SWELLING AND FOLDING AS MECHANISMS OF 3D SHAPE FORMATION IN THIN ELASTIC SHEETS

A Dissertation Presented

by

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To my dear loving wife Joyce E. Dias
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ABSTRACT

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We work with two different mechanisms to generate geometric frustration on thin elastic sheets; isotropic differential growth and folding. We describe how controlled growth and prescribing folding patterns are useful tools for designing three-dimensional objects from information printed in two dimensions. The first mechanism is inspired by the possibility to control shapes by swelling polymer films, where we propose a solution for the problem of shape formation by asking the question, “what 2D metric should be prescribed to achieve a given 3D shape?”, namely the reverse problem. We choose two different types of initial configurations of sheets, disk-like with one boundary and annular with two boundaries. We demonstrate our technique by choosing four examples of 3D axisymmetric shapes and finding the respective swelling factors to achieve the desired shape. Second, we present a mechanical
model for a single curved fold that explains both the buckled shape of a closed fold and its mechanical stiffness. The buckling arises from the geometrical frustration between the prescribed crease angle and the bending energy of the sheet away from the crease. This frustration increases as the sheet’s area increases. Stiff folds result in creases with constant space curvature while softer folds inherit the broken symmetry of the buckled shape. We extend the application of our numerical model to show the potential to study multiple fold structures.
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INTRODUCTION

A source of great inspiration is found in the work of the Scottish biologist and mathematician D’Arcy Wentworth Thompson (2 May 1860 – 21 June 1948), in his masterpiece *On Growth and Form* [1], “For the harmony of the world is made manifest in Form and Number, and the heart and soul and all the poetry of Natural Philosophy are embodied in the concept of mathematical beauty” ([1] pp. 1096-7). When thinking about the classification of the morphology of living organisms – for instance the complex world of plants – one finds a very rich scenario for studying surface geometries that emerge in many different length scales, such as the wrinkles, buckles, and folded structures. It is remarkable how differential growth plays a role in driving shape formation and how its mechanism, once understood, could potentially be manipulated to design and control features of new materials. Inspired by this scenario, we pursue a mathematical and physical characterization of membrane elasticity by means of differential growth and folding, which we believe to be important mechanisms for shape formation.

Throughout this thesis we use the concept of geometric frustration as the basis to generate 3D shapes. We define an elastic material to be frustrated if it retains residual stresses. In the examples we shall discuss, this happens either because of excess of material due to differential growth or by means of incompatibility between imposed and realizable distances among material points (in other words metrics). In our context, these two ways in which frustration happens can be seen in a unified picture. Figure 1 illustrates how folding, top row, and growth, bottom row, can both be seen to yield a transient state in which the geometry is
incompatible to a flat configuration. Here incompatibility is a synonym of frustration, which is released by allowing out-of-plane buckling.

**Figure 1.** Example of geometric frustration. Two ways to generate geometric frustration, by adding an excess angle in a cut annulus and a folded structure. In the top row, folding yields a higher center line curvature, which reduces the gap. In both the top and bottom row a wedge is inserted into the gap resulting in a frustrated transient state. Frustration is realized by allowing out-of-plane buckling.

In Chapter 1 we present the two problems, swelling (or growth) and folding, in a more general context.

Recent experiments have imposed controlled swelling patterns on thin polymer films, which subsequently buckle into three-dimensional shapes. In Chapter 2, we focus on our first project, where we develop a solution to the design problem suggested by such systems,
namely, if and how one can generate particular three-dimensional shapes from thin elastic sheets by mere imposition of a two-dimensional pattern of locally isotropic growth. Not every shape is possible. Several types of obstruction can arise, some of which depend on the sheet thickness. We provide some examples using the axisymmetric form of the problem, which is analytically tractable.

Despite an almost two thousand year history, origami, the art of folding paper, remains a challenge both artistically and scientifically. Traditionally, origami is practiced by folding along straight creases. A whole new set of shapes can be explored, however, if, instead of straight creases, one folds along arbitrary curves. In Chapter 3, we present a mechanical model for curved fold origami in which the energy of a plastically-deformed crease is balanced by the bending energy of developable regions on either side of the crease. Though geometry requires that a sheet buckle when folded along a closed curve, its shape depends on the elasticity of the sheet. Along these same lines, we discuss in Chapter 4 a natural extension by considering a kinematical construction of multiple folded structures.

We close this thesis in Chapter 5 with conclusions and a brief discussion of future work.
CHAPTER 1
WHY WOULD WE RATHER NOT GO FLAT?

1.1 Growth and the role of mechanics

For many years it was a common belief among biologists that growth and pattern formation in nature was primarily controlled by chemical processes. This seems to be a natural trend of thought, especially when one considers the great advancements in molecular biology and biochemistry during the last century. In this viewpoint, biologists believe that the main regulating factor during plant development, for instance the curls at the edge of a leaf or flower, is dominated by a very detailed and complex set of instructions given by the genetic code, telling every part of the plant how to bend and where and when to take action [2]. Despite strong evidence that chemicals control pattern formation in nature as a transduction mechanism [3], it has been a growing belief among some biologists, as well as physicists, that mechanics should play a central role in such processes and that patterns in nature can be spontaneously generated [4–8]. This mechanist way of thinking about development of living organisms gained its first well established formulation in 1917 in the work On Growth and Form by D’Arcy Wentworth Thompson – “Cell and tissue, shell and bone, leaf and flower, are so many portions of matter, and it is in obedience to the laws of physics that their particles have been moved, moulded and conformed. They are no exceptions to the rule that Θεὸς ἄει γεωμετρεῖ.1 Their problems of form are in the first instance mathematical

1 “God always geometrizes.”
problems, their problems of growth are essentially physical problems, and the morphologist is, *ipso facto*, a student of physical science.” ([1] pp. 10).

**Figure 1.1.** Image published by Green [5]. RightsLink License Number: 2913790988627

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Figure of text and diagrams explaining the differential equation for the bar and the process of transduction via complex integration. The conversion changing a non-periodic surface to an undulating one occurs in the formation of a potato chip (A), as it makes a saddle. Such a change could be viewed as the orchestrated summation of multiple simple transductions (B) or as a single transduction of a single continuous unit (C). We treat it as the latter. As in class I, time is eliminated by considering the transduction to be a step. For the similar case of a bar under compression (C), the sense of the transduction is that applied stress (cause) will tend to increase deflection of any curved region until the increasing structural consequences of deflection (effect), i.e. resistance to bending and/or being displaced, equalize the stress. When the effect cancels the cause, the step is over. D. (1) At the differential level, cause and effect is addressed in terms of a point (i.e., whether it goes up or down). The immediate effector is applied stress. (2) The immediate responding system involves the shape and physical properties of the bar (or plate). (E) For the long-term process, the before state is the initial nearly flat topography. The after state, the integral, is the final topography (3). The three categories of input for co-variation study can be recognized. They are terms in the function, limits of integration, and constant of integration. The latter two categories are new functions, not simple parameters as in class III. Deduction of this type of differential function by co-variation appears exceedingly unlikely.
Among modern biologists, who shared that mechanics is also a fundamental driving mechanism to how patterns are formed, we cite the work of Paul B. Green. In his work [5], Green suggests that some geometrical transitions in the form of living organisms, for instance change of phyllotaxy pattern and gastrulation, could be understood using the concept of mechanical instability instead of millions of chemically controlled transductions. The diagram in Figure 1.1 illustrates the way that Green thought about these problems by the end of the last century, where patterns could be explained with only one mechanical transduction through buckling instability. As a simplified example, he mentioned the potato chip undergoing shape change from a flat slice to a saddle-shape, which happens because of inhomogeneous change of surface area – when the potato is cooking its rim gets stiffer, under compression, while its center shrinks, therefore under tension.

![Figure 1.2. Image published by Dumais [7]. RightsLink License Number: 2913790988627](https://s100.copyright.com/CustomerAdmin/PLF.jsp?IID=2012051_1337630768307)

Figure 2. Tissue patterning by mechanical buckling. (a) Rippling pattern of a grass blade. (b) Paul Green’s tri-partite knitted band, showing rippling of the central region.

Another buckling phenomenon is the ripples that appear on a grass blade, Figure 1.2. These ripples happen due to homogeneous growth of an initially flat plate under constraints
applied to its boundaries by the relatively slower growing veins that run along the blade \[7\]. This problem has been modeled from the physical viewpoint in \[9\] using the hypothesis of multiplicative decomposition \[10\], a formalism which has been shown to be equivalent \[11\] to the one we shall discuss in this thesis (Chapter 2). The pattern shown in Figure 1.2 (a) is a consequence of a purely mechanical response, which can also be experimented with by knitting a ribbon that has twice as many stitches along the central region than on its outer boundaries, Figure 1.2 (b).

Let us now turn our attention to the rhetorical question posed in the title of this chapter: “Why would we rather not go flat?” Here, by flat, we mean zero gaussian curvature\(^2\). Sometimes nature would rather choose flat surfaces, such as insect wings and leaves of many plants. Nevertheless, growing flat structures seems to be a very difficult task for nature, since there are many ways this could go wrong. Once again, we quote D’Arcy Wentworth Thompson: “An organism is so complex a thing, and growth so complex a phenomenon, that for growth to be so uniform and constant in all the parts as to keep the whole shape unchanged would indeed be an unlikely and an unusual circumstance. Rates vary, proportions change, and the whole configuration alters accordingly.” (\[1\] pp. 205). For instance, in order for a leaf to remain flat, it would probably need a very specific set of instructions while the structure is undergoing growth. It turns out that these instructions, given by the genetic code of the plant \[13, 14\], are responsible for prescribing the local geometries. Nath et al. \[13\] provided us with an example of how nature goes about controlling the local curvature of a leaf in the \textit{Antirrhinum} genus by mutating the \textit{CINCINNATA} gene (CIN). They observed that the wild-type leaves are usually flat, while the CIN mutant leaves are wrinkled due to an inhomogeneous excess of material that grows inside the leaf, Figure 1.3 (the figure shows a sequence of flattened CIN leaves, where the dark spots are regions

---

\(^2\)The definition of gaussian curvature is the product of the two principal curvatures \[12\]
where material overlaps due to excess of material). During the growth process, the size of the cells remain small while they are still dividing, but start to increase their size after the arrest of division, Figure 1.4. Opposite to what happens to the wild-type leaves, the mutation in the CIN gene causes the front of arrest to move more slowly from the center of the leaf to its periphery. Therefore, the shape of the front of arrest becomes concave, Figure 1.4, allowing the outer region of the CIN leaf to grow for a longer period of time, which causes the buckling observed in Figure 1.3.
The work on the CIN gene is not only clear evidence of the role of mechanics in shape formation in nature, but also suggests a way to formulate the problem from a mathematical viewpoint. If, for a moment, we think further about the possible implications of the form of the front of arrest in Figure 1.4, we could very well be led to conclude that the front of arrest determines approximately how distances are programmed into the final shape of the leaf or, in other words, the prescribed metric, $\bar{g}$, of the leaf should be a function of the front of arrest of the cellular division. The stresses in the leaf, $\sigma$, will be proportional to the difference between the target, $g$, and the prescribed metrics, which reads, $\sigma \sim g - \bar{g}$. The
final shape of the leaf will depend on the internal force balance due to its stress distribution. This natural laboratory remarkably shows us the fundamental role of mechanics in pattern formation. Such a biological implication has its analogies with inhomogeneous growth and swelling of thin elastic sheets, where controlled experiments have been made [15–17] using thin cross-linked polymer sheets that undergo thermoactive shape changes. This is a powerful method that has opened a new way to explore shape formation as well as experimental geometry. The understanding of these techniques allows controlling of swelling on many different scales [16,17]. Its theoretical framework is based on predicting the buckled shape of a sheet that results from a given imposed pattern of growth or swelling [6,18–24].

Allow us to end this section by asking the following question: Does the actual final shape of a leaf fulfill the prescribed metric imposed by the arrest front? If, on one hand, the answer to this question is “yes”, the problem of finding the shape of a leaf will be purely geometrical and the condition called isometric\(^3\) embedding will be satisfied. If, on the other hand, the answer is “no”, the leaf will have some residual stress and, therefore, its shape will be determined by mechanics.

1.2 Art and science of folding paper

Some would say that the origin of origami (composed Japanese word, *oru* = *to fold* and *kami* = *paper*) [25,26], also known as the art of folding paper, dates back to the year of 105 A.D. in China during the time paper was invented. However, it was only around the 6th century that paper was brought to Japan by Buddhist monks and finally used by them as a practice of folding exclusively for religious purposes and ceremonies due to its high cost. This practice only became popular as an art form at the beginning of the 17th century.

\(^3\)If the metric does not change, the mapping between the initial and final states is called an *isometry* and the in-plane strain is zero.
The first ever written document containing instructions on how to fold an origami figure, entitled “Senbazuru Orikata” (Figure 1.5), was published in 1797. In the western culture there was also an appearance of paper folding as an art, which started in Spain during the 12th century, possibly brought by the Moors, who were also responsible for the Islamic architecture of North Africa and parts of Spain and Portugal.

![Figure 1.5. Senbazuru Orikata. First known origami document, titled “Senbazuru Orikata”, published in 1797. This page shows the popular crane origami figure. (This is a image of public domain (PD-1923) from the Wikimedia Commons: http://commons.wikimedia.org/wiki/File:Hiden_Senzazuru_Orikata.jpg)](image)

Although we may sometimes in this document misuse the word “origami” when talking about paper folding in the general sense, traditional origami is strictly made by folding a single square piece of paper, without cutting or gluing separate parts together. In the
past, this practice was poorly transmitted from generation to generation, since its knowledge was mostly passed by word-of-mouth. Not until the 1930’s, when the father of modern origami, Akira Yoshizawa, developed a set of diagrammatical rules that gave this art a common language [27], did origami gain a more effective way of teaching, but also remarkably incorporated mathematics into the heart of its communication followed by significant advancements in the complexity of the sculptures. Mathematics allowed origamists to further explore the full potential of this art form and mathematicians to find an intellectual laboratory by folding paper. Before we attempt to describe an endless list of practical applications of origami, it is important to point out that paper folding is a very broad and interesting subject on its own artistic right as well as from the purely mathematical viewpoint. Nowadays, paper folding incorporates several fields of mathematics, where the rules within the art itself are still being explored. To cite a few examples [28], calculus, number theory, algebra [29], combinatorics [30], topology, geometry [31], computational complexity theory [32], and etc., are all related to paper folding. As an example, let us consider flat foldable origami. In this case, for every isolated vertex there are $n$ straight creases and their angles with respect to each other are indicated by $\varphi_n$, Figure 1.6, where the condition $\varphi_1 + \varphi_2 + \cdots + \varphi_n = 2\pi$ is always satisfied. An example of a flat foldable crease pattern is shown in Figure 1.6, where the dashed lines represent mountains and the dotted line a valley. Generally speaking [28], the flat-foldability is guaranteed if, and only if,

- Crease patterns are two colored

- Theorem due to Kawasaki and Justin:

\[ \varphi_1 + \varphi_3 + \cdots + \varphi_{n-1} = \varphi_2 + \varphi_4 + \cdots + \varphi_n \]  \hspace{1cm} (1.1)

- Theorem due to Maekawa and Justin:
\# mountains − \# valleys = \pm 2. \hspace{1cm} (1.2)

- A sheet can never penetrate a fold.

**Figure 1.6.** (a) Flat fold vertex. Fold pattern of straight creases coming out of a vertex. (b) example of a flat foldable vertex.

With this set of rules, some natural questions could be asked: Given a crease pattern of mountains and valleys, can it be flat foldable? If so, how would that process take place? It turns out that answering these questions, for arbitrary crease patterns, is a NP-complete problem [28, 32]. Before the crease pattern is folded into the final object, no matter how complicated the network of creases is, one should always achieve the final goal respecting the above rules at every step. This problem was first solved not so long ago, in 1996 by Bern and Hayes [32], where many other challenging questions were raised by the authors in that same work, such as: “How many different flat origamis can there be with the same crease pattern?” This question is also known as the “map problem”, discussed in [28]. Many other related questions are still open problems.
One particular mathematical question of interest is related to pleated folding, in other words, corrugated structures where folds are placed in sequence by alternating mountains and
valleys. In order to illustrate this problem, let us think about the “hyperbolic paraboloid” model, or simply “hypar”. This model was first introduced in the late 1920’s at the German school of art and design named Bauhaus [33]. Since their first appearance in the 1920’s, the question of whether or not hypars are indeed true isometric embeddings remained unsolved until the year 2009. Demaine et al. [34] proved that the crease pattern that is shown in Figure 1.7 is not possible to fold without adding extra creases, Figure 1.8. In this same work, they also conjectured that the existence of the pleated concentric circles structure, Figure 1.9, was possible; however, a hole needed to be cut out of the center of the sheet. In this work, we hope to contribute to clarify whether or not these pleated curved structures exist without needing stretching, problem which still remains open. Solving this problem, besides opening new directions for artistic explorations, has a lot of potential to be further explored in the construction of (quasi) isometric embeddings applied to real materials, which have been of interest for quite some time [35,36].
Besides having its own artistic value and pure mathematical relevance, folding techniques have also been widely applied to fields in structural engineering, architecture, and design, contributing to many insights to solve practical problems of these fields. Let us, for instance, consider deployable devices, which are objects that use folding techniques to dynamically change their shape and at the same time preserve their parts in a more compact form. A very simple example is the useful umbrella. Although the umbrella was first invented in China around the year 21 A.D. [37], its concept, design, and usage remain robust through time. Deployable structures are very functional and useful for solving various structural problems in engineering, and it is through a combination of art, design, and science that origami techniques have been shown to be a practical way to exploit the potential of these devices. A modern, and quite remarkable, application of origami to deployable structures was given by Koryo Miura in his design solution of a space solar panel [38]. Originally, Miura was interested in determining how an idealized infinite elastic plate, therefore infinitely thin, would deform if subject to uniform compression [39]. Analogous with the pattern that emerges when buckling a cylindrical shell, as was shown in 1955 by Y. Yoshimura [40], Miura realized that his problem should have similar structure and it could also be identified with an origami pattern [41]. Miura named his solution to the planar case developable double corrugation [42], which later became known as the miura-ori pattern, Figure 1.10. Fascinating features of the miura-ori – and other folding patterns in general – are its deployability, increase of stiffness in some directions, and material properties like effective negative poisson ratio, such results that are applied to study textured shells structures and meta-materials [43, 44]. These led Miura to realize how to apply the features of his origami pattern to find a practical solution to the problem of how to pack and deploy a solar panel in a space shuttle and sent it out in space. Miura-ori is also found in biological systems, where nature has used this concept as solutions for energy optimization to the problem of deployment of leaves inside buds [45, 46] and insect wing folding [47]. The interplay between origami and deployable
structures is a very active field of research and it plays an important role in interdisciplinary collaborations. To cite a few more applications of this field, we refer to the following: space telescope deployment [48], a project that has been developed by the scientist and origami artist Robert J. Lang and the Lawrence Livermore National Laboratory (LLNL); airbag folding simulation [49]; wrapping of solar sails [50], which is still a subject in development by the Japanese Space Agency JAXA, where the general interest is to look for solutions to the problem of packing of very large solar sails for inter-planetary travel; relating growth and pattern formation in nature, a beautiful application of origami to understand stages of plant development [8], where the final unfolded state relates to the initial folded embryonic stage, which grows under constraint inside the bud; and finally, in biomedical sciences, origami stent craft has been used as a solution to minimize invasive surgery procedures [51].

Figure 1.10. Miura-ori origami pattern. These images are licensed under the Creative Commons Attribution-Share Alike 3.0 Unported license.
Although the idea that a sheet of paper can be folded along an arbitrary curve is unfamiliar to many, performing this activity has been a form of art for quite some time. Bauhaus, the school of art and design, was a pioneer in the concept curved folding structures by the end of the 1920’s [33]. The result of this practice is often severely buckled and mechanically stiff sculptures, which provides interesting structural properties and reveals new ways to explore designs. Traditional origami has already been a strong influence in architecture and design [53, 54], but the extrapolation of this long established art form still has a lot of potential. In 2003, the visionary architect, Frank Gehry, designed the fabulous Walt Disney Concert Hall using concepts of developable surfaces and straight and curved creases. Since the work by David Huffman in 1976 [55], where he described the geometry of curved creases for the first time, more attention has been given to this subject [36, 56–58]. However, little is still understood about this new class of folds, especially from a physical perspective. We shall present a mechanical model for a single curved fold that explains both the buckled shape of a closed fold and its mechanical stiffness. The buckling arises from the geometrical between the crease and the stretching energy of the sheet away from the crease.

1.3 Folding, crumpling, and creasing

This section is devoted to the subject of creasing a sheet of paper. The details and complexity of the physics of paper and its internal structure, which is related to elasticity and plasticity of fibered network, is a very rich field [59, 60] and the details of such matter is outside the scope of this discussion. Nevertheless, we are looking for just enough evidence that, in general, a very thin creased elastic sheet should indeed be treated differently from an uncreased one owing to changes in boundary conditions. In other words, we want to distinguish between two situations, the first being the deforming of a paper sheet which will eventually localize stress under load or confinement, and the second being in regard to the deformation of a paper sheet that has already passed through its point of yield stress. The
former, as we shall briefly discuss, pays stretching energy at localized regions as stretching ridges and apexes of developable cones as well as bending energy [61–64]. The latter can be understood as mechanical equilibria coming from the balance between a hinge and an unstretchable region, where developability is kept throughout the sheet except for a singular creased region, where the sheet behaves like a hinge [60, 69].

A rich problem emerges in elasticity and pattern formation when crumpling paper [65–68]. Its complexity is reflected in the appearance of random stretching ridges and apexes of developable cones leading to permanent deformations. Although this experiment might seem very uncontrolled, elasticity and confinement reveal a pattern that is basically made of creases connected by vertexes only to form a polyhedral surface pattern, as shown in Figure 1.12. This complex texture shows that most of the initial surface areas of the paper sheet are preserved, therefore, favoring isometric deformations by stacking layers of material without

Figure 1.11. Walt Disney Concert Hall. The photograph on the left is from the Carol M. Highsmith Archive at the Library of Congress. These images were taken from the Wikimedia Commons and have been released into the public domain by respective authors.
focusing [68]. To understand how stretching comes into play during the initial stage of crease formation, as in Figure 1.12 (b), we now look at an isolated fold, as shown in Figure 1.13, where the sheet is bent around an undetermined radius of curvature $R$. Because bending energy is proportional to the curvature square of the fold, if $R$ becomes infinitely small, as it would be apparent in a sharp fold, the amount of bending energy should diverge. In order to avoid infinities, the sheet finds a better energy balance by allowing stretching to happen within a very localized region along the entire fold. This problem, also known as the stretching ridge, has been extensively studied [61, 62, 64, 66], and we will only review a few basic elements of this theory. The parameters of this problem are the thickness of the sheet $t$, the length of the fold $L$, the Young’s modulus $tE$ (responsible for resistance to stretching), and the bending modulus $B$, also known as flexural rigidity (measures resistance to bending). If the ridge were straight, in other words, sharp, its length would have to be equal to the base line $L$. However, in a real situation $R$ never becomes zero, therefore, for any $R \neq 0$, there must exist an opposite sign curvature determined by the sagging of size $\zeta$. 

**Figure 1.12.** Crumpled paper. (a) Crumpled sheet of paper. (b) Texture left on the sheet due to permanent deformations.
of the ridge, Figure 1.13 (c), in other words, the fold must have a finite amount of gaussian curvature (due to its hyperbolic geometry), which implies that stretching must occur. The strain along the ridge line can be estimated to be $\gamma \sim (\zeta/L)^2$, and is distributed within a perpendicular region of size $\delta$. Therefore, the stretching energy associated with this strain is estimated to be

$$\mathcal{E}_s \sim E t A \gamma^2 \sim E t L \delta (\zeta/L)^4,$$

(1.3) where $A \sim L \delta$ is the local area. Since for large deflections $R \sim \zeta \sim \delta$, stretching energy is of order

$$\mathcal{E}_s \sim E t R^5/L^3,$$

(1.4) which shows that it is minimized when the radius is small. On the other hand, bending energy can be roughly calculated to be

$$\mathcal{E}_b \sim B A R^{-2} \sim B L \delta R^{-2} \sim B L/R,$$

(1.5) which is minimized with large values of $R$. The competition between these two energies gives us an optimal radius of curvature, where through a simple minimization calculation we arrive at $R \sim L (t/L)^{1/3}$, where we have used $B/(Et) \sim t^2$. The energy for the formation of a ridge is, therefore, given by

$$\mathcal{E}_T = \mathcal{E}_s + \mathcal{E}_b \sim B (L/t)^{1/3}.$$

(1.6) Hence, most of the energy spent when crumpling a sheet of paper would have to come from counting all contributions from each fold, $B (L/t)^{1/3}$.

Let us now introduce the second approach, which passes the limit of yield stress of the sheet. Consider a much simpler set up than the crumpling experiment, in which we crease a single sheet of paper along a straight line. Figure 3.6 schematically represents a thin elastic
Figure 1.13. Fold and stretching ridge. (a) sharp fold, $R \to 0$, of length $L$ without stretching. (b) stretching ridge fold, where $L$ is the base line. (c) cross section of the fold, where $R$ is the radius of curvature of the fold, $\zeta$ the sagging size, and $\delta$ the typical size of the stretching region.

sheet symmetrically bent and confined between two plates. The plates are subjected to the application of a force $F$ bending the sheet in between. We consider that the material is thin enough so that it will at first bend without stretching. As the sheet bends more and more under compression, its curvature at $S = 0$ increases until a critical curvature $K_0$, above which the deformation becomes plastic. In reality, before the localized plastic hinge is developed, the material will stretch in the same way discussed previously, however, here we only want to describe what happens before and after the critical curvature is reached (which happens within a finite range of deformation, where stress focusing phenomena plays a role). Therefore, the way we distinguish both limits is by setting different boundary conditions for before and after the critical curvature to be
Figure 1.14. Folding a paper sheet. (a) Thin elastic sheet confined between two plates under compression. The inset shows the force, $F$, and moment, $M$ balance for an infinitesimal material element. The curvature, $K$, of the sheet at $S = 0$ is less than a critical value $K_0$. (b) After the curvature at $S = 0$ overcomes the critical value $K_0$ the sheet forms a crease at that point.

\[
\theta(0) = \frac{\pi}{2}, \quad \theta(S^*) = \frac{d\theta}{dS}(S^*) = 0, \quad \text{for } K \leq K_0 \tag{1.7}
\]

\[
\theta(0) = \theta_0, \quad \theta(S^*) = \frac{d\theta}{dS}(S^*) = 0, \quad \text{for } K > K_0, \tag{1.8}
\]

where $S^*$ is the contact point with the plate and $\theta_0$ is some finite angle. In this analysis we use a simple elastic-perfectly plastic model [69], where the moment is related to the curvature as follows

\[
M = \begin{cases} 
B K & K \leq K_0 \\
BK_0 & K > K_0 
\end{cases} \tag{1.9}
\]

where $B$ is the bending modulus. Before we proceed, let us write everything in terms of dimensionless quantities. Lengths are multiplied by the critical curvature $K_0$, forces are divided by $BK_0^2$, and moments are divided by $BK_0$. Therefore, we have
\[ x = XK_0, \quad y = YK_0, \quad s = SK_0 \quad (1.10) \]
\[ \kappa = \frac{K}{K_0}, \quad f = \frac{F}{BK_0^2}, \quad m = \frac{M}{BK_0}, \quad (1.11) \]

where
\[ \frac{dx}{ds} = \cos \theta, \quad \frac{dy}{ds} = \sin \theta. \quad (1.12) \]

The balance equation can be derived by looking at the force balance as represented in the inset of the Figure 3.6 (a), which is given by
\[ \frac{dm}{ds} = f \cos \theta \quad (1.13) \]
\[ \Rightarrow \frac{dm}{ds} = f \cos \theta. \quad (1.14) \]

From the equation (1.9) we have that \( m = d\theta/ds \), therefore, we have to solve the following equation
\[ \frac{d^2\theta(s)}{ds^2} = f \cos [\theta(s)]. \quad (1.15) \]

Integrating the equation (1.15) once gives us
\[ \frac{d\theta(s)}{ds} = -\sqrt{2f} \sin [\theta(s)]. \quad (1.16) \]

During the elastic phase we have that \( \kappa = |d\theta/ds| \leq 1 \), which under the boundary conditions (1.7) at \( s = 0 \) leads us to conclude that the plastic hinge appears at the critical force \( f = 0.5 \), since \( d\theta(0)/ds = -\sqrt{2f} \). When \( \kappa > 1 \), the angle at \( s = 0 \) becomes discontinuous when the force continues to act beyond its critical value \( f = 0.5 \) and the new boundary condition becomes \( d\theta(0)/ds = -1 \), or
\[ \theta(0) = \theta_0 = \arcsin \left( \frac{1}{2f} \right). \quad (1.17) \]

On a more technical note, in this model we have an abrupt change on the differentiability of the sheet, by going from a \( C^2 \) to a \( C^0 \) manifold (Figures 1.15 (c) and (a)). These boundary
conditions, for both regimes, are plotted as a function of the acting force $f$ as shown in Figure 1.16. It is worth pointing out that taking stress focusing into account would have resulted in a smoother transition in between the two regimes, allowing us to have a slightly smoother surface of class $C^1$, (Figure 1.15 (b)). We can also find full analytical solutions of the equation (1.16) for the two regimes, $\kappa \leq 1$ and $\kappa > 1$. They are given in terms of the amplitude $am(\phi|m)$ for the Jacobi elliptic functions and the elliptic integral of the first kind $F(\phi|m)$,

$$\theta(s) = \frac{\pi}{2} - 2am \left( \frac{\sqrt{fs}}{\sqrt{2}} \right), \quad \kappa \leq 1 \quad (1.18)$$

$$\theta(s) = \frac{\pi}{2} - 2am \left( \frac{\sqrt{fs}}{\sqrt{2}} + F \left( \frac{1}{2} \sec^{-1}(2f) \left| \frac{2}{2} \right. \right) \right), \quad \kappa > 1. \quad (1.19)$$

By integrating the equations (1.12), we can get the solutions for the actual shapes. These shapes are shown in Figure 1.17.

Figure 1.15. Surface classes. Example of different orders of differentiability. (a) is made by two piecewise surfaces, a plane and a cylinder, joined by an arbitrary angle; (b) is also by made the same two piecewise surfaces shown in (a), however the plane and the cylinder are lined up; (c) a smooth plane that is bent. The reference for this figure is found in http://math.univ-lyon1.fr/ borrelli/Hevea/Presse/index–en.html
Figure 1.16. Discontinuity at the crease. (a) Moment at the creasing point. (b) Discontinuity at the crease Angle at the creasing point. They both show the discontinuity at $f = 0.5$, where blue is for $\kappa \leq 1$ and red for $\kappa > 1$.

In this research we assume that origami, a pre-creased paper sheet, behaves approximately as a surface of class $C^0$ (Figure 1.15 (a)). That means we consider that the crease will be looked at as an elastic hinge where the derivatives of the embedding are discontinuous along the crease lines. We shall formulate in later chapters how this assumption is imposed in the theory of curved folds by adding a phenomenological crease energy that simulates this sudden change in the boundary conditions on the sheet.
Figure 1.17. Shape for a folded sheet. Rescaled solutions for the shapes within the range of both regimes, $\kappa \leq 1$ and $\kappa > 1$, for different values of forces.
CHAPTER 2
CAN ONE SWELL THE SHAPE OF A DRUM?

2.1 Introduction

The inhomogeneous growth of thin elastic sheets is emerging as a powerful method for the design of three-dimensional structures from two-dimensional templates [16,17,70]. Most of the associated theory has focused on predicting the buckled shape of a sheet that results from a given imposed pattern of growth or swelling [6,18–23]. This work has led to a number of insights and continues to present challenges for theorists and experimentalists alike. For most applications, however, one knows the desired surface but not the growth pattern that generates it. In this work, we pose this reverse buckling problem and solve its axisymmetric form.

The reverse problem has received little attention thus far, and one might mistakenly assume that the solution is trivial. Certainly, it is an elementary exercise in differential geometry to determine the unique metric associated with a surface. With recourse to through-thickness variations in material properties, extrinsic curvatures may also be programmed to select a unique shape. However, real material systems may be more limited. Thus, we wish to explore the question of which shapes can be made with thin elastic sheets when only the two-dimensional midsurface metric can be prescribed. In a practical sense, these are the
shapes programmable by encoding a single, spatially-dependent scalar property into a sheet, namely an isotropic swelling ratio\(^1\).

How could programming the metric of a shape fail to reproduce the shape? A metric may have families of shapes from which to choose its buckled configuration. One sees this possibility immediately using a flat metric; a continuum of a cylindrical, a truncated conical, and other developable immersions exist, but a piece of the plane has lower bending energy among all choices. Beyond this, the actual metric realized by the sheet often differs from the prescribed metric because accommodating a finite-thickness bending energy may induce an in-plane strain through geometric compatibility conditions. Finally, there is no guarantee that a given metric can satisfy all of the boundary conditions at the sheet edges. Such a sheet presumably forms a boundary layer which, while vanishing in the zero thickness limit [22], may have nontrivial effects at finite thickness.

Thus, we wish to know how to prescribe a metric on a sheet of given thickness to produce a desired shape, and what limits there are to the shapes that can be prescribed exactly, up to and including their boundaries. We will make some progress towards answering these questions in what follows.

\section{2.2 Mathematical formulation}

\subsection{2.2.1 Elasticity of isotropic solids}

Our starting point is an energy functional measuring the energy cost for deformations in the spaces of metrics. Recalling Truesdell’s hyper-elastic principle, which says that a deformed body endowed with a metric tensor \( g \) stores some elastic energy that can be written in terms of the integral over the body volume of a local elastic energy density. Such

\footnote{This follows from the existence of some conformal coordinate system such that the metric is expressible as \( \Omega(u, v) \left( du^2 + dv^2 \right) \). Such conformal coordinate systems are guaranteed to exist in a neighborhood of any point for sufficiently well-behaved metrics [79].}
elastic energy depends only on the metric and the material properties. Therefore, the energy functional takes the form

\[ \mathcal{E} = \int_B dV \sqrt{g} h(g), \]  

(2.1)

where the elastic energy density obeys the constitutive relationship

\[ h(g) = \frac{1}{2} A^{ijkl} e_{ij} e_{kl} \]  

(2.2)

and the material properties being captured in the elastic tensor \( A^{ijkl} \) defined as

\[ A^{ijkl} \equiv \lambda g^{ij} g^{kl} + \mu (g^{ik} g^{jl} + g^{il} g^{kj}) \]  

(2.3)

where \( \lambda \) and \( \mu \) are the Lamé coefficients. The elements

\[ e_{ij} = \frac{1}{2} (g_{ij} - \bar{g}_{ij}) \]  

(2.4)

define the components of the strain tensor, \( e \), which are the exact measure of the deformation of a programmed metric, \( \bar{g} \), into the actual metric, \( g \). The elastic tensor, defined in terms of the metric \( g \), is a fourth order tensor that gives us the internal symmetries of the material points of the deformed body. The contravariant components of the metric in the elastic tensor, \( g^{ij} = g^{ik} g^{jl} g_{kl} \), raise indexes of the strain making invariants of the body. The energy density in terms of those invariants can be written as

\[ h(g) = \frac{1}{2} \left[ \lambda \text{tr}(e)^2 + 2\mu \text{tr}(e)^2 \right]. \]  

(2.5)

At this level we state the following problem: given a prescribed metric \( \bar{g} \), what is the metric of the equilibrium configuration, \( g \), that minimizes the energy functional \( \mathcal{E} \)? Notice
that, in order to have a minimizer metric $g$ that describe a real configuration $\mathbf{R}(x)$, in other words, an embedding, we must have geometrical constraints satisfied (if we have a three dimensional body embedded in the euclidian space, $\mathbb{E}^3$, the curvature tensor must vanish).

The answer for this question is given by finding the equations of equilibrium and solving them for specific boundary conditions. Considering that the strains in the body are small, a perturbation on the configuration, $\delta \mathbf{R}$, leads to the following equations of equilibrium when we vary (2.1),

$$\nabla_i S^{ij} + \mathcal{O}(e^2) = 0 \quad \text{in} \quad \mathcal{B}$$

$$n_i S^{ij} + \mathcal{O}(e^2) = 0 \quad \text{on} \quad \partial \mathcal{B},$$

(2.6)

where $n_i$ is the covariant component of the unit vector normal to the body, $\nabla_i$ is the covariant derivative having its connection defined with respect to the metric $g$, $\Gamma^i_{jk} = g^{il} (\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk})$, and the components of the stress tensor measure on the body are defined as

$$S^{ij} = \frac{\delta h}{\delta \epsilon_{ij}} = A^{ijkl} e_{kl}.$$  

(2.7)

### 2.2.2 Dimensional reduction

Our goal in this section is to generally describe a theory for the deformation of thin bodies, in other words, a theory of isotropic elasticity under the assumption that the body $\mathcal{B}$ has one of its dimensions much smaller than the other dimensions of the body. By considering the decomposition $\mathcal{B} = \mathcal{S} \times [-t/2, t/2]$, where $t$ is the thickness of the body, it is possible to integrate out the dimension $z \in [-t/2, t/2]$ in (2.1) writing it in terms of an integral over the middle surface $\mathcal{S}$. Simplifications of this integral can be done if we consider a set of physical conditions called Kirchhoff-Love or membrane assumptions [21]. These assumptions state that the body $\mathcal{B}$ is in a state of plane stress, $S_i^3 = 0$, and that there is no shear of the parallel surfaces to the middle surface $\mathcal{S}$, $e_{i3} = 0$ (weaker condition and it will only be considered later). Throughout this paper we shall use the notation where latin indexes
\(i, j, k, \ldots = \{1, 2, 3\}\) refer to the components of tensors defined on points of the body \(B\), while greek indexes \(\alpha, \beta, \gamma, \ldots = \{1, 2\}\) refer to the components of tensors defined on points of the surfaces in the stack, constant \(z\). Let us first consider only the condition \(S^{i3} = 0\), specifically when \(i = 3\), which gives us

\[
S^{33} = 0 \Rightarrow e^{33} = -\frac{\lambda}{\lambda + 2\mu} e^{\alpha \alpha}.
\]

The last equation allows us to rewrite the energy density (2.2) only in terms of quantities living on the surfaces of constant \(z\),

\[
h = \mu \left( \frac{\lambda}{\lambda + 2\mu} e^{\alpha \beta} e^{\beta \beta} + e^{\alpha \beta} e^{\beta \alpha} \right) = \frac{1}{2} A^{\alpha \beta \gamma \delta} e_{\alpha \beta} e_{\gamma \delta}.
\]

At this point we redefine the constants that depend on the material as

\[
2\mu \equiv \frac{E}{1 + \nu} \quad \text{and} \quad \frac{\lambda}{\lambda + 2\mu} \equiv \frac{\nu}{1 - \nu},
\]

where \(E\) and \(\nu\) are the Young modulus and Poisson ratio, respectively.

According to the construction of the figure (2.1), two parallel surfaces, the embedding of an arbitrary surface for \(z \neq 0\) can be written in terms of the embedding of the middle surface, \(S\), and the normal vector \(N\) in the following way,

\[
R(x^1, x^2, z) = S(x^1, x^2) + zN(x^1, x^2),
\]

where \((x^1, x^2)\) are the local coordinates of \(S\). The embedding (4.6) suggests that the metric is decomposed in a block diagonal form,

\[
(g_{ij}) = (\partial_i R, \partial_j R) = \begin{pmatrix}
(g_{\alpha \beta})_{2 \times 2} & \circ \\
\circ & 1
\end{pmatrix}.
\]

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Figure 2.1. Foliation of a thin plate. Construction of the stack of surfaces with respect to the middle plane. $S$ is the embedding of the middle surface $S$ defined at $z = 0$ and $R$ maps points of the body $B$. $\partial_\alpha S$ and $\partial_\alpha R$ are the tangent vectors to the middle surface and to any parallel surface to $S$, respectively.

The tangent vectors $\partial_\alpha S$ and $\partial_\alpha R$ can also be written one as a function of the other,

$$\partial_\alpha R = \partial_\alpha S + z\partial_\alpha N = \left(\delta_\alpha^\beta - zb^\beta_\alpha\right)\partial_\beta S,$$

(2.13)

where the definition of the components of the second fundamental form has been used, $b_{\alpha\beta} = -\partial_\alpha N.\partial_\beta S$. The equation (2.13) suggests to us the definition of tensor components, $\pi_\alpha^\beta$, that map objects living on the middle surface to its parallel stacks. Therefore, we define

$$\pi_\alpha^\beta \equiv \delta_\alpha^\beta - zb_\alpha^\beta.$$

(2.14)

The projection tensor (2.14) allows us to conveniently write the components of the metric of the surfaces in the stack with respect to the components of the metric (first fundamental form), of $S$, $a_{\alpha\beta} \equiv \partial_\alpha S.\partial_\beta S$. Hence
\[ g_{\alpha \beta} = \pi^\gamma _\alpha \pi^\delta _\beta a_{\gamma \delta} \quad (2.15) \]

and

\[ g^{\alpha \beta} = \rho^\alpha _\gamma \rho^\beta _\delta a^{\gamma \delta}, \quad (2.16) \]

where \( \rho^\alpha _\gamma \pi^\gamma _\beta = \delta^\alpha _\beta \), which implies the following expansion for the components of the inverse of the projection,

\[ \rho^\beta _\alpha = \delta^\beta _\alpha + zb^\beta _\alpha + z^2 b^\beta _\gamma b^\gamma _\alpha + \mathcal{O}(z^3). \quad (2.17) \]

Before we attempt to rewrite the energy functional for a thin body, let us consider the second Kirchhoff-Love condition. The condition says that \( e_{i3} = 0 \), which sets a form for the prescribed metric,

\[ (\bar{g}_{ij}) = \begin{pmatrix} (\bar{a}_{\alpha \beta})_{2 \times 2} & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.18) \]

where such an assumption is \( z \) independent. That only means that we shall consider here only the cases in which we do not have variation across the thickness of the body.

Using the projections (2.14) and (2.17) we can rewrite the energy density (2.2) as an expansion of the thickness variable \( z \), and by integrating it over the limit \([-t/2, t/2]\) we get the energy functional (2.1) only in terms of objects living in \( S \). The two dimensional components of the elastic tensor, \( A^{\alpha \beta \gamma \delta} \), can be expanded as

\[ A^{\alpha \beta \gamma \delta} = \rho^\alpha _\kappa \rho^\beta _\lambda \rho^\gamma _\mu \rho^\delta _\nu A^{\kappa \lambda \mu \nu} \]

\[ = A^{\kappa \lambda \mu \nu} \sum_{i+j+k+l=0}^{\infty} z^i b^\gamma _\alpha (b^\beta )^\lambda _\alpha (b^\gamma )^\mu _\beta (b^\delta )^{i \delta \mu} \]

\[ = A_{(0)}^{\alpha \beta \gamma \delta} + z A_{(1)}^{\alpha \beta \gamma \delta} + z^2 A_{(2)}^{\alpha \beta \gamma \delta} + \mathcal{O}(z^3), \quad (2.19) \]

where we define the notation \( (b^\mu )^\alpha _\kappa = \underbrace{b^\alpha _\lambda b^\lambda _\delta \ldots b^\delta _\kappa}_i \) and the elastic tensor, \( A^{\alpha \beta \gamma \delta} \), in terms of the first fundamental form.
\[ A^{\alpha\beta\gamma\delta} \equiv A_{(0)}^{\alpha\beta\gamma\delta} = \frac{E}{1 + \nu} \left( \frac{\nu}{1 - \nu} a^{\alpha\beta} a^{\gamma\delta} + a^{\alpha\gamma} a^{\beta\delta} \right). \] (2.20)

Finally, let us consider the measure of integration \( dV \sqrt{g} \) using the projection (2.14). We conveniently write \( \det (\pi^{\alpha\beta}) \) in terms of the mean and gaussian curvature, respectively \( H = \text{tr}(b)/2 \) and \( K = \det b \),

\[ \det (\pi^{\alpha\beta}) = 1 - 2zH + z^2K, \] (2.21)

which gives us

\[ \sqrt{g} = (1 - 2zH + z^2K)\sqrt{a}. \] (2.22)

Substituting the results (2.18), (2.14), (2.19), (2.22), and (2.9) into the energy functional (2.1), it follows that

\[
\mathcal{E} = \int_S dS \sqrt{a} \int_{-t/2}^{t/2} dz \left( 1 - 2zH + z^2K \right) \left( A_{(0)}^{\alpha\beta\gamma\delta} + zA_{(1)}^{\alpha\beta\gamma\delta} + z^2A_{(2)}^{\alpha\beta\gamma\delta} + O(z^3) \right) \times
\]

\[ \times \left[ \varepsilon_{\alpha\beta\varepsilon_{\gamma\delta}} - 2z\varepsilon_{\alpha\beta}b_{\gamma\delta} + z^2 (b_{\alpha\beta}b_{\gamma\delta} + \varepsilon_{\alpha\beta}c_{\gamma\delta}) - z^3b_{\alpha\beta}c_{\gamma\delta} \frac{z^4}{4} c_{\alpha\beta}c_{\gamma\delta} \right], \] (2.23)

where the components \( c_{\alpha\beta} \equiv b_{\alpha\gamma}b^{\gamma\beta} \) define the third fundamental form and \( \varepsilon_{\alpha\beta} \equiv \frac{1}{2}(a_{\alpha\beta} - \bar{a}_{\alpha\beta}) \) refers to the components of the in-plane strain. Notice that the integration over the thickness only gets even power contributions of \( z \), since the limits are symmetric. After the integration we are going to leave out terms of the order \( t^3||\varepsilon||^2||b|| \) and higher. The last approximation tells us that the only term in the expansion (2.19), besides \( A_{(0)}^{\alpha\beta\gamma\delta} \), that will matter to us is the one followed by the first order in \( z \), \( A_{(1)}^{\alpha\beta\gamma\delta} \), which is

\[ A_{(1)}^{\alpha\beta\gamma\delta} = 2 \left( A_{(0)}^{\kappa\beta\gamma\delta} b^{\kappa\alpha} + A_{(0)}^{\alpha\beta\kappa\delta} b^{\gamma\kappa} \right). \] (2.24)

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Therefore, after some manipulation, we get the energy functional only in terms of the middle surface,

\[
\mathcal{E} = \frac{t}{2} \int dS \sqrt{a} A^{\alpha \beta \gamma \delta} \left[ \varepsilon_{\alpha \beta} \left( \varepsilon_{\gamma \delta} - \frac{t^2}{4} c_{\gamma \delta} + \frac{t^2}{3} H b_{\gamma \delta} \right) - \frac{t^2}{3} b^\kappa_{\alpha \varepsilon \kappa \beta} b_{\gamma \delta} \right] + \\
+ \frac{t^3}{24} \int dS \sqrt{a} A^{\alpha \beta \gamma \delta} b_{\alpha \beta} b_{\gamma \delta}.
\] (2.25)

### 2.2.3 Energy variation

The energy functional gives the energy landscape for the deformation of a thin body of thickness \( t \). A question that could be addressed at this stage is, how thin does the body have to be such that the energy (2.25) is the right one? We have obviously restricted ourselves to describe a limited set of possible deformations. Nevertheless, more than finding which type of deformation can be described by such energetic cost, we are also interested in keeping all the approximations self-consistent with the overall picture of this paper. Looking at (2.25) we may recognize the total energy as a sum of two terms, \( \mathcal{E} \equiv \mathcal{E}_s + \mathcal{E}_b \), defining the stretching, which contains new terms that go with the curvature, and bending energies in the following convenient forms,

\[
\mathcal{E}_s = \frac{t}{2} \int dS \sqrt{a} A^{\alpha \beta \gamma \delta} \left[ \varepsilon_{\alpha \beta} \left( \varepsilon_{\gamma \delta} - \frac{t^2}{4} c_{\gamma \delta} + \frac{t^2}{3} H b_{\gamma \delta} \right) - \frac{t^2}{3} b^\kappa_{\alpha \varepsilon \kappa \beta} b_{\gamma \delta} \right] \] (2.26)

\[
\mathcal{E}_b = \frac{t^3 E}{24(1 - \nu^2)} \int dS \sqrt{a} \left[ 4H^2 - 2(1 - \nu)K \right], \] (2.27)

where \( H = b^\alpha_{\alpha} / 2 \) is the mean curvature and \( K = b^\alpha_{\gamma \gamma} b^\beta_{\delta \delta} \epsilon^{\alpha \beta} \epsilon_{\gamma \delta} / 2 \) is the gaussian curvature (here we use the definition of the anti-symmetric tensors as \( \epsilon_{\alpha \beta} = \text{off-diag}\{ \sqrt{a}, -\sqrt{a} \} \) and \( \epsilon^{\alpha \beta} = \text{off-diag}\{ 1/\sqrt{a}, -1/\sqrt{a} \} \)).
Applying the virtual work principle, we shall consider a variation of the configuration such that \( \{ \delta u^\alpha \}_{\alpha=1} \) and \( \delta \zeta \) represent virtual displacements on the in-plane and normal directions, respectively. Therefore, the current configuration variation is given by

\[
\delta S = \delta u^\alpha \partial_\alpha S + \delta \zeta N,
\]  

(2.28)

which is a vector living in \( \mathbb{E}^3 \). Taking its derivative with respect to the coordinates on the surface, it follows that

\[
\partial_\alpha \delta S = \partial_\alpha (\delta u^\beta) \partial_\beta S + \delta u^\beta \partial_\alpha \partial_\beta S + \partial_\alpha (\delta \zeta) N + \delta \zeta \partial_\alpha N.
\]  

(2.29)

When we look at the effects of the variation (2.28) on the first and second fundamental forms, using (2.29), we have

\[
\delta a_{\gamma\delta} = (\nabla_\gamma \delta u_\delta + \nabla_\delta \delta u_\gamma - 2\delta \zeta b_{\gamma\delta})
\]  

(2.30)

and

\[
\delta b_{\alpha\beta} = (\nabla_\beta \delta u^\gamma - b_{\beta}{}^\gamma \delta \zeta) b_{\gamma\alpha} + \nabla_\alpha \nabla_\beta \delta \zeta + \nabla_\alpha (\delta u^\gamma b_{\gamma\beta}).
\]  

(2.31)

Before writing the full variation of the energy (2.25), let us first consider the variation of its parts. After some calculation, using the variations (2.30) and (2.31), we arrive at the following expressions,

\[
\delta A^\alpha{}_{\beta\gamma\delta} = - (A^\lambda{}_{\beta\gamma\delta} a_{\alpha\kappa} + A^a_{\alpha\beta\gamma\lambda} a_{\kappa\delta}) \delta a_{\kappa\lambda}
\]  

(2.32)

\[
\delta \sqrt{a} = \sqrt{a} (\nabla_\alpha \delta u^\alpha - 2\delta \zeta H)
\]  

(2.33)

\[
\delta H = (2H^2 - K) \delta \zeta + \frac{1}{2} \nabla^2 \delta \zeta + \delta u^\alpha \nabla_\alpha H.
\]  

(2.34)
The only remaining term is the gaussian curvature $K$. Note that, owing to the Gauss-Bonnet theorem, the integration of the gaussian curvature over the whole surface presents itself as a boundary contribution. Let us pay some special attention to the variation of $K$ and perform the calculations with a little more detail. Writing the gaussian curvature as $K = b_{\alpha\gamma} b_{\beta\delta} \epsilon^{\alpha\beta} \epsilon^{\gamma\delta} / 2$ and using the variations (2.30) and (2.31), we have

$$\delta K = -K a^{\alpha\beta} \delta a_{\alpha\beta} + (\delta b_{\alpha\beta}) b_{\gamma\delta} \epsilon^{\alpha\gamma} \epsilon^{\beta\delta}$$

$$= -2K \nabla_\gamma \delta u^\gamma + 4HK \delta \zeta - b_{\beta\gamma} b_{\gamma\alpha} b_{\lambda\delta} \epsilon^{\alpha\lambda} \epsilon^{\beta\delta} \delta \zeta + b_{\lambda\delta} \epsilon^{\alpha\lambda} \epsilon^{\beta\delta} \nabla_\alpha \nabla_\beta \delta \zeta +$$

$$+ b_{\gamma\alpha} b_{\lambda\delta} \epsilon^{\alpha\lambda} \epsilon^{\beta\delta} \left\{ \nabla_\beta \delta u^\gamma \right\} + b_{\lambda\delta} \epsilon^{\alpha\lambda} \epsilon^{\beta\delta} \nabla_\alpha \left\{ \delta u^\gamma b_{\gamma\beta} \right\}, \quad (2.35)$$

where the fifth term right-hand-side can be expressed as $2KH$. Hence, rewriting some total derivatives and applying the Gauss-Codazzi relationship,

$$\nabla_\gamma b_{\alpha\beta} = \nabla_\alpha b_{\gamma\beta}, \quad (2.36)$$

we have

$$\delta K = -2K \nabla_\gamma \delta u^\gamma + 2HK \delta \zeta - \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} \nabla_\beta \left( b_{\alpha\gamma} b_{\lambda\delta} \right) \delta u^\gamma +$$

$$+ \nabla_\alpha \left[ (2b_{\beta\gamma} \delta u^\gamma + \nabla_\beta \delta \zeta) b_{\lambda\delta} \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} \right]. \quad (2.37)$$

Now we use the equation (2.33) in order to write the variation of $\sqrt{aK}$,

$$\delta \left( \sqrt{aK} \right) = \nabla_\alpha \left[ (2b_{\beta\gamma} \delta u^\gamma + \nabla_\beta \delta \zeta) b_{\lambda\delta} \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} - \sqrt{aK} \delta u^\alpha \right] +$$

$$+ \sqrt{a} \frac{1}{2} \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} \nabla_\gamma \left( b_{\alpha\beta} b_{\lambda\delta} \right) \delta u^\gamma - \sqrt{a} \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} \nabla_\beta \left( b_{\alpha\gamma} b_{\lambda\delta} \right) \delta u^\gamma. \quad (2.38)$$
The third term on the right-hand-side on the above equation can be expressed as

\[
\epsilon^\alpha\epsilon^\beta \nabla_\gamma (b_{\alpha\beta}) b_{\lambda\delta} = \epsilon^\alpha\epsilon^\beta \nabla_\beta (b_{\alpha\gamma}) b_{\lambda\delta}
\]

where Gauss-Codazzi, (2.36), has been used. After some manipulations we arrive at the variation of \(\sqrt{a}K\) as a total divergence of a vector,

\[
\delta(\sqrt{a}K) = \nabla_\alpha \left\{ \sqrt{a} \left[ K\delta u^\alpha + \left( 2Ha^{\alpha\beta} - b^\alpha_\gamma a^{\gamma\beta} \right) \nabla_\beta \delta \zeta \right] \right\},
\]

which is the expected result, knowing from the Gauss-Bonnet theorem that such a term only gives a surface contribution.

Therefore, the variation of the stretching and bending energies are, respectively,

\[
\delta E_s = \int dS \sqrt{a} s^{\alpha\beta} \delta a_{\alpha\beta} + O(t^3 \| \varepsilon \| \| b \|) + O(t \| \varepsilon \|^2)
\]

\[
\approx \int dS \sqrt{a} \left[ (\nabla_\gamma s^\gamma\delta) \delta u_\delta + s^{\alpha\beta} b_{\alpha\beta} \delta \zeta \right]
\]

\[
\delta E_b = 2B \int dS \sqrt{a} \left\{ \nabla_\alpha (\delta u^\alpha H^2) + 2H(2H^2 - K) \delta \zeta + H \nabla^2 \delta \zeta \right\}
\]

\[
- \frac{1}{2} \nabla_\alpha \left[ K\delta u^\alpha + \left( 2Ha^{\alpha\beta} - b^\alpha_\gamma a^{\gamma\beta} \right) \nabla_\beta \delta \zeta \right]
\]

\[
= B \int dS \sqrt{a} \left\{ \frac{1}{2} \left[ 4H^2 - 2(1 - \nu) K \right] \delta u^\alpha - 2a^{\alpha\beta} \nabla_\beta H \delta \zeta \right\}
\]

\[
+ B \int dS \sqrt{a} \left[ \nabla^2 H + 2H (H^2 - K) \right] \delta \zeta + B \int dS \sqrt{a} m^{\alpha\beta} \partial_\beta \delta \zeta,
\]
where we used the definition for the following tensors, respectively called pure in-plane stress, generalized stress, and bend moment,

\[
\sigma^\alpha{}^\beta \equiv t A^{\alpha\beta\gamma\delta} \varepsilon_{\gamma\delta}, \tag{2.43}
\]

\[
s^\alpha{}^\beta \equiv \sigma^\alpha{}^\beta - \frac{t^3}{8} A^{\alpha\beta\gamma\delta} c_{\gamma\delta} + H m^\alpha{}^\beta - b^\alpha{}^{\gamma} m^\gamma{}^\beta, \tag{2.44}
\]

and

\[
m^\alpha{}^\beta \equiv \frac{t^3}{12} A^{\alpha\beta\gamma\delta} b_{\gamma\delta} = B \left[ 2\nu H a^{\alpha\beta} + (1 - \nu) b^\alpha{}^{\gamma} a^\gamma{}^\beta \right]. \tag{2.45}
\]

\(B \equiv E t^3 / (12(1 - \nu^2))\) defines the bending modulus. Notice that the generalized definition of the stress comes from explicit couplings between strain and curvatures.

The last term in (2.42) we rewrite in the following way

\[
\int_C dl \sqrt{a} n_\alpha m^\alpha{}^\beta \partial_\beta \delta \zeta
= \int_C dl \sqrt{a} n_\alpha m^\alpha{}^\beta (n_\beta \partial_n + l_\beta \partial_l) \delta \zeta
= \int_C dl \sqrt{a} n_\alpha n_\beta m^\alpha{}^\beta \partial_\beta \delta \zeta
+ \int_C dl \sqrt{a} n_\alpha l_\beta m^\alpha{}^\beta \partial_\beta \delta \zeta
= \int_C dl \sqrt{a} n_\alpha n_\beta m^\alpha{}^\beta \partial_\beta \delta \zeta
- \int_C dl \partial_l (\sqrt{a} n_\alpha l_\beta m^\alpha{}^\beta) \delta \zeta
= \int_C dl \sqrt{a} \nabla_l (n_\alpha l_\beta m^\alpha{}^\beta) \delta \zeta. \tag{2.46}
\]

Setting

\[
\delta E = \delta E_s + \delta E_b = 0 \tag{2.47}
\]

for arbitrary variations of \(\delta u^\alpha\) and \(\delta \zeta\), we have, dropping the orders \(t^3 \|\varepsilon\| \|b\|\) and \(t \|\varepsilon\|^2\), leads to equations of equilibrium

\[
2B \left[ \nabla_\alpha \nabla^\alpha H + 2H (H^2 - K) \right] - s^\alpha{}^\beta b_{\alpha\beta} = 0, \tag{2.48}
\]

\[
\nabla_\alpha s^\alpha{}^\beta = 0, \tag{2.49}
\]

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free smooth boundary conditions

\[2n_\alpha \nabla^\alpha H + (1 - \nu) l_\gamma \nabla^\gamma (b^{\alpha \beta} n_\alpha l_\beta) \bigg|_{\partial S} = 0,\]  

(2.50)

\[n_\beta \left[ B \left( 2H^2 - (1 - \nu) K \right) a^{\alpha \beta} + s^{\alpha \beta} \right] \bigg|_{\partial S} = 0,\]  

(2.51)

\[n_\alpha n_\beta \left[ 2\nu Ha^{\alpha \beta} + (1 - \nu) b^{\alpha \beta} \right] \bigg|_{\partial S} = 0,\]  

(2.52)

and a corner jump condition for free piecewise-smooth boundaries

\[\left[ n_\alpha l_\beta \right] \left[ 2\nu Ha^{\alpha \beta} + (1 - \nu) b^{\alpha \beta} \right] \bigg|_{\partial \partial S} = 0.\]  

(2.53)

Here, \( \nabla_\alpha \) is a covariant derivative constructed from the realized surface metric, \( n \) and \( l \) are surface tangents normal and tangent, respectively, to the boundary, \([ ]\) denotes a jump in the enclosed quantities, and \( B \equiv \frac{Y t^3}{12(1 - \nu^2)} \) is a bending modulus. We have neglected terms of orders \( t^3 \| b \| ^2 \| \varepsilon \| \) and \( t \| \varepsilon \| ^2 \), with the implicit assumption that derivatives do not affect order. The “effective” stress tensor given by

\[s^{\alpha \beta} = t A^{\alpha \beta \gamma \delta} \varepsilon_{\gamma \delta} + \frac{t^3}{12} \left( H A^{\alpha \beta \gamma \delta} b_{\gamma \delta} - b^\kappa A^{\kappa \beta \gamma \delta} b_{\gamma \delta} \right) - \frac{t^3}{8} A^{\alpha \beta \gamma \delta} b^\kappa b_{\kappa \delta},\]  

(2.54)

shows that an unstretched midsurface, that is, one free of in-plane strain, does not imply an unstressed finite-thickness sheet. This extrinsic contribution to the stress is the only result of our retention of coupled strain-curvature terms in the energy. The tensors \( t A^{\alpha \beta \gamma \delta} \varepsilon_{\gamma \delta} \) and \( \frac{t^3}{12} A^{\alpha \beta \gamma \delta} b_{\gamma \delta} \) are the stress and moment tensors of Efrati et al. [21]; after explicitly raising indices, the latter becomes the bracketed term in the torque boundary condition (2.52) and

\[2\text{Our normal force boundary condition (2.50) differs from that of Efrati et al. [21] but agrees with those of other sources [82–85] in the appropriate limits.}\]
corner condition (2.53). The final term in the definition of the effective stress is an analogous application of the elastic tensor to the “third fundamental form” $c_{\gamma\delta} \equiv b^\kappa_\gamma b_{\kappa\delta}$.

This stress plays the same role as that of the Lagrange multipliers of Guven & Müller [75], whose equilibrium equations for paper coincide with ours when $K = 0$. Actually, the definition of “stress” is rather malleable. Our definition’s inclusion of the extrinsic terms that manifest in the divergence-free quantity in (2.49) seems natural, especially as they arise from variation of the energy with respect to the in-plane strain. Operationally, any scalar $T$ that does not vary, or varies such that $\delta T = T^{\alpha\beta} \delta a_{\alpha\beta}$, will produce only terms that may be tucked into $s^{\alpha\beta}$. We note also the absorption of gravitational forces into the Lagrange multipliers in [75], reminiscent of the definition of “dynamic pressure” in problems involving isochoric fluids. For more ambiguities, see [76] and the discussion of “null stresses” in [77].

If these equations are to be solved for the six terms $a_{\alpha\beta}$ and $b_{\alpha\beta}$, rather than an explicit immersion $S$, they must be supplemented by the Peterson-Mainardi-Codazzi and Gauss equations:

$$\nabla_\alpha b_{\beta\gamma} = \nabla_\beta b_{\alpha\gamma}, \quad (2.55)$$

$$K = a^{\alpha\beta} \left( \partial_\gamma \Gamma_\alpha^\gamma - \partial_\alpha \Gamma_\beta^\gamma + \Gamma_\delta^\gamma \Gamma_{\alpha\beta}^{\delta\gamma} - \Gamma_\gamma^\gamma \Gamma_{\delta\beta}^{\delta\gamma} \right). \quad (2.56)$$

The $\Gamma_\alpha^\gamma$ are the usual Christoffel symbols. These auxiliary equations are automatically satisfied by any immersion $S$ or $S + \delta S$.

Many shapes cannot satisfy the boundary conditions (2.50-2.52). For example, the normal force and torque boundary conditions (2.50) and (2.52) are incompatible for minimal surfaces of the helicoid-catenoid family. Such shapes require either a boundary layer or applied boundary forces and torques.
A consequence of the corner condition (2.53) may be observed by bending two adjacent sides of a piece of paper towards each other. Curvature must vanish “across” the sheet at the corner so the tip remains flat.

2.3 Making shapes

2.3.1 Construction of axisymmetric solutions

When the prescribed metric is given and one is solving “forwards” for the shape, one must solve the equilibrium equations (2.48-2.49) and geometric integrability conditions (2.55-2.56) for the six components $a_{\alpha\beta}$ and $b_{\alpha\beta}$. In the reverse problem, we choose a shape $S$ that satisfies two boundary conditions, (2.50) and (2.52); integrability is automatically satisfied. After solving the equilibrium equations (2.48-2.49), along with boundary condition (2.51), for the components of the stress tensor $s^{\alpha\beta}$, we recover via (2.54) the target metric $\bar{a}_{\alpha\beta}$ in whatever buckled coordinate system we chose for our initial convenience on $S$. Finally, we must determine a coordinate transformation back into an appropriate laboratory frame for assigning the swelling factor $\Omega$ to the unbuckled sheet [78].

Though many shapes do not satisfy all of the boundary conditions, in principle, only a boundary layer is needed to balance the normal force (2.50) and torque (2.52) conditions. Note that this layer may be incorporated into the prescribed swelling factor, so it need not share the characteristic width of spontaneously formed layers [22]. The tangential force condition (2.51) is more involved. In general, the integration constants of the first-order equation (2.49) may be insufficient to balance these in-plane forces, which may require global changes in the metric. Below, we will explore how these boundary conditions affect the construction of axisymmetric shapes, for which we solve the equations of equilibrium analytically.

The system of equations (2.48-2.49) plus its boundary conditions (2.50) and (2.52) give us equilibrium configurations that minimize the energy functional (2.25). Solving those equations in what we call the “forward” way can be very complicated, since we are dealing with
a coupled system of fourth order non-linear differential equations. By “forward” we mean, giving the prescribed metric we want to find the embedding of the equilibrium configuration. Another way of looking at the problem is to ask, “Given an embedding of an equilibrium configuration, what is the metric we need to prescribe in order to get the desirable shape?” We call the last one the “backwards” problem. Whether one is interested in the “forward” or the “backwards” it all depends on the experimental setup one wants to perform.

In this section we will study the “backwards” problem. For simplicity we will first consider the classification of axisymmetric configurations. In order to do that we conveniently choose the geodesic coordinate system, or arc-length parameterization, to write the embedding, \( S : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \), of the desirable configuration as follows,

\[
S(u,v) = \left( \rho(u) \cos v, \rho(u) \sin v, \int \sqrt{1 - \rho'(u)^2} du \right). \tag{2.57}
\]

Using these coordinates, first and second fundamental forms are given by

\[
ds_I^2 = a_{\alpha\beta} dx^\alpha dx^\beta = du^2 + \rho(u)^2 dv^2 \tag{2.58}
\]

and

\[
ds_{II}^2 = \frac{\rho''(u)}{\sqrt{1 - \rho'(u)^2}} du^2 + \rho(u) \sqrt{1 - \rho'(u)^2} dv^2, \tag{2.59}
\]

respectively. It follows directly from (2.58) and (B.7) mean and gaussian curvatures,

\[
H = \frac{1 - \rho'(u)^2 - \rho(u)\rho''(u)}{2\rho(u)\sqrt{1 - \rho'(u)^2}} \quad \text{and} \quad K = \frac{-\rho''(u)}{\rho(u)}, \tag{2.60}
\]

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and the Christoffel symbols

\[
[\Gamma^u_{\alpha\beta}] = \begin{pmatrix} 0 & 0 \\ 0 & -\rho'(u)\rho(u) \end{pmatrix} \quad \text{and} \quad [\Gamma^v_{\alpha\beta}] = \begin{pmatrix} 0 & \rho'(u) \\ \rho'(u) & 0 \end{pmatrix}. \tag{2.61}
\]

We are free to specify the function \( \rho \), as long as \( (\partial_u \rho)^2 < 1 \). If we assume a diagonalized, axisymmetric target metric, the equilibrium equations reduce to one differential equation

\[
\partial_u s^{uu} + s^{uu} \partial_u \ln \left[ \rho \sqrt{1 - (\partial_u \rho)^2} \right] + g(u) = 0, \tag{2.62}
\]

and one algebraic equation

\[
s^{vv} = \frac{\partial^2 \rho}{\rho \left[ 1 - (\partial_u \rho)^2 \right]} s^{uu} - \frac{g(u)}{\rho \partial_u \rho}, \tag{2.63}
\]

where we have defined

\[
g(u) \equiv -2B \frac{\partial_u \rho}{\sqrt{1 - (\partial_u \rho)^2}} \left( \partial_u^2 H + \frac{\partial_u \rho}{\rho} \partial_u H + 2H(H^2 - K) \right). \tag{2.64}
\]

We can integrate equation (2.62) to yield

\[
s^{uu} = \frac{1}{\rho \sqrt{1 - (\partial_u \rho)^2}} \left[ C - \int_{u_b}^u dy g(y) \rho(y) \sqrt{1 - [\partial_y \rho(y)]^2} \right], \tag{2.65}
\]

where \( C \) is an integration constant, and \( u_b > u \) lies on one boundary of the sheet. Given the stress tensor, equation 2.54 is now an algebraic equation for the strain tensor and, thus, the prescribed metric \( \bar{a}_{\alpha\beta} \).

Our coordinates \((u, v)\) are natural for the buckled object, but not for the laboratory. We must perform a change of variables to a coordinate system convenient for programming an
isotropic swelling factor \( \Omega(r) \). A natural choice for axisymmetric shapes is to use cylindrical polar coordinates \((r, \theta)\) and identify \( v \) with \( \theta \), so that the metric becomes \( \Omega(r)(dr^2 + r^2d\theta^2) \). The coordinate transformation \( u(r) \) is determined by the solution to the differential equation

\[
[\partial_r u(r)]^2 = \frac{\bar{a}_{vv}[u(r)]}{r^2 \bar{a}_{uu}[u(r)]}, \tag{2.66}
\]

and the swelling factor by

\[
\Omega(r) = \frac{\bar{a}_{vv}[u(r)]}{r^2}. \tag{2.67}
\]

### 2.3.2 The axisymmetric boundary conditions

Our construction relies on our ability to find an \( S \) that satisfies the free boundary conditions. Two of these, (2.50) and (2.52), are simply two conditions on \( \rho(u) \) on the boundaries. On the boundary \( u_b \), the first of these is

\[
\rho''(u_b) = \nu \frac{1 - \rho^2(u_b)}{\rho(u_b)}, \tag{2.68}
\]

which implies \( K(u_b) = -\rho''(u_b)/\rho(u_b) < 0 \). The second is

\[
\rho'''(u_b) = \frac{(1 + \nu + \nu^2)}{\nu} \rho'(u_b) K(u_b). \tag{2.69}
\]

These conditions can be easily satisfied using an arbitrarily narrow region near the boundary \( u_b \). If there is another boundary \( u_a < u_b \), the same considerations apply there.

We can also satisfy the boundary condition (2.51) at \( u_b \) by choosing the integration constant

\[
C = -B \left[ 2H(u_b)^2 - (1 - \nu)K(u_b) \right] \rho(u_b) \sqrt{1 - \rho'(u_b)^2}. \tag{2.70}
\]
Figure 2.2. Ziggurat. (a) Swelling factor $\Omega(r)$ that swells a disk into the shape in (b) for thicknesses $t = 1/100$ (solid black), $t = 1/200$ (dashed black), and $t = 1/500$ (solid grey). The outer radii needed are $\approx 0.9$, $\approx 1.2$ and $\approx 1.3$, respectively. Lengths are in units of the radial arc length of the final shape.

If the surface has only one boundary, this procedure is sufficient. However, with two boundaries we must also satisfy an integral constraint,

$$
\int_{u_a}^{u_b} du \, g(u) \rho(u) \sqrt{1 - \rho'(u)^2} = -C - B \left[ 2H(u_a)^2 - (1 - \nu)K(u_a) \right] \rho(u_a) \sqrt{1 - \rho'(u_a)^2}.
$$

This is a nonlocal constraint as it involves both boundaries, located at $u_a$ and $u_b$. This global balance may require a global change in $\rho(u)$ to accommodate. Our procedure for this accommodation, by no means unique, is shown below in the example of the asymmetric annular sheet.
2.3.3 Examples

We start our survey of examples with a topological disk. The swelled shape is shown in Figure 2.2(b) and the swelling factor used to produce it is shown in Figure 2.2(a) for three different sheet thicknesses. This figure was produced using a metric of the form

$$\rho(u) = u + A_1 u^4 e^{-10 [e^{10u} - 1]} + A_2 u^5 e^{-10 [e^{10u} - 1]} - \frac{0.3 u^5}{1 + 10 u^4} - \frac{0.3 u^5}{0.01 + u^4/\left[\sin(7\pi u)/(7\pi) + 0.5\right]}.$$

where $A_1$ and $A_2$ were chosen to satisfy the normal force and torque boundary conditions at $u = u_b = 1$. We find $A_1 \approx -0.829$ and $A_2 \approx -0.723$. Despite its seeming absurdity, equation (2.72) underscores the flexibility we have in choosing a metric. Moreover, there is some method to our choice. Since stresses cannot diverge, we require that $\rho(u)$ asymptotically flatten at the center. It is a straightforward calculation, by expanding the stresses in a power series in $u$, to show that this requires $\rho(0) = 0$, $\rho'(0) = 1$, $\rho''(0) = 0$, $\rho'''(0) < 0$, and $\rho''''(0) = 0$. Following the procedure described in the previous two sections, the swelling factor is easily obtained.

The process of choosing coefficients for equation (2.72) reveals some of the potential pitfalls of designing a shape. Some $A_1$ and $A_2$ that satisfy the boundary conditions require $|\rho'(u)| > 1$ at one or more places within the sheet. Moreover, one could find that the resulting prescribed metric is not positive definite everywhere within sheets that are too thick.

We now consider a pair of topological annuli, shapes with two free boundaries. Satisfying conditions on both boundaries is simple for a surface symmetric about a fixed $u$. By choosing $C$ to satisfy one boundary, we automatically satisfy the other. An example is shown in Figure 2.3. The metric we use is
\[
\rho(u) = A_1 e^{-10} [e^{10u} - 1] + A_2 (1 - u) e^{-10} [e^{10u} - 1] + B_1 [e^{-10u} - e^{-10}] + B_2 u [e^{-10u} - e^{-10}] + 1 + 0.4 \frac{\pi}{5} \sin(5\pi u).
\] (2.72)

We find \(A_1 = B_1 \approx 0.205\) and \(A_2 = B_2 \approx 1.013\) in order to satisfy the normal force and torque boundary conditions at \(u = u_a = 0\) and \(u = u_b = 1\).

![Swelling Factor](image1.png) ![Sheave](image2.png)

**Figure 2.3.** Sheave. (a) Swelling factor \(\Omega(r)\) that swells an annulus with inner radius \(r = 0.5\) and outer radius \(r \approx 1.28\) into the shape in (b) for thicknesses \(t = 1/100\) (solid black) and \(t = 1/500\) (dashed black). Lengths are in units of the radial arc length of the final shape.

An asymmetric annulus is significantly more complicated. It is no longer sufficient to simply choose a metric appropriately on the boundaries, because the integral constraint (2.71) depends on the value of \(\rho(u)\) throughout the sheet. After satisfying one boundary, it will generally be impossible to satisfy the other without modifying the metric. For the example in Figure 2.4, we use
\[
\rho(u) = A_1 e^{-10} \left[ e^{10u} - 1 \right] + A_2 (1-u) e^{-10} \left[ e^{10u} - 1 \right] + B_1 \left[ e^{-10u} - e^{-10} \right] + B_2 u \left[ e^{-10u} - e^{-10} \right] \\
-\eta u + 0.5 + \frac{1}{16} e^{-32(u-3/4)^2},
\]

(2.73)

adjusting \( A_1, A_2, B_1 \) and \( B_2 \) according to the normal force and torque boundary conditions at \( u = u_a = 0 \) and \( u = u_b = 1 \), choosing \( C \) to satisfy the tangential force boundary condition at \( u = u_b = 1 \) and the parameter \( \eta \) to satisfy the tangential force boundary condition at \( u = u_a = 0 \). We find \( A_1 \approx -0.044, A_2 \approx 0.029, B_1 \approx -0.18 \) and \( B_2 \approx 0.10 \). Although there is some weak thickness dependence in \( \eta \), we find \( \eta \approx -0.051 \) for thicknesses from \( t = 1/20 \) to \( t = 1/500 \). There is also a weaker dependence on thickness for \( \Omega(r) \) in this example, presumably because this swelled shape has less curvature than the others.

![Diagram](attachment:image.png)

(a) Swelling Factor
(b) Compression fitting

**Figure 2.4.** Compression fitting. (a) Swelling factor \( \Omega(r) \) that swells an annulus with inner radius \( r = 0.3 \) and outer radius \( r \approx 1.81 \) into the shape in (b) for thicknesses \( t = 1/20 \) (dashed black) and \( t = 1/100 \) (solid black). Lengths are in units of the radial arc length of the final shape.
Our final example is the disk shown in Figure 2.5, with

$$\rho(u) \approx (6.22 \times 10^{-11}) e^{-10} (e^{10u} - 1) u^4 + (-2.1 \times 10^{-11}) e^{-10} (e^{10u} - 1) u^5$$

$$+ \frac{1}{8} [((5.966 - 7.02u) \text{erf}(7.07u - 6.01) + (u[1.2u - 2.16] + 0.9855) \text{erf}(6.67u - 6)$$

$$- \frac{0.13u^3}{125u^3 + 1} + (3.39 \times 10^{-12}) u^4 + 1.2u^2 - 1.18u + 6.95 + e^{-44.44(u - 0.9)^2}(0.1u - 0.09)$$

$$- 0.56e^{-50(u - 0.85)^2}]$$  \quad (2.74)$$

This “drum” requires a disk of radius 60 to produce, though it has a total center-to-edge arc length of only 2.75. This requires significant shrinking, up to a local swelling factor of $\approx 5 \times 10^{-5}$. This, of course, is another obstruction to swelling a shape: the required swelling factor may be beyond the capabilities of any existing experimental system. So one can swell the shape of a drum in principle, though perhaps not currently in practice. We note, however, that conformal transformations of the prescribed metric into some atypical coordinate system may be a way to improve the range of required swelling factors.
Figure 2.5. Drum. (a) Swelling factor $\Omega(r)$ swells a disk with outer radius $r \approx 60$ into a “drum” with a total radial arc length of 2.75 units, for thicknesses $t = 1/100$ (solid black), $t = 1/200$ (dashed black), and $t = 1/500$ (solid grey). We show $\Omega(r)$ only up to $r = 1.5$, beyond which it simply approaches zero.
CHAPTER 3

GEOMETRIC MECHANICS OF CURVED CREASE ORIGAMI

3.1 Introduction

In this work we suggest a generalized framework to understanding the mechanics of folding papers. This has been a challenge for quite some time at both the artistic and scientific level. Our motivation lies in the art of folding paper, origami, that is traditionally done by folding along straight creases. A whole new set of shapes can be explored if, instead of straight creases, one folds along arbitrary curves. Such structures resulting from these curved creases is a fairly new subject and it has received little scientific exploration. The first mathematical description for this problem was suggested by David Huffman [55], who took a geometrical approach, describing the local shapes of developable surfaces when creased along curves. The question that we want to address goes beyond the purely geometrical arguments [36, 86, 87], taking into account the physical questions that are also part of the problem. We aim to understand the shape formation owing to mechanical equilibria coming from the balance between an unstretchable region, where the sheet remains developable, and a singular creased region, where the sheet is plastically deformed. The equilibrium shape is obtained by minimizing the elastic energy of the system. The elastic energy considered in our model has two main terms: the bending energy, which tells us how much energetic cost is needed to deform the developable part from its preferred flat shape; the phenomenological contribution, which is responsible for the energetic cost needed for the creased angle to deviate from its preferred angle set by plastically deforming the paper along the curve.
3.2 Kinematics of curved folds

3.2.1 Curves as creases on developables

Consider a surface with the parametrization $S(s, v) = c(s) + v\hat{g}(s)$ (see appendix C equation (C.1)) and the condition that guarantees developability (C.9), which tell us how to construct the embedding of a developable surface by choosing a particular one-parameter family of straight lines called generators. In this section we shall describe the problem of folding as an arbitrary curve drawn on a flat sheet of paper. We assume the strong condition that paper is an inextensible material, implying that deformations are isometries almost everywhere, except for localized singularities that may occur.

![Before Folding](image1.png) ![After Folding](image2.png)

**Figure 3.1.** Before and after folding. In (a) we consider a flat paper sheet where one prescribes the curvature $\kappa_g$ of the line one draws, $c_0$. After folding (b) the initial flat line is now a curve in space, $c$ connected by two developables on either side. The Frenet-Serret and Darboux (for one side) frames are also shown.

Before folding or deforming the surface, we first draw a curved crease $c_0(s)$ on a flat paper sheet. The crease pattern has a prescribed curvature $\kappa_g(s)$, which is the boundary line that divides up the plane into two regions, $S_{0+}(s, v)$ and $S_{0-}(s, v)$ (Figure 3.1 (a)). While the paper remains planar (before folding) the crease pattern remains flat as well, which can
be expressed by having its torsion equal to zero, \( \tau(s) = 0 \), and curvature \( \kappa(s) = \kappa_g(s) \), in other words, there is no projection of the curvature of the crease pattern along the normal to the surface. Folding the paper along the crease pattern means isometrically deforming each surface,

\[
\varphi : S_0(s, v) \rightarrow S_\pm(s, v),
\]

so that the deformed surfaces match up along a new boundary curve given by \( c(s) \) (Figure 3.1 (b)), having curvature and torsion in space, \( \kappa(s) \) and \( \tau(s) \) respectively. We call the new boundary curve \( c(s) \), or directrix, the *fold*. On the other hand, the problem of folding can also be seen as a deformation of the crease pattern into the fold,

\[
\varphi : c_0(s) \rightarrow c(s).
\]

Starting off with only one curve drawn on a flat sheet, simple paper model experiments tell us some facts, such as: the fold has a higher curvature than the crease pattern; inflection points on the crease pattern remain inflection points on the fold; if we have a closed crease pattern, we end up with a fold that buckles out of plane, which means it has a nonzero torsion; if we have an open crease pattern, the fold remains planar (in absence of external forces) and the angle between the two surfaces \( S_+(s, v) \) and \( S_-(s, v) \) stays constant throughout the fold. Fuchs and Tabachnikov [86] have already demonstrated that these facts are true using geometrical facts. Nevertheless, questions related to the mechanical properties of curved folds are still open and we wish to explore them later. For now, we are going to concentrate our attention into the geometry of the curved folds.

Let us first look at only one side of the surface, i.e. a developable surface, (C.1), having \( c(s) \) as one of its boundaries. Since the deformation \( \varphi \) is an isometry and the prescribed geodesic curvature is an intrinsic property of the surface, we conclude that \( \kappa_g(s) \) is invariant under the map \( \varphi \), in other words, \( (\varphi^* \kappa_g)(s) = \kappa_g(s) = \kappa_0(s) \). We recall the formalism derived
for the geometry of curves on surfaces, specifically, sitting at a point \( p \), on the fold with the help of (B.16) we can define the principal directions on our developable surface. Because the straight lines on our developable surface, zero curvature \( \kappa_{\parallel} = 0 \), are in the parallel direction to the generators, we clearly have one of the principal directions at \( p \) given by \( \hat{g}|_p \), while the other principal direction is perpendicular to \( \hat{g}|_p \), defining a new vector \( \hat{e}|_p \). It is important to notice that the above conclusions are valid for every \( p \) if our surface is smooth everywhere, and the relations can be written as a function of the arc-length. Defining the angle between the generator \( \hat{g}(s) \) and the tangent \( \hat{t}(s) \) to the fold as being \( \gamma(s) \), we have

\[
\begin{pmatrix}
\hat{t} \\
\hat{u}
\end{pmatrix} =
\begin{pmatrix}
\cos \gamma & -\sin \gamma \\
\sin \gamma & \cos \gamma
\end{pmatrix}
\begin{pmatrix}
\hat{g} \\
\hat{e}
\end{pmatrix}.
\]  

(3.2)

Knowing that \( d\hat{N}(\hat{g}) = -\kappa_{\parallel}\hat{g} = 0 \), defining \( d\hat{N}(\hat{e}) \equiv -\kappa_e\hat{e} \), and using the results in (B.17) and (B.18) we can write the normal curvature and the geodesic torsion of the fold with respect to the surface,

\[
\kappa_N(s) = \kappa_e(s) \sin^2 \gamma(s) \tag{3.3}
\]

and

\[
\tau_g(s) = -\frac{\kappa_e(s)}{2} \sin(2\gamma(s)) = -\frac{\kappa_N(s)}{\tan \gamma(s)}, \tag{3.4}
\]

respectively. Using the expression (B.11), we derive the principal curvature in the \( \hat{e}(s) \) direction,

\[
\kappa_e(s) = \frac{\kappa(s) \cos \alpha(s)}{\sin^2 \gamma(s)} = \kappa(s) \cos \alpha(s) + \frac{(\tau(s) + \alpha'(s))^2}{\kappa(s) \cos \alpha(s)} \tag{3.5}
\]

The above results provide us with the curve restriction of the invariants of the surface, gaussian and mean curvatures, which are respectively given by

\[
K = \kappa_{\parallel}\kappa_e = 0 \tag{3.6}
\]
and
\[
H = \frac{1}{2}(\kappa_{||} + \kappa_{e}) = \frac{1}{2} \left( \kappa(s) \cos \alpha(s) + \left( \frac{\tau(s) + \alpha'(s)}{\kappa(s) \cos \alpha(s)} \right)^2 \right).
\] (3.7)

We finally recall the expressions (C.5) in order to explicitly calculate the components of the first and second fundamental forms, (C.3) and (C.4) respectively. Inverting (3.2) we write the generator \( \hat{g}(s) \) in terms of the Darboux frame,
\[
\hat{g}(s) = \hat{t}(s) \cos \gamma(s) + \hat{u}(s) \sin \gamma(s). \] (3.8)

Using the Darboux frame equations (B.15), we can take derivatives of the last equation,
\[
\begin{align*}
\hat{g}'(s) &= (\kappa_{g}(s) + \gamma'(s)) \hat{e}(s) \\
\hat{g}''(s) &= (\kappa_{g}'(s) + \gamma''(s)) \hat{e}(s) - (\kappa_{g}(s) + \gamma'(s))^2 \hat{g}(s) + \\
&\quad - (\kappa_{g}(s) + \gamma'(s)) (\kappa_{N}(s) \sin \gamma(s) - \tau_{g}(s) \cos \gamma(s)) \hat{N}(s).
\end{align*}
\]

The normal field to the surface, \( \mathbf{N}(s, v) \), and its restriction to the curve, \( \mathbf{N}(c(s)) = \mathbf{N}(s) \), are related as follows
\[
\begin{align*}
\mathbf{N}(s, v) &= \frac{\partial_s \mathbf{S} \times \partial_v \mathbf{S}}{|\partial_s \mathbf{S} \times \partial_v \mathbf{S}|} \\
&= \text{sgn} \left[ \sin \gamma(s) - v (\kappa_{g}(s) + \gamma'(s)) \right] \mathbf{N}(c(s)). \quad (3.9)
\end{align*}
\]

From (C.3) and (C.4) we have
\[
\begin{align*}
g_{ss} &= 1 + v (\kappa_{g}(s) + \gamma'(s)) [v (\kappa_{g}(s) + \gamma'(s)) - 2 \sin \gamma(s)] \\
g_{sv} &= g_{vs} = \cos \gamma(s) \\
g_{vv} &= 1 \quad (3.10)
\end{align*}
\]
and

\[
\begin{align*}
    b_{ss} &= \text{sgn} [\sin \gamma(s) - v (\kappa_g(s) + \gamma'(s))] \\
          &\quad \times [\kappa_N(s) - v (\kappa_g(s) + \gamma'(s)) (\kappa_N(s) \sin \gamma(s) - \tau_g(s) \cos \gamma(s))] \\
    b_{sv} &= b_{us} = 0 \\
    b_{uv} &= 0.
\end{align*}
\] (3.11)

Gaussian and mean curvatures are respectively given by

\[
K(s,v) = 0 \quad (3.12)
\]

and

\[
\begin{align*}
    H(s,v) &= \text{sgn} [\sin \gamma(s) - v (\kappa_g(s) + \gamma'(s))] \\
          &\quad \times \frac{\kappa_N(s)}{2 \sin \gamma(s) (\sin \gamma(s) - v (\kappa_g(s) + \gamma'(s)))}.
\end{align*}
\] (3.13)

3.2.2 Geometrical constraints

In this section we shall derive all the necessary geometrical constraints that consistently connect two developable surfaces that share the same directrix curve. In other words, we consider that the embedding of the two developable surfaces, \( S_+(s,v) \) and \( S_-(s,v) \), give us the parameterization of the fold in space such that \( S_+(s,0) = S_-(s,0) = c(s) \). At this point we recall the definition of an osculating plane, \( P_p^{osc} \), which is the plane at each point \( p \) along the curve that contains the tangent and the normal vectors to the curve. It is convenient to choose \( P_p^{osc} \) to be the common plane in between two surfaces that helps us to do the proper projections (Figure 3.2). Therefore, we also define the angles \( \beta_+(s) \) and \( \beta_-(s) \).
between the osculating plane $\mathcal{P}_{c(s)}^{osc}$ and the tangent planes $T_{c(s)}(S_+)$ and $T_{c(s)}(S_-)$ (Figure 3.2), respectively, such that

$$\alpha_\pm = \frac{\pi}{2} \mp \beta_\pm(s).$$

(3.14)

**Figure 3.2.** Edge on view of the fold. The tangent to the fold is represented by a vector coming out of the plane.

As already mentioned, a very important conclusion can be drawn from the fact that folding along the prescribed crease pattern $c_0(s)$, done by the mapping $\varphi$ (3.1), is an isometry. Therefore, the geodesic curvature, given by the projection of the curvature on the surface, $Dc'(s)/ds$, must be equal to the curvature of the crease pattern, which can be written as follows

$$\kappa_{g\pm}(s) = \kappa(s) \sin \alpha_\pm(s),$$

(3.15)

then

$$\kappa_{g_+}(s) = \kappa_{g_-}(s) = \kappa_g(s).$$

(3.16)
The equation (3.16) leads us to the conclusion that the osculating plane $\mathcal{P}_p^{osc}$ bisects the tangent planes $T_p(S_+)$ and $T_p(S_-)$ for every point $p$ on the fold,

$$\beta_+(s) = \beta_-(s) \equiv \beta(s). \quad (3.17)$$

Based on previous conclusion, we define what we call the folding angle, given by the angle in between the tangent planes to the surfaces on either side of the fold, $2\beta(s)$. It is useful to notice that the folding angle is related to the dihedral angle by $\theta_D(s) = \pi - 2\beta(s)$ (the subindex $D$ will be suppressed in future calculations). The relation (3.15) can be interpreted as the geometrical constraint between the folding angle and the curvature of the curve in space, Figure 3.3,

$$\theta_D(s) = 2 \sin^{-1}\left(\frac{\kappa_g}{\kappa(s)}\right). \quad (3.18)$$

**Figure 3.3.** Constraint between the dihedral angle and the curvature of the curve in space. When the paper is unfolded the ratio between curvature is equal to 1. It can be seen that folding the sheet at zero dihedral angle one needs an infinite curvature.
Another important conclusion, although quite obvious from the experimental models, is that the fold must be connected by convex and concave parts of the surface. The last statement can be proved by substituting each of the angles (3.14) combined with (3.17) into the equation for the normal curvature, which gives us the following,

$$\kappa_{N\pm}(s) = \kappa(s) \cos \alpha_{\pm}(s) = \mp \kappa(s) \sin \beta(s). \quad (3.19)$$

Using the definition (B.14) and the expressions (3.14) and (3.17) to write

$$\tau_{g\pm}(s) = \tau(s) \pm \beta'(s). \quad (3.20)$$

The diagram in Figure 3.4 shows the angles $\gamma_{+}(s)$ and $\gamma_{-}(s)$ between the tangent $\hat{t}(s)$ to the fold and the generators $\hat{g}_{+}(s)$ and $\hat{g}_{-}(s)$ on either side of the fold, $S_{+}(s, v)$ and $S_{-}(s, v)$. According to Figure 3.4, principal directions are rotations of the basis $\{\hat{t}, \hat{u}_{+}\}$ and $\{\hat{t}, \hat{u}_{-}\}$ with the respective angles $\gamma_{+}(s)$ and $\gamma_{-}(s)$ on either side,

$$\begin{pmatrix} \hat{t} \\ \hat{u}_{\pm} \end{pmatrix} = \mathcal{R}_{\pm} \begin{pmatrix} \hat{g}_{\pm} \\ \hat{e}_{\pm} \end{pmatrix}, \quad (3.21)$$

where the rotation matrices $\mathcal{R}_{\pm}$ are given by

$$\mathcal{R}_{\pm} \equiv \begin{pmatrix} \cos \gamma_{\pm} & \pm \sin \gamma_{\pm} \\ \mp \sin \gamma_{\pm} & \cos \gamma_{\pm} \end{pmatrix}. \quad (3.22)$$

Therefore, we can use (B.14) and show that the direction of the generators are dependent on the relation between geodesic torsion and normal curvature,

$$\cot \gamma_{\pm}(s) = \pm \frac{\tau_{g\pm}(s)}{\kappa_{N\pm}(s)}. \quad (3.23)$$
The above equation, (3.45), gives us the following relationships,

\[
\frac{-2\tau(s)}{\kappa(s) \sin \beta(s)} = \frac{\sin (\gamma_+(s) + \gamma_-(s))}{\sin \gamma_+(s) \sin \gamma_-(s)} \tag{3.24}
\]

and

\[
\frac{2\beta'(s)}{\kappa(s) \sin \beta(s)} = \frac{\sin (\gamma_-(s) - \gamma_+(s))}{\sin \gamma_+(s) \sin \gamma_-(s)}. \tag{3.25}
\]

The relation (3.47) tells us that we have constant folding angle if the generators on both sides of the surface make the same angle with the tangent, \( \gamma_+(s) = \gamma_-(s) \). If the torsion of the fold is zero, from (3.46), we must have the angles aligned, \( \gamma_+(s) + \gamma_-(s) = \pi \).

![Figure 3.4. Generators on a surface. Piece of a fold showing the generators at a point. The angles of the generators on the sheet are defined, \( \gamma_+ \equiv \angle(\hat{g}_+, \hat{t}) \).](image)

### 3.3 Mechanics of curved folds

#### 3.3.1 Bending energy

Paper folding is basically done in two steps: (i) drawing a crease pattern with some prescribed geodesic curvature \( \kappa_g(s) \) and then creasing the paper along this arbitrary pattern
of curves; (\(ii\)) isometrically deforming pieces of surfaces in between folds. We will address step \((i)\) in more detail during the next section, but anticipating a little of that discussion, the question we need to ask is related to what we mean when we talk about creasing a paper. As discussed in Chapter 1, creasing paper implies local plastic deformation – deformation beyond yield-stress threshold – which sets the preferred angle between joined surfaces by a fold should naturally be. Although one could think that steps \((i)\) and \((ii)\) do not happen independently, in this section we shall look at isometrically deforming parts of the surface by itself. Further corrections to deviation from the preferred angle will be added later.

In the theory of elasticity of thin membranes, deformations from the ground state pay an energy cost given schematically by

\[
\mathcal{E}_{el} = t \int dA \sqrt{g} \ e_s + t^3 \int dA \sqrt{g} \ e_b, \tag{3.26}
\]

where \(e_s\) and \(e_b\) are respectively stretching and bending energy densities and \(t\) is the thickness of the membrane. In order to write down the elastic energy associated with the deformation described in \((ii)\), we have to consider two important constraining conditions already mentioned. One of these is inextensibility, which is guaranteed in the limit \(t \rightarrow 0\), and the other is developability, \(K = 0\). Therefore, a good approximation for elastically deforming the surface into the curved fold is only due to bending energy on either side, which is given by integration of the total mean square curvature,

\[
\mathcal{E}_b = \frac{B}{2} \int dA \sqrt{g} \ H^2, \tag{3.27}
\]

where \(B\) is the bending modulus. From equation (3.10) we have

\[
g \equiv \det(g_{ij}). \tag{3.28}
\]
Since the embedding of the surface is known to have the form (C.1), we can compute the mean curvature everywhere in \((s, v)\) coordinates as

\[
H_\pm(s, v) = \frac{1}{2} \frac{\kappa_{N\pm}(s) \csc \gamma_\pm(s)}{\sin \gamma_\pm(s) \pm v(\kappa_g(s) \mp \gamma'_\pm(s))}.
\] (3.29)

Therefore, the total bending energy has two contributions, one for each surface, and can be written as,

\[
E_b = \frac{B}{2} \left( \int_0^L ds \int_0^{w_+} dv \sqrt{g_+(H_+)^2} + \int_0^L ds \int_{-w_-}^0 dv \sqrt{g_-(H_-)^2} \right)
\] (3.30)

where \(w_\pm\) are how far to go along the generators on either surface, \(S_{\pm}\), until one hits the boundary. After integrating (3.30) we have

\[
E_b = \frac{B}{8} \int_0^L ds \sum_{\pm} \frac{\kappa_{N\pm}^2 \csc^2 \gamma_\pm}{\kappa_g \mp \gamma'_\pm} \ln \left( \frac{\sin \gamma_\pm}{\sin \gamma_\pm - w_\pm (\kappa_g \mp \gamma'_\pm)} \right).
\] (3.31)

If we specialize in curves with constant curvature and a finite constant width \(w\) on either side of the crease, simple geometry yields

\[
\kappa_g w_\pm = \mp \sin \gamma \pm \sqrt{\kappa_g^2 w^2 \pm 2 \kappa_g w + \sin^2 \gamma}. \] (3.32)

The radicand in this expression must be positive, yielding a lower bound on \(\gamma_\pm\). When this bound is violated, a generator no longer reaches the inner boundary. Instead, it must contact the crease again further, leading to another type of singularity.

When the crease is a straight line, \(\kappa_g = 0\). Equation (3.42) simplifies to

\[
E_b = \frac{B}{2} \int_0^L ds \frac{\kappa^2 (1 + (\tau/\kappa)^2)^2}{w (\tau/\kappa)^2} \ln \left( \frac{1 + w (\tau/\kappa)}{1 - w (\tau/\kappa)} \right),
\] (3.33)

64
the energy for a straight, developable strip of width $w$. This is a particular case of our model was used in [91]. Alternatively, when we consider the case of a narrow strip folding by taking the limit $w\pm \approx w / \sin \gamma \pm \to 0$ in (3.42), we are able to expand the full expression (3.42) in powers of the surface width $w$ and it becomes

$$E_b \approx \frac{Bw}{8} \int_0^L ds \left[ \kappa_{N+}(s)^2 \csc^4 \gamma_+(s) + \kappa_{N-}(s)^2 \csc^4 \gamma_-(s) \right]. \quad (3.34)$$

Another way of writing the approximate energy functional (3.34) is

$$E_b \approx \frac{Bw}{8} \int_0^L ds \left[ \kappa_{N+}(s)^2 \left( 1 + \frac{\tau^2_{g+}(s)}{\kappa_{N+}(s)^2} \right)^2 + \kappa_{N-}(s)^2 \left( 1 + \frac{\tau^2_{g-}(s)}{\kappa_{N-}(s)^2} \right)^2 \right], \quad (3.35)$$

which is comparable to Sadowsky’s functional [94], a model for an elastic strip having the directrix curve to be a geodesic, $\kappa_g(s) = 0$.

The important conclusion from the above procedure that leads to the form (3.42) is that we have reduced our surface energy to a curve energy, or a one-dimension variational problem. If we were only to consider the bending energy as the full energy to describe curved folds, finding minimizers of (3.30) for free boundary conditions case would have lead us to trivial flat solutions. In the next section, we shall see how to add a new term to compete with bending energy which adds geometric frustration to the system such that the solutions are no longer trivial.

### 3.3.2 Phenomenological energy

Let us get back to the afore mentioned step (i). Here we deal with the fact that, besides bending either side of the surface into the desirable shape, creasing the paper involves some sort of plastic deformation. As mentioned in the discussion in the first chapter, it appears to be rather complicated to deal with the details of such a deformation [59, 60]. We shall,
instead, model this local plastic deformation as an extra term that fixes a preferred dihedral angle, $\theta_0$, when the paper is creased and any deviation from such angle to the actual angle, $\theta(s)$, has a local energy cost. We suggest a phenomenological energy functional that is harmonic in this deviation,

$$\mathcal{E}_c = \frac{K}{2} \int_0^L ds \left[ \cos \left( \frac{\theta(s)}{2} \right) - \cos \left( \frac{\theta_0}{2} \right) \right]^2,$$

(3.36)

where $K$ denotes the stiffness of the fold. Therefore, a better model for the total energy is given by the sum of the bending energy derived in the previous section and the phenomenological term,

$$\mathcal{E}_T = \mathcal{E}_b + \mathcal{E}_c.$$

(3.37)

Therefore, as suggested in [57], the equilibrium shape is a result of the mechanical balance between an inextensible region and a creased region modeled by (3.36).

### 3.4 Prototype model

#### 3.4.1 Out-of-plane response

Qualitative experiments with a complete circular annulus of paper having a concentric, circular crease show that folding buckles the crease into a saddle (Figure 3.5a), while the same crease along a cut annulus remains planar (Figure 3.5b). This behavior is a consequence of a fundamental incompatibility between the geometry of the fold and the stretching elasticity of the sheet. As we will see, and it is apparent in Figure 3.5 (b), the sheet responds to folding by wrapping around itself to eliminate in-plane mechanical stresses. The closed annulus, on the other hand, can expel these stresses by buckling. In the limit where the thickness of the sheet is much smaller than the width, which is itself smaller than the length of the crease, the shape that arises is a balance between the bending energy of the sheet on either side of
the crease, the energy at the crease itself, and the geometrical constraints arising from the sheet’s closed topology. Topological constraints are also crucial to shape and mechanics in the small deformations of shells [88] and non-Euclidean plates [23].

![Figure 3.5](image)

**Figure 3.5.** Paper folding of the prototype model. A photograph of the model built by cutting a flat annulus of width $2w$ from a flat sheet of paper with central circle of radius $r$. (a) Folding along its center line buckles the structure out-of-plane. However, if we cut the annulus, (b), the structure collapses to an overlapping planar state with curvature given by equation (3.38). (c) The inset shows a cross section of the fold, where the right and left planes, $S_+$ and $S_-$, define the dihedral angle $\theta$.

We consider an annulus of uniform thickness $t$, width $2w$ folded along a central circular crease of radius $r$ ($t \ll w < r$). In the deformed state, the crease is a space curve parametrized by arc length $s$, with curvature $\kappa(s)$ and torsion $\tau(s)$, and the surfaces on either side of it come together at a finite fold (or dihedral) angle $\theta(s)$. Assuming isometric deformations away from the crease, the mid-surface of the sheet on either side of the crease.
is developable. Then any point on it can be characterized in terms of a set of coordinates $(s,v)$, corresponding to the arc length and the generators of the developable on the inside and outside of the crease, $g_{\pm}$ (Figure 3.5c), with the coordinates: $S_\pm(s,v) = S(s,0) + vg_{\pm}(s)$.

For developables, the generators must also satisfy the condition that $g_{\pm}(s) \times dg_{\pm}(s)/ds$ is perpendicular to the crease [12]. Since folding does not induce in-plane strains, the projection of the crease curvature onto the tangent plane on either side of the sheet must remain $1/r$. This leads to two geometrical conditions [86] that relate the dihedral angle of the crease to its spatial curvature and the angle of the generators of the developable surface on either side of it. These are

\[ \sin \left( \frac{\theta}{2} \right) = \frac{1}{\kappa}, \]  
\[ \cot \gamma_{\pm} = -\frac{1}{2} \left( 2\tau \pm \kappa \frac{d\theta}{ds} \right) \tan \left( \frac{\theta}{2} \right), \]  

where $\kappa/r$ and $\tau/r$ are the curvature and torsion of the crease respectively; $\gamma_{\pm}$ is the angle between the unit tangent vector of the crease $dS(s)/ds$ and the generator. We see that $\kappa(s) \geq 1$, with equality only when $\theta = \pi$. For a circular crease concentric with a circular annulus of constant dimensionless half-width $\omega = w/r$, we find

\[ v_{\pm}^{max}(\xi) = \pm \sin \gamma_{\pm}(\xi) \pm \sqrt{\omega^2 \mp 2\omega + \sin^2 \gamma_{\pm}(\xi)} \]  

(3.40)

to be the dimensionless distance to the boundary along a generator leaving the crease from a point labeled by the dimensionless arc length $\xi = s/r$.

The energy of the sheet is the sum of the energy of deforming the sheet on either side of the crease and that of the fold that connects them. Since the creased folded surface is piecewise developable, the energy per unit surface is proportional to the square of the mean curvature [89]. The mean curvature on either side of the sheet is
\[ H_\pm(\xi, v) = \pm \frac{\cot(\theta/2) \csc \gamma_\pm}{2r [\sin \gamma_\pm \mp v(1 \mp \gamma'_\pm)]}, \]  

(3.41)

where \((.,.)' = d/d\xi(.)\). Then the energy of each surface \(E_b = B \int_0^{2\pi} \int_{v_{\pm}^{\text{max}}}^0 H_\pm^2 dv d\xi\), where \(B\) is the bending stiffness of the material of the sheet. Carrying out the integral along the generators, \(v\), explicitly leads to the following scaled bending energy for the two surfaces

\[
\frac{E_b}{B} = \frac{1}{8} \int_0^{2\pi} d\xi \sum_{\pm} \frac{\cot^2(\theta/2) \csc^2 \gamma_\pm}{1 \pm \gamma'_\pm} \times \ln \left[ \frac{\sin \gamma_\pm}{\sin \gamma_\pm - v_{\pm}^{\text{max}}(\xi)(1 \pm \gamma'_\pm)} \right].
\]

(3.42)

We see that (3.42) is determined entirely in terms of the geometry of the crease. To model the fold itself, we use a phenomenological energy functional measuring the deviation of \(\theta(\xi)\) from an equilibrium angle \(\theta_0\), which we assume to be constant, so that the scaled crease energy

\[
\frac{E_c}{B} = \frac{\sigma}{2} \int_0^{2\pi} d\xi \left[ \cos \left( \frac{\theta(\xi)}{2} \right) - \cos \left( \frac{\theta_0}{2} \right) \right]^2,
\]

(3.43)

where \(\sigma = Kr/B\) is the ratio of the crease stiffness \(K\) and the bending stiffness \(B\). This energy reduces to an expression quadratic in the difference \(\theta - \theta_0\) when \(\theta \sim \theta_0\); although the precise form of this term does not affect our analytic results, it conforms to our numerical model [90].

### 3.4.2 Perturbative calculation

In this section, we discuss details of the perturbative calculation used to construct an \textit{ansatz} for the shape of the crease. We start with a circular crease of radius \(r\) between two concentric, circular boundaries with radii \(r - w\) and \(r + w\) (being \(w/r \sim 0.1\)), as shown in Figure 3.6. The preferred angle is set by \(E_c\) to be \(\theta_0\). The geometrical constraint between
the dihedral angle of the fold and curvature of the crease allows us to associate a curvature 

\( (1 + \epsilon)/r \) with this angle, where

\[
\sin\left(\frac{\theta_0}{2}\right) = \frac{1}{1 + \epsilon}.
\]  

(3.44)

Since \( \epsilon > 0 \), the preferred curvature of the crease is always larger than \( 1/r \); as discussed in the text, this requires us to accommodate this additional curvature by buckling the crease out of the plane.

\[\text{Figure 3.6. Crease pattern of the prototype model. The crease pattern of the prototype model. The parameters of the problem, width } w \text{ and radius of curvature } r \text{ of the crease, are indicated.}\]

The generator angles with respect to the tangent, \( \gamma_{\pm} \) can be expressed in terms of the dimensionless curvature, torsion, and the rate of change of the dihedral angle of the fold with respect to \( \xi = s/r \) as follows [36,86],

\[
cot \gamma_{\pm}(s) = -\left[\frac{\tau(s) \pm \theta'(s)/2}{\cot[\theta(s)/2]}\right].
\]  

(3.45)
equation (3.45) provides the relationships,

$$\tau(s) = -\cot\left(\frac{\theta(s)}{2}\right) \frac{\sin [\gamma_+(s) + \gamma_-(s)]}{\sin \gamma_+(s) \sin \gamma_-(s)}$$

(3.46)

and

$$\theta'(s) = -\cot\left(\frac{\theta(s)}{2}\right) \frac{\sin [\gamma_+(s) - \gamma_-(s)]}{2 \sin \gamma_+(s) \sin \gamma_-(s)}.$$  

(3.47)

We can write an expansion of the curvatures and torsions around a planar state,

$$\kappa(s) = \kappa_0 + \delta \kappa(s) + \mathcal{O}(\delta \kappa^2)$$

(3.48)

$$\tau(s) = \delta \tau(s) + \mathcal{O}(\delta \tau^2),$$

and solve the Euler-Lagrange equations for the fold order-by-order.

To zeroth order, we obtain the relationship between $\kappa_0$ and the control parameters $\sigma$, $\epsilon$, and $\omega = w/r$,

$$\sigma = \frac{\kappa_0^2 (1 + \epsilon) (1 + \kappa_0^2) \ln(1 - \omega)/4}{2 (\kappa_0^2 - 2) \kappa_0 \sqrt{\frac{2 \epsilon + \epsilon^2}{\kappa_0^2 - 1} + (1 + \epsilon) (3 - 2 \kappa_0^2) + \kappa_0^2/(1 + \epsilon)}}.$$  

(3.49)

The parameter $\kappa_0$ is the curvature of lowest energy that would be achieved on an incomplete, and therefore torsionless, fold. Figure 3.7 shows the solutions for $\kappa_0$ as a function of $\sigma$ and dimensionless width $\omega$ for different values of $\epsilon$. For large $\sigma$ ($1/\sigma \lesssim 0.1$), $\kappa_0$ approaches the preferred curvature $1 + \epsilon$ as expected. We can find an approximate solution, valid when $\sigma$ is large, by expanding in powers of $1/\kappa_0 - 1/(1 + \epsilon)$ to obtain

$$\kappa_0 \approx 1 + \epsilon + \frac{\epsilon (1 + \epsilon)(2 + \epsilon) [2 + \epsilon(2 + \epsilon)] \ln(1 - \omega)}{8\sigma}.$$  

(3.50)
Figure 3.7. Zeroth order solution. Dependence $\sigma$ as a function of the solutions for the zeroth-order curvature, $\kappa_0$. The vertical lines represent three different preferred angles set in the phenomenological energy, $\theta_0 = \{2\pi/3, 3\pi/4, 5\pi/6\}$. The color scheme from red to purple represent the normalized widths, $\omega \equiv w/r$, from 0.02 to 0.2.

To first order, the Euler-Lagrange equations give us coupled equations for $\delta\kappa(s)$ and $\delta\tau(s)$,

$$A_0 \delta\kappa + A_2 \partial_s^2 \delta\kappa + A_4 \partial_s^4 \delta\kappa + A_6 \partial_s^6 \delta\kappa + B_1 \partial_s \delta\tau + B_3 \partial_s^3 \delta\tau = 0 \quad (3.51)$$

$$\bar{A}_2 \partial_s^2 \delta\kappa + \bar{A}_4 \partial_s^4 \delta\kappa + \bar{B}_1 \partial_s \delta\tau + \bar{B}_3 \partial_s^3 \delta\tau + \bar{B}_5 \partial_s^5 \delta\tau = 0,$$

with constant coefficients,
\[ A_0 \equiv -\sigma \left( \frac{3}{\kappa_0^2} - \frac{2\sqrt{2\epsilon + \epsilon^2\kappa_0^3}}{(1 + \epsilon)(\kappa_0^2 - 1)^{3/2}} - \frac{1}{(1 + \epsilon)^2 + 2} \right) - \frac{(1 + 3\kappa_0^2) \ln(1 - \omega)}{4} \]

\[ A_2 \equiv -\frac{2\sigma}{\kappa_0^4(1 + \epsilon)} \left[ 3(1 + \epsilon) - \left( \frac{1}{\kappa_0^2 - 1} + 3 \right) \kappa_0 \sqrt{\frac{2\epsilon + \epsilon^2}{\kappa_0^2 - 1}} \right] \]

\[ A_4 \equiv -\frac{1}{\kappa_0^2(\kappa_0^2 - 1)} \omega^3 (1 + 7\kappa_0^2 + \omega^2\kappa_0^2 - 2\omega (1 + 3\kappa_0^2)) + 2(\omega + 1)(\omega - 1)^2 (1 - 3\kappa_0^2) \ln(1 - \omega) \]

\[ A_6 \equiv \frac{(2 - 3\omega)\omega + 2(\omega - 1)^2 \ln(1 - \omega)}{4(\omega - 1)^2\kappa_0^3(\kappa_0^2 - 1)} \]

\[ B_1 \equiv \frac{\omega^2\kappa_0}{4(\omega - 1)^2(\omega + 1)\sqrt{\kappa_0^2 - 1}} \]

\[ B_3 \equiv \frac{B_1}{\kappa_0^2} = \frac{\omega^2}{4(\omega - 1)^2(\omega + 1)\kappa_0\sqrt{\kappa_0^2 - 1}} \]

\[ \bar{A}_2 \equiv \frac{\bar{A}_2}{\kappa_0^2} = \frac{\omega^2}{4(\omega - 1)^2(\omega + 1)\kappa_0\sqrt{\kappa_0^2 - 1}} \]

\[ \bar{A}_4 \equiv \frac{\bar{A}_4}{\kappa_0^2} = \frac{\omega^2}{4(\omega - 1)^2(\omega + 1)\kappa_0\sqrt{\kappa_0^2 - 1}} \]

\[ \bar{B}_1 \equiv \frac{\kappa_0}{4(\omega - 1)^2(\omega + 1)} \frac{\omega(\omega^2 - 2)}{4(\omega - 1)^2(\omega + 1)} - \frac{2\sigma}{\kappa_0^3} \left( \kappa_0 \sqrt{\frac{2\epsilon + \epsilon^2}{\kappa_0^2 - 1}(1 + \epsilon)} - 1 \right) \]

\[ \bar{B}_3 \equiv \frac{1}{\kappa_0} \frac{\omega^3 (1 + 3\kappa_0^2 + \omega^2\kappa_0^2 - 2\omega (1 + \kappa_0^2)) + 2(\omega + 1)(\omega - 1)^2 (1 - \kappa_0^2) \ln(1 - \omega)}{4(\omega - 1)^2(\omega + 1)} \]

\[ \bar{B}_5 \equiv \frac{1}{4\kappa_0} \left[ \omega(3\omega - 2) - 2 \ln(1 - \omega) \right] , \]

Since equations (3.51) are linear, they can be solved with a linear superposition of complex exponentials,

\[ \delta \kappa = \sum_{n=1}^{11} C_n e^{k_n \xi} \]

\[ \delta \tau = \sum_{n=1}^{11} \bar{C}_n e^{k_n \xi} . \]
for some set of complex wave numbers $k_n$ which must be determined numerically, and coefficients $C_n$ and $\bar{C}_n$. Thus, we search for oscillating curvature and torsion solutions. Based on the observed shapes and a four-vertex theorem for convex space curves [96], we expect four points of vanishing torsion. Thus, we decompose a single closed fold into four segments of dimensionless length $\pi/2$ with vanishing torsion at their endpoints and choose boundary conditions such that the four pieces can be reassembled into a closed shape. From equation (3.46), points of vanishing torsion will have $\gamma_+(\xi) + \gamma_-(\xi) = \pi$ while equation (3.47) implies that points of extremal angle, at which $\theta' = 0$, have $\gamma_+ = \gamma_-$. In equations (refeq:odes1), the number of derivatives of torsion is odd while those of curvature is even. Thus, we expect that the points of vanishing torsion coincide with those of extremal curvature and, therefore, $\gamma_+(\xi) = \gamma_-(\xi) = \pi/2$ so that the generators are aligned and perpendicular to the curve at these points. These assumptions are consistent with observations of simulated and paper closed folds.

In principle, we can set additional boundary conditions for $\tau'$ at both ends of the simulated pieces. As a practical matter, the solutions to equations (3.51) have several extremely small length scales, $k_n^{-1}$ owing to the small size of $A_6 \propto \omega^3$ and $\bar{B}_5 \propto \omega^3$. This makes determining the solution from boundary conditions numerically stiff. Assuming that the shape of the torsion is of sufficiently long length scale, we neglect the highest order derivatives in equations (3.51) in determining our solution. Thus, we are able to set $\tau'$ at one end of a piece of fold. Thus setting the overall scale of the oscillating torsion but results in a solution shape that is in continuity class $C^4$. Nevertheless, this small non-analyticity does not cause any change in the computation of the energy.

To summarize, we set $\theta' = \tau = 0$ at $\xi = 0$, $\pi/2$, $\pi$, and $3\pi/2$, and set $\theta(\pi) = \theta(0)$ and $\theta(\pi/2) = \theta(3\pi/2)$. Finally, the perturbative solution is expressible in terms of three parameters only, $\theta(0)$, $\theta(\pi/2)$ and $\tau'(0)$. Finally, we set $\tau'(0)$ by requiring that the fold, resulting from gluing all four pieces together, is closed. This can be imposed by integration.
of the Frenet-Serret system under periodic boundary conditions,

\[
\frac{d}{ds} \begin{pmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{pmatrix},
\]

(3.54)

where the triad of vectors \( \{\hat{t}, \hat{n}, \hat{b}\} \) represents the moving tangent, normal, and binormal on the curve. Therefore, the moduli space consistent with closed origami is a manifold defined by \( \delta \tau'(0) = \delta \tau'|_{\xi=0} \{\theta(0), \theta(\pi/2)\} \). In order to explore the landscape of energy for the allowed configurations, we substitute the solutions for (3.51) and (3.52) into the total energy density and integrate it over the domain \( \xi \in [0, 2\pi] \). The result gives the total energy as a function of the two remaining parameters, \( \theta(0) \) and \( \theta(\pi/2) \), \( E = E[\theta(0), \theta(\pi/2)] \).

### 3.4.3 Minimal energy configurations

The equilibrium shape of the curved crease results from minimizing \( E = E_b + E_c \) and is characterized by three parameters: the scaled natural width of the ribbon \( \omega \), the natural dihedral angle between the two surfaces adjoining the crease \( \theta_0 \) and the dimensionless crease-surface energy scale \( \sigma \), subject to appropriate boundary conditions. For example, an open circular crease has free ends and thus remains planar with \( \tau = 0 \) since non-planarity would increase both the curvature and torsion [71]. A closed crease, however, is frustrated by geometry, forcing it to buckle, a fact that follows from the inequality \( \kappa = 1/\sin(\theta/2) > 1 \) when \( \theta < \pi \) which requires \( \int d\xi \kappa > 2\pi \), and is incompatible with a planar crease with \( \tau = 0 \) [86].
Figure 3.8. Minimal energy configurations. (a) Perturbative fold of width $\omega = 0.1$ and $\sigma = 2/\sqrt{3}$ shaded by mean curvature. The generators are indicated by the lines on the surface. The inset shows the dimensionless torsion and curvature of the crease. (b) A simulated (results of L. Dudte and L. Mahadevan) fold of width $\omega \approx 0.0994$ shaded by local area change relative to the flat state.
Though geometrical constraints induce buckling, the resulting fold shapes are determined by minimizing the total elastic energy consisting of contributions from the sheet (3.42) and the fold (3.43), expressed entirely in terms of the curvature and torsion of the crease [93,95]. For relatively narrow, but stiff, folds \( \omega \ll 1 \) and \( \sigma \gg 1 \) that are weakly folded, i.e. so that the dihedral angle \( \theta_0 \sim \pi \), and thence \( \epsilon \equiv 1/ \sin(\theta_0/2) - 1 \ll 1 \). Then, we find that the total scaled energy \( E = (E_b + E_c)/B \) simplifies to [71]

\[
E \approx \int_0^{2\pi} d\xi \left\{ \frac{\sigma}{4\epsilon} (\kappa - 1 - \epsilon)^2 + \frac{\omega}{2} \tau^2 \right\},
\]

(3.55)
in terms of the scaled curvature \( \kappa \) and torsion \( \tau \). We see that as \( \sigma \to \infty \), the rescaled curvature \( \kappa \to 1/ \sin(\theta_0/2) = 1/(1+\epsilon) \), the prescribed curvature. The minimal energy crease shape, therefore, minimizes \( \tau^2 \) subject to the constraints of fixed length and curvature. In this limit, the Euler-Lagrange equations become \( [\tau'' + (1 + \epsilon)^2 \tau]' \approx 0 \) at constant curvature [71]. Unless \( \epsilon = 0 \) – corresponding to dihedral angle \( \theta_0 = \pi \) – there is no completely smooth solution to these equations. However, we note that a solution of continuity class \( C^4 \) may be obtained to these equations with \( \kappa = 1 + \epsilon \) and oscillating torsion,

\[
\delta \tau = \begin{cases} 
\tau_0 \left[ 1 - \frac{\cos((\xi - \pi/2)(1+\epsilon))}{\cos((\pi/2)(1+\epsilon))} \right], & 0 \leq \xi \leq \pi \\
-\tau_0 \left[ 1 - \frac{\cos((\xi - 3\pi/2)(1+\epsilon))}{\cos((\pi/2)(1+\epsilon))} \right], & \pi \leq \xi \leq 2\pi.
\end{cases}
\]

(3.56)
The absolute magnitude of the torsion \( \tau_0 \) is then chosen so that the curved fold has arc length \( 2\pi r \). Consistent with the four-vertex theorem for closed convex space curves, there are four points with vanishing torsion [96].

An asymptotic analysis of the full Euler-Lagrange equations can be performed by expanding the shape of the crease around a planar curve of constant curvature, \( \kappa_0 \). Following [93,95], we write \( \kappa = \kappa_0 + \delta \kappa \) and \( \tau = \delta \tau \) and compute the Euler-Lagrange equations. To lowest order, we obtain an algebraic expression determining the ideal curvature of the crease, \( \kappa_0 \) in
terms of arbitrary $\sigma$, $\epsilon$ and $\omega$ [71]. This is the curvature that would be obtained by a cut, annular fold having zero torsion. To next order, we find that both the curvature and torsion oscillate. A typical analytical solution for general arbitrary parameters is shown in Figure 3.8a, shaded by mean curvature, with the inset showing the oscillating torsion vanishing at the extrema of curvature (see Figure 3.5). We choose the overall amplitude of $\tau$ to close the curve, with $\theta(0)$ and $\theta(\pi/2)$ parametrizing the solutions [71].

These qualitative features are also confirmed by direct numerical minimization done by our collaborators L. Dudte and L. Mahadevan. Their model considers the energy of a triangular mesh model for the curved origami structure in which adjacent triangles across the crease prefer a fixed, non-planar dihedral angle [90], adjacent triangles in each sheet prefer a planar dihedral angle and each edge is treated as a linear spring, with the scaled ratio of the bending stiffness to the stretching stiffness $B/Sl^2 \approx 10^{-3}$. These simulations relax the isometry of the folding process and thus allow us to capture how extension and shear arise in wide folds (Figure 3.8b); we find that they typically localize where the mean curvature, based on our isometric analytic theory (shown in Figure 3.8a), becomes large.
Figure 3.9. Minimal energy states and comparison between the perturbation theory and numerics. (a) Angle differences $|\theta(\pi/2) - \theta(0)|$ as a function of $\omega$ with $\theta_0 = 2\pi/3$. The red curve (diamonds) are computed from first-order perturbation theory with $\sigma = 2/\sqrt{3}$ and $\theta_0 = 2\pi/3$. Corresponding energy landscapes, as a function of $\theta(0)$ and $\theta(\pi/2)$ respectively, are shown for (b) $\omega = 0.01$, (c) 0.05, and (d) 0.1, with energy minima drawn as white dots. In (a), numerical simulations (results of L. Dudte and L. Mahadevan) are shown (blue – dashed lines) with $\sigma = 2\sqrt{3}$ (circles) and $\sigma = 160/\sqrt{3}$ (squares) and are compared with a nonperturbative variational ansatz (green – solid line), $\kappa(2)$ and $\tau(2)$, described in the text with $\sigma = \sqrt{3}/40$ (circles) and $\sigma = \sqrt{3}/2$ (squares).

Moving beyond the simple asymptotic theory for narrow folds, we consider the dependence of the solution on the scaled width by using the perturbative shapes as a variational ansatz in the exact, analytical energy. Since the shapes have a 4-fold symmetry, we plot the energy as a function of $\theta(0)$ and $\theta(\pi/2)$ in Figure 3.9b-d. When $\omega \lesssim 0.1$, annuli with
large $\sigma$ have a nearly constant dihedral angle around the entire length of the fold, with $\theta(0) \approx \theta(\pi/2)$ for the narrowest fold widths. For small $\sigma$, however, the energy minimum generically has $\theta(0) \neq \theta(\pi/2)$. To understand this, we plot $\theta(0) - \theta(\pi/2)$ for the minimal energy configuration as a function of the scaled width $\omega$.

Plotting the associated energy in Figure 3.9b-d for some representative values of $\omega$, we see that the energy contours develop forks because a range of $\theta(0)$ and $\theta(\pi/2)$ are forbidden by the geometric constraints that the generators of our two surfaces can intersect only outside the actual surface, else the bending energy diverges. To avoid the intersection of generators inside the outer surface requires

$$\gamma_+' < \frac{\sin \gamma_+}{v_{\max}^+} - 1 \quad \text{and} \quad \gamma_-'> \frac{\sin \gamma_-}{v_{\max}^+} + 1,$$

which reduces to $|\tau'| < (1 - \omega) \cot(\theta/2)/\omega$, at points in which $\tau = 0$. Similarly, to avoid the intersection of the generators on the inner surface inside the inner boundary requires the discriminant in equation (4.7) to be positive, implying a bound on the torsion,

$$\left| \tau + \frac{\theta'}{2} \right| < \frac{1 - \omega}{\sqrt{2\omega - \omega^2}} \cot \left( \frac{\theta}{2} \right).$$

These geometrical bounds restrict the range of allowed torsion and thus the buckling of the crease, which requires torsion. In particular, wide folds will become stiff to deformations as the sheet quickly reaches a regime in which stretching is required and the generators do not have too much freedom to move around $\xi = \pi/2$. In the perturbation theory underlying Figure 3.9d, this is manifested by the presence of large forks carved out by the forbidden configurations. Since the energy minima occur close to the singularities, the perturbative expansion of the shape is not likely to be valid. Even at intermediate widths, however, where the perturbative expansion should be at least qualitatively valid, the bifurcation of the minima are the shadows of the prominent forks observed in Figure 3.9d.
These singularity bounds suggest a second ansatz: \( \kappa(2) = \kappa_0 + \kappa_1 \cos(2\xi) \) and \( \tau(2) = \tau_0[\sin(2\xi) + \eta \sin(4\xi)] \), choosing \( \tau_0 \) to close the fold and \( \eta \) to minimize the energy. When \( \eta = 0 \) we find very good agreement with the perturbative ansatz previously considered. However, we find that \( \eta \approx -0.45 \) for large widths, which lowers the maximum of the torsion and better satisfies the singularity bounds in equations (4.16a - 4.17). Using \( \sigma \) as a fitting parameter, we see that \( \theta(\pi/2) - \theta(0) \) agrees quite well with the numerical solutions for small \( \omega \) and only diverges from numerical simulations for large widths, around \( \omega \sim 0.08 \) as shown in Figure 3.9a.
CHAPTER 4

KINEMATICAL CONSTRUCTION OF CURVED PLEATS

4.1 Introduction

Origami, the art of folding paper, has been developing throughout the years into a scientific field. In this form of art, two fundamental questions can be raised, “How and why can we fold structures?” and “What shapes can be folded?”. The former is a question of foldability and all that needs to be found is an algorithm for folding. For instance, a more specific question would be, “Which crease patterns can fold flat?”, which turns out to be an NP-hard problem [32]. The latter question, which is about design, has shown to have a lot of potential to be applied to a wide range of disciplines, from architecture to engineering science [33,44,52,57].

In this article we are concerned with the question of design of pleated structures, in other words, corrugated structures where creases are folded by alternating mountains and valleys. We are particularly interested in exploring a class of pleated patterns suggested in the late 1920’s at the extinct German school of crafts and fine arts, Bauhaus, where in an art project, paper models of “hyperbolic paraboloids”, or simply “hypar”, were folded by alternating mountains and valleys of concentric squares and circles [33]. These are particularly interesting objects because once they are creased along closed paths they are in a frustrated state [97], therefore, in order to balance their internal forces, they go through a buckling process which leads to self-folding. This suggests that mechanics should be fundamental to determine the equilibrium configurations. However, before we attempt to answer
questions concerned with the mechanics of hypar paper models, we focus on their kinematical constructions, where all geometrical constraints that tell us about the existence of these structures are derived. Such constraints emerge from the condition that paper deforms isometrically almost everywhere, away from creases [97]. This question of existence, in other words, whether or not these structures admit isometric embeddings, remained completely open until 2009. Demaine et al., [34] showed that the crease patterns of concentric squares are not possible to fold without inserting extra creases. Moreover, they also showed that, even after adding the extra creases that allowed the folding process, the final object was not actually a true hyperbolic paraboloid. In that same work, Demaine et al. conjectured that circular pleat folds would exist if a hole were cut out of the center. This remains a challenge until today.

Curved folding is a relatively new subject of origami research. Though its geometry has been studied [36, 55, 86, 87], mechanics of curved origami remains largely unexplored. We have explored this subject from the mechanical viewpoint [97], where geometric frustration on an elastic sheet allows us to study shape formation by prescribing circular folds. One possible natural extension of the single curved fold project is to consider multiple folded structures. More specifically, the tools we already developed will be used to formulate the problem of multiple concentric circles prescribed on a plane that are folded by alternating mountains and valleys (Figure 4.1).

4.2 Construction of multiple concentric circles

Let us start by defining coordinates and relationships among vectors for mountains and valleys. Let $c$ be the embedding of a set of curves in space, such that $c : (s, i) \rightarrow \mathbb{R}^3$, where $s \in [0, 2\pi]$ and $i \in \mathbb{N}^*$, where $s(i)$ is the arc-length correspondent to $ith$ curve and we define $s \equiv s(1)$. $c$ can also be interpreted as the embedding of a semi-discrete surface (Ref.).
Consider the diagram in Figure 4.2 showing three consecutive folds. For the $i$th fold we can decompose the generators (straight directions on the surface) in the following way

\[ \hat{g}_{\pm}(s, i) = \hat{t}(s, i) \cos[\gamma_{\pm}(s, i)] \mp \hat{u}_{\pm}(s, i) \sin[\gamma_{\pm}(s, i)], \]

(4.1)

where $\hat{t}(s, i)$ is the tangent to the curve, $\hat{u}_{\pm}(s, i)$ are the tangent vectors to the surface on either side of the fold,

\[ \hat{u}_{\pm}(s, i) = \hat{n}(s, i) \sin\left[\frac{\theta(s, i)}{2}\right] \pm \hat{b}(s, i) \cos\left[\frac{\theta(s, i)}{2}\right], \]

(4.2)

and $\hat{n}(s, i)$ and $\hat{b}(s, i)$ the Frenet-Serret normal and binormal, respectively. The angles between the tangent vector and the generators, $\gamma_{\pm}(s, i)$, can be expressed in terms of
geodesic torsions, $\tau_{g\pm}(s,i) = \tau(s,i) \mp \partial_s \theta(s,i)/2$, and normal curvatures, $\kappa_{N\pm}(s,i) = \mp \kappa_g(i) \cot [\theta(s,i)/2]$, which allows us to define the following new variable

$$\eta_{\pm}(s,i) \equiv \cot [\gamma_{\pm}(s,i)] = \pm \frac{\tau_{g\pm}(s,i)}{\kappa_{N\pm}(s,i)}. \quad (4.3)$$

It follows from the above definitions that $\tau_{g+}(s,i) = \tau_{g-}(s,i) - \partial_s \theta(s,i)$ and $\kappa_{N+}(s,i) = -\kappa_{N-}(s,i)$, therefore

$$\eta_{+}(s,i) = \eta_{-}(s,i) + \frac{\partial_s \theta(s,i)}{\kappa_g(i) \cot [\theta(s,i)/2]}, \quad (4.4)$$

which gives us a prescription on how to go from one side to the other over the crease. In other words, if we approach the crease from the ($-$) side, we can write the quantities on the ($+$) side by knowing the dihedral angle in between the two surfaces.

**Figure 4.2.** Side view of multiple folds. Here we schematically represent the geometry of the generators in the multiple folded configuration.
Figure 4.3. Top view of multiple folds. Here we schematically represent the geometry of the generators in the multiple folded configuration. Top view, showing points on different curves connected by a generator line.

Figure 4.3 shows a schematic representation of the geometry in consideration, which is constructed such that the generators of consecutive curves are related to each other by the following constraint relationship

\[ \hat{g}_+(i) + \hat{g}_-(i) = 0. \] \hspace{1cm} (4.5)

Moreover, these curves are related to each other by geometry in such a way that the curve at position \( i \) can be written in terms of the one at \( i - 1 \) as follows

\[ c(s, i) = c(s, i - 1) + v_{\max}^+(s, i - 1)\hat{g}_+(s, i - 1). \] \hspace{1cm} (4.6)
We shall consider the case in which the crease pattern is given by a sequence of concentric circles evenly spaced by a distance $\Delta w$ and having $\kappa_g(1) > \kappa_g(2) > \cdots > \kappa_g(i) > \cdots$ (Figure 4.4). In that case, simple planar geometry allows us to write the maximum distance along the generator until the next curve,

$$v^\text{max}_\pm = \frac{1}{\kappa_g} \left( \mp \sin \gamma \pm \sqrt{\kappa_g \Delta w (\kappa_g \Delta w \pm 2) + \sin^2 \gamma} \right). \quad (4.7)$$

**Figure 4.4.** Multiple fold pattern. Folding pattern evenly spaced by concentric circles $\Delta w$ apart. $\kappa_g(i-1)$'s are the prescribed geodesic curvatures and $R_0$ is the radius of the hole cut in the center of the sheet. Dashed and dotted lines represent alternating mountain and valley patterns.

Due to the inextensibility condition, we can determine the evolution from fold to fold of the geodesic curvature by using planar geometry, which is given by

$$\kappa_g(i) = \frac{\kappa_g(i-1)}{1 + \Delta w \kappa_g(i-1)}. \quad (4.8)$$
The equation (4.6) gives us a prescription to write recursion relations for the geometric quantities that determine the shape of the creases. In other words, we are looking for the evolution of the arc-length, the angle \( \theta(s, i) \), and the variable \( \eta_\pm(s, i) \). The arc-length evolution can be calculated by normalized tangent, \( \hat{t}(s, i) = \partial_{s(i)} c(s, i) \), where the operator \( \partial_{s(i)} \) is defined by

\[
\partial_{s(i)} \equiv \frac{1}{l(s, i)} \partial_s,
\]

and

\[
l(s, i) \equiv (1 + \Delta w \kappa_g(i - 1)) \left( 1 + \frac{(\partial_{s(i-1)} \eta_+(s, i - 1)) v_+^{\max}(s, i - 1) / \sqrt{1 + \eta_+(s, i - 1)^2}}{1 + \kappa_g(i - 1)v_+^{\max}(s, i - 1) / \sqrt{1 + \eta_+(s, i - 1)^2}} \right) l(s, i - 1).
\]

Using the operator (4.9) on the embedding (4.6), we can calculate the rate of rotation of the Darboux frame for the curve \( i \),

\[
\partial_{s(i)} \left( \begin{array}{c} \hat{t} \\ \hat{u}_\pm \\ \hat{N}_\pm \end{array} \right) = \left( \begin{array}{ccc} 0 & \kappa_g(i) & \kappa_{N_\pm} \\ -\kappa_g(i) & 0 & \tau_{g_\pm} \\ -\kappa_{N_\pm} & -\tau_{g_\pm} & 0 \end{array} \right) \left( \begin{array}{c} \hat{t} \\ \hat{u}_\pm \\ \hat{N}_\pm \end{array} \right),
\]

and identify the quantities \( \kappa_{N_\pm}(s, i) \) and \( \tau_{g_\pm}(s, i) \) in terms of geometrical quantities of the curve \( i - 1 \),

\[
\kappa_{N_\pm}(s, i) = \frac{1 + \kappa_g(i - 1)v_+^{\max}(s, i - 1) \sqrt{1 + \eta_+(s, i - 1)^2}}{1 + \frac{(\partial_{s(i-1)} \eta_+(s, i - 1)) v_+^{\max}(s, i - 1) / \sqrt{1 + \eta_+(s, i - 1)^2}}{1 + \kappa_g(i - 1)v_+^{\max}(s, i - 1) / \sqrt{1 + \eta_+(s, i - 1)^2}}} \kappa_{N_\pm}(s, i - 1)
\]

and

\[
\tau_{g_\pm}(s, i) = \frac{1/ (1 + \Delta w \kappa_g(i - 1))^2}{1 + \frac{(\partial_{s(i-1)} \eta_+(s, i - 1)) v_+^{\max}(s, i - 1) / \sqrt{1 + \eta_+(s, i - 1)^2}}{1 + \kappa_g(i - 1)v_+^{\max}(s, i - 1) / \sqrt{1 + \eta_+(s, i - 1)^2}}} \tau_{g_\pm}(s, i - 1).
\]
Using the equation (4.3) yields
\[ \eta_-(s,i) = \frac{-\eta_+(s,i-1)}{1 + \kappa_g(i-1)v_+(s,i-1)\sqrt{1 + \eta_+(s,i-1)^2}}, \tag{4.14} \]
and in addition to \( \theta(s,i) = 2\cot^{-1} \left[ \frac{\kappa_{N_-}(s,i)}{\kappa_g(i)} \right] \), we have the evolution for all the quantities. For simplicity, we explicitly write these relationships up first order expansion in \( \Delta w \),

\[ \kappa_g(i) = \kappa_g(i-1) - \Delta w \kappa_g(i-1)^2 + \mathcal{O}(\Delta w^2) \tag{4.15a} \]

\[ c(s,i) = c(s,i-1) + \Delta w \sqrt{1 + \eta_+(s,i-1)^2} g_+(s,i-1) + \mathcal{O}(\Delta w^2) \tag{4.15b} \]

\[ l(s,i) = l(s,i-1) \left[ 1 + \Delta w \left( \kappa_g(i-1) + \partial_s(s,i-1) \eta_+(s,i-1) \right) \right] + \mathcal{O}(\Delta w^2) \tag{4.15c} \]

\[ \eta_-(s,i) = -\eta_+(s,i-1) \left[ 1 - \kappa_g(i-1) \Delta w \left( 1 + \eta_+(s,i-1)^2 \right) \right] + \mathcal{O}(\Delta w^2) \tag{4.15d} \]

\[ \theta(s,i) = 2\pi - \theta(s,i-1) + \Delta w \sin(\theta(s,i-1)) \left( \kappa_g(i-1) \eta_+(s,i-1)^2 - \partial_s(s,i-1) \eta_+(s,i-1) \right) + \mathcal{O}(\Delta w^2). \tag{4.15e} \]

Figure 4.5. Side and top view of a multiple fold structure – Concentric circles. Surface given by solving the full constraint equations.

By fixing the shape of the inner most curve, we can solve all the above constraints to construct a curved fold structure. In order to exemplify our construction, we solve the above
recursion relations, where the solution for three consecutive folds are shown in Figures 4.5 (a)-(b). We can solve for the case of open folds, Figures 4.6 (a)-(f), which turns out to be either flat, zero torsion, or helixes multiple folded structures.

![Discrete Helicoid](image)

**Figure 4.6.** Discrete helicoid. In this case, we have the initial inner curves having space curvatures set as (a) $\kappa/\kappa_g = 1.001$ and $\tau/\kappa_g = 0$, (b) $\kappa/\kappa_g = 1.2$ and $\tau/\kappa_g = 0$ ((c) half fold with $\kappa/\kappa_g = 1.2$ and $\tau/\kappa_g = 0$), (d) $\kappa/\kappa_g = 1.2$ and $\tau/\kappa_g = 0.05$, (e) $\kappa/\kappa_g = 1.2$ and $\tau/\kappa_g = 0.25$, and (f) $\kappa/\kappa_g = 1.2$ and $\tau/\kappa_g = 0.5$.

Singularities can emerge in between the folds in one of two ways. First, a singularity occurs if two generators cross on the sheet, resulting in both a diverging bending energy if stretching is completely expelled as in our model. These types of singularities, given by bounds on the rate of change of the generator angles,
\[ \gamma' < \kappa_g(i) \left( \frac{\sin(\gamma_-)}{\epsilon_{\text{max}}^{-}} - 1 \right) \]  

\[ \gamma' > \kappa_g(i) \left( -\frac{\sin(\gamma_+)}{\epsilon_{\text{max}}^{+}} + 1 \right). \]

The equation (4.16a) can be understood as bounds on the rate of change of the torsion, which results in an expression for limits on the slope of the torsion, \(|\tau'| < \kappa^2_g ((1 - w\kappa_g)/w\kappa_g) \cot (\theta/2))\), at points \(s^*\) where \(\gamma_\pm = \pi/2\) and the curvature is an extremum. Second, generators on the inside sheet may emerge from the crease at a value of \(\gamma_-\) sufficiently small that the generator fails to meet the inner boundary. Again, this is pre-empted by a diverging energy at angles \(\gamma_-\) such that the generators are tangent to the inner boundary. This geometric constraint, which occurs when the discriminant in equation (4.7) is negative, can be translated into a bound for the torsion,

\[ \left(1/\kappa_g\right) \left| \tau + \frac{\theta'}{2} \right| < \frac{1 - w\kappa_g}{\sqrt{2w\kappa_g - w^2\kappa^2_g}} \cot \left( \frac{\theta}{2} \right). \]

These constraints shape the fundamental mechanical behavior of the folded annulus. In particular, wide folds will become stiff to deformations as the sheet quickly reaches a regime in which stretching is required.

### 4.3 Continuum limit

In this section we aim to write a continuous limit for a surface that approximates the corrugated structure at the limit in which \(\Delta w\) tends to zero and the number of folds is very large. In this construction, as schematically represented in Figure 4.2, we write the embedding of the fold \(c(s, i)\) with respect to the two adjacent folds \(c(s, i \pm 1)\),

\[ c(s, i \pm 1) = c(s, i) + \nu_{\pm}^{\text{max}} \mathbf{g}_{\pm}(s, i). \]

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A proper continuum limit can be accomplished if we expand the geometrical quantities in powers of $\Delta w$, keeping only terms up to the first order. The expansion of the equation (4.7) yields the following simplified form for the distances along the generators in between folds,

$$v_{\pm}^{\text{max}}(s, i) = \sqrt{1 + \eta_{\pm}(s, i)^2} \Delta w + O(\Delta w^2).$$  

(4.19)

Using the equation (4.1), we can rewrite the equation (4.18),

$$c(s, i \pm 1) = c(s, i) + \Delta w \left[ \eta_{\pm}(s, i) \hat{t}(s, i) \mp \hat{u}_{\pm}(s, i) \right] + O(\Delta w^2).$$  

(4.20)

Therefore, we define the difference between the embeddings,

$$\Delta c(s, i) \equiv c(s, i+1) - c(s, i-1) = \Delta w \left\{ [\eta_+(s, i) - \eta_-(s, i)] \hat{t}(s, i) - [\hat{u}_+(s, i) + \hat{u}_-(s, i)] \right\} + O(\Delta w^2).$$  

(4.21)

We use the equations (4.2) and (4.4) to show that the vector $\Delta c(s, i)$ lies on the osculating plane. In other words $\Delta c(s, i)$ is expanded only in terms of the tangent and the normal of the creases, $\{\hat{t}(s, i), \hat{n}(s, i)\}$, suggesting a discrete form for the variation of the embedding from mountain to mountain (or valley to valley) as a central finite difference formula,

$$\frac{\Delta c(s, i)}{2\Delta w} = \frac{\partial s(i) \theta(s, i)}{2\kappa_g(i) \cot \left[ \frac{\theta(s, i)}{2} \right]} \hat{t}(s, i) - \sin \left[ \frac{\theta(s, i)}{2} \right] \hat{n}(s, i) + O(\Delta w).$$  

(4.22)

Taking the continuum limit by allowing $\Delta w \to 0$, for a large number of folds, $N$, we arrive at the following expression,

$$\frac{\partial c(s, w)}{\partial w} = \lim_{\Delta w \to 0} \frac{\Delta c(s, i)}{2\Delta w} = \frac{\partial s(w) \theta(s, w)}{2\kappa_g(w) \cot \left[ \frac{\theta(s, w)}{2} \right]} \hat{t}(s, w) - \sin \left[ \frac{\theta(s, w)}{2} \right] \hat{n}(s, w),$$  

(4.23)
where the discrete coordinate \(i\) is now represented by the continuum independent variable \(w\) and \(\partial_{s(w)} \equiv l(s, w)^{-1} \partial_s\). It is convenient to define the following functions

\[
\begin{align*}
\alpha(s, w) &\equiv \frac{\partial_{s(w)} \theta(s, w)}{2 \kappa_g(w) \cot \left[\theta(s, w)/2\right]} \\
\beta(s, w) &\equiv -\sin \left[\theta(s, w)/2\right] 
\end{align*}
\]  
(4.24)

as components of a “velocity” field, \(V(s, w)\), defined by

\[
V(s, w) \equiv \frac{\partial c(s, w)}{\partial w} = \alpha(s, w) \mathbf{\hat{t}}(s, w) + \beta(s, w) \mathbf{\hat{n}}(s, w).
\]  
(4.26)

As noted before, the discrete variation (4.22) lies on the osculating plane, therefore, the velocity (4.26) lies on the tangent plane to the approximate surface which turns out to be the osculating plane after the limit is taken.

Regarding the other geometric quantities, the continuum equations can be derived in two different ways, however equivalent. The first, a more natural approach, would be to use the recursion relations, analogously to the construction we have done before, in order to derive the following equations,

\[
\begin{align*}
\kappa_g(i \pm 1) &= \kappa_g(i) \mp \Delta w \kappa_g(i)^2 + \mathcal{O}\left(\Delta w^2\right) \\
l(s, i \pm 1) &= \left[1 \mp \Delta w \left(\kappa_g(i) \pm \partial_{s(i)} \eta_\pm(s, i)\right)\right] l(s, i) + \mathcal{O}\left(\Delta w^2\right) \\
\kappa_{N_\mp}(s, i \pm 1) &= \left[1 \mp \Delta w \left(\kappa_g(i) \left(1 - \eta_\pm(s, i)^2\right) \pm \partial_{s(i)} \eta_\pm(s, i)\right)\right] \kappa_{N_\pm}(s, i) + \mathcal{O}\left(\Delta w^2\right) \\
\tau_{g_\mp}(s, i \pm 1) &= \left[1 \mp 2\Delta w \left(\kappa_g(i) \pm \frac{\partial_{s(i)} \eta_\pm(s, i)}{2}\right)\right] \tau_{g_\pm}(s, i) + \mathcal{O}\left(\Delta w^2\right).
\end{align*}
\]  
(4.27)

Using the relationship \(\kappa_{N_\pm}(s, i) = \mp \kappa_g(i) \cot [\theta(s, i)/2]\), we can write a recursion expression for the two adjacent dihedral angle,

\[
\theta(s, i \pm 1) = 2\pi - \theta(s, i) \pm \Delta w \sin [\theta(s, i)] \left(\kappa_g(i) \eta_\pm(s, i)^2 \mp \partial_{s(i)} \eta_\pm(s, i)\right) + \mathcal{O}\left(\Delta w^2\right).
\]  
(4.28)
After some algebraic manipulations, the above discrete equations turn into discrete evolutions, similarly to (4.22), for the geodesic curvature, correction of the arc-length, space curvature, and torsion, respectively given by

\[
\Delta \kappa_g(i) = -\kappa_g(i)^2 + \mathcal{O}(\Delta w) \quad (4.29a)
\]

\[
\Delta l(s,i) = l(s,i) \left\{ \partial_s(i) \left[ \frac{\partial_s(i) \theta(s,i)}{2\kappa_g(i) \cot \left[ \theta(s,i)/2 \right]} \right] + \kappa_g(i) \right\} + \mathcal{O}(\Delta w) \quad (4.29b)
\]

\[
\Delta \kappa(s,i) = - \left[ \kappa(s,i)^2 - \tau(s,i)^2 \right] \sin \left[ \theta(s,i)/2 \right] - \partial_s^2 \sin \left[ \theta(s,i)/2 \right]
\]

\[
+ \frac{\partial_s(i) \theta(s,i)}{2\kappa_g(i) \cot \left[ \theta(s,i)/2 \right]} \partial_s(i) \kappa(s,i) + \mathcal{O}(\Delta w) \quad (4.29c)
\]

\[
\Delta \tau(s,i) = - \partial_s(i) \sin \left[ \theta(s,i)/2 \right] \tau(s,i) + 2 \frac{\tau(s,i)}{\kappa(s,i)} \partial_s(i) \sin \left[ \theta(s,i)/2 \right]
\]

\[
- \tau(s,i) \kappa(s,i) \sin \left[ \theta(s,i)/2 \right] - \frac{\partial_s(i) \theta(s,i)}{2\kappa_g(i) \cot \left[ \theta(s,i)/2 \right]} \partial_s(i) \tau(s,i) + \mathcal{O}(\Delta w) \quad (4.29d)
\]

The second approach, a more elegant one, uses the interpretation that the velocity field, equation (4.26), drives the evolution of the initial curve in space tracing the approximate surface. We use the compatibility conditions, \( \partial_s \partial_w = \partial_w \partial_s \) in the equation (4.26) and Frenet-Serret equations,

\[
\partial_s(w) \begin{pmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s,w) & 0 \\ -\kappa(s,w) & 0 & \tau(s,w) \\ 0 & -\tau(s,w) & 0 \end{pmatrix} \begin{pmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{pmatrix}, \quad (4.30)
\]

in order to obtain the following evolution equations.
\[ \partial_w \kappa_g(w) = -\kappa_g(w)^2 \]  
(4.31a)

\[ \partial_w l(s, w) = l(s, w) (\partial_{s(w)} \alpha(s, w) - \kappa(s, w) \beta(s, w)) \]  
(4.31b)

\[ \partial_w \kappa(s, w) = \partial_{s(w)}^2 \beta(s, w) + (\kappa(s, w)^2 - \tau(s, w)^2) \beta(s, w) + \alpha(s, w) \partial_{s(w)} \kappa(s, w) \]  
(4.31c)

\[ \partial_w \tau(s, w) = \partial_{s(w)} \left[ \frac{\beta(s, w)}{\kappa(s, w)} \partial_{s(w)} \tau(s, w) + 2 \frac{\tau(s, w)}{\kappa(s, w)} \partial_{s(w)} \beta(s, w) \right] \]  
\[ + 2 \kappa(s, w) \tau(s, w) \beta(s, w) - \alpha(s, w) \partial_{s(w)} \kappa(s, w), \]  
(4.31d)

which agree with the equations (4.29) when the limit \( \Delta w \to 0 \) is taken. Besides the equations (4.31), we also get the evolution of the Frenet-Serret frame

\[
\partial_w \begin{pmatrix}
\hat{t} \\
\hat{n} \\
\hat{b}
\end{pmatrix}
= \begin{pmatrix}
0 & \partial_{s(w)} \beta + \kappa \alpha & \beta \tau \\
-\partial_{s(w)} \beta - \kappa \alpha & 0 & \frac{\partial_{s(w)} (\beta \tau)}{\kappa} + \frac{\tau}{\kappa} (\partial_{s(w)} \beta + \kappa \alpha) \\
-\beta \tau & -\frac{\partial_{s(w)} (\beta \tau)}{\kappa} - \frac{\tau}{\kappa} (\partial_{s(w)} \beta + \kappa \alpha) & 0
\end{pmatrix}
\begin{pmatrix}
\hat{t} \\
\hat{n} \\
\hat{b}
\end{pmatrix},
\]  
(4.32)

which are also known as the Weingarten equations.

### 4.4 Geometry of the approximate surface

The continuum limit construction suggests that we treat multiple corrugated structures as a problem of curve evolution. The solution for this evolution, given by solving the equations (4.31), traces out a surface in space in which, in the limit that the spacing between creases goes to zero, should approximate the multiple folded structure. One way to explore the space of possible solutions of this nonlinear system given by the equations (4.31), is to understand the specific geometries that can arise from these systems. For instance, one could in principle look for solutions that lead to minimal surfaces, which is the class of surfaces that have zero mean curvature, \( H(s, w) = 0 \). In order to accomplish that, one should be able to calculate the mean curvature in terms of the components of the velocity field (4.26), curvature, and
torsion and then impose the condition $H(s, w) = 0$ as a constraint. In face of that challenge, we shall here calculate the geometrical properties of such approximate surfaces starting from the first and second fundamental forms, which are defined as follows,

$$ I(ds, dw) = a_{ij} dx^i dx^j = \langle \partial_i c, \partial_j c \rangle dx^i dx^j $$

$$ \equiv E(ds)^2 + 2F(dsdw) + G(dw)^2 \quad (4.33) $$

and

$$ II(ds, dw) = b_{ij} dx^i dx^j = -\langle N, \partial_i \partial_j c \rangle dx^i dx^j $$

$$ \equiv e(ds)^2 + 2f(dsdw) + g(dw)^2. \quad (4.34) $$

In order to calculate the components of the fundamental forms we use the equations (A.7) and (4.32) to write explicit form for the derivatives of the embedding $c(s, w)$,

$$ E \equiv \langle \partial_s c(s, w), \partial_s c(s, w) \rangle = l(s, w)^2 \quad (4.35a) $$

$$ F \equiv \langle \partial_s c(s, w), \partial_w c(s, w) \rangle = l(s, w)\alpha(s, w) \quad (4.35b) $$

$$ G \equiv \langle \partial_w c(s, w), \partial_w c(s, w) \rangle = \alpha(s, w)^2 + \beta(s, w)^2 \quad (4.35c) $$

and

$$ e \equiv \langle N(s, w), \partial^2_s c(s, w) \rangle = 0 \quad (4.36a) $$

$$ f \equiv \langle N(s, w), \partial_s \partial_w c(s, w) \rangle = -l(s, w)\beta(s, w)\tau(s, w) \quad (4.36b) $$

$$ g \equiv \langle N(s, w), \partial^2_w c(s, w) \rangle = -2\alpha(s, w)\beta(s, w)\tau(s, w) + $$

$$ -\frac{\beta(s, w)}{l(s, w)} [\partial_s (\beta(s, w)\tau(s, w)) + \tau(s, w)\partial_s \beta(s, w)] \quad (4.36c) $$

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where \( N(s, w) = \partial_s c \times \partial_w c / |\partial_s c \times \partial_w c| = -\hat{b}(s, w) \). Therefore, gaussian and mean curvatures are respectively given by

\[
K(s, w) = -\tau(s, w)^2
\]

and

\[
H(s, w) = \frac{2\partial_s \kappa(s, w) \tau(s, w) - \kappa(s, w) \partial_s \tau(s, w)}{2l(s, w) \kappa(s, w)^2}.
\]

An important consequence of this construction is that all surfaces have negative gaussian curvature, in other words, we have shown that circular corrugation always approximates surfaces that are locally hyperbolic.

### 4.5 Continuum limit solution – the helicoid

Now our task is to solve the nonlinear coupled system of partial differential equations, (4.31). As a first attempt to solve the equations, we look for solutions that have the following simple form

\[
\kappa(s, w) = \kappa_0 a(w) \quad (4.39a)
\]

\[
\tau(s, w) = \tau_0 b(w) \quad (4.39b)
\]

\[
l(s, w) = c(w), \quad (4.39c)
\]

with initial conditions

\[
a(w_0) = 1, \quad b(w_0) = 1, \quad \text{and} \quad c(w_0) = 1. \quad (4.40)
\]

The evolution of the geodesic curvature can be solved independently, where its solution is given by

\[
\kappa_g(w) = \frac{1}{r + w - w_0}, \quad (4.41)
\]
where $\kappa_g(w_0) = 1/r$. The other three equations become,

$$a'(w) = -\kappa_g(w)a(w) + \frac{\tau_0^2}{\kappa_0^2} b(w)^2 \kappa_g(w)^2$$  \hspace{1cm} (4.42a)\\
$$b'(w) = -2\kappa_g(w)b(w)$$  \hspace{1cm} (4.42b)\\
$$c'(w) = \kappa_g(w)c(w),$$  \hspace{1cm} (4.42c)

having the following solutions

$$b(w) = r^2\kappa_g(w)^2$$  \hspace{1cm} (4.43a)\\
$$a(w) = \frac{r\kappa_g(w)}{\kappa_0} \sqrt{\tau_0^2 (1 - r^2\kappa_g(w)^2) + \kappa_0^2}$$  \hspace{1cm} (4.43b)\\
$$c(w) = \frac{1}{r\kappa_g(w)}.$$  \hspace{1cm} (4.43c)

Integrating the Frenet-Serret frame both along the arc-length $s$ and the variable $w$, equations (A.7) and (4.32) respectively, we can write exactly the embedding for the approximate surface,

$$c(s, w) = \begin{pmatrix}
\frac{\tau_0^2}{\kappa_0^2 + \tau_0^2} s + \frac{\kappa_0 \sqrt{\kappa_0^2 w^2 + \tau_0^2 (w^2 - 1)} \sin\left(\frac{s \sqrt{\kappa_0^2 + \tau_0^2}}{\kappa_0 + \tau_0^2}\right)}{(\kappa_0^2 + \tau_0^2)^{3/2}} \\
\kappa_0 - \frac{\sqrt{\kappa_0^2 w^2 + \tau_0^2 (w^2 - 1)} \cos\left(\frac{s \sqrt{\kappa_0^2 + \tau_0^2}}{\kappa_0 + \tau_0^2}\right)}{(\kappa_0^2 + \tau_0^2)^{3/2}} \\
\kappa_0 \tau_0 \frac{s}{\kappa_0^2 + \tau_0^2} - \frac{\tau_0 \sqrt{\kappa_0^2 w^2 + \tau_0^2 (w^2 - 1)} \sin\left(\frac{s \sqrt{\kappa_0^2 + \tau_0^2}}{\kappa_0 + \tau_0^2}\right)}{(\kappa_0^2 + \tau_0^2)^{3/2}} \\
\end{pmatrix},$$  \hspace{1cm} (4.44)

where we set $r = 1$ for simplicity. We can also calculate the mean and gaussian curvatures for this surface, which are given by

$$H(s, w) = 0$$  \hspace{1cm} (4.45a)\\
$$K_G(s, w) = -\frac{\tau_0^2}{w^4}.$$  \hspace{1cm} (4.45b)

Therefore, the surfaces shown in Figure 4.7 are true helicoids, in other words, they are all minimal surfaces.
Figure 4.7. Approximate surface – Helicoid. Approximate surfaces, \(c(s, w)\), for (a) \(\kappa_0 = 1.5\) and \(\tau_0 = 2\), (b) \(\kappa_0 = 2.5\) and \(\tau_0 = 3\), (c) \(\kappa_0 = 3.5\) and \(\tau_0 = 4\), and (d) \(\kappa_0 = 4.5\) and \(\tau_0 = 5\).
5.1 Swelling of thin sheets

We have presented equations that govern the design of shapes by isotropic growth of a thin elastic sheet and solved them analytically for axisymmetric cases. There are two relevant cases: disk-like sheets with one boundary, and annular sheets with two boundaries. For disk-like sheets, the boundary conditions can be satisfied locally by choosing appropriate gradients in the metric near the edges. This implies, among other things, that a negative Gaussian curvature lip must appear. For a generic annular sheet, not only must we satisfy the local boundary conditions at two boundaries, but a difficult nonlocal condition resulting from in-plane force balance. We have found that an additional term in $\rho(u)$ linear in $u$ with adjustable coefficient can be used to satisfy this boundary condition without changing the surface dramatically. Once the assumption of axisymmetry is lifted, it is not at all clear what would be required to satisfy this last boundary condition.

Several obstacles to swelling a shape may arise. A chosen functional form for $\rho(u)$ may, after boundary conditions are applied, fail to satisfy $|\rho'(u)| < 1$ at one or more places on the sheet. This implies that an axisymmetric shape of the desired form cannot satisfy the boundary conditions. For a sufficiently thick sheet, the required prescribed metric may fail to remain positive definite at one or more places. Thus, the shape would not be swellable, at least not by an axisymmetric target metric, at the desired thickness. This is not a surprising result, as sufficiently thick sheets will not buckle at all in response to an $O(1)$ inhomogeneity.
in swelling. Yet another problem – one that we did not encounter – may arise because our method of search for an isotropic swelling factor rests on an implicit assumption about the existence of a global conformal coordinate system. Such a coordinate system is only guaranteed to exist locally [79]. Failure to find a coordinate system of this type globally suggests, again, that a particular shape may not be swellable. Finally, we note that we have not investigated the stability of the generated surfaces, only whether axisymmetric extrema exist. The azimuthal stress $s^{\nu\nu}$, Figure 5.1, in the surface from Figure 2.4 oscillates between tensile and compressive, which suggests the possibility of a wrinkling instability in the compressed regions of sufficiently thin sheets. Wrinkling patterns in thin sheet elasticity is a subject of broad interest. These kind of phenomena emerge in a variety of problems, ranging from larger scales, like finger tip pruning and the shape of a leaf edge, to the micron scales [98]. Recent experiments that control wrinkling patterns at small scales [99] are of increasing interest in the scientific community. At the same time, much effort has been spent from a theoretical side to classify and understand these problems [100]. Due to the presence of compressive stresses in our problem, exploring wrinkling instability can also be a subject of future research.

Figure 5.1. Azimuthal stress for the compression fitting. Nondimensionalized azimuthal stress, $s^{\nu\nu}/(tY)$, in the swelled surface of Figure 2.4, with $t = 1/20$. 

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Another possible extension for this project is related to questions on nematic liquid crystal elastomers undergoing prescribed macroscopic shape changes. A thin strip of elastomer with a nonuniform crosslink density will, upon exposure to an external stimulus such as a solvent, undergo nonuniform deformations. Predicting the state of the deformed material becomes even more interesting when the polymer strands have an anisotropic shape of gyration as is the case in nematic elastomers. We would like to pursue the case in which the pattern of the crosslinks defines a metric, and hence a desired shape upon relaxation. Starting from a phenomenological formalism of the strain-order coupling, we can try to understand how the presence of the nematic degree of freedom frustrates the shape selection expected for the case of a non-uniformly swelled isotropic elastomer.

5.2 Curved folds and origami

Curved crease origami is a consequence of the fundamental frustration between folding along a curve and the avoidance of singularities and in-plane stretching. The avoidance of in-plane stresses impose geometric constraints on the shape that are reflected in a bifurcation of the curvature of a closed crease of large width. Indeed, the coupling between shape and in-plane stretching endows these structures with a stiffness and response that is unusual, as we have demonstrated in the simplest of situations - a closed circular fold. Moving forward, our approach may be generalized to more complex curves with variable dihedral angles in folded structures with curved creases and thus sets the stage for the analysis and design of these objects.

Within our goals during this project, we have spent part of our efforts to identify in geometry and elasticity theory a common language that could be used to study shape formation by prescribing arbitrary folds, which is seen here as a mechanism to generate geometric frustration on elastic sheets. We hope that with the basis of this, we will be able to further explore questions that generalize the problems reported in this thesis for both single and
multiple folds structures. Our next step for multiple curved folds is to go beyond the kinematical construction, shown in Chapter 4. In order to do that, we need to formulate the mechanics by adding contributions of the energies (3.36) and (3.42) for all folds in consideration. We can gain some insights in this problem by performing this sum when taking the limit $\Delta w \to 0$. As we did before, in the continuous limit treatment, it is more convenient to consider as a discrete unit the sum of two adjacent folds, such that the recursion relations (4.27) become useful here and we also avoid double counting facets when summing over $i$. Therefore, we have that

\[ E_b(i) \approx \Delta w B \frac{1}{8} \int_{0}^{2\pi} ds l(s, i) \sum_{\pm} \frac{\kappa_{N\pm}^2(s, i)}{\kappa_{N\pm}^2(s, i)} \left( 1 + \frac{\tau_{g\pm}^2(s, i)}{\kappa_{N\pm}^2(s, i)} \right)^2 \] (5.1)

and

\[ E_b = \frac{1}{2} \lim_{\Delta w \to 0} \sum_{i=1}^{n'} [E_b(i + 1) + E_b(i - 1)]. \] (5.2)

In this project we want to explore the possibility that the form (5.2) will give back an effective surface energy that when minimized should yield the configuration of the approximate surface. As for the phenomenological energy, responsible for deviations from prescribed preferred angles for each fold, we believe that this should be measured as a strain energy for the in-plane deformation of the corrugations. In the continuous limit, this would imply that the “strain energy” is given in terms of the target metric, given by (4.35), and the reference metric, which is flat.
APPENDIX A

GEOMETRY OF CURVES IN $\mathbb{R}^3$

Let us first consider a curve $c$ in space $\mathbb{R}^3$ parametrized by its arc-length, $c : I \rightarrow \mathbb{R}^3$, where $s \in I$. Using the arc-length parameterization, the tangent vector to the curve, $c'(s) \equiv \hat{t}(s)$, also called velocity of the curve, has unit length, or $\langle c'(s), c'(s) \rangle = \langle \hat{t}(s), \hat{t}(s) \rangle = 1$. Here we use the convention of primed quantities, $(...)'(s)$, to mean derivatives with respect to the arc-length $s$ and $\langle ..., ... \rangle$ represents the inner product. The unitarity of the tangent implies that the second derivative of the curve with respect to the arc-length, or acceleration, has to be normal to its velocity,

$$\frac{d}{ds} \langle c'(s), c'(s) \rangle = 2 \langle c'(s), c''(s) \rangle = 0. \quad (A.1)$$

We call the magnitude of $c''(s)$ curvature of the curve, $\kappa(s) \equiv |c''(s)|$, and, as $c''(s)$ is itself normal to the velocity, we define the unit normal vector to the curve, $\hat{n}(s)$, as follows

$$c''(s) = \hat{t}'(s) = \kappa(s)\hat{n}(s). \quad (A.2)$$

The plane defined by the unit tangent and unit normal vectors at a point $p$ on the curve is called the osculating plane. Normal to the osculating plane, we define the binormal unit vector by

$$\hat{b}(s) \equiv \hat{t}(s) \times \hat{n}(s). \quad (A.3)$$

The rate of change of the binormal, $\hat{b}'(s)$, measures how fast the curve goes off the osculating plane. It is easy to see that $\hat{b}'(s)$ is normal to $\hat{b}(s)$. We can also conclude that $\hat{b}'(s)$ and
\( \hat{t}(s) \) are normal to each other by taking the derivative of (A.3) with respect to the arc-length and using (A.2),

\[
\hat{b}'(s) = \hat{t}'(s) \times \hat{n}(s) + \hat{t}(s) \times \hat{n}'(s) = \hat{t}(s) \times \hat{n}'(s).
\]

(A.4)

Therefore, we can show that the rate of change of the binormal is parallel (or anti-parallel) to the unit normal \( \hat{n}(s) \) and its magnitude defines the torsion of the curve, \( \tau(s) \equiv |\hat{b}'(s)| \).

In this way we can write the formula

\[
\hat{b}'(s) = -\tau(s)\hat{n}(s),
\]

(A.5)

where the negative sign is arbitrary at this point and it only means that we choose \( \hat{b}'(s) \) anti-parallel to \( \hat{n}(s) \). Now, for completeness, we look at \( \hat{n}(s) = \hat{b}(s) \times \hat{t}(s) \) and differentiate it with respect to the arc-length in order to get the rate of change of the normal,

\[
\hat{n}'(s) = \hat{b}'(s) \times \hat{t}(s) + \hat{b}(s) \times \hat{t}'(s)
= -\kappa(s)\hat{t}(s) + \tau(s)\hat{b}(s),
\]

(A.6)

where we have used (A.2) and (A.5). For each value of \( s \) along the curve we associate an orthonormal fame \( \{\hat{t}(s), \hat{n}(s), \hat{b}(s)\} \). The frame rotates as we move along \( s \) according to (A.2), (A.5), and (A.6). This is known as the Frenet-Serret moving frame, and (A.2) , (A.5), and (A.6) can be grouped together in the following matrix formulation

\[
\frac{d}{ds} \begin{pmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \hat{t} \\ \hat{n} \\ \hat{b} \end{pmatrix}.
\]

(A.7)
The Frenet-Serret system, (A.7), tells us that, knowing functional form of curvature and torsion of the curve, we can determine the embedding of the curve, $c(s)$, up to rigid rotations and translations in space.
In the previous section we saw how curvature and torsion define a curve in space \( \mathbb{R}^3 \). Nevertheless, if we were to consider that our curve also lies on a surface \( S \in \mathbb{R}^3 \), \( \kappa(s) \) and \( \tau(s) \) would not be the right invariants for such formulation. We need new scalar functions that also capture the properties of the surface \( S \). Let \( \hat{N}(\mathbf{c}(s)) \) (\( = \hat{N}(s) \) for simplicity) be the restriction to the surface of the normal field to the curve \( \mathbf{c}(s) \). It is very intuitive that the tangent to the curve \( \mathbf{c}'(s) = \hat{t}(s) \) is also tangent to the surface, in other words, \( \hat{t}(s) \) lies on a tangent plane to the surface, \( T_{\mathbf{c}(s)}(S) \). The acceleration \( \mathbf{c}''(s) \), however, in general could be pointing anywhere in space \( \mathbb{R}^3 \). Therefore, let us consider the normal and the tangential projection of \( \mathbf{c}''(s) \) with respect to the surface. These can be written as

\[
\mathbf{c}_\perp''(s) = \langle \mathbf{c}''(s), \hat{N}(s) \rangle \hat{N}(s) \equiv \tilde{\kappa}_N(s) \quad (B.1)
\]

and

\[
\mathbf{c}_\parallel''(s) = \frac{D}{ds} \mathbf{c}'(s) \equiv \tilde{\kappa}_g(s), \quad (B.2)
\]

where the symbol \( (D/ds) = D_{\hat{t}(s)} \) represents the covariant derivative relative to the tangent \( \hat{t}(s) \). The equations (B.1) and (B.2) define the normal curvature vector, \( \tilde{\kappa}_N(s) \), and the geodesic curvature vector, \( \tilde{\kappa}_g(s) \), respectively. At this point it is important to notice that the magnitude of the geodesic curvature, or the numerical value of the covariant derivative, is an intrinsic property of the surface and it is preserved under isometries (maps that preserve the metric on the surface).
Now, let us choose a unit vector \( \mathbf{u}(c(s)) \in T_{c(s)}(\mathcal{S}) \) that is also normal to the tangent vector \( \mathbf{t}(s) \), such that we can fully expand the restriction of the tangent plane \( T_{c(s)}(\mathcal{S}) \) to the curve and \( \mathbf{N}(s) \equiv \mathbf{t}(s) \times \mathbf{u}(s) \). While considering that the curve is parametrized by the arc-length, we have

\[
\langle c'(s), c'(s) \rangle = 1 \Rightarrow \left\langle \frac{D}{ds} c'(s), c'(s) \right\rangle = 0,
\]

which implies that

\[
\kappa_g(s) \sim \mathbf{u}(s).
\]

Therefore we can rewrite (B.1) and (B.2) as

\[
\kappa_N(s) = \kappa_N(s)\mathbf{N}(s)
\]

and

\[
\kappa_g(s) = \kappa_g(s)\mathbf{u}(s),
\]

where \( \kappa_N(s) \equiv \langle c''(s), \mathbf{N}(s) \rangle \) and \( \kappa_g(s) \equiv \langle c''(s), \mathbf{u}(s) \rangle \) are the normal and geodesic curvatures, respectively. It can be proven that the normal curvature \( \kappa_N(s) \) depends only on the direction that the tangent to the curve pointing as it follows,

\[
\kappa_N(s) = \langle c''(s), \mathbf{N}(s) \rangle
\]

\[
= \langle c'(s), \mathbf{N}(s) \rangle' - \left\langle \frac{d}{ds} \mathbf{N}(c(s)), c'(s) \right\rangle
\]

\[
= -\left\langle d\mathbf{N}(c'(s)), c'(s) \right\rangle
\]

\[
\equiv II(c'(s), c'(s)),
\]

where we have used \( \langle c'(s), \mathbf{N}(s) \rangle = 0 \) and the map \( II(\quad,\quad) \) defines the second fundamental form. We now look back to the equation (A.2) and write the following expression.
\[ c''(s) = \mathbf{t}'(s) = \kappa(s)\mathbf{n}(s) \]
\[ = \kappa_g(s)\mathbf{u}(s) + \kappa_N(s)\mathbf{N}(s), \quad (B.8) \]
then
\[ \kappa(s)^2 = \kappa_g(s)^2 + \kappa_N(s)^2, \quad (B.9) \]
which relates the curvature of the curve in space and its curvatures with respect to the surface. If we define \( \alpha(s) \) as the angle between the normal to the curve and the normal to the surface, we can write the normal to the curve \( \mathbf{n}(s) \) in terms of \( \mathbf{u}(s) \) and \( \mathbf{N}(s) \),
\[ \mathbf{n}(s) = \mathbf{u}(s) \sin \alpha(s) + \mathbf{N}(s) \cos \alpha(s), \quad (B.10) \]
which, together with (B.8), gives us
\[ \kappa_N(s) = \kappa(s) \cos \alpha(s), \quad (B.11) \]
and
\[ \kappa_g(s) = \kappa(s) \sin \alpha(s). \quad (B.12) \]
Recalling (B.10), we can also write the basis \( \{\mathbf{u}(s), \mathbf{N}(s)\} \) in terms of \( \{\mathbf{n}(s), \mathbf{b}(s)\} \) by rotating these vectors with respect to the angle \( \alpha(s) \). Hence, we may write
\[ \begin{pmatrix} \mathbf{u} \\ \mathbf{N} \end{pmatrix} = \begin{pmatrix} \sin \alpha(s) & -\cos \alpha(s) \\ \cos \alpha(s) & \sin \alpha(s) \end{pmatrix} \begin{pmatrix} \mathbf{n} \\ \mathbf{b} \end{pmatrix}. \quad (B.13) \]
We have already proven that the normal curvature, a component of the second fundamental form, is given by \( \kappa_N(s) = \langle \mathbf{N}'(s), \mathbf{t}'(s) \rangle \). For completeness, we can calculate \( \langle \mathbf{N}(s), \mathbf{u}'(s) \rangle \), which is another component of the second fundamental form and it is given by,
\[ \langle \mathbf{N}(s), \mathbf{u}'(s) \rangle = \tau(s) + \alpha'(s) \equiv \tau_g(s), \quad (B.14) \]
where we have used (B.13) and (A.7). The equation (B.14) defines the geodesic torsion, which measures the arc-rate of rotation of the normal $\hat{\mathbf{N}}(s)$ along the curve.

Similarly to what was done in the previous section, when looking at the curve from the surface point of view, we have an orthonormal fame $\{\hat{\mathbf{t}}(s), \hat{\mathbf{u}}(s), \hat{\mathbf{N}}(s)\}$ that follows the curve along the arc-length and its rate of change is expected to be just a rotation, in other words, an anti-symmetric matrix times the basis gives us the rate of rotation. Looking at (B.8) and (B.14) and the fact that we need an anti-symmetric matrix, we can conclude that

$$
\frac{d}{ds} \begin{pmatrix}
\hat{\mathbf{t}} \\
\hat{\mathbf{u}} \\
\hat{\mathbf{N}}
\end{pmatrix}
= \begin{pmatrix}
0 & \kappa_g(s) & \kappa_N(s) \\
-\kappa_g(s) & 0 & \tau_g(s) \\
-\kappa_N(s) & -\tau_g(s) & 0
\end{pmatrix}
\begin{pmatrix}
\hat{\mathbf{t}} \\
\hat{\mathbf{u}} \\
\hat{\mathbf{N}}
\end{pmatrix}.
$$

(B.15)

The system (B.15) is called the Darboux frame. It can also be checked that (B.15) is consistent with (B.13).

If we sit at a point $p$ on the curve, say when $s = 0$, we can write the normal curvature and the geodesic torsion at that point in terms of the principal curvatures. Letting $\{\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_2\} \in T_{p=c(0)}(\mathcal{S})$ be the principal directions on the surface, $\{\kappa_1, \kappa_2\}$ are define as the principal curvatures, which are the lines of minimum and maximum curvature at the point $p$. Rotating the principal axes, we can always write the expansion of $T_{p=c(0)}(\mathcal{S})$ in terms of $\{\hat{\mathbf{t}}(0), \hat{\mathbf{u}}(0)\}$,

$$
\begin{pmatrix}
\hat{\mathbf{t}} \\
\hat{\mathbf{u}}
\end{pmatrix}
= \begin{pmatrix}
\cos \gamma & -\sin \gamma \\
\sin \gamma & \cos \gamma
\end{pmatrix}
\begin{pmatrix}
\hat{\mathbf{X}}_1 \\
\hat{\mathbf{X}}_2
\end{pmatrix},
$$

(B.16)

where $\gamma$ is the angle between $\hat{\mathbf{t}}$ and $\hat{\mathbf{X}}_1$. Using (B.7) and (B.14) we can write
\[\kappa_N = -\left\langle d\hat{N}(\hat{t}), \hat{t}\right\rangle_p\]
\[= -\left\langle d\hat{N}(\hat{X}_1 \cos \gamma - \hat{X}_2 \sin \gamma), \hat{X}_1 \cos \gamma - \hat{X}_2 \sin \gamma\right\rangle_p\]
\[= \kappa_1 \cos^2 \gamma + \kappa_2 \sin^2 \gamma,\]  \hspace{1cm} (B.17)

known as Euler formula, and

\[\tau_g = -\left\langle d\hat{N}(\hat{t}), \hat{u}\right\rangle_p\]
\[= -\left\langle d\hat{N}(\hat{X}_1 \cos \gamma - \hat{X}_2 \sin \gamma), \hat{X}_1 \sin \gamma + \hat{X}_2 \cos \gamma\right\rangle_p\]
\[= \sin \gamma \cos \gamma (\kappa_1 - \kappa_2),\]  \hspace{1cm} (B.18)

where we have used the fact that the linear map \(d\hat{N}_p : T_p(S) \rightarrow T_p(S)\) is well defined and operates on the orthonormal basis \(\{\hat{X}_1, \hat{X}_2\}\) such that \(d\hat{N}_p(\hat{X}_1) = -\kappa_1 \hat{X}_1\) and \(d\hat{N}_p(\hat{X}_2) = -\kappa_2 \hat{X}_2\).
APPENDIX C
DEVELOPABLE SURFACES

In this section we shall construct a special class of surfaces that will be of interest later. Let us start considering a one-parameter family of straight lines given by the pair \( \{c(s), \hat{g}(s)\} \), where \( c(s) \) is an arbitrary curve parametrized by arc-length and \( \hat{g}(s) \) is a unit vector field, \( |\hat{g}(s)| = 1 \). Therefore, for each \( s \in I \), we have a straight line that passes through the point \( c(s) \in \mathbb{R}^3 \) on the curve and it is parallel to the vector \( \hat{g}(s) \in \mathbb{R}^3 \). Given this one-parameter family, we can construct a surface given by

\[
S(s,v) = c(s) + v\hat{g}(s),
\]

where we have added the parameter \( v \in \mathbb{R} \). The surface (C.1) is known as ruled surface, where the curve \( c(s) \) is called the directrix and the straight lines are called the rulings or generators of the surface. Calculating the gaussian curvature, \( K \), of (C.1) gives us a classification of these surfaces, which is given in terms of the components of the first and second fundamental forms by

\[
K = \frac{eg - f^2}{EG - F^2},
\]

where we define in general

\[
I(du, dv) = g_{ij}dx^i dx^j = \langle \partial_i S, \partial_j S \rangle dx^i dx^j \\
\equiv E(du)^2 + 2F(dudv) + G(dv)^2
\]
\[ II(du, dv) = b_{ij} dx^i dx^j = -\langle N, \partial_i \partial_j S \rangle dx^i dx^j \]
\[ \equiv e(du)^2 + 2f(dudv) + g(dv)^2. \]  
(C.4)

For the surface (C.1) we have

\[ \partial_s S = c'(s) + v \hat{g}'(s), \quad \partial_v S = \hat{g}(s), \]
\[ \partial_s^2 S = c''(s) + v \hat{g}''(s), \quad \partial_s \partial_v S = \partial_v \partial_s S = \hat{g}'(s), \]
\[ \partial_v^2 S = 0, \]  
(C.5)

and its normal

\[ N(s, v) = \frac{\partial_s S \times \partial_v S}{|\partial_s S \times \partial_v S|} \sim c'(s) \times \hat{g}(s) + v \hat{g}'(s) \times \hat{g}(s). \]  
(C.6)

Therefore

\[ g = 0 \quad \text{and} \quad f = \frac{\langle \hat{g}(s) \times \hat{g}'(s), c'(s) \rangle}{|\partial_s S \times \partial_v S|}, \]  
(C.7)

which gives us the gaussian curvature

\[ K = -\frac{\langle \hat{g}(s) \times \hat{g}'(s), c'(s) \rangle^2}{|\partial_s S \times \partial_v S|^2}. \]  
(C.8)

The above expression tells us that the gaussian curvature of a ruled surface can be negative or zero, \( K \leq 0 \). In this work, we are interested in cases where gaussian curvature is strictly
equal to zero for every regular point on the surface. For this case, $K = 0$, the surface is known as a developable surface, and the condition

$$\langle \hat{g}(s) \times \hat{g}'(s), c'(s) \rangle = 0,$$  \hspace{1cm} (C.9)

has to be satisfied, which means that $\hat{g}(s)$, $\hat{g}'(s)$, and $c'(s)$ are linearly dependent everywhere. Such surfaces are locally isometric to the plane, in other words, they preserve the planar metric structure. The last statement is a direct consequence of the *theorema egregium* of Gauss.

The equation (C.9) gives us a classification of developable surfaces. We first recall the condition $|\hat{g}(s)| = 1$, which implies that $\langle \hat{g}(s), \hat{g}'(s) \rangle = 0$. Considering that $\hat{g}(s)$ and $\hat{g}'(s)$ are linearly dependent everywhere, the only possible consistent solution is $\hat{g}'(s) = 0$. When $\hat{g}(s) = \hat{g}_0$ is a constant vector, our surface is always made up of parts of cylinders,

$$S(s, v) = c(s) + v\hat{g}_0.$$  \hspace{1cm} (C.10)

When $\hat{g}(s)$ and $\hat{g}'(s)$ are linearly independent everywhere, there exist functions $f_1(s)$ and $f_2(s)$ such that, in order to satisfy (C.9), $c'(s) = f_1(s)\hat{g}(s) + f_2(s)\hat{g}'(s)$. Let us also define $\tilde{c}(s) \equiv c(s) - f_2(s)\hat{g}(s)$. Therefore, we have

$$\tilde{c}'(s) = c'(s) - f_2(s)\hat{g}'(s) - f_2'(s)\hat{g}(s) \hspace{1cm} (f_1(s) - f_2'(s))\hat{g}(s).$$  \hspace{1cm} (C.11)

From the above equation, we have two cases to analyze. If $f_1(s) - f_2'(s) = 0$, $\tilde{c}(s) = \tilde{c}_0$ is a constant vector and the surface is given by
\[ S(s, v) = \tilde{c}_0 + (v + f_2(s))\tilde{g}(s) \]
\[ = S_{cone}(s, v) + f_2(s)\tilde{g}(s), \]  \hspace{1cm} (C.12)

where \( S_{cone}(s, v) \equiv \tilde{c}_0 + v\tilde{g}(s) \) is a conical part of the surface. If \( f_1(s) - f_2'(s) \neq 0 \), we can write

\[ \tilde{g}(s) = \frac{\tilde{c}'(s)}{f_1(s) - f_2'(s)}, \]  \hspace{1cm} (C.13)

which gives us the following surface

\[ S(s, v) = \tilde{c}(s) + \frac{v + f_2(s)}{f_1(s) - f_2'(s)}\tilde{c}'(s) \]
\[ = S_{tan}(s, v) + \frac{f_2(s)}{f_1(s) - f_2'(s)}\tilde{c}'(s), \]  \hspace{1cm} (C.14)

where \( S_{tan}(s, v) \equiv \tilde{c}(s) + v\tilde{c}'(s) \) is known as the tangent developable surface. Finally, we conclude that developable surfaces are made of pieces of planes, cylinders, cones, and tangent developable.
BIBLIOGRAPHY


[71] This follows from the existence of some conformal coordinate system such that the metric is expressible as $\Omega(u, v)(du^2 + dv^2)$. Such conformal coordinate systems are guaranteed to exist in a neighborhood of any point for sufficiently well-behaved metrics [79].


[74] Our normal force boundary condition (2.50) differs from that of Efrati et al. [21] but agrees with those of other sources [82–85] in the appropriate limits.


