\mu\} - m\right]\psi \tag{36} $$

and involves the vierbein $e^{\mu}{}_{a}$ which links global coordinates with those in a locally flat space. The vierbein is in some sense the "square root" of the metric tensor $g_{\mu\nu}$ and satisfies the relations

$$ \begin{aligned} e_\mu{}^a e_\nu{}^b \eta_{ab} &= g_{\mu\nu} & \qquad e^\mu{}_a e^\nu{}_b \eta^{ab} &= g^{\mu\nu} \\ e_\mu{}^a e_\nu{}^b g^{\mu\nu} &= \eta_{ab} & \qquad e^\mu{}_a e^\nu{}_b g_{\mu\nu} &= \eta^{ab}. \end{aligned} \tag{37} $$

The covariant derivative is

$$ D_\mu = \frac{1}{2}\partial_\mu^{LR} + \frac{i}{4}\omega_\mu{}^a{}_b\,\eta_{ac}\,\sigma^{cb} \tag{38} $$

with $\sigma^{cb} = \frac{i}{2}\left[\gamma^c, \gamma^d\right]$ and the partial derivative ∂_μ^{LR} acts only on spinors and in such a way that

$$ \bar{\psi}\partial_\mu^{LR}\psi = \bar{\psi}\,\partial_\mu\psi - \left(\partial_\mu\bar{\psi}\right)\psi. \tag{39} $$

Putting everything together, we find then

$$\sqrt{-g}\,\mathcal{L}_m = \sqrt{-g}\,\bar{\psi}\left[\frac{i}{2}\gamma^a e^\mu{}_a \partial_\mu^{LR} - \frac{1}{8} e^\mu{}_{a'}\;\omega_\mu{}^a{}_b\,\eta_{ac}\,\{\gamma^{a'},\sigma^{cb}\} - m\right]\psi. \qquad (40)$$

The spin connection $\omega_\mu{}^a{}_b\,\eta_{ac}$ can be derived in terms of vierbeins by requiring $D_\mu e_\nu{}^a = 0$ and by antisymmetrization in $\mu \leftrightarrow \nu$ in order to get rid of Christoffel symbols[9]. The result is:

$$\omega_\mu{}^a{}_b\,\eta_{ac} = \left(\frac{\eta_{ab}}{2}\,e^\nu{}_c\,(\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a) + \frac{\eta_{af}}{2}\,e^\nu{}_c e^\rho{}_b e_\mu{}^f\,\partial_\rho e_\nu{}^a\right) - \Big(b \leftrightarrow c\Big) \qquad (41)$$

In order to derive the Feynman rules we expand the ingredients in Eq. (40) that contain graviton couplings, that is we need $e^\mu{}_a$ and $\omega_\mu{}^a{}_b\,\eta_{ac}$ expanded up to $\mathcal{O}(\kappa^2)$

$$\begin{aligned} e_\mu{}^a &= \delta^a_\mu + \frac{\kappa}{2}h^a_\mu - \frac{\kappa^2}{8}h_{\mu\rho}h^{a\rho} + \dots \\ e^\mu{}_a &= \delta^\mu_a - \frac{\kappa}{2}h^\mu_a + \frac{3\kappa^2}{8}h_{a\rho}h^{\mu\rho} + \dots \\ \omega_\mu{}^a{}_b\,\eta_{ac} &= \frac{\kappa}{2}\partial_b h_{\mu c} + \frac{\kappa^2}{8}h^\rho_b\partial_\mu h_{c\rho} - \frac{\kappa^2}{4}h^\rho_b\partial_\rho h_{\mu c} + \frac{\kappa^2}{4}h^\rho_b\partial_c h_{\mu\rho} - \Big(b \leftrightarrow c\Big) \end{aligned} \qquad (42)$$

After these expansions are employed, we no longer need to distinguish between Latin Lorentz indeces and Greek covariant indices and can use the Minkowski metric to lower and raise all indices.

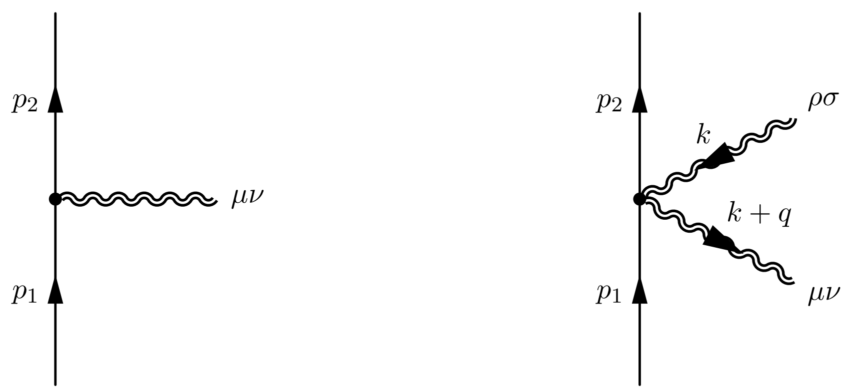
The matter Lagrangian then has the expansion—(note here that our conventions are $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\epsilon^{0123} = +1$)

$$\begin{aligned} \sqrt{-g}\mathcal{L}_m^{(0)} &= \bar{\psi}\left(\frac{i}{2}\,\overleftrightarrow{\partial\!\!\!/}^{LR} - m\right)\psi \\ \sqrt{-g}\mathcal{L}_m^{(1)} &= \frac{\kappa}{2}h\,\bar{\psi}\left(\frac{i}{2}\,\partial\!\!\!/^{LR} - m\right)\psi - \frac{\kappa}{2}h^{\mu\nu}\,\bar{\psi}\,\frac{i}{2}\partial_\mu^{LR}\gamma_\nu\psi \\ \sqrt{-g}\mathcal{L}_m^{(2)} &= \frac{\kappa^2}{8}\Big(h^2 - 2h_{\alpha\beta}h^{\alpha\beta}\Big)\,\bar{\psi}\left(\frac{i}{2}\,\partial\!\!\!/^{LR} - m\right)\psi \end{aligned}$$

[9]For our purposes we shall use only the symmetric component of the vierbein matrices, since these are physical and can be connected to the metric tensor, while their antisymmetric components are associated with freedom of homogeneous transformations of the local Lorentz frames and do not contribute to nonanalyticity [16].

$$+ \quad \frac{\kappa^2}{8}\Big(3h^{\mu\rho}h^{\nu}_{\rho} - 2hh^{\mu\nu}\Big)\,\bar{\psi}\,\frac{i}{2}\partial^{LR}_{\mu}\gamma_{\nu}\psi$$
$$+ \quad \frac{i\kappa^2}{16}\epsilon^{\alpha\beta\gamma\delta}\,h^{\rho}_{\alpha}(i\partial_{\beta}h_{\rho\gamma})\,\bar{\psi}\gamma_{\delta}\gamma_5\psi \tag{43}$$

and the corresponding one- and two-graviton vertices are found to be



$$\tfrac{1}{2}\tau^{(1)}_{\mu\nu}(p_2,p_1,m) = \frac{-i\kappa}{2}\left[\frac{1}{4}\Big(\gamma_\mu(p_1+p_2)_\nu + \gamma_\nu(p_1+p_2)_\mu\Big) - \eta_{\mu\nu}\left(\frac{1}{2}(\not{p}_1+\not{p}_2)-m\right)\right]$$

$$\begin{aligned}\tfrac{1}{2}\tau^{(2)}_{\mu\nu,\rho\sigma}(p_2,p_1,m) = i\kappa^2\Bigg[&-\frac{1}{2}\left(\frac{1}{2}(\not{p}_1+\not{p}_2)-m\right)P_{\mu\nu,\rho\sigma}\\ &-\frac{1}{16}\Big[\eta_{\mu\nu}\Big(\gamma_\rho(p_1+p_2)_\sigma + \gamma_\sigma(p_1+p_2)_\rho\Big)\\ &\quad+\eta_{\rho\sigma}\Big(\gamma_\mu(p_1+p_2)_\nu + \gamma_\nu(p_1+p_2)_\mu\Big)\Big]\\ &+\frac{3}{16}(p_1+p_2)^\epsilon\rho^\xi(I_{\xi\phi,\mu\nu}I^{\phi}{}_{\epsilon,\rho\sigma} + I_{\xi\phi,\rho\sigma}I^{\phi}{}_{\epsilon,\mu\nu})\\ &+\frac{i}{16}\epsilon^{\epsilon\phi\eta\lambda}\gamma_\lambda\gamma_5\Big(I_{\rho\sigma,\phi\xi}I_{\mu\nu,\eta}{}^{\xi}k_\epsilon - I_{\mu\nu,\phi\xi}I_{\rho\sigma,\eta}{}^{\xi}(k+q)_\epsilon\Big)\Bigg].\end{aligned} \tag{44}$$

The tree level transition amplitude from one-graviton exchange is then

$$\begin{aligned}\tfrac{1}{2}\mathcal{M}^{(1)}(q) \;=\; \frac{-16\pi G m_a m_b}{\sqrt{2E_1 2E_2 E_3 E_4}}\Bigg[&-\frac{m_a m_b}{q^2}\,\bar{u}(p_4)u(p_3)\\ &+\frac{s-m_a^2-m_b^2+\frac{1}{2}q^2}{q^2}\,\frac{1}{m_a}\bar{u}(p_4)\,\not{p}_1 u(p_3)\Bigg].\end{aligned} \tag{45}$$

Defining the spin vector as

$$S_b^\mu = \frac{1}{2}\bar{u}(p_4)\gamma_5\gamma^\mu u(p_3) \tag{46}$$

we find the identity

$$\bar{u}(p_4)\gamma_\mu u(p_3) = \left(\frac{1}{1-\frac{q^2}{4m_b^2}}\right)\left[\frac{(p_3+p_4)_\mu}{2m_b}\bar{u}(p_4)u(p_3) - \frac{i}{m_b^2}\epsilon_{\mu\beta\gamma\delta}q^\beta p_3^\gamma S_b^\delta\right] \tag{47}$$

whereupon the nonanalytic part of the transition amplitude in the thereshold limit $s \longrightarrow s_0 = (m_a+m_b)^2$ can be written in the form

$$\tfrac{1}{2}\mathcal{M}^{(1)}(q) \simeq -\frac{4\pi G m_a m_b}{q^2}\left[\bar{u}(p_4)u(p_3) + \frac{2i}{m_a m_b^2}\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta\right]. \tag{48}$$

In order to define the potential we require the nonrelativistic amplitude in the symmetric center of mass frame ($\vec{p}_1 = -\vec{p}_3 = \vec{p} + \vec{q}/2$) where

$$S_b^\alpha \stackrel{NR}{\longrightarrow} (0, \vec{S}_b) \qquad \text{with} \qquad \vec{S}_b = \frac{1}{2}\chi_f^{b\dagger}\vec{\sigma}\chi_i^b, \tag{49}$$

$$\bar{u}(p_4)u(p_3) \stackrel{NR}{\longrightarrow} \chi_f^{b\dagger}\chi_i^b - \frac{i}{2m_b^2}\vec{S}_b\cdot\vec{p}\times\vec{q} \tag{50}$$

and

$$\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta \stackrel{NR}{\longrightarrow} (m_a+m_b)\left(1+\frac{\vec{p}^2}{2m_a m_b}\right)\vec{S}_b\cdot\vec{p}\times\vec{q}, \tag{51}$$

so that

$$\tfrac{1}{2}\mathcal{M}^{(1)}(\vec{q}) \simeq \frac{4\pi G m_a m_b}{\vec{q}^2}\left[\chi_f^{b\dagger}\chi_i^b + \frac{i(3m_a+4m_b)}{2m_a m_b^2}\vec{S}_b\cdot\vec{p}\times\vec{q} + \ldots\right] \tag{52}$$

and the lowest order potential becomes

$$\begin{aligned}
\tfrac{1}{2}V_G^{(1)}(\vec{r}) &= -\int \frac{d^3q}{(2\pi)^3}\,\tfrac{1}{2}\mathcal{M}^{(1)}(\vec{q})\,e^{-i\vec{q}\cdot\vec{r}} \\
&= -\frac{G m_a m_b}{r}\chi_f^{b\dagger}\chi_i^b - \frac{3m_a+4m_b}{2m_a m_b^2}\vec{S}_b\cdot\vec{p}\times\vec{\nabla}\left(-\frac{G m_a m_b}{r}\right) \\
&= -\frac{G m_a m_b}{r}\chi_f^{b\dagger}\chi_i^b + \frac{G}{r^3}\frac{3m_a+4m_b}{2m_b}\vec{L}\cdot\vec{S}_b
\end{aligned} \tag{53}$$

where $\vec{L} = \vec{r} \times \vec{p}$ is the angular momentum—the modification of the leading spin-independent potential has a spin-orbit character.

In order to determine the second order potential, we must evaluate the one loop diagrams in Fig. 2. A subtlety that arises in the calculation involving spin is that *two* independent kinematic variables arise: the momentum transfer q^2 and $s - s_0$, which is to leading order proportional to p_0^2 (where $p_0^2 \equiv \vec{p}_i^{\,2},\ i = 1, 2, 3, 4$) in the center of mass frame. We find that our results differ if we perform an expansion first in $s - s_0$ and then in q^2 or vice versa. This ordering issue occurs only for the box diagram, diagram (d) of Fig. 2, where it stems from the reduction of vector and tensor box integrals. Their reduction in terms of scalar integrals involves the inversion of a matrix whose Gram determinant vanishes in the nonrelativistic threshold limit $q^2, s - s_0 \to 0$. More precisely, the denominators or the vector and tensor box integrals (see Appendix A in [9]) involve a factor of $(4p_0^2 - \vec{q}^{\,2})$ when expanded in the nonrelativistic limit. Since $q^2 = 4p_0^2 \sin^2 \frac{\theta}{2}$ with θ the scattering angle, we notice that $4p_0^2 > \vec{q}^{\,2}$ unless we consider backward scattering where $\theta = \pi$ and where the scattering amplitude diverges. And since p_0^2 originates from the relativistic structure $s - s_0$, it is clear that we must first expand our vector and tensor box integrals in q^2 and then in $s - s_0$. With this procedure, the contributions of the loop diagrams in Fig. 2 to the spin-0 – spin-1/2 scattering amplitude read

$$
\begin{aligned}
{}^{\frac{1}{2}}\mathcal{M}^{(2)}_{2a}(q) &= G^2 m_a m_b \Big[\bar{u}(p_4)u(p_3)(-10L) + \frac{1}{m_a}\bar{u}(p_4)\,\not{p}_1 u(p_3)(-24L)\Big] \\
{}^{\frac{1}{2}}\mathcal{M}^{(2)}_{2b}(q) &= G^2 m_a m_b \Big[\bar{u}(p_4)u(p_3)(-m_a S + 3L) \\
&\qquad + \frac{1}{m_a}\bar{u}(p_4)\,\not{p}_1 u(p_3)(8m_a S + 20L)\Big] \\
{}^{\frac{1}{2}}\mathcal{M}^{(2)}_{2c}(q) &= G^2 m_a m_b \Big[\bar{u}(p_4)u(p_3)(4m_b S + 2L) \\
&\qquad + \frac{1}{m_a}\bar{u}(p_4)\,\not{p}_1 u(p_3)(4m_b S + 16L)\Big] \\
{}^{\frac{1}{2}}\mathcal{M}^{(2)}_{2d}(q) &= G^2 m_a m_b \Big[\bar{u}(p_4)u(p_3)\Big(- \frac{4m_a m_b}{q^2}L - \frac{m_a m_b(3m_a + 4m_b)}{s - s_0}S \\
&\qquad - (6m_a + 11m_b)S - \frac{42m_a^2 + 23m_a m_b - 4m_b^2}{6m_a m_b}L\Big) \\
&\qquad + \frac{1}{m_a}\bar{u}(p_4)\,\not{p}_1 u(p_3)\Big(\frac{8m_a m_b}{q^2}L + \frac{m_a m_b(3m_a + 4m_b)}{s - s_0}S
\end{aligned}
$$

$$
\begin{aligned}
& \qquad\qquad +(9m_a+13m_b)S \\
& \qquad\qquad +\frac{153m_a^2-44m_am_b+88m_b^2}{12m_am_b}L\bigg)\bigg] \\
& - i4\pi G^2m_a^2m_b^2\,\frac{L}{q^2}\sqrt{\frac{m_am_b}{s-s_0}}\left(2\,\frac{1}{m_a}\bar{u}(p_4)\,\not{p}_1 u(p_3)-\bar{u}(p_4)u(p_3)\right)
\end{aligned}
$$

$$
\begin{aligned}
\tfrac{1}{2}\mathcal{M}^{(2)}_{2_e}(q) \;=\; & G^2m_am_b\bigg[\bar{u}(p_4)u(p_3)\bigg(\frac{4m_am_b}{q^2}L \\
& +\frac{19m_a}{4}S+\frac{42m_a^2+21m_am_b-4m_b^2}{6m_am_b}L\bigg) \\
& +\frac{1}{m_a}\bar{u}(p_4)\,\not{p}_1 u(p_3)\bigg(-\frac{8m_am_b}{q^2}L-\frac{33m_a+16m_b}{4}S \\
& -\frac{153m_a^2+268m_am_b+88m_b^2}{12m_am_b}L\bigg)\bigg] \\
\tfrac{1}{2}\mathcal{M}^{(2)}_{2_f}(q) \;=\; & G^2m_am_b\bigg[\bar{u}(p_4)u(p_3)\bigg(-\frac{3m_a}{2}S+10L\bigg) \\
& +\frac{1}{m_a}\bar{u}(p_4)\,\not{p}_1 u(p_3)\bigg(-\frac{m_a+4m_b}{2}S+4L\bigg)\bigg] \\
\tfrac{1}{2}\mathcal{M}^{(2)}_{2_g}(q) \;=\; & G^2m_am_b\bigg[\bar{u}(p_4)u(p_3)\bigg(-\frac{1}{15}L\bigg)+\frac{1}{m_a}\bar{u}(p_4)\,\not{p}_1 u(p_3)\bigg(-\frac{42}{15}L\bigg)\bigg]. \qquad (54)
\end{aligned}
$$

Again, we have calculated the form factor diagrams in Fig. 2(f) both directly and via the results from [7, 8]. Summing, we find

$$
\begin{aligned}
\tfrac{1}{2}\mathcal{M}^{(2)}_{tot}(q) \;=\; & G^2m_am_b\bigg[L\left(\frac{23}{5}\bar{u}(p_4)u(p_3)-\frac{64}{5}\frac{1}{m_a}\bar{u}(p_4)\,\not{p}_1 u(p_3)\right) \\
& +S\bigg(-\frac{15m_a+28m_b}{4}\,\bar{u}(p_4)u(p_3) \\
& +\frac{11(3m_a+4m_b)}{4}\frac{1}{m_a}\bar{u}(p_4)\,\not{p}_1 u(p_3)\bigg) \\
& -\frac{(3m_a+4m_b)m_am_bS}{s-s_0}\left(\bar{u}(p_4)u(p_3)-\frac{1}{m_a}\bar{u}(p_4)\,\not{p}_1 u(p_3)\right)\bigg] \\
& - i4\pi G^2m_a^2m_b^2\,\frac{L}{q^2}\sqrt{\frac{m_am_b}{s-s_0}}\left(-\bar{u}(p_4)u(p_3)+2\,\frac{1}{m_a}\bar{u}(p_4)\,\not{p}_1 u(p_3)\right)
\end{aligned}
$$

(55)

Using the identity Eq. (47) and

$$p_1 \cdot (p_3 + p_4) = 2m_a m_b + s - s_0 + \frac{q^2}{2}$$

Eq. (55) becomes

$$\begin{aligned}
\tfrac{1}{2}\mathcal{M}^{(2)}_{tot}(q) \;=\; & G^2 m_a m_b \Big[\bar{u}(p_4)u(p_3)\Big(6(m_a+m_b)S - \frac{41}{5}L\Big) \\
& + \frac{i}{m_a m_b^2}\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta \Big(\frac{11(3m_a+4m_b)}{4}S - \frac{64}{5}L\Big) \\
& + \frac{iS(3m_a+4m_b)}{m_b(s-s_0)}\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta\Big] \\
& - i4\pi G^2 m_a^2 m_b^2 \frac{L}{q^2}\sqrt{\frac{m_a m_b}{s-s_0}}\Big(\bar{u}(p_4)u(p_3) + \frac{2i}{m_a m_b^2}\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta\Big)
\end{aligned} \tag{56}$$

Finally, working in the center of mass frame and taking the nonrelativistic limit, we find

$$\begin{aligned}
\tfrac{1}{2}\mathcal{M}^{(2)}_{tot}(\vec{q}) \;\simeq\; & \Big[G^2 m_a m_b\Big(6(m_a+m_b)S - \frac{41}{5}L\Big) - i4\pi G^2 m_a^2 m_b^2 \frac{L}{q^2}\frac{m_r}{p_0}\Big]\chi_f^{b\dagger}\chi_i^b \\
& + \Big[G^2\Big(\frac{12m_a^3 + 45m_a^2 m_b + 56 m_a m_b^2 + 24 m_b^3}{2(m_a+m_b)}S - \frac{87m_a + 128 m_b}{10}L\Big) \\
& + \frac{G^2 m_a^2 m_b^2(3m_a+4m_b)}{(m_a+m_b)}\Big(-i\frac{2\pi L}{p_0 q^2} + \frac{S}{p_0^2}\Big)\Big]\frac{i}{m_b}\vec{S}_b\cdot\vec{p}\times\vec{q}
\end{aligned} \tag{57}$$

We note from Eq. (57) that the scattering amplitude consists of two pieces—a spin-independent component proportional to $\chi_f^{b\dagger}\chi_i^b$ whose functional form

$$G^2 m_a m_b\Big(6(m_a+m_b)S - \frac{41}{5}L\Big) - i4\pi G^2 m_a^2 m_b^2\frac{L}{q^2}\frac{m_r}{p_0} \tag{58}$$

is *identical* to that of spinless scattering—together with a spin-orbit component proportional to

$$\frac{i}{m_b}\vec{S}_b\cdot\vec{p}\times\vec{q}$$

whose functional form is

$$
\begin{aligned}
& G^2\left(\frac{12m_a^3+45m_a^2m_b+56m_am_b^2+24m_b^3}{2(m_a+m_b)}S-\frac{87m_a+128m_b}{10}L\right)\\
+\quad & \frac{G^2m_a^2m_b^2(3m_a+4m_b)}{(m_a+m_b)}\left(-i\frac{2\pi L}{p_0q^2}+\frac{S}{p_0^2}\right) \qquad (59)
\end{aligned}
$$

We note in Eq. (59) the presence in the spin-orbit potential of an imaginary final state rescattering term proportional to i/p_0, similar to that found in the case of spin-independent scattering, together with a *completely new* type of kinematic form, proportional to $1/p_0^2$ which diverges at threshold. The presence of *either* term would prevent us from writing down a well defined second order potential.

The solution to this problem is, as before, to properly subtract the iterated first order potential—

$$
{}^{\frac{1}{2}}\mathrm{Amp}_G^{(2)}(\vec{q}) \;=\; -\int\frac{d^3\ell}{(2\pi)^3}\frac{\left\langle\vec{p}_f\left|{}^{\frac{1}{2}}\hat{V}_G^{(1)}\right|\vec{\ell}\right\rangle\left\langle\vec{\ell}\left|{}^{\frac{1}{2}}\hat{V}_G^{(1)}\right|\vec{p}_i\right\rangle}{\frac{p_0^2}{2m_r}-\frac{\ell^2}{2m_r}+i\epsilon} \qquad (60)
$$

where we now use the one-graviton exchange potential ${}^{\frac{1}{2}}V_G^{(1)}(\vec{r})$ given in Eq. (53). Splitting this lowest order potential into spin-independent and spin-dependent components—

$$
\left\langle\vec{p}_f\left|{}^{\frac{1}{2}}\hat{V}_G^{(1)}\right|\vec{p}_i\right\rangle=\left\langle\vec{p}_f\left|{}^{\frac{1}{2}}\hat{V}_{S-I}^{(1)}\right|\vec{p}_i\right\rangle+\left\langle\vec{p}_f\left|{}^{\frac{1}{2}}\hat{V}_{S-O}^{(1)}\right|\vec{p}_i\right\rangle \qquad (61)
$$

where

$$
\begin{aligned}
\left\langle\vec{p}_f\left|{}^{\frac{1}{2}}\hat{V}_{S-I}^{(1)}\right|\vec{p}_i\right\rangle &= -\frac{4\pi Gm_am_b}{\vec{q}^2}\chi_f^{b\dagger}\chi_i^b=-\frac{4\pi Gm_am_b}{(\vec{p}_i-\vec{p}_f)^2}\chi_f^{b\dagger}\chi_i^b\\
\left\langle\vec{p}_f\left|{}^{\frac{1}{2}}\hat{V}_{S-O}^{(1)}\right|\vec{p}_i\right\rangle &= -\frac{4\pi Gm_am_b}{\vec{q}^2}\frac{3m_a+4m_b}{2m_am_b}\frac{i}{m_b}\vec{S}_b\cdot\vec{p}\times\vec{q}\\
&= -\frac{4\pi Gm_am_b}{(\vec{p}_i-\vec{p}_f)^2}\frac{3m_a+4m_b}{2m_am_b}\frac{i}{m_b}\vec{S}_b\cdot\frac{1}{2}(\vec{p}_i+\vec{p}_f)\times(\vec{p}_i-\vec{p}_f)
\end{aligned} \qquad (62)
$$

we find that the iterated amplitude splits also into spin-independent and spin-dependent pieces. The leading spin-independent amplitude is

$$
{}^{\frac{1}{2}}\mathrm{Amp}_{S-I}^{(2)}(\vec{q}) \;=\; -\int\frac{d^3\ell}{(2\pi)^3}\frac{\left\langle\vec{p}_f\left|{}^{\frac{1}{2}}\hat{V}_{S-I}^{(1)}\right|\vec{\ell}\right\rangle\left\langle\vec{\ell}\left|{}^{\frac{1}{2}}\hat{V}_{S-I}^{(1)}\right|\vec{p}_i\right\rangle}{\frac{p_0^2}{2m_r}-\frac{\ell^2}{2m_r}+i\epsilon}
$$

$$= i\sum_{s_\ell}\int \frac{d^3\ell}{(2\pi)^3}\frac{c_G^2\chi_f^{b\dagger}\chi_{s_\ell}^b}{|\vec{\ell}-\vec{p}_f|^2+\lambda^2}\frac{i}{\frac{p_0^2}{2m_r}-\frac{\ell^2}{2m_r}+i\epsilon}\frac{c_G^2\chi_{s_\ell}^{b\dagger}\chi_i^b}{|\vec{p}_i-\vec{\ell}|^2+\lambda^2}$$

$$= \chi_f^{b\dagger}\chi_i^b H = -i4\pi G^2 m_a^2 m_b^2\frac{L}{q^2}\frac{m_r}{p_0}\chi_f^{b\dagger}\chi_i^b \tag{63}$$

where we defined $c_G^2 \equiv -4\pi G m_a m_b$, and the leading spin-dependent term is

$$\begin{aligned}
\tfrac{1}{2}\mathrm{Amp}_{S-O}^{(2)}(\vec{q}) &= -\int\frac{d^3\ell}{(2\pi)^3}\frac{\left\langle\vec{p}_f\left|\tfrac{1}{2}\hat{V}_{S-I}^{(1)}\right|\vec{\ell}\right\rangle\left\langle\vec{\ell}\left|\tfrac{1}{2}\hat{V}_{S-O}^{(1)}\right|\vec{p}_i\right\rangle}{\frac{p_0^2}{2m_r}-\frac{\ell^2}{2m_r}+i\epsilon} \\
&\quad-\int\frac{d^3\ell}{(2\pi)^3}\frac{\left\langle\vec{p}_f\left|\tfrac{1}{2}\hat{V}_{S-O}^{(1)}\right|\vec{\ell}\right\rangle\left\langle\vec{\ell}\left|\tfrac{1}{2}\hat{V}_{S-I}^{(1)}\right|\vec{p}_i\right\rangle}{\frac{p_0^2}{2m_r}-\frac{\ell^2}{2m_r}+i\epsilon} \\
&= \frac{i(3m_a+4m_b)}{2m_a m_b^2}\vec{S}_b\cdot \\
&\quad\left(i\int\frac{d^3\ell}{(2\pi)^3}\frac{c_G^2}{|\vec{\ell}-\vec{p}_f|^2+\lambda^2}\frac{i}{\frac{p_0^2}{2m_r}-\frac{\ell^2}{2m_r}+i\epsilon}\frac{c_G^2\,\frac{1}{2}(\vec{p}_i+\vec{\ell})\times(\vec{p}_i-\vec{\ell})}{|\vec{p}_i-\vec{\ell}|^2+\lambda^2}\right. \\
&\quad\left.+i\int\frac{d^3\ell}{(2\pi)^3}\frac{c_G^2\,\frac{1}{2}(\vec{\ell}+\vec{p}_f)\times(\vec{\ell}-\vec{p}_f)}{|\vec{\ell}-\vec{p}_f|^2+\lambda^2}\frac{i}{\frac{p_0^2}{2m_r}-\frac{\ell^2}{2m_r}+i\epsilon}\frac{c_G^2}{|\vec{p}_i-\vec{\ell}|^2+\lambda^2}\right) \\
&= \frac{i(3m_a+4m_b)}{2m_a m_b^2}\,\vec{S}_b\cdot\vec{H}\times\vec{q} \\
&= \frac{G^2 m_a^2 m_b^2(3m_a+4m_b)}{(m_a+m_b)}\left(-i\frac{2\pi L}{p_0 q^2}+\frac{S}{p_0^2}\right)\frac{i}{m_b}\vec{S}_b\cdot\vec{p}\times\vec{q}
\end{aligned} \tag{64}$$

(In principle we would also have to iterate the leading order spin-orbit piece twice. However this procedure yields only terms higher order in p_0^2.) We observe that when the amplitudes Eqs. (64) and (63) are subtracted from the full one loop scattering amplitude Eq. (57) both the terms involving $1/p_0^2$ and those proportional to i/p_0 disappear leaving behind a well-defined second order potential

$$\begin{aligned}
\tfrac{1}{2}V_G^{(2)}(\vec{r}) &= -\int\frac{d^3q}{(2\pi)^3}e^{-i\vec{q}\cdot\vec{r}}\left[\tfrac{1}{2}\mathcal{M}_{tot}^{(2)}(\vec{q})-\tfrac{1}{2}\mathrm{Amp}_G^{(2)}(\vec{q})\right] \\
&= \int\frac{d^3q}{(2\pi)^3}e^{-i\vec{q}\cdot\vec{r}}\Bigg[G^2 m_a m_b\left(-6(m_a+m_b)S+\frac{41}{5}L\right)\chi_f^{b\dagger}\chi_i^b \\
&\qquad+G^2\Bigg(-\frac{12m_a^3+45m_a^2m_b+56m_am_b^2+24m_b^3}{2(m_a+m_b)}S
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad + \frac{87m_a + 128m_b}{10} L \bigg) \frac{i}{m_b} \vec{S}_b \cdot \vec{p} \times \vec{q} \bigg] \\
= & \left[-\frac{3G^2 m_a m_b (m_a + m_b)}{r^2} - \frac{41 G^2 m_a m_b \hbar}{10\pi r^3} \right] \chi_f^{b\dagger} \chi_i^b \\
+ & \frac{1}{m_b} \vec{S}_b \cdot \vec{p} \times \vec{\nabla} \bigg[\frac{G^2 (12m_a^3 + 45m_a^2 m_b + 56 m_a m_b^2 + 24 m_b^3)}{4(m_a + m_b) r^2} \\
&\qquad + \frac{G^2 (87 m_a + 128 m_b)\hbar}{20\pi r^3} \bigg] \\
= & \left[-\frac{3G^2 m_a m_b (m_a + m_b)}{r^2} - \frac{41 G^2 m_a m_b \hbar}{10\pi r^3} \right] \chi_f^{b\dagger} \chi_i^b \\
+ & \bigg[\frac{G^2 (12m_a^3 + 45m_a^2 m_b + 56 m_a m_b^2 + 24 m_b^3)}{2m_b (m_a + m_b) r^4} \\
& + \frac{3G^2 (87 m_a + 128 m_b)\hbar}{20\pi m_b r^5} \bigg] \vec{L} \cdot \vec{S}_b
\end{aligned} \tag{65}
$$

We observe then that the second order potential for long range gravitational scattering of a spinless and spin-1/2 particle consists of two components: one which is independent of the spin of particle b and is identical to the potential found for the case of spinless scattering, accompanied by a spin-orbit interaction involving a new form for its classical and quantum potentials. It is tempting to speculate that the form of this new spin-orbit potential is also universal. In order to check this hypothesis we consider the case of spin-0 – spin-1 scattering.

3.2 Spin-0 – Spin-1

The dynamics of a neutral spin-1 field ϕ_μ having mass m is described by the Proca Lagrangian which, when coupled to gravity via minimal substitution, takes the form

$$
\sqrt{-g}\mathcal{L}_m = \sqrt{-g} \left[-\frac{1}{4} U_{\mu\nu} U_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} + \frac{1}{2} m^2 \phi_\mu \phi_\nu g^{\mu\nu} \right] \tag{66}
$$

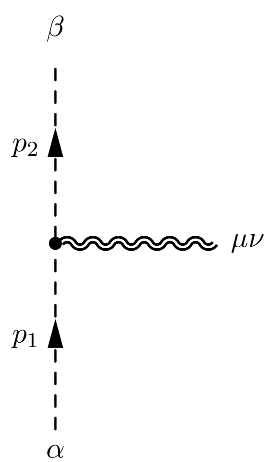
where

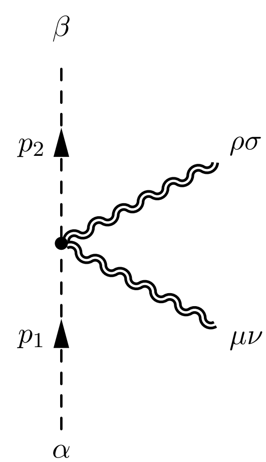
$$
U_{\mu\nu} = D_\mu \phi_\nu - D_\nu \phi_\mu = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu \tag{67}
$$

is the spin-1 field tensor. The last equality in Eq. (67) follows from the symmetry of the connection coefficients $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$. Expanded in terms of the graviton field, the matter Lagrangian then has the form

$$
\begin{aligned}
\sqrt{-g}\mathcal{L}_m^{(0)} &= -\frac{1}{2}\partial_\mu\phi_\nu\partial^\mu\phi^\nu + \frac{1}{2}\partial_\mu\phi_\nu\partial^\nu\phi^\mu + \frac{1}{2}m^2\phi_\mu\phi^\mu \\
\sqrt{-g}\mathcal{L}_m^{(1)} &= \frac{\kappa}{2}h\left(-\frac{1}{2}\partial_\mu\phi_\nu\partial^\mu\phi^\nu + \frac{1}{2}\partial_\mu\phi_\nu\partial^\nu\phi^\mu + \frac{1}{2}m^2\phi_\mu\phi^\mu\right) \\
&- \kappa h^{\mu\nu}\left(-\frac{1}{2}\partial_\mu\phi_\alpha\partial_\nu\phi^\alpha - \frac{1}{2}\partial_\alpha\phi_\mu\partial^\alpha\phi_\nu + \partial_\mu\phi^\alpha\partial_\alpha\phi_\nu + \frac{1}{2}m^2\phi_\mu\phi_\nu\right) \\
\sqrt{-g}\mathcal{L}_m^{(2)} &= \frac{\kappa^2}{8}\left(h^2 - 2h_{\alpha\beta}h^{\alpha\beta}\right)\left(-\frac{1}{2}\partial_\mu\phi_\nu\partial^\mu\phi^\nu + \frac{1}{2}\partial_\mu\phi_\nu\partial^\nu\phi^\mu + \frac{1}{2}m^2\phi_\mu\phi^\mu\right) \\
&+ \frac{\kappa^2}{2}\left(2h^{\mu\rho}h^\nu_\rho - hh^{\mu\nu}\right) \\
&\qquad \times\left(-\frac{1}{2}\partial_\mu\phi_\alpha\partial_\nu\phi^\alpha - \frac{1}{2}\partial_\alpha\phi_\mu\partial^\alpha\phi_\nu + \partial_\mu\phi^\alpha\partial_\alpha\phi_\nu + \frac{1}{2}m^2\phi_\mu\phi_\nu\right) \\
&+ \kappa^2 h^{\mu\rho}h^{\nu\sigma}\left(-\frac{1}{2}\partial_\mu\phi_\nu\partial_\rho\phi_\sigma + \frac{1}{2}\partial_\mu\phi_\nu\partial_\sigma\phi_\rho\right) \qquad (68)
\end{aligned}
$$

and the one- and two-graviton vertices are





$$
\begin{aligned}
{}^1\tau^{(1)}_{\beta,\alpha,\mu\nu}(p_2,p_1,m) &= -\frac{i\kappa}{2}\eta_{\mu\nu}\Big[(p_1\cdot p_2 - m^2)\eta_{\alpha\beta} - p_{1\beta}p_{2\alpha}\Big] \\
&\quad + i\kappa I_{\mu\nu,\kappa\lambda}\Big[(p_1\cdot p_2 - m^2)I_{\alpha\beta,}{}^{\kappa\lambda} + \frac{1}{2}(p_1^\kappa p_2^\lambda + p_1^\lambda p_2^\kappa)\eta_{\alpha\beta} \\
&\qquad -(p_1^\kappa p_{2\alpha}\delta^\lambda_\beta + p_2^\kappa p_{1\beta}\delta^\lambda_\alpha)\Big]
\end{aligned}
$$

$$
{}^1\tau^{(2)}_{\beta,\alpha,\mu\nu,\rho\sigma}(p_2,p_1,m) = \frac{i\kappa^2}{2}P_{\mu\nu,\rho\sigma}\Big[(p_1\cdot p_2 - m^2)\eta_{\alpha\beta} - p_{1\beta}p_{2\alpha}\Big]
$$

$$
\begin{aligned}
& - i\kappa^2\Big(I_{\mu\nu,}{}^{\kappa\delta} I_{\rho\sigma,\delta}{}^{\lambda} + I_{\rho\sigma,}{}^{\kappa\delta} I_{\mu\nu,\delta}{}^{\lambda} - \frac{1}{2}\Big(\eta_{\mu\nu} I_{\rho\sigma,}{}^{\kappa\lambda} + \eta_{\rho\sigma} I_{\mu\nu,}{}^{\kappa\lambda}\Big)\Big) \\
& \qquad \times\Big[(p_1\cdot p_2 - m^2) I_{\alpha\beta,}{}^{\kappa\lambda} + \frac{1}{2}(p_1^\kappa p_2^\lambda + p_1^\lambda p_2^\kappa)\eta_{\alpha\beta} \\
& \qquad\quad -(p_1^\kappa p_{2\alpha}\delta_\beta^\lambda + p_2^\kappa p_{1\beta}\delta_\alpha^\lambda)\Big] \\
& - \frac{i\kappa^2}{2}\left(I_{\mu\nu,}{}^{\eta\theta} I_{\rho\sigma,}{}^{\kappa\lambda} + I_{\rho\sigma,}{}^{\eta\theta} I_{\mu\nu,}{}^{\kappa\lambda}\right) \\
& \qquad \times\Big[p_{1\eta}\eta_{\alpha\kappa}(p_{2\theta}\eta_{\beta\lambda} - p_{2\lambda}\eta_{\beta\theta}) + p_{2\eta}\eta_{\beta\kappa}(p_{1\theta}\eta_{\alpha\lambda} - p_{1\lambda}\eta_{\alpha\theta})\Big].
\end{aligned}
\tag{69}
$$

If we take the incoming spin-1 particle to have polarization vector ϵ_i^b satisfying $\epsilon_i^b\cdot p_3 = 0$ and the outgoing particle to have polarization ϵ_f^b satisfying $\epsilon_f^b\cdot p_4 = 0$, then the one-graviton exchange amplitude can then be written as

$$
\begin{aligned}
{}^1\mathcal{M}^{(1)}(q) &= -\frac{8\pi G}{\sqrt{2E_1 2E_2 2E_3 2E_4}} \\
&\quad \times\Big[-\epsilon_f^{b*}\cdot\epsilon_i^b\left(\frac{(s-m_a^2-m_b^2+\frac{1}{2}q^2)^2 - 2m_a^2 m_b^2 + m_a^2 q^2 - \frac{1}{4}q^4}{q^2}\right) \\
&\quad\quad - \left(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1 - \epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q\right)\frac{2(s-m_a^2-m_b^2+\frac{1}{2}q^2)}{q^2} \\
&\quad\quad - \left(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1 + \epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q\right) \\
&\quad\quad + 2\,\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\,\frac{m_a^2}{q^2} + 2\,\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot p_1\Big] \\
&\simeq -\frac{4\pi G m_a m_b}{q^2}\Big[-\epsilon_f^{b*}\cdot\epsilon_i^b - \frac{2}{m_a m_b}\left(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1 - \epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q\right) \\
&\qquad + \frac{1}{m_b^2}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\Big]
\end{aligned}
\tag{70}
$$

where the first expression is exact while the second expression contains only the nonanalytic part in the threshold limit $s \to s_0 = (m_a+m_b)^2$. Now we rewrite this expression using the identity

$$
\epsilon_{f\mu}^{b*}\,\epsilon_i^b\cdot q - \epsilon_{i\mu}^b\,\epsilon_f^{b*}\cdot q = \frac{1}{1-\frac{q^2}{4m_b^2}}\left[\frac{i}{m_b}\epsilon_{\mu\beta\gamma\delta}\,p_3^\beta q^\gamma S_b^\delta - \frac{(p_3+p_4)_\mu}{2m_b^2}\epsilon_f^{b*}\cdot q\,\epsilon_i^b\cdot q\right] \tag{71}
$$

where we have defined the spin vector

$$S_{b\mu} = \frac{i}{2m_b}\epsilon_{\mu\beta\gamma\delta}\,\epsilon_f^{b*\beta}\epsilon_i^{b\gamma}(p_3+p_4)^\delta \tag{72}$$

The leading one-graviton exchange amplitude can then be written as

$$^1\mathcal{M}^{(1)}(q) \simeq -\frac{4\pi G m_a m_b}{q^2}\left[-\epsilon_f^{b*}\cdot\epsilon_i^b + \frac{2i}{m_a m_b^2}\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta - \frac{1}{m_b^2}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\right] \tag{73}$$

In the nonrelativistic limit we have

$$\epsilon_i^{b0} \simeq \frac{1}{m_b}\hat{\epsilon}_i^b\cdot\vec{p_3}, \qquad \epsilon_f^{b0} \simeq \frac{1}{m_b}\hat{\epsilon}_f^b\cdot\vec{p_4} \tag{74}$$

so that

$$\begin{aligned}\epsilon_f^{b*}\cdot\epsilon_i^b &\simeq -\hat{\epsilon}_f^{b*}\cdot\hat{\epsilon}_i^b + \frac{1}{m_b^2}\hat{\epsilon}_f^{b*}\cdot\vec{p}_4\,\hat{\epsilon}_i^b\cdot\vec{p}_3 \\ &\simeq -\hat{\epsilon}_f^{b*}\cdot\hat{\epsilon}_i^b + \frac{1}{2m_b^2}\hat{\epsilon}_f^{b*}\times\hat{\epsilon}_i^b\cdot\vec{p}_4\times\vec{p}_3 \\ &\quad +\frac{1}{2m_b^2}\left(\hat{\epsilon}_f^{b*}\cdot\vec{p}_4\hat{\epsilon}_i^b\cdot\vec{p}_3 + \hat{\epsilon}_f^{b*}\cdot\vec{p}_3\hat{\epsilon}_i^b\cdot\vec{p}_4\right)\end{aligned} \tag{75}$$

Since

$$-i\hat{\epsilon}_f^{b*}\times\hat{\epsilon}_i^b = \left\langle 1, m_f\left|\vec{S}_b\right|1, m_i\right\rangle, \tag{76}$$

Eq. (75) becomes

$$\begin{aligned}\epsilon_f^{b*}\cdot\epsilon_i^b &\simeq -\hat{\epsilon}_f^{b*}\cdot\hat{\epsilon}_i^b - \frac{i}{2m_b^2}\vec{S}_b\cdot\vec{p}_3\times\vec{p}_4 + \frac{1}{2m_b^2}\left(\hat{\epsilon}_f^{b*}\cdot\vec{p}_4\,\hat{\epsilon}_i^b\cdot\vec{p}_3 + \hat{\epsilon}_f^{b*}\cdot\vec{p}_3\,\hat{\epsilon}_i^b\cdot\vec{p}_4\right) \\ &\simeq -\hat{\epsilon}_f^{b*}\cdot\hat{\epsilon}_i^b + \frac{1}{m_b^2}\hat{\epsilon}_f^{b*}\cdot\vec{p}\,\hat{\epsilon}_i^b\cdot\vec{p} + \frac{i}{2m_b^2}\vec{S}_b\cdot\vec{p}\times\vec{q} - \frac{1}{4m_b^2}\hat{\epsilon}_f^{b*}\cdot\vec{q}\,\hat{\epsilon}_i^b\cdot\vec{q}\end{aligned} \tag{77}$$

and the transition amplitude assumes the form

$$\begin{aligned}^1\mathcal{M}^{(1)}(\vec{q}) \simeq \frac{4\pi G m_a m_b}{\vec{q}^2}\Big[\,&\hat{\epsilon}_f^{b*}\cdot\hat{\epsilon}_i^b - \frac{1}{m_b^2}\hat{\epsilon}_f^{b*}\cdot\vec{p}\,\hat{\epsilon}_i^b\cdot\vec{p} + \frac{i(3m_a+4m_b)}{2m_a m_b^2}\vec{S}_b\cdot\vec{p}\times\vec{q} \\ &- \frac{3}{4m_b^2}\hat{\epsilon}_f^{b*}\cdot\vec{q}\,\hat{\epsilon}_i^b\cdot\vec{q}\Big]\end{aligned} \tag{78}$$

The spin-independent and spin-orbit terms here are identical in form to those found in the spin-0 – spin-1/2 case but now are accompanied by new terms which are quadrupole in nature, as can be seen from the identity

$$\begin{aligned} T^b_{cd} &\equiv \frac{1}{2}\left(\hat{\epsilon}^{b*}_{fc}\hat{\epsilon}^b_{id} + \hat{\epsilon}^b_{ic}\hat{\epsilon}^{b*}_{fd}\right) - \frac{1}{3}\delta_{cd}\hat{\epsilon}^{b*}_f\cdot\hat{\epsilon}^b_i \\ &= -\left\langle 1, m_f \left| \frac{1}{2}(S_cS_d + S_dS_c) - \frac{2}{3}\delta_{cd} \right| 1, m_i \right\rangle \end{aligned} \tag{79}$$

The corresponding lowest order potential is then

$$\begin{aligned} {}^1V^{(1)}_C(\vec{r}) &= -\int \frac{d^3q}{(2\pi)^3}\, {}^1\mathcal{M}^{(1)}(\vec{q})\, e^{-i\vec{q}\cdot\vec{r}} \\ &\simeq -\frac{Gm_am_b}{r}\left(\hat{\epsilon}^{b*}_f\cdot\hat{\epsilon}^b_i - \frac{1}{m_b^2}\hat{\epsilon}^{b*}_f\cdot\vec{p}\;\hat{\epsilon}^b_i\cdot\vec{p}\right) \\ &\quad -\frac{3m_a+4m_b}{2m_am_b^2}\,\vec{S}_b\cdot\vec{p}\times\vec{\nabla}\left(-\frac{Gm_am_b}{r}\right) \\ &\quad +\frac{3}{4m_b^2}\,\hat{\epsilon}^{b*}_f\cdot\vec{\nabla}\;\hat{\epsilon}^b_i\cdot\vec{\nabla}\;\left(-\frac{Gm_am_b}{r}\right) \\ &\simeq -\frac{Gm_am_b}{r}\left(\hat{\epsilon}^{b*}_f\cdot\hat{\epsilon}^b_i - \frac{1}{m_b^2}\;\vec{p}:T^b:\vec{p}\right) + \frac{G}{r^3}\frac{3m_a+4m_b}{2m_b}\vec{L}\cdot\vec{S}_b \\ &\quad -\frac{G}{r^5}\frac{9m_a}{4m_b}\;\vec{r}:T^b:\vec{r} \end{aligned} \tag{80}$$

where we have defined

$$\vec{w}:T^b:\vec{s} \equiv w_cT^b_{cd}s_d$$

and which agrees precisely with its spin-1/2 analog in Eq. (53) up to quadrupole and tensor corrections.

The calculation of the one loop corrections proceeds as before, but with increased complexity due to the unit spin. Evaluating the diagrams in Fig. 2, we find then

$$\begin{aligned} {}^1\mathcal{M}^{(2)}_{2a}(q) = G^2m_am_b\Big[&28L\epsilon^{*b}_f\cdot\epsilon^b_i - \frac{16L}{m_a^2}\epsilon^{b*}_f\cdot p_1\epsilon^b_i\cdot p_1 \\ &+\frac{16L}{m_am_b}(\epsilon^{b*}_f\cdot q\epsilon^b_i\cdot p_1 - \epsilon^{b*}_f\cdot p_1\epsilon^b_i\cdot q) \\ &+\frac{8L}{m_a^2}(\epsilon^{b*}_f\cdot q\epsilon^b_i\cdot p_1 + \epsilon^{b*}_f\cdot p_1\epsilon^b_i\cdot q) \end{aligned}$$

$$
\begin{aligned}
&\qquad + \frac{4L}{m_b^2}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\Big] \\
{}^1\mathcal{M}_{2b}^{(2)}(q) \;=\;& G^2 m_a m_b\Big[(-4m_aS-16L)\epsilon_f^{*b}\cdot\epsilon_i^b + \frac{8m_aS+16L}{m_a^2}\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot p_1 \\
&-\frac{8L}{m_am_b}(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1 - \epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q) \\
&-\frac{4m_aS+8L}{m_a^2}(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1 + \epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q) \\
&-\left(\left(2m_a-\frac{3m_b^2}{2m_a}\right)S+\left(2-\frac{2m_b^2}{m_a^2}\right)L\right)\frac{1}{m_b^2}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\Big] \\
{}^1\mathcal{M}_{2c}^{(2)}(q) \;=\;& G^2 m_a m_b\Big[(-8m_bS-12L)\epsilon_f^{*b}\cdot\epsilon_i^b + \frac{16L}{m_a^2}\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot p_1 \\
&-\frac{4m_bS+8L}{m_am_b}(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1 - \epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q) \\
&-\frac{8L}{m_a^2}(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1 + \epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q) \\
&-\frac{m_bS+8L}{m_b^2}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\Big] \\
{}^1\mathcal{M}_{2d}^{(2)}(q) \;=\;& G^2 m_a m_b\Big[\frac{4m_am_b}{q^2}L\Big(-\epsilon_f^{*b}\cdot\epsilon_i^b - \frac{2}{m_am_b}(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1 - \epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q) \\
&\qquad + \frac{1}{m_b^2}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\Big) \\
&+\frac{S}{s-s_0}\Big(-(3m_a+4m_b)(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1 - \epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q) \\
&\qquad + \frac{m_a(5m_a+7m_b)}{2m_b}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\Big) \\
&-\Big((6m_a+4m_b)S + \frac{12m_a^2-6m_am_b+24m_b^2}{3m_am_b}L\Big)\epsilon_f^{*b}\cdot\epsilon_i^b \\
&+\Big(-4m_aS+\frac{22m_a-24m_b}{3m_b}L\Big)\frac{1}{m_a^2}\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot p_1 \\
&-\Big(13(m_a+m_b)S+\frac{25m_a^2+7m_am_b+22m_b^2}{3m_am_b}L\Big) \\
&\quad\times\frac{1}{m_am_b}(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1 - \epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q)
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(2m_aS-\frac{11m_a-12m_b}{3m_b}L\Big)\frac{1}{m_a^2}(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1+\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q)\\
&+\Big(\Big(7m_a+9m_b-\frac{3m_b^2}{4m_a}\Big)S+\Big(5\frac{m_a}{m_b}+2+9\frac{m_b}{m_a}-\frac{m_b^2}{m_a^2}\Big)L\Big)\\
&\quad\times\frac{1}{m_b^2}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\Big]\\
&- i4\pi G^2m_a^2m_b^2\frac{L}{q^2}\sqrt{\frac{m_am_b}{s-s_0}}\Big(-\epsilon_f^{*b}\cdot\epsilon_i^b-\frac{2}{m_am_b}(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1-\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q)\\
&\qquad+\frac{1}{m_b^2}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\Big)
\end{aligned}
$$

$$
\begin{aligned}
{}^1\mathcal{M}_{2e}^{(2)}(q) \;=\; & G^2m_am_b\Big[\frac{4m_am_b}{q^2}L\Big(\epsilon_f^{*b}\cdot\epsilon_i^b+\frac{2}{m_am_b}(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1-\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q)\\
&\qquad-\frac{1}{m_b^2}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\Big)\\
&+\Big((2m_a+4m_b)S+\frac{12m_a^2+52m_am_b+24m_b^2}{3m_am_b}L\Big)\epsilon_f^{*b}\cdot\epsilon_i^b\\
&-\Big(4m_aS+\frac{22m_a+24m_b}{3m_b}L\Big)\frac{1}{m_a^2}\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot p_1\\
&+\Big(\Big(\frac{17}{4}m_a+4m_b\Big)S+\frac{25m_a^2+49m_am_b+22m_b^2}{3m_am_b}L\Big)\\
&\quad\times\frac{1}{m_am_b}(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1-\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q)\\
&+\Big(2m_aS+\frac{11m_a+12m_b}{3m_b}L\Big)\frac{1}{m_a^2}(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1+\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q)\\
&-\Big(\Big(\frac{13}{8}m_a+\frac{31}{8}m_b+\frac{3m_b^2}{4m_a}\Big)S+\Big(5\frac{m_a}{m_b}+\frac{16}{3}+9\frac{m_b}{m_a}+\frac{m_b^2}{m_a^2}\Big)L\Big)\\
&\quad\times\frac{1}{m_b^2}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\Big]
\end{aligned}
$$

$$
\begin{aligned}
{}^1\mathcal{M}_{2f}^{(2)}(q) \;=\; & G^2m_am_b\Big[(2(m_a+m_b)S-14L)\epsilon_f^{*b}\cdot\epsilon_i^b\\
&+\Big(\Big(\frac{1}{2}m_a+2m_b\Big)S-4L\Big)\frac{1}{m_am_b}(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1-\epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q)\\
&+\Big(\Big(-\frac{1}{4}m_a+\frac{1}{2}m_b\Big)S+4L\Big)\frac{1}{m_b^2}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\Big]
\end{aligned}
$$

$$
\begin{aligned}
{}^1\mathcal{M}^{(2)}_{2_g}(q) \;=\; G^2 m_a m_b \Big[& \frac{43L}{15}\epsilon_f^{*b}\cdot\epsilon_i^b + \frac{14L}{5m_a m_b}(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1 - \epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q) \\
& -\frac{7L}{5m_b^2}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\Big]
\end{aligned}
\tag{81}
$$

Combining, we find the full one loop amplitude

$$
\begin{aligned}
{}^1\mathcal{M}^{(2)}_{tot}(q) \;=\; G^2 m_a m_b \Big[& \frac{S}{s-s_0}\Big(-(3m_a+4m_b)(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1 - \epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q) \\
& \qquad + \frac{m_a(5m_a+7m_b)}{2m_b}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\Big) \\
& + \Big(6(m_a+m_b)S - \frac{41}{5}L\Big)\Big(-\epsilon_f^{b*}\cdot\epsilon_i^b\Big) \\
& + \Big(-\frac{11(3m_a+4m_b)}{4}S + \frac{64}{5}L\Big) \\
& \quad \times\frac{1}{m_a m_b}(\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot p_1 - \epsilon_f^{b*}\cdot p_1\epsilon_i^b\cdot q) \\
& + \Big(\frac{25m_a+37m_b}{8}S - \frac{101}{15}L\Big)\frac{1}{m_b^2}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\Big]
\end{aligned}
\tag{82}
$$

which, using the identity Eq. (71), becomes

$$
\begin{aligned}
{}^1\mathcal{M}^{(2)}_{tot}(q) \;=\; G^2 m_a m_b \Big[& -\epsilon_f^{b*}\cdot\epsilon_i^b\Big(6(m_a+m_b)S - \frac{41}{5}L\Big) \\
& + \frac{i}{m_a m_b^2}\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta\Big(\frac{11(3m_a+4m_b)}{4}S - \frac{64}{5}L\Big) \\
& + \frac{1}{m_b^2}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\Big(-\frac{53m_a+67m_b}{8}S + \frac{91}{15}L\Big) \\
& + \frac{i(3m_a+4m_b)S}{m_b(s-s_0)}\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta \\
& - \frac{m_a(m_a+m_b)S}{2m_b(s-s_0)}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\Big] \\
- \; i4\pi G^2 m_a^2 m_b^2\frac{L}{q^2}\sqrt{\frac{m_a m_b}{s-s_0}} & \Big(-\epsilon_f^{*b}\cdot\epsilon_i^b + \frac{2i}{m_a m_b}\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta \\
& \qquad -\frac{1}{m_b^2}\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q\Big)
\end{aligned}
\tag{83}
$$

Notice here that without the $\epsilon_f^{b*}\cdot q\epsilon_i^b\cdot q$ terms, Eq. (83) has an identical structure to that of the case of spin-0 – spin-1/2 scattering—Eq. (56)—provided we substitute $\bar{u}(p_4)u(p_3)\longrightarrow -\epsilon_f^{b*}\cdot\epsilon_i^b$.

Finally, taking the nonrelativistic limit we find

$$
\begin{aligned}
{}^1\mathcal{M}_{tot}^{(2)}(\vec{q}) \simeq & \left[G^2m_am_b\left(6(m_a+m_b)S-\frac{41}{5}L\right)-i4\pi G^2m_a^2m_b^2\frac{L}{q^2}\frac{m_r}{p_0}\right] \\
& \times\left(\hat{\epsilon}_f^{b*}\cdot\hat{\epsilon}_i^b-\frac{1}{m_b^2}\,\vec{p}:T^b:\vec{p}\right) \\
& +\left[G^2\left(\frac{12m_a^3+45m_a^2m_b+56m_am_b^2+24m_b^3}{2(m_a+m_b)}S-\frac{87m_a+128m_b}{10}L\right)\right. \\
& \left.+\frac{G^2m_a^2m_b^2(3m_a+4m_b)}{(m_a+m_b)}\left(-i\frac{2\pi L}{p_0q^2}+\frac{S}{p_0^2}\right)\right]\frac{i}{m_b}\vec{S}_b\cdot\vec{p}\times\vec{q} \\
& +\left[G^2m_am_b\left(-\frac{21m_a^2+47m_am_b+28m_b^2}{4(m_a+m_b)}S+\frac{241}{60}L\right)\right. \\
& \left.+\frac{G^2m_a^3m_b^3}{2(m_a+m_b)}\left(i\frac{6\pi L}{p_0q^2}-\frac{S}{p_0^2}\right)\right]\frac{1}{m_b^2}\,\vec{q}:T^b:\vec{q} \qquad (84)
\end{aligned}
$$

As found in the earlier calculations, there exist terms involving both i/p_0 and $1/p_0^2$ which prevent the defining of a simple second order potential. The solution now is well known—subtraction of the iterated first order potential. Since the form of the spin-independent—$\hat{\epsilon}_B^*\cdot\hat{\epsilon}_A$—and spin-orbit—$\vec{S}_b\cdot\vec{p}\times\vec{q}$—terms is identical to that found for the case of spin-1/2, it is clear that the subtraction goes through as before and that the corresponding pieces of the second order potential have the *same* form as found for spin-1/2. In addition, there are now two new pieces of the amplitude, the quadrupole structure $\vec{q}:T^b:\vec{q}$ which multiplies terms involving both i/p_0 and $1/p_0^2$ and the tensor structure $\vec{p}:T^b:\vec{p}$ multiplying only i/p_0. In order to remove these we must iterate the full first order potential including these quadrupole and tensor components. However, we find that our simple nonrelativistic iteration fails to remove them! We suspect the reason to be the presence of the tensor structure $\vec{p}:T^b:\vec{p}$ in the lowest order potential which is in some sense a relativistic correction but which when iterated yields also quadrupole pieces $\vec{q}:T^b:\vec{q}$. A fully relativistic iteration should thus be performed which is under study but is beyond the scope of this paper. It would be interesting to investigate if the requirement of canceling all i/p_0 and $1/p_0^2$ forms in the quadrupole and tensor pieces could clarify the ambiguity in the iteration of the leading order potential as discussed in [15, 9].

Since we did not perform the proper iteration of the quadrupole and tensor pieces we include only the spin independent and spin-orbit pieces in the resulting second order potential

$$\begin{aligned}
{}^1V_G^{(2)}(\vec{r}) &= -\int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \left[{}^1\mathcal{M}_{tot}^{(2)}(\vec{q}) - {}^1\mathrm{Amp}_G^{(2)}(\vec{q})\right] \\
&= \left[-\frac{3G^2 m_a m_b(m_a+m_b)}{r^2} - \frac{41G^2 m_a m_b \hbar}{10\pi r^3}\right] \hat{\epsilon}_f^{b*}\cdot\hat{\epsilon}_i^b \\
&+ \frac{1}{m_b}\vec{S}_b\cdot\vec{p}\times\vec{\nabla}\left[\frac{G^2(12m_a^3+45m_a^2m_b+56m_am_b^2+24m_b^3)}{4(m_a+m_b)r^2}\right. \\
&\qquad\left.+\frac{G^2(87m_a+128m_b)\hbar}{20\pi r^3}\right] + {}^1V_T^{(2)}(\vec{r}) \\
&= \left[-\frac{3G^2 m_a m_b(m_a+m_b)}{r^2} - \frac{41G^2 m_a m_b \hbar}{10\pi r^3}\right] \hat{\epsilon}_f^{b*}\cdot\hat{\epsilon}_i^b \\
&+ \left[\frac{G^2(12m_a^3+45m_a^2m_b+56m_am_b^2+24m_b^3)}{2m_b(m_a+m_b)r^4}\right. \\
&\quad\left.+\frac{3G^2(87m_a+128m_b)\hbar}{20\pi m_b r^5}\right]\vec{L}\cdot\vec{S}_b + {}^1V_T^{(2)}(\vec{r}) \qquad (85)
\end{aligned}$$

where ${}^1V_T^{(2)}(\vec{r})$ denotes the tensor pieces not explicitly shown. Comparison with the corresponding form of $\frac{1}{2}V^{(1)}(\vec{r})$ given in Eq. (65) confirms the universality which we have suggested—the spin-independent and spin-orbit terms have identical forms. The next task is to see whether this universality applies when both scattered particles carry spin. For this purpose we consider the case of spin-1/2 – spin-1/2 scattering.

4 Spin-Dependent Scattering: Spin-Spin Interaction

4.1 Spin-1/2 – Spin-1/2

In the case of scattering of a pair of spin-1/2 particles, the basic vertices have already been developed in the spin-0 – spin-1/2 scattering section, so we can proceed directly to our calculation. The one-graviton exchange amplitude at tree level reads

$$\begin{aligned}{}^{\frac{1}{2}\frac{1}{2}}\mathcal{M}^{(1)}(q) \;=\; -\frac{4\pi G m_a m_b}{q^2}\Bigg[&\frac{s-m_a^2-m_b^2+\frac{1}{2}q^2}{2m_am_b}\bar{u}(p_1)\gamma_\alpha u(p_2)\,\bar{u}(p_3)\gamma^\alpha u(p_4)\\ &+\frac{1}{m_am_b}\bar{u}(p_1)\,\not{p}_3\,u(p_2)\,\bar{u}(p_3)\,\not{p}_1\,u(p_4)\\ &-\bar{u}(p_1)u(p_2)\,\bar{u}(p_3)u(p_4)\Bigg]\times\sqrt{\frac{m_a^2m_b^2}{E_1E_2E_3E_4}}\end{aligned} \tag{86}$$

and with the spin identities Eq. (47) for system b and

$$\bar{u}(p_2)\gamma_\mu u(p_1) = \left(\frac{1}{1-\frac{q^2}{4m_a^2}}\right)\left[\frac{(p_1+p_2)_\mu}{2m_a}\bar{u}(p_2)u(p_1)+\frac{i}{m_a^2}\epsilon_{\mu\beta\gamma\delta}q^\beta p_1^\gamma S_a^\delta\right] \tag{87}$$

for system a, Eq. (86) can be written as

$$\begin{aligned}{}^{\frac{1}{2}\frac{1}{2}}\mathcal{M}^{(1)}(q) = -\frac{4\pi G m_a m_b}{q^2}\Bigg[&\bar{u}(p_2)u(p_1)\bar{u}(p_4)u(p_3)\\ &+\frac{2i}{m_am_b^2}\bar{u}(p_2)u(p_1)\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta\\ &+\frac{2i}{m_a^2m_b}\bar{u}(p_4)u(p_3)\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_a^\delta\\ &+\frac{1}{m_am_b}(S_a\cdot qS_b\cdot q-q^2S_b\cdot S_a)\Bigg].\end{aligned} \tag{88}$$

In the symmetric center of mass frame we take the nonrelativistic limit yielding

$$\begin{aligned}{}^{\frac{1}{2}\frac{1}{2}}\mathcal{M}^{(1)}(\vec{q}) \;\simeq\; \frac{4\pi G m_a m_b}{\vec{q}^2}\Bigg[&\chi_f^{a\dagger}\chi_i^a\,\chi_f^{b\dagger}\chi_i^b+\frac{i(3m_a+4m_b)}{2m_am_b^2}\vec{S}_b\cdot\vec{p}\times\vec{q}\,\chi_f^{a\dagger}\chi_i^a\\ &+\frac{i(4m_a+3m_b)}{2m_a^2m_b}\vec{S}_a\cdot\vec{p}\times\vec{q}\,\chi_f^{b\dagger}\chi_i^b\\ &+\frac{1}{m_am_b}\left(\vec{S}_a\cdot\vec{q}\,\vec{S}_b\cdot\vec{q}-\vec{q}^2\vec{S}_a\cdot\vec{S}_b\right)\Bigg]\end{aligned} \tag{89}$$

whereby the lowest order potential for spin-1/2 – spin-1/2 scattering becomes

$${}^{\frac{1}{2}\frac{1}{2}}V_G^{(1)}(\vec{r}) \;=\; -\int\frac{d^3q}{(2\pi)^3}\;{}^{\frac{1}{2}\frac{1}{2}}\mathcal{M}^{(1)}(\vec{q})\,e^{-i\vec{q}\cdot\vec{r}}$$

$$
\begin{aligned}
&\simeq -\frac{Gm_am_b}{r}\chi_f^{a\dagger}\chi_i^a\chi_f^{b\dagger}\chi_i^b \\
&\quad -\frac{(3m_a+4m_b)}{2m_am_b^2}\vec{S}_b\cdot\vec{p}\times\vec{\nabla}\Big(-\frac{Gm_am_b}{r}\Big)\,\chi_f^{a\dagger}\chi_i^a \\
&\quad -\frac{(4m_a+3m_b)}{2m_a^2m_b}\vec{S}_a\cdot\vec{p}\times\vec{\nabla}\Big(-\frac{Gm_am_b}{r}\Big)\,\chi_f^{b\dagger}\chi_i^b \\
&\quad -\frac{1}{m_am_b}\vec{S}_a\cdot\vec{\nabla}\vec{S}_b\cdot\vec{\nabla}\Big(-\frac{Gm_am_b}{r}\Big) \\
&\simeq -\frac{Gm_am_b}{r}\chi_f^{a\dagger}\chi_i^a\chi_f^{b\dagger}\chi_i^b+\frac{G}{r^3}\frac{(3m_a+4m_b)}{2m_b}\vec{L}\cdot\vec{S}_b\,\chi_f^{a\dagger}\chi_i^a \\
&\quad +\frac{G}{r^3}\frac{(4m_a+3m_b)}{2m_a}\vec{L}\cdot\vec{S}_a\,\chi_f^{b\dagger}\chi_i^b+\frac{G}{r^5}\Big(3\vec{S}_a\cdot\vec{r}\,\vec{S}_b\cdot\vec{r}-r^2\vec{S}_a\cdot\vec{S}_b\Big)\,.
\end{aligned}
\tag{90}
$$

Note that since the piece proportional to $\vec{S}_a\cdot\vec{S}_b$ in Eq. (89) is analytic in $\vec{q}^2$ it yields only a short distance contribution which is omitted in the potential Eq. (90).

In this case when we evaluate the loop diagrams in Fig. 2, we notice that part of the spin-spin structure piece contains the form $q^2S_a\cdot S_b$ multiplying the nonanalytic structures L and S. Due to the extra factor of q^2 in this form, we must expand all loop integrals to *one order higher* in q^2 than before in order to be consistent. This has been done and does make a difference in our results for the spin-spin interaction piece. The results for the individual diagrams are then found to be

$$
\begin{aligned}
{}^{\frac12\frac12}\mathcal{M}_{2_a}^{(2)}(q) &= G^2m_am_b\Big[\mathcal{U}_a\mathcal{U}_b\,(-26L)+i\frac{\mathcal{E}_a\mathcal{U}_b}{m_a^2m_b}\,(-17L)+i\frac{\mathcal{U}_a\mathcal{E}_b}{m_am_b^2}\,(-17L) \\
&\qquad +\frac{S_a\cdot qS_b\cdot q}{m_am_b}\,(-9L)-\frac{q^2S_a\cdot S_b}{m_am_b}\,\Big(-\frac{15}{2}L\Big)\Big] \\
{}^{\frac12\frac12}\mathcal{M}_{2_b}^{(2)}(q) &= G^2m_am_b\Big[\mathcal{U}_a\mathcal{U}_b\,(15L+7m_aS) \\
&\qquad +i\frac{\mathcal{E}_a\mathcal{U}_b}{m_a^2m_b}\,\Big(\frac{21}{2}L+4m_aS\Big)+i\frac{\mathcal{U}_a\mathcal{E}_b}{m_am_b^2}\,(13L+8m_aS) \\
&\qquad +\frac{S_a\cdot qS_b\cdot q}{m_am_b}\,\Big(\frac{17}{3}L+\frac{5}{2}m_aS\Big) \\
&\qquad -\frac{q^2S_a\cdot S_b}{m_am_b}\,\Big(\frac{11}{3}L+\frac{3}{2}m_aS\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
{}^{\frac{1}{2}\frac{1}{2}}\mathcal{M}^{(2)}_{2c}(q) \;=\; & G^2 m_a m_b\Big[\mathcal{U}_a\mathcal{U}_b\left(15L+7m_bS\right) \\
& +i\frac{\mathcal{E}_a\mathcal{U}_b}{m_a^2 m_b}\left(13L+8m_bS\right)+i\frac{\mathcal{U}_a\mathcal{E}_b}{m_a m_b^2}\left(\frac{21}{2}L+4m_bS\right) \\
& +\frac{S_a\cdot qS_b\cdot q}{m_a m_b}\left(\frac{17}{3}L+\frac{5}{2}m_bS\right) \\
& -\frac{q^2 S_a\cdot S_b}{m_a m_b}\left(\frac{11}{3}L+\frac{3}{2}m_bS\right)\Big] \\
{}^{\frac{1}{2}\frac{1}{2}}\mathcal{M}^{(2)}_{2d}(q) \;=\; & G^2 m_a m_b\Big[\mathcal{U}_a\mathcal{U}_b\Big(L\left(\frac{4m_a m_b}{q^2}+\frac{31m_a^2-16m_a m_b+31m_b^2}{4m_a m_b}\right) \\
& +S\,\frac{9}{2}(m_a+m_b)\Big) \\
& +i\frac{\mathcal{E}_a\mathcal{U}_b}{m_a^2 m_b}\Big(L\left(\frac{8m_a m_b}{q^2}+\frac{124m_a^2-11m_a m_b+177m_b^2}{12m_a m_b}\right) \\
& +S\left(\frac{m_a m_b(4m_a+3m_b)}{s-s_0}+(13m_a+9m_b)\right)\Big) \\
& +i\frac{\mathcal{U}_a\mathcal{E}_b}{m_a m_b^2}\Big(L\left(\frac{8m_a m_b}{q^2}+\frac{177m_a^2-11m_a m_b+124m_b^2}{12m_a m_b}\right) \\
& +S\left(\frac{m_a m_b(3m_a+4m_b)}{s-s_0}+(9m_a+13m_b)\right)\Big) \\
& +\frac{S_a\cdot qS_b\cdot q}{m_a m_b}\Big(L\left(\frac{4m_a m_b}{q^2}+\frac{39m_a^2+25m_a m_b+39m_b^2}{6m_a m_b}\right) \\
& +S\left(\frac{m_a m_b(m_a+m_b)}{s-s_0}+\frac{39}{4}\left(m_a+m_b\right)\right)\Big) \\
& -\frac{q^2 S_a\cdot S_b}{m_a m_b}\Big(L\left(\frac{2m_a m_b}{q^2}+\frac{153m_a^2+218m_a m_b+153m_b^2}{24m_a m_b}\right) \\
& +S\left(\frac{m_a m_b(m_a+m_b)}{s-s_0}+\frac{41}{4}\left(m_a+m_b\right)\right)\Big) \\
& +\Big(2S_a\cdot p_3 S_b\cdot p_1+S_a\cdot qS_b\cdot p_1-S_a\cdot p_3 S_b\cdot q\Big)\frac{22L}{3m_a m_b}\Big] \\
- \quad & i4\pi G^2 m_a^2 m_b^2\frac{L}{q^2}\sqrt{\frac{m_a m_b}{s-s_0}}\Big(\mathcal{U}_a\mathcal{U}_b+2i\frac{\mathcal{E}_a\mathcal{U}_b}{m_a^2 m_b}+2i\frac{\mathcal{U}_a\mathcal{E}_b}{m_a m_b^2}
\end{aligned}
$$

$$+\frac{S_a\cdot qS_b\cdot q-\frac{1}{2}q^2S_a\cdot S_b}{m_am_b}\Bigg)$$

$$
\begin{aligned}
{}^{\frac12\frac12}\mathcal{M}^{(2)}_{2_e}(q) \;=\; G^2m_am_b\Bigg[&\mathcal{U}_a\mathcal{U}_b\Bigg(L\left(-\frac{4m_am_b}{q^2}-\frac{93m_a^2+232m_am_b+93m_b^2}{12m_am_b}\right)\\
&+S\left(-\frac{7}{2}(m_a+m_b)\right)\Bigg)\\
&+i\frac{\mathcal{E}_a\mathcal{U}_b}{m_a^2m_b}\Bigg(L\left(-\frac{8m_am_b}{q^2}-\frac{124m_a^2+235m_am_b+177m_b^2}{12m_am_b}\right)\\
&+S\left(-4m_a-\frac{33}{4}m_b\right)\Bigg)\\
&+i\frac{\mathcal{U}_a\mathcal{E}_b}{m_am_b^2}\Bigg(L\left(-\frac{8m_am_b}{q^2}-\frac{177m_a^2+235m_am_b+124m_b^2}{12m_am_b}\right)\\
&+S\left(-\frac{33}{4}m_a-4m_b\right)\Bigg)\\
&+\frac{S_a\cdot qS_b\cdot q}{m_am_b}\Bigg(L\left(-\frac{4m_am_b}{q^2}-\frac{39m_a^2+23m_am_b+39m_b^2}{6m_am_b}\right)\\
&+S\left(\frac{39}{4}\left(m_a+m_b\right)\right)\Bigg)\\
&-\frac{q^2S_a\cdot S_b}{m_am_b}\Bigg(L\left(-\frac{2m_am_b}{q^2}-\frac{153m_a^2+158m_am_b+153m_b^2}{24m_am_b}\right)\\
&+S\left(-\frac{3}{2}\left(m_a+m_b\right)\right)\Bigg)\\
&+\Big(2S_a\cdot p_3S_b\cdot p_1+S_a\cdot qS_b\cdot p_1-S_a\cdot p_3S_b\cdot q\Big)\frac{-22L}{3m_am_b}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
{}^{\frac12\frac12}\mathcal{M}^{(2)}_{2_f}(q) \;=\; G^2m_am_b\Bigg[&\mathcal{U}_a\mathcal{U}_b\Big(14L-2(m_a+m_b)S\Big)\\
&+i\frac{\mathcal{E}_a\mathcal{U}_b}{m_a^2m_b}\left(4L+\left(-2m_a-\frac{1}{2}m_b\right)S\right)\\
&+i\frac{\mathcal{U}_a\mathcal{E}_b}{m_am_b^2}\left(4L+\left(-\frac{1}{2}m_a-2m_b\right)S\right)\\
&+\frac{S_a\cdot qS_b\cdot q}{m_am_b}\left(-2L-(m_a+m_b)S\right)
\end{aligned}
$$

$$-\frac{q^2 S_a\cdot S_b}{m_a m_b}\left(-2L-(m_a+m_b)S\right)\Bigg]$$

$$\begin{aligned}
{}^{\frac{1}{2}\frac{1}{2}}\mathcal{M}^{(2)}_{2g}(q) \;=\; G^2 m_a m_b\Bigg[&\mathcal{U}_a\mathcal{U}_b\left(-\frac{43}{15}L\right)+i\frac{\mathcal{E}_a\mathcal{U}_b}{m_a^2 m_b}\left(-\frac{14}{5}L\right)+i\frac{\mathcal{U}_a\mathcal{E}_b}{m_a m_b^2}\left(-\frac{14}{5}L\right)\\
&+\frac{S_a\cdot qS_b\cdot q}{m_a m_b}\left(-\frac{7}{5}L\right)-\frac{q^2 S_a\cdot S_b}{m_a m_b}\left(-\frac{7}{5}L\right)\Bigg]. \qquad (91)
\end{aligned}$$

where we have defined

$$\mathcal{U}_a=\bar{u}(p_2)u(p_1)\qquad\qquad \mathcal{U}_b=\bar{u}(p_4)u(p_3) \qquad (92)$$

and

$$\mathcal{E}_i=\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_i^\delta \qquad (93)$$

with $i=a,b$ to keep our notation compact. The sum of all diagrams is found to be

$$\begin{aligned}
{}^{\frac{1}{2}\frac{1}{2}}\mathcal{M}^{(2)}_{tot}(q)=G^2 m_a m_b\Bigg[&\mathcal{U}_a\mathcal{U}_b\left(6(m_a+m_b)S-\frac{41}{5}L\right)\\
&+i\frac{\mathcal{E}_a\mathcal{U}_b}{m_a^2 m_b}\left(\frac{11(4m_a+3m_b)}{4}S-\frac{64}{5}L+\frac{m_a m_b(4m_a+3m_b)}{s-s_0}S\right)\\
&+i\frac{\mathcal{U}_a\mathcal{E}_b}{m_a m_b^2}\left(\frac{11(3m_a+4m_b)}{4}S-\frac{64}{5}L+\frac{m_a m_b(3m_a+4m_b)}{s-s_0}S\right)\\
&+S(m_a+m_b)\,\frac{S_a\cdot qS_b\cdot q-q^2S_a\cdot S_b}{m_a m_b}\left(\frac{37}{4}+\frac{m_a m_b}{s-s_0}\right)\\
&-L\,\frac{11S_a\cdot qS_b\cdot q-16\,q^2S_a\cdot S_b}{15m_a m_b}\Bigg]\\
-i4\pi G^2 m_a^2 m_b^2\frac{L}{q^2}\sqrt{\frac{m_a m_b}{s-s_0}}&\left(\mathcal{U}_a\mathcal{U}_b+2i\frac{\mathcal{E}_a\mathcal{U}_b}{m_a^2 m_b}+2i\frac{\mathcal{U}_a\mathcal{E}_b}{m_a m_b^2}\right.\\
&\left.+\frac{S_a\cdot qS_b\cdot q-\frac{1}{2}q^2S_a\cdot S_b}{m_a m_b}\right). \qquad (94)
\end{aligned}$$

Comparing with our finding for spin-0 – spin-1/2 scattering in Eq. (56), we notice the by now familiar universality of the amplitude: The form of the component proportional to $\bar{u}(p_4)u(p_3)$ of Eq. (56) is found here in the component proportional to $\mathcal{U}_a\mathcal{U}_b$, and the form of the structure proportional to $\epsilon_{\alpha\beta\gamma\delta}p_1^\alpha p_3^\beta q^\gamma S_b^\delta$ of Eq. (56) is now found in the component proportional to $\mathcal{U}_a\mathcal{E}_b$ in Eq. (94). Moreover, the amplitude is symmetric in $a \leftrightarrow b$ and new gravitational spin-spin interaction corrections arise. The quantum part of the spin-spin component has been calculated previously by Kirilin [17] whose result disagrees with our result for the numerical prefactors. For the quantum terms in the spin-orbit components however, we fully agree with Kirilin's results in [17]. In the nonrelativistic limit we obtain the expression

$$\begin{aligned}
{}^{\frac{1}{2}\frac{1}{2}}\mathcal{M}^{(2)}_{tot}(\vec{q}) \simeq & \left[G^2 m_a m_b\Big(6(m_a+m_b)S - \frac{41}{5}L\Big) - i4\pi G^2 m_a^2 m_b^2 \frac{L}{q^2}\frac{m_r}{p_0}\right]\chi_f^{a\dagger}\chi_i^a\chi_f^{b\dagger}\chi_i^b \\
& + \left[G^2\left(\frac{24m_a^3+56m_a^2m_b+45m_am_b^2+12m_b^3}{2(m_a+m_b)}S - \frac{128m_a+87m_b}{10}L\right)\right. \\
& \left. + \frac{G^2m_a^2m_b^2(4m_a+3m_b)}{(m_a+m_b)}\left(-i\frac{2\pi L}{p_0q^2}+\frac{S}{p_0^2}\right)\right]\frac{i}{m_a}\vec{S}_a\cdot\vec{p}\times\vec{q}\,\chi_f^{b\dagger}\chi_i^b \\
& + \left[G^2\left(\frac{12m_a^3+45m_a^2m_b+56m_am_b^2+24m_b^3}{2(m_a+m_b)}S - \frac{87m_a+128m_b}{10}L\right)\right. \\
& \left. + \frac{G^2m_a^2m_b^2(3m_a+4m_b)}{(m_a+m_b)}\left(-i\frac{2\pi L}{p_0q^2}+\frac{S}{p_0^2}\right)\right]\chi_f^{a\dagger}\chi_i^a\frac{i}{m_b}\vec{S}_b\cdot\vec{p}\times\vec{q} \\
& + G^2m_am_b\,\frac{19m_a^2+36m_am_b+19m_b^2}{2(m_a+m_b)}\,S\,\frac{\vec{S}_a\cdot\vec{q}\,\vec{S}_b\cdot\vec{q}-\vec{q}^2\vec{S}_a\cdot\vec{S}_b}{m_am_b} \\
& - G^2m_am_b\,L\,\frac{11\vec{S}_a\cdot\vec{q}\,\vec{S}_b\cdot\vec{q}-16\vec{q}^2\vec{S}_a\cdot\vec{S}_b}{15m_am_b} \\
& + \frac{G^2m_a^3m_b^3}{m_a+m_b}\,\frac{S}{p_0^2}\,\frac{\vec{S}_a\cdot\vec{q}\,\vec{S}_b\cdot\vec{q}-\vec{q}^2\vec{S}_a\cdot\vec{S}_b}{m_am_b} \\
& + \frac{G^2m_a^3m_b^3}{m_a+m_b}\left(-i\frac{4\pi L}{p_0q^2}\right)\frac{\vec{S}_a\cdot\vec{q}\,\vec{S}_b\cdot\vec{q}-\frac{1}{2}\vec{q}^2\vec{S}_a\cdot\vec{S}_b}{m_am_b} \qquad (95)
\end{aligned}$$

which obviously has to exhibit the universalities of the spin-independent and the spin-orbit pieces. Therefore, we know that the subtraction of the second Born iteration successfully removes the unwanted i/p_0 and $1/p_0^2$ structures for the spin-independent and the spin-orbit components.

The leading spin-spin term of the second Born iteration amplitude is new, however, and we compute

$${}^{\frac{1}{2}\frac{1}{2}}\mathrm{Amp}^{(2)}_{S-S}(\vec{q}) \;=\; -\int\frac{d^3\ell}{(2\pi)^3}\,\frac{\left\langle\vec{p}_f\left|{}^{\frac{1}{2}\frac{1}{2}}\hat{V}^{(1)}_{S-I}\right|\vec{\ell}\right\rangle\left\langle\vec{\ell}\left|{}^{\frac{1}{2}\frac{1}{2}}\hat{V}^{(1)}_{S-S}\right|\vec{p}_i\right\rangle}{\frac{p_0^2}{2m_r}-\frac{\ell^2}{2m_r}+i\epsilon}$$

$$
\begin{aligned}
& -\int \frac{d^3\ell}{(2\pi)^3} \frac{\left\langle \vec{p}_f \left| {}^{\frac{1}{2}\frac{1}{2}}\hat{V}^{(1)}_{S-S} \right| \vec{\ell} \right\rangle \left\langle \vec{\ell} \left| {}^{\frac{1}{2}\frac{1}{2}}\hat{V}^{(1)}_{S-I} \right| \vec{p}_i \right\rangle}{\frac{p_0^2}{2m_r} - \frac{\ell^2}{2m_r} + i\epsilon} \\
= \; & \frac{1}{m_a m_b} S_a^r S_b^s \\
& \Bigg(i\int \frac{d^3\ell}{(2\pi)^3} \frac{c_G^2}{|\vec{\ell}-\vec{p}_f|^2+\lambda^2} \frac{i}{\frac{p_0^2}{2m_r} - \frac{\ell^2}{2m_r} + i\epsilon} \frac{c_G^2 (p_i-\ell)^r (p_i-\ell)^s}{|\vec{p}_i - \vec{\ell}|^2+\lambda^2} \\
& + i\int \frac{d^3\ell}{(2\pi)^3} \frac{c_G^2 (\ell-p_f)^r (\ell-p_f)^s}{|\vec{\ell}-\vec{p}_f|^2+\lambda^2} \frac{i}{\frac{p_0^2}{2m_r} - \frac{\ell^2}{2m_r} + i\epsilon} \frac{c_G^2}{|\vec{p}_i - \vec{\ell}|^2+\lambda^2} \Bigg) \\
\xrightarrow{\lambda\to 0} \; & \frac{1}{m_a m_b} \Bigg[\left(\vec{S}_a\cdot\vec{p_i}\,\vec{S}_b\cdot\vec{p_i} + \vec{S}_a\cdot\vec{p_f}\,\vec{S}_b\cdot\vec{p_f} \right) H \\
& \qquad - \vec{S}_a\cdot(\vec{p_i}+p_f)\vec{S}_b\cdot\vec{H} - \vec{S}_a\cdot\vec{H}\,\vec{S}_b\cdot(\vec{p_i}+p_f) \\
& \qquad + 2\, S_a^r S_b^s\, H^{rs} \Bigg] \\
= \; & \frac{G^2 m_a^2 m_b^2}{m_a+m_b} \frac{S}{p_0^2} (\vec{S}_a\cdot\vec{q}\,\vec{S}_b\cdot\vec{q} - \vec{q}^{\,2}\vec{S}_a\cdot\vec{S}_b) \\
+ \; & \frac{G^2 m_a^2 m_b^2}{m_a+m_b} \left(-i\frac{4\pi L}{p_0 q^2} \right) (\vec{S}_a\cdot\vec{q}\,\vec{S}_b\cdot\vec{q} - \frac{1}{2}\vec{q}^{\,2}\vec{S}_a\cdot\vec{S}_b).
\end{aligned} \tag{96}
$$

where we again defined $c_G^2 \equiv -4\pi G m_a m_b$. With this, the full second Born iteration amplitude becomes

$$
\begin{aligned}
{}^{\frac{1}{2}\frac{1}{2}}\mathrm{Amp}_G^{(2)}(\vec{q}) \; = \; & {}^{\frac{1}{2}\frac{1}{2}}\mathrm{Amp}_{S-I}^{(2)}(\vec{q}) + {}^{\frac{1}{2}\frac{1}{2}}\mathrm{Amp}_{S-O}^{(2)}(\vec{q}) + {}^{\frac{1}{2}\frac{1}{2}}\mathrm{Amp}_{S-S}^{(2)}(\vec{q}) \\
= \; & -i4\pi G^2 m_a^2 m_b^2 \frac{L}{q^2} \frac{m_r}{p_0} \chi_f^{a\dagger}\chi_i^a\, \chi_f^{b\dagger}\chi_i^b \\
+ \; & \frac{G^2 m_a^2 m_b^2 (2m_a+m_b)}{m_a+m_b} \left(-i\frac{2\pi L}{p_0 q^2} + \frac{S}{p_0^2} \right) \frac{i}{m_a} \vec{S}_a\cdot\vec{p}\times\vec{q}\; \chi_f^{b\dagger}\chi_i^b \\
+ \; & \frac{G^2 m_a^2 m_b^2 (m_a+2m_b)}{m_a+m_b} \left(-i\frac{2\pi L}{p_0 q^2} + \frac{S}{p_0^2} \right) \chi_f^{a\dagger}\chi_i^a\; \frac{i}{m_b} \vec{S}_b\cdot\vec{p}\times\vec{q} \\
+ \; & \frac{G^2 m_a^2 m_b^2}{m_a+m_b} \frac{S}{p_0^2} (\vec{S}_a\cdot\vec{q}\,\vec{S}_b\cdot\vec{q} - \vec{q}^{\,2}\vec{S}_a\cdot\vec{S}_b) \\
+ \; & \frac{G^2 m_a^2 m_b^2}{m_a+m_b} \left(-i\frac{4\pi L}{p_0 q^2} \right) (\vec{S}_a\cdot\vec{q}\,\vec{S}_b\cdot\vec{q} - \frac{1}{2}\vec{q}^{\,2}\vec{S}_a\cdot\vec{S}_b)
\end{aligned} \tag{97}
$$

and we observe that when this amplitude is subtracted from the full one loop scattering amplitude Eq. (95), *all* terms involving $1/p_0^2$ and i/p_0 disappear leaving behind a well-defined second order potential

$$
\begin{aligned}
{}^{\frac{1}{2}\frac{1}{2}}V_G^{(2)}(\vec{r}) &= -\int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \left[{}^{\frac{1}{2}\frac{1}{2}}\mathcal{M}_{tot}^{(2)}(\vec{q}) - {}^{\frac{1}{2}\frac{1}{2}}\mathrm{Amp}_G^{(2)}(\vec{q})\right] \\
&\simeq \left[-\frac{3G^2 m_a m_b (m_a+m_b)}{r^2} - \frac{41 G^2 m_a m_b \hbar}{10\pi r^3}\right] \chi_f^{a\dagger}\chi_i^a\,\chi_f^{b\dagger}\chi_i^b \\
&+ \frac{1}{m_a}\vec{S}_a\cdot\vec{p}\times\vec{\nabla}\Bigg[\frac{G^2(24m_a^3+56m_a^2m_b+45m_am_b^2+12m_b^3)}{4(m_a+m_b)r^2} \\
&\qquad + \frac{G^2(128m_a+87m_b)\hbar}{20\pi r^3}\Bigg]\chi_f^{a\dagger}\chi_i^a \\
&+ \chi_f^{a\dagger}\chi_i^a\,\frac{1}{m_b}\vec{S}_b\cdot\vec{p}\times\vec{\nabla}\Bigg[\frac{G^2(12m_a^3+45m_a^2m_b+56m_am_b^2+24m_b^3)}{4(m_a+m_b)r^2} \\
&\qquad + \frac{G^2(87m_a+128m_b)\hbar}{20\pi r^3}\Bigg] \\
&+ \frac{\vec{S}_a\cdot\vec{\nabla}\vec{S}_b\cdot\vec{\nabla} - \vec{\nabla}^2\vec{S}_a\cdot\vec{S}_b}{m_am_b}\left[\frac{G^2m_am_b(19m_a^2+36m_am_b+19m_b^2)}{4(m_a+m_b)r^2}\right] \\
&+ \frac{11\vec{S}_a\cdot\vec{\nabla}\vec{S}_b\cdot\vec{\nabla} - 16\vec{\nabla}^2\vec{S}_a\cdot\vec{S}_b}{15m_am_b}\left[\frac{G^2m_am_b\hbar}{2\pi r^3}\right] \\
&\simeq \left[-\frac{3G^2 m_a m_b (m_a+m_b)}{r^2} - \frac{41 G^2 m_a m_b \hbar}{10\pi r^3}\right] \chi_f^{a\dagger}\chi_i^a\,\chi_f^{b\dagger}\chi_i^b \\
&+ \Bigg[\frac{G^2(24m_a^3+56m_a^2m_b+45m_am_b^2+12m_b^3)}{2m_a(m_a+m_b)r^4} \\
&\qquad + \frac{3G^2(128m_a+87m_b)\hbar}{20\pi m_a r^5}\Bigg]\vec{L}\cdot\vec{S}_a\,\chi_f^{b\dagger}\chi_i^b \\
&+ \Bigg[\frac{G^2(12m_a^3+45m_a^2m_b+56m_am_b^2+24m_b^3)}{2m_b(m_a+m_b)r^4} \\
&\qquad + \frac{3G^2(87m_a+128m_b)\hbar}{20\pi m_b r^5}\Bigg]\chi_f^{a\dagger}\chi_i^a\,\vec{L}\cdot\vec{S}_b \\
&+ \left[\frac{2G^2(19m_a^2+36m_am_b+19m_b^2)}{(m_a+m_b)r^4}\right]\left(\vec{S}_a\cdot\vec{r}\,\vec{S}_b\cdot\vec{r}/r^2 - \frac{1}{2}\vec{S}_a\cdot\vec{S}_b\right) \\
&+ \frac{11G^2\hbar}{2\pi r^5}\left(\vec{S}_a\cdot\vec{r}\,\vec{S}_b\cdot\vec{r}/r^2 - \frac{43}{55}\vec{S}_a\cdot\vec{S}_b\right) \qquad (98)
\end{aligned}
$$

which besides the universal spin-independent and spin-orbit components displays a new (and presumably universal) gravitational spin-spin interaction.

5 Conclusions

Above we have analyzed the gravitational scattering of two particles having nonzero mass. In lowest order the interaction arises from one-graviton exchange and leads at threshold to the well known Newtonian interaction $V(r) = -Gm_am_b/r$. Inclusion of two-graviton exchange effects means adding the contribution from box, cross-box, triangle, and bubble diagrams, which have a rather complex form. The calculation can be simplified, however, by using ideas from effective field theory. The point is that if one is interested only in the leading long-range behavior of the interaction, then one need retain only the leading nonanalytic small momentum-transfer piece of the scattering amplitude. Specifically, the terms which one retains are those which are nonanalytic and behave as either $1/\sqrt{-q^2}$ or $\log -q^2$. When Fourier transformed, the former leads to classical ($\hbar$-independent) terms in the potential of order G^2M^3/r^2 while the latter generates quantum mechanical ($\hbar$-dependent) corrections of order $G^2M^2\hbar/r^3$. (Of course, there are also shorter range nonanalytic contributions than these that are generated by scattering terms of order $q^{2n}/\sqrt{-q^2}$ or $q^{2n}\log -q^2$. However, these pieces are higher order in momentum transfer and thus lead to shorter distance effects than those considered above and are therefore neglected in our discussion.)

Specific calculations were done for particles with spin $0-0$, $0-1/2$, $0-1$, and $1/2-1/2$ and various universalities were found. In particular, we found that in each case there was a spin-independent contribution of the form

$$\begin{aligned}{}^{S_aS_b}\mathcal{M}^{(2)}_{tot}(q) = &\left[G^2m_am_b\left(6(m_a+m_b)S - \frac{41}{5}L\right) - i4\pi G^2m_a^2m_b^2\frac{L}{q^2}\sqrt{\frac{m_am_b}{s-s_0}}\,\right] \\ &\times \langle S_a, m_{af}|S_a, m_{ai}\rangle \langle S_b, m_{bf}|S_b, m_{bi}\rangle \end{aligned} \qquad (99)$$

where $L = \log -q^2$ and $S = \pi^2/\sqrt{-q^2}$ and with S_a the spin of particle a and S_b the spin of particle b with projections m_a and m_b on the quantization axis. The imaginary component of the amplitude, which would not, when Fourier-transformed lead to a real potential, is eliminated when the iterated lowest order potential contribution is subtracted, leading to a well defined spin-independent second order potential of universal form

$$
\begin{aligned}
{}^{S_aS_b}V^{(2)}_{S-I}(\vec{r}) &= \left[-\frac{3G^2m_am_b(m_a+m_b)}{r^2} - \frac{41G^2m_am_b\hbar}{10\pi r^3}\right] \\
&\quad \times \langle S_a, m_{af}|S_a, m_{ai}\rangle \langle S_b, m_{bf}|S_b, m_{bi}\rangle \qquad (100)
\end{aligned}
$$

whose classical component depends on the way the iteration of the leading order potential is performed. This ambiguity shows that the second order potential itself is not an observable, but we use it as a nice way to display the long distance components of the scattering amplitude in coordinate space.

If either scattering particle carries spin then there exists an additional spin-orbit contribution, whose form is also universal

$$
\begin{aligned}
{}^{S_aS_b}V^{(2)}_{S-O}(\vec{r}) &= \left[\frac{G^2(24m_a^3+56m_a^2m_b+45m_am_b^2+12m_b^3)}{2m_a(m_a+m_b)r^4}\right. \\
&\quad \left. + \frac{3G^2(128m_a+87m_b)\hbar}{20\pi m_a r^5}\right] \times \vec{L}\cdot\vec{S}_a \langle S_b, m_{bf}|S_b, m_{bi}\rangle \\
&+ \left[\frac{G^2(12m_a^3+45m_a^2m_b+56m_am_b^2+24m_b^3)}{2m_b(m_a+m_b)r^4}\right. \\
&\quad \left. + \frac{3G^2(87m_a+128m_b)\hbar}{20\pi m_b r^5}\right] \times \langle S_a, m_{af}|S_a, m_{ai}\rangle \vec{L}\cdot\vec{S}_b \qquad (101)
\end{aligned}
$$

where we have defined

$$
\vec{S}_a = \left\langle S_a, m_{af}\left|\vec{S}\right|S_a, m_{ai}\right\rangle \quad \text{and} \quad \vec{S}_b = \left\langle S_b, m_{bf}\left|\vec{S}\right|S_b, m_{bi}\right\rangle .
$$

In this case a well defined second order potential required the subtraction of infrared singular terms behaving as both i/p_0 *and* $1/p_0^2$ which arise from the iterated lowest order potential.

In the calculation of spin-0 – spin-1 scattering we encountered new tensor structures including a quadrupole interaction. Unfortunately, the subtraction of the i/p_0 and $1/p_0^2$ tensor pieces in the two-graviton exchange amplitude was not successful with our simple nonrelativistic iteration of the leading order potential so that we cannot at this time give the form of the quadrupole component of the potential. Further work is needed to clarify this issue. The corrections to the spin-spin interaction have only been calculated in spin-1/2 – spin-1/2 scattering where we found their contributions to the scattering amplitude and to the potential. Since we verified these forms only for a single spin configuration we have not confirmed its universality which we, however, strongly suspect. Of course, for higher spin configurations, there also exist quadrupole-quadrupole interactions, spin-quadrupole interactions, etc. However, the calculation of such forms becomes increasingly cumbersome as the spin increases, and the phenomenological importance becomes smaller. Thus we end our calculations here.

One point of view to interpret the universalities of the long distance components of the scattering amplitudes and the resulting potentials is that if we increase the spins of our scattered particles, all we do is to add additional multipole moments. The spin-independent component can then be viewed as a monopole-monopole interaction, the spin-orbit piece as a dipole-monopole interaction etc. As long as we do not change the quantum numbers that characterize the lower multipoles, an increase in spin of the scattered particles merely adds new interactions that are less important at long distances. The same kind of universalities were also found in long distance effects in electromagnetic scattering [9] and in the long range components of mixed electromagnetic-gravitational scattering [18].

Acknowledgements

We would like to thank John Donoghue for many clarifying discussions and Walter Goldberger for important comments. This work was supported in part by the National Science Foundation under award PHY05-53304 (BRH and AR) and by the the US Department of Energy under grant DE-FG-02-92ER40704 (AR).

A Iteration Integrals

In this appendix we give the integrals

$$[H;H_r;H_{rs}] = i\int \frac{d^3\ell}{(2\pi)^3}\frac{-4\pi G m_a m_b}{|\vec{\ell}-\vec{p}_f|^2+\lambda^2}\frac{i[1;\ell_r;\ell_r\ell_s]}{\frac{p_0^2}{2m_r}-\frac{\ell^2}{2m_r}+i\epsilon}\frac{-4\pi G m_a m_b}{|\vec{p}_i-\vec{\ell}|^2+\lambda^2} \quad (102)$$

which are needed in order to perform the iteration of the lowest order Newton potentials. Here we list only the results; for a more detailed derivation, albeit with a different prefactor, see [9]. The leading expressions for the iteration integrals read

$$
\begin{aligned}
H &\simeq -i4\pi G^2 m_a^2 m_b^2 \, \frac{L}{q^2}\frac{m_r}{p_0} \\
H_r &\simeq (p_i + p_f)_r \; G^2 m_a^2 m_b^2 \left(-i2\pi \, \frac{L}{q^2}\frac{m_r}{p_0} + S\frac{m_r}{p_0^2} \right) \\
H_{rs} &\simeq \delta_{rs} \; \vec{q}^{\,2} \; G^2 m_a^2 m_b^2 \left(i\pi \frac{L}{q^2}\frac{m_r}{p_0} - \frac{1}{2} S\frac{m_r}{p_0^2} \right) \\
&+ (p_i + p_f)_r (p_i + p_f)_s \; G^2 m_a^2 m_b^2 \left(-i\pi \frac{L}{q^2}\frac{m_r}{p_0} + S\frac{m_r}{p_0^2} \right) \\
&+ (p_i - p_f)_r (p_i - p_f)_s \; G^2 m_a^2 m_b^2 \left(-i\pi \frac{L}{q^2}\frac{m_r}{p_0} + \frac{1}{2} S\frac{m_r}{p_0^2} \right). \qquad (103)
\end{aligned}
$$

B Classical Equations of Motion

Above we have argued that the scattering amplitude which is ultimately related to observables in quantum field theory is a physical result while the potential we have given is not an observable and depends both on the gauge, *i.e.*, the choice of coordinates used, and on the way we perform the iteration, *i.e.*, on the way we perform the matching. While the classical component of our potential is in fact plagued by these ambiguities, the quantum part is unique since it is unaffected by how we perform the matching and since a quantum field theory calculation in any gauge would result in the same result [13].

In this appendix we will demonstrate how we can recover the classical equations of motion from our scattering amplitudes by setting up the Einstein-Infeld-Hoffmann (EIH) Lagrangian [2]. The EIH Lagrangian is itself dependent on the choice of coordinates, but can be expressed in the center of mass frame ($\vec{P} \equiv \vec{p}_a = -\vec{p}_b$, $\vec{r} \equiv \vec{r}_a - \vec{r}_b$) in a general way as [19, 20]

$$L_{EIH} = T - V \qquad (104)$$

where the kinetic energy to NLO in the nonrelativistic expansion reads

$$T = \frac{\vec{P}^{\,2}}{2m_a} + \frac{\vec{P}^{\,2}}{2m_b} - \frac{\vec{P}^{\,4}}{8m_a^3} - \frac{\vec{P}^{\,4}}{8m_b^3} \qquad (105)$$

and the potential is

$$V = V^{(1)} + V^{(2)} \qquad (106)$$

with

$$V^{(1)} = -\frac{Gm_am_b}{r}\left\{1+\left[\frac{1}{2}+\left(\frac{3}{2}-\alpha\right)\frac{(m_a+m_b)^2}{m_am_b}\right]\frac{\vec{P}^2}{m_am_b} + \left[\frac{1}{2}+\alpha\frac{(m_a+m_b)^2}{m_am_b}\right]\frac{\left(\vec{P}\cdot\hat{r}\right)^2}{m_am_b}\right\} \tag{107}$$

$$V^{(2)} = (1-2\alpha)\frac{G^2m_am_b(m_a+m_b)}{2r^2}. \tag{108}$$

The parameter α parameterizes the choice of coordinates used, where $\alpha = 0$ was the gauge of the original EIH result. The coordinate change

$$\vec{r} \to \vec{r}\left(1-\alpha\frac{G(m_a+m_b)}{r}\right) \tag{109}$$

which implies

$$\vec{P} \to \vec{P}+\alpha\frac{G(m_a+m_b)}{r}\left[\vec{P}-\left(\vec{P}\cdot\hat{r}\right)\hat{r}\right] \tag{110}$$

brings the original EIH Lagrangian into the form above, which is the most general result.

Since we perform our matching on-shell, *i.e.*, we use the on-shell one-graviton exchange amplitude to define the leading order $\mathcal{O}(G)$ potential, terms proportional to $\vec{P}\cdot\hat{r}$ would never arise. Clearly, our result must be in a gauge such that the coefficient of the structure

$$\frac{\left(\vec{P}\cdot\hat{r}\right)^2}{m_am_b}$$

in Eq. (107) vanishes. That is the case if and only if the gauge parameter is

$$\alpha = -\frac{m_am_b}{2(m_a+m_b)^2} \tag{111}$$

whereby the EIH potential becomes

$$V^{(1)} = -\frac{Gm_am_b}{r}\left\{1+\left[1+\frac{3}{2}\frac{(m_a+m_b)^2}{m_am_b}\right]\frac{\vec{P}^2}{m_am_b}\right\} \tag{112}$$

$$V^{(2)} = \left(1+\frac{m_am_b}{(m_a+m_b)^2}\right)\frac{G^2m_am_b(m_a+m_b)}{2r^2}. \tag{113}$$

Comparing the EIH potential $V^{(1)}$ in this gauge of Eq. (112) with the long distance component of the leading order spin-independent potential in Eq. (22) we find full agreement for the relativistic corrections to the $\mathcal{O}(G)$ potential. However, comparing the EIH potential $V^{(2)}$ in this gauge of Eq. (113) with the classical component of our spin-independent potential in Eq. (33) we see that the two do not agree! The reason for this discrepancy is that we elected to use a nonrelativistic iteration when we performed the second Born iteration of the leading order potential in Eq. (32). This procedure, however, is not self-consistent when we are interested in equations of motion at NLO, for which we must account for the leading relativistic corrections in the iteration. In particular, we must use expressions for the potential and the propagator in Eq. (30) which include the leading relativistic corrections[10]

$$\left\langle \vec{p}_f \left| {}^0\hat{V}^{(1)}_{NLO} \right| \vec{p}_i \right\rangle \simeq -\frac{4\pi G m_a m_b}{\vec{q}^2}\left[1+\frac{\vec{p}_i^{\,2}+\vec{p}_f^{\,2}}{2m_a m_b}\left(1+\frac{3(m_a+m_b)^2}{2m_a m_b}\right)\right] \quad (114)$$

$$G^{(0)}_{NLO}(\ell) = \frac{i}{\frac{p_0^2}{2m_r}-\frac{\ell^2}{2m_r}+i\epsilon}\times\left[1+\left(\frac{p_0^2}{4m_r^2}+\frac{\ell^2}{4m_r^2}\right)\left(1-3\frac{m_r^2}{m_a m_b}\right)\right] \quad (115)$$

which yields a second Born iteration amplitude

$$\begin{aligned} {}^0\mathrm{Amp}^{(2)}_{NLO}(\vec{q}) &\simeq -\int \frac{d^3\ell}{(2\pi)^3}\frac{4\pi G m_a m_b}{|\vec{p}_f-\vec{\ell}|^2}\frac{1}{\frac{p_0^2}{2m_r}-\frac{\ell^2}{2m_r}+i\epsilon}\frac{4\pi G m_a m_b}{|\vec{\ell}-\vec{p}_i|^2} \\ &\quad \times\left[1+\frac{(p_0^2+\ell^2)}{m_a m_b}\left(\frac{1}{4}+\frac{7}{4}\frac{m_a m_b}{m_r^2}\right)\right] \\ &\simeq H+\frac{1}{m_a m_b}\left(p_0^2 H+\delta_{rs}H_{rs}\right)\left(\frac{1}{4}+\frac{7}{4}\frac{m_a m_b}{m_r^2}\right) \\ &\simeq -i4\pi G^2 m_a^2 m_b^2\frac{L}{q^2}\frac{m_r}{p_0}+\frac{G^2 m_a^2 m_b^2}{m_a+m_b}\left(1+\frac{7(m_a+m_b)^2}{m_a m_b}\right)S. \end{aligned} \quad (116)$$

Subtracting this iterated amplitude which includes all corrections to NLO from the scattering amplitude ${}^0\mathcal{M}^{(2)}_{tot}(\vec{q})$ of Eq. (21) we find then the second order potential

$${}^0V^{(2)}_{NLO}(\vec{r}) = -\int \frac{d^3q}{(2\pi)^3}e^{-i\vec{q}\cdot\vec{r}}\left[{}^0\mathcal{M}^{(2)}_{tot}(\vec{q})-{}^0\mathrm{Amp}^{(2)}_{NLO}(\vec{q})\right]$$

[10]The subscript NLO in this sections refers to the iteration being performed at NLO in the relativistic expansion.

$$= \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} G^2 m_a m_b \left[\left((m_a + m_b) + \frac{m_a m_b}{m_a + m_b} \right) S + \frac{41}{5} L \right]$$
$$= \left(1 + \frac{m_a m_b}{(m_a + m_b)^2} \right) \frac{G^2 m_a m_b (m_a + m_b)}{2r^2} - \frac{41 G^2 m_a m_b \hbar}{10\pi r^3} \quad (117)$$

and observe that now the classical component agrees with the $\mathcal{O}(G^2)$ EIH potential of Eq. (113).

Thus we have shown that if we consistently take into account the v^2 and GM/r corrections beyond Newtonian physics we reproduce the EIH Lagrangian in a certain gauge. From the resulting EIH Lagrangian we could evaluate observables such as the precession of the perihelion of Mercury which must clearly be independent of the gauge used. The inclusion of the v^2 corrections is required since the equations of motion can be used to describe bound states where $v^2 \sim GM/r$ by the virial theorem.

However, our methods are clearly clumsy for the calculation of classical observables. Recently, Goldberger and Rothstein have developed an effective field theory of gravity which is optimized for calculating classical observables of bound states called NRGR [21, 22, 23, 24, 25, 26]. Here the external particles are static sources so that no loops are to be calculated in their theory when calculating classical observables since the only propagating particles present are gravitons which are massless and thus the loop expansion in NRGR corresponds to an expansion in $\hbar$. In the NRGR framework the spin-dependent classical equations of motion were calculated recently to NLO by Porto and Rothstein [27, 28, 29, 30] so that we will not continue here to evaluate the corresponding spin-dependent classical potentials consistently taking into account all relativistic $\mathcal{O}(v^2)$ effects in the iteration.

References

[1] See, *e.g.*, S. Weinberg,**Gravitation and Cosmology**, Wiley, New York (1972).

[2] A. Einstein, L. Infeld and B. Hoffmann, Annals Math. **39**, 65 (1938).

[3] J. F. Donoghue, Phys. Rev. Lett. **72**, 2996 (1994) [arXiv:gr-qc/9310024].

[4] J. F. Donoghue, Phys. Rev. D **50**, 3874 (1994) [arXiv:gr-qc/9405057].

[5] J. F. Donoghue, arXiv:gr-qc/9512024.

[6] C. P. Burgess, Living Rev. Rel. **7**, 5 (2004) [arXiv:gr-qc/0311082].

[7] N. E. J. Bjerrum-Bohr, J. F. Donoghue and B. R. Holstein, Phys. Rev. D **68**, 084005 (2003) [Erratum-ibid. D **71**, 069904 (2005)] [arXiv:hep-th/0211071].

[8] B. R. Holstein, Phys. Rev. D **74**, 084030 (2006).
B. R. Holstein, arXiv:gr-qc/0607051.

[9] B.R. Holstein, A. Ross, arXiv:0802.0715 [hep-ph].

[10] Y. Iwasaki, Prog. Theo. Phys. **46**, 1587 (1971).

[11] I. B. Khriplovich and G. G. Kirilin, J. Exp. Theor. Phys. **95**, 981 (2002) [Zh. Eksp. Teor. Fiz. **95**, 1139 (2002)] [arXiv:gr-qc/0207118].

[12] I. B. Khriplovich and G. G. Kirilin, eConf **C0306234**, 1361 (2003) [J. Exp. Theor. Phys. **98**, 1063 (2004 ZETFA,125,1219-1228.2004)] [arXiv:gr-qc/0402018].

[13] N. E. J. Bjerrum-Bohr, J. F. Donoghue and B. R. Holstein, Phys. Rev. D **67**, 084033 (2003) [Erratum-ibid. D **71**, 069903 (2005)] [arXiv:hep-th/0211072].

[14] G. 't Hooft and M. J. G. Veltman, Annales Poincare Phys. Theor. A **20**, 69 (1974).

[15] J. Sucher, Phys. Rev. D **49**, 4284 (1994).

[16] S. Deser and P. van Nieuwenhuizen, Phys. Rev. D **10**, 411 (1974).

[17] G. G. Kirilin, Nucl. Phys. B **728**, 179 (2005) [arXiv:gr-qc/0507070].

[18] B.R. Holstein, A. Ross, arXiv:0802.0717 [hep-ph].

[19] K. Hiida and H. Okamura, Prog. Theor. Phys. **47**, 1743 (1972).

[20] B. M. Barker and R. F. O'Connell, Phys. Rev. D **12**, 329 (1975).

[21] W. D. Goldberger and I. Z. Rothstein, Phys. Rev. D **73**, 104029 (2006) [arXiv:hep-th/0409156].

[22] W. D. Goldberger and I. Z. Rothstein, Gen. Rel. Grav. **38**, 1537 (2006) [Int. J. Mod. Phys. D **15**, 2293 (2006)] [arXiv:hep-th/0605238].

[23] W. D. Goldberger, arXiv:hep-ph/0701129.

[24] W. D. Goldberger and I. Z. Rothstein, Phys. Rev. D **73**, 104030 (2006) [arXiv:hep-th/0511133].

[25] B. Kol and M. Smolkin, arXiv:0712.2822 [hep-th].

[26] B. Kol and M. Smolkin, arXiv:0712.4116 [hep-th].

[27] R. A. Porto, Phys. Rev. D **73**, 104031 (2006) [arXiv:gr-qc/0511061].

[28] R. A. Porto and I. Z. Rothstein, Phys. Rev. Lett. **97**, 021101 (2006) [arXiv:gr-qc/0604099].

[29] R. A. Porto, arXiv:0710.5150 [hep-th].

[30] R. A. Porto and I. Z. Rothstein, arXiv:0712.2032 [gr-qc].