PRODUCTION OF COSMOLOGICAL
OBSERVABLES DURING THE
INFLATIONARY EPOCH

A Dissertation Presented
by
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“I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.”

–Issac Newton
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ACKNOWLEDGEMENTS

This thesis represents theoretical work I have carried out over the past three years. Ostensibly, it would seem as if I am solely responsible for it (due to my name being the only one on it). However, the knowledge, experience, and insight needed to write it were in fact accrued not in isolation but in aggregate throughout my life involving many different people. Just like it takes a village to raise a child, it takes a village (both academically and personally) to write a PhD thesis.

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are graduate school. (Note: this metaphor implies that graduate students are somehow descending into some type of nightmarish hellscape – this is only partially true.)

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Finally, I would like to thank Einstein whose constant companionship and support was greatly appreciated while I wrote this thesis, even though sometimes he would interrupt my writing to inform me that he needed to go outside. Einstein – by the way – is our dog. He’s a Westie with wiry, white hair much like the eponymous human after which he is named.
This dissertation proposal explores the production of present day cosmological observables which might have been produced during the inflationary era. The first observable is the current net electric charge of our observable universe produced by charge fluctuations during inflation. Next, we examine the possibility of a signal in the primordial gravitational wave power spectrum produced by a scalar field with a time dependent mass. Finally, we examine primordial magnetic fields produced during inflation through the Ratra model coupling with the Schwinger effect.
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CHAPTER 1

INFLATION: AN OVERVIEW

“The Initial Mystery that attends any journey is: how did the traveler reach his starting point in the first place?”

–Louise Bogan

1.1 Prolegomenon

Cosmological inflation is a period of rapid expansion theorized to have occurred during the incipiency of our universe. During this era of accelerated expansion, our universe grew linearly by at least $\sim 10^{29}$ (or in terms of volume $\sim 10^{87}$) over a time scale of only $\sim 10^{-35}$ s. This is the same as imagining the volume of an atom is increased to the same volume encompassing our nearest stellar neighbor Proxima Centauri. Even though this expansion occurred over an infinitesimally small time scale compared to the age of our universe, inflation is capable of explaining several key cosmological features. These include: the near homogeneity and isotropy of the universe on large scales; the lack of curvature or flatness on large scales which additionally implies that our universe is very big
and very old\(^1\); the CMB measurements which include the uniformity of the temperature across the sky, its associated near-scale invariant spectrum of temperature fluctuations, the non-Gaussianity of these fluctuations, and the ascoustic peaks in the CMB; and the inhomogeneities in the matter distribution of the universe with its associated non-Gaussianity. Inflation has the additional capacity of generating a stochastic background of gravitational waves\(^2\) and perhaps most tantalizing of all inflation leads to the idea that our universe might be part of a larger multiverse.

Inflation arose in the 1980s when Alan Guth was trying to come up with a way to get rid of magnetic monopoles which should have been created in the early universe [1]. His initial inflationary theory solved not only this problem, but two additional ones that had been plaguing cosmologists for many years. These had to do with the initial conditions of the universe which in the standard Hot Big Bang model had to simply be put in by hand. These problems can be stated in the form of the following questions: why is the universe so uniform and isotropic on large scales (the horizon problem), why isn’t there significant curvature on large scales (the flatness problem), why don’t we observe topological defects such as magnetic monopoles (the relic density problem), and how did our universe come to exist? We will address the first two issues in the following sections 1.2 and 1.3. In section 1.4, we will give the overall picture of the inflationary paradigm and discuss how it can be thought of as a hierarchy of increasing theoretical sophistication. In sections 1.5 and 1.6, we will show how inflation is realized in terms of microphysics by studying both the background and perturbations of a single scalar field called the inflation which

\(^1\)You might protest and say, “Big and old compared to what?” However, it could very well be that our universe is all there is, and so by default it is the oldest and biggest thing around. But for the sake of comparison, it is old ($\sim 10^{17}$ s) relative to the life of a muon $\tau_{\mu^-} \sim 10^{-6}$ s or a giraffe $\tau_{\text{giraffe}} \sim 10^8$ s and it is big ($\sim 10^{27}$ m) relative to the radius of a proton $r_p \sim 10^{-15}$ m or the radius of the Milky Way $r_{\text{MW}} \sim 10^{20}$ m.

\(^2\)These, as will be discussed in Chapter 3, have not been observed as of yet. However if (or hopefully when) they are eventually measured they would provide both evidence for the quantization of gravity and strong evidence in favor of inflation.
is responsible for the inflationary expansion. Finally in section 1.7, we will detail the current cosmological data that any proposed inflationary model must satisfy.

After the above generalities of the inflationary paradigm have been detailed, the main body of this thesis will focus on generating three different cosmological observables during inflation: a net electric charge for our universe, primordial gravitational waves, and primordial magnetic fields. These three mechanisms do not involve the study of inflation per se, but instead assume a period of inflationary expansion took place in the early universe and for the most part will be independent of how exactly inflation is realized. We provide now (for those cursory readers) a brief description of each of the three mechanisms and their overall result:

– Chapter 2: Net Electric Charge for the Observable Universe – A net electric charge is generated for our observable universe by considering the amplification of charge fluctuations during inflation. Electric charge is found to be conserved globally (in the ‘entire’ universe), but a non-zero charge distribution is created by considering the charge variance in a finite region (such as our observable universe). We find that if a charged massive fermion is around during inflation it can generate a charge density several orders of magnitude less than the current observational bound, \( \rho_0 \lesssim 10^{-33} n_B \). However, we find that a charge density can accumulate in very light charged scalars \( (m \ll H) \) which can exceed the same observational bound thus producing a net electric charge for our observable universe, \( \rho_0 \lesssim 10^{-26} n_B \).

– Chapter 3: Production of Primordial Gravitational Waves – A general feature of inflation is that it produces a stochastic background of gravitational waves, \( \mathcal{P}_T = \frac{2}{\pi} \frac{H^2}{M_p^2} \). In addition, gravitational waves can also be sourced by any inhomogeneous fields that are around during inflation, \( h_{ij} \propto \partial_i \chi \partial_j \chi \). We discuss such a mechanism which would provide an additional feature in the primordial
gravitational wave power spectrum. Specifically, we study PGW production by a scalar field that becomes and stays massless during inflation. We find that our mechanism is capable of producing a tensor-to-scalar ratio of $r \sim 10^{-5}$ and an energy density of $\Omega_{\text{GW}} h^2 \sim 10^{-13}$. Both of which might have the capacity of being detected with future experiments such as the CMB-S4 or LISA.

– Chapter 4: Production of Primordial Magnetic Fields – Cosmological magnetic fields exist throughout the universe in galaxies, galaxy clusters, and potentially in the intergalactic medium (the voids between galaxies). A possible solution to how these fields originated is primordially. The Ratra model is capable of generating a substantial primordial magnetic field during inflation ($B_0 \sim 10^{-10}$), however it additionally produces an electric field which violates energy conservation, $\rho_{\text{EM}} \gg \rho_{\text{inflation}}$. We calculate how the Schwinger effect might lessen this electric field while still maintaining the magnetic field. We find that the Schwinger mechanism does lessen the overall electric field which allows a magnetic field strength of $B_0 \sim 10^{-27}$ which is still many orders of magnitude less than present day large scale fields.

To conclude this introductory section, we introduce some of the mathematical notation and physics nomenclature used throughout this thesis. First, we use for simplicity natural units throughout $c = \hbar = k_B = 1$. This implies that mass, energy, temperature, inverse length, and inverse time have the same dimensions. The Fourier transforms for both position and momentum space are defined as,

$$F(x, t) = \int \frac{d^3p}{(2\pi)^{3/2}} e^{ip\cdot x} F(p, t), \quad F(p, t) = \int \frac{d^3x}{(2\pi)^{3/2}} e^{-ip\cdot x} F(x, t). \quad (1.1)$$
This is turn means that the Dirac delta function takes the following form,

$$\int \frac{d^3p}{(2\pi)^3} e^{ip(x-x')} = \delta^3(x - x'). \tag{1.2}$$

The power spectrum in momentum space for the same quantity is defined as,

$$\langle \mathcal{F}(k)\mathcal{F}(k') \rangle = \frac{2\pi^2}{k^3} \delta^3(k + k') P_F(k). \tag{1.3}$$

Finally, the general relativity metric we will use throughout is the flat ($\kappa = 0$) FLRW metric which is the unique metric for a perfectly homogeneous and isotropic universe,

$$ds^2 = -dt^2 + a^2(t)dx_i dx^i, \tag{1.4}$$

where $t$ is the cosmological time measured by a comoving observer with zero peculiar velocity, $a(t)$ is the scalar factor signifying how distances change between comoving observers, and $x_i$ are comoving coordinates which do not change as the universe expands.

## 1.2 Flatness Problem

The central idea behind the flatness problem relates to the curvature (or lack of curvature) of the universe on large scales. Our universe in general can have three basic configurations for its spatial curvature: positive, negative, or zero curvature. The zero curvature option is an unstable equilibrium meaning that if the universe starts off close to flat it will naturally become less flat over time and thus have an increasingly non-zero curvature as time progresses. We know today however that the universe is flat to 1 part in 1,000 [2] and so the universe must have started out very close to flat in order to account for the present lack of curvature. This brings us to the central question in the flatness problem:
why should the universe start off so arbitrary close to flat such that today we measure an essentially flat universe?

Of course, there is a simple way to account for this observation, the universe could have started off with an arbitrarily small curvature and so not enough time has elapsed to enable the curvature to deviate from its initial value. This argument however implies that the curvature must be very finely tuned – that is close to zero but not exactly zero. While it is on the one hand possible, on the other it is both aesthetically and theoretically unsatisfactory. And so we posit another question: is there a process in our early universe that would favor a spatially flat universe? As it turns out, one of the general features of inflation is that it actually drives the spatial flatness of the universe not away from zero as the conventional Hot Big Bang model does but \emph{towards zero}.

To illustrate the flatness problem in a more mathematically rigorous way, let’s start with the first Friedmann equation which relates the expansion of the universe to what is in the universe,

$$H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{1}{3M_P^2} \left[ \frac{\rho_M}{a^3} + \frac{\rho_r}{a^2} + \ldots \right] - \frac{k}{a^2}. \quad (1.5)$$

$H$ is the Hubble parameter which is a measure of the rate of expansion, $M_P$ is the Planck mass which in natural units is $M_P = (8\pi G)^{-1/2}$, $k$ represents the spatial curvature which can be $+1,-1,0$ for positive, negative, or zero curvature respectively, and $\rho$ represents various energy densities such as for matter or radiation. As you can see, the energy density of matter falls off as the volume of the universe increases ($\propto a^{-3}$). Radiation falls off faster due to the same volume dilution but also by another power of the scale factor due to wavelengths being stretched and thus lowering their energy, $p \sim (a\lambda)^{-1}$. The energy associated with the curvature of the universe however falls off the slowest and so should eventually come to dominate the energy of the universe. We can rewrite the
Friedmann equation in a more suggestive way as,

\[ \sum_i \Omega_i - 1 = \Omega_M + \Omega_\gamma - 1 = \frac{k}{a^2 H^2}, \]  

(1.6)

where \( \Omega \) represents the fraction of each energy sector in terms of the critical energy density today, \( \rho_0 = 3M_P^2 H_0^2 \), and a perfectly flat universe corresponds to \( \Omega = 1 \) implying \( k = 0 \). If we assume that the universe evolved as either matter or radiation dominated up to the present era which is the standard Hot Big Bang paradigm then the scale factor will read,

\[
a(t) = \begin{cases} 
\left( \frac{t}{t_0} \right)^{\frac{1}{2}} & ; \text{radiation dominated} \\
\left( \frac{t}{t_0} \right)^{\frac{2}{3}} & ; \text{matter dominated}
\end{cases},
\]

(1.7)

where \( t_0 \) is the present time, \( t \) is the cosmological time which starts off at \( t_i = 0 \), and the scale factor has been set equal to unity today. The comoving Hubble horizon, \((aH)^{-1}\), for these two cases will read,

\[
(aH)^{-1} = (\dot{a})^{-1} = \begin{cases} 
2H_0^{-1}a(t) & ; \text{radiation dominated} \\
\frac{3}{2}H_0^{-1}\sqrt{a} & ; \text{matter dominated}
\end{cases}.
\]

(1.8)

Thus, the comoving Hubble horizon decreases as we go further back in time and the right hand side of equation 1.6 is smaller in the past than it is today since \( a(t) \) ranges from 0 to 1. This again illustrates the crux of the flatness problem: the universe must have started off very finely tuned in order for the curvature measured today to be so close to zero since the further we go back in time the smaller the comoving Hubble radius becomes. For example, if we go back to the Planck era or BBN (Big Bang Nucleosynthesis) assuming radiation domination we have,

\[
\frac{|1 - \Omega|_{Pl}}{|1 - \Omega|_0} = \frac{(\dot{a}_0)^2}{(\dot{a}_{Pl})^2} = \frac{T_{Pl}^2}{T^2_0} \sim 10^{-64}, \quad \frac{|1 - \Omega|_{BBN}}{|1 - \Omega|_0} = \frac{T_{BBN}^2}{T^2_0} \sim 10^{-20}.
\]

(1.9)
As you can see, the universe had to have started off very close to but not exactly equal to zero in order to account for the present day value of $\Omega_0$. However, if we now postulate a period of inflation we will not only provide a justification for why our universe is flat, but inflation actually drives the universe towards flatness. Thus a general prediction of inflation is that $\Omega_0 = 1$. As you can see from equation 1.6, in order for this to happen we want the comoving Hubble radius to increase, not decrease as it does for both a matter or radiation dominated period. To see how this happens, we again calculate the comoving Hubble radius but now for an accelerating universe, $(aH)^{-1} \sim He^{Ht}$. We can then calculate how long inflation must last in order to achieve a flat universe today,

$$\frac{|1 - \Omega|_{t_f}}{|1 - \Omega|_{t_i}} = \frac{(\dot{a}_f)^2}{(\dot{a}_i)^2} = e^{2H(t_f-t_i)} = e^{-2N}, \quad (1.10)$$

where $t_i$ and $t_f$ are the start and end of inflation and $N$ are the total number of e-folds during inflation, $N = H(t_f - t_i)$. The estimated total number of e-folds during inflation in order to account for the present day flatness will be,

$$e^{-2N} \gtrsim 10^{-53} \implies N \approx 61, \quad (1.11)$$

where $10^{-53}$ is derived as in equation 1.9 but where the temperature is taken to be the current upper bound of the temperature at the end of inflation. As you can see, we need to only have about 61 e-folds of inflation to account for the present day value of the spatial curvature. We could of course have more than 61 e-folding of inflation but 61 is all we need to solve the flatness problem.

Perhaps the above discussion was not very convincing, and you want to claim that the
universe just started off exactly\(^3\) at \(\Omega = 1\). There is another compelling reason to positulate a phase of inflationary expansion in our early universe. It has to do with the spectacular uniformity of the oldest light we can see and it is the subject of the next section.

1.3 Horizon Problem

Another issue arising in the standard Big Bang model concerns the observation that distance parts of the universe which are presently not in causal contact suspiciously have the same temperature. The temperature of the CMB (cosmic microwave background) is incredibly uniform to 1 part in 100,000 \((T_0 = 2.72548 \pm 0.000057 \text{ K} [3])\) implying that at the time of last scattering the universe was very homogeneous and had thermalized. This would be analogous to considering a room which has a variety of gases all at vastly different temperatures. If you allowed these gases to interact it would take a certain amount of time for them to come to an equilibrium temperature. The central problem is that there was simply not enough time in the traditional Hot Big Bang model for the universe to properly thermalize and so a priori there is no reason to assume it should have the same temperature everywhere just as we would not expect the gases to be at a common temperature without them first being able to sufficiently interact.

Figure 1.1 is a rendering illustrating the general idea behind the horizon problem. At the time of last scattering, light rays free streamed from the points \(r_1\) and \(r_2\) to us today. However, their spheres of influence (light gray circles) are only now just coming into causal contact meaning they have had no way to communicate in the past. However, due to the

\(^3\)You are free to do this, and in fact if the universe starts off being perfectly flat (\(\Omega = 1\)) then it will always be flat. However, this would be similar to considering the case of a pencil that is perfectly balanced on its tip. Yes, it is a perfectly valid solution for the equation of motion of the pencil in a classical theory (just as General Relativity is), but it leaves an unpleasant taste in one’s mouth.
CMB light having the same temperature today it would seem as if they were in causal contact at the time of photon decoupling so that “everyone” knew what temperature to be at today.

Just as with the flatness problem, we could make the assumption that the early universe just happened to be very smooth and uniform. But again, we ask a different question: is there a mechanism by which our observable universe could have been in causal contact for a sufficiently long period of time in the early universe? As you might suspect, inflation once again provides a way of accounting for this uniformity by allowing the universe to shrink to a much smaller size than is traditionally assumed in the Big Bang model which would in turn allow it to be in causal contact in the distant past.

To see why this is the case, let’s calculate the maximum distance a photon could have traveled from the beginning of the universe to the surface of last scattering in both a matter and radiation dominated epoch. We start with the general formula for the comoving distance, $\chi$, that a photon has traveled at time, $t$,

$$\chi - \chi_i = \eta - \eta_i = \int_{t_i}^{t} \frac{dt'}{a(t')} ,$$

\hspace{1cm} (1.12)
where $\eta$ is the conformal time or equally the comoving distance and $t_i$ is the initial time. Conformal time is convenient since light rays always move at $45^\circ$ angles on the $\chi - \eta$ coordinate system. We find that in both matter or radiation dominated periods a photon covers the finite distance given by,

$$
\eta(t) = \begin{cases} 
H_0^{-1} a; & \text{radiation dominated} \\
H_0^{-1} a^\frac{1}{2}; & \text{matter dominated}
\end{cases}.
$$

(1.13)

This implies that more of the universe comes into causal contact over time or conversely that less of the universe is in causal contact for earlier times. The comoving Hubble patch at the time of last scattering that was in causal contact will simply be, $\eta_{LS} = H_0^{-1} a_{LS}$. Thus we would expect that the CMB is only homogeneous on comoving scales on the order of $\eta_{LS}$ which since $a_{LS} \ll a_0$ further implies that only a small portion of the sky should be thermalized. We can estimate this patch by considering the ratio of the current Hubble surface to the surface subtended by $\eta_{LS}$,

$$
\frac{S_0}{S_{LS}} = \frac{4\pi (H_0^{-1})^2}{4\pi (H_0^{-1} a_{LS})^2} = a_{LS}^{-2} \approx 10^6.
$$

(1.14)

So in theory there should be $10^6$ causally disconnected regions in the CMB, but we of course know that the CMB is incredibly uniform.

What we need in order to have all of these disconnected regions to be in causal contact is for the scale associated with the current Hubble radius ($\lambda_{H_0}$) to be within the Hubble radius during inflation, thus making all observable length scales today causally connected in the past. We can calculate what the length scale associated with the current Hubble radius would be during inflation using,

$$
\lambda_{H_0} = H_0^{-1} \left( \frac{a_{end}}{a_0} \right) \left( \frac{a_i}{a_{end}} \right) = H_0^{-1} \left( \frac{T_0}{T_{end}} \right) e^{-N},
$$

(1.15)
where we have taken the period from the end of inflation, $a_{\text{end}}$, to the present to be a radiation epoch. The above scale should be smaller than the Hubble radius during inflation, $H_I^{-1}$, which leads us to the relation,

$$H_0^{-1} \left( \frac{T_0}{T_{\text{end}}} \right) e^{-N} < H_I^{-1} \quad \Rightarrow \quad N > \ln \left( \frac{T_0}{H_0} \right) - \ln \left( \frac{T_I}{H_I} \right) \approx 70 - \frac{1}{2} \ln \left( \frac{M_P}{H_I} \right).$$

We find that we need around 70 e-folds of inflation though the exact number will be less depending on the energy scale of inflation. Again, as we pointed out in the last section inflation could have lasted $N \gg 70$, but we need at least this number in order to solve the horizon problem.

An equally valid way of thinking about how inflation solves the horizon problem is in terms of the comoving Hubble radius during inflation, $\lambda_{\text{com}} = (aH)^{-1}$. As we saw in the last section, the comoving Hubble radius increases (equation 1.8) during both a matter and radiation dominated era, $\lambda_{\text{com}} \propto a^n(t)$ where $n > 0$. However, during inflation the comoving Hubble radius decreases at an exponential rate, $\lambda_{\text{com}} = (aH)^{-1} = H^{-1}e^{-Ht}$.

This allows modes that are coming into the horizon today to have been in causal contact early in our universe.

Figure 1.2 shows how the comoving Hubble radius changes in both cases with it decreasing during inflation and subsequently increasing afterwards. For a particular comoving scale $\bar{\lambda}$, it is initially well within the horizon, $\bar{\lambda} \ll \lambda_{\text{com}}(t < t_1)$. Then, it exits the horizon at a particular time $t_1$ during inflation, $\bar{\lambda} \simeq \lambda_{\text{com}}(t_1)$. $\bar{\lambda}$ then goes superhorizon, $\bar{\lambda} \gg \lambda_{\text{com}}(t_1 < t < t_2)$. Before finally coming back into the horizon at time $t_2$, $\bar{\lambda} < \lambda_{\text{com}}(t > t_2)$.

This picture reinforces the already mentioned fact that modes coming into the horizon today could not have been within the horizon in the past if the universe only under went periods of either matter or radiation domination.
1.4 The Inflationary Paradigm

We have shown that both the horizon and flatness problems can be explained by postulating a period of accelerated expansion in our early universe. This is in the broadest sense the key feature of the inflationary paradigm. During a sufficiently long enough period in the nascent universe, space itself rapidly expanded resulting in a nearly flat and homogeneous universe. The next logical question is: how exactly does such a state come about? In light of the universe’s current accelerating expansion, it might seem reasonable to assume that a similar mechanism such as a cosmological constant is responsible for both periods of acceleration. This, however, is not likely to be the case. If the universe’s current acceleration is due to a cosmological constant, then the acceleration will in fact increase over time as matter (which gravitationally opposes expansion) dilutes away and dark energy (which gravitationally enhances expansion) stays constant. Thus, there is no end to our current period of acceleration since more and more dark energy fills the universe as it expands. Inflation, however, must stop at a certain point (the ‘graceful exit’).
so that the traditional Big Bang model can take over. And so the cosmological constant which represents a constant energy density (and thus unchanging) does not seem plausible. Additionally, the energy scale associated with both expansions is vastly different and so it would be natural to assume different mechanisms produce the two effects.

The theory of cosmological inflation itself has evolved since its infancy in the 1980s from old inflation [5], new inflation [6], chaotic inflation [7], and today into a whole zoo of inflationary models [8]. The overall premise of inflation can be divided into roughly three levels of increasing complexity along with increasing assumptions. It is illustrative to think of it as an inflationary pyramid where the bottom level represents the minimal assumptions needed in order to achieve inflation while the top represents the fully fleshed out theory. The first level of this pyramid (and the most important both theoretically and structurally) is that there was a period of accelerated expansion that took place in our early universe capable of resolving the problems previously mentioned with the Big Bang model. There is no assumptions about what exactly drove this expansion other than it occurred for a long enough time period to satisfy the flatness and horizon problems. The second level of this inflationary pyramid makes certain assumptions about what drove inflation. In particular, the canonical view (though certainly not the only one) is that the energy density of the early universe was dominated by a single scalar field called the inflaton (obviously, what else would it be called!) whose nearly constant potential energy drove the expansion. The formalism associated with the evolution of the inflaton under this regime is called slow roll due to the inflaton slowly rolling down its nearly flat

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4 The universe during inflation doubles in size about every $10^{-35}$ s, while the doubling time today is roughly 14 billion years.

5 There is the possibility, called quintessence [4], that the universe’s current expansion is driven by a similar mechanism (that is a scalar field slowly rolling down its potential) as in the standard inflationary paradigm.

6 As we will soon discuss there is no exact model of inflation, and so it is usually better to think of inflation as more of a paradigm (just like the title of this section does), than as an exact theory with a specific realization. That being said, I will occasionally take a page out of the evolution deniers’ playbook and abuse the term theory by putting it next to the term inflation.
potential. There is no need to specify a particular potential other than it should be flat for a sufficiently long enough period of time. The top of this “theoretical” pyramid (and thus the last part to be assembled) is the specific microphysical incarnation of inflation. This would be the fully detailed mechanism for what exactly drives inflation. If inflation is caused by the inflaton for example, the theory should: explain how the inflaton originates for example through some phase transition or by a mechanism in string theory; what the exact potential is for the inflaton; it should provide a way for inflation to end (the graceful exit problem); and it must be in line with all of the observational evidence we know about the universe in particular measurements of the CMB and matter distribution.

There has been much work done in constructing the top of this pyramid with a whole myriad of inflationary models posited ranging from: single field to multiple field models, small field to large field models, models with non-canonical kinetic terms, models stemming from string theory, models stemming from SUSY, and in actuality “any” model appropriately fine tuned can produce the cosmological observables we know of today\(^7\). Despite this inflationary hydra that has grown since the 1980s, observations seems to point to ‘simple’ scalar models involving one scalar field. For example, a scalar field rolling down a “plateau” like potential such as exponential SUSY inflation (ESI) or Starobinsky inflation (SI) whose potential takes the form,

\[
V(\phi) = M^4 \left[1 - e^{-\frac{\phi}{M_P}}\right]^n,
\]

where \(M\) is a mass parameter, and \((\gamma, n)\) are \((\sqrt{2}, 1)\) and \((\sqrt{2/3}, 2)\) corresponding to the ESI and SI models respectively [10]. Since the three mechanisms we will discuss do not rely on us knowing how exactly inflation took place, we will illustrate the basic idea of how

\(^7\)This last point has lead some in the cosmological community to question the legitimacy of inflation as a scientific theory due to its seemingly inexhaustibility when confronted with observations [9]. The overall claim is that since inflation is capable of predicting every possible incarnation of our universe (due to fine tuning the parameters of inflation such so), then it has no real power as a predictive theory.
inflation works by detailing the second level of our pyramid. That is, we will focus on a simple version of inflation using a single-field scalar model undergoing slow-roll inflation.

### 1.5 Background Evolution of $\phi$

We begin, since inflation is a theory of how the universe evolves, with Einstein’s field equations,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G N T_{\mu\nu},$$

(1.18)

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the metric, $R$ is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor. The above equation relates what is in the universe, $T_{\mu\nu}$, to the geometry of the universe, $g_{\mu\nu}$. In order to be in line with isotropy and homogeneity, both the geometry and matter/energy content of the universe should reflect this. For the geometry, the Friedmann-Lemaître-Robertson-Walker (FLRW) metric is the unique metric which achieves these conditions and is given by

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right],$$

(1.19)

where $a(t)$ is the scale factor and $\kappa = -1, 0, +1$ corresponding respectively to a negatively, flat, and positively curved universe. As we previously discussed, the universe today as far as we can tell is flat\(^8\) and so we take $\kappa = 0$. This fixes the background geometry on to which matter and energy can evolve.

Next, we define the matter content of the universe as a perfect fluid with energy density, $\rho(t)$, and pressure density, $p(t)$, which results in an energy-momentum tensor reading,

$$T_{\mu\nu} = \text{diag} [\rho(t), p(t), p(t), p(t)].$$

(1.20)

\(^8\)In fact, this will be even more pronounced as we go further back in time due to the decreasing comoving Hubble radius as we discussed in section 1.2.
This definition for the energy and matter of the universe is again in line with homogeneity and isotropy. It has no spatial variations, \( \partial_i \rho = \partial_i p = 0 \), but it can vary with time.

Plugging the above relations into Einstein’s equations, the Friedmann equations which govern the evolution of a homogeneous and isotropic universe can be derived,

\[
H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{\kappa}{a^2}, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p), \tag{1.21}
\]

where \( H \) is the Hubble parameter. We can further define the equation of state parameter for our perfect fluid which relates the pressure density to the energy density. The equation reads, \( p = w \rho \), where matter (usually called dust) is a pressureless fluid, \( p = 0 \), while radiation’s equation of state reads, \( p = \frac{1}{3} \rho \). Both cases imply a negative acceleration, \( \ddot{a} < 0 \), and so are not capable of achieving a period of acceleration. Inflation however is a period of \textit{accelerated} expansion and so we need to have a substance with an equation of state parameter, \( w < -\frac{1}{3} \), in order to achieve \( \ddot{a} > 0 \).

As previously state, this mysterious substance with a \textit{negative pressure} will be a scalar field called the inflaton. To see how we can achieve the correct equation of state, we start by deriving both the energy and pressure density of the inflaton starting with the action,

\[
S_{\text{inflaton}} = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right], \tag{1.22}
\]

where \( \sqrt{-g} \) is the determinant of the metric and \( V(\phi) \) is the potential for the inflaton which we leave unspecified. The equation of motion for the inflation can be found from the Euler-Lagrange equation,

\[
\partial_\mu \left( \frac{\delta(\sqrt{-g} \mathcal{L})}{\delta \partial_\mu \phi} \right) - \frac{\delta(\sqrt{-g} \mathcal{L})}{\delta \phi} = 0, \tag{1.23}
\]
with a resulting equation of motion for the inflaton reading,

\[ \ddot{\phi} + 3H\dot{\phi} - \frac{\Delta\phi}{a^2} + \frac{dV(\phi)}{d\phi} = 0. \]  

(1.24)

We can split \( \phi \) into a classical solution \( \phi_0(t) \) and fluctuations about the classical solution \( \delta\phi(x,t) \) through the decomposition,

\[ \phi(x,t) = \phi_0(t) + \delta\phi(x,t). \]  

(1.25)

For the remainder of this section, we will consider only the homogeneous case \( (\phi_0) \) and for notational simplicity drop the subscript ‘0’. In order to incorporate the field \( \phi \) into the Friedmann equations, we also calculate its energy and pressure density starting with its energy-momentum tensor,

\[ T_{\mu\nu}^{\text{scalar}} \equiv -\frac{2}{\sqrt{g}} \frac{\delta S_{\text{inflaton}}}{\delta g_{\mu\nu}} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left[ \frac{1}{2} \partial_\sigma \phi \partial^\sigma \phi - V(\phi) \right]. \]  

(1.26)

Using the expression for the energy and pressure densities equation (1.20), we can derive what the inflaton’s energy and pressure densities are,

\[ \rho_\phi \equiv -T_0^0 = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p_\phi \equiv \frac{1}{3} T^i_i = \frac{1}{2} \dot{\phi}^2 - V(\phi). \]  

(1.27)

The above system does not necessarily afford a period of accelerated expansion. For example, if we have a homogeneous field with a dominant kinetic term, we would have \( \rho_\phi \approx p_\phi \), thus not achieving an accelerated expansion, \( \rho_\phi \not\approx -3p_\phi \). However, if we work in the so-called slow roll regime, meaning that the scalar field slowly rolls down its potential, then we can have a period of accelerated expansion. The slow roll regime is
usually quantified in terms of the slow roll parameters,

\[
\epsilon = \frac{M_P^2}{2} \left( \frac{V'}{V} \right)^2, \quad \eta = M_P^2 \frac{V''}{V},
\]

(1.28)

where both parameters are small for most of the duration of inflation, \( \epsilon \ll 1 \) and \( |\eta| \ll 1 \).

The first slow roll parameter, \( \epsilon \), should be small during inflation to ensure that the field \( \phi \) slowly rolls down its potential. If this is the case, then we can neglect the velocity of \( \phi \) in both the energy and pressure densities as well as neglect the acceleration of \( \phi \) in its equation of motion. The second slow roll parameter, \( \eta \), should additionally be small to ensure that the potential is not only flat, but flat for a sufficient period of time.

Another important quantity which we have already mentioned is the number of e-folds during inflation defined as,

\[
N = \int H(t) dt = \int \frac{H}{\dot{\phi}} d\phi \simeq -3 \int \frac{H^2}{V'(\phi)} d\phi \simeq -\frac{1}{M_P^2} \int \frac{V(\phi)}{V'(\phi)} d\phi,
\]

(1.29)

where we have assumed \( \phi \) is evolving under slow roll to make certain approximations.

From sections 1.2 and 1.3, we saw that inflation should last around \( N \approx 65 \) in order to solve both the horizon and flatness problems. Of course the actual number of e-folds that inflation lasted could be much larger, but it at least has to last this long. The number of e-folds between when inflation ends and those modes that left the horizon during inflation corresponding to CMB scales is \( N_{CMB} \approx 40 - 60 \) which depends on the energy scale of inflation. The lower the temperature at which reheating occurs, the fewer e-folds you need, and conversely the higher the temperature the more e-folds you need.

Figure 1.3 show an example of the inflaton slowly rolling down a potential based on the SI potential mentioned in equation 1.17. During inflation \( \epsilon < 1 \) and inflation ends when \( \epsilon = 1 \) or \( \ddot{a} < 0 \). After inflation, the acceleration term in equation 1.24 can no longer be neglected and the inflaton oscillates about the minimum of its potential in a process
referred to as reheating. During this period of reheating, the inflaton through couplings to the standard model decays into particles releasing its inflationary energy which ‘reheats’ the universe.

Returning to our expressions for the energy and pressure densities we find that due to the flatness of $V(\phi)$, the field $\phi$ will very slowly roll down its potential. Because it is slowly rolling we now assume $\dot{\phi}$ can be neglected with respect to the potential $V(\phi)$, and so the energy and pressure densities will read,

$$\rho_{\phi} \simeq V(\phi) \quad p_{\phi} \simeq -V(\phi) \quad \Rightarrow \quad \rho_{\phi} \simeq -p_{\phi},$$  \hspace{1cm} (1.30)

thus achieving the necessary condition ($p < -\frac{1}{3}\rho$) to ensure an accelerated expansion, $\ddot{a} > 0$. We can now return to the first Friedmann equation and calculate the behavior of the scale factor,

$$\left(\frac{\ddot{a}}{a}\right)^2 \simeq \frac{V(\phi)}{3M_P^2} = H_I^2 \quad \Rightarrow \quad a(t) = a(t_i) e^{H_I(t-t_i)},$$  \hspace{1cm} (1.31)

where $a(t_i)$ is the scale factor at some initial time, $t_i$, and $H_I$ is the Hubble parameter.
during inflation. Unlike matter or radiation which dilutes as the universe expands, the energy of the inflaton grows exponentially as it encompasses more and more volume.

A pertinent question to ask is: how fast is this expansion? The physical distance a photon travels is given by the comoving distance multiplied by the scale factor. The comoving distance (equation 1.12) will be,

\[
\chi(t) = \int_{t_i}^{t} \frac{dt'}{a(t')} = \frac{H^{-1}}{a_i} \left[ 1 - e^{-H(t-t_i)} \right],
\]

(1.32)

where we find that the comoving distance quickly approaches the Hubble distance for \( t - t_i \gg H^{-1} \). For the physical distance we multiply the above by the scale factor,

\[
d_p = a(t)\chi(t) = H^{-1} \left[ e^{H(t-t_i)} - 1 \right],
\]

(1.33)

where now we find that the physical distance grows exponentially, \( d_p \gg H^{-1} \). In particular, after only one Hubble time we can calculate the velocity of the photon which will be,

\[
\frac{\Delta x}{\Delta t} = \frac{d_p(H^{-1})}{H^{-1}} = e - 1 \approx 1.78 > 1.
\]

(1.34)

This would imply a superluminal\(^9\) velocity! There is however no contradiction with Einstein. Special Relativity dictates that nothing can travel faster than the speed of light through space. However, it is space itself which is undergoing this exponential expansion, while particles stay essentially motionless with respect to their comoving coordinate. Thus, information can still not travel faster than the speed of light even if particles are moving superluminally with respect to each other in terms of physical distance.

The above considerations for how inflation is achieved will be enough in order to discuss

\(^9\)Recall we are using natural units where \( c = 1 \) so a velocity greater than 1 is faster than the speed of light.
the three mechanisms for the main body of this thesis. The three mechanisms are not testing specific inflationary models nor are they studying inflation itself. They simply assume that a sufficiently long \((N \gtrsim 65)\) period of accelerated expansion took place and that this period is an exact de Sitter expansion meaning that the Hubble parameter stays exactly constant\(^{10}\). While an exact de Sitter expansion is not realistic since we want inflation to end, it will suffice for the calculations we wish to perform. We next consider quantum fluctuations in the inflaton field which will be a nice primer for the calculations in the main body of this thesis.

### 1.6 Fluctuations in \(\phi\)

In the previous section, we used the homogenous solution for \(\phi\) undergoing slow roll inflation to show how a period of accelerated expansion could be achieved and demonstrated how both the horizon and flatness problems are ameliorated when such an expansion occurs. It turns out that inflation is capable of producing a much richer set of cosmological observables if we examine how quantum fluctuations evolve during an inflationary expansion. In fact, even though the background evolution of \(\phi\) is capable of explaining the uniformity for the temperature of last scattering, it does not provide a mechanism which explains the fluctuations for this same temperature. They are however capable of being explained if we examine how quantum fluctuations evolve during the inflationary epoch. At first glance, this would seem to be a nonsensical way of trying to explain temperature fluctuations in the CMB. How could quantum fluctuations produced during inflation have anything to do with subtle changes in temperature of the CMB? Both events occur at starkly different time periods for the universe and at vastly different energy scales.

\(^{10}\)With the exception of Chapter 4 where we do take into account both a slowly varying potential for the inflaton as well as fluctuations of the inflaton field both of which in a full inflationary theory must be included anyway.
The first link supporting this seemingly unconnected chain, starts with the fluctuations themselves which we take to be fluctuations of the inflation, \( \delta \phi \). As we know from quantum mechanics and in particular from Heisenberg’s uncertainty principle, no field is truly motionless but always has fluctuations. These fluctuations characterized by a particular wavelength, \( \lambda \), which are initially well within the horizon, \( \lambda \ll H_i^{-1} \), are stretched due to the exponential expansion to cosmological scales on the order of the horizon itself \( \lambda \approx H^{-1} \). Once these fluctuations leave the horizon, they essentially stop evolving and their amplitude is “frozen in” , \( |\delta \dot{\phi}|_{\lambda \gg H^{-1}}^2 \approx 0 \). From General Relativity, we know that perturbations in the stress-energy tensor, \( \delta T_{\mu \nu} \), produce perturbations in the metric, \( \delta g_{\mu \nu} \). We can relate the fluctuations of the inflaton, \( \delta \phi \), which are perturbations of a scalar field to scalar perturbations in the metric through the comoving curvature perturbation \( \delta R \), which is a measure of the spatial curvature on comoving hypersurfaces. Just like fluctuations of \( \phi \), \( R \) is constant for those modes outside of the horizon and so is not influenced by the unknown microphysics of for example reheating, \( R_{\lambda \gg H^{-1}} \approx 0 \).

For the next link in our theoretical chain, the scalar perturbations of the metric can again be related to changes in the energy density \( \delta \rho \) once modes start to reenter the horizon, \( R \Rightarrow \delta \rho \). These areas of slight over and under density can be measured today by mapping galaxy distributions in the universe [11]. Finally, changes in the energy density corresponding to areas of slight over or under densities can be related to fluctuations in temperature, \( \delta \rho \Rightarrow \delta T \). These fluctuations in the temperature correspond to subtle changes in the CMB that we know of today. Taken in aggregate, the above amounts to relating fluctuations that occur during inflation to fluctuations in the CMB, \( \delta \phi \Rightarrow \delta T \).

To see how this is carried out in practice, we first start off by calculating the fluctuations in the \( \phi \) field. The main quantity we will be interested in for determining these fluctuations

\[ \text{We could equally use } \zeta \text{ which is the curvature perturbation on uniform-density hypersurfaces. This corresponds to a particular time slicing where is no perturbation in the energy density, unlike } R \text{ which corresponds to a particular time slicing of constant } \phi. \]
here and for most of the thesis will be various power spectra. The power spectrum describes how the amplitude for a particular quantity changes as a function of wavelength. To start, we take the equation of motion for $\phi$ (equation 1.24) and perform a Fourier transform into momentum space,

$$\delta\phi(x, t) = \int \frac{d^3p}{(2\pi)^{3/2}} e^{ip\cdot x} \delta\phi(p, t),$$

which enables us to solve each of the modes separately since they decouple\textsuperscript{12}. Using the above, we can write the fluctuations of $\phi$ in momentum space as,

$$\langle \delta\phi(x, t) \delta\phi(x, t) \rangle = \int \frac{d^3pd^3p'}{(2\pi)^3} e^{ix\cdot(p-p')} \langle \delta\phi(p, t) \delta\phi(p', t) \rangle,$$

where we have assumed $\phi$ is a real field, $\delta\phi^* (p) = \delta\phi(-p)$. The power spectrum is defined as,

$$\langle \delta\phi(k) \delta\phi(k') \rangle = \frac{2\pi^2}{k^3} \delta^{(3)}(k + k') P_\phi(k),$$

which when inserted into equation 1.36 yields,

$$\langle \delta\phi(x, t) \delta\phi(x, t) \rangle = \int \frac{d^3pd^3p'}{(2\pi)^3} e^{ix\cdot(p-p')} \frac{2\pi^2}{p^3} \delta^{(3)}(p + p') P_\phi(p) = \int \frac{dp}{p} P_\phi(k),$$

thus as advertised the power spectrum for $\phi$ directly relates to the fluctuations in $\phi$, $\langle \delta\phi^2 \rangle$. In order to explicitly calculate the ensemble average of the field $\phi$ in momentum space, we must first quantize the field and then solve for the mode functions themselves. Quantization proceeds as is usually done in QFT by decomposing the field in momentum space along a set of creation and annihilation operators, $a(p)/a^\dagger(p)$, which when applied\textsuperscript{12} in QFT, each point in space is represented by a harmonic oscillator. In position space, all of these oscillators are linked together due to the spatial derivatives causing adjacent points to affect each other. The reason we perform this Fourier transform into momentum space is that we can study the evolution of each particular mode independently from the rest. Another way of phrasing this is that all the modes “decouple” from each other. This should not be confused with a coupling between two fields such as $\chi^2\phi^2$.\textsuperscript{24}
to the vacuum state, \( |0 \rangle \), either create/annihilation a state with a particular momentum, \( |p \rangle \), from its associated mode function being \( \phi_p(t) \). The decompose itself is given by,

\[
\delta \phi(p, t) = \phi_p(t) a(p) + \phi^*_p(t) a^\dagger(-p),
\]

(1.39)

where the mode functions are found by solving \( \phi \)'s equation of motion (equation 1.24) whose inhomogeneous solution will read,

\[
\delta \ddot{\phi}(x, t) + 3H \delta \dot{\phi}(x, t) - \frac{\Delta(\delta \phi(x, t))}{a^2} + V''(\phi_0(t)) \delta \phi(x, t) = 0,
\]

(1.40)

where the third term is now included signifying that \( \phi \) can in general have spatial fluctuations and we have expanded the potential assuming the fluctuations are small, \( |\phi_0(t)| \gg |\delta \phi(x, t)| \). Transforming the above to momentum space will yield,

\[
\delta \ddot{\phi}(p, t) + 3H \delta \dot{\phi}(p, t) - \frac{p^2 \delta \phi(p, t)}{a^2} + V''(\phi_0(t)) \delta \phi(p, t) = 0.
\]

(1.41)

In order to make our task of calculating the power spectrum easier, we perform two transformations: one for our time coordinate, \( t \), and another by redefining our field \( \phi \) to bring it into canonical form in order to properly quantize its modes. For the first, we define the time coordinate, \( \tau \), called the conformal time which is related to the cosmological time, \( t \), through

\[
dt = a \, d\tau.
\]

(1.42)

The meaning of the phrase conformal time can be seen by inserting the above into the FLRW metric which will transform as,

\[
d\tau^2 = a^2(\tau) \left[ -d\tau^2 + dx_i dx^i \right].
\]

(1.43)

The metric now is the usual Minkowski metric, \( \eta_{\mu\nu} = \text{diag}[-1,1,1,1] \), multiplied by an
overall conformal factor, \( a^2(\tau) \), hence the name conformal time. This temporal redefinition enables us to study the dynamics of \( \phi \) as if it were in a Minkowski background, but as we will see with a time dependent mass term. The scale factor in conformal time can be derived from equation 1.42 and assuming an exponential expansion in cosmological time, \( a(t) \propto e^{Ht} \), will then read,

\[
\int d\tau = \int \frac{dt}{a(t)} \quad \Rightarrow \quad a(\tau) = -\frac{1}{H\tau},
\]

where in general \( \tau \in (-\infty, -H^{-1}] \) with \(-H^{-1}\) corresponding to the end of inflation. As an aside, inflation solves the horizon problem by extending \( \tau \) back to arbitrarily far times. For both matter and radiation dominated universes, the amount of conformal time that elapses since the beginning of the universe is finite as can be seen from equation 1.13. Inflation thus allows the conformal time to extend to arbitrary earlier times before \( \tau = 0 \) enabling a greater volume of the universe to be in causal contact at early times.

Our second transform is of \( \delta \phi \) itself which will be,

\[
\psi(p, \tau) = a(\tau)\delta\phi(p, \tau).
\]

Performing these two transformations brings the equation of motion for \( \phi \) into the canonical form,

\[
\psi''_p(\tau) + \left[ p^2 - \frac{a''}{a} + a^2(\tau)V''(\phi_0(t)) \right] \psi_p(\tau) = 0,
\]

where a prime now denotes derivatives with respect to \( \tau \).

There are a couple of observations to make concerning the above equation. First, is that the field \( \psi \) is sometimes referred to as the comoving field since in comoving coordinates a particular comoving scale (here the momentum) is unchanged. The “physical” field \( \phi \) in cosmological time does have a decreasing momentum as evidenced by the scale factor.
in the third term of equation 1.41. The second observation (if we take $V(\phi) = 0$) is that the above equation is the same as a free field in Minkowski space, but now with a time dependent frequency $\omega(\tau)$. The expansion now plays the part of a negative mass term $-\frac{a''}{a}$ which accounts for the amplification of modes during inflation\textsuperscript{13}. Indeed, if we substitute $\phi = \frac{\psi}{a}$ into equation 1.22 and perform an integration by parts, we find,

$$S = \int d^4x \left[ \frac{1}{2} \psi' \psi' - \frac{1}{2} (\partial_i \psi)(\partial_i \psi) + \frac{a''}{2a} \psi^2 \right],$$

(1.47)

which is the action for a scalar field in Minkowski with mass $-\frac{a''}{a}$. Finally, because of the evolving frequency there is now an issue with properly defining what we mean by the vacuum state since the mode functions themselves are time dependent. A state created at a particular time for example $|p_1\rangle$ might not be the same state associated with the creation operator $a^\dagger(p)$ at a later time since the mode functions themselves are changing. This in turn raises the question of how to properly define the vacuum $|0\rangle$ and how to properly define the number density of particles $n_\phi$ and also correlators $\langle \phi(k)\phi(k') \rangle$ since it would seem there are now a set of different vacuum states associated with the evolving mode functions. The formalism associated with properly addressing this issue is to use the Bogolyubov formalism which is discussed in Appendix A. It amounts to defining an adiabatic vacuum state which is valid when the mode functions are not evolving very rapidly or adiabatically. This adiabatic vacuum has its own associated adiabatic operators and so it is possible for example to define an initial state with no particles and a final state where particle production has occurred assuming these states correspond to adiabatic states where their associated mode functions are evolving slowly over time. We do not discuss this formalism further in the introduction since it is detailed in Appendix A and there are examples of it extensively throughout the main body of this thesis, but

\textsuperscript{13}Recall that the solutions for a free field with a time independent frequency are plane waves, $e^{\pm i\omega t}$. If there is a negative mass term then the frequency can in general be $\pm |\omega|$ which can produce both a growing and decay solution, $e^{\pm |\omega| t}$. 

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only bring it up here to highlight the proper way of addressing particle production in a
time dependent background.

We now return to equation 1.37 in order to calculate the power spectrum. Using our de-
composition of the field $\phi$ in momentum space (equation 1.39), we find that the correlator
for $\phi$ reads,

$$
\langle \phi(k, \tau) \phi(k', \tau) \rangle = \phi_k(\tau) \phi_{-k}(\tau) \delta^3(\mathbf{k} + \mathbf{k}') = \frac{\psi_k(\tau) \psi_{-k}(\tau)}{a^2(\tau)} \delta^3(\mathbf{k} + \mathbf{k}'),
$$

(1.48)

which when compared to equation 1.37 yields the power spectrum for the fluctuations of
$\phi$,

$$
P_{\phi}(k) = \frac{k^3}{2\pi^2 a^2(\tau)} |\psi_k|^2.
$$

(1.49)

The next task is to solve for the mode functions themselves using equation 1.46. The
exact solution depends on the particular potential, $V(\phi)$, that governs the evolution of
$\phi$. The exact solution will then in general have two undetermined constants since it is
a second order differential equation. One of these constants can be determined through
the canonical quantization of the field $\psi$ (which is the field we quantize since it is in the
proper canonical form) with its conjugate momentum $\Pi = \dot{\psi}$,

$$
[\psi(x), \psi(y)] = i\delta^3(x - y).
$$

(1.50)

Using the above, we find that the mode functions of $\psi$ must obey the normalization
condition,

$$
\psi_p \psi_{p'}^* - \psi_p^* \psi_{p'} = i,
$$

(1.51)

which will fix one of the integration constants. The other constant is determined by
specifying the initial mode function for $\psi_p$. As previously mentioned, it is a non-trivial
task defining what we mean by a vacuum state in the presence of a time dependent
background. However, there is a unique initial state called the Bunch-Davies vacuum which is the high energy limit ($k \to \infty$) to the mode functions or alternatively the state that you would obtain in Minkowski space for a free field, $\psi_{p}^{Min}$. This initial state, $\psi_{p}^{Min}$, makes sense in light of the fact that modes are initially well within the horizon ($k \gg H$) and so the modes do not “feel” the curvature and locally it “appears” to be a Minkowski spacetime. We also impose that this initial state does not contain any particles and so is in fact a vacuum state. To accomplish this, we identify the initial Minkowski state with its positive frequency solution ($+\omega$) and if upon subsequent evolution it picks up a non-zero negative frequency contribution this is interpreted as particle production of the associated field. The exact solution for the mode function, $\psi_{p}(\tau)$, is determined by matching it in the distant past ($\tau \to -\infty$) with this initial Minkowski state, $\psi_{p}^{Min}$. The mode functions corresponding to a free field in Minkowski space are simply plane waves,

$$\psi_{p}^{Min}(\tau) \approx c_{k}e^{-ik\tau},$$

where the constant $c_{k}$ is found through the normalization condition, $c_{k} = (2k)^{-1/2}$.

Now that we know how to properly define the initial conditions for the mode functions, let’s examine two of the simplest cases: a massless field, $V(\phi) = 0$, and a massive field, $V(\phi) = \frac{1}{2}m^{2}\phi^{2}$. The potential for the massive field is referred to as chaotic inflation and was first proposed by Linde in 1983 [7] where the term ‘chaotic’ means that $\phi$ does not have to start off close to zero, but can start off with any value assuming the initial energy is less than the Planck energy density, $-\frac{M_{P}^{2}}{m^{2}} < \phi_{i} < \frac{M_{P}^{2}}{m^{2}}$. Even though this model seems to be ruled out by the recent Planck data [2] it is still a useful model for illustrative purposes due to its simplicity.
The mode functions for the massive field obey the equation of motion,
\[ \psi''_p(\tau) + \left[p^2 - \frac{a''}{a} + m^2 a^2\right] \psi_p(\tau) = \psi''_p(\tau) + \left[p^2 - \frac{1}{\tau^2} \left(\nu^2 - \frac{1}{4}\right)\right] \psi_p(\tau) = 0, \quad (1.53) \]
where we have defined \( \nu = \sqrt{\frac{a''}{a} - \frac{m^2}{\tau^2}} \). The general solution to the above equation reads,
\[ \psi_p(\tau) = A_p \sqrt{-\tau H_\nu^{(1)}(-p\tau)} + B_p \sqrt{-\tau H_\nu^{(2)}(-p\tau)}, \quad (1.54) \]
where \( H_\nu^{(1)} \) and \( H_\nu^{(2)} \) are Hankel functions of the first and second kind. \( A_p \) and \( B_p \) are constants to be determined by matching the above solution to equation 1.52 and we find that \( A_p = \sqrt{\frac{\pi}{4}} e^{\frac{i\pi}{4} \left(\nu + \frac{1}{2}\right)} \) and \( B_p = 0 \) which leads to,
\[ \psi_p(\tau) = \sqrt{-\frac{\tau \pi}{4}} e^{\frac{i\pi}{4} \left(\nu + \frac{1}{2}\right)} H_\nu^{(1)}(-p\tau). \quad (1.55) \]

Now that we have the solution for mode functions, we can finally calculate the power spectrum,
\[ \mathcal{P}_\phi(p, \tau; \nu) = \frac{p^3}{2\pi^2 a^2(\tau)} |\psi_p'|^2 = \left(\frac{H}{2\pi}\right)^2 \frac{\pi |p^3 \tau^3|}{2} H_\nu^{(1)}(-p\tau) H_\nu^{(2)}(-p\tau), \quad (1.56) \]
where we have used \( (H_\nu^{(1)}(z))^* = H_{\nu'}^{(2)}(z^*) \).

Before discussing features for the above power spectrum, we first discuss which modes are actually excited during inflation. In general the momentum, \( p \), can take on any value ranging from 0 to \( \infty \). Usually, however, there are either UV and/or IR cutoffs in order to obtain finite results which are pertinent to the scales of interest. For example, a UV cutoff might be imposed, \( p_{UV} \), corresponding to some high energy scale such as the Planck scale, \( p_{UV} = \Lambda_P \), or an IR cutoff might be imposed corresponding to a particle’s mass, \( p_{IR} = m \). In the same way, we need to determine what are the appropriate limits on the momenta of those fluctuations generated during inflation.
For Minkowski space, we can determine the vacuum fluctuations and the typical amplitude for these fluctuations from the same expression we already derived (equation 1.49), but taking \(a \to 1\) and the mode functions as a free field in Minkowski space, \(\psi_p = e^{-ip\tau}/\sqrt{2p}\), which results in a power spectrum of

\[
P_{Min}(p) = \left(\frac{p}{2\pi}\right)^2,
\]

and so the fluctuations, \(\delta_{Min}\), are simply proportional to the momentum, \(\delta_{Min} \sim \sqrt{P_{Min}} \sim p\). The typical length scale, \(\lambda\), on which these fluctuations are generated can be estimated from the uncertainty principle, \(\Delta x \Delta p \sim 1\), and will simply be the inverse of the momentum \(\lambda \sim p^{-1}\). These fluctuations can in theory be quite large\(^{14}\) for example on the Planck scale they correspond to vanishingly small scales, \(L_P \sim 10^{-35}\) m. Thus, it seems quite reasonable to assume that fluctuations on such small length scales play no part in cosmology\(^{15}\) where scales of interest are on the order of \(\text{Mpc} \sim 10^{22}\) m.

The amazing and rather radical prediction of inflation is that quantum fluctuations are not only important on cosmological scales, but they are directly responsible for the large scale structure that we see in the universe and are part of today! This is due to length scales being exponential stretched during inflation, and so scales initially corresponding to quantum scales can very quickly grow to cosmological scales. Returning again to the expression for the typical length scale of a quantum fluctuation, \(\lambda \sim p^{-1}\), for inflation we must now account for the expansion which will stretch length scales by the scale factor, \(a\). For a particular momentum \(p^*\) the length scale associated with this momentum \(\lambda^*\) will

\(^{14}\)The procedure for removing these seemingly large fluctuations is to properly renormalize them away which amounts to subtracting these inherent fluctuations present even in Minkowski space. We do not detail it here, but it is explained in Appendices A and B.

\(^{15}\)However, one possible way of explaining Dark Energy is precisely from considering the energy associated with these fluctuations [12]. However, this in turn leads to the so called Cosmological Constant Problem which is a fine tuning in regards to the near cancellation between the inherent fluctuations (which as stated can be quite large) in all fields and their classical values on the order of \(\mathcal{O}(10^{-120})\).
grow as,
\[ \lambda_{p_*} \sim \frac{a}{p_*}, \quad (1.58) \]
and grows to cosmological scales when \( \lambda_{p_*} \) is on the order of the Hubble scale during inflation,
\[ \lambda_{p_*} \sim H^{-1}_*, \quad (1.59) \]
where \( H^{-1}_* \) is the value of the Hubble parameter when that particular mode “crosses” the horizon. This will occur at a time \( \tau_* \) which corresponds to a particular value of the scale factor \( a(\tau_*) \equiv a_* \). Putting everything together, we arrive at the expression describing when a particular mode leaves the horizon,
\[ \lambda_{p_*} = \frac{a_*}{p_*} = H^{-1}_* \quad \Rightarrow \quad p_* = a_* H_. \quad (1.60) \]

And so we are now in the position to determine what are the relevant momenta during inflation. Only those momenta which are initially within the horizon, \( p \gg aH \), and then subsequently leave the horizon, \( p \ll aH \), will contribute to for example the power spectrum since these are the momenta that are amplified during inflation. With this in mind, we can then estimate the momentum scales of interest,
\[ a_i H_i < p < a_f H_f. \quad (1.61) \]

With the allowed momentum scales in hand, we can now return to our expression for the power spectrum (equation 1.56) and calculate its amplitude based on these momenta. Let’s first examine the massless case, \( \nu = \frac{3}{2} \), which yields,
\[ P_\phi(p, \tau; 3/2) = \left( \frac{H}{2\pi} \right)^2 \left( 1 + |p\tau|^2 \right). \quad (1.62) \]

If we select a particular momentum in our allowed range \( p_* \), then we can relate this back
to the scale factor and Hubble parameter when it crosses the horizon $a_*H_*$. Further, we can exchange $\tau$ in the above equation for the scale factor since $|\tau| = (Ha)^{-1}$. This leads to the massless power spectrum reading,

$$P_\phi(p, \tau; 3/2) = \left(\frac{H}{2\pi}\right)^2 \left(1 + \left(\frac{a_*}{a(\tau)}\right)^2\right),$$  \hspace{1cm} (1.63)

We find that for $\tau < \tau_*$ (times when modes are subhorizon) the power spectrum is exponentially enhanced,

$$P_\phi(p, \tau \ll \tau_*; 3/2) = \left(\frac{H}{2\pi}\right)^2 e^{2H(t_* - t)},$$  \hspace{1cm} (1.64)

but for times $\tau > \tau_*$ (times when modes are superhorizon) it goes to a constant

$$P_\phi(p, \tau \gg \tau_*; 3/2) = \left(\frac{H}{2\pi}\right)^2.$$  \hspace{1cm} (1.65)

This is one of the most substantial predictions of inflation and so we will point out two key properties of inflation that can be inferred from this expression. The first is that this expression is independent of the particular momentum which generated it. Thus all the momenta that are excited during inflation (equation 1.61) will have (nearly) the same amplitude for their fluctuations,

$$\delta_\phi = \sqrt{P_\phi} = \frac{H}{2\pi}.$$  \hspace{1cm} (1.66)

The second feature is that the above statement is not exactly true but nearly true.

Even though we have assumed so up to now that the Hubble parameter is constant\textsuperscript{17},

\begin{footnotesize}
\textsuperscript{16}This result would also be obtained when properly renormalizing the power spectrum which would amount to subtracting off the Minkowski part of the mode function, $\psi^{Min}_p = e^{-ip\tau}/\sqrt{2p}$.

\textsuperscript{17}In fact, if $H$ is exactly constant, then inflation would never end since $\dot{\phi}$ would have no way of stopping the expansion since it would have no dynamical properties, $\dot{\phi} \propto V' \propto \epsilon \propto H = 0$. This idea of a never ending inflationary era is similar to the idea of eternal inflation [13] where inflation continues \textit{ad infinitum}
\end{footnotesize}
during inflation it has a slight time dependence, \( H(t) \). This time dependence can be parameterized in terms of the first slow roll parameter if we consider the equation of motion for the inflation and also use the first Friedmann equation with a time derivative applied

\[
\frac{d}{dt} \left[ H^2 \simeq \frac{V(\phi)}{3M_P^2} \right] \Rightarrow 2H \dot{H} = \frac{V'(\phi)}{3M_P^2 \phi} \quad 3H \dot{\phi} \simeq -V'(\phi) \quad \epsilon = \frac{M_P^2}{2} \left( \frac{V'}{V} \right)^2 = -\frac{\dot{H}}{H^2}.
\]  

(1.67)

This in turn will cause the scale factor, \( a(\tau) \), to be slightly perturbed from its exact solution,

\[
a(\tau) = -\left( \frac{1 - \epsilon}{H\tau} \right)^{-1},
\]

(1.68)

which we can incorporate into the power spectrum through the index of the Hankel function,

\[
\nu = \sqrt{\frac{9}{4} + 3\epsilon} \simeq \frac{3}{2} + \epsilon.
\]

(1.69)

Expanding the power spectrum (equation 1.56) for small \( \epsilon \) we find,

\[
\mathcal{P}_\phi(p_*, \tau \gg \tau_*; \epsilon) = \left( \frac{H_*}{2\pi} \right)^2 \left[ \frac{|p_* \tau|^{6+2\epsilon}}{9} + |p_* \tau|^{-2\epsilon} \right] + \mathcal{O}(\epsilon) \simeq \left( \frac{H_*}{2\pi} \right)^2 |p_* \tau|^{-2\epsilon},
\]

(1.70)

where we have also expanded for \( |p\tau| \ll 1 \) and where \( H_* \) is the value of the Hubble parameter when \( p_* = a_* H_* \). As you can see, the power spectrum is nearly the same as before but with a slightly growing amplitude. This means that those modes that exit the horizon later during inflation will have a slightly larger amplitude meaning there is a larger amplitude for less energetic modes. We can also define the spectral index as,

\[
\frac{d\ln \mathcal{P}_\phi}{d\ln p} \equiv n_\phi - 1 = -2\epsilon \quad \Rightarrow \quad n_\phi = 1 - 2\epsilon,
\]

(1.71)

but can stop in certain “pocket” universes such as our own. This is as you might suspect the basic premise behind the multiverse which is another cudgel used against the idea of inflation.
where $n_\phi > 1$ corresponds to a ‘blue’ spectrum, $n_\phi < 1$ to a ‘red’ spectrum, and $n_\phi = 1$ corresponds to a flat spectrum which is what we found in equation 1.65 when we did not take into account an evolving Hubble parameter. This result, $n_\phi = 1 - 2\epsilon$, is another general prediction of single field inflationary models. The spectral tilt should be slightly red with an almost constant amplitude.

The case for a massive scalar in the superhorizon limit will yield a similar power spectrum but with a different value for the index $\nu = \sqrt{\frac{q}{4} - 3\eta + 3\epsilon}$ where we have traded the mass of $\phi$ for the second slow roll parameter, $\eta = \frac{m^2}{3H^2}$. Using the above, we find

$$P_\phi(p_*, \tau \gg \tau_*; \epsilon, \eta) \simeq \left( \frac{H_*}{2\pi} \right)^2 \left( \frac{a_*}{a(\tau)} \right)^{2\eta - 2\epsilon}, \quad (1.72)$$

where we have expanded for both small mass, $m \ll H$ (or equally small $\eta$), and small $\epsilon$. As you can see the addition of the mass term can cause the spectrum to be either red or blue depending upon on the relative magnitudes of $\epsilon$ and $\eta$. In theory, the spectrum could even change during inflation due to either an evolving mass or the increasing $\epsilon$. However, whereas $\epsilon$ causes a slightly larger amplitude for modes at the end inflation, the mass term suppresses these same modes. In the case of a very massive scalar, $m \gg H$, then the power spectrum is severely damped after a particular mode crosses the horizon,

$$P_\phi(p_*, \tau \gg \tau_*; m \gg H) \simeq \left( \frac{H_*}{2\pi} \right)^2 \frac{H}{m} \left( \frac{a_*}{a(\tau)} \right)^3. \quad (1.73)$$

This shows that in general only light fields ($m \lesssim H$) have the change of being appreciably produced during inflation since it is ‘easier’ to amplify a lighter field than a heavier one. The above analysis of the power spectra for $\phi$, can equally be applied to other scalar fields that are around during inflation with the important caveat that any extra fields around should not interfere very much with the slow roll evolution of $\phi$ which if they do could result in spoiling inflation.
In full disclosure, up to this point we have employed a sort of legerdemain in regard to how we have treated the fluctuations in the field $\phi$. We have assumed that fluctuations $\delta \phi(x,t)$ exist in a homogeneous background $g_{\mu\nu}(t)$, but this cannot be true based on Einstein’s field equations. According to equation 1.18, the geometry of spacetime is linked to what is in spacetime and vice versa, $g_{\mu\nu} \Leftrightarrow T_{\mu\nu}$. Thus fluctuations in the energy-momentum tensor such as $\delta \phi(x,t)$ must then produce fluctuations, $\delta g_{\mu\nu}(x,t)$, in the metric itself, $\delta g_{\mu\nu} \Leftrightarrow \delta T_{\mu\nu}$. The full mathematical machinery necessary for adequately calculating these relativistic cosmological perturbations will not be detailed here. In fact, all of the important calculations in this thesis take place during inflation and so the main computational thrust is to see if they can in fact be generated in the first place. We will however provide a very brief overview of how cosmological perturbation theory works at linear order for the specific case of temperature anisotropies in the CMB.

We first consider an action involving our old friend $\phi$ and the Einstein-Hilbert action involving the Ricci scalar,

$$S = \int d^4p \sqrt{-g} \left[ \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) + \frac{R M_p^2}{2} \right],$$

where the now perturbed metric reads, $ds^2 = a^2 [-d\tau^2 + (1 - 2R)dx_i dx^i]$. We will be interested in formulating our results in terms of the gauge invariant scalar $R$ called the comoving curvature perturbation. As its name implies, $R$ measures the spatial curvature of comoving hypersurfaces and represents the spatial curvature for a particular slicing of spacetime, i.e. comoving slicing. This means that along these hypersurfaces observes measure $\delta \phi = 0$. Through proper and tedious mathematical massaging the above action

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18This is not the most general perturbed metric one can write down. The most general perturbed metric has 10 degrees of freedoms (dofs) representing 4 scalar, 4 vector, and 2 tensor dofs. The perturbed scalar sector (including an additional dof from $\phi$ itself) can be whittled down to just one dof by specifying a particular gauge (this eliminates 2 dofs) and then using constraint equations (this eliminates an additional 2 dofs) with a resultant one true dof. Or, if you prefer, there is only one physical dof in the matter sector, $\delta \phi$, which we can trade for one dof in the metric sector, $R$. 

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can be written in the surprising simple form,

\[ S = \int d^4p \left[ \frac{1}{2} v'v' - \frac{1}{2} (\partial_i v)(\partial_i v) + \frac{z''}{2z} v^2 \right], \quad (1.75) \]

where \( v = z\mathcal{R} \) and \( z^2 = a^2 \dot{\phi}^2 / \pi^2 \). The astute reader might recognize that this action looks very similar to equation 1.47, and indeed it is simply the action for a canonically normalized scalar. Thus, we can immediately find the power spectrum for \( \mathcal{R} \) by first identifying the correlator,

\[ \langle \mathcal{R}(k)\mathcal{R}(k') \rangle = \frac{\langle v(k)v(k') \rangle}{z^2} = \frac{H^2 |\psi_k|^2}{\dot{\phi}^2 a^2}, \quad (1.76) \]

which leads to a power spectrum reading,

\[ P_{\mathcal{R}}(k) = \frac{H^2}{\dot{\phi}^2} \left( \frac{H_*}{2\pi} \right)^2 \left( \frac{k}{a_* H_*} \right)^{n_{\mathcal{R}}-1}, \quad (1.77) \]

where we have introduced a general spectral index \( n_{\mathcal{R}} \) similar to equation 1.71 and \( * \) is the value for a given quantity when a particular mode \( k \) crosses the horizon. Finally, we must make contact with things we can actually measure if this whole exercise of relating fluctuations in \( \phi \) to fluctuations in \( \mathcal{R} \) is to be meaningful. The two main cosmological observables\(^\text{19}\) that we can relate directly to the power spectrum of \( \mathcal{R} \) are the matter power spectrum, \( P_\delta \), and the power spectrum for the CMB temperature anisotropies, \( C_{\ell}^{TT} \). The first reads,

\[ P_\delta(k, \tau) = \frac{4}{25} \left( \frac{k}{aH} \right)^4 T^2_{\delta \rho}(k, \tau) P_{\mathcal{R}}(k), \quad (1.78) \]

where \( T_{\delta \rho} \) is the transfer function which takes into account the time evolution of both \( \mathcal{R} \) and \( \delta \rho \). We focus however on the CMB anisotropies whose expression relating them to

\(^{19}\)For a more thorough discussion of how these are actually derived see for example [14, 15].
\( \mathcal{R} \) reads,

\[
C_{\ell}^{TT} \simeq \int \frac{dk}{k} P_R(k) \Delta_T(k) \Delta_{T\ell}(k), \tag{1.79}
\]

where \( \Delta_{T\ell} \) is another transfer function this time relating changes in the temperature \( \Delta T \) to \( \mathcal{R} \) and \( C_{\ell}^{TT} \) are the famed multipole moments of the angular power spectra for the CMB temperature fluctuations. These two quantities \( P_{\delta \phi} \) and \( C_{\ell}^{TT} \) are in general non-trivial to calculate due to the time evolution of the fields once they reenter the horizon. In particular, the late-time evolution of the matter distribution of the universe is very challenging and not understood very well today since non-linear effects must be taken into account for late stage evolution [16].

Despite these challenges, we can make some approximations in certain regimes. For example, if we consider only large angular scales\(^{20} \) \( (2 \leq \ell \lesssim 100) \) then the transfer function is a Bessel function,

\[
\Delta_{T\ell}(k) = \frac{1}{3} J_{\ell}(k|\tau_0 - \tau_{rec}|), \tag{1.80}
\]

where \( \tau_0 \) is the current conformal time and \( \tau_{rec} \) is the time at recombination. This is a nice regime since the CMB was not effected by subhorizon evolution since the modes were still outside of the horizon which allows us to write the transfer function in a relatively simple form. The Bessel function essentially acts as a delta function since the integral is peaked for those values of \( k \simeq \frac{\ell}{\tau_0 - \tau_{rec}} \). This in turn leads to the integral in equation 1.79 becoming,

\[
C_{\ell}^{TT} \simeq [P_R(k)]_{k=\frac{\ell}{\tau_0 - \tau_{rec}}} \int \frac{dz}{z} [J_{\ell}(z)]^2, \tag{1.81}
\]

where since the integral is peaked at \( k \simeq \frac{\ell}{\tau_0 - \tau_{rec}} \) we can approximated it by its value at the peak,

\[
\int \frac{dz}{z} [J_{\ell}(z)]^2 \simeq \frac{1}{\ell(\ell + 1)}. \tag{1.82}
\]

---

\(^{20}\)This is referred to as the Sachs-Wolfe regime.
Finally, we arrive at the expression for the temperature fluctuation multipoles for large scales,

\[ \ell(\ell + 1)C^T_T \simeq [\mathcal{P}_R(k)]_k = \frac{\epsilon}{\epsilon_{0 - \text{rec}}} \cdot \]

(1.83)

The power spectrum for curvature perturbations is usually parameterized in the literature as,

\[ \mathcal{P}_R(k) = A_s \left( \frac{k}{k_s} \right)^{n_s - 1 + \cdots} \]

(1.84)

where \( k_s = 0.05 \text{ Mpc}^{-1} \) corresponds to scales that left the horizon at decoupling, \( n_s \) is the spectral index, \( A_s \) is the amplitude for scalar perturbations, and \( \cdots \) represents higher order corrections such as the running of the spectral index, \( |\alpha_s| \simeq 0.003 \ll n_s \) [2]. We can then immediately relate the above equation to equation 1.77 and we find,

\[ A_s = \frac{H_s^2}{\dot{\phi}_s^2} \left( \frac{H_s}{2\pi} \right)^2 , \quad n_s = n_R . \]

(1.85)

The current values for these quantities are \( A_s \approx 2.21 \times 10^{-9} \) and \( |n_s - 1| \approx 0.965 \) [2].

And so, we have completed the mathematical chain we discussed at the beginning of the section relating \( \delta \phi \Rightarrow R \Rightarrow \Delta T \). We again stress the significance of the above relationship.

By measuring very subtle changes in the temperature of light created over 13 billion years ago, we are able to infer certain properties about the universe when it was only a fraction of a second old. We next summarize all of the relevant cosmological data that pertains to constraining different models of inflation.
1.7 Cosmological Observables Constraining Inflationary Models

The mathematical relations and observational data needed to constrain the inflationary landscape is presented below. Some of these are measured with incredible precision for example the average temperate of the CMB and so can provide tight constraints on certain inflationary models. Other observations like the tensor-to-scalar ratio have not yet been measured, but do have upper bounds which nevertheless provide a way of constraining the allowed parameter space.

- Curvature of the universe: $\Omega_k = 0.000 \pm 0.005$ [2]
  - This measurement corresponds to the energy associated with the curvature of the universe. As you can see from the first Friedmann equation (eq. 1.21), the curvature of the universe can equally be treated as an energy which decays as $a^{-2}$. Thus you can add up all of the known sources of energy such as Dark Energy, Dark Matter, etc. and subtract it from 1 to arrive at $\Omega_k$. A value of $\Omega_k = 0$ corresponds to a perfectly flat universe, and as you can see to within error bars we live in a flat universe. This means that parallel lines (such as two beams of light) will remain parallel and not curve in or away from each other. It could very likely be the case that the universe is so big that we simply can not ‘see’ the curvature since we are looking at a finite portion of it – just as it is hard to ‘see’ the curvature of the Earth on small scales. If we take $\Omega_k = 0.005$, we can estimate the radius of a spatially closed universe as $R = H_0^{-1} \Omega_k^{\frac{1}{2}} \simeq 14H_0^{-1}$. This means that the ‘entire’ universe would be a factor $14^3$ bigger than our observable one.
• Temperature homogeneity: $T_0 = 2.72548 \pm 0.000057$ K [3]
  
  - The average temperature of the CMB we measure is incredibly uniform implying that the universe was very homogeneous and isotropic at the time of last scattering.

• Anisotropic expansion: overwhelming disfavored at a ratio 121,000:1 [17]
  
  - If the universe’s expansion was not isotropic then photon coming from different directions in the sky would redshift at different rates. This is found not to be the case implying that the universe is isotropic.

• Amplitude for scalar perturbations, $\ln(10^{10}A_s) = 3.094 \pm 0.034$ [2]
  
  - As discussed at the end of the last section, this corresponds to the amplitude for the fluctuations in the spatial curvature which can be related to the anisotropies in the CMB. This amplitude has been known since the COBE satellite measured it, but we provide the most recent measurement of it from the Planck satellite.

• Slightly red scalar index $n_s = 0.9645 \pm 0.0049$ [2]
  
  - Even though the amplitude for scalar perturbations is nearly constant for all wavelengths, there is a slight tilt to the ‘red’ end of the spectrum. This means that there is slightly more power for longer wavelength than for shorter ones. One of the general predictions of inflation is a slightly red spectrum.

• Non-gaussantites: $f_{NL}^{local} = 0.8 \pm 5.0$ [75]
  
  - The scalar perturbations predicted by single field inflationary models are very close to being exactly Gaussian. $f_{NL}$ is a way of measuring the level of non-Gaussianity with $f_{NL} = 0$ corresponding to a perfect Gaussian distribution.
The level of non-Gaussanity assuming $f_{NL} = 5$ implies that the perturbations are Gaussian to a 0.01% level.

- **Tensor-to-scalar ratio:** $r < 0.07$ at 95% confidence [45]
  
  - This is a dimensionless parameterization of the strength of tensor modes, $r = \mathcal{P}_T/\mathcal{P}_R$. We only have an upper bound on $r$ which can provide a maximum energy scale during inflation. We will discuss $r$ more in Chapter 3.

- **Tensor spectral index:** $n_T = ?$
  
  - Since the power spectrum of the tensor modes has not been observed, we do not yet have information concerning its spectral index. However if it is measured, $n_T$ would provide yet another way of constraining the various inflationary models.

- **Consistency relation:** $r = -8n_T \Rightarrow -n_T \lesssim 0.009$
  
  - If the tensor power spectrum is measured in the future and we are able to determine $n_T$, this relation would provide a way of verifying that the measured background PGWs are indeed caused by the inflationary amplification of the graviton and not from some other source.
CHAPTER 2

NET ELECTRIC CHARGE FOR THE OBSERVABLE UNIVERSE

2.1 Introduction

The conservation of electric charge is one of the best established, least questioned laws of physics [20]. While scenarios where charge is not conserved have been proposed (see below for an incomplete list), these correspond to exotic situations, and the charge of the Universe is usually assumed to vanish.

We focus on the fact that, even if electric charge is exactly conserved as a global quantity, during inflation with Hubble parameter $H$ large scale charge fluctuations are generated if there exist charged particles with mass $m \lesssim H$. As a consequence, even if the entire Universe is electrically neutral, any finite portion (including our observable one) of it can have a net charge. We will estimate the typical magnitude of the average charge density $\rho_R$ in a volume of radius $R$ by computing its variance right after inflation.

The constraints on the electric charge density $\rho_0$ of the Universe are tight, $\rho_0 \lesssim 10^{-26} n_B$, where the $n_B$ is the number density of baryons [21] (see also [22, 23] for previous analyses that did not account for the large conductivity of the primordial plasma). For charged
massive fermions of mass \( m = \mathcal{O}(H) \) we will find a charge density that is orders of magnitude smaller, \( \rho_0 \lesssim 10^{-33} n_B \). The charge density in scalars with \( m = \mathcal{O}(H) \) will be comparable to that of fermions. However, the charge density accumulated in very light charged scalar particles can be much larger, and can exceed by several orders of magnitude the bounds of [21] in the limit of a massless scalar species.

We also wish to stress that in no case do we expect (and indeed we do not obtain) large values of charge densities on superhorizon scales. Our central question is: given that fluctuations of charged fields might occur during inflation, how large (or better, how small) can the corresponding charge fluctuations be? And how do they compare to another very small number, the upper bound [21] set by observations on the charge density of our observable Universe?

One might worry that the electric field produced by these charge inhomogeneities during inflation can oppose charge separation or annihilate charges via Schwinger effect. As we will see, this is typically not the case, even if in some instances the Schwinger effect can be relevant.

The idea that the Universe might carry a net electric charge dates back to the work of Lyttleton and Bondi [24], who assumed \( \rho_0/n_B \simeq 10^{-18} \) to explain the recession of distant galaxies, while in the '60s Alfvén and Klein [25] considered a cosmology where charge separation would play a central role. The possibility of a charge imbalance, analogous to that of [24], but confined to dark matter was discussed more recently in [26]. References [27–30] considered the generation of a net charge caused by the spontaneous breaking of the electromagnetic gauge symmetry (used in [31] to generate cosmological magnetic fields), and [32] has shown that the same effect is produced by a photon mass. The authors of [33] discussed the possibility that electric charge is not conserved in brane world models. Closer to our work, a massive charged scalar during inflation was discussed in [34], whose focus, however, was on the generation of magnetic fields. The system of [34] was
reanalyzed in [35], where the current charge density of the Universe was also estimated in the case of a massless charged scalar. A charged curvaton was considered in [36], where it was argued that the charge density should not survive until the end of inflation because of Schwinger pair production. In [37], a mechanism analogous to ours, with the electric charge replaced by the baryon number, was proposed to produce the observed baryon asymmetry of our Universe. As we discuss in Section 2.5, our results differ significantly from those of [35–37].

To conclude this introductory section, we provide a brief description of the sections in this chapter. First in Section 2.2, we introduce the general procedure for defining the charge variance in a finite volume and apply it to both a massive fermion and massive, charged scalar and in particular we calculate the power spectrum in both cases. In Section 2.3, we consider what constraints will need to be taken into account when an electric field induced by charge separation occurs during inflation. In Section 2.4, we check that our calculation is consistent with charge conservation and in Section 2.5 we compare our results with previous calculations. Finally in Section 2.6, we compare our results for a charge asymmetry in our universe with the current observational constraint and in Section 2.7 we summarize our overall results. The material presented in this chapter follows closely with the paper associated with this work [38].

2.2 Charge density during inflation

We define the average charge density $\rho_R$ in a volume of radius $R$ as

$$\rho_R \equiv \int \frac{d^3 \mathbf{x}}{(\sqrt{\pi R})^3} e^{-x^2/R^2} \rho(\mathbf{x}),$$ \hspace{1cm} (2.1)
correlations are possible\(^1\). This situation is identical to that that would be realized in a version of the model of Affleck-Dine baryogenesis [39] where the Affleck-Dine field has a mass that is tuned in such a way that it starts rolling some efoldings before the end of inflation. In such a situation the baryon number of the Universe is generated \textit{during} inflation, and then one invokes baryon number conservation to deduce the magnitude of the \textit{current} baryon asymmetry of the Universe.

To our knowledge, the present paper is the first one where the charge variance in Dirac fermions is computed at the end of inflation. On the other hand, the charge variance of complex scalar fields during inflation was considered in [35–37], that however did not account for the effects of renormalization. Because of this, our results differ significantly from those of [35–37]. We believe that our analysis, which gives unambiguously finite results, is the appropriate one for this problem. In Appendices A and B we motivate our renormalization procedure. In Section 2.5 we compare our analysis with that of [35–37] and we describe the reason of our different results.

2.2.1 Fermions

The action for a massive fermion in an FLRW geometry will be

\[
S = \int d^4x \sqrt{-g} \bar{\psi} \left[ i \frac{\gamma^\mu}{a} \partial_\mu + \frac{3a'}{2a} \gamma^0 - M \right] \psi, \quad (2.4)
\]

\(^1\)We insist however that, even if the correlations are generated during inflation, the \textit{computation} of the charge density should be performed only after the end inflation, when the relevant modes are evolving adiabatically.
where $\rho(x)$ is the charge density operator for the form of matter under consideration. Since electric charge is conserved, and any initial charge density will be rapidly driven to zero by inflation, $\langle \rho_R \rangle = 0$. However the variance of $\rho_R$ will not vanish and its square root will give the typical size of the charge density in a sphere of radius $R$. If we define the charge power spectrum $P_\rho(k)$

$$
\langle \rho(k) \rho(k') \rangle \equiv \frac{2\pi^2}{k^3} \delta(k + k') P_\rho(k),
$$

with $\rho(k) \equiv \int \frac{d^3x}{(2\pi)^3} e^{-ikx} \rho(x)$, it is then straightforward to prove that

$$
\langle \rho_R^2 \rangle = \int \frac{dk}{k} P_\rho(k) e^{-k^2 R^2 / 2}.
$$

We will be caring about the limit of large $R$, so that we will need to compute only $P_\rho(k)$ for $k \to 0$.

The charge density is proportional to the number density of particles, that is well defined only when the frequency of the mode functions is evolving adiabatically, so that the mode functions can be expressed by the solutions of the equations of motion in the WKB approximation and there is a clear distinction between positive- and negative-energy modes. The Bogolyubov coefficients are thus especially well suited for this problem (see Appendix A), as they effectively measure the relative amplitude of the negative frequency modes with respect to that of the positive frequency ones. Since the Bogolyubov coefficients are not constant for super-horizon modes during inflation, the concept of particle is not well defined at that stage. For this reason one might conclude that charge imbalances cannot be created before the end of inflation and that large scale charge correlations are thus forbidden by causality. However, the field correlations that are associated to a net charge density are defined also for superhorizon fluctuations. As a consequence, charge correlations are really created during inflation, and large scale charge
where we use conformal time, $\tau$, in an exact de Sitter space, $a(\tau) = -(H\tau)^{-1}$, primes are w.r.t conformal time, and $\bar{\psi} = \psi^\dagger \gamma^0$. The gamma matrices, $\gamma^\mu$, are defined as

$$
\gamma^0 = \text{diag}(1, 1, -1, -1) \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},
$$

(2.5)

where $\sigma^i$ are the Pauli matrices. We can solve for the canonically normalized field’s, $\Psi \equiv \frac{3}{2}a^3 \psi$, equation of motion which obeys

$$(i \gamma^\mu \partial_\mu - m a) \Psi = 0$$

(2.6)

that we solve by decomposing

$$
\Psi(k, \tau) = \sum_{r=\pm 1} \left[ u_r(k, \tau) a_r(k) + v_r(k, \tau) b^\dagger_r(-k) \right],
$$

(2.7)

with (using the conventions of [40])

$$
\begin{align*}
 u_r(k, \tau) &= \frac{1}{\sqrt{2}} \begin{pmatrix} U_+(k, \tau) \psi_r(\hat{k}) \\ r U_-(k, \tau) \psi_r(\hat{k}) \end{pmatrix}, \\
 v_r(k, \tau) &= \frac{1}{\sqrt{2}} \begin{pmatrix} V_+(k, \tau) \psi_r(k) \\ r V_-(k, \tau) \psi_r(k) \end{pmatrix},
\end{align*}
$$

(2.8)

where $\psi_r$ is an eigenfunction of the helicity operator with eigenvalue $r/2$. The equations of motion read

$$
U_+^\prime = -ik U_+ \mp im a U_+.
$$

(2.9)

Given that the system is invariant under charge conjugation, we have $V_+ = -U_-^*$, $V_- = U_+^*$. Moreover, the normalization $|U_+|^2 + |U_-|^2 = 2$ is preserved by the equations of motion.
In a de Sitter geometry $a(\tau) = - (H\tau)^{-1}$, eqs. (2.9) are solved by

$$U_\pm = \sqrt{-\frac{\pi k\tau}{2}} e^{\pm \frac{\pi m}{H}} H_{\frac{1}{2} + i \frac{m}{H}} (-k\tau),$$

(2.10)

where $H_{\nu}(x)$ denotes the Hankel function of the first kind.

In order to compute the renormalized two point function of the charge operator we compute the Bogolyubov coefficients for this system. To do so we decompose $\Psi(k, \tau)$ on a different set of creation/annihilation operators $\tilde{a}_r^{(1)}(k, \tau), \tilde{b}_r^{(1)}(-k, \tau)$ and mode functions $\tilde{U}_\pm(k, \tau)$ that are the adiabatic solutions of eqs. (2.9)

$$\tilde{U}_\pm = \left(1 \pm \frac{ma}{\sqrt{k^2 + m^2}} \right)^{1/2} e^{-i \int \sqrt{k^2 + m^2} a^2 \tau}$$

(2.11)

and are linearly related to the functions $U_\pm$ by

$$U_+(k, \tau) = \alpha(k, \tau) \tilde{U}_+(k, \tau) - \beta(k, \tau) \tilde{U}_-(k, \tau)$$

$$U_-(k, \tau) = \alpha(k, \tau) \tilde{U}_-(k, \tau) + \beta(k, \tau) \tilde{U}_+(k, \tau).$$

(2.12)

We can also relate the adiabatic operators to the original operators by,

$$\hat{a}_r(k) = \alpha \hat{a}_r(k) - \beta^* \hat{b}_r^\dagger(-k)$$

$$\hat{b}_r^\dagger(-k) = \beta \hat{a}_r(k) + \alpha^* \hat{b}_r^\dagger(-k)$$

(2.13)

By definition, during adiabatic evolution, $\omega' \ll \omega^2$, the Bogolyubov coefficients $\alpha(k, \tau)$ and $\beta(k, \tau)$ are constant, and the occupation number for modes with momentum $k$ is given by $\langle 0 | \hat{a}(k) \hat{a}(k) | 0 \rangle = |\beta(k)|^2$, where the vacuum $|0\rangle$ is annihilated by the $a_r(k), b_r(k)$ operators.
For modes with $k \ll a$ the adiabaticity condition reads $a'/a^2 \ll m$. During inflation this condition is not satisfied for the fermions with $m \lesssim H$ we are considering, but it is after inflation ends, when the Hubble parameter $a'/a^2$ decreases. Therefore to compute the Bogolyubov coefficients we join the inflationary period to a radiation dominated\(^2\) one with $a(\tau) = H\tau + 2$ for $\tau > -1/H$. The equations of motion for $U_\pm$ during radiation domination can be solved in terms of parabolic cylinder functions and yield the final value of the Bogolyubov coefficients, whose main feature is that $k^3|\beta(k)|^2$ is peaked at $k \simeq m$. Their explicit expression, which is long and not very illuminating, will not be presented here.

The normal ordered (in terms of the tilded operators) two point function of the charge is

$$
\langle \rho(k)\rho(k') \rangle = e^2 \int \frac{d^3x d^3y}{(2\pi)^3} e^{-ikx - ik'y}
\times \langle \Psi^\dagger(x, \tau)\Psi(x, \tau) \Psi^\dagger(y, \tau)\Psi(y, \tau) \rangle,
$$

that, in the limit $k, k' \rightarrow 0$, gives

$$
P_f^\rho(k \rightarrow 0) = -e^2 \frac{k^3}{2^2 \pi^5} \int d^8q |\beta|^2 \equiv -e^2 k^3 H^3 f^f \left( \frac{m}{H} \right),
$$

where the function $f^f(m/H)$, plotted in figure 2.1, shows that, for $m \sim H$, $P_f^\rho(k) \sim 10^{-5} e^2 k^3 H^3$.

\(^2\)The description of reheating as a sudden transition between an exact de Sitter stage and an exact radiation dominated one is clearly an approximation that is however commonly made in the literature (e.g., in [34, 35]), and that is expected to be valid for the long wavelength modes $k \ll H$ of isocurvature fields we are considering.
2.2.2 Scalars

The case of a complex scalar is treated similarly, but, due to the absence of Pauli blocking, will lead to a richer set of possibilities. The canonically normalized field $\varphi$ satisfies

$$\varphi'' + \left( k^2 + m^2 a^2 - \frac{a''}{a} \right) \varphi = 0 \quad (2.16)$$

that is solved by decomposing

$$\varphi(k, \tau) \equiv \phi(k, \tau) a(k) + \phi^*(-k, \tau) b^1(-k), \quad (2.17)$$

where the mode functions read

$$\phi(k, \tau) = \sqrt{\frac{-\pi \tau}{4}} H^{(1)}_{\nu}(-k \tau), \quad \nu \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}, \quad (2.18)$$
where we assume $m < \frac{3}{2} H$. As we did for fermions, we then decompose $\varphi(k, \tau)$ using a different set of operators $\tilde{a}(k, \tau)$ and $\tilde{b}(k, \tau)$ and the adiabatic mode functions

$$
\tilde{\varphi}(k, \tau) = e^{-i \int \omega_k d\tau} \sqrt{2 \omega_k}, \quad \omega_k^2 \equiv k^2 + m^2 a^2 - \frac{a''}{a}.
$$

We then join the solutions during inflation to those obtained during a radiation dominated phase, so that the adiabatic condition is satisfied at late times. The process of computing the Bogolyubov coefficients and matching the solutions during inflation to the radiation dominated phase is detailed below.

1. **Calculation of Bogolyubov Coefficients**

The exact solution for the mode functions of a scalar field of mass $m$ during inflation with Hubble parameter $H$ is

$$
\varphi_I = \sqrt{-k\tau} \left[ A_k H^{(1)}(k\tau) + B_k H^{(2)}(-k\tau) \right],
$$

where $A_k$ and $B_k$ are arbitrary constants. The adiabatic solution to the mode functions is

$$
\varphi^{WKB}(k, \tau) = \alpha_k \tilde{\varphi}(k, \tau) + \beta_k \tilde{\varphi}^*(k, \tau),
$$

$$
\tilde{\varphi}(k, \tau) \equiv \frac{1}{\sqrt{2\omega_k}} e^{-i \int \omega_k d\tau}.
$$

If our initial state contains no particles, then $\alpha_k^{IN} = 1$ and $\beta_k^{IN} = 0$, which leads to

$$
A_k = \sqrt{\frac{\pi}{4k}}, \quad B_k = 0.
$$
The adiabatic condition \( \left( \frac{\dot{\omega}}{\omega^2} \right) \ll 1 \) at the end of inflation reads \( m \gg \frac{a'}{a^2} \), where we have assumed that we are looking at long wavelength modes for which \( \omega \simeq ma \). For masses on the order of \( H \) or less, the modes are not evolving adiabatically at the end of inflation, and the number of particles is therefore not a well-defined quantity. We can however join the end of inflation to a radiation epoch with \( a = H \tau + 2 \), where the adiabatic condition reads
\[
\frac{\dot{\omega}}{\omega^2} = \frac{Hm^2(H\tau + 2)}{(m^2(H\tau + 2)^2 + k^2)^{3/2}} \ll 1, \quad (2.23)
\]
showing that a well defined concept of particle will exist assuming we wait long enough, \( \tau \gg \frac{1}{\sqrt{mH}} \).

The equation of motion of a massive scalar during the radiation epoch is
\[
\varphi''_R + \left( k^2 + m^2(H\tau + 2)^2 \right) \varphi_R = 0, \quad (2.24)
\]
whose solution can be written in terms of parabolic cylinder functions as
\[
\varphi_R(\tau) = a_k D_{-\frac{1}{2} - i \frac{k^2}{2Hm}} \left( e^{\frac{3}{2} \frac{2m}{H} (H\tau + 2)} \right) + b_k D_{-\frac{1}{2} + i \frac{k^2}{2Hm}} \left( e^{\frac{3}{2} \frac{2m}{H} (H\tau + 2)} \right). \quad (2.25)
\]
The constants \( a_k \) and \( b_k \) are determined by joining the exact solutions during inflation to those during the radiation dominated era, that is, by imposing \( \varphi_I(\tau_R) = \varphi_R(\tau_R) \) and \( \varphi'_I(\tau_R) = \varphi'_R(\tau_R) \), where \( \tau_R = -1/H \) denotes the time of the end of inflation. The adiabatic solution for the mode functions after inflation will have the form of eq. (2.21) with \( \omega_k^2 = k^2 + m^2(H\tau + 2)^2 \). We can solve for the Bogolyubov coefficients by matching
the exact solution to the adiabatic solution for late times \((\tau \to +\infty)\)

\[
\varphi_R(\tau \to +\infty) \approx \frac{e^{-\frac{\tilde{k}^2}{2\tilde{m}^2}}}{\sqrt{\tilde{\tau}}\sqrt{2\tilde{m}}} \left[ a_k e^{\frac{\pi k^2}{2\tilde{m}}} + b_k e^{-\frac{\pi k^2}{2\tilde{m}} e^{\frac{i\pi}{4}}} \right] e^{-i\tilde{m}\tau^2/2\tilde{m}} + b_k e^{-\frac{\pi k^2}{2\tilde{m}}} e^{\frac{i\pi}{4}} \Gamma \left( \frac{1}{2} - i\frac{k^2}{2\tilde{m}} \right) e^{-\frac{i\pi k^2}{8\tilde{m}^2} e^{\frac{i\pi}{4}}},
\]

\[
\varphi_{\text{WKB}}(\tau \to \infty) \approx \frac{\alpha_k}{\sqrt{2H\tilde{m}}} e^{-\frac{i\tilde{m}^2 \tau^2}{4\tilde{m}}} + \frac{\beta_k}{\sqrt{2H\tilde{m}}} e^{\frac{i\tilde{m}^2 \tau^2}{4\tilde{m}}},
\]

(2.26)

obtaining

\[
\alpha_k = \sqrt{H} (2\tilde{m})^{1/4} e^{\frac{\pi k^2}{2\tilde{m}}} \left( a_k + b_k e^{\frac{\pi k^2}{2\tilde{m}}} \frac{\sqrt{2\pi} e^{i\pi/4}}{\Gamma \left( \frac{1}{2} - i\frac{k^2}{2\tilde{m}} \right)} \right),
\]

\[
\beta_k = \sqrt{H} (2\tilde{m})^{1/4} e^{-\frac{3\pi k^2}{2\tilde{m}}} b_k,
\]

(2.27)

where for notational simplicity \(\tilde{k} = k/H, \tilde{m} = m/H\) and \(\tilde{\tau} = H\tau\). We will be interested in nonrelativistic \((k \ll ma)\), superhorizon \((-k\tau_R \ll 1)\) modes, for which we find

\[
\beta_k \approx -e^{\frac{i\pi}{8}} \frac{\Gamma(1/4)}{2\sqrt{2\pi}} \left( \frac{H}{m} \right)^{1/4} \left( \frac{k}{H} \right)^{-\sqrt{9/4-m^2}/H^2},
\]

(2.28)

where \(\Gamma(1/4)/(2\sqrt{2\pi}) \approx .72\). Since \(\beta_k\) is non-zero we interpret this as the de Sitter expansion causing quanta of \(\varphi\) to be created.

An analogous study can be performed in the case of massless scalars, and the exact Bogolyubov coefficients take a much simpler form

\[
\alpha_k = e^{\frac{ik}{H}} \left( 1 + i \frac{H}{k} - \frac{H^2}{2k^2} \right),
\]

\[
\beta_k = e^{\frac{ik}{H}} \frac{H^2}{2k^2}.
\]

(2.29)

Now that we have the expressions for the Bogolyubov coefficients we can proceed with calculating the charge variance.
2.2.2.2 Calculation of Charge Variance for Scalars

Returning to our calculation for the charge variance, the charge operator for scalars reads

$$\rho(x, \tau) = -ie \left[ \varphi^\dagger(x, \tau) \varphi'(x, \tau) - \varphi'(x, \tau) \varphi(x, \tau) \right],$$  \hspace{1cm} (2.30)

which when inserted into equation 2.2 will lead to a power spectrum of the form

$$P_\rho(k) = e^2 \frac{k^3}{(2\pi)^5} \int \frac{d^3q}{\omega_{k+q} \omega_q} \left\{ 2 |\beta_q|^2 |\beta_{k+q}|^2 (\omega_q^2 + \omega_{k+q}^2) ight. \\
space \space \space \space \space \space \space \space \space \space - (\omega_q + \omega_{k+q})^2 \text{Re} \left[ \beta_q \beta^*_{k+q} \alpha_{k+q} \alpha_{k+q} e^{2i \int \omega_q d\tau - 2i \int \omega_{k+q} d\tau} \right] \\
\space \space \space \space \space \space \space \space \space \space - (\omega_q - \omega_{k+q})^2 \text{Re} \left[ \beta^*_{k+q} \beta_{k+q} \alpha_{k+q} \alpha_{k+q} e^{-2i \int \omega_q d\tau - 2i \int \omega_{k+q} d\tau} \right] \\
\space \space \space \space \space \space \space \space \space \space + 2(\omega_{k+q}^2 - \omega_q^2) \left( |\beta_{k+q}|^2 \text{Re} \left[ \beta^*_{k+q} \alpha_{k+q} e^{-2i \int \omega_{k+q} d\tau} \right] \right) \\
\space \space \space \space \space \space \space \space \space \space - |\beta_q|^2 \text{Re} \left[ \beta^*_{k+q} \alpha_{k+q} e^{-2i \int \omega_{k+q} d\tau} \right] \right\},$$  \hspace{1cm} (2.31)

The general expression of the Bogolyubov coefficients is rather cumbersome, but in the regime $q \ll H, \ m \gtrsim H$, which is of interest for us, it simplifies to

$$|\beta_q|^2 \simeq \begin{cases} \frac{5 \sqrt{H/m} \times (H/q)^{2\nu}}{H^4/(4q^4)}, & q \lesssim \sqrt{mH} \ \\
\frac{\sqrt{H/m}}{q}, & \sqrt{mH} \gtrsim q \lesssim H. \end{cases} \hspace{1cm} (2.32)$$

For nonrelativistic massive scalars the phase $\int \omega d\tau \simeq mH \tau^2/2$ (remember that $a(\tau) \simeq H \tau$ well after the end of inflation) oscillates rapidly after inflation, so that the second and third lines of equation (2.31) can be neglected. Also, one can take $k \to 0$ in the first line of that equation since, as we will see, one obtains a finite result. As a consequence, for scalars with a mass that is large enough, using the relation $|\alpha_q|^2 - |\beta_q|^2 = 1$, the
charge power spectrum can be written in the simple form

\[ P_\rho(k) = -\frac{e^2}{2^3 \pi^5} k^3 \int d^3q |\beta_q|^2 \]

\[ \simeq -\frac{3e^2}{8\pi^4} k^3 H^3 \left( \frac{H}{m} \right)^{5/2} \left[ \left( \frac{m}{H} \right)^{\frac{m^2}{2m^2}} - \left( \frac{\Lambda_{IR}}{H} \right)^{\frac{2m^2}{2m^2}} \right] \]  

(2.33)

where we have assumed \( m \ll H \) and used the first of eqs. (2.32). In eq. (2.33), \( \Lambda_{IR} \) corresponds to the scales that left the horizon at the beginning of inflation, so that the total number of efoldings of inflation is given by \( N_{\text{Tot}} \equiv \log(H/\Lambda_{IR}) \).

Depending on the total duration of inflation, eq. (2.33) simplifies to two different expressions. If \( N_{\text{Tot}} \gg \frac{3H^2}{2m^2} \) (the case which includes the limit \( \Lambda_{IR} \to 0 \)), then

\[ P_\rho(k) \simeq -\frac{3e^2}{8\pi^4} k^3 H^3 \left( \frac{H}{m} \right)^{5/2} . \]  

(2.34)

If, on the contrary, inflation did not last for too long and \( N_{\text{Tot}} \ll \frac{3H^2}{2m^2} \) then

\[ P_\rho(k) \simeq -\frac{e^2}{4\pi^4} k^3 H^3 \left( \frac{H}{m} \right)^{1/2} \log \left( \frac{m}{\Lambda_{IR}} \right) . \]  

(2.35)

Finally, we note that eq. (2.33) was obtained assuming that the dominant contribution to eq. (2.31) is given by the regime of integration of lowest \( q \), \( \Lambda_{IR} \lesssim q \lesssim \sqrt{mH} \), i.e., by using the expression for \( |\beta_q|^2 \) given by the first line of eq. (2.32). However if the scale of interests, characterized by the wave number \( k \), are such that \( k > \sqrt{mH} \), then the scalar field will be effectively massless. The exact Bogolyubov coefficients for a massless scalars read

\[ \alpha_q = -\frac{H^2 + 2iq + 2q^2}{2q^2} e^{i\eta/H}, \quad \beta_q = \frac{H^2}{2q^2} e^{i\eta/H} . \]  

(2.36)
Introducing these expressions into eq. (2.31) we obtain the simple expression, valid for one massless scalar species

\[
P_\rho(k) = -e^2 H^4 \frac{k^3}{2^5 \pi^5} \int \frac{d^3q}{q^3 |k + q|} \]

\[
\simeq -e^2 H^4 \frac{k^2}{2^3 \pi^2} (N_{\text{Tot}} - N_k),
\]

(2.37)

where \(N_k\) corresponds to the number of efoldings before the end of inflation at which the scale \(k\) left the horizon, so that \(N_k \simeq 50\).

### 2.3 Effects of the electric field during inflation

One might worry that the charge fluctuations generated during inflation produce an electric field which might either oppose further charge separation or annihilate charge via Schwinger pair production. Here we discuss why, in general, this is not the case.

The rate of change of a physical momentum \(p\), due to the expansion of the Universe, is given by \(H p\). For the effect of the electric field to be negligible with respect to that of cosmological expansion we then require \(e E_p \ll H p\), where \(E_p\) is the typical intensity of the electric field in modes with wavelength larger that \(1/p\). In other words, the acceleration due to the electric field should be negligible with respect to the proper deceleration due to the expansion of the Universe. We estimate \(E_p\) using Gauss’s law

\[
E_p^2 \simeq \langle E^2 \rangle_p = \int_{p} \frac{dk}{k^3} P_\rho(k). \tag{2.38}
\]

Since sub-horizon charge fluctuations are negligible, we assume \(p \lesssim H\), and for fermions we obtain \(e E_p \simeq 3 \times 10^{-3} e^2 H^{3/2} p^{1/2}\) so that only very low momentum modes with \(p \lesssim 10^{-5} e^4 H \simeq 10^{-7} H\) are affected by the electric field. Since most of the charge is in
modes with $p = \mathcal{O}(m) \gg 10^{-7} H$, the effect of the electric field on fermions can be safely neglected.

For scalars things are more complicated. A charged scalar $\phi$ with mass $m \lesssim H$ gets large fluctuations with variance $\langle |\phi|^2 \rangle = \frac{3 H^4}{4 \pi^2 m^2}$ and with a correlation length $\sim \int d^3 k k^{-1} |\phi_k|^2$ that is IR-divergent. This implies that $\phi$ acts as a uniform Higgs field, and that the photon gets a mass $m_\gamma \sim e \langle |\phi|^2 \rangle^{1/2} \approx 0.3 e H^2 / m$, which therefore imposes an infrared cutoff in the integral (2.38)$^3$. As a consequence, for $p \lesssim m_\gamma$ the range of integration in eq. (2.38) is vanishing and the electric field is negligible. On the other hand, the discussion of section 2.2.2 above shows that most of the contribution to the electric charge of the Universe comes from the very infrared modes with $p \sim 1/R \ll m_\gamma$. Therefore, the effect of the electric field is negligible.

Another possibility is that the electric field produced by the charge fluctuations ends up annihilating the fluctuations themselves via Schwinger effect. Schwinger pair production is effective if a charged particle $\chi_{\text{Schw}}$ with mass $m_{\text{Schw}}^2 \lesssim e \mathcal{E}/\pi$ exist, provided the coherence length of the electric field $\lambda$ satisfies $\lambda > 2 \pi m_{\text{Schw}}/(e \mathcal{E})$ [42]. Both conditions give an upper bound on $m_{\text{Schw}}$ and must both be satisfied for Schwinger pair production to be effective.

In the case of fermionic charge, the electric field will have a typical intensity $\mathcal{E} \sim 3 \times 10^{-3} e H^2$ and its coherence length is approximately $2 \pi / H$, so that the Schwinger phenomenon is effective if $m_{\text{Schw}} \lesssim e \mathcal{E}/H \approx 3 \times 10^{-4} H$.

In the case of charge generated by scalars the coherence length of the electric field is set by the mass of the photon, $\lambda = 2 \pi / m_\gamma$. As a consequence, if $m_\gamma \gtrsim H$ then the electric field will be negligible, as the infrared cutoff $\sim m_\gamma$ of the electric field is larger than its ultraviolet cutoff $\sim H$ determined by the absence of charge fluctuations at subhorizon

\[3\text{In the case of effectively massless scalars one gets } \langle |\phi|^2 \rangle = \frac{H^2}{4 \pi^2} N_{\text{Tot}}, \text{ so that } m_\gamma \approx 0.15 e H \sqrt{N_{\text{Tot}}} \]
scales. The mass of the photon will be larger than $H$ for $m \lesssim .1 H$. Therefore as long as there is a charged scalar with mass smaller than $.1 H$ we should not worry about the Schwinger effect. For scalars with $.1 H \lesssim m \lesssim H$ we insert eq. (2.33) into eq. (2.38) and take $p \simeq H$ as ultraviolet cutoff. We thus obtain $E \simeq .05 e H^2 (H/m)^{5/4}$. By evaluating numerically the condition that $m_{\text{Schw}}$ be smaller both than $e E/m_{\gamma}$ and than $\sqrt{e E/\pi}$ we obtain that, for $.1 H \lesssim m \lesssim H$, the Schwinger effect can be efficient if $m_{\text{Schw}} \lesssim .1 H$. As we stated above, if the field $\chi_{\text{Schw}}$ is a scalar, then its large scale fluctuations will contribute to $m_{\gamma}$ via a Higgs effect, yielding $m_{\gamma} \gtrsim H$. Therefore, the effect will be important only if $\chi_{\text{Schw}}$ is a fermion.

To sum up, Schwinger effect will affect the charge fluctuations only if there exists during inflation a fermion whose mass is smaller than $3 \times 10^{-4} H$, if the charges originate from the fluctuations of a fermion, or $10^{-1} H$, if they originate from a scalar with $.1 H \lesssim m \lesssim H$. It is worth noting that, since we do not know what is the expectation value of the Higgs field (or of any other scalar field that carries Standard Model charge) during inflation, the mass of the particles of the Standard Model will generally have mass that is different from the one measured today.

### 2.4 Consistency with current conservation

In this section, we check that the results presented above for scalars, and in particular eq. (2.31), which gives the charge density after the end of inflation, is consistent with the continuity equation for electric current.
Let $j^\mu$ be a covariantly conserved current, $\nabla_\mu j^\mu = 0$. Then in a FRW Universe with conformal time one has

$$\partial_0 j^0 + \partial_i j^i + 4 \frac{a'}{a} j^0 = 0.$$  \hspace{1cm} (2.39)

We define $J^\mu = a^4 j^\mu$ so that $\partial_0 J^0 + \partial_i J^i = 0$. Next, consistently with eq. (2.1) we define the charge density within a radius $R$ as

$$\rho_R(\tau) = \int \frac{d\mathbf{x}}{(\sqrt{\pi R})^3} e^{-x^2/R^2} J^0(\mathbf{x}, \tau),$$ \hspace{1cm} (2.40)

and, using twice the continuity equation, we obtain

$$\frac{d}{d\tau} \langle \rho_R(\tau)^2 \rangle = \int_{\tau_0}^{\tau} d\theta \int \frac{d\mathbf{x} \, d\mathbf{y}}{(\sqrt{\pi R})^6} e^{-(\mathbf{x}^2+\mathbf{y}^2)/R^2}$$

$$\times \frac{\partial^2}{\partial x^i \partial y^j} \left[ \langle J^i(\mathbf{x}, \tau) J^j(\mathbf{y}, \theta) \rangle + \langle J^i(\mathbf{x}, \theta) J^j(\mathbf{y}, \tau) \rangle \right],$$ \hspace{1cm} (2.41)

or, in momentum space,

$$\frac{d}{d\tau} \langle \rho_R(\tau)^2 \rangle = - \int_{\tau_0}^{\tau} d\theta \int \frac{d\mathbf{k} \, d\mathbf{q}}{(2\pi)^3} e^{-(k^2+q^2)/R^2/4} k_i q_j$$

$$\times \left[ \langle J^i(\mathbf{k}, \tau) J^j(\mathbf{q}, \theta) \rangle + \langle J^i(\mathbf{k}, \theta) J^j(\mathbf{q}, \tau) \rangle \right].$$ \hspace{1cm} (2.42)

Now, our current $j^\mu = -ie \left[ (\partial^\mu \Phi)^\dagger - \Phi^\dagger (\partial^\mu \Phi) \right]$ reads, in momentum space

$$J^i(\mathbf{k}) = -e a^2 \int \frac{d\mathbf{p}}{(2\pi)^{3/2}} \left[ k^i + 2 p^i \right] \Phi(\mathbf{k} + \mathbf{p})^\dagger \Phi(\mathbf{p}),$$ \hspace{1cm} (2.43)

where $\Phi$ is the “physical”, not the canonically normalized field. Using Wick’s theorem and dropping the disconnected diagram we obtain
\[
\frac{d}{d\tau} \langle \rho_R(\tau)^2 \rangle = -e^2 a^2(\tau) \int_{\tau_0}^{\tau} d\theta a^2(\theta) \int \frac{d\mathbf{k} d\mathbf{q}}{(2\pi)^3} \int \frac{d\mathbf{p}_1 d\mathbf{p}_2}{(2\pi)^3} e^{-(k^2+q^2) \tau^2/4} \times \\
\times (k^2 + 2k \cdot \mathbf{p}_1) (q^2 + 2q \cdot \mathbf{p}_2) \left\{ \langle \Phi(\mathbf{k} + \mathbf{p}_1, \tau) \Phi(\mathbf{q} + \mathbf{p}_2, \theta) \rangle \right\} \left\{ \langle \Phi(\mathbf{p}_1, \tau) \Phi(\mathbf{p}_2, \theta) \rangle \right\} \\
+ \langle \Phi(\mathbf{k} + \mathbf{p}_1, \tau) \Phi(\mathbf{p}_2, \theta) \rangle \langle \Phi(\mathbf{p}_1, \tau) \Phi(\mathbf{q} + \mathbf{p}_2, \theta) \rangle \right\} + (\tau \leftrightarrow \theta) \right\}. 
\]

(2.44)

To compute the two point function we decompose the field \( \Phi(\mathbf{p}, \tau) \) as \( \Phi(\mathbf{p}, \tau) = \phi(\mathbf{p}, \tau) a + \phi^*(\mathbf{p}, \tau) b^\dagger_p \), so that

\[
\langle \Phi(\mathbf{q}, \tau) \Phi(\mathbf{p}, \theta) \rangle = \langle \Phi^\dagger(\mathbf{q}, \tau) \Phi^\dagger(\mathbf{p}, \theta) \rangle = 0, \\
\langle \Phi(\mathbf{q}, \tau) \Phi^\dagger(\mathbf{p}, \theta) \rangle = \langle \Phi^\dagger(\mathbf{q}, \tau) \Phi(\mathbf{p}, \theta) \rangle = \delta(p - q) \phi(\mathbf{p}, \tau) \phi^*(\mathbf{p}, \theta). 
\]

(2.45)

Therefore, remembering that the definition of charge power spectrum is given by eq. (2.3), we obtain

\[
\frac{d P_\rho(k)}{d\tau} = e^2 a^2(\tau) \frac{k^3}{2 \pi^2} \int_{\tau_0}^{\tau} d\theta a^2(\theta) \int \frac{d\mathbf{p}_1}{(2\pi)^3} \times \\
\times (k^2 + 2k \cdot \mathbf{p}_1)^2 \left[ \phi(|\mathbf{k} + \mathbf{p}_1|, \tau) \phi^* (|\mathbf{k} + \mathbf{p}_1|, \theta) \right] \phi (|\mathbf{p}_1|, \tau) \phi^* (|\mathbf{p}_1|, \theta) + h.c.] .
\]

(2.46)

Using the property

\[
2 \text{Re} \left\{ \int_{x_0}^{x} f(y) \left( \int_{x_0}^{y} f^*(z) dz \right) dy \right\} = \left| \int_{x_0}^{y} f(z) dz \right|^2, 
\]

(2.47)
we can integrate over \(d\tau\) the equation for \(P_\rho\) and obtain

\[
P_\rho(k, \tau) = e^2 \frac{k^3}{2 \pi^2} \int \frac{dp_1}{(2\pi)^3} (k^2 + 2kp_1)^2 \left| \int_{\tau_0}^{\tau} d\theta a^2(\theta) \phi(|k + p_1|, \theta) \phi(|p_1|, \theta) \right|^2.
\]

(2.48)

For the scalar field in the system we are considering (de Sitter space followed by radiation domination) the function \(a(\tau) \phi(k, \tau)\) is increasing during the inflationary period and oscillating during the following radiation dominated phase. As a consequence, the integral in \(d\theta\) in the equation above will be dominated by the later times. After the end of inflation, when the modes are evolving adiabatically, the mode functions are given by

\[
a(\tau) \phi(k, \tau > -H^{-1}) = \frac{\alpha(k)}{\sqrt{2\omega_k}} e^{-i \int \omega_k d\tau} + \frac{\beta(k)}{\sqrt{2\omega_k}} e^{i \int \omega_k d\tau},
\]

(2.49)

where \(\alpha\) and \(\beta\) are the Bogolyubov coefficients. In the regime of adiabatic evolution \(|\omega'| \ll \omega^2\) one has \(\int d\theta e^{i \int \omega dr} \simeq (i \omega)^{-1} e^{i \int \omega dr}\). Using this fact, the integral in \(d\theta\) in eq. (2.48) above can be computed explicitly, yielding a final result which is precisely eq. (B.9), i.e., the same result that we would obtained by directly computing the two point function of the charge density, once we consider that in this section we have not performed any normal ordering. We thus conclude that the result (2.31) is consistent with current conservation.

2.5 Comparison with existing results

In this section, we examine how our results compare and differ from those previously found [16-18].
Comparison with references [36, 37]. The analyses presented in references [36] and [37] are rather similar to each other. We refer here to the notation of [37]. The charge density operator is defined in terms of two real fields that correspond respectively to the real and the imaginary part of \( \phi \), and reads

\[
\hat{\rho}(\eta, x) = \hat{\Phi}_2 \partial_\tau \hat{\Phi}_1 - \hat{\Phi}_1 \partial_\tau \hat{\Phi}_2,
\]

so that the charge variance reads

\[
\langle \rho(x) \rho(x') \rangle = 2e^2 \left[ G(x, x') \frac{\partial^2}{\partial \tau \partial \tau'} G(x, x') - \frac{\partial}{\partial \tau} G(x, x') \frac{\partial}{\partial \tau} G(x, x') \right],
\]

with

\[
G(x, x') = \langle \hat{\Phi}_1(x) \hat{\Phi}_1(x') \rangle = \langle \hat{\Phi}_2(x) \hat{\Phi}_2(x') \rangle = \int \frac{d^3k}{(2\pi)^3} \phi_k(\tau) \phi_k^*(\tau') e^{ik \cdot (x-x')}.
\]

The main difference with respect to our formalism lies in the mode functions used in eq. (2.52): in [36, 37] those mode functions are computed during inflation, for superhorizon scales. However, as we discuss at the beginning of Section 2.2, in order to calculate particle densities one must be in a regime where the concept of particle is well defined, i.e., in a regime where the energy of the quanta of the relevant states are evolving adiabatically. This is not the case for superhorizon modes during inflation, so that the direct use of the super-horizon mode functions in [36, 37] is not appropriate for the quantities we are interested in.

Note that the charge variance found in [37] scales as \( R^{-4} \), see eq. (8) in that paper, whereas that found in [36] scales as \( R^{-1} \), see eq. (3.19) in that paper. This difference can be seen to originate from the ambiguity in evaluating the divergent integral eq. (3.18) of [36], where \( \int d^3k k e^{ik \cdot x} \), is evaluated to scale as \( H^3/|x| \), whereas [37] evaluates the same integral by looking only at the scaling at small momenta \( \int d^3k k e^{ik \cdot x} \sim 1/|x|^4 \). Again, this fact emphasizes the need for a renormalization procedure that leads to finite quantities.

Reference [37] computes the baryon variance instead of the charge variance, but other than this the calculation is almost identical.
Comparison with reference [35]. In this paper the charge variance is calculated in a fashion that is closer to ours (and that follows the derivation of [34]), since it emphasizes the need to compute quantities related to number densities after the end of inflation, when the energies of the relevant modes are evolving adiabatically. The main difference with respect to our calculation is the absence of normal ordering of the operator whose expectation value should be computed. In [35] equation (2.52) is found to be given by

\[
G(x, x') = \int \frac{d^3k}{(2\pi)^3} e^{-ik\cdot(x-x')} \left[ |\alpha_k|^2 g_k(\tau)g_k^*(\tau') + |\beta_k|^2 g_k^*(\tau)g_k(\tau') \right. \\
+ \alpha_k\beta_k^* g_k(\tau)g_k(\tau') + \alpha_k^*\beta_k g_k^*(\tau)g_k^*(\tau') \right],
\]

(2.53)

where \(g_k(\tau)\) is the mode function after the end of inflation that converges to positive frequency modes only in the future, i.e., in the adiabatic regime. The main difference with respect to our calculation lies in the fact that [35] does not normal order the adiabatic operators with respect to the adiabatic vacuum. In fact, the charge variance obtained in [35] coincides with the expression (B.10) that one obtains in our formalism if we do not perform normal ordering, as one can see from eq. (3.12) in that paper. Normal ordering changes the \(|\alpha_k|^2\) to a \(|\beta_k|^2\) in eq. (2.53), which if done results in an unambiguously finite result for \(G(x, x')\) since for large momentum \(\beta_k\) provides a suppression at large \(k\), while \(\alpha_k\) goes to unity.

To see explicitly that the result in [35] is divergent in the ultraviolet, we note that in this regime the mode functions \(g_k(\tau)\) must go as \(g_k(\tau) \to e^{-ik\tau}/\sqrt{2k}\), and \(\alpha_k \to 1, \beta_k \to 0\). In this regime, eq. (3.8) of [35] yields

\[
\langle \rho(x) \rho(y) \rangle \simeq \frac{e^2}{4} \int \frac{dp \, dq \, (p - q)^2}{(2\pi)^6 \, pq} \, e^{i(p+q)(x-y)}.
\]

(2.54)
Changing variables to \( q = k - p \) this can be rewritten as

\[
\langle \rho(x, t) \rho(y, t) \rangle = \int \frac{dk}{4 \pi k^3} e^{i k(x-y)} P(k),
\]

(2.55)

where

\[
P(k) = \frac{e^2}{2} \frac{k^3}{(2\pi)^5} \int \frac{dq}{q|k-q|} (|k-q|-q)^2
\]

(2.56)

that can be seen to diverge in the UV, as the integrand goes as \( q^2 (k \cdot q)^2 / q^4 \) for \( q \gg k \).

We therefore conclude that the non-normal ordered function is divergent in the ultraviolet (even when computed for different values of the coordinates \( x \) and \( y \)). For this reason normal ordering is necessary to give finite, renormalized values of observables such as the charge density.

### 2.6 Constraints from observations

After inflation ends, the charged fermions and scalars considered in Section 2.2 will decay into ordinary matter. However, since electric charge is conserved, the charge density produced during inflation will not be affected. Reference [21] has shown that primordial charge fluctuations are associated to magnetic fields and to vorticity. This is due to the fact that in the post-inflationary Universe the conductivity is very large. As a consequence, as soon as a charge excess and a charge deficit are in causal contact with each other, an electric current annihilates them. The electric current is however associated to a magnetic field and an anisotropy of the metric. Therefore, observational constraints on vorticity and on the intensity of cosmological magnetic fields impose then an upper

\[6\]

For the same reason, the presence of an exponential cutoff \( \sim e^{-k^2 R^2 / 2} \) in eq (2.3) is not sufficient to guarantee the finiteness of the two point function, since, in the absence of normal ordering, the power spectrum \( P_\rho \) that appears in that equation is in itself a divergent quantity.
bound on the charge density in the Universe. To see how the values of $\rho_R$ derived above compare to the constraints of [21], we define the quantity

$$y_R = \frac{\sqrt{\langle \rho_R^2 \rangle}}{e n_B},$$  

(2.57)

where $n_B$ is the number density of baryons, $n_B \simeq 1.5 \times 10^{-10} T^3$. The bound [21] depends somehow on the spectral index of the magnetic field, but reads approximately $y_{R,.1\,h^{-1}\,\text{Mpc}} \lesssim 10^{-26}$. We will assume, as we did above, that reheating is instantaneous.

### 2.6.1 Fermions

We insert eq. (2.15) into eq. (2.3), we use the fact that $\rho_R$ scales as the inverse of the volume element, and that $R$ in eq. (2.3) is a comoving distance. Assuming $g_* \simeq 10^2$ at the time of reheating, setting $T = T_0 \simeq 3 \times 10^{-4} \text{ eV}$, and taking $R \simeq .1 \, h^{-1} \, \text{Mpc}$, we find

$$y_{R,.1\,h^{-1}\,\text{Mpc}} \simeq 3 \times 10^{-33} \sqrt{f f(m/H)} \left( \frac{H}{9 \times 10^{13} \text{ GeV}} \right)^{3/4},$$  

(2.58)

where we have normalized $H$ to its maximum possible value, that is determined by the non-observation of tensor modes in the CMB. Fermions fall short of the constraint by at least 7 orders of magnitude.

### 2.6.2 Scalars

For massive scalars with a “long” inflation ($N_{\text{Tot}} \gg H^2/m^2$), an analogous computation yields

$$y_{R,.1\,h^{-1}\,\text{Mpc}} \simeq 4 \times 10^{-32} \left( \frac{H}{9 \times 10^{13} \text{ GeV}} \right)^{3/4} \left( \frac{H}{m} \right)^{5/4},$$  

(2.59)
so that the constraint $y_{R,1} h^{-1} \text{Mpc} \lesssim 10^{-26}$ is satisfied unless the scalar is very light $m \lesssim 5 \times 10^{-5} H$. For these small values of $m$, however, the condition $N_{\text{Tot}} \gg H^2/m^2$ is easily violated, and we should rather use eq. (2.35) to compute $y_R$, yielding the bound $m \gtrsim 10^{-23} \log^2 (m/\Lambda_{IR}) H$. If this bound is violated, however, $m$ will be so small that it is more natural to consider the exactly massless case, that gives

$$y_{R,1} h^{-1} \text{Mpc} \simeq 10^{-20} \left( \frac{H}{9 \times 10^{13} \text{GeV}} \right) \sqrt{N_{\text{Tot}}}, \quad (2.60)$$

that, for the maximal allowed value of $H$, exceeds the observational limit by at least 7 orders of magnitude even in the case of short inflation, $N_{\text{Tot}} = \mathcal{O}(10^2)$.

### 2.7 Conclusions

If one or more electrically charged species have a mass smaller than the Hubble parameter during inflation, then our Universe will typically carry a net electric charge. We have found that each species whose mass is of the order of the Hubble parameter contributes a charge density that is $5 \div 7$ order of magnitude below the observational limits. Very light scalars, however, can contribute much more charge density, and in the limit of massless scalars the resulting charge density can exceed by seven (or more, depending on the duration of inflation) orders of magnitude the constraints of [21], unless the Hubble parameter during inflation is well below the “high scale inflation” regime $H \simeq 10^{13}$ GeV.

We should point out that our analysis concerns the simpler regime of constant mass particles during inflation with constant Hubble parameter. While it is straightforward to extend our conclusions to the case of adiabatically evolving $m$ or $H$, it would be especially interesting to consider the case where the parameters in the theory are rapidly evolving, for instance as a consequence of a phase transition. Finally, it would be interesting to
study whether charged scalars that are experiencing a period of tachyonic evolution can generate large charge fluctuations.
CHAPTER 3

PRIMORDIAL GRAVITATIONAL WAVE PRODUCTION

3.1 Introduction

Inflation generates a isotropic, homogeneous, flat Universe with a spectrum of scalar perturbations. On the top of this, inflation also produces a spectrum of primordial gravitational waves (PGWs) – see [43] for a recent review. The contribution to the graviton power spectrum produced by a pure de Sitter expansion with constant expansion rate $H$ is [44]

$$P_T^{\text{vacuum}} = \frac{2}{\pi^2} \frac{H^2}{M_p^2}. \quad (3.1)$$

The above relation provides a simple and yet powerful prediction of inflation, which allows us to connect the energy scale of inflation to an observable quantity.

While the stochastic background amplitude (3.1) of the spectrum of PGWs generated during inflation is model dependent and might be too small to be observable, the detection of PGWs through the Cosmic Microwave Background would certainly represent a major result in support of inflation. The current upper bound on tensor modes produced during inflation for a single field model is provided by the BICEP/Keck collaboration, that, after including other constraints from cosmological measurements, finds the limit $r < .07$ [45],

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where \( r \) is the tensor-to-scalar ratio defined as \( r \equiv \mathcal{P}_T / \mathcal{P}_C \simeq 4.5 \times 10^8 \mathcal{P}_T \). Future CMB experiments aim at pushing this limit further. In particular, the next generation CMB-S4 experiment aims at a tensor-to-scalar ratio sensitivity of \( r \sim 10^{-4} \) [46]. Direct detection, in the near future, of the stochastic PGW background generated during inflation from amplification of vacuum fluctuations is unlikely due to CMB constraints [45] which yield an upper bound \( \Omega_{GW} h^2 \lesssim 10^{-15} \) on the energy density of PGWs. Far future experiments such as BBO or DECIGO, however, aim at sensitivities of the order \( \Omega_{GW} h^2 \sim 10^{-15} - 10^{-17} \) [47, 48].

There has been an increasing interest in the possibility of disentangling the value of \( r \) from the energy scale of inflation by adding new sources of tensor modes. Such an interest was partly motivated by an early belief [49] that models of inflation in String Theory generally take place at such low energies that \( r \) is small and unobservable. Moreover, alternative mechanisms producing gravitational waves lead in general to a phenomenology that is much richer than that of the “standard” PGWs generated by the amplification of vacuum fluctuations, which have a featureless, slightly red power spectrum, do not violate parity, and do not present any detectable nongaussianities. In particular, models where the inflaton is coupled to gauge fields through a parity-violating interaction have been shown to be able to generate a spectrum of PGWs where all those properties of vacuum tensors are violated to some degree [50–63]. Reference [64] has considered the case where chiral fermions are sourcing PGWs. The possibility that the PGW spectrum shows some features implies, in particular, that those PGWs might even be directly detectable by interferometers, as first proposed in [65] and also discussed in [51, 53, 54, 61] (the work [66] refers to much of the literature on this topic).

Models generating additional tensor modes usually assume the existence of a sector whose finite momentum modes are for some reason excited during inflation and act as a classical source of tensors [65, 67]. One the simplest and most studied systems where a sector gets
excited during inflation is that of a scalar field $\chi$ that interacts with the inflaton $\phi$ through the coupling [90, 92]

$$\mathcal{L}_{\phi\chi} = -\frac{g^2}{2} (\phi - \phi_*)^2 \chi^2,$$  \hspace{1cm} (3.2)

with $\phi_*$ a constant. If, as is the case during inflation, the spatial gradients of $\phi$ are negligible, the coupling (3.2) can be seen as an effective mass $m_{\chi} = g |\phi - \phi_*|$ for $\chi$. When $m_{\chi}$ crosses zero (that is, when $\phi$ crosses $\phi_*$), quanta of $\chi$ with momenta up to $\sim \sqrt{g |\dot{\phi}|}$ are excited [70–72]. Those quanta act in their turn as a source of gravitational waves, whose amplitude was first computed in [65, 67] and was found not to be competitive with that of the PGWs generated by the amplification of vacuum fluctuations, eq. (3.1). More specifically, by choosing the coupling $g = 1$ to maximize the effect, reference [65] found that the tensor-to-scalar ratio $r_{\text{sourced}}$ of the induced tensors was satisfying the condition

$$\frac{r_{\text{sourced}}}{r_{\text{vacuum}}} \lesssim 5 \times 10^{-7} \left(\frac{r_{\text{vacuum}}}{.07}\right),$$  \hspace{1cm} (3.3)

which was leading to a small and unobservable $r_{\text{sourced}} \lesssim 10^{-8}$ even for the largest allowed $r_{\text{vacuum}} \simeq .07$.

The fact that the coupling (3.2) does not induce a sufficiently large amplitude of gravitational waves was interpreted [53] as a consequence of the fact that, after crossing 0, the value of $m_{\chi}$ obtained from eq. (3.2) starts growing again, rapidly turning the excited modes of $\chi$ into nonrelativistic ones, which are a very inefficient source of gravitational waves. Based on this observation, in the present paper we will consider gravitational waves produced by a scalar $\chi$ that becomes massless during inflation through its coupling to a secondary field $\sigma$ and stays massless afterwards (reference [67] studied, in a construction different from ours, the case where the mass $m_{\chi}$ converged to a constant
after the event of particle creation). In our model the mass of the field $\chi$ is controlled by a third field $\sigma$ that undergoes symmetry restoration as a consequence of the dynamics of the inflaton. The mass of $\chi$ linearly decreases during its early evolution parameterized by a mass term $-\Lambda_\chi^3(t - t_*)$, and then becomes massless from the time $t_*$ through the end of inflation.

After subtracting unphysical divergences and applying appropriate constraints from CMB observations, we find that for a single scalar field $\chi$ the value of $r_{\text{sourced}}$ is subdominant with respect to the vacuum contribution $r_{\text{vacuum}}$ and can be at most of the order of $\sim 10^{-5}$. This value can be boosted by a factor $N_\chi$ equal to the number of $\chi$ species.

On the other hand, at the smaller scales probed by interferometers, where we can ignore the constraints that originate from CMB observations, we find an absolute upper bound on the energy density of gravitational waves of $\Omega_{GW} h^2 \lesssim 10^{-12}$ (which again can be enhanced by a factor $N_\chi$) which is obtained by saturating a number of inequalities. For “natural” choices of parameters, however, we expect to find values of $\Omega_{GW} h^2$ are a few orders of magnitude smaller. For comparison, amplitudes of the order of $\Omega_{GW} h^2 \sim 10^{-13}$ would be detectable by LISA [66].

This chapter is organized as followed. In section 3.2, we discuss the model for both scalars $\chi$ and the spectator field $\sigma$ as well as the equations for the gravitational waves. In section 3.3, we discuss and show results for our mechanism in the simpler Minkowski background. We find a GW spectrum that is parametrically the same as what we will find in de Sitter. In section 3.4.1, we calculate the evolution of $\chi$ and $\sigma$ in the de Sitter background and $\chi$’s contribution to the graviton’s power spectrum while also discussing our renormalization procedure for removing the UV-divergence introduced when $\chi$ becomes massless. In section 3.5, we constrain the parameters of our model by imposing both observational constraints such as COBE normalization as well as taking into account backreaction
effects. In section 3.6, we discuss how our results might be applied to future experiments such as ELisa or CMB-S4. Finally in section 3.7, we summarize our results.

### 3.2 Set Up

We examine graviton production in a $3 + 1$ dimensional FLRW Universe with metric

$$g_{\mu\nu} = a^2(\tau) \left[ -d\tau^2 + (\delta_{ij} + h_{ij}) \, dx^i dx^j \right], \quad (3.4)$$

where $\tau$ is conformal time and $h_{ij}$ is the transverse and traceless tensor which defines the gravitational waves, and whose equation of motion\(^1\) reads

$$h''_{ij} + 2\frac{a'}{a}h'_{ij} - \Delta h_{ij} = \frac{2}{M_P^2} \Pi_{ij}^{ab} T_{ab}, \quad (3.5)$$

where $M_P = (8\pi G)^{-1/2}$ is the reduced Planck mass, $\Pi_{ij}^{lm} = \Pi_{ij}^{l} \Pi_{ij}^{m} - \frac{1}{2} \Pi_{ij} \Pi_{lm}$ is the transverse, traceless projector, $\Pi_{ij} = \delta_{ij} - \partial_i \partial_j / \Delta$, and a prime denotes derivatives with respect to conformal time $\tau$.

As we have discussed in the introduction, our goal is to consider a scenario where the mass of a scalar field $\chi$ goes from a nonvanishing to a vanishing value during inflation. In order to realize such a situation, we consider a second field $\sigma$ which controls the mass of $\chi$ and that behaves as order parameter in a phase transition describing a symmetry restoration. More specifically, we will consider a system where a field $\sigma$ and the inflaton $\varphi$ are subject to a potential of the form

$$\mathcal{L}_{\varphi\sigma} = -\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - \frac{\mu}{2} \varphi^2 - \frac{\lambda}{4} \sigma^4 - V(\varphi), \quad (3.6)$$

\(^1\)A derivation of this equation can be found in Appendix C.
where $V(\phi)$ is some flat potential able to support inflation, $\lambda$ is a dimensionless coupling constant, and $\mu$ is a mass dimension-1 coupling constant. The coupling between $\varphi$ and $\sigma$ would generally take the form $\frac{\mu}{2} (\phi - \phi_s) \sigma^2$, where $\phi_s$ is some constant value crossed by the expectation value of $\phi$ during inflation. However, we can always set $\phi_s = 0$ by an appropriate shift of $\phi$.

We will assume without loss of generality that $\dot{\varphi} > 0$, so that the term proportional to $\mu$ in the Lagrangian (3.6) behaves like a negative mass squared term for $\sigma$ at early times, triggering symmetry breaking, while at later times it behaves like a positive mass term, enforcing $\sigma = 0$. More explicitly, for $\varphi < 0$ the minimum of the potential for $\sigma$ is $\sigma_{\min} = \pm \sqrt{\frac{\mu \varphi}{\lambda}}$, while for $\varphi > 0$, the minimum is $\sigma_{\min} = 0$. We will assume that some early inflationary dynamics has chosen one specific minimum, say $\sigma_{\min} = +\sqrt{\frac{\mu \varphi}{\lambda}} > 0$ for the early value of the zero mode of the $\sigma$ field.

Let us now introduce a third field $\chi$, that will be our source of gravitational waves. The field $\chi$ interacts with $\sigma$ through the lagrangian

$$\mathcal{L}_\chi = -\frac{1}{2} \partial_\mu \chi \partial^\mu \chi - \frac{h^2}{2} \sigma^2 \chi^2, \quad (3.7)$$

where $h$, is a dimensionless coupling constant. If $\sigma$ tracks the minimum of its potential (we will see in subsection 3.5.3 under which conditions this requirement is satisfied) then the mass of $\chi$ will be given by

$$m_\chi = \begin{cases} \sqrt{-\frac{h^2 \mu}{\lambda}} \varphi & \text{for } t < t_*, \\ 0 & \text{for } t > t_*, \end{cases} \quad (3.8)$$
where $t_\ast$ corresponds to the time when $\varphi$ crosses 0. If the inflaton evolves under the usual slow-roll conditions then we can model its time evolution around $t_\ast$ as,

$$\varphi(t) \simeq \dot{\varphi}_\ast (t - t_\ast).$$  

(3.9)

We will consider for definiteness the configuration where $V'(\varphi) < 0$ and $\dot{\varphi}_\ast > 0$. The mass of $\chi$ reads

$$m_\chi = \begin{cases} 
\Lambda_\chi^3 \sqrt{t_\ast - t} & \text{for } t < t_\ast, \\
0 & \text{for } t > t_\ast 
\end{cases} \quad \Lambda_\chi^3 \equiv \frac{h^2 \mu}{\lambda} \dot{\varphi}_\ast.  

(3.10)

In the following section, we discuss how the above mechanism generates GWs in the simpler Minkowski background to see how the overall effect works and to see how our results in de Sitter will compare.

### 3.3 Gravitational Wave Production in Minkowski

#### 3.3.1 Set Up in Minkowski

Before detailing how our mechanism works in the more complicated (due to the time dependent background) of de Sitter space, we calculate the exact same quantities of interest (the GW power spectrum) in the time independent background of Minkowski space. We again start by examining graviton production in a $3 + 1$ dimensional spacetime with will be Minkowski whose background metric reads,

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1),$$  

(3.11)
and whose perturbed metric which parameterizes the gravitational wave reads,

$$\delta g_{\mu\nu} = h_{\mu\nu},$$  \hspace{1cm} (3.12)$$

where $h_{ij}$ is traceless ($h_{ij}\delta^{ij} = 0$) and transverse ($\partial_i h^{ij} = 0$). The full metric is given by,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \hspace{1cm} (3.13)$$

The equation of motion for the graviton in Minkowski will read,

$$\ddot{h}_{ij} - \Delta h_{ij} = \frac{2}{M_P^2} \Pi_{ij}^{ab} T_{ab}, \hspace{1cm} (3.14)$$

where $M_P = (8\pi G)^{-1/2}$ is the reduced Planck mass, $\Pi_{ij}^{lm} = \Pi_i^l \Pi_j^m - \frac{1}{2} \Pi_{ij} \Pi^{lm}$ is the transverse, traceless projector, and $\Pi_{ij} = \delta_{ij} - \partial_i \partial_j / \Delta$. Notice that the Hubble expansion has been removed due to a non-expanding background. The source for the graviton will be the same field $\chi$ characterized by the Lagrangian,

$$\mathcal{L}_\chi = -\frac{1}{2} \partial^\mu \chi \partial_\mu \chi - \frac{h^2}{2} m_{\chi}(t)^2 \chi^2, \hspace{1cm} (3.15)$$

where $h$ is a dimensionless coupling constant and $m_{\chi}$ is the time dependent mass of $\chi$ characterized in the same way as before

$$m_{\chi} = \begin{cases} \Lambda^2 \sqrt{t_0 - t} & \text{for } t < t_0 \\ 0 & \text{for } t > t_0 \end{cases}. \hspace{1cm} (3.16)$$

We do not mention in this section nor consider the effects of how exactly the mass of $\chi$ originates since our main impetus is to find out what the power spectrum for the GWs will look. Thus, unlike in de Sitter case where we calculation the evolution of $\sigma$, here will
only consider how $\chi$ evolves.

### 3.3.2 Production of quanta of $\chi$

To simplify formulae in this section we set $t_0 = 0$ in eq. (3.10). We decompose the field $\chi$ as

$$\hat{\chi}(x, t) = \int \frac{d^3p}{(2\pi)^{3/2}} e^{i\mathbf{p} \cdot \mathbf{x}} \left[ \chi_p(t) \hat{a}_p + \chi^{*}_{-p}(t) \hat{a}_{-p}^\dagger \right] ,$$

(3.17)

where the mode functions satisfy

$$\ddot{\chi}_p + [p^2 + m^2_\chi(t)] \chi_p = 0 .$$

(3.18)

Let us first examine for what parameters the field $\chi$ is evolving adiabatically. The parameter relevant for adiabacity is $|\dot{\omega}/\omega^2|$ which is a measure of how fast the field evolves. For $t < 0$, the condition reads,

$$\left| \frac{\dot{\omega}}{\omega^2} \right| = \frac{1}{2} (K^2 - T)^{-3/2} ,$$

(3.19)

where we have introduced the dimensionless variables, $K = \frac{k}{\Lambda}$ and $T = \Lambda t$. The field $\chi$ evolves adiabatically when,

$$\frac{1}{2} (K^2 - T)^{-3/2} \ll 1 ,$$

(3.20)

thus there will always be an initial adiabatic vacuum for the field $\chi$ assuming we go back far enough in the past. The adiabatic condition for $t > 0$ is always satisfied since $\dot{\omega}(t > 0) = 0$, thus ensuring that an adiabatic vacuum exists in the future.

We now write the solution of $\chi$ for $t < 0$ in general as

$$\chi_p(t < 0) = A_k \sqrt{z} H_1^{(1)} \left( \frac{2}{3} z^{3/2} \right) + B_k \sqrt{z} H_1^{(2)} \left( \frac{2}{3} z^{3/2} \right) , \quad z \equiv \frac{p^2}{\Lambda^2} - \Lambda t ,$$

(3.21)
where $H^{(1)}_\nu(z)$ and $H^{(2)}_\nu(z)$ denotes the Hankel function of the first and second kind. To determine the constants $A_k$ and $B_k$ we match the general solution to the WKB solution,

$$
\chi_p^{WKB}(t) = \alpha_p \frac{e^{-i \int \omega(t') dt'}}{\sqrt{2 \omega}} + \frac{\beta_p}{\sqrt{2 \omega}} e^{i \int \omega(t') dt'}, \quad \omega(t) = p^2 - \Lambda^2 t, \tag{3.22}
$$

in the initial adiabatic vacuum ($t \to -\infty$) where $\alpha_p = 1$ and $\beta_p = 0$. We find for the WKB solution

$$
\chi_p^{WKB}(t \to -\infty) \to e^{-i \int \omega(t') dt'} \frac{1}{\sqrt{2 \lambda \sqrt{z}}} e^{i \frac{2}{3} z^{3/2}}, \tag{3.23}
$$

while the general solution becomes,

$$
\chi_p(t \to -\infty) = A_k \sqrt{\frac{1}{2 \lambda \sqrt{z}}} e^{i \frac{2}{3} z^{3/2}} + B_k \sqrt{\frac{1}{2 \lambda \sqrt{z}}} e^{-i \frac{2}{3} z^{3/2}}, \tag{3.24}
$$

and we find

$$
\chi_p(t < 0) = \sqrt{\frac{\pi z}{6 \lambda}} \frac{H^{(1)}_{\frac{1}{2}}}{H^{(1)}_{\frac{1}{2}} \left( \frac{2}{3} z^{3/2} \right)}. \tag{3.25}
$$

The solution of eq. (3.18) for $t > 0$ is simply the massless Klein-Gordon solution, which we write as

$$
\chi_p(t > 0) = \frac{\alpha_p}{\sqrt{2 \lambda}} e^{-i p t} + \frac{\beta_p}{\sqrt{2 \lambda}} e^{i p t}, \tag{3.26}
$$

where the parameters $\alpha_p$ and $\beta_p$ are the Bogolyubov coefficients, which are determined by matching the mode functions and their first derivatives at time $t = 0$, and are given by

$$
\alpha_p = \sqrt{\frac{\pi}{12}} \left( \frac{p}{\lambda} \right)^{3/2} \left( H^{(1)}_{\frac{3}{2}} \left( \frac{2 p^3}{3 \lambda^3} \right) - i H^{(1)}_{-\frac{3}{2}} \left( \frac{2 p^3}{3 \lambda^3} \right) \right),
$$

$$
\beta_p = \sqrt{\frac{\pi}{12}} \left( \frac{p}{\lambda} \right)^{3/2} \left( H^{(1)}_{\frac{3}{2}} \left( \frac{2 p^3}{3 \lambda^3} \right) + i H^{(1)}_{-\frac{3}{2}} \left( \frac{2 p^3}{3 \lambda^3} \right) \right). \tag{3.27}
$$
The number density for $\chi$ is then given by

$$n_\chi = \int \frac{d^3 p}{(2\pi)^3} |\beta_p|^2 = \frac{27\Lambda^3}{96\pi} \int_0^\infty dx \ x^5 \left( H_1^{(1)} (x^3) + i H_2^{(1)} (x^3) \right) \left( H_1^{(2)} (x^3) - i H_2^{(2)} (x^3) \right)$$

$$= \frac{\Lambda^3}{48\sqrt{3}\pi^2},$$

(3.28)

whereas the energy density reads

$$\rho_\chi = \int \frac{d^3 p}{(2\pi)^3} p |\beta_p|^2 \simeq 8 \times 10^{-4} \Lambda^4.$$  

(3.29)

### 3.3.3 Gravitational Wave Amplitude in Minkowski

The equation of motion for the graviton in Minkowski space and the energy-momentum tensor for a scalar reads

$$\ddot{h}_{ij} - \Delta h_{ij} = \frac{2}{M_p^2} \Pi_{ij}^{ab} T_{ab}, \quad T_{ab} = -\frac{2}{\sqrt{-g}} \delta S_{\chi}^{\mu\nu} = \partial_a \chi \partial_b \chi - g_{ab} \left[ \frac{1}{2} \partial_\sigma \chi \partial^\sigma \chi + V(\chi) \right].$$

(3.30)

The projection operator will project away terms proportional to $\delta_{\alpha\beta}$ in $T_{ab}$, thus the relevant portion of the energy-momentum tensor will be,

$$T_{ab} = \partial_a \chi \partial_b \chi + h_{ab} \left[ \frac{1}{2} (\dot{\chi})^2 - \frac{1}{2} (\nabla \chi)^2 - V(\chi) \right].$$

(3.31)

Going to momentum space, we find

$$\ddot{h}_{ij}(p) + p^2 h_{ij}(p) = \frac{2}{M_p^2} \Pi_{ij}^{ab}(p) T_{ab}(p),$$

(3.32)

$$T_{ij}^{(1)}(p) = -\int \frac{d^3 k}{(2\pi)^3/2} (k_i)(p - k)j \chi(k) \chi(p - k),$$

(3.33)
\[ T_{ij}^{(2)}(p) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[ \frac{1}{2} \int \frac{d^3k'}{(2\pi)^{3/2}} \left( h_{ij}(p - k - k') \left[ \dot{\chi}(k)\dot{\chi}(k') + (k \cdot k')\chi(k)\chi(k') \right] - h_{ij}(p - k)V(\chi(k)) \right) \right] \]  
\[ \text{where we have split the energy-momentum tensor into two parts, one proportional to} \ h_{ij} \ \text{and one that is not. We can now split the solution for the graviton into a vacuum solution,} \]
\[ \dot{h}_{ij}^{(0)}(p, t) + p^2 h^{(0)}_{ij}(p, t) = 0, \]  
\[ \text{and a perturbed solution,} \]
\[ h_{ij}^{(1)}(p, t) = \frac{2}{M_p^2} \int dt' G_p(t, t') \Pi_{ab}^{ij}(p) T_{ab}(p, t'), \quad G_p(t, t') = \frac{\sin(p(t - t'))}{p} \Theta(t - t'), \]  
\[ \text{where} \ G_p(t, t') \ \text{is the retarded propagator which solves the homogeneous form of the graviton equation of motion. The equation for the graviton then reads,} \]
\[ h_{ij}(p, t) = h_{ij}^{(0)}(p, t) + h_{ij}^{(1)}(p, t), \]  
\[ \text{where for the perturbed solution we will further split up the relevant power spectrum by using either} \ T_{ij}^{(1)} \ \text{or} \ T_{ij}^{(2)}. \ \text{The equal-time correlator for the graviton will read,} \]
\[ \langle \dot{h}_{ij}(k, t) \dot{h}_{ij}(k', t) \rangle = \langle \dot{h}_{ij}^{(0)}(k, t) \dot{h}_{ij}^{(0)}(k', t) \rangle + \langle \dot{h}_{ij}^{(0)}(k, t) \dot{h}_{ij}^{(1)}(k', t) \rangle + \langle \dot{h}_{ij}^{(1)}(k, t) \dot{h}_{ij}^{(0)}(k', t) \rangle + \langle \dot{h}_{ij}^{(1)}(k, t) \dot{h}_{ij}^{(1)}(k', t) \rangle. \]  
\[ \text{The power spectrum in general reads,} \]
\[ \langle h_{ij}(k)h_{ij}(k') \rangle = \frac{2\pi^2}{k^3} \delta^{(3)}(k + k') P_T(k) \]
\[ = \frac{2\pi^2}{k^3} \delta^{(3)}(k + k') \left[ P_0^0(k) + P_0^1(k) + P_1^0(k) + P_1^1(k) \right]. \]
\( \mathcal{P}_T^{00}(k) \) corresponds to the vacuum contribution to the power spectrum, \( \mathcal{P}_T^{01}(k) \) and \( \mathcal{P}_T^{10}(k) \) correspond to the cross terms of the vacuum solution with \( T_{ij}^{(2)} \), and finally \( \mathcal{P}_T^{11}(k) \) corresponds to the product of \( T_{ij}^{(1)} \) with itself. We solve for the above power spectra in the following sections.

### 3.3.3.1 Vacuum Power Spectrum, \( \mathcal{P}_T^{00} \)

We first calculate the vacuum contribution to the power spectrum. The vacuum power spectrum will read,

\[
\mathcal{P}_T^{00}(k) = \frac{k^3}{2\pi^2 \delta^{(3)}(k + k')} \langle \hat{h}_{ij}^{(0)}(k, t) \hat{h}_{ij}^{(0)}(k', t) \rangle ,
\]

where we decompose the graviton along a set of creation/annihilation operators with their associated mode functions as,

\[
\hat{h}_{ij}^{(0)}(p, t) = \frac{2}{M_P} \sum_{\lambda = +, \times} \left[ v_p(t, \lambda) e_{ij}(\hat{p}, \lambda) \hat{a}_p(\lambda) + v_p^*(t, \lambda) e_{ij}^*(\hat{-p}, \lambda) \hat{a}_p^+(\lambda) \right] ,
\]

where \( e_{ij}(\hat{p}, \lambda) \) are a basis of helicity vectors and \( v_p(t, \lambda) \) are the vacuum mode functions satisfying,

\[
\ddot{v}_p + p^2 v_p = 0 , \quad v_p(t) = \frac{e^{-ip t}}{\sqrt{2p}} .
\]

Plugging in the above, we find the vacuum correlator yielding,

\[
\langle \hat{h}_{ij}^{(0)}(k, t) \hat{h}_{ij}^{(0)}(k', t) \rangle = \frac{4 \delta^{(3)}(k + k')}{M_P^2} |v_k|^2 \sum_\lambda e_{ij}(\hat{p}, \lambda) e_{ij}(\hat{-p}, \lambda) = \frac{4 \delta^{(3)}(k + k')}{k M_P^2} .
\]

Which finally translates into the vacuum power spectrum of,

\[
\mathcal{P}_T^{00}(k) = \frac{2k^2}{\pi^2 M_P^2} .
\]
This will presumably be the main contribution to the power spectrum with the remaining terms providing subdominant corrections. By the way, the above result demonstrates that (just like any other field) the graviton naturally has fluctuations even in the routine spacetime of Minkowski.

### 3.3.3.2 \( \mathcal{P}_{11}^{11} \) Power Spectrum

Now, we calculate the higher order terms to the power spectrum which will be relevant for us to compare with our results in de Sitter. First, we calculate \( \mathcal{P}_{11}^{11} \),

\[
\mathcal{P}_{11}^{11}(k) = \frac{k^3}{2\pi^2 \delta^{(3)}(k + k')} \langle \hat{h}_{ij}^{(1)}(k, t) \hat{h}_{ij}^{(1)}(k', t) \rangle .
\] (3.45)

The main task will be to calculate the correlator of the perturbed solution of the graviton. Using equations 3.33 and 3.36, we find

\[
\langle \hat{h}_{ij}^{(1)}(k, t) \hat{h}_{ij}^{(1)}(k', t) \rangle = \frac{4}{M_p^4} \int dt' G_k(t, t') \int dt'' G_{k'}(t, t'') \Pi_{ij}^{ab}(k) \Pi_{ij}^{cd}(k') \times \\
\times \int \frac{d^3p d^3p'}{(2\pi)^3} p_a(k_b - p_b)p'e(k_d' - p_d') \langle \hat{\chi}(p, t') \hat{\chi}(k - p, t') \hat{\chi}(p', t'') \hat{\chi}(k' - p', t'') \rangle .
\] (3.46)

To properly calculate particle production in a time-dependent background we use the Bogolyubov formalism as presented in Appendix A, and decompose \( \chi \) using adiabatic mode functions and operators,

\[
\hat{\chi}(p, t) = \tilde{\chi}_p(t) \hat{a}(p) + \tilde{\chi}_p^*(-p) \hat{a}^\dagger(-p), \quad \tilde{\chi}_p(t) = \frac{1}{\sqrt{2p}} e^{-ipt}, \quad \hat{a}(p) = \alpha_p \hat{a}(p) + \beta_p^\dagger \hat{a}^\dagger(-p),
\] (3.47)

where \( \alpha_p \) and \( \beta_p \) are the Bogolyubov coefficients which diagonalize the Hamiltonian which we evaluate when \( \chi \) is massless. Using Wick’s Theorem to reduce the four-point function
of $\chi$ we find,

$$\langle \chi(p, t') \chi(k - p, t') \chi(p', t'') \chi(k' - p', t'') \rangle = \langle \chi(p, t') \chi(k - p, t') \chi(p', t'') \rangle \langle \chi(k' - p', t'') \chi(p', t'') \rangle + \langle \chi(p, t') \chi(k' - p', t'') \rangle \langle \chi(k - p, t') \chi(p', t'') \rangle + \langle \chi(p, t') \chi(k - p', t''') \rangle \langle \chi(p', t') \chi(p', t'') \rangle.$$  \hspace{1cm} (3.48)

The first term produces a disconnected term, $\delta^{(3)}(k)\delta^{(3)}(k')$, which we remove \footnote{We actually want the graviton variance $\sigma_h^2 = \langle (h - \langle h \rangle)(h - \langle h \rangle) \rangle$ so that the $\langle h \rangle \langle h \rangle$ term also produces a disconnected term which cancels with this one.} and the remaining two terms are equivalent. The normal-ordered scalar correlator reads,

$$\langle :\chi(p, t')\chi(q, t'') : \rangle = \delta^{(3)}(p + q) \left[ |\beta_p|^2 \cos(p(t' - t'')) + \text{Re} \left[ \alpha_p \beta_p^* e^{-ip(t' + t'')} \right] \right].$$  \hspace{1cm} (3.49)

It is important, and will be relevant for the discussion in section 3.4.4.2 where we work in de Sitter space, to note that the prescription (3.49) above is equivalent to setting

$$\langle \chi(p, t')\chi(q, t'') \rangle = \delta^{(3)}(p + q) \left[ \chi(p, t') \chi(p, t'') - \tilde{\chi}(p, t') \tilde{\chi}(p, t'') \right],$$  \hspace{1cm} (3.50)

where $\tilde{\chi}(p, t) = e^{-ipt \sqrt{2p}}$ corresponds to the mode function in absence of particle creation, $\Lambda \to 0$.

We can reduce the projection operators and momentum summation as,

$$\Pi_{ij}^{ab}(k)\Pi_{cd}^{bc}(k')p_a(k_b - p_b)(k_c - p_c)p_d = \frac{1}{2} \left( p^2 - \frac{(p \cdot k)^2}{k^2} \right)^2.$$  \hspace{1cm} (3.51)
The graviton two-point function now reads,

\[ \langle h^{(1)}_{ij}(k, t) h^{(1)}_{ij}(k', t) \rangle = \frac{\delta^{(3)}(k + k')}{2\pi^3 k^2 M_p^4} \int_0^t dt' \sin(k(t - t')) \int_0^t dt'' \sin(k(t - t'')) \int d^3 p \frac{p^4 \sin^4 \theta}{p|k - p|} \times \]

\[ \left[ |\beta_{k-p}|^2 \cos(|k - p|(t' - t'')) + \text{Re} \left[ \alpha_{k-p} \beta_{k-p}^* e^{-i(k-p)(t'+t'')} \right] \right] \times \]

\[ \left[ |\beta_p|^2 \cos(p(t' - t'')) + \text{Re} \left[ \alpha_p \beta_p^* e^{-iP(y'+z')} \right] \right] , \]  

(3.52)

where we have used \( k \cdot p = kp \cos \theta \). We can convert to dimensionless variables: \( x = kt \), \( y = kt' \), \( z = kt'' \), \( P = \frac{|k - p|}{k} \), and \( Q = \frac{|k - p|}{k} \), and using

\[ \int_0^\pi d\theta \sin^5 \theta = \int_{|P-1|}^{P+1} \frac{dQ}{P} \frac{Q}{P} \left( 1 - \frac{(P^2 - Q^2 + 1)^2}{4P^2} \right) , \]  

(3.53)

the graviton correlator yields,

\[ \langle h^{(1)}_{ij}(k, t) h^{(1)}_{ij}(k', t) \rangle = \frac{k^4 \delta^{(3)}(k + k')}{\pi^2 M_p^4} \int_0^x dy \sin(x - y) \int_0^x dz \sin(x - z) \times \]

\[ \int_0^{\infty} dP \int_{|P-1|}^{P+1} dQ \left( 1 - \frac{(P^2 + 1 - Q^2)^2}{4P^2} \right) \left[ |\beta_P|^2 \cos(P(y - z)) + \text{Re} \left[ \alpha_P \beta_P^* e^{-iP(y+z)} \right] \right] \times \]

\[ \left[ |\beta_Q|^2 \cos(Q(y - z)) + \text{Re} \left[ \alpha_Q \beta_Q^* e^{-iQ(y+z)} \right] \right] . \]  

(3.54)

and so we finally arrive at the expression for the contribution, \( \mathcal{P}_{11} \), to the power spectrum,

\[ \mathcal{P}_{11}(k) = \frac{k^4}{2\pi^3 M_p^4} \int_0^x dy \sin(x - y) \int_0^x dz \sin(x - z) \int_0^{\infty} dP \int_{|P-1|}^{P+1} dQ \left( 1 - \frac{(P^2 + 1 - Q^2)^2}{4P^2} \right) \left[ |\beta_P|^2 \cos(P(y - z)) + \text{Re} \left[ \alpha_P \beta_P^* e^{-iP(y+z)} \right] \right] \times \]

\[ \left[ |\beta_Q|^2 \cos(Q(y - z)) + \text{Re} \left[ \alpha_Q \beta_Q^* e^{-iQ(y+z)} \right] \right] . \]  

(3.55)
Figure 3.1: Numerical evaluation of the power spectrum for $\mathcal{P}^{11}$ for $\lambda = 10$ (red), $\lambda = 20$ (orange), and $\lambda = 30$ (blue). The y-axis is in log scale.

We can further simplify notation by introducing the dimensionless variable $\lambda = \frac{1}{k}$,

$$
\mathcal{P}^{11}_T(k) = \frac{k^4}{M_p^4} \frac{1}{2 \pi^4} \int_0^\infty dP \int dQ \frac{P^{P+1}}{P-1} \left( 1 - \left( \frac{P^2 + 1 - Q^2}{4P^2} \right)^2 \right)^2 \int_0^x dy \int_0^x dz \sin(x-y) \sin(x-z) \times \left[ |\beta_P|^2 \cos(P(y-z)) + \text{Re} \left[ \alpha_P \beta_P^* e^{-iP(y+z)} \right] \right] \left[ |\beta_Q|^2 \cos(Q(y-z)) + \text{Re} \left[ \alpha_Q \beta_Q^* e^{-iQ(y+z)} \right] \right] .
$$

We can compute the above integrals numerically. Plots of the above power spectrum (excluding $\frac{k^4}{M_p^4}$) for $\lambda = \frac{1}{k} = 10$, 20, 30 are shown in figure 3.1. The maximum occurs around $x = kt \approx 2.65$. Setting $x = 2.65$, we determine that the numerical integral goes,

$$
\mathcal{P}_h(k) = 1.25 \times 10^{-4} \frac{\Lambda^5}{M_p^4 k}, \quad x = kt \approx 2.65 .
$$

The above result shows that $\mathcal{P}^{11}_T \propto \Lambda^5_X/M_p^4$, in accordance with the result we will obtain on a de Sitter background in the next section.
Finally, in Minkowski space we calculate $\mathcal{P}_{T}^{10}$,

$$
\mathcal{P}_{T}^{10}(k) = \frac{k^3}{2\pi^2 \delta^{(3)}(k + k')} \langle \hat{h}_{ij}^{(1)}(k, t) \hat{h}_{ij}^{(0)}(k', t) \rangle ,
$$

(3.58)

where for the perturbed graviton we use 3.34 we leads to the correlator being,

$$
\langle \hat{h}_{ij}^{(1)}(k, t) \hat{h}_{ij}^{(0)}(k', t) \rangle = \frac{1}{M_{p}^2} \int dt' G_{k}(t, t') \Pi_{ij}^{ab}(k) \int \frac{d^3 \mathbf{p} d^3 \mathbf{p}'}{(2\pi)^3} \langle h_{ab}^{(0)}(k - \mathbf{p} - \mathbf{p}', t') \hat{h}_{ij}^{(0)}(k', t) \rangle
$$

$$
\left[ \langle \dot{\chi}(\mathbf{p}, t') \dot{\chi}(\mathbf{p}', t') \rangle + (\mathbf{p} \cdot \mathbf{p}') \langle \chi(\mathbf{p}, t) \chi(\mathbf{p}', t') \rangle \right].
$$

(3.59)

Since $h_{ij}^{(0)}$ and $\chi$ are uncorrelated, we have been able to split the expectation value into the product $\langle hh \rangle \langle \chi \chi \rangle$ where we can calculate the correlators independently. We first solve for the graviton’s correlator using the same decomposition we had before,

$$
\hat{h}_{ij}^{(0)}(\mathbf{p}, t) = \frac{2}{M_{p}} \sum_{\lambda} \left[ v_{\mathbf{p}}(t, \lambda) e_{ij}(\hat{\mathbf{p}}, \lambda) \hat{a}_{\mathbf{p}}(\lambda) + v_{-\mathbf{p}}(t, \lambda) e_{ij}(\hat{\mathbf{p}}, \lambda) \hat{a}^\dagger_{-\mathbf{p}}(\lambda) \right],
$$

(3.60)

where $e_{ij}(\hat{\mathbf{p}}, \lambda)$ is the helicity$–\lambda$ projector. This will lead to the correlator reading,

$$
\langle h_{ab}^{(0)}(\mathbf{q}, t') h_{ij}^{(0)}(\mathbf{q}', t) \rangle = \frac{4}{M_{p}^2} \sum_{\sigma = +, x} v_{\mathbf{q}}(t', \gamma) v^{*}_{-\mathbf{q}}(t, \sigma) e_{ab}(\hat{\mathbf{q}}, \gamma) e_{ij}(\hat{\mathbf{q}}, \gamma) \langle \hat{a}_{\mathbf{q}}(\gamma) \hat{a}_{-\mathbf{q}}^\dagger(\sigma) \rangle
$$

$$
= \frac{2\delta^{(3)}(\mathbf{q} + \mathbf{q}')}{M_{p}^2 q} e^{-i q(t' - t)} \sum_{\sigma = +, x} e_{ab}(\hat{\mathbf{q}}, \sigma) e_{ij}(\hat{\mathbf{q}}, \sigma)
$$

$$
= \frac{2\delta^{(3)}(\mathbf{q} + \mathbf{q}')}{M_{p}^2 q} e^{-i q(t' - t)} \Pi_{ab}^{ij}(\hat{\mathbf{q}}),
$$

(3.61)
where we have used the relation,

\[
\sum_{\sigma=+,x} e_{ab}(\hat{q}, \sigma) e_{ij}(\hat{q}, \sigma) = \Pi_{ab}^{ij}(\hat{q}) .
\]  

(3.62)

Now that we have the graviton’s correlator, we evaluate \(\chi\)’s correlators. The second one we already found in the previous section while the correlator with time derivatives will read,

\[
\langle : \dot{\chi}(p, t') \dot{\chi}(q, t') : \rangle = p \delta^{(3)}(p + q) \left[ |\beta_p|^2 - \text{Re} \left[ \alpha_p \beta_p^* e^{-2ipt'} \right] \right] .
\]

(3.63)

Plugging all of the above into the graviton’s correlator we find,

\[
\langle \hat{h}^{(1)}_{ij}(k, t) \hat{h}^{(0)}_{ij}(k', t) \rangle = -\frac{\delta^{(3)}(k + k')}{2\pi^4 k M_p^4} \int dt' \ G_k(t, t') e^{ik(t-t')} \Pi_{ij}^{ab}(k) \Pi_{ij}^{ab}(k) \times
\]

\[
\int d^3 p \ p \text{Re} \left[ \alpha_p \beta_p^* e^{-2ipt'} \right] ,
\]

(3.64)

where we can use the following relation to reduce the projection operators as,

\[
\Pi_{ij}^{ab}(k) \Pi_{ij}^{ab}(k) = 3 ,
\]

(3.65)

which leads us to the expression for the power spectrum,

\[
P_{10}^{10}(k) = -\frac{3k^3}{4\pi^2 M_p^4} \int_0^t dt' \ \sin(k(t-t')) e^{ik(t-t')} \int d^3 p \ p \text{Re} \left[ \alpha_p \beta_p^* e^{-2ipt'} \right] .
\]

(3.66)

We also need to calculate \(P_{10}^{01}\),

\[
P_{10}^{01}(k) = \frac{k^3}{2\pi^2 \delta^{(3)}(k + k')} \langle \hat{h}^{(0)}_{ij}(k, t) \hat{h}^{(1)}_{ij}(k', t) \rangle ,
\]

(3.67)
Figure 3.2: Numerical evaluation of the power spectrum for $P^{01} + P^{10}$ for $\lambda = 10$ (blue), $\lambda = 20$ (orange), and $\lambda = 30$ (red). The first maximum occurs around $x \approx 2.37$. which will be entirely the same as the above so we simply state the result,

$$ P^{10}_T(k) = -\frac{3k}{4\pi^5 M_P^4} \int_0^t dt' \sin(k(t-t')) e^{-ik(t-t')} \int d^3p \ p \ \text{Re} \left[ \alpha_p \beta_p^* e^{-2ip't'} \right]. \quad (3.68) $$

Combining both $P^{01}_T$ and $P^{10}_T$, we find

$$ P^{10}_T(k) + P^{01}_T(k) = -\frac{3k}{4\pi^5 M_P^4} \int_0^t dt' \sin(2k(t-t')) \int d^3p \ p \ \text{Re} \left[ \alpha_p \beta_p^* e^{-2ip't'} \right], \quad (3.69) $$

where again we switch to dimensionless variables $x = kt$, $P = \frac{p}{k}$, and $\lambda = \frac{A}{k}$,

$$ P^{10}_T(k) + P^{01}_T(k) = \frac{k^4}{M_P^4} \left( -\frac{3}{2\pi^4} \int_0^\infty dP \ P^3 F(x, \lambda, P) \right). \quad (3.70) $$

Examining the UV behavior of the above integral, we find that it diverges logarithmically since in the UV it goes as $\frac{1}{P}$. We integrate up to $P_{UV} = 50\lambda$ to estimate the integral numerically. A plot for $\lambda = 10, 20, 30$ is shown in figure 3.2 and as you can see the first peak for all three values of $\lambda$ occurs around $x \approx 2.37$. Using this value of $x$, we determine
the behavior of quantity in (...) above goes as $1.6 \times 10^{-2} \lambda^3$ the power spectrum is then,

$$P_h(k) = 1.6 \times 10^{-2} \frac{k \Lambda^3}{M_p^3}, \quad x = k t \approx 2.37,$$

where we again note that the above result has the same scaling $\propto \Lambda^3_\chi / M_p^3$ as that obtained in the de Sitter calculation of the next section.

We have calculated our proposed mechanism of producing PGWs in the less taxing Minkowski background. The results obtained in particular $P_{11}^1$ and $P_{10}^\perp$ will be compared to the same calculation in de Sitter. As we will see, the two results will be parametrically the same.

### 3.4 Gravitational Wave Production in de Sitter

#### 3.4.1 Production of gravitational waves

In order to compute the amplitude of gravitational waves produced by quanta of the $\chi$ field, we must first characterize the production of quanta of the field $\chi$ induced by the time-dependence (3.10) of its mass. For our subsequent analysis we will also need to study the fluctuations of the field $\sigma$ which follows a similar calculation to have as $\chi$. The fluctuations of $\sigma$ can be studied using the standard formalism of Bogolyubov coefficients. Since the field $\chi$ stay massless after particle production, on the other hand, its superhorizon modes will not be evolving adiabatically after production of its quanta, and we will need a more subtle analysis, which we will present in subsection 3.4.3.
3.4.2 Production of quanta of $\sigma$

We will assume that the parameters of the model are such that the field $\sigma$ is heavy (in units of the Hubble scale) for most of the evolution of the system. However, when $\varphi$ crosses zero the field $\sigma$ becomes temporarily massless, and quanta of $\sigma$ are created by resonant effects. We decompose $\sigma$ into a homogeneous $\sigma_0(\tau)$ and perturbed $\delta\sigma(x, \tau)$ part

$$\sigma(x, \tau) = \sigma_0(\tau) + \delta\sigma(x, \tau), \quad (3.72)$$

and we further decompose the fluctuations as

$$\delta \hat{\sigma}(x, \tau) = \frac{1}{a(\tau)} \int \frac{d^3p}{(2\pi)^3/2} e^{i\mathbf{p} \cdot \mathbf{x}} \left[ \delta\sigma_p(\tau) \hat{c}_p + \delta\sigma^{*}_p(\tau) \hat{c}^\dagger_p \right], \quad (3.73)$$

where $\hat{c}^{(1)}$ are the ladder operators for $\delta \hat{\sigma}$, and where the equation of motion for the canonically normalized fluctuations reads

$$\delta\sigma''_p + \left[ p^2 - \frac{a''}{a} + m^2_\sigma(\tau)a^2 \right] \delta\sigma_p = 0, \quad (3.74)$$

with $m^2_\sigma(\tau < \tau_*) = -2\mu \varphi(\tau)$ which shows that as $\varphi$ approaches 0 the field $\sigma$ becomes massless and the WKB approximation is not a good one for the evolution of its mode functions. This implies a resonant amplification of the quantum fluctuations of $\sigma$, that we study as it is usual [72] by switching to physical time, approximating $\varphi(t) \simeq \varphi_*(t - t_*)$, and neglecting the expansion of the Universe during the period of nonadiabaticity. In this regime, the equation for the mode functions of the rescaled field $\delta\sigma_p = a^{-\frac{3}{2}}\delta\sigma_c$ reads

$$\delta\ddot{\sigma}_c + \left[ \frac{p^2}{a^2_c} - \Lambda^3_\sigma(t - t_*) \right] \delta\sigma_c = 0, \quad (3.75)$$
where we have defined
\[ \Lambda_3^3 = 2 \mu \dot{\phi} = \frac{2 \lambda}{h^2} \Lambda_3^3. \]  
(3.76)

The assumption that the expansion of the Universe is negligible during the nonadiabatic period is equivalent to
\[ \Lambda_\sigma \gg H, \]  
(3.77)

which also implies that, as stated above, the mass of \( \sigma \) is much larger than the Hubble parameter for most of the time\(^3\). The solution of eq. (3.75) reads
\[
\delta \sigma_c(t < t_*) = \sqrt{\frac{\pi}{6 \Lambda_\sigma}} H^{(1)}_{\frac{2}{3}} \left( \frac{2}{3} z^2 \right), \quad z \equiv \frac{p^2}{a^2 \Lambda_\sigma^2} - \Lambda_\sigma (t - t_*),
\]  
(3.78)

where \( H^{(1)}_\nu(z) \) denotes the Hankel function of the first kind and where we have determined the integration constants assuming that the modes of \( \sigma \) are in their adiabatic vacuum at early times. For \( t > t_* \) the quanta of \( \sigma \) become massive again, and their equation of motion reads
\[
\delta \sigma_c'' + \left[ p^2 - \frac{a''}{a} + \mu a^2 \dot{\varphi} \right] \delta \sigma_c = 0.
\]  
(3.79)

Proceeding as we did for \( t < t_* \), we obtain
\[
\delta \sigma_c(t > t_0) = c_1 \sqrt{\tilde{z}} H^{(1)}_{\frac{2}{3}} \left( \frac{2}{3} \tilde{z}^\frac{1}{2} \right) + c_2 \sqrt{\tilde{z}} H^{(2)}_{\frac{2}{3}} \left( \frac{2}{3} \tilde{z}^\frac{1}{2} \right), \quad \tilde{z} \equiv 2^{2/3} \frac{p^2}{a^2 \Lambda_\sigma^2} + 2^{-\frac{1}{3}} \Lambda_\sigma (t - t_*),
\]  
(3.80)

where \( c_1 \) and \( c_2 \) are determined by matching the solution at \( t = t_* \),
\[
c_1 = i \frac{\pi}{3 \times 2^{1/3}} \sqrt{\frac{\pi}{6 \Lambda_\sigma}} \frac{2}{a^3 \Lambda_\sigma^3} \left[ H^{(1)}_{-\frac{1}{3}} \left( \frac{2}{3} \right) \frac{p^3}{a^3 \Lambda_\sigma^3} H^{(2)}_{-\frac{1}{3}} \left( \frac{4}{3} \right) \frac{p^3}{a^3 \Lambda_\sigma^3} \right] + H^{(1)}_{\frac{1}{3}} \left( \frac{2}{3} \right) \frac{p^3}{a^3 \Lambda_\sigma^3} H^{(2)}_{\frac{1}{3}} \left( \frac{4}{3} \right) \frac{p^3}{a^3 \Lambda_\sigma^3},
\]  
\[ c_2 = -i \frac{\pi}{3 \times 2^{1/3}} \sqrt{\frac{\pi}{6 \Lambda_\sigma}} \frac{2}{a^3 \Lambda_\sigma^3} \left[ H^{(1)}_{-\frac{1}{3}} \left( \frac{4}{3} \right) \frac{p^3}{a^3 \Lambda_\sigma^3} H^{(2)}_{-\frac{1}{3}} \left( \frac{2}{3} \right) \frac{p^3}{a^3 \Lambda_\sigma^3} \right] + H^{(1)}_{\frac{1}{3}} \left( \frac{2}{3} \right) \frac{p^3}{a^3 \Lambda_\sigma^3} H^{(2)}_{\frac{1}{3}} \left( \frac{4}{3} \right) \frac{p^3}{a^3 \Lambda_\sigma^3}.
\]  
(3.81)

\(^3\)This is also discussed in Appendix A where the period of nonadiabaticity is shown to be much less than a Hubble time.
The occupation number of $\sigma$ particles $|\beta_\sigma|^2$ as a function of the momentum $p$ expressed in units of $a_\ast \Lambda_\sigma$.

By matching the above exact solution to the WKB solution for large $\tilde{z}$ we read off the Bogolyubov coefficients as

$$
\alpha_\sigma = -i \frac{\pi}{3\sqrt{2}} e^{i\left(\frac{\pi}{2} \Lambda_\sigma^3 + \frac{\pi}{2} \Lambda_\sigma\right)} \left[ \frac{p^3}{a_\ast^3 \Lambda_\sigma^3} \left( \frac{4 \ H^{(1)}_{\frac{1}{2}}}{3 a_\ast^3 \Lambda_\sigma^3} \right) - \frac{2 \ H^{(1)}_{\frac{1}{2}}}{3 a_\ast^3 \Lambda_\sigma^3} + \frac{2 \ p^3}{3 a_\ast^3 \Lambda_\sigma^3} \right] \right], \\
\beta_\sigma = i \frac{\pi}{3\sqrt{2}} e^{i\left(\frac{\pi}{2} \Lambda_\sigma^3 - \frac{\pi}{2} \Lambda_\sigma\right)} \left[ \frac{p^3}{a_\ast^3 \Lambda_\sigma^3} \left( \frac{2 \ H^{(1)}_{\frac{1}{2}}}{3 a_\ast^3 \Lambda_\sigma^3} \right) - \frac{2 \ H^{(1)}_{\frac{1}{2}}}{3 a_\ast^3 \Lambda_\sigma^3} + \frac{2 \ p^3}{3 a_\ast^3 \Lambda_\sigma^3} \right] \right].
$$

The occupation number $|\beta_\sigma|^2$ is plotted in figure 3.3. We will use the above coefficients to determine the effects of the produced $\sigma$ particles in Subsections 3.5.5 and 3.5.7 below.

### 3.4.3 Production of quanta of $\chi$

The analysis of the production of quanta of $\chi$ is similar to that of the previous section, with the additional complication that the $\chi$ particles will remain massless (and the superhorizon modes will therefore not be evolving adiabatically) after the event of particle production. We decompose the field $\chi$ as

$$
\hat{\chi}(\mathbf{x}, \tau) = \frac{1}{a(\tau)} \int \frac{d^3p}{(2\pi)^{3/2}} e^{i\mathbf{p} \cdot \mathbf{x}} \left[ \chi_{\mathbf{p}}(\tau) \hat{a}_\mathbf{p} + \chi_{-\mathbf{p}}^*(\tau) \hat{a}^\dagger_{-\mathbf{p}} \right],
$$

(3.83)
where the (canonically normalized) mode functions satisfy

$$\chi''_p + \left[p^2 - \frac{a''}{a} + m^2_\chi(\tau) a^2\right] \chi_p = 0, \quad (3.84)$$

with \(m_\chi(\tau)\) given by eq. (3.10). We will assume that the parameters of the system are such that \(m_\chi\) evolves adiabatically, \(\left|(a m_\chi)'\right| \ll a^2 m^2_\chi\) for most of the time, and that the period in which the adiabaticity condition is violated, close to the time when \(m_\chi = 0\), is much shorter than an Hubble time, which implies the condition

$$\Lambda_\chi \gg H. \quad (3.85)$$

We can now determine the mode functions during the nonadiabatic regime by switching from conformal to physical time, neglecting the expansion of the Universe during this epoch, and introducing the rescaled field \(\chi = a^{-\frac{1}{2}} \chi_c\). Then the equation for \(\chi_c\) simplifies to

$$\ddot{\chi}_c + \left[p^2 + \Lambda^2_\chi (t_* - t)\right] \chi_c = 0, \quad (3.86)$$

whose solution for \(t < t_*\), reducing to the adiabatic vacuum at early times, can be written as

$$\chi_c(t < t_*) = \frac{\pi}{6 \Lambda_\chi} \Pi^{(1)}_1\left(\frac{2}{3} z^\frac{3}{2}\right), \quad z \equiv \frac{p^2}{a^2_* \Lambda^2_\chi} + \Lambda_\chi(t_* - t). \quad (3.87)$$

For \(\tau > \tau_*\) the scalar is massless, so that its mode functions are given by

$$\chi_c(\tau > \tau_*) = c_+ \frac{e^{-ip\tau}}{\sqrt{2p}} \left(1 - \frac{i}{p \tau}\right) + c_- \frac{e^{ip\tau}}{\sqrt{2p}} \left(1 + \frac{i}{p \tau}\right), \quad (3.88)$$
where the constants $c_+$ and $c_-$ are determined by imposing continuity of $\chi_c$ and of its first derivative at $\tau_*$, so that

$$c_+ = \sqrt{\frac{\pi a_*^2 H^4}{12 p \Lambda^3}} \left[ \left(1 - i \frac{p}{a_* H} + \left(\frac{p}{a_* H}\right)^2 \right) H^{(1)}_{\frac{3}{2}} \left(\frac{2 p^3}{3 a_*^3 \Lambda^3}\right) \right. \\
- \left. \left(\frac{p}{a_* H} + i \left(\frac{p}{a_* H}\right)^2 \right) H^{(1)}_{-\frac{3}{2}} \left(\frac{2 p^3}{3 a_*^3 \Lambda^3}\right) \right],$$

$$c_- = \sqrt{\frac{\pi a_*^2 H^4}{12 p \Lambda^3}} e^{i \frac{p}{a_* H}} \left[ \left(1 + i \frac{p}{a_* H} + i \left(\frac{p}{a_* H}\right)^2 \right) H^{(1)}_{\frac{3}{2}} \left(\frac{2 p^3}{3 a_*^3 \Lambda^3}\right) \right. \\
+ \left. \left(\frac{p}{a_* H} - \left(\frac{p}{a_* H}\right)^2 \right) H^{(1)}_{-\frac{3}{2}} \left(\frac{2 p^3}{3 a_*^3 \Lambda^3}\right) \right].$$

Note that in the Minkowski limit $H \to 0$, $\tau_* = -H^{-1} \to \infty$ the coefficients $c_+$ and $c_-$ converge to the Bogolyubov coefficients, respectively $\alpha$ and $\beta$, in Minkowski space as we discussed in Section 3.3. In a de Sitter background, however, the interpretation $c_+$ and $c_-$ as Bogolyubov coefficients associated to the occupation number of $\chi$ particles is not rigorous, since for $\tau > \tau_*$ the superhorizon modes of $\chi$ are not evolving adiabatically. Moreover, we do not expect the coefficients (3.89) to be accurate for modes that were superhorizon, $k/a_* < H$, at the time $\tau_*$, since the solution (3.87) has been found under the assumption that the Hubble parameter is negligibly small, so that those modes are not accounted for. Since $\Lambda \chi \gg H$, however, we believe that the error is negligible, and, as we will show below, we will still be able to effectively study the production of $\chi$ through a subtraction method.
3.4.4 Production of gravitational waves

We now calculate the spectrum of gravitational waves sourced by the field $\chi$. The equation of motion for the graviton $h_{ij}$ and the energy-momentum tensor for the field $\chi$ reads,

$$h_{ij}'' + \frac{2a'}{a} h_{ij}' - \Delta h_{ij} = \frac{2}{M_P^2} \Pi_{ij}^{ab} T_{ab},$$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_\chi}{\delta g^{\mu\nu}} = \partial_\mu \hat{\chi} \partial_\nu \hat{\chi} - g_{\mu\nu} \left( \frac{1}{2} \partial^{\alpha} \hat{\chi} \partial_{\alpha} \hat{\chi} + V(\hat{\chi}) \right),$$

(3.90)

so that, focusing on the spatial components only and expanding to first order in $h_{ab}$,

$$T_{ab} = \partial_a \hat{\chi} \partial_b \hat{\chi} + \hat{h}_{ab} \left[ \frac{1}{2} \hat{\chi}'^2 - \frac{1}{2} (\nabla \hat{\chi})^2 - a^2 V(\hat{\chi}) \right] + \ldots,$$

(3.91)

where the dots denote terms that are second or higher order in $\hat{h}_{ij}$ and the terms that are proportional to $\delta_{ij}$ and are projected out by $\Pi_{ij}^{ab}$. We write the equation of motion for the graviton in momentum space as,

$$h_{ij}''(p, \tau) + 2 \frac{a'}{a} h_{ij}'(p, \tau) + p^2 h_{ij}(p, \tau) = \frac{2}{M_P^2} \Pi_{ij}^{ab}(p) T_{ab}(p, \tau),$$

(3.92)

and just as we can in the previous Minkowski section we split the two contributions to the energy-momentum tensor into two parts,

$$T_{ij}^{(1)}(p, \tau) = -\frac{1}{a^2(\tau)} \int \frac{d^3k}{(2\pi)^{3/2}} (k_i)(p - k)_j \chi(k, \tau) \chi(p - k, \tau),$$

(3.93)

$$T_{ij}^{(2)}(p, \tau) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[ \frac{1}{2} \int \frac{d^3k'}{(2\pi)^{3/2}} \left( h_{ij}^{(0)}(p - k - k') \left[ \left( \frac{\chi(k)}{a(\tau)} \right)' \left( \frac{\chi(k')}{a(\tau)} \right)' + \frac{(k \cdot k')}{a^2(\tau)} \chi(k) \chi(k') \right] \right) \right.$$

$$\left. - a^2(\tau) h_{ij}^{(0)}(p - k) V(\chi(k)) \right].$$

(3.94)
We now split the solution for the graviton into a vacuum solution and a perturbed solution,

\[ h_{ij}(p, \tau) = h_{ij}^{(0)}(p, \tau) + h_{ij}^{(1)}(p, \tau), \]  

(3.95)

where the vacuum solution satisfies,

\[ (h_{ij}^{(0)}(p, \tau))'' + 2 \frac{a'}{a}(h_{ij}^{(0)}(p, \tau))' + p^2 h_{ij}^{(0)}(p, \tau) = 0, \]  

(3.96)

and a perturbed solution,

\[ h_{ij}^{(1)}(p, \tau) = \frac{2}{M_*^2} \int d\tau' G_p(\tau, \tau') \Pi_{ij}^{ab}(p) T_{ab}(p, \tau'), \]  

(3.97)

where the Green’s function satisfies the homogeneous equation of motion for the graviton and whose solution reads,

\[ G_p(\tau, \tau') = \frac{1}{p^3 \tau'^2} \left[ (1 + p^2 \tau \tau') \sin(p(\tau - \tau')) + p(\tau' - \tau) \cos(p(\tau - \tau')) \right] \Theta(\tau - \tau'). \]  

(3.98)

Note that either \( T_{ij}^{(0)} \) or \( T_{ij}^{(1)} \) can be used in the expression for the energy-momentum tensor in equation 3.97. The equal-time correlator for the graviton will read,

\[ \langle \hat{h}_{ij}(k, \tau) \hat{h}_{ij}(k', \tau) \rangle = \langle \hat{h}_{ij}^{(0)}(k, \tau) \hat{h}_{ij}^{(0)}(k', \tau) \rangle + \langle \hat{h}_{ij}^{(0)}(k, \tau) \hat{h}_{ij}^{(1)}(k', \tau) \rangle \\
+ \langle \hat{h}_{ij}^{(1)}(k, \tau) \hat{h}_{ij}^{(0)}(k', \tau) \rangle + \langle \hat{h}_{ij}^{(1)}(k, \tau) \hat{h}_{ij}^{(1)}(k', \tau) \rangle, \]  

(3.99)

which will lead to an expression for the power spectrum from all components being,

\[ \langle h_{ij}(k) h_{ij}(k') \rangle = \frac{2\pi^2}{k^3} \delta^{(3)}(k + k') \left[ P_T^{00}(k) + P_T^{01}(k) + P_T^{10}(k) + P_T^{11}(k) \right]. \]  

(3.100)
\( \mathcal{P}_T^{00}(k) \) corresponds to the vacuum contribution to the power spectrum, \( \mathcal{P}_T^{01}(k) \) and \( \mathcal{P}_T^{10}(k) \) correspond to the cross terms of the vacuum solution with \( T_{ij}^{(2)} \), and finally \( \mathcal{P}_T^{11}(k) \) corresponds to the product of \( T_{ij}^{(1)} \) with itself. We solve for the above power spectra in the following sections.

### 3.4.4.1 Vacuum Power Spectrum, \( \mathcal{P}_T^{00}(k) \)

The standard gravitational wave power spectrum produced solely due to the de Sitter expansion is found using equation 3.96. To solve for the vacuum solution, we promote the graviton to an operator whose solution reads,

\[
\hat{h}^{(0)}_{ij}(x, \tau) = \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot x} \hat{h}^{(0)}_{ij}(p, \tau) = \frac{1}{a(\tau) M_P^2} \int \frac{d^3p}{(2\pi)^3} e^{ip \cdot x} \sum_{s=\pm, \times} \left[ v_p(\tau, s) e_{ij}(p, s) \hat{a}_p(s) + v^*_p(\tau, s) e^{*}_{ij}(-p, s) \hat{a}^\dagger_{-p}(s) \right],
\]

(3.101)

where the factor of \( a^{-1} \) comes from canonically quantizing the field, the factor \( 2M_P^{-2} \) comes from normalizing the graviton with respect to other sources of gravitational waves, \( e_{ij} \) are the polarization vectors for the 'plus' (+) and 'cross' (\( \times \)) polarizations, and \( v_p \) are the mode functions satisfying,

\[
v_p'' + \left[ p^2 - \frac{a''}{a} \right] v_p = 0,
\]

(3.102)

with solution,

\[
v_p = \frac{e^{-ip\tau}}{\sqrt{2p}} \left( 1 - \frac{i}{p\tau} \right).
\]

(3.103)

The two point function will be,

\[
\langle h^{(0)}_{ij}(k, \tau) h^{(0)}_{ij}(k', \tau) \rangle = 2 \times \frac{\delta^{(3)}(k + k')}{a^2(\tau) M_P^4} |v_k|^2,
\]

(3.104)
where the factor of 2 out front comes from summing over the two polarizations. The power spectrum is then,

\[ P_0^0(k) = \frac{2H^2}{\pi^2 M_p^2} \left( 1 + k^2 \tau^2 \right). \]  

(3.105)

Considering only superhorizon modes (those modes that leave the horizon during inflation) we take \(|k\tau| \ll 1\), and obtain

\[ P_0^0(k) = \frac{2}{\pi^2 M_p^2} H^2. \]  

(3.106)

Note, we could also obtain the same answer by subtracting the Minkowski contribution to the mode functions,

\[ \langle h_{ij}^{(0)}(k, \tau) h_{ij}^{(0)}(k', \tau) \rangle = \frac{8\delta^{(3)}(k + k')}{a^2(\tau) M_p^2} \left( |v_k|^2 - |v_{k_\text{MIN}}|^2 \right) = \frac{4H^2 \delta^{(3)}(k + k')}{k^3 M_p^2}. \]  

(3.107)

### 3.4.4.2 Obtaining finite quantities

To evaluate \( P_{01}^0 \) and \( P_{11}^1 \) we use Wick’s theorem, so that we have to evaluate correlators of the form \( \langle \hat{\chi}(k_1, \tau_1) \hat{\chi}(k_2, \tau_2) \rangle \), that need to be renormalized. If we were to perform this calculation on a Minkowski background, we would have a straightforward and physically transparent way of performing such a renormalization. On a Minkowski background, in fact, the frequency of the field \( \hat{\chi} \) (which is just its momentum) evolves adiabatically after \( \tau_* \), so that we can fully use the formalism of Bogolyubov coefficients. This means that we would decompose the field in terms of new creation/annihilation operators \( \hat{b}_k^{(f)} \), where the \( \hat{b}_k \) operator multiplies the positive frequency component of the mode functions of \( \hat{\chi} \) for times \( \tau > \tau_0 \), when the the frequency of the modes of \( \hat{\chi} \) is adiabatically evolving. Then we would normal order the \( \hat{\chi}(k_1, \tau_1) \hat{\chi}(k_2, \tau_2) \) operator in terms of the \( \hat{b}_k^{(f)} \) ladder operators, and not in terms of the original \( \hat{a}_k^{(f)} \) ones, that were used to quantize \( \hat{\chi} \) for \( \tau < \tau_* \). This means that observers born after \( \tau = \tau_* \) would renormalize away the vacuum
fluctuations of the mode functions defined for \( \tau > \tau_s \) by normal ordering the operators \( \hat{b}_k^\dagger \), that correspond to their notion of particle. Writing the relationship between the \( \hat{b}_k^\dagger \) and the \( \hat{a}_k^\dagger \) operators as

\[
\hat{b}_k(\tau) = \alpha(k, \tau) \hat{a}_k + \beta^*(-k, \tau) \hat{a}_k^\dagger,
\]

we would obtain

\[
\langle \hat{\chi}(p, \tau') \hat{\chi}(q, \tau'') \rangle = \frac{\delta^{(3)}(p + q)}{2 \sqrt{\omega_p(\tau') \omega_p(\tau'')}} \left[ \left( e^{i \int' \omega_p \beta^*(-p, \tau') \beta(-p, \tau'')} + \text{h.c.} \right) + \left( e^{-i \int' \omega_p \beta^*(p, \tau') \beta(p, \tau'')} + \text{h.c.} \right) \right].
\]

As we discussed in the previous section, the prescription (3.109) above is equivalent to setting

\[
\langle \chi(p, \tau') \chi(q, \tau'') \rangle = \delta^{(3)}(p + q) \left[ \chi(p, \tau') \chi(p, \tau'') - \hat{\chi}(p, \tau') \hat{\chi}(p, \tau'') \right],
\]

where, in the case of particles on a Minkowskian background, \( \hat{\chi}(p, \tau) = \frac{e^{-ip\tau}}{\sqrt{2p}} \) corresponds to the mode function in absence of particle creation, \( \Lambda \to 0 \). In its turn, this means that the procedure presented above corresponds precisely to that of adiabatic regularization, where one subtracts from the UV-divergent propagator its adiabatic part to obtain a finite result.

Let us now go back to the production of quanta of \( \hat{\chi} \) in quasi de Sitter space. As discussed above, since we are talking about massless particles in de Sitter space, we have \( \omega^2 = k^2 - 2/\tau^2 \) that for \( k \lesssim -1/\tau \) is not evolving adiabatically. Therefore the prescription (3.109) cannot be applied to this case. However, the prescription (3.110), that on a Minkowski background is equivalent to (3.109), can be applied to our de Sitter
background, once we set
\[
\tilde{\chi}(k, \tau) = \chi(k, \tau)\bigg|_{\Lambda \to 0}.
\] (3.111)

Based on the above considerations, we will use
\[
\langle \chi(p, \tau') \chi(q, \tau'') \rangle = \delta^{(3)}(p + q) \left[ \chi(p, \tau') \chi(p, \tau'') - \tilde{\chi}(p, \tau') \tilde{\chi}(p, \tau'') \right],
\] (3.112)

where the function \( \chi(p, \tau) \) is given by eq. (3.88) with the integration constants given by eq. (3.89), whereas \( \tilde{\chi}(p, \tau) \) is obtained by setting \( \Lambda \to 0 \) in \( \chi(p, \tau) \), so that
\[
\tilde{\chi}(p, \tau) = b_+ \frac{e^{-ip\tau}}{\sqrt{2p}} \left[ 1 - \frac{i}{p\tau} \right] + b_- \frac{e^{ip\tau}}{\sqrt{2p}} \left[ 1 + \frac{i}{p\tau} \right],
\] (3.113)

with
\[
b_+ = 1 - \frac{i}{y} - \frac{1}{2y^2}, \quad b_- = -\frac{e^{2iy}}{2y^2}.
\] (3.114)

where, we recall, \( y = -p\tau_* \). This prescription is analogous to that used for instance in [92].

### 3.4.4.3 Contributions to the Power Spectrum from \( \chi, \mathcal{P}_{ij}^{11}, \mathcal{P}_{ij}^{01} \) and \( \mathcal{P}_{ij}^{10} \)

With the correct renormalizing method in hand, we next compute the correlators using for the power spectrum from \( T_{ij}^{(1)} \) with itself, which will yield
\[
\langle \hat{h}_{ij}^{(1)}(k, \tau) \hat{h}_{ij}^{(1)}(k', \tau) \rangle = \frac{4}{M_p^4} \int \frac{d\tau'}{a^2(\tau')} G_k(\tau, \tau') \int \frac{d\tau''}{a^2(\tau'')} G_{k'}(\tau, \tau'') \Pi_{ij}^{ab}(k) \Pi_{ij}^{cd}(k')
\times \int d^8p d^3p' p_{a_i}(k_{b_i} - p_{b_i}) p_{c_j}(k_{d_j} - p'_{d_j}) \langle \tilde{\chi}(p, \tau') \tilde{\chi}(k - p, \tau') \tilde{\chi}(p', \tau'') \tilde{\chi}(k' - p', \tau'') \rangle.
\] (3.115)
The contribution from $T_{ij}^{(2)}$ with the vacuum contribution to the graviton reads,

$$
\langle \hat{h}_{ij}^{(0)}(k, \tau) \hat{h}_{ij}^{(1)}(k', \tau) \rangle = \frac{2}{M_p^2} \int d\tau' G_{k}(\tau, \tau') \Pi_{ij}^{ab}(k') \times \int \frac{d^3p \, d^3p'}{(2\pi)^3} \times
$$

$$
\langle \hat{h}_{ij}^{(0)}(k, \tau) \hat{h}_{ab}^{(0)}(k' - p - p', \tau') [\hat{\chi}'(p, \tau') \hat{\chi}'(p', \tau') + (p \cdot p' - m_{\chi}^2 a(\tau')^2) \hat{\chi}(p, t') \hat{\chi}(p', \tau')] \rangle,
$$

(3.116)

Notice that, since we are evaluating the amplitude of the tensors produced after the production of quanta of $\chi$ field, we will set $m_{\chi} = 0$.

Let us now evaluate $P_{11} T$ and $P_{01} T$ where the existence of the two contributions themselves can also be derived in a different way in the context of the in-in formalism, and originates from the two different diagrams presented in [74].

- $P_{11} T$. To calculate $P_{11} T$ we need the following expression, that allows to compute the factor proportional to the transverse-traceless projectors

$$
\Pi_{ij}^{ab}(k) \Pi_{ij}^{cd}(k') p_a(k_b - p_b)(k_c - p_c) p_d = \frac{1}{2} \left( p^2 - \frac{(p \cdot k)^2}{k^2} \right)^2.
$$

(3.117)

After taking the limit $\tau \rightarrow 0$, so that we evaluate the effects at the end inflation, when the relevant scales are well outside of the horizon, we obtain

$$
P_{11} T = \frac{H^4}{2 \pi^3 k^3 M_p^2} \int d^3p \left( p^2 - \frac{(p \cdot k)^2}{k^2} \right)^2
$$

$$
\times \int d\tau' d\tau'' [- \sin(k\tau') + k\tau' \cos(k\tau')] [- \sin(k\tau") + k\tau" \cos(k\tau")]
$$

$$
\times \left[ \chi_p(\tau') \chi_p^*(\tau'') - \tilde{\chi}_p(\tau') \tilde{\chi}_p^*(\tau'') \right] \left[ \chi_{k-p}(\tau') \chi_{k-p}^*(\tau'') - \tilde{\chi}_{k-p}(\tau') \tilde{\chi}_{k-p}^*(\tau'"") \right].
$$

(3.118)

We have integrated the above expression numerically and a plot for $\Lambda = 10 H, 20 H$ and $30 H$ is shown in the left panel of figure 3.4 as a function of $-k \tau_*$. In the right
\begin{align}
\mathcal{P}_{T}^{11}(k) &= 2.5 \times 10^{-6} \frac{H^4 \Lambda^5}{M_P^4 H^5}.
\end{align}

• $\mathcal{P}_{T}^{01}$. The relevant correlator is computed from

\begin{align}
\frac{2 \pi^2}{k^3} &\delta(k_1 + k_2) \times \langle 2 \text{Re} \{ \mathcal{P}_{T}^{01} \} \rangle = -\frac{1}{M_P^2} \int d\tau' \int d^3q \, d^3q' \frac{1}{(2\pi)^3} \\
&\times \left[ G_{k'}(\tau, \tau') \langle h^{(0)}_{ij}(k, \tau) h^{(0)}_{ij}(k' - q - q', \tau') \rangle + G_k(\tau, \tau') \langle h^{(0)}_{ij}(k - q - q', \tau') h^{(0)}_{ij}(k', \tau) \rangle \right] \\
&\times \left[ \chi'(q, \tau') \chi'(q', \tau') + (q \cdot q') \chi(q, \tau') \chi(q', \tau') \right],
\end{align}

where the graviton correlator is given by

\begin{align}
\langle h^{(0)}_{ij}(k, \tau) h^{(0)}_{ij}(k', \tau') \rangle &= \frac{4 \delta^{(3)}(k + k')}{a(\tau) a(\tau')} \frac{1}{M_P^2} \frac{1}{k k' \tau \tau'} \left( 1 + \frac{i(\tau - \tau')}{k \tau \tau'} + \frac{1}{k^2 \tau \tau'} \right) e^{-i k (\tau - \tau')},
\end{align}

Taking $\tau \to 0$, we obtain an expression for $\mathcal{P}_{T}^{01}$ that can be integrated numerically for various values of $\Lambda/H$. Spectra for $\Lambda = 10H$, $20H$ and $30H$ are given in the left panel of figure 3.5. As we see, the spectra have a peak at $-k \tau_0 \simeq 2$. In the right
Figure 3.5: Left: Numerical plot of $\frac{M^4_p}{\pi^2} \text{Re} \{ P_{11}^{01} \}$ as a function of $-k \tau_0$ for (top to bottom) $\Lambda = 30 H, 20 H, 10 H$. Right: The amplitude of $\frac{M^4_p}{\pi^2} \text{Re} \{ P_{11}^{01} \}$ at its peak, $-k \tau_s \approx 2$, as a function of $\Lambda / H$. The solid line corresponds to the analytical fit $P_{11}^{11}(-k \tau_s = 2) = 4 \times 10^{-3} \frac{\Lambda^3 H}{M^4_p}$, the red bullets correspond to numerical evaluation of the integral (3.120).

panel of figure 3.5 we show the amplitude of $2 \text{Re} \{ P_{11}^{01} \} (-k \tau_s = 2)$ as a function of $\Lambda$. The numerical fit shows that the amplitude of $2 \text{Re} \{ P_{11}^{01} \}$ at its peak is well approximated by

$$2 \text{Re} \{ P_{11}^{01} \} \sim 4 \times 10^{-3} \frac{\Lambda^3 H}{M^4_p}. \quad (3.122)$$

Since this is a factor $\sim H^2/\Lambda^2$ smaller than the amplitude of $P_{11}^{11}$, we conclude that $P_{11}^{01}$ gives a negligible contribution to the spectrum of gravitational waves produced by the quanta of $\chi$.

3.5 Constraints on the parameter space of the model

We have seen in the previous Section that the amplitude of gravitational waves induced by the quanta of $\chi$ goes, for large values of $\Lambda_\chi$, as $\sim 10^{-6} \frac{\Lambda_\chi^2}{M^4_p H}$, with $\Lambda_\chi^3 \equiv \frac{h^2 \mu}{\Lambda} \dot{\varphi}_s$. The energy scale $\Lambda_\chi$ can be in principle very large and might lead to a very large amplitude of induced gravitational waves. In this section we focus on the specific model described
in Section ?? to evaluate the constraints on the parameter space of this scenario and the maximum possible amplitude of $\mathcal{P}_{T}^{II}$.

Consistency of our analysis will require a number of conditions that we will now detail.

### 3.5.1 Perturbativity of Coupling Constants

This condition is simple:

$$h < 1, \quad \lambda < 1.$$  \hfill (3.123)

### 3.5.2 Masses of $\sigma$ and $\chi$

We require that $\sigma$ and $\chi$ be massive and cosmologically irrelevant for most of the evolution of the system, with the exception of a short period (less than one efold) around the time $t_\ast$. Since the masses of those fields are proportional to $\Lambda_{\sigma,\chi}^3 |t - t_\ast|$, this condition is equivalent to requiring

$$\Lambda_\sigma \gg H, \quad \Lambda_\chi \gg H.$$  \hfill (3.124)

### 3.5.3 Evolution of the zero modes

We require the validity of the approximate dynamics described in Section 3.2. Therefore the dynamics of the zero mode $\varphi$ should not be affected significantly by the interactions with $\sigma$, and $\sigma$ should follow the instantaneous minimum of its potential, $\sigma \simeq \sqrt{-\mu \varphi / \Lambda}$. 

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The equations of motion for the zero modes $\varphi_0$ and $\sigma_0$ are

$$\ddot{\varphi}_0 + 3 H \dot{\varphi}_0 + \frac{\mu}{2} \sigma_0^2 + V'(\varphi) = 0,$$

$$\ddot{\sigma}_0 + 3 H \dot{\sigma}_0 + \mu \varphi_0 \sigma_0 + \lambda \sigma_0^3 = 0,$$

$$H^2 = \frac{1}{3 M_p^2} \left( \frac{\varphi_0^2}{2} + \frac{\sigma_0^2}{2} + \frac{\mu}{2} \varphi_0 \sigma_0^2 + \frac{\lambda}{4} \sigma_0^4 + V(\varphi) \right). \quad (3.125)$$

We parametrize the potential near $\varphi = 0$ as

$$V(\varphi) \simeq 3\bar{H}^2 M_p^2 - 3\sqrt{2} \epsilon \bar{H}^2 M_p \varphi + 3\eta \bar{H}^2 \varphi^2 \quad (3.126)$$

where $\epsilon$ and $\eta$ are the (constant) slow roll parameters. Then, performing the following redefinitions

$$\tilde{H} = \frac{H}{\bar{H}}, \quad \tilde{\varphi} = \frac{\varphi_0}{M_p}, \quad \tilde{\sigma} = \sigma \sqrt{\frac{\lambda}{\mu M_p}},$$

$$g_\sigma \equiv \frac{\mu M_p}{\bar{H}^2}, \quad g_\phi \equiv \frac{\mu^2}{\lambda \bar{H}^2}, \quad (3.127)$$

we can rewrite the background equations as

$$\ddot{\tilde{\varphi}} + 3 \tilde{H} \dot{\tilde{\varphi}}' + \frac{g_\phi}{2} \tilde{\varphi}^2 - 3 \sqrt{2} \epsilon + 3 \eta \tilde{\varphi} = 0,$$

$$\ddot{\tilde{\sigma}} + 3 \tilde{H} \dot{\tilde{\sigma}}' + g_\sigma \tilde{\phi} \tilde{\sigma} + g_\sigma \tilde{\sigma}^3 = 0,$$

$$\tilde{H}^2 = 1 + \frac{1}{3} \left( \frac{\tilde{\varphi}'^2}{2} + \frac{g_\phi}{g_\sigma} \tilde{\varphi}^2 + \frac{g_\phi}{2} \tilde{\phi} \tilde{\varphi}^2 + \frac{g_\phi}{4} \tilde{\varphi}^4 - \sqrt{2} \epsilon \tilde{\phi} + \frac{3}{2} \eta \tilde{\phi}^2 \right), \quad (3.128)$$

where a prime denotes a derivative with respect to $\tilde{H} t$. Then the conditions

$$g_\phi \ll 6 |\eta|, \quad g_\sigma \sqrt{2} \epsilon \gg 1, \quad (3.129)$$

are sufficient to guarantee that $\tilde{\phi} \simeq \sqrt{2} \epsilon \tilde{H} (t - t_*)$, $\tilde{\sigma} \simeq \sqrt{-\tilde{\phi}}$, $\tilde{H} \simeq 1$ provide good
approximations to the actual solutions for our system for a few efoldings around $t = t_\ast$.

The equality $g_\sigma = \Lambda_\sigma^3/(2\sqrt{2} \epsilon \bar{H}^3)$ implies that the condition $g_\sigma \sqrt{2} \epsilon \gg 1$ is identically verified since we are already assuming $\Lambda_\sigma \gg H$.

Therefore we conclude that the only new condition required for the background dynamics is just $g_\phi < 6|\eta|$, which is equivalent to

$$\frac{\mu^2}{\lambda H^2} \ll 6|\eta|.$$  \hfill (3.130)

We show in figure 3.6 the numerical solutions of equation 3.128, which we label $\tilde{\phi}_N$, $\tilde{\sigma}_N$, and $\tilde{H}_N$, along with the analytical approximations: $\tilde{\phi} = \sqrt{2\epsilon t}$, $\tilde{\sigma} = \sqrt{-\sqrt{2}\epsilon t}$, and $\tilde{H} = 1/(\epsilon t + 1)$ where $\tilde{t} = \bar{H}(t - t_\ast)$. The constants chosen for the numerical solutions are

$$\epsilon = 0.0045, \quad \eta = \frac{n_s - 1 + 6\epsilon}{2} \simeq -0.0065,$$

$$g_\phi = 6|\eta| \simeq 0.04, \quad g_\sigma = \frac{\Lambda_\sigma^3}{(2\sqrt{2} \epsilon \bar{H}^3)} \simeq 2 \times 10^6,$$  \hfill (3.131)

where we have fixed $\epsilon \simeq 0.07/16$ by imposing that the “vacuum” tensor spectrum has maximum amplitude, while $\eta$ is determined by setting the spectral index $n_s = .96$. Finally, $g_\phi$ and $g_\sigma$ are determined by saturating the inequalities that appear below in Section 3.6.

Even though, for this choice of parameters, the inequalities of Section 3.6 are fully saturated, figure 3.6 shows that the analytical approximation for the evolution of the zero modes $\sigma_0$ and $\varphi_0$ does in fact provide an excellent approximation of the exact evolution of the system.

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3.5.4 Scalar perturbations before the event of $\chi$ production

We require that the metric perturbations are simply given by the usual single field formula

$$\zeta = -H \frac{\delta \varphi}{\dot{\varphi}_0}.$$  

Well before the event of particle production, that is for $t \ll t_\ast - \text{Max} \{ \Lambda_\chi^{-1}, \Lambda_\sigma^{-1} \}$ our system is in general described by two fields, $\varphi$ and $\sigma$, since $\chi$ is vanishing and irrelevant at this stage. The general expression for the curvature perturbation is

$$\zeta = -H \frac{\delta \rho}{\dot{\rho}} = -H \frac{\dot{\varphi}_0 \delta \varphi + \dot{\sigma}_0 \delta \sigma + [V'(\varphi_0) + \frac{\mu}{2} \sigma_0^2] \delta \varphi + [\mu \sigma_0 \varphi_0 + \lambda \sigma_0^3] \delta \sigma}{-3 H \dot{\varphi}_0^2 - 3 H \dot{\sigma}_0^2}. \ (3.132)$$

Since the fluctuations of the field $\sigma$ are heavy at those times, $m_\sigma^2 \simeq \Lambda_\sigma^3 (t_\ast - t) \gg \Lambda_\chi^2 \gg H^2$, we can neglect $\delta \sigma$ in the equation above. Moreover, since the fluctuations of the field $\varphi$ become constant in the super horizon limit, we can neglect the term in $\delta \dot{\varphi}$. As a consequence, the expression for $\zeta$ simplifies to

$$\zeta = -H \frac{[V'(\varphi_0) + \frac{\mu}{2} \sigma_0^2] \delta \varphi}{-3 H \dot{\varphi}_0^2 - 3 H \dot{\sigma}_0^2}. \ (3.133)$$
In order to simplify this expression to its standard single field form we will then impose the following two requirements

\[(i) \quad \frac{\mu}{2} \sigma_0^2 \ll |V'(\phi_0)|,\]

\[(ii) \quad |\dot{\sigma}_0| \ll |\dot{\phi}_0|. \quad (3.134)\]

It is straightforward to see that condition \((i)\) is equivalent to the requirement that that term in \(\sigma_0\) in eq. (3.128) be negligible. This implies that condition \((i)\) is satisfied whenever eq. (3.130) holds.

Condition \((ii)\) is equivalent to

\[\frac{\mu^4}{4\lambda} \ll \dot{\phi}_* |t - t_*|, \quad (3.135)\]

which we want to be satisfied for at least \(|t - t_*| \gtrsim \Lambda_\sigma^{-1}\) leading to,

\[\frac{\mu^4}{2^5 \lambda^3} \ll 2 \epsilon H^2 M_P^2. \quad (3.136)\]

### 3.5.5 Energy density in the fluctuations of \(\sigma\)

Since the fluctuations of \(\sigma\) are sourced by the inflaton, energy conservation requires the energy density in those fluctuations to be smaller than the inflaton’s kinetic energy. The expression for the energy density in terms of the canonical field \(\sigma\) after \(t_\ast\) and for \(|t - t_*| > \Lambda_\sigma^{-1}\) (since we want to be in the region of nonadiabaticity in order to use the Bogolyubovs) is given by,

\[
\rho_\sigma = \frac{1}{2a^4} \left[ \sigma'\sigma_c' + \partial_i \sigma_c \partial_i \sigma_c + \left[-\frac{a''}{a} + a^2 m_\sigma^2(\tau)\right] \sigma_c \sigma_c \right]. \quad (3.137)
\]
Going to momentum space and taking the expectation value, we find

\[
\langle \rho \sigma \rangle = \frac{1}{2a^4} \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{d^3p} e^{i\mathbf{p} \cdot \mathbf{x} + i\mathbf{p}' \cdot \mathbf{x}} \left[ \langle \sigma'_c(p)\sigma'_c(p') \rangle + \omega^2_\sigma \langle \sigma_c(p)\sigma_c(p') \rangle \right]
\]

\[
\omega^2_\sigma = -\mathbf{p} \cdot \mathbf{p}' - \frac{a''}{a} + a^2(\tau)m^2_\sigma(\tau).
\]

(3.138)

The two correlators for \( \sigma \) will be,

\[
\langle \sigma_c(p)\sigma_c(p') \rangle = \frac{\delta^{(3)}(p + p')}{\omega_p(\tau)} \left[ |\beta_p|^2 + \text{Re} \left[ \alpha_p \beta_p^* e^{-2i\int \omega d\tau} \right] \right],
\]

\[
\langle \sigma'_c(p)\sigma'_c(p') \rangle = \omega_p(\tau) \delta^{(3)}(p + p') \left[ |\beta_p|^2 + \text{Re} \left[ \alpha_p \beta_p^* e^{-2i\int \omega d\tau} \right] \right],
\]

(3.139)

which results in the energy for the \( \sigma \) particles yielding,

\[
\langle \rho \sigma \rangle = \frac{1}{a^3} \int \frac{d^3p}{(2\pi)^3} \omega_p(\tau) \left[ |\beta_p|^2 + \text{Re} \left[ \alpha_p \beta_p^* e^{-2i\int \omega d\tau} \right] \right].
\]

(3.140)

Approximating \( \omega_p(\tau) \approx a(\tau)\sqrt{\mu \varphi} \) and neglecting the oscillating term, we then have

\[
\langle \rho \sigma \rangle = \frac{\sqrt{\mu \varphi}}{a^3} \int \frac{d^3p}{(2\pi)^3} |\beta_p|^2 \simeq 1.7 \times 10^{-3} \Lambda^3 a^3_\sigma \sqrt{\mu \varphi}.
\]

(3.141)

Approximating \( \varphi \approx \dot{\varphi}_* (t - t_*) \) and \( a_*/a \approx e^{-H(t - t_*)} \), we see that \( \rho_\sigma \) is maximized for \( t - t_* \approx (6H)^{-1} \), where it evaluates to

\[
\rho_\sigma^{\text{max}} \simeq 6.0 \times 10^{-4} \Lambda^3 \sqrt{\frac{\mu \dot{\varphi}_*}{H}},
\]

(3.142)

that we require to be smaller than the kinetic energy of the inflaton \( \dot{\varphi}_*^2/2 \), leading to the constraint

\[
\frac{\mu^3}{\sqrt{2e} H^2 M_P} \ll 1.7 \times 10^5.
\]

(3.143)
3.5.6 Energy density in the fluctuations of $\chi$

By an argument analogous to that of the previous subsection we require the energy in the $\chi$ particles to be smaller than the kinetic energy in the zero mode of the $\sigma$ field.

Inserting eqs. (3.88) and (3.89) into the expression for the energy in modes of $\chi$,

$$\langle \rho_\chi \rangle = \frac{1}{2} \frac{d^3 p}{(2\pi)^3} \left[ \chi'_p \chi'^*_p + \chi_p \chi^*_p \left( p^2 - \frac{2}{r^2} \right) \right],$$

we obtain an expression that is ultraviolet divergent. To make it finite we subtract off the energy in the mode functions computed for $\Lambda = 0$,

$$\langle \rho_\chi \rangle \rightarrow \frac{1}{2} \frac{d^3 p}{(2\pi)^3} \left[ \left( \chi'_p \chi'^*_p - \bar{\chi}'_p \bar{\chi}'^*_p \right) + \left( \chi_p \chi^*_p - \bar{\chi}_p \bar{\chi}^*_p \right) \left( p^2 - \frac{2}{r^2} \right) \right],$$

where the functions $\chi_p$ and $\bar{\chi}_p$ are given in equations 3.88 and 3.113 above. Numerical integration then gives the result

$$\langle \rho_\chi \rangle = 8 \times 10^{-4} \Lambda^4 \frac{a^4}{a^4},$$

that is maximal when $a = a_*$. Since quanta of $\chi$ are produced by the rolling of the field $\sigma$, energy conservation requires

$$\langle \rho_\chi \rangle \ll \frac{1}{2} \dot{\sigma}(t_{prod})^2$$

where $t_{prod}$ is the time at which the production of most of the quanta of $\chi$ occurs. We note that, for $t < t_*$, $\sigma(t) \sim \sqrt{\frac{r}{\chi}} \dot{\varphi}_* (t_* - t)$, so that $\dot{\sigma}$ is divergent as $t \to t_*$. However, the production happens at a typical time of the order $t \simeq t_* + O(\Lambda^{-1}_\chi)$. As a consequence, we will impose

$$\langle \rho_\chi \rangle \ll \frac{1}{2} \dot{\sigma}(t_* + O(\Lambda^{-1}_\chi))^2 \sim \Lambda^4_\chi / h^2,$$

leading to the constraint $h \ll 30$ that is always satisfied since we will require $h \lesssim 1$ by perturbativity.
3.5.7 Effect of the fluctuations of $\sigma$ on the metric perturbations

We next consider how the fluctuations in $\sigma$ will affect the fluctuations in $\varphi$. In particular, we want to make sure that those induced fluctuations in $\varphi$ are small compared to the scalar perturbations measured in the CMB. Since the fluctuations of $\sigma$ are significant only after $t_*$ we set $\sigma_0 = 0$ so that we are left with the following equation for the fluctuations in $\varphi$:

$$
\delta \varphi'' + 2 \frac{a'}{a} \delta \varphi' - \Delta \delta \varphi + \frac{\mu^2}{2} a^2 \delta \sigma^2 = 0,
$$

(3.147)

that we can solve the above using the Green’s function

$$
\delta \varphi(k, \tau) = \frac{\mu^2}{2} \int d\tau' a^2(\tau') G_k(\tau, \tau') \int \frac{d^3p}{(2\pi)^3} \delta \sigma(p, \tau') \delta \sigma(k - p, \tau'),
$$

(3.148)

where the Green’s function reads

$$
G_k(\tau, \tau') = \frac{1}{k^3 \tau'^2} \left[ (1 + k^2 \tau \tau') \sin(k(\tau - \tau')) + k(\tau' - \tau) \cos(k(\tau - \tau')) \right] \Theta(\tau - \tau').
$$

(3.149)

We are ultimately interested in the power spectrum for the fluctuations in $\varphi$ so we first calculate the correlator,

$$
\langle \delta \varphi(k, \tau) \delta \varphi(k', \tau) \rangle = \frac{\mu^2}{4} \int \frac{d\tau'}{H^2 \tau'^2} \frac{d\tau''}{H^2 \tau''^2} G_k(\tau, \tau') G_{k'}(\tau, \tau'') \times \int \frac{d^3p}{(2\pi)^3} \langle \delta \sigma(p, \tau') \delta \sigma(k - p, \tau') \delta \sigma(p', \tau'') \delta \sigma(k' - p', \tau'') \rangle.
$$

(3.150)
The correlator for $\sigma$ is given by an equation analogous to eq. (3.109), with the modes for $\delta\sigma$ quickly becoming nonrelativistic after $t_*$ since $m_\sigma \sim \sqrt{\mu \dot{\phi}_* (t - t_*)}$ continues to grow. As a consequence, dropping the terms that are quickly oscillating, we obtain

$$
\langle \delta\sigma(p, \tau) \delta\sigma(p - k, \tau') \delta\sigma(p', \tau'') \delta\sigma(p' - k', \tau'') \rangle \approx \frac{\delta^3(k + k')}{2 a^2(\tau') a^2(\tau'') \omega_\sigma(\tau') \omega_\sigma(\tau'')} \times \int dp \{ |\beta_\sigma(p)|^2 |\beta_\sigma(|k - p|)|^2 + Re [\alpha_\sigma(p) \beta_\sigma^*(p) \alpha_\sigma^*(|k - p|) \beta_\sigma(|k - p|)] \}.
$$

(3.151)

Also, we will compute the correlator for modes that are well outside of the horizon at the end of inflation, so that we can set $\tau \to 0$ in eq. (3.150). Collecting everything we get

$$
\langle \delta\varphi(k, \tau) \delta\varphi(k', \tau') \rangle = \frac{\mu^2 H^3 \delta^{(3)}(k + k')}{32 \pi^3 k^3 \Lambda_\sigma^3} \left[ \frac{1}{k^3} \left[ \int_{\tau_*}^{\tau_* + \hbar} d\tau' \frac{\sin k\tau' - k\tau' \cos k\tau'}{\sqrt{\ln \left( \frac{\tau_*}{\tau'} \right)}} \right]^2 \right]
\times \int dp \{ |\beta_\sigma(p)|^2 |\beta_\sigma(|k - p|)|^2 + Re [\alpha_\sigma(p) \beta_\sigma^*(p) \alpha_\sigma^*(|k - p|) \beta_\sigma(|k - p|)] \}.
$$

(3.152)

To calculate the momentum integral, we note that the integral in $dp$ gets most of its contributions by $p = O(\Lambda_\sigma a_*)$. On the other hand, the temporal function of $\tau_*$ forces $k = O(|\tau_*|^{-1}) \ll O(\Lambda_\sigma a_*)$. Therefore we can neglect the $k$-dependence inside the momentum integral, so that the second line of eq. (3.152) can approximated by

$$
\int dp \{ 2 |\beta_\sigma(p)|^4 + |\beta_\sigma(p)|^2 \} \approx .51 a_*^2 \Lambda_\sigma^3.
$$

(3.153)

Thus we finally find the power spectrum of fluctuations in $\delta\varphi$ induced by the fluctuations in $\delta\sigma$

$$
\mathcal{P}_{\delta\varphi}^{sourced} = \mu^2 f(-k \tau_*),
$$

(3.154)
where
\[
f(y) \simeq \frac{.51}{64\pi^5 y^4} \left[ \int_0^y \frac{dx}{x} - \sin(x) + x \cos(x) \right]_2^2,
\]

is plotted in figure 3.7 and is maximized at \( y \simeq 3 \) where it evaluates to \( 1.7 \times 10^{-5} \).

To sum up, the requirement that the metric perturbations induced by the interactions between the inflaton and the field \( \sigma \) do not exceed the measured amplitude of the scalar power spectrum leads to the constraint
\[
\frac{\dot{\varphi}_s^2}{H^2} P_\zeta \simeq \left( \frac{H}{2\pi} \right)^2 P_{S_{\delta\varphi}}(k) \Rightarrow 25 \times 10^{-10} \frac{\dot{\varphi}_s^2}{H^2} \gg \mu^2 f(-k\tau_s),
\]
or, equivalently,
\[
\frac{\mu^2}{2\epsilon M_P^2} \ll 1.5 \times 10^{-4}.
\]

Before concluding this Section, we note that the sourced scalar perturbations will obey non-gaussian statistics and would be in principle subject to the strong constraints from the Planck satellite on the amplitude of equilateral bispectra [75]. However, those strong constraints hold when the nongaussian component of the scalar perturbations has a (quasi) scale invariant component. Since the contribution (3.154) is strongly scale dependent, we do not expect the non-observation of equilateral nongaussianities to be constrain the parameter space of the model more strongly than eq. (3.157), similarly to what found in [58].
3.6 How large of a spectrum for induced PGWs can be generated?

The spectrum of produced gravitational waves is proportional to $\Lambda_\chi^5$, and in this Section we estimate how large $\Lambda_\chi$ can be once the constraints of the previous section are enforced. The constraints found in the previous section can be summarized as follows

\begin{align}
(i) \quad & 2\mu \sqrt{2\epsilon} M_P \gg H^2 \quad \text{Subsection 4.2,} \\
(ii) \quad & \frac{h^2 \mu \sqrt{2\epsilon} M_P}{\lambda} \gg H^2 \quad \text{Subsection 4.2,} \\
(iii) \quad & \frac{\mu^2}{\lambda H^2} \ll 6 |\eta| \quad \text{Subsection 4.3,} \\
(iv) \quad & \frac{\mu^4}{2^5 \lambda^3} \ll 2 \epsilon H^2 M_P^2 \quad \text{Subsection 4.4,} \\
v) \quad & \frac{\mu^3}{\sqrt{2\epsilon} H^2 M_P} \ll 1.7 \times 10^5 \quad \text{Subsection 4.5,} \\
vii) \quad & \frac{\mu^2}{2\epsilon M_P^2} \ll 1.5 \times 10^{-4} \quad \text{Subsection 4.7,} \\
\end{align}

besides the perturbativity requirements $h, \lambda < 1$. 

\hspace{1cm} (3.158)
3.6.1 Detectability at CMB scales

Let us first consider the maximal amplitude of the sourced gravitational waves at CMB scales, where the dynamics of the inflaton is constrained by CMB observations. First, we trade $M_p$ for $H$ and $\epsilon$ using COBE normalization $2\epsilon \simeq 10^7 H^2/M_p^2$:

\[(i) \quad \mu \gg 1.7 \times 10^{-4} H,\]
\[(ii) \quad \frac{h^2}{\lambda} \mu \gg 1.7 \times 10^{-4} H,\]
\[(iii) \quad \frac{\mu^2}{\lambda} \ll 6|\eta| H^2,\]
\[(iv) \quad \frac{\mu^4}{\lambda^3} \ll 3 \times 10^8 H^4,\]
\[(v) \quad \mu^3 \ll 5 \times 10^8 H^3,\]
\[(vi) \quad \mu^2 \ll 1.5 \times 10^3 H^2.\]

(3.159)

We now note, first, that $h$ appears only in condition (ii). In order to maximize the volume of our parameter space while remaining within the perturbative regime we set from now on $h = 1$. Moreover, we note that, once the perturbativity requirement $\lambda < 1$ is imposed, conditions (i), (iii), (iv), (v) and (vi) reduce simply to conditions (i) and (iii). Therefore we are left just with

\[(i) \quad \mu \gg 1.7 \times 10^{-4} H,\]
\[(iii) \quad \frac{\mu^2}{\lambda} \ll 6|\eta| H^2.\]

(3.160)
We remember that we are seeking to maximize $\Lambda \chi / H$, which is proportional to $(\mu / (\lambda H))^{1/3}$.

Trading $\mu$ for $\Lambda \chi$ in the equations above by using

$$\Lambda^3 \chi = \frac{h^2 \mu}{\lambda} \sqrt{2 \epsilon H M_P} \rightarrow 3 \times 10^3 \frac{\mu}{\lambda} H^2 , \quad (3.161)$$

we obtain the following constraints

\begin{align*}
(i) \quad & \lambda \Lambda^3 \chi \gg 0.5 H^3 , \\
(iii) \quad & \lambda^{1/2} \Lambda^3 \chi \ll 8 \times 10^3 \sqrt{|\eta|} H^3 , \quad (3.162)
\end{align*}

where $\Lambda \chi$ is maximized by setting

$$\lambda \simeq 4 \times 10^{-9} |\eta|^{-1} , \quad \Lambda \chi \simeq 500 |\eta|^{1/3} H . \quad (3.163)$$

From now on we set $|\eta| = 0.02$ to fix ideas (this is the value one obtains if one assumes that $\epsilon$ gives a negligible contribution to the scalar spectral index $n_s = 1 + 2 \eta - 6 \epsilon \simeq 0.96$).

Then, trading $\mu$ for $r_{\text{sourced}} = 10^3 \Lambda^5 \chi / (H M^4_P)$ and for $r_{\text{vacuum}} = 0.8 \times 10^8 H^2 / M^2_P$, so that

$$\frac{r_{\text{sourced}}}{r_{\text{vacuum}}} \simeq \left( \frac{r_{\text{vacuum}}}{0.07} \right) \left( \frac{\Lambda \chi}{620 H} \right)^5 \ll 5 \times 10^{-4} \left( \frac{r_{\text{vacuum}}}{0.07} \right) . \quad (3.164)$$

We conclude therefore that the sourced component, in the case of a single $\chi$ species, can give at most a $O(0.1\%)$ contribution to the vacuum contribution to the primordial spectrum of tensors, and that such a situation is obtained in the regime where the vacuum contribution is maximal while in agreement with the existing observational constraints.

Figure 3.8 shows the constraint plot for the allowed value of $\mu$ and $\lambda$ with constant lines $r$ using the above relations.

We finally note that figure 3.6 shows an excellent agreement between the analytical approximation and the actual numerical solution to the background evolution equations.
Figure 3.8: Constraint plot for equation (3.162) showing the allowed parameter space for $\mu$ and $\lambda$. Lines of constant $r$ are showed ranging from $r = 10^{-4} - 10^{-10}$.

In that figure, the constraint (iii) above, which limits the amplitude of the sourced tensors, is fully saturated. Therefore, it is possible that the constraint (iii) might even be violated by a factor 10 or so without changing significantly the dynamics of the system. This in turn implies that the bound (3.164) might be slightly too restrictive. We do not expect, however, this consideration to significantly affect our conclusion that the sourced component is well subdominant with respect to the vacuum one, at least in the case of a single (or a few) $\chi$ species.

3.6.2 Direct detectability by gravitational wave interferometers

Next, we ask the question of whether it would be possible to obtain tensors with a larger amplitude at smaller scales directly probed by gravitational interferometers such as Advanced LIGO or LISA [66, 76]. While, on the one hand, the sensitivity of those experiments to primordial gravitational waves is much weaker than that of CMB polarization, on the other hand the system is not subject to the constraints imposed by CMB
observations. In the case of production of primordial gravitational waves by the amplification of vacuum fluctuations of gauge fields, this allows for observable gravitational waves at interferometer scales [65].

In the case of the present model, however, the upper bound (3.164) is imposed by conditions (i) and (iii) above, which in turn derive just from the requirement of the consistency of the background dynamics, and do not depend strongly on the CMB constraints. The only difference is that we can disentangle $\epsilon$ and $H$ by not imposing the COBE relation $2\epsilon \simeq \frac{10^{7} H^{2}}{M_{P}^{2}}$. More explicitly, by imposing $h = 1$, we obtain

$$
\Omega_{GW} h^{2} \ll 1.2 \times 10^{-11} \lambda^{-5/3} \mu^{5/3} \frac{H^{4/3}}{M_{P}^{7/3}}.
$$

(3.165)

Now, all inequalities (3.158) are best satisfied when $\epsilon$ and $|\eta|$ are largest. When both slow roll parameters are of the order of the unity, the most stringent among those inequalities are again (i) and (iii). If we saturate them we obtain

$$
\Omega_{GW} h^{2} \ll 1.4 \times 10^{-9} (\epsilon |\eta|)^{5/3} \left( \frac{H}{M_{P}} \right)^{2/3}.
$$

(3.166)

Finally, we note that energy conditions require that the value of the Hubble parameter at the smaller scales probed by CMB interferometers must be smaller than the value of the same quantity at CMB scales, which is constrained by observations to $H = 1.1 \times 10^{-4} \sqrt{\frac{r_{\text{vacuum}}}{M_{P}}} < 3 \times 10^{-5} M_{P}$. As a consequence we get the absolute upper bound

$$
\Omega_{GW} h^{2} \ll 1.3 \times 10^{-12} (\epsilon |\eta|)^{5/3}.
$$

(3.167)

While this figure, for large values of $\eta, \epsilon = \mathcal{O}(1)$, is above the projected sensitivity of LISA, $\Omega_{GW} \simeq 10^{-13}$ in its most optimistic configuration, we should stress that it has been obtained by saturating a few “much larger” inequalities: $X \gg Y \rightarrow X = Y$ and
that the “natural” value of the slow roll parameters, at these scales that exit the horizon closer to the end of inflation, is of the order of $\sim 10^{-1}$. Therefore we are led to the conclusion that our scenario will generally not lead to a detectable effect at the LISA level. It is worth stressing, however, that a feature in the inflationary potential leading to larger values of $\epsilon$ and $\eta$, along with the presence of a number $N_\chi > 1$ of $\chi$ species that will enhance our effect by a factor $N_\chi$, might be able to bring it into the observable window without requiring a huge stretch of parameters.

3.7 Conclusion

Production of inflationary gravitational waves by resonant production of scalars, first studied in [65] (see also [53, 67, 74]) is known to be inefficient. The main cause of such inefficiency is attributed to the fact that, in the model of [65], the scalar that sources gravitational waves becomes very massive soon after the event of particle production. This observation has motivated us to study a similar mechanism in the case of a system that undergoes symmetry restoration: a scalar $\chi$, whose mass is controlled by an order parameter $\sigma$ suddenly goes from massive to massless.

We have found that the amplitude for the spectrum of gravitational waves obtained in this scenario is indeed larger by some orders of magnitude than that of [65]. More specifically, if we impose CMB constraints, the sourced component of tensors can yield a sourced tensor-to-scalar ratio as large as $r_{\text{sourced}} \sim 10^{-5}$ per $\chi$ species (in [65] the corresponding figure was $\sim 10^{-8}$). Since the largest value of $r_{\text{sourced}} \sim 10^{-5}$ is obtained under the assumption that the vacuum contribution to the tensor spectrum is the largest one compatible with current observations, $r_{\text{vacuum}} = .07$, we expect that detection of the sourced component, even in the most optimistic scenario, and possibly assuming a $\mathcal{O}(10)$ boost factor to account for multiple $\chi$ species, will be extremely challenging. The
main signature of the sourced component would be an oscillating feature on the top of the spectrum of the smooth vacuum component.

If we relax the constraints from CMB and simply require that the inflaton is still in slow roll ($\epsilon, |\eta| < 1$) then we can get a contribution to the energy density on gravitational waves $\Omega_{GW} h^2$ that in our case can be as large as $10^{-12}$ per $\chi$ species (in [65] the corresponding figure was $\sim 10^{-20}$). This situation is relevant for smaller scales, where the constraints from CMB do not hold, and that would be of interest for gravitational interferometers. For reference, the projected sensitivity of LISA is $\Omega_{GW} h^2 \sim 10^{-13}$ [66]. We stress, however, that the maximum amplitude $\Omega_{GW} h^2 \sim 10^{-12}$ is obtained in our model by looking at a very narrow, and unlikely even if not forbidden, portion of parameter space. It would be interesting to perform a more detailed, possibly fully numerical, analysis of the resulting spectrum in a concrete model of inflation.

Finally, we note that if the symmetry that gets restored is a gauge symmetry, then the mechanism discussed in this paper would lead to the generation of gauge bosons. In [53] it was shown that, for models where the produced particle becomes massive shortly after production, the spectrum of gravitational waves sourced by vectors had the same amplitude as that sourced by scalars (times a factor that accounted for the different number of degrees of freedom). It is not obvious, however, that such a result would hold also for vectors. Moreover, one of the two most constraining factors ($i$) and ($iii$) of Section 3.5 that limits the amplitude of the sourced tensor component does emerge from the requirement of the consistency of the background dynamics for this very specific model. It would be interesting to study whether other mechanisms that lead to symmetry restoration during inflation could give different results. We plan to attack these questions in future work.
CHAPTER 4

PRODUCTION OF PRIMORDIAL MAGNETIC FIELDS VIA THE RATRA AND SCHWINGER MECHANISMS

4.1 Introduction

There is considerable evidence for the existence of magnetic fields for most astrophysical objects. The Earth has a magnetic field around half a Gauss [77] while the Sun produces a magnetic field as well [78]. On scales much larger, magnetic fields existence within galaxies and galaxy clusters on the order of $\mu G$ [79]. The exact origin of galactic magnetic fields is unknown, but a proposed mechanism for sustaining and amplifying these fields is called the galactic dynamo which takes an initial seed magnetic field and amplifies it [80]. The initial value needed for these seed magnetic field to generate the present-day magnetic fields is still not known but are estimated to be on the order $\sim 10^{-23} G$ [81]. However, evidence of magnetic fields in galaxies with $z \sim 3$ have also been found with strengths comparable to those found today possibility contradicting the dynamo mechanism since the strength of the magnetic field should increase over time as a galaxy goes through move rotations [82]. It is important to note that even if the dynamo mechanism correctly describes the evolution of magnetic fields in galaxies, there would still need to be an
initial magnetic field which the dynamo mechanism itself does not address. Of even greater interest is evidence of magnetic fields existing extragalacticly in the cosmic void with a strength of at least \( \sim 10^{-18} - 10^{-16} \text{G} \) [83].

Though a variety of mechanisms have been proposed to explain these magnetic fields one of the more favored cases is their production occurred very early in the universe. The production of magnetic fields very early in the universe (or primordial magnetic fields PMFs) solves the issue of explaining the presence of extragalactic magnetic fields while also providing an initial seed magnetic field for galaxies. A natural candidate for generating such PMFs would be during inflation where amplification of fluctuations of the gauge field, \( A_\mu \), associated with the electromagnetic field might lead to the generation of significant magnetic fields. To test this, we consider the standard Maxwell kinetic term associated with the photon a U(1) gauge field in an expanding universe,

\[
S = \int d^4 x \sqrt{-g} \left[ \frac{-1}{4} F_{\mu \nu} F^{\mu \nu} \right] = \int d^4 x \sqrt{-g} \eta^{\mu \sigma} \eta^{\nu \gamma} \left[ \frac{-1}{4} F_{\mu \nu} F_{\sigma \gamma} \right], \tag{4.1}
\]

where \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the covariant derivative and \( \sqrt{-g} \) is the determinant of the metric, \( g_{\mu \nu} = -dt^2 + a^2(t) d^2x \). If we work in conformal time \( dt = a(\tau) d\tau \), then the metric becomes

\[
ds^2 = a^2(\tau) \left[ -d\tau^2 + d^2x \right], \quad g^{\mu \sigma} = a^{-2}(\tau) \eta^{\mu \sigma}, \quad \sqrt{-g} = a^4(\tau), \tag{4.2}
\]

where \( \eta_{\mu \nu} \) is the Minkowski metric. We now see that the metric is conformal\(^1\) to the Minkowski metric and so we find that the action for the gauge field simply becomes,

\[
S = \int d^4 x \eta^{\mu \sigma} \eta^{\nu \gamma} \left[ \frac{-1}{4} F_{\mu \nu} F_{\sigma \gamma} \right]. \tag{4.3}
\]

\(^1\)This means that the metric is just the Minkowski metric \( \eta_{\mu \nu} \) multiplied by an overall conformal factor \( a^2(\tau) \).
This is nothing but the usual Maxwell kinetic term in Minkowski space whose resultant equation of motion is the massless Klein-Gordon equation,

\[ A_i'' - \Delta A_i = 0, \]  

(4.4)

with plane wave solutions, \( A_i \sim e^{-ik\tau} \). Thus we have no generation or amplification of magnetic fields due to the conformal invariance of an FLRW universe. This conformal invariance for massless, spin-1 fields [84] will need to be broken in order to generate magnetic fields.

The standard ways of breaking the conformal invariance is by either making the gauge field massive (\( \sim m^2 A_\mu A^\mu \)) [85] or by introducing a time-dependent coupling constant for the Maxwell kinetic term. Both options serve to break the conformal invariance, but the introduction of a massive gauge field additionally breaks gauge invariance, something that a time-dependent coupling constant still preserves. It is the latter option that we will employ usually referred to as the Ratra model in order to generate PMFs [86]. However as we will show in addition to the sought after magnetic fields, electric fields can also be generated due to the time-dependency of the gauge field and their energy can easily exceed the background energy. Thus, the Ratra model alone is not enough to generate significant PMFs while simultaneously obeying energy conservation. In order to dissipate the induced electric field, we couple the Schwinger mechanism (pair production of charged particles in the presence of a strong electric field) with the Ratra model to see if appreciable PMFs can be obtained. The ultimate goal being that the generated Schwinger current will in turn lead to its own production of magnetic fields while reducing the electric field. As we will show, the Schwinger effect does have the intended effect of dissipating the electric field, but not to the extent where a significant magnetic field can also be produced.
This chapter will be outlined as followed: first, we will demonstrate how the Ratra model can generate magnetic fields and why (due either to the generation of an excessive electric field or imposing a non-strongly coupled regime) it fails to generate a significant magnetic field during inflation; next, we will detail how the Schwinger mechanism works by calculating the Schwinger current in Minkowski space; finally, we combine both mechanisms and show to what extent the Schwinger effect ameliorates the electric field.

4.2 Ratra Model During Inflation

4.2.1 Set-Up

We will first examine how the Ratra model works during inflation to break the conformal invariance of electromagnetism and produce magnetic fields. We start with the action for the Ratra model which is simply the Maxwell kinetic term with a time-dependent coupling constant,

\[ S_{\text{Ratra}} = \int dx^4 \sqrt{-g} L_{\text{Ratra}} = \int dx^4 \sqrt{-g} \left( \frac{-I(\tau)^2}{4} F_{\mu \nu} F^{\mu \nu} \right), \tag{4.5} \]

where the electromagnetic tensor is defined as \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and \( A_\mu \) is a spin-1 gauge field. The extra Ratra term, \( I(\tau) \), can in general be any time dependent coupling constant which for now we leave unspecified. We also define our metric which is the FRLW metric in conformal time,

\[ ds^2 = a^2(\tau) \left( -d\tau^2 + dx^2 \right), \tag{4.6} \]
where $a(\tau) = -(H\tau)^{-1}$ that is we have assumed an exact de Sitter expansion ($\dot{H} = 0$). The equations of motion for the gauge field are derived from the Euler-Lagrange equations

$$\partial_\mu \left( \frac{\delta(\sqrt{-g}L_{Ratra})}{\delta (\partial_\mu A_\nu)} \right) - \frac{\delta(\sqrt{-g}L_{Ratra})}{\delta A_\nu} = 0,$$

(4.7)

and we find,

$$\partial_\mu (\sqrt{-g}I^2(\tau)F^{\mu\nu}) = 0.$$  

(4.8)

Working in the Coulomb gauge ($A_\tau = \partial_i A_i = 0$) we find for the equation of motion,

$$A''_i + 2\frac{I'(\tau)}{I(\tau)}A'_i - \Delta A_i = 0.$$  

(4.9)

To solve the above equation, we promote the gauge field to an operator, $\hat{A}(x, \tau)$, and decompose it along a set of creation/annihilation operators,

$$\hat{A}(x, \tau) = \int \frac{d^3p}{(2\pi)^{3/2}} e^{ip \cdot x} \hat{A}(p, \tau)$$

$$= \frac{1}{I(\tau)} \int \frac{d^3p}{(2\pi)^{3/2}} e^{ip \cdot x} \sum_{\lambda = \pm} \left[ e_{\lambda}(p) \hat{A}_\lambda(p, \tau)\hat{a}_{\lambda}(p) + e^*_\lambda(-p) A^*_\lambda(p, \tau)\hat{a}^\dagger_{\lambda}(-k) \right],$$

(4.10)

where the factor of $I^{-1}(\tau)$ has been included to bring the mode functions into canonical form. $e_{\lambda}(p)$ are a basis of helicity vectors with the following properties,

$$p \cdot e_{\pm}(p) = 0, \quad e_{\sigma}(p) \cdot e_{\sigma'}(p) = \delta_{\sigma,-\sigma'}, \quad e^*_\pm(p) = e_{\mp}(p),$$

$$e_{\pm}(-p) = -e_{\mp}(p), \quad p \times e_{\pm}(p) = \mp ip e_{\pm}(p),$$

(4.11)

and the creation/annihilation operators satisfy the usual commutation relation,

$$[\hat{a}_{\lambda}(p), \hat{a}^\dagger_{\lambda'}(p')] = 0, \quad [\hat{a}^\dagger_{\lambda}(p), \hat{a}^\dagger_{\lambda'}(p')] = 0, \quad [\hat{a}_{\lambda}(p), \hat{a}^\dagger_{\lambda'}(p')] = \delta_{\lambda,\lambda'} \delta^{(3)}(p - p').$$  

(4.12)
After plugging in equation 4.10 into equation 4.9, the decoupled equation of motion for the mode functions, \( A_\lambda(p, \tau) \), satisfy

\[
A''_\pm(p, \tau) + \left[ p^2 - \frac{I''(\tau)}{I(\tau)} \right] A_\pm(p, \tau) = 0,
\]

(4.13)

where both polarizations satisfy the same equation of motion as so we drop the subscript for subsequent equations. We now specify the form of the time dependent coupling constant which will be in terms of the scale factor,

\[
I(\tau) = \left( \frac{a(\tau)}{a_{\text{end}}} \right)^n = \left( \frac{1}{H\tau} \right)^n,
\]

(4.14)

where \( n \) is a dimensionless, free parameter of our system and we take the scale factor at the end of inflation to be unity. We also switch to the dimensionless variable \( x = -k\tau \) and so our equation for the mode functions is now,

\[
x^2 \frac{\partial^2 A(p, \tau)}{\partial x^2} + (x^2 - n(n + 1)) A(p, \tau) = 0.
\]

(4.15)

The solution to the above differential equation can be written in terms of the Riccati–Bessel functions, \( S_n(x) \), \( C_n(x) \), \( \xi_n(x) \), and \( \zeta_n(x) \) which can also be expressed as,

\[
A(p, \tau) = \sqrt{\frac{\pi x}{2}} \left( a_p H^{(1)}_{n+\frac{1}{2}}(x) + b_p H^{(2)}_{n+\frac{1}{2}}(x) \right),
\]

(4.16)

where \( H^{(1)}_n(x) \) and \( H^{(2)}_n(x) \) are the Hankel functions of the first and second kind respectively. To solve for the constants of integration, \( a_p \) and \( b_p \), we match the above solution to the Minkowski solution which it approximates in the distant past,

\[
A(p, \tau \to -\infty) \approx \frac{1}{\sqrt{2p}} e^{-ip\tau}.
\]

(4.17)
The asymptotic behavior of the Hankel functions [87] are,

\[ H^{(1)}_{\nu}(z) \approx \sqrt{\frac{2}{\pi z}} e^{i \left( z - \frac{\nu \pi}{2} - \frac{\pi}{4} \right)} , \quad \text{for} \quad -\pi < \text{Arg}(z) < 2\pi , \quad (4.18) \]

\[ H^{(2)}_{\nu}(z) \approx \sqrt{\frac{2}{\pi z}} e^{-i \left( z - \frac{\nu \pi}{2} - \frac{\pi}{4} \right)} , \quad \text{for} \quad -2\pi < \text{Arg}(z) < \pi . \quad (4.19) \]

For \( x \to \infty \), \( \text{Arg}(x) = 0 \) and so we find that,

\[ a_p = \frac{1}{\sqrt{2p}} e^{i \frac{x}{2}(n+1)} , \quad b_p = 0 , \quad (4.20) \]

and the exact solution will be,

\[ A(p, \tau) = \sqrt{\frac{\pi x}{4p}} e^{i \frac{x}{2}(n+1)} H^{(1)}_{n+\frac{1}{2}}(x) . \quad (4.21) \]

We will be interested in those modes that are excited during inflation, that is those modes which are initially well within the horizon \( (k \gg H) \) and are then stretched by the accelerating expansion and subsequently leave the horizon \( (k \ll H) \) during inflation. For a given mode \( \tilde{k} \) this will occur at a specific time \( \tilde{\tau} \), namely when \( x \approx 1 \) or the wavelength associated with \( \tilde{k} \) \( (\tilde{\lambda} \sim \tilde{k}^{-1}) \) is approximately the Hubble scale. The physical wavelength in terms of the comoving wavelength is \( \lambda_p = a \lambda_c \) and so for a given wavelength we have

\[ -\frac{\tilde{\tau} a(\tilde{\tau})}{\lambda_p} = \frac{\tilde{\lambda}_p^{-1}}{H} \simeq 1 , \quad (4.22) \]
or as previously stated the wavelength is on the order of the Hubble radius. We this in mind, we expand the mode functions for \( x \ll 1 \),
\[
H_{n+\frac{1}{2}}^{(1)}(x \ll 1) \approx x^{n+\frac{1}{2}} \left( \frac{2^{-(n+\frac{1}{2})}}{\Gamma(n+\frac{3}{2})} + \frac{e^{-\frac{i\pi}{2}}2^{-n-\frac{1}{2}}\cos\left(\pi\left(n+\frac{1}{2}\right)\right)\Gamma\left(-n-\frac{1}{2}\right)}{\pi}\right) + \\
x^{-(n+\frac{1}{2})}e^{-\frac{i\pi}{2}2^{n+\frac{1}{2}}\Gamma\left(n+\frac{1}{2}\right)}.
\]
(4.23)

Thus the expression for the gauge field for modes that exit the horizon will be,
\[
A(p, \tau) = \frac{c_1(n)}{p^{1/2}} x^{n+1} + \frac{c_2(n)}{p^{1/2}} x^{-n},
\]
\[
c_1(n) = \sqrt{\pi} e^{\frac{i\pi}{4}} 2^{-(n+\frac{3}{2})} \left( \frac{e^{\frac{i\pi}{2}}}{\Gamma(n+\frac{3}{2})} + \frac{\cos\left(\pi\left(n+\frac{1}{2}\right)\right)\Gamma\left(-n-\frac{1}{2}\right)}{\pi} \right),
\]
\[
c_2(n) = \frac{e^{\frac{i\pi}{4}} 2^{n+\frac{1}{2}} \Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}}.
\]
(4.24)

Finally, remembering that our expression for the magnetic field equation 4.35 also includes factors of the time dependent coupling constant we have,
\[
\frac{A(p, \tau)}{I(\tau)} = \frac{c_1(n)}{p^{1/2}} \left( \frac{p}{H} \right)^{n+1} (a(\tau))^{-2n-1} + \frac{c_2(n)}{p^{1/2}} \left( \frac{p}{H} \right)^{-n}.
\]
(4.25)

From the above expression, we see that we have in general two cases: one for \( n > -\frac{1}{2} \) where the second term dominates and \( n < -\frac{1}{2} \) where the first term dominates. Before we discuss each of the two cases, we first examine how both the magnetic and electric fields energies and the magnetic field power spectrum should be properly defined.
4.2.2 Energy in the Electric and Magnetic Fields 
and Magnetic Field Power Spectrum

To properly identify the magnetic and electric fields, let’s calculate the stress-energy tensor for the action in equation 4.5,

\[ T_{\mu\nu}^{\text{atra}} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{atra}}}{\delta g^{\mu\nu}} = I^2(\tau) F_{\mu\alpha} F^{\alpha}_\nu + g_{\mu\nu} \left[ -\frac{I(\tau)^2}{4} F^{\alpha\beta} F_{\alpha\beta} \right], \]  

(4.26)

and the corresponding energy density will be,

\[ \rho_{\text{EM}} = -T^0_0 = \frac{I^2(\tau)}{2a^4(\tau)} F_{0i} F^{0i} + \frac{I^2(\tau)}{4a^4(\tau)} F_{ij} F^{ij} = \frac{I^2(\tau)}{2a^4(\tau)} A_i A'_i + \frac{I^2(\tau)}{2a^4(\tau)} \left[ \partial_i A_j \partial_i A_j - \partial_i A_j \partial_j A_i \right], \]  

(4.27)

where the sum over repeated indices is implied. The electric and magnetic fields and their associated energy in classical E&M can be defined through the electric potential and the vector potential as,

\[ E = -\nabla A_0 + \frac{\partial A}{\partial t}, \quad B = \nabla \times A, \quad \rho_{\text{EM}} = \frac{E^2}{2} + \frac{B^2}{2}. \]  

(4.28)

Using the above equations we can identify the electric and magnetic fields as,

\[ E^2(\tau) = \frac{I^2(\tau)}{a^4(\tau)} \langle A' \cdot A' \rangle, \quad B^2(\tau) = \frac{I^2(\tau)}{a^4(\tau)} \langle (\nabla \times A) \cdot (\nabla \times A) \rangle, \]  

(4.29)

where we have defined \( A \equiv A_i e_i \) and the factor of \( a^{-4}(\tau) \) is due to the dilution from the expansion of the universe. Now that we have the expression for the electromagnetic energy, let’s calculate the power spectrum for the magnetic field using,

\[ B(x, \tau) = \frac{I(\tau)}{a^2(\tau)} (\nabla \times A(x, \tau)), \]  

(4.30)
where we decompose the magnetic field in momentum space as,

\[
B(x, \tau) = \int \frac{d^3p}{(2\pi)^{3/2}} e^{ip\cdot x} B(p).
\]

(4.31)

Using equation 4.10, we find in momentum space,

\[
B(p, \tau) = \frac{i}{a^2(\tau)} (p \times A(p, \tau)).
\]

(4.32)

Finally, the power spectrum can be defined as,

\[
\langle B(k, \tau) \cdot B(k', \tau) \rangle \equiv \frac{2\pi^2}{k^3} \delta^{(3)}(k + k')\mathcal{P}_B(k) = -\frac{1}{a^4(\tau)} \langle (k \times A(k, \tau)) \cdot (k' \times A(k', \tau)) \rangle.
\]

(4.33)

4.2.3 Magnetic Field Production

Now that we know how to properly define both the magnetic and electric fields, let’s calculate the energy in the magnetic field. The magnetic field is defined as,

\[
B^2(\tau) = \frac{I(\tau)^2}{a^4(\tau)} \langle (\nabla \times A(x, \tau))^2 \rangle.
\]

(4.34)

We do not detail all of the intermediate steps, but using the above expression along with the commutation relations for the creation/annihilation operators in equation 4.12, the curl of the vector field will be

\[
\langle (\nabla \times A(x, \tau))^2 \rangle = \frac{2}{I^2(\tau)} \int \frac{d^3p}{(2\pi)^3} p^2 |A(p, \tau)|^2,
\]

(4.35)
where we have dropped the distinction between the two polarizations since the mode functions (equation 4.13) are independent of them. Next, we calculate the expression for the power spectrum for the magnetic field,

\[ P_B(k) = \frac{k^3}{2 \pi^2 \alpha^4(\tau) \delta^{(3)}(k + k')} \langle (k \times A(k, \tau)) \cdot (k' \times A(k', \tau)) \rangle , \]  

(4.36)

whose computation is similar to the above,

\[ P_B(k) = \frac{|A(k, \tau)|^2 k^5}{\pi^2 \alpha^4(\tau)} . \]  

(4.37)

We now have all the necessary ingredients to calculate the electromagnetic energy and the power spectrum for the two cases \( n < -\frac{1}{2} \) and \( n > -\frac{1}{2} \).

### 4.2.3.1 Magnetic Field for \( n > -\frac{1}{2} \)

For \( n > -\frac{1}{2} \), we can neglect the first term in equation 4.25 since it is a decreasing mode and our expression becomes,

\[ \frac{A(p, \tau, n > -\frac{1}{2})}{I(\tau)} = \frac{c_2(n)}{p^{1/2}} \left( \frac{p}{H} \right)^{-n} . \]  

(4.38)

Equation 4.35 now becomes,

\[ \langle (\nabla \times A(x, \tau))^2 \rangle = \frac{|c_2(n)|^2 H^{2n}}{\pi^2} \int_{H_{a_i}}^{H_{a_f}} dp \, p^{-2n+3} , \]  

(4.39)

where we integrate from those modes that left the horizon at the beginning of inflation \( (p_i = H_{a_i}) \) to those modes that left the horizon at the end of inflation \( (p_f = H_{a_f}) \). Our
equation for the magnetic field energy now becomes,

$$B^2 = \frac{I(\tau)^2}{a^4(\tau)} \langle (\nabla \times A(x, \tau))^2 \rangle = \left| c_2(n) \right|^2 \frac{H^4}{2\pi^2(2 - n)} \left[ 1 - \left( \frac{a_i}{a_f} \right)^{-2n+4} \right],$$

which we evaluate at the end of inflation. We can further simply the above by assuming that $a_i \ll a_f$ and splitting the expression for the magnetic field at $n = 2$,

$$B^2(n) = \frac{H^4}{2\pi^2} \times \begin{cases} \frac{\left| c_2(n) \right|^2}{2 - n} & n < 2 \\ 2 \ln\left( \frac{a_f}{a_i} \right) & n = 2 \\ \frac{\left| c_2(n) \right|^2}{n-2} \left( \frac{a_i}{a_f} \right)^{-2n+4} & n > 2 \end{cases}.$$  \hspace{1cm} (4.41)

The expression for the power spectrum for $n > -\frac{1}{2}$ will read,

$$P_B(k) = \frac{\left| A(k, \tau) \right|^2 k^5}{\pi^2 a^4(\tau)} = \frac{H^4}{\pi^2} \left| c_2(n) \right|^2 \left( \frac{k}{H} \right)^{-2n+4},$$

where we do not divide the spectrum into values larger or smaller than $n = 2$, since it will be the same expression but with an overall enhancement or suppression of $\left( \frac{k}{H} \right)^{-2n+4}$.

4.2.3.2 Magnetic Field for $n < -\frac{1}{2}$

For $n < -\frac{1}{2}$, the first term in equation 4.25 will dominate,

$$\frac{A(p, \tau, n < -\frac{1}{2})}{I(\tau)} = \frac{c_1(n)}{p^{1/2}} \left( \frac{p}{H} \right)^{n+1} (a(\tau))^{-2n-1},$$

which leads to

$$\langle (\nabla \times A(x, \tau))^2 \rangle = \frac{c_3(n)H^{2n} \pi^{4n+2}}{\pi^2} \int_{H_i a_i}^{H_f a_f} dp \, p^{2n+5},$$

(4.44)
where
\[ c_3(n) = 2^{-(2n+3)} \left( \frac{\pi}{\Gamma^2(n + \frac{3}{2})} + \frac{\cos^2\left(\pi\left(n + \frac{1}{2}\right)\right)}{\pi} \Gamma^2\left(-n - \frac{1}{2}\right) \right) \]. \quad (4.45)

The equation for the magnetic field reads,
\[ B^2(n) = \frac{I(\tau)^2}{a^4(\tau)} \left\langle (\nabla \times \mathbf{A}(\mathbf{x}, \tau))^2 \right\rangle = \frac{c_3(n)H^4}{2\pi^2(n+3)} \left[ 1 - \left(\frac{a_i}{a_f}\right)^{2n+6} \right] , \quad (4.46)\]

where we have again evaluated our expression at the end of inflation. Again, the above expression can be simplified as,
\[ B^2(n) = \frac{H^4}{2\pi^2} \left\{ \begin{array}{ll}
\frac{c_3(n)}{n+3}, & n > -3 \\
2 \ln\left(\frac{a_i}{a_f}\right) & n = -3 \\
-\frac{c_3(n)}{n+3} \left(\frac{a_i}{a_f}\right)^{2n+6} & n < -3 
\end{array} \right. \quad (4.47)\]

The power spectrum in this case will be,
\[ P_B(k) = \frac{H^4c_3(n)}{\pi^2} \left(\frac{k}{H}\right)^{2n+6} , \quad (4.48)\]

where again we do not divide the spectrum into values larger or smaller than \( n = -3 \), since it will be the same expression but with an overall enhancement or suppression of \( \left(\frac{k}{H}\right)^{2n+6} \).

### 4.2.4 Electric Field Production

The electric field energy is given in equation 4.29 and reads,
\[ E^2(\tau) = \frac{I^2(\tau)}{a^4(\tau)} \left\langle (\mathbf{A}'(\mathbf{x}, \tau))^2 \right\rangle . \quad (4.49)\]
The expression for the temporal derivative of the vector field eventually reduces to

\[ \langle A(x, \tau)^2 \rangle = 2 \int \frac{dp^3}{(2\pi)^3} \left| \left( \frac{A(p, \tau)}{I(\tau)} \right) \right|^2. \tag{4.50} \]

Using again equation 4.25 for the mode functions, we proceed as we did before discussing the two separate cases: \( n < -\frac{1}{2} \) and \( n > -\frac{1}{2} \).

### 4.2.4.1 Electric Field for \( n > -\frac{1}{2} \)

For \( n > -\frac{1}{2} \) the mode function for the gauge field reads,

\[ \frac{A(p, \tau)}{I(\tau)} = \frac{c_2(n)}{p^{1/2}} \left( \frac{p}{H} \right)^{-n}, \tag{4.51} \]

which when inserted into equation 4.50 yields zero, resulting in a vanishing electric field since the mode function is not time dependent.

### 4.2.4.2 Electric Field for \( n < -\frac{1}{2} \)

Unlike the above, the case for \( n < -\frac{1}{2} \) will yield an electric field. The mode function now will be,

\[ \frac{A(p, \tau)}{I(\tau)} = \frac{c_1(n)}{p^{1/2}} \left( \frac{p}{H} \right)^{n+1} (a(\tau))^{-2n-1}, \tag{4.52} \]

which when inserted into equation 4.50 will produce,

\[ \langle A(x, \tau)^2 \rangle = \frac{c_4(n) H^{2n} \rho^{4n}}{\pi^2} \int_{H_{1a}}^{H^{\alpha_f}} dp p^{2n+3}, \]

\[ c_4(n) = 2^{-(2n+3)} (2n+1)^2 \left( \frac{\pi}{\Gamma^2 \left( n + \frac{3}{2} \right)} + \frac{\cos^2 \left( \pi \left( n + \frac{1}{2} \right) \right)}{\pi} \right) \Gamma^2 \left( -n - \frac{1}{2} \right). \tag{4.53} \]
After performing the integration our expression for the electric field reads,

\[
E^2(n) = \frac{c_4(n)H^4}{2\pi^2(n + 2)} \left[ 1 - \left( \frac{a_i}{a_f} \right)^{2n+4} \right]. \tag{4.54}
\]

Being explicit about our value of \( n \) and for \( a_i \ll a_f \), we find

\[
E^2(n) = \frac{H^4}{2\pi^2} \times \begin{cases} 
\frac{c_4(n)}{n+2}, & n > -2 \\
2 \ln \left( \frac{a_i}{a_f} \right), & n = -2 \\
-\frac{c_4(n)}{n+2} \left( \frac{a_i}{a_f} \right)^{2n+4}, & n < -2 
\end{cases} \tag{4.55}
\]

### 4.2.5 Energy Constraints and Strong Coupling

We are now in the position to calculate how strong of a magnetic field can be produced at the end of inflation. Equations 4.41 and 4.47 show that in order to have a significant value for the magnetic field, then the value of the Hubble parameter during inflation should be maximized. The present upper limit on \( H \) comes from the non observation of primordial gravitational waves: \( H < 2.76 \times 10^{-5} M_P = 2.72 \times 10^{13} \text{ GeV} \). Also, in order to translate our results into the appropriate units, Gauss, from our current natural units we use,

\[
1 \text{ Gauss} = 6.92 \times 10^{-2} \text{ eV}^2 = 6.92 \times 10^{-20} \text{ GeV}^2 = 1.17 \times 10^{-56} M_P^2,
\]

which leads to \( H^2 < 6.5 \times 10^{46} \text{ G} \). We also define the number of e-folds, \( N \), during inflation using equation 1.16,

\[
N \simeq 70 - \frac{1}{2} \ln \left( \frac{M_P}{H_I} \right), \tag{4.57}
\]
Figure 4.1: A graph of $\frac{\rho_{EM}}{\rho_{tot}}$ vs $n$ for $n > -\frac{1}{2}$ with three different energy scales during inflation. The vertical dashed lines correspond to when the electromagnetic energy dominates over the background energy.

which we can relate to the scale factors using $N = \int H dt = \ln \left( \frac{a_f}{a_i} \right)$ since we are assuming a constant Hubble parameter. Before computing the magnetic and electric fields, we must ensure that the energy of the produced fields does not exceed the energy in the universe. The total energy is given by the first Friedmann equation,

$$\rho_{tot} = 3 M_P^2 H^2,$$

(4.58)

and the electromagnetic energy will be,

$$\rho_{EM} = \frac{E^2}{2} + \frac{B^2}{2},$$

(4.59)

thus we want to ensure that the energy in the electric and magnetic fields does not exceed the total energy,

$$\rho_{EM} < \rho_{tot} = 3 H^2 M_P^2.$$

(4.60)
Computing the electromagnetic energy using equation 4.41 for $n > -\frac{1}{2}$ we find,

$$\frac{\rho_{EM}}{\rho_{tot}} = \frac{H^2}{6\pi^2 M^2_P} \begin{cases} \frac{|c_2(n)|^2}{2-n} & n < 2 \\ 2N & n = 2 \\ \frac{|c_2(n)|^2}{n-2} e^{2N(n-2)} & n > 2 \end{cases}, \quad (4.61)$$

where $N$ is the number of e-folds that we defined in equation 4.57. The above is plotted in figure 4.1 where it shows the energy in the electromagnetic field is subdominant to the background energy for increasing values of $n$ as the energy scale is lowered. Thus as long as we satisfy this constraint the magnetic field will not dominate the background energy.

The magnetic field power spectrum for $n > -\frac{1}{2}$ will be,

$$\mathcal{P}_B(k) = \frac{H^4 |c_2(n)|^2}{\pi^2} \left( \frac{2\pi H^{-1}}{\lambda_{end}} \right)^{-2n+4}, \quad (4.62)$$

where we have exchanged the momentum for its associated wavelength $k = 2\pi/\lambda$ where $\lambda_{end}$ is a particular length scale at the end of inflation. To calculate what $\lambda_{end}$ should be in terms of cosmological scales today ($\sim$ Mpc) we use,

$$\lambda_{today} \sim 1 \text{ Mpc} \quad \Rightarrow \quad \lambda_{end} = \frac{a_{end}}{a_0} \times \lambda_{today} \approx 3.4 \times 10^{-29} \text{ Mpc} \sim 3.6 \times 10^{23} H^{-1}, \quad (4.63)$$

where we used $1 \text{ m} \approx 5.07 \times 10^6 \text{ eV}^{-1}$ and $1 \text{ Mpc} \approx 3.1 \times 10^{22} \text{ m}$.

We next plot in figure 4.2 the magnetic field strength at the end of inflation using equation 4.62 for Mpc scales. To see how these field strengths translate into a magnetic field strength today we use that magnetic fields decay as the square of the scale factor,

$$B(t) = B(t_{end}) \left( \frac{a_{end}}{a(t)} \right)^2, \quad (4.64)$$

where $a_{end}$ is the scale factor at the end of inflation. To obtain a reasonable estimate for
Figure 4.2: A graph of the magnetic field at the end of inflation is shown in Gaussian units. The dashed vertical lines corresponds to when the electromagnetic energy dominates over background energy. The value $10^{40}$ G is shown in orange which corresponds to a field strength today of $B_0 \sim 10^{-17}$ G. A maximum value of $B_0 \sim 10^{-7}$ G is achieved by all three values of $H$.

how much the scale factor changes after inflation, we take the end of inflation up to the present-day as a radiation epoch and using that during a radiation epoch the scale factor is inversely proportional to the temperature, $a \propto T^{-1}$, we have,

$$\frac{a_{\text{end}}}{a_0} = \frac{T_0}{T_{\text{end}}}. \quad (4.65)$$

The present background temperature is $T_0 \approx 2.73 K \simeq 2.35 \times 10^{-13}$ GeV [2] and the temperature at the end of inflation is found using,

$$\rho(T) = g_*(T) \frac{\pi^2 T^4}{30} = 3M_p^2 H^2 \quad \Rightarrow \quad T_{\text{end}} = \left( \frac{90M_p^2 H^2}{g_*(T_{\text{end}})} \right)^{\frac{1}{4}} \approx \left( 7 \times 10^{15} \text{ GeV} \right) \left( \frac{H}{10^{-5} M_p} \right), \quad (4.66)$$

where the temperature will be $T_{\text{end}} < 7 \times 10^{15}$ GeV which is based on the energy scale during inflation. For example assuming $H$ is maximized, we find that the value of the magnetic field strengths today for both a flat spectrum ($n = 2$) and the largest allowed
The value of $n$ will yield,

$$B_0(n = 2) \sim 10^{-11} \text{ G}, \quad B_0(n = 2.2) \sim 10^{-7} \text{ G}. \quad (4.67)$$

The above strengths are certainly more than enough to account for the lower limit on magnetic fields in the cosmic voids as well as enough to provide the seed magnetic fields for galaxies. However, even though we can achieve a significant magnetic field on cosmological scales day while obeying energy constraints there is an additional constraint for values of $n > 0$. If we couple the gauge field to a charged scalar field through the Lagrangian,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - (\partial_{\mu} \phi^* - ig \phi^* A_{\mu})(\partial^{\mu} \phi + ig \phi A^{\mu}) - V(\phi), \quad (4.68)$$

where we are free to redefine the gauge field as $\tilde{A}_{\mu} = g A_{\mu}$ which leads to,

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} - (\partial_{\mu} \phi^* - i \phi^* \tilde{A}_{\mu})(\partial^{\mu} \phi + i \phi \tilde{A}^{\mu}) - V(\phi). \quad (4.69)$$

Thus, the time dependent coupling constant for the Ratra model can be interpreted as the charge associated with some field coupled to the gauge field $g = (I(\tau))^{-1}$, which for $n < 0$ implies that $g \gg 1$ since $I(\tau)$ decreases to unity at the end of inflation. Thus in order to void a regime of strong coupling we must enforce $n < 0$. Restricting $n < 0$, the magnetic field strength is now given by

$$\mathcal{P}_B(k) = \frac{H^4 c_3(n)}{\pi^2} \left( \frac{2\pi H^{-1}}{\lambda_{\text{end}}} \right)^{2n+6}, \quad (4.70)$$

which for $n = -3$ corresponding to a flat spectrum (equal amplitude at all wavelengths) yields a field strength today of $B_0 \sim 10^{-10}$ G where $H$ has been maximized. Unfortunately, the induced electric field from the time dependent gauge field easily exceeds the inflationary energy as is shown in figure 4.3 for $n = -3$. If we try to lower $n$ in order to
Figure 4.3: A graph of $\frac{\rho_{EM}}{\rho_{tot}}$ vs $n$ for $n < -\frac{1}{2}$. The energy in the electromagnetic field is subdominant to the background energy for values of $n$ based on the energy scale, $H$, during inflation.

satisfy energy constraints to $n \simeq -2.1$, this only leads to a magnetic field strength today of $B_0 \sim 10^{-33}$ G which is certainly not enough to explain either the magnetic fields in the voids nor those associated with the seed fields for galaxies. And we do not benefit from lowering the energy scale of inflation since this also lowers the magnetic field strength as well. In light of the these findings, we would ideally like to decrease the energy in the electric field somehow while simultaneously keeping the magnetic field. We will try to accomplish this by incorporating the Schwinger effect with the Ratra model. We first calculate the Schwinger effect in Minkowski to demonstrate how the effect works in a simpler background.
4.3 Schwinger Effect in Minkowski

4.3.1 Introduction and Set-Up

The Schwinger Effect is the creation of a charged particle and its anti-particle in the presence of a strong electric field [88]. Just as a dielectric has a breakdown voltage at which the insulator can fail producing an electric discharge, the vacuum itself can become polarized leading to pair production. Before combining the Schwinger Effect with the Ratra mechanism in de Sitter space, we first calculate the Schwinger Effect in Minkowski space to demonstrate how it works in a simpler background.

To start, we give a heuristic derivation\(^2\) to show how strong of an electric field is needed in order for the effect to be seen. To maximize the effect we consider the production of the lightest charged pair, the electron-positron. The energy needed to produce such a pair will need to be at least their rest mass energies,

\[\Delta E = 2m_e c^2,\]  

(4.71)

with the ensuing pair created on a time scale given by the energy-time uncertainty principle, \(\Delta t \Delta E \sim \hbar\). The created pair will be produce a distance \(\Delta x\) apart with energy,

\[\Delta E = eE \Delta x\]  

(4.72)

where \(\Delta x \sim c\Delta t\). Equating the above energies we find,

\[2m_e c^2 = eE \Delta x \quad \Rightarrow \quad E = \frac{4m_e^2 c^3}{\hbar e} \approx 10^{18} \frac{V}{m}.\]  

(4.73)

\(^2\)This derivation is very similar to that found on page 16 of [89].
For comparison, an electric field of only $E_{air} \sim 10^6 \frac{V}{m}$ is needed for the dielectric breakdown of air.

To demonstrate the Schwinger Effect in Minkowski, we take as our pair particles a charged massive scalar field, $\phi$ ($\phi^*$), which will be coupled to a U(1) gauge field, $A_\mu$, whose Lagrangian will be,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - (D_\mu \phi)^* (D^\mu \phi) - m^2 \phi^* \phi$$

(4.74)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor, $D_\mu = \partial_\mu + ie A_\mu$ is the covariant derivative, $e$ is the charge associated with U(1) gauge invariance, and our metric is just Minkowski: $g_{\mu\nu} = \eta_{\mu\nu} = \text{Diag}[-1, 1, 1, 1]$.

The equation of motion for $\phi$ is found using the Euler-Lagrange equation,

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^*)} \right) - \frac{\partial \mathcal{L}}{\partial \phi^*} = 0,$$

(4.75)

with the resulting equation of motion,

$$\partial^\mu \partial_\mu \phi + ie \partial_\mu (A^\mu \phi) + ie A^\mu \partial_\mu \phi - e^2 A_\mu A^\mu \phi - m^2 \phi = 0.$$  

(4.76)

Since we are trying to produce pairs of $\phi/\phi^*$ in the presence of an electric field, we define through the gauge field a constant electric field which will produce the charged pairs. We could specify for the gauge field,

$$A_\mu = (0, 0, 0, -Et) \quad \text{or} \quad A_\mu = (Ez, \vec{0}),$$  

(4.77)
with a resulting electric field in either case of,

\[ \mathbf{E} = -\nabla A_0 - \frac{\partial \mathbf{A}}{\partial t} = (0, 0, E). \]  

(4.78)

Choosing the former definition for the gauge field, the equation of motion for \( \phi \) now reads,

\[ \ddot{\phi} - \Delta \phi + 2i e t (E \hat{\mathbf{z}} \cdot \nabla \phi) + e^2 E^2 t^2 \phi + m^2 \phi = 0. \]  

(4.79)

To solve the above, we promote \( \phi \) to an operator \( \hat{\phi} \) and decompose it along a set of creation/annihilation operators,

\[ \hat{\phi}(\mathbf{x}, t) = \int \frac{d^3 \mathbf{p}}{(2\pi)^{3/2}} e^{i \mathbf{p} \cdot \mathbf{x}} \left[ \phi_p(t) \hat{a}(\mathbf{p}) + \phi_p^*(t) \hat{b}^\dagger(-\mathbf{p}) \right]. \]  

(4.80)

The commutation relations for the operators read,

\[ [\hat{a}(\mathbf{p}), \hat{a}(\mathbf{p}')] = 0, \quad [\hat{a}^\dagger(\mathbf{p}), \hat{a}^\dagger(\mathbf{p}')] = 0, \quad [\hat{a}^\dagger(\mathbf{p}), \hat{b}^\dagger(\mathbf{p}')] = 0, \]

\[ [\hat{a}(\mathbf{p}), \hat{a}^\dagger(\mathbf{p})] = [\hat{b}(\mathbf{p}), \hat{b}^\dagger(\mathbf{p})] = \delta^{(3)}(\mathbf{p} - \mathbf{p}'). \]  

(4.81)

The mode functions satisfy the equations of motion,

\[ \ddot{f}_p + ((eEt - p_z)^2 + p_x^2 + p_y^2 + m^2) f_p = 0. \]  

(4.82)

\[ \ddot{g}_{-p}^* + ((eEt + p_z)^2 + p_x^2 + p_y^2 + m^2) g_{-p}^* = 0. \]  

(4.83)

Note that \( f_p = g_p \) so for simplicity of notation we set \( f_p = g_p = \phi_p \),

\[ \hat{\phi}(\mathbf{x}, t) = \int \frac{d^3 \mathbf{p}}{(2\pi)^{3/2}} e^{i \mathbf{p} \cdot \mathbf{x}} \left[ \phi_p(t) \hat{a}(\mathbf{p}) + \phi_p^*(t) \hat{b}^\dagger(-\mathbf{p}) \right]. \]  

(4.84)

We want to calculate the number density for the created \( \phi \) particles in the presence of a
constant electric field and so we will use the Bogolyubov formalism for particle creation in a time-dependent background as discussed in Appendix A. To start, we decompose the field $\phi$ along a different set of creation/annihilation operators,

$$
\hat{\phi}(x, t) = \int \frac{d^3p}{(2\pi)^{3/2}} e^{ipx} \left[ \tilde{\phi}_p(t) \hat{a}(p) + \tilde{\phi}_{-p}^*(t) \hat{b}^\dagger(-p) \right], \quad (4.85)
$$

where $\tilde{\phi}$ are the adiabatic mode functions,

$$
\tilde{\phi}_p(t) = \frac{1}{\sqrt{2\omega_p}} e^{-i\int \omega_p dt}, \quad (4.86)
$$

which are related to the original mode functions through a Bogolyubov transformation,

$$
\phi_p(t) = \alpha_p \tilde{\phi}_p(t) + \beta_p \tilde{\phi}_{-p}^*(t), \quad (4.87)
$$

where the new operators are related to the old operators through the Bogolyubov coefficients,

$$
\hat{a}_p = \alpha_p \hat{a}(p) + \beta_p \hat{b}^\dagger(-p) \quad \hat{b}^\dagger(-p) = \beta_p \hat{a}(p) + \alpha_p \hat{b}^\dagger(-p). \quad (4.88)
$$

To solve the equation of motion for $\phi$, we introduce the dimensionless variables,

$$
z = \sqrt{|eE|} \left( t - \frac{p_z}{eE} \right), \quad a = \frac{p_x^2 + p_y^2 + m^2}{|eE|}, \quad \omega(z) = \sqrt{|eE|} \sqrt{z^2 + a}, \quad (4.89)
$$

which transforms the equation of motion as,

$$
\phi''_p(z) + (z^2 + a)\phi_p(z) = 0. \quad (4.90)
$$

The solution to the above equation is parabolic cylinder functions (PCF) which take the general form,

$$
\frac{\partial^2 u}{\partial z^2} + (z^2 + \lambda)u = 0, \quad (4.91)
$$

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with corresponding solution,

\[
\begin{align*}
  u(z) &= A_\lambda D_{-\frac{1+i\lambda}{2}} ((1+i)z) + B_\lambda D_{-\frac{1+i\lambda}{2}} (- (1+i)z), \\
\end{align*}
\]  

(4.92)

where \(A_\lambda\) and \(B_\lambda\) are constants. The adiabatic solution for the mode functions will be,

\[
\tilde{\phi}_p(t) = \frac{1}{\sqrt{2\omega_p}} e^{-i \omega_p(t')dt'} = \frac{1}{\sqrt{2\sqrt{|eE|}\sqrt{z^2 + a}}} e^{-\frac{i}{2}(z\sqrt{z^2 + a} + a \ln(z + \sqrt{z^2 + a})}. 
\]  

(4.93)

To determine the two unknown constants, we match the general and the adiabatic solution for early times \((t \to -\infty)\). The adiabatic solution for early times becomes,

\[
\tilde{\phi}_p(t \to -\infty) \approx \frac{1}{\sqrt{2\sqrt{|eE|}}} |z|^{-\frac{1+ia}{2}} e^{\frac{i\pi}{2}} e^{\frac{\pi a}{2}}. 
\]  

(4.94)

For the general solution we use equation 4.92 but with \(A_\lambda = P_a\), \(B_\lambda = Q_a\), and \(p = \frac{-1+ia}{2}\). For the general solution, as \(t \to -\infty\) then \(z \to -\infty\) and so we have the first PCF in equation 4.92 going like \(D_p(-(1+i)|z|)\) with \(\text{Arg}[-(1+i)|z|] = -\frac{3\pi}{4}\) and the second PCF going like \(D_p((1+i)|z|)\) with \(\text{Arg}[(1+i)|z|] = \frac{\pi}{4}\). Since the arguments are different they will have different behaviors asymptotically. Thus we find,

\[
\phi_p(z \to -\infty) \approx (2|z|)^{-p+1} e^{\frac{i\pi}{2}} \left( -P_a \frac{\sqrt{2\pi} e^{-i\pi p}}{\Gamma(-p)} \left( -\frac{1+i}{2} \right)^{-(p+1)} \right) + \\
(2|z|)^p e^{-\frac{i\pi}{2}} \left( P_a \left( -\frac{1+i}{2} \right)^p + Q_a \left( \frac{1+i}{2} \right)^p \right). 
\]  

(4.95)

We match the solutions for early times using \(\alpha_p = 1\) and \(\beta_p = 0\) in equation 4.87 where we find,

\[
P_a = -\frac{\Gamma(-p)e^{i\pi p}}{\sqrt{2\pi \sqrt{|eE|}}} \left( -\frac{1+i}{2} \right)^{p+1} Q_a = -P_a e^{i\pi p}. 
\]  

(4.96)

Now that we have the constants \(P_a\) and \(Q_a\) for the general solution, we can solve for
the Bogolyubov coefficients in the distant future \((t \to -\infty)\). For the adiabatic solution (equation 4.87) in the future we find,

\[
\phi_p(t \to \infty) \approx \frac{\alpha_p}{\sqrt{2\sqrt{|eE|}}} 2z^p e^{-iz^2} + \frac{\beta_p}{\sqrt{2\sqrt{|eE|}}} z^{-(p+1)} e^{iz^2}.
\]

(4.97)

For the general solution, first PCF goes as \(D_p((1 + i)z)\) with \(\text{Arg}[(1 + i)z] = \frac{\pi}{4}\), and the second PCF as \(D_p(-(1 + i)z)\) with \(\text{Arg}[-(1 + i)z] = -\frac{3\pi}{4}\). This leads to,

\[
\phi_p(z \to \infty) \approx (2z)^p e^{-iz^2} \left( P_a \left( \frac{1 + i}{2} \right)^p + Q_a \left( -\frac{1 + i}{2} \right)^p \right) + (2z)^{-(p+1)} e^{iz^2} \left( -Q_a \sqrt{2\pi e^{-iz^2}} \Gamma(-p) \left( -\frac{1 + i}{2} \right)^{-(p+1)} \right).
\]

(4.98)

Matching coefficients for the above equations, we find

\[
\alpha_p = P_a \left( \frac{1 + i}{2} \right)^p + Q_a \left( -\frac{1 + i}{2} \right)^p \quad \beta_p = -Q_a \sqrt{2\pi e^{-iz^2}} \Gamma(-p) \left( -\frac{1 + i}{2} \right)^{-(p+1)}.
\]

(4.99)

In order to calculate the number density for the produced Schwinger pairs, we will be interested in the coefficient for the positive-frequency solutions, \(\beta\), whose non-zero value is interpreted as particle production,

\[
|\beta_p|^2 = e^{-a\pi} = e^{-\pi(m^2 + p^2 + p_y^2)} |eE|.
\]

(4.100)

### 4.3.2 Number of Particles and Rate of Production

We can now calculate the number density of \(\phi\) particles,

\[
n_\phi = \int \frac{d^3p}{(2\pi)^3} |\beta_p|^2 = e^{-\pi m^2 |eE|} \int_{-\infty}^{+\infty} e^{-\pi p^2} dp_x \int_{-\infty}^{+\infty} e^{-\pi p_y^2} dp_y \int_{-\infty}^{+\infty} dp_z.
\]

(4.101)
The $p_x$ and $p_y$ integrals are simple Gaussian integrals, while the $p_z$ integral ostensibly seems to be divergent. However, we must remember that particle production occurs when $\phi$ goes through its period of nonadiabaticity which happens around $z \simeq 0$, and so we only have $\phi$ particles for $p_z < |eE|t$. For the lower limit, we introduce a regulator $p = |eE|t_0$ corresponding to when the electric field is switched on, $t_0 < t$. If this is not included then we would have infinite particle production corresponding to the case of an electric field being on for an infinite amount of time. With the $p_z$ limits set, we are now in the position to calculate the number density of $\phi$,

$$n_\phi = \frac{|eE|e^{-\frac{m^2}{2|eE|t}}} {8\pi^3} \int_{|eE|t_0}^{|eE|t} dp_z = \frac{e^{-\frac{m^2}{2|eE|}}} {4\pi^3} (eE)^2 \Delta t,$$

(4.102)

where we have set $\Delta t = t - t_0$. Again we note that the above expression diverges for $t \to \infty$ due to having a constant electric field producing pair particles over an infinite length of time. We see that production of pair particles is enhanced for,

$$\frac{m^2}{|eE|} \ll 1 \Rightarrow |E| \gg \frac{m^2}{|e|},$$

(4.103)

which makes sense in light of our heuristic derivation in equation 4.73. A more useful quantity will be how the electric field will change over time which we will use in the next section,

$$\dot{n}_\phi = \frac{(eE)^2} {(2\pi)^3} e^{-\frac{m^2}{|eE|}}.$$

(4.104)
4.3.3 Energy Conservation and Dissipating the Electric Field

Let’s now find how the energy density changes over time. The energy density of the electric field is given by

\[ \rho_E = \frac{E^2}{2}. \]  

(4.105)

The time derivative of \( \rho_E \) assuming \( E \) is a function of time will then be,

\[ \dot{\rho}_E = E \dot{E}. \]  

(4.106)

The energy density for the pair production of \( \phi \) is,

\[ \rho_\phi = \int \frac{d^3p}{(2\pi)^3} \omega |\beta_p|^2, \quad \omega_p = \sqrt{(eEt - p_z)^2 + p_x^2 + p_y^2 + m^2}. \]  

(4.107)

Plugging in for \( \beta_p \) we find,

\[ \rho_\phi = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{|\epsilon|\epsilon_0}^{|\epsilon|t} \frac{dp_x dp_y dp_z}{(2\pi)^3} \sqrt{(eEt - p_z)^2 + p_x^2 + p_y^2 + m^2} e^{-\frac{\pi}{|\omega_p|}(p_x^2 + p_y^2 + m^2)}. \]  

(4.108)

Again, let’s calculate how the energy changes with respect to time,

\[ \dot{\rho}_\phi = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{|\epsilon|\epsilon_0}^{|\epsilon|t} dp_x dp_y e^{-\frac{\pi}{|\omega_p|}(p_x^2 + p_y^2 + m^2)} \times \]  

\[ \times \frac{d}{dt} \left( \int_{|\epsilon|\epsilon_0}^{|\epsilon|t} dp_z \sqrt{(eEt - p_z)^2 + p_x^2 + p_y^2 + m^2} \right). \]  

(4.109)
\[
\dot{\rho}_\phi = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \, dp_x dp_y \, e^{-\frac{\pi}{|eE|} (p_x^2 + p_y^2 + m^2)} \left( eE \sqrt{(f(p_x, p_y))^2 + p_x^2 + p_y^2 + m^2} + eE \int_{|eE|t}^{\infty} \, dp_z \, \frac{eEt - p_z}{\sqrt{(eEt - p_z)^2 + p_x^2 + p_y^2 + m^2}} \right). \tag{4.110}
\]

Since we will be interested in late times, we neglect the first term which is independent of time,

\[
\dot{\rho}_\phi = \frac{eE}{(2\pi)^3} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \, dp_x dp_y \, e^{-\frac{\pi}{|eE|} (p_x^2 + p_y^2 + m^2)} \sqrt{(eEt - eEt_0)^2 + p_x^2 + p_y^2 + m^2}. \tag{4.111}
\]

Converting to polar coordinates we have,

\[
\dot{\rho}_\phi = \frac{eE}{(2\pi)^2} \int_0^{+\infty} \, dp_r \, p_r \, e^{-\frac{\pi}{|eE|} (p_r^2 + m^2)} \sqrt{(eEt - eEt_0)^2 + p_r^2 + m^2}. \tag{4.112}
\]

This yields,

\[
\dot{\rho}_\phi = \frac{(eE)^2 \sqrt{eE}}{\sqrt{\pi} (2\pi)^2} \left( \frac{\sqrt{\pi}}{2} e^{\frac{\pi}{|eE|} (eEt - eEt_0)^2} \text{Erfc} \left[ \sqrt{\frac{\pi}{eE} (eEt - eEt_0)^2} \right] + e^{-\frac{\pi m^2}{|eE|}} \sqrt{\frac{\pi}{eE} (eEt - eEt_0)^2} \right), \tag{4.113}
\]

which for large \( t \) yields,

\[
\dot{\rho}_\phi = \frac{(eE)^2}{(2\pi)^2} \left| eEt - eEt_0 \right| e^{-\frac{\pi m^2}{|eE|}}. \tag{4.114}
\]

If our system is solely the electric field and the \( \phi \) particles then energy conservation will yield,

\[
\rho_E + \rho_\phi = \text{Constant} \quad \Rightarrow \quad \dot{\rho}_E = -\dot{\rho}_\phi \quad \Rightarrow \quad E \dot{E} = -\frac{(eE)^2}{(2\pi)^2} \left| eEt - eEt_0 \right| e^{-\frac{\pi m^2}{|eE|}}. \tag{4.115}
\]
Figure 4.4: Plot of $\bar{E}(\tilde{t})$ vs. $\tilde{t}$ for various values of $\tilde{m}$. The plots from the top (red) to the bottom (blue) correspond to $\tilde{m} = 10, 2, 1, 0.1$ respectively.

Figure 4.5: Plot of $\bar{E}(\tilde{t})$ vs. $\tilde{m}$ for $\tilde{t} = 20$. For those values of $\tilde{m} \lesssim O(1)$ there is an effective discharge of the electric field, while for those values $\tilde{m} \gtrsim O(1)$ the electric field does not discharge.

We set our initial conditions to be $E(0) = E_0$ and for simplicity $t_0 = 0$. Calculating the above explicitly,

$$\int_{E_0}^{E(t)} \frac{e^{\pi m^2}}{E|E|} dE = -\frac{e^3}{(2\pi)^2} \int_0^t t' dt',$$

which yields,
\[
\bar{E}(t) \equiv \frac{E(t)}{E_0} = \left(1 + \frac{1}{\tilde{m}^2} \ln \left(1 + \tilde{m}^2 \tilde{t}^2 e^{-\tilde{m}^2} \right) \right)^{-1}, \quad \tilde{m}^2 = \frac{\pi m^2}{eE_0}, \quad \tilde{t}^2 = \frac{E_0 \epsilon^2 t^2}{8\pi^2},
\]

(4.117)

where we have introduced dimensionless variables. A plot of \( \bar{E} \) for various values of \( \tilde{m} \) is shown in figure 4.4. As you can see, the electric field effectively discharges for values \( \tilde{m} \lesssim O(1) \) which is in line with the critical electric field we calculated in equation 4.73. If we take \( \tilde{m} \to \infty \) then we find \( \bar{E} \to 1 \) meaning the electric field fails to discharge which makes sense for pair production of particles whose mass is much larger than the critical electric field. For a massless field, the electric field reads

\[
\bar{E}(\tilde{t}, \tilde{m} \to 0) = \frac{1}{1 + \tilde{t}^2},
\]

(4.118)

which is the most effective way of dissipating the electric field since the produced particles are massless. Finally, we plot \( \bar{E} \) vs \( \tilde{m} \) for \( \tilde{t} = 20 \) in figure 4.5. For those values of \( \tilde{m} \lesssim O(1) \) there is an effective discharge of the electric field, while for those values \( \tilde{m} \gtrsim O(1) \) the electric field does not discharge. Again, this makes sense since for those values of the electric field from equation 4.73 less than the critical value there is an ineffective creation of pair particles.

Now that we have examined the Schwinger mechanism in the simpler Minkowski background, we now couple the Ratra model with the Schwinger mechanism in a de Sitter background.
4.4 Magnetic and Electric Field Production: Ratra Model and Schwinger Effect

We finally turn to the task of combining both mechanisms previously discussed which are the Ratra model with the Schwinger effect. This is done by taking the gauge field $A_\mu$ of equation 4.5 and minimally coupling it to a charged, massive scalar field $\phi$ in de Sitter space. The fields $\phi$ and $\phi^*$ are the Schwinger pairs generating a current that will in turn help to reduce the overall electric field since they produce their own electric field which will counter and thus decrease the electric field sourcing their generation. The action combining both mechanisms is given by,

$$S = \int d^4x \sqrt{-g} \left( -\frac{I(\tau)^2}{4} F^{\mu\nu} F_{\mu\nu} - (D_\mu \phi)^*(D^\mu \phi) - m^2 \phi \phi^* - \frac{R}{6} \phi \phi^* \right),$$

(4.119)

where $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative, $m$ is the mass of $\phi/\phi^*$ and $R = \frac{6a''}{a}$ is the Ricci scalar. This last term is a conformal coupling for $\phi$ to gravity and is included so that the additional mass term generated by the gravitational field is exactly canceled by this added term making the equations of motion easier to solve. The equation of motion for $\phi$ reads,

$$\phi'' + 2\frac{a'}{a} \phi' - \Delta \phi - i2eA_i \partial_i \phi + e^2 A_i^2 \phi + a^2 m^2 \phi + \frac{a''}{a} \phi = 0,$$

(4.120)

where we have chosen to work in the Coloumb gauge, $A_r = \partial_r A_i = 0$. The equation of motion for the gauge field reads,

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} I^{\mu\nu}) = J^\nu = g'^\alpha \left[ ie(\partial_\alpha \phi^*) \phi - ie(\partial_\alpha \phi) \phi^* + e^2 A_\alpha (\phi^* \phi + \phi \phi^*) \right].$$

(4.121)
The spatial equation of motion for the gauge field will be,

\[
\frac{I^2}{a^2} \left[ A_i'' + \frac{2}{I} A_i' - \Delta A_i \right] = J_i \equiv i e (\partial_i \phi) \phi^* - i e (\partial_i \phi^*) \phi - e^2 A_i (\phi^* \phi + \phi \phi^*).
\] (4.122)

We solve for the gauge field in the classical (long wavelength) limit with the current term providing a source from the Schwinger effect which is calculated as an ensemble average, \(\langle \hat{J}_i \rangle\). To obtain the expression for the current we must first solve \(\phi\)'s equation of motion.

We first promote \(\phi\) to an operator \(\hat{\phi}\) and decompose it along a set of creation/annihilation operators,

\[
\hat{\phi}(x, \tau) = \frac{1}{a(\tau)} \int \frac{d^3k}{(2\pi)^{3/2}} e^{ik \cdot x} \hat{\phi}(k, \tau) = \frac{1}{a(\tau)} \int \frac{d^3k}{(2\pi)^{3/2}} e^{ik \cdot x} \left[ \phi_p(\tau) \hat{a}(p) + \phi_p^*(\tau) \hat{b}^\dagger(p) \right],
\] (4.123)

where the mode functions satisfy,

\[
\phi_p'' + \left[ p^2 + 2e A_i p_i + e^2 A_i^2 + a^2 m^2 \right] \phi_p = 0.
\] (4.124)

As we saw in the last section, the electric field is more effectively discharged for a lighter field and so we take \(\phi\) to be massless \((m = 0)\) in order to maximize the overall effect. Since we already know that the Ratra model produces an electric field, we model this electric field by again taking \(A_z = -E \tau\) in order to estimate the produced Schwinger current,

\[
\phi_p'' + \left[ (p_z - e E \tau)^2 + p_x^2 + p_y^2 \right] \phi_p = 0.
\] (4.125)

The expression for the current will read,

\[
\langle \hat{J}_z(x, \tau) \rangle = \langle \left[ i e (\partial_z \phi(x, \tau)) \phi^*(x, \tau) - i e (\partial_z \phi^*(x, \tau)) \phi(x, \tau) - e^2 A_z (\phi^*(x, \tau) \phi(x, \tau) + \phi(x, \tau) \phi^*(x, \tau)) \right] \rangle.
\] (4.126)
where again we use the decomposition of $\phi$ in equation 4.123 to write,

$$
\langle : \hat{J}_z(x, \tau) : \rangle = \frac{1}{a^2(\tau)} \int \frac{d^3p d^3p'}{(2\pi)^3} e^{i(p-p') \cdot x} \left[ - e p_z \phi(p) \phi^\dagger(p') - e p'_z \phi^\dagger(p') \phi(p) \right. \\
\left. - e^2 A_z \left( \phi^\dagger(p') \phi(p) + \phi(p) \phi^\dagger(p') \right) \right] .
$$

(4.127)

To solve the above, we normal-order with respect to the adiabatic operators defined through the adiabatic expression for the field in momentum space,

$$
\hat{\phi}(p, \tau) = \tilde{\phi}_p(\tau) \hat{a}(p) + \tilde{\phi}_p^*(\tau) \hat{b}^\dagger(-p), \quad \tilde{\phi}_p(\tau) = \frac{1}{\sqrt{2\omega_p}} e^{-i \int \omega(\tau') d\tau'},
$$

(4.128)

where $\hat{a}$ and $\hat{b}$ are the adiabatic operators defined through the Bogolyubov transformation,

$$
\begin{pmatrix}
\tilde{a}_p(\tau) \\
\tilde{b}^\dagger_{-p}(\tau)
\end{pmatrix}
= \begin{pmatrix}
\alpha_p(\tau) & \beta_p^*(\tau) \\
\beta_p(\tau) & \alpha_p^*(\tau)
\end{pmatrix}
\begin{pmatrix}
a_p \\
b^\dagger_p
\end{pmatrix}.
$$

(4.129)

The main quantity we need to find is the correlator,

$$
\langle : \phi(p) \phi^\dagger(p') : \rangle = \tilde{\phi}_p \tilde{\phi}^*_p \langle : \hat{a}(p) \hat{a}^\dagger(p') : \rangle + \tilde{\phi}_p \tilde{\phi}^*_p \langle : \hat{a}(p) \hat{b}(-p') : \rangle + \tilde{\phi}^*_p \tilde{\phi}^*_p \langle : \hat{b}(-p) \hat{a}^\dagger(p') : \rangle + \tilde{\phi}^*_p \tilde{\phi}^*_p \langle : \hat{b}(-p) \hat{b}^\dagger(-p') : \rangle,
$$

(4.130)

where we normal-order w.r.t the adiabatic operator but act on the vacuum with the operators defined during the period of nonadiabaticity since this is when particle production occurs,

$$
\langle : \phi(p) \phi^\dagger(p') : \rangle = \delta^{(3)}(p - p') \left[ |\tilde{\phi}_p|^2 |\beta_p|^2 + \tilde{\phi}_p \tilde{\phi}_p^* \alpha_p \beta_p^* + \tilde{\phi}_p \tilde{\phi}_p^* \alpha_p \beta_p + |\tilde{\phi}_p|^2 |\beta_p|^2 \right].
$$

(4.131)
Using the above, our equation for the Schwinger current becomes,

\[
\langle \hat{J}_z(x, \tau) \rangle = \frac{-2e}{a^2(\tau)} \int \frac{d^3p}{(2\pi)^3} \frac{(p_z - eE\tau)|\beta_p|^2}{\omega_p} = -2e \int \frac{d^3p}{(2\pi)^3} \frac{(p_z - eE\tau)e^{-\frac{(p_x^2 + p_y^2)}{\omega_p^2}}}{\sqrt{p_x^2 + p_y^2 + (p_z - eE\tau)^2}},
\]

(4.132)

where we have neglected the oscillating terms for the \( \phi \) correlator. In the adiabatic regime when the Bogolyubov coefficients are constant the frequency for \( \phi \) will be \( \omega_p \simeq |p_z - eE\tau| \) and we have the greatest production of \( \phi \) for \( p_z \simeq eE\tau \). This leads to our expression for the current as,

\[
\langle \hat{J}_z(x, \tau) \rangle = -\frac{e^3E|E|}{4a^2(\tau)\pi^3}(\tau - \tau_*),
\]

(4.133)

where \( \tau_* \) is the time of production for \( \phi \). We now want to introduce the Schwinger current back into the equation of motion for the gauge field, but we wish to do so in a way that incorporates the current in a time-dependent manner since we want to see how the gauge field responds to such a current. Even though we calculated the Schwinger current for a constant electric field, we now take the electric field for the Schwinger current to be time dependent and write it as,

\[
\langle \hat{J}_z(x, \tau) \rangle = -\frac{e^3E|E|}{4\pi^3a^2(\tau)} \int d\tau |E(\tau)|,
\]

(4.134)

where for a constant electric field, we recover our previous expression. The above equation is valid as long as the electric field is evolving adiabatically. We can now use the Schwinger current in the equation of motion for the gauge field,

\[
(I^2E)' = -\frac{e^3}{4\pi^3} \int E|E|d\tau,
\]

(4.135)
where in keeping with how we defined the electric field for the Schwinger effect we define the electric field in terms of the gauge field, $E = -A'$. We can eliminate the integral from the above equation by taking the derivative of the equation,

$$E'' + 4 \frac{I'}{I} E' + 2 \left( \frac{(I')^2}{I^2} + \frac{I''}{I} \right) E = - \frac{e^3 E|E|}{4\pi^3 I^2}. \quad (4.136)$$

In order to solve the above equation numerically, we will need the initial value for the electric field at the start of magnetogenesis which we can obtain from equation 4.53

$$E^2(\tau) = \frac{c_4(n)H^{2n}e^{4n}}{\pi^2} \int_{H_{a_i}}^{Ha} dp p^{2n+3} = \frac{H^4 c_4(n)a^{4-2n}}{\pi^2(2n+4)} \left[ 1 - \left( \frac{a_i}{a} \right)^{2n+4} \right], \quad (4.137)$$

where the main contribution for $n < -\frac{1}{2}$ comes from those modes that leave the horizon at the beginning of inflation. Since we are interested in how the energy in the electric field compares to the background energy, we dimensionize the electric field with respect to the background energy,

$$\tilde{E} = \frac{E}{\sqrt{3}M_P H} \quad (4.138)$$

and introduce the time variable,

$$T = \ln \left( \frac{a}{a_i} \right), \quad dT = \frac{da}{a} = \frac{a'}{a} d\tau, \quad (4.139)$$

where $T$ is defined for $T \in [0, N]$ and $N$ is the number of e-foldings during inflation. The expressions involving derivatives of the coupling constant can be simplified using the following relations: $I' = naH I$ and $I'' = (n + n^2)(Ha)^2 I$. Using the above relations, we can write the equation for the electric field as,

$$\frac{d^2 \tilde{E}}{dT^2} + (4n + 1) \frac{d\tilde{E}}{dT} + 2n(2n + 1) \tilde{E} = - \frac{e^3 \sqrt{3} M_P}{4\pi^3 Ha} |\tilde{E}| e^{2(n+1)(N-T)}, \quad (4.140)$$
where the initial condition now reads

\[
\tilde{E}(T_i) = \frac{H}{M_P} \sqrt{\frac{c_4(n)}{6\pi^2(n + 2)}} \left(1 - e^{-2T_i (n+2)}\right),
\]

where we have remembered that a factor of \(a^4/I^2\) should be multiplied to the overall expression. We can estimate when the Schwinger effect becomes important by comparing the Schwinger term with the third term for the Ratra model,

\[
\frac{2n(2n+1)\tilde{E}_c}{4\pi^3} = \frac{e^3\sqrt{3} M_P}{H} \tilde{E}_c |\tilde{E}_c| e^{2(n+1)N-T_c} |\tilde{E}_c|e^{-3nN}.
\]

We can use equation 4.137 to estimate the electric field when the Schwinger effect becomes important,

\[
\tilde{E}(T_c) = \frac{H}{M_P} e^{-2nT_c-N(2-n)} \sqrt{\frac{c_4(n)}{6\pi^2(n + 2)}}.
\]

We find that the critical value when the Schwinger effect becomes important is

\[
T_c = \frac{1}{2 - 4n} \ln \left( \frac{8\pi^4 n(2n+1)}{e^3} \sqrt{\frac{2|n+2|}{c_4(n)}} e^{-3nN} \right).
\]

Figure 4.6 shows the ratio of the energy in the electric field to the background energy both with and without the Schwinger effect. We find that there is an effect with the Schwinger mechanism allowing the value of \(n\) to be decreased to lower values before the electric field energy dominates once again. The change in \(n\) is only marginal though going from \(n = -2.2\) to a value \(n = -2.3\). This lowering of \(n\) will in turn increase the magnetic field generated by the Ratra model which can be estimated using 4.48. In figure 4.7, we plot the magnetic field for cosmological scales today (\(\lambda_{today} \sim \text{Mpc}\)) as a function of \(n\) for \(n < -2\). As you can see, the Schwinger mechanism does allow the
Figure 4.6: Plot of the energy density in the electric field both with the Schwinger effect (blue line) and without (orange line). The Schwinger mechanism lowers the energy in the electric field with the allowed value of $n$ going from $-2.2$ to $-2.3$.

Figure 4.7: Plot of the magnetic field generated by the Ratra model with the Schwinger effect (red dotted line) and without (blue dotted line). The Schwinger mechanism increases the magnetic field by 2 orders of magnitude, $B_0 \sim 10^{-27}$ G.

The magnetic field increases from $B_0 \sim 10^{-29}$ G to $B_0 \sim 10^{-27}$ G which is an increase in two orders of magnitude. However, this falls far short of producing a substantial magnetic field strength today of around $B_0 \sim 10^{-17}$ G.
4.5 Conclusion

This Chapter has studied the generation of magnetic fields in the early universe using the Ratra model coupled with the Schwinger effect during inflation as a source for present day large scale magnetic fields. The exact origin of these cosmological magnetic fields is still unclear, but the Ratra model is one possible way of generating them early in the Universe in order to explain both magnetic fields in the cosmic voids as well as provide a seed magnetic field for galaxies. The Ratra model alone is not capable of generating a sufficient magnetic field due to the additional generation of a large electric field. We used the Schwinger mechanism to siphon off energy from the electric field in hopes of generating a more substantial magnetic field. We found that the Schwinger mechanism does have an effect and lowers the electric field to an extend. However, since this mechanism only kicks in towards the end of inflation its overall effect is not very substantial. The magnetic field generated is increased by two orders of magnitude over that produced without the Schwinger mechanism. This translates into a magnetic field strength today of $B_0 \sim 10^{-27}$ which is 10 orders of magnitude less than the estimated field strength in the cosmic voids.
CHAPTER 5

FINAL REMARKS

This thesis has explored the production of three large scale observables which might have been created during inflation. In Chapter 2, we found that a net charge density can be generated for our observable universe through the amplification of charged scalar fields. This net charge density is produced by only considering the variance associated with the charge density in a finite region of space. We found that the produced charge density is capable of explaining the present upper bound for large scale charge densities. In Chapter 3, the generation of a feature in the primordial gravitational wave power spectrum was discussed which was sourced by a scalar field with a rapid time dependent mass. Our results indicate that unlike previous research which produced negligible GWs sourced by scalar fields, our mechanism is capable of producing a significantly larger amplitude which might even be capable of detection with future experiments. Finally in Chapter 4, we discussed the generation of primordial magnetic fields by considering both the Ratra and Schwinger mechanisms with the goal of siphoning off energy from the electric field into the magnetic field. We found that the two mechanisms together are capable of transferring some of this energy to the magnetic field, but not to the degree whereby the generated magnetic field is capable of explaining present day fields.
All of the above mechanisms were produced in the first moments of our universe during inflation. Inflation itself is far from being a fully-formed, self-consistent theory capable of accurately predicting the nature of the universe in which we find ourselves. It does – however – do a rather remarkable job even in its present form of explaining key features about our universe and represents our best “guess” for what came before the traditional Hot Big Bang. As is always the case, future experiments will provide the much needed observations which will enable us to both hone in on how exactly inflation transpired and also narrow down the specific microphysical incarnation which drove its expansion.

There is the unfortunate possibility that we will never have a fully formed theory for how inflation took place, or even worse (depending on your perspective) that we will not be able to fully discriminate between inflation and some other early universe model. The “experiment” that we are testing inflation against took place billions of years ago when “no one” was around, and so we can only make inferences based off of how inflation affected other things. We will probably never be able to reproduce the same high energy conditions of inflation again leaving us with a measly sample size of one. So even if we do one day have a theory of inflation in which we believe, from a rigidly scientific point of view where falsifiability and reproducibility are paramount, we will probably never be able to test the veracity of such a theory.

That being said, it should not be considered a fool’s errand to try and work towards a better if imperfect understanding of the early universe. There might come a time (and some would say it has already come) where the evidence overwhelming favors an inflationary expansion over any alternative. If this occurs, it will likely be due to inflation accruing more and more circumstantial evidence, much like the work presented in this thesis, which would show that only an inflationary expansion is capable of producing and explaining the cosmological observables we see around us. It could never be said that inflation (or for that matter any scientific theory) has been proven or disproven, but we
might find that it is much more reasonable to assume an inflationary expansion than to not.

Finally, we note that inflation represents a remarkable period in the evolution of our universe which joined two disparate branches of physics (quantum mechanics and cosmology) by stretching quantum fluctuations through its exponential expansion to cosmological scales. These stretched fluctuations represent small ripples in spacetime from which the first galaxies and stars formed. Carl Sagan once said that, “We are made of star-stuff.” If inflation is indeed how our universe started out, a more apt adage might be, “We are made of quantum-stuff.”
In this appendix, we discuss the Bogolyubov formalism which is used in all of the mechanisms discussed in the main body. It is a procedure for renormalization and appropriately defining particle densities in a time-dependent background. We first provide some motivation for why this is necessary by exploring the arguments used in [90] for fermions and in [40] for real scalars, but we generalize them to complex scalars since they are used in Chapters 2 and 4. We then discuss the specific case of a scalar with a time dependent mass during inflation.

To start, we examine the structure of the Hamiltonian for a complex scalar field, starting from its action in terms of the comoving field

$$S = \int d^4x \left\{ \phi' \phi'' + \phi^* \left[ \Delta + \frac{a''}{a} - a^2 m^2 \right] \varphi \right\},$$  \hspace{1cm} (A.1)

where the conjugate momenta of $\phi$ and $\phi^*$ are $\Pi_\phi = \phi''$ and $\Pi_{\phi^*} = \phi'$. We can decompose the scalar field using,

$$\phi(k, \tau) \equiv \phi(k, \tau) a(k) + \phi^*(-k, \tau) b^*(-k),$$  \hspace{1cm} (A.2)

which enables us to write the Hamiltonian as

$$H = \int d^3k \left[ (a_k a_k^\dagger + b_{-k} b_{-k}) g(k, \tau) + a_k b_{-k} f(k, \tau) + b_{-k} a_k^\dagger f^*(k, \tau) \right],$$  \hspace{1cm} (A.3)

where

$$f(k, \tau) = \phi'(k)^2 + \omega_k^2 \phi(k)^2,$$

$$g(k, \tau) = |\phi'(k)|^2 + \omega_k^2 |\phi(k)|^2.$$  \hspace{1cm} (A.4)

The Hamiltonian in the above form is generally not diagonal and the definition of the number operator is unclear since the $a$ and $b$ operators do not annihilate energy eigenstates. We can however diagonalize it by performing a Bogolyubov transformation on the
operators,
\[
\begin{pmatrix}
\tilde{a}_k(\tau) \\
\tilde{b}^\dagger_{-k}(\tau)
\end{pmatrix} = \begin{pmatrix} \alpha_k(\tau) & \beta_k^*(\tau) \\ \beta_k(\tau) & \alpha_k^*(\tau) \end{pmatrix} \begin{pmatrix} a_k \\
b_k \end{pmatrix},
\]
(A.5)
where \(\alpha_k\) and \(\beta_k\) are the Bogolyubov coefficients, and \(\tilde{a}_k\) and \(\tilde{b}_k\) are new annihilation operators. By imposing that both the \(a_k, b_{-k}\) and the \(\tilde{a}_k, \tilde{b}_{-k}\) operators satisfy canonical commutation relations, we find the constraint \(|\alpha_k|^2 - |\beta_k|^2 = 1\). We can invert the transformation (A.5) as
\[
\begin{pmatrix} a_k(\tau) \\
b_{-k}^\dagger(\tau)
\end{pmatrix} = \begin{pmatrix} \alpha_k^*(\tau) & -\beta_k^*(\tau) \\ -\beta_k(\tau) & \alpha_k(\tau) \end{pmatrix} \begin{pmatrix} \tilde{a}_k \\
\tilde{b}^\dagger_{-k} \end{pmatrix},
\]
(A.6)
which allows us to write the Hamiltonian as
\[
H = \int d^3k \omega_k \left[ \tilde{a}_k \tilde{a}^\dagger_k + \tilde{b}^\dagger_k \tilde{b}_k \right],
\]
(A.7)
where \(f(k, \tau)\) and \(g(k, \tau)\) must satisfy
\[
f(k, \tau) = 2\omega_k \alpha_k \beta_k, \quad g(k, \tau) = \omega_k \left(|\alpha_k|^2 + |\beta_k|^2\right),
\]
(A.8)
in order for eq. (A.7) to hold. In contrast to the original operators \(a_k\) and \(b_k\), that do not annihilate energy eigenstates, the \(\tilde{a}_k, \tilde{b}_{-k}\) operators are associated to physical particles, so that we can unambiguously define the number operators, \(\tilde{N}_a^k = \tilde{a}_k \tilde{a}^\dagger_k\) and \(\tilde{N}_b^k = \tilde{b}^\dagger_k \tilde{b}_k\).

In particular, since we are working in the Heisenberg picture, the creation/annihilation operators will evolve, but the states will be constant. This implies that if the system was initially in its vacuum \(|0\rangle\), the occupation number will be given by \(\langle 0|\tilde{N}_a^k|0\rangle = \langle 0|\tilde{N}_b^k|0\rangle = |\beta_k|^2\). The number of particles will not change (other than for trivial dilution effects in the expanding Universe) when eqs. (A.8) are satisfied with \(\alpha_k\) and \(\beta_k\) constant. This is precisely what occurs when the mode functions satisfy a WKB-type solution,
\[
\varphi(k, \tau) = \alpha_k \hat{\varphi}(k, \tau) + \beta_k \hat{\varphi}^*(k, \tau),
\]
\[
\hat{\varphi}(k, \tau) \equiv \frac{1}{\sqrt{2\omega_k}} e^{-i \int \omega_k d\tau},
\]
(A.9)
with the condition for adiabaticity \(|\alpha'| \ll \omega^2\).

To summarize, an adiabatic vacuum exists (along with its associated adiabatic operators)
for early times during which our Hamiltonian is diagonalized allowing us to clearly define the number of particles\(^1\). Subsequently, a period of nonadiabacity occurs, and then our system evolves adiabatically again. The Bogolyubov transform in eq. (A.5) allows us to relate the initial and the final adiabatic stages, and the Bogolyubov coefficients enable us to calculate the number of particles created during the period of nonadiabaticity.

The above considerations were for a general complex scalar field during inflation. We now use the above formalism to explore the specific case of a real scalar field with a time dependent mass and relate it to specific instances used in the main body of the thesis. We begin with the equation of motion for a scalar field,

\[
\ddot{\phi}(x, t) + 3\frac{\dot{a}}{a}\dot{\phi}(x, t) - \frac{\Delta \phi(x, t)}{a^2} + m^2_\phi(t)\phi = 0,
\]

(A.10)

where \(m_\phi(t)\) is some arbitrary function with mass dimension 2. Writing the above in momentum space using the following Fourier transform,

\[
\phi(x, \tau) = \frac{1}{a(\tau)}\int \frac{d^3p}{(2\pi)^{3/2}} e^{ip\cdot x} \phi(p, \tau),
\]

(A.11)

the equation of motion in momentum space then reads,

\[
\phi''(p, \tau) + \left[p^2 - \frac{a''}{a} + a^2 m^2_\phi(\tau)\right] \phi(p, \tau) = \phi''(p, \tau) + \omega^2(\tau)\phi(p, \tau) = 0,
\]

(A.12)

where we have switched to conformal time. In conformal time, the canonically normalized field \(\phi(p, \tau)\) acts like a scalar field in Minkowski with the gravitational field encapsulated in the effective mass

\[
m^2_{\text{eff}}(\tau) = a^2 m(\tau)^2 - \frac{a''}{a} = \frac{1}{\tau^2} \left[\frac{m^2(\tau)}{H^2} - 2\right].
\]

(A.13)

If \(m(\tau) > \sqrt{2}H\), then the effective mass of \(\phi\) is positive definite and so is the frequency \(\omega^2\). However, if \(m(\tau) < \sqrt{2}H\), then the effective mass of \(\phi\) can be imaginary for late times and so will the frequency leading to a growing solution for the mode functions.

\(^1\)We assume that the initial charge density vanishes, but even if there is an initial density it will quickly be diluted by the de Sitter expansion
A proposed solution for equation A.12 could be,

\[
\phi(p, \tau) = \frac{1}{\sqrt{2W_p(\tau)}} e^{-i \int W_p(\tau') d\tau'},
\]

(A.14)

where \(W_p\) is some arbitrary function of \(p\) and \(\tau\) with mass dimension 1. Plugging this into the equation of motion, we find

\[
W_p'^2 = \omega_p^2 - \frac{1}{2} \left[ \frac{W_p''}{W_p} - \frac{3}{2} \left( \frac{W_p'}{W_p^2} \right)^2 \right].
\]

(A.15)

As previously mentioned, we would like to have \(W_p = \omega_p\), so that we can approximate the mode functions using equation A.9. This is achieved if

\[
\frac{W_p''}{W_p^3} \ll 1, \quad \left( \frac{W_p'}{W_p^2} \right)^2 \ll 1,
\]

(A.16)

which yields the requested relation \(W_p \simeq \omega_p\). This in turn implies,

\[
\frac{\omega_p''}{\omega_p^3} \ll 1, \quad \left( \frac{\omega_p'}{\omega_p^2} \right)^2 \ll 1.
\]

(A.17)

For the case of a massless particle, the adiabatic expressions are,

\[
\frac{\omega_p''}{\omega_p^3} = -\frac{2a^4 \left( 3a^2_* - 4a^2 \right)}{(a^2_* - 2a^2)^3}, \quad \left( \frac{\omega_p'}{\omega_p^2} \right)^2 = \frac{4a^6}{(a^2_* - 2a^2)^3},
\]

(A.18)

where there is a clear period of nonadiabaticity centered around \(a(\tau_{nad}) \equiv a_{adi} = 2^{-1/2}a_*\). Plots of \(\omega_p''/\omega_p^3\) and \((\omega_p''/\omega_p^2)^2\) are shown in figure A.1 with \(a_* = e^{-3}\). For times \(\tau \ll \tau_*\), the adiabatic conditions are satisfied with both terms \(\ll 1\). However, around \(a_{nad} = a_*\) and afterwards both conditions are violated implying that an adiabatic vacuum does not exist at the end of inflation for a massless particle and so calculating the Bogolybov coefficients at this time would be inappropriate.

The massless case is similar to our discussion in Chapter 2 for very light or massless particles sourcing charge fluctuations. In order to ensure an adiabatic vacuum does exist, we join the end of inflation to a radiation epoch where the scale factor and frequency will
Figure A.1: Plots of $\omega''_p/\omega^3_p$ (left) and $(\omega''_p/\omega^2_p)^2$ (right) are shown for a massless particle during inflation with $a_* = e^{-3}$.

be,

$$a(\tau) = H\tau + 2, \quad \omega^2_p = p^2 - \frac{a''}{a} + m^2(\tau)a^2(\tau) = p^2 + m^2(\tau)(H\tau + 2)^2.$$  \(A.19\)

For a scalar with constant mass ($M = m/H$) the adiabatic conditions now read,

$$\left(\frac{\omega'_p}{\omega^2_p}\right)^2 = \frac{a^2 M^4}{(a^2 M^2 + a^2_*)^3}, \quad \frac{\omega''_p}{\omega^3_p} = \frac{a_*^2 M^2}{(a^2 M^2 + a^2_*)^3},$$  \(A.20\)

where assuming we wait long enough the system will evolve adiabatically again. As equation A.19 shows, for a massless particle the adiabatic conditions are always satisfied during a radiation epoch. The adiabatic conditions for the case of a massive particle with mass $M = 0.1$ and $p = a_* H = e^{-3}H$ are shown on the left in figure A.2 where as the graphs indicate the adiabatic conditions are eventually satisfied which again is the reason we must join the end of inflation to a radiation era to ensure the modes are evolving adiabatically.

In Chapter 3, we discuss the production of $\sigma$ which becomes massless for a brief period during inflation. The frequency for $\sigma$ particles reads

$$\omega^2 = p^2 - \frac{a''}{a} + m^2_\sigma(\tau)a^2(\tau),$$  \(A.21\)
Figure A.2: Left: The adiabatic conditions are shown with \( a_* = e^{-3} \) for a massive particle after inflation during a radiation epoch where both adiabatic conditions are quickly satisfied. Right: The adiabatic conditions for \( \sigma \) which becomes momentarily massless at \( a_* \). The period of nonadiabaticity is much less than a Hubble time.

where the mass of \( \sigma \) is given by

\[
m_{\sigma}^2(\tau) = \begin{cases} 
-2\mu \varphi(\tau) & \text{for } \tau < \tau_* \\
\mu \varphi(\tau) & \text{for } \tau > \tau_* ,
\end{cases}
\tag{A.22}
\]

and \( \varphi \) is approximated as \( \varphi(t) = \dot{\varphi}_*(t - t_*) = \dot{\varphi}_* \ln \left( \frac{\tau_*}{\tau} \right) \). Figure A.2 shows plots of the adiabatic conditions (on the right) for \( p = e^{-3}H \) and \( a_* = e^{-4} \) which is when \( \sigma \) becomes massless. Since \( \sigma \) is very massive (\( m_\sigma \gg H \)) for times either before or after \( \tau_* \), it is evolving adiabatically. However, around \( a_* \) these conditions are violated and production of \( \sigma \) becomes efficient. The period of nonadiabaticity lasts much less than a Hubble time, \((\ln a \ll 1)\), which reinforces the approximations used in Chapter 3 about the Hubble parameter being approximately constant during this period of particle production.
In this Appendix, we discuss the procedure of normalizing ordering using the Bogolyubov formalism. We will again consider the case of a charged scalar and in particular the charge density associated with the scalar, but the general procedure can be applied to any of the three mechanisms discussed in the main body. Based on the discussion of Appendix A above, when the system is evolving adiabatically, all we need to do to compute the expectation value of physical quantities such as the charge density is to evaluate them as operators built out of the physical \( \tilde{a}_k \), \( \tilde{b}_{-k} \) annihilation operators on the constant vacuum \( |0\rangle \) of the system. To get rid of infinities that occur even when the system is in its vacuum, however, we need to normal order the relevant operators before we compute their expectation value. To clarify why this is necessary it is sufficient to work in the limit where particles are not created, so that \( \tilde{a}_k = a_k \), \( \tilde{b}_k = b_k \).

Let us consider the operator that corresponds to the total charge of a complex scalar field \( \rho_{\text{Tot}} = \int d\mathbf{x} \rho(\mathbf{x}) \), where \( \rho(\mathbf{x}) \) is given by (2.30). Then

\[
\rho_{\text{Tot}} = e \int d\mathbf{p} \left( a_{\mathbf{p}}^\dagger a_{\mathbf{p}} - b_{-\mathbf{p}}^\dagger b_{-\mathbf{p}} \right). \tag{B.1}
\]

It is well known that the expectation value on the vacuum of this quantity is divergent, and that this divergence is renormalized away by normal ordering the operator to: \( \rho_{\text{Tot}} := e \int d\mathbf{p} \left( a_{\mathbf{p}}^\dagger a_{\mathbf{p}} - b_{-\mathbf{p}}^\dagger b_{-\mathbf{p}} \right) \). Let us note in passing that this is precisely what is done when declaring that the occupation number of particles and of antiparticles with momentum \( \mathbf{p} \) is given respectively by \( \langle 0 | a_{\mathbf{p}}^\dagger a_{\mathbf{p}} | 0 \rangle \) and \( \langle 0 | b_{\mathbf{p}}^\dagger b_{\mathbf{p}} | 0 \rangle \). Without normal ordering the second quantity should have been \( \langle 0 | b_{\mathbf{p}} b_{\mathbf{p}}^\dagger | 0 \rangle \).

Moving on to the operator \( \rho_{\text{Tot}}^2 \), a direct calculation gives

\[
\langle 0 | \rho_{\text{Tot}}^2 | 0 \rangle = \left( e \delta(\mathbf{p} = 0) \int d\mathbf{p} \right)^2, \tag{B.2}
\]

so that, after remembering that \( \delta(\mathbf{p} = 0) = V/(2\pi)^3 \) (where \( V \) is the (infinite) volume of space), the square of the charge density takes the divergent value \( \langle 0 | \rho_{\text{Tot}}^2 | 0 \rangle / V^2 = 169 \).
\[ e^2 \left[ \int \frac{dp}{(2\pi)^2} \right]^2. \] Again, by normal ordering one obtains the (physically more sensible) result 
\[ \langle 0 | : \rho^2_{\text{Tot}} : |0 \rangle / V^2 = 0. \]

At a practical level, the procedure used here (Bogolyubov transformations accompanied by normal ordering) is equivalent to the standard procedure of adiabatic regularization, as we will now show.

Normal ordering is equivalent, by use of Wick’s theorem, to computing the relevant operator in terms of the propagator 
\[ \langle 0 | : \phi(k, \tau) \phi^\dagger(-q, \tau') : |0 \rangle, \] where we remind the reader that normal ordering is performed in terms of the \( \tilde{a}_k \), \( \tilde{b}_k \) operators that annihilate energy eigenstates at late times. We decompose

\[ \phi(k, \tau) \equiv \phi(k, \tau) a_k + \phi^*(-k, \tau) b^\dagger_k \]
\[ \equiv \tilde{\phi}(k, \tau) \tilde{a}_k + \tilde{\phi}^*(-k, \tau) \tilde{b}^\dagger_k, \tag{B.3} \]

where

\[ \tilde{\phi}(k, \tau) = e^{-i \int \omega_k d\tau} \sqrt{2 \omega_k} \tag{B.4} \]

is the mode function in the adiabatic approximation, whereas \( \phi(k, \tau) \) is the actual solution to the equations of motion for our field. Then, as described above, one uses the relations (A.5), (A.6) and (A.9) to obtain

\[ \langle : \phi(k, \tau) \phi^\dagger(q, \tau') : \rangle = \delta(k - q) \left| \beta_k \right|^2 \left( \tilde{\phi}_k \tilde{b}_{-q} - \right) + \tilde{\phi}_k \tilde{a}^\dagger_{-q} \right) \tag{B.5} \]

with

\[ \langle \tilde{a}_k^\dagger \tilde{a}_k \rangle = \langle \tilde{b}_{-k}^\dagger \tilde{b}_{-q} \rangle = \delta(k - q) \left| \beta_k \right|^2, \quad \langle \tilde{a}_k \tilde{b}_{-q} \rangle = \langle \tilde{b}_{-k}^\dagger \tilde{a}_q^\dagger \rangle = \delta(k - q) \alpha_k \beta_k^* \tag{B.6} \]

so that

\[ \langle : \phi(k, \tau) \phi^\dagger(q, \tau') : \rangle = \delta(k - q) \left[ \left| \beta_k \right|^2 \left( \tilde{\phi}_k \tilde{b}_{-q} + \tilde{\phi}_k \tilde{a}_q^\dagger \right) \right] + \tilde{\phi}_k \tilde{b}_{-q} \tilde{a}_q^\dagger + \tilde{\phi}_k \tilde{a}_q^\dagger \tilde{b}_{-q} \right) \tag{B.7} \]
Now, using $|\alpha_k|^2 - |\beta_k|^2 = 1$, we can rewrite this as

$$\langle : \phi(k, \tau) \phi^\dagger(q, \tau') : \rangle = \delta(k - q) \left[ |\alpha_k|^2 \phi(k, \tau) \phi^*(-k, \tau') + |\beta_k|^2 \phi^*(-k, \tau) \phi(-k, \tau') \right.
\left. + \phi(-k, \tau') \phi(-k, \tau') \alpha_k \beta_k^* + \phi^*(k, \tau') \phi^*(k, \tau') \alpha_k^* \beta_k \right]$$

$$- \delta(k - q) \phi(k, \tau) \phi^*(k, \tau'), \quad (B.8)$$

where the first two lines give the non-normal ordered expression $\langle \phi(k, \tau) \phi^\dagger(q, \tau') \rangle$ whereas the last line is equivalent to $-\langle \phi_{WKB}(k, \tau) \phi_{WKB}^\dagger(q, \tau') \rangle$, with $\phi_{WKB}(q, \tau')$ is the field computed in adiabatic approximation. This shows that normal ordering is equivalent to adiabatic regularization, i.e., to computing all the relevant operators after replacing the propagator $\langle \phi(x) \phi(x') \rangle$ with its renormalized value $\langle \phi(x) \phi(x') \rangle - \langle \phi(x) \phi(x') \rangle_{\text{adiab}}$ where $\langle \phi(x) \phi(x') \rangle_{\text{adiab}}$ corresponds to the propagator computed in the adiabatic approximation [91]. This method is for instance used in [92].

To see explicitly the effect of normal ordering on our system we report here the expression for the charge power spectrum in the case of a complex scalar:

$$P_{\rho}^{\text{nonren}}(k) = e^2 \frac{k^3}{(2\pi)^5} \int \frac{d^3q}{\omega_{k+q}} \left\{ \frac{1}{2} \left( |\alpha_q|^2 + |\beta_q|^2 \right) \left( |\alpha_{k+q}|^2 + |\beta_{k+q}|^2 \right) (\omega_q^2 + \omega_{k+q}^2) \right. 
- \omega_{k+q} \omega_q - (\omega_q + \omega_{k+q})^2 \text{Re} \left[ \beta_q \beta_{k+q}^* \alpha_k \alpha_{k+q}^* e^{2i \int \omega_q d\tau - 2i \int \omega_{k+q} d\tau} \right] 
+ (\omega_{k+q}^2 - \omega_q^2) \left( |\alpha_{k+q}|^2 + |\beta_{k+q}|^2 \right) \text{Re} \left[ \beta^*_q \alpha_q e^{-2i \int \omega_q d\tau} \right] 
- (|\alpha_q|^2 + |\beta_q|^2) \text{Re} \left[ \beta_{k+q}^* \alpha_{k+q} e^{-2i \int \omega_{k+q} d\tau} \right] 
- (\omega_q - \omega_{k+q})^2 \text{Re} \left[ \beta^*_q \beta_{k+q}^* \alpha_q \alpha_{k+q} e^{-2i \int \omega_{k+q} d\tau} \right] \left\}, \quad (B.9) \right.$$
evaluate eq. (B.9) in the adiabatic regime \( \omega \rightarrow \infty \), we have

\[
P_{\rho}^{\text{nonren}}(k) = e^2 \frac{k^3}{(2\pi)^5} \int d^3 q \left\{ -4 \text{Re} \left[ \beta_q \beta_{k+q}^* \alpha_{k+q} \alpha_{q}^* \right] 
+ \left( |\alpha_q|^2 + |\beta_q|^2 \right) \left( |\alpha_{k+q}|^2 + |\beta_{k+q}|^2 \right) - 1 \right\},
\]

that should be compared to the renormalized expression

\[
P_{\rho}(k) = e^2 \frac{k^3}{(2\pi)^5} \int d^3 q \left\{ -4 \text{Re} \left[ \beta_q \beta_{k+q}^* \alpha_{k+q} \alpha_{q}^* \right] 
+ 4 |\beta_q|^2 |\beta_{k+q}|^2 \right\}.
\]

Finally, taking \(|k| \ll |q|\) and using the condition \(|\alpha_k|^2 - |\beta_k|^2 = 1\), we obtain our eq. (2.33), \(P_{\rho}(k) \propto k^3\), whereas, taking the same limit in eq. (B.10), one obtains \(P_{\rho}^{\text{nonren}}(k) = O(k^4)\).
APPENDIX C

DERIVATION OF THE GRAVITON’S EQUATION OF MOTION

To derive the equation of motion for the graviton, we start with the perturbed metric,

\[ g_{\mu\nu} = a^2(\tau) \left[ -d\tau^2 + (\delta_{ij} + h_{ij}) dx^i dx^j \right], \]  

(C.1)

where \( h_{ij} \) is a perturbation \( |h_{ij}| \ll |g_{\mu\nu}| \), traceless \( h_{ii} = 0 \), and transverse \( \partial_i h^{ij} = 0 \). The equation of motion is derived using Einstein’s field equations,

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{M_P^2} T_{\mu\nu}. \]  

(C.2)

We start by calculating the Christoffel symbols,

\[ \Gamma^c_{ab} = \frac{1}{2} g^{cd} \left[ \partial_a g_{bd} + \partial_b g_{da} - \partial_d g_{ab} \right], \]  

(C.3)

\[ \Gamma^0_{00} = \frac{a'}{a}, \quad \Gamma^i_{00} = \Gamma^0_{i0} = 0, \quad \Gamma^0_{ij} = H \delta_{ij} + \left( H + \frac{1}{2} \partial_\tau \right) h_{ij} \]  

(C.4)

\[ \Gamma^j_{i0} = \delta^j_i H + \frac{1}{2} h^j_i, \quad \Gamma^k_{ij} = \frac{1}{2} \left[ \partial_i h^k_j + \partial_j h^k_i - \partial^k h_{ij} \right]. \]  

(C.5)

The Riemann and Ricci Tensors will be,

\[ R^a_{\ bca} = \partial_c \Gamma^a_{db} - \partial_d \Gamma^a_{cb} + \Gamma^a_{ce} \Gamma^e_{db} - \Gamma^a_{de} \Gamma^e_{cb}, \quad R_{\mu\nu} = R^a_{\mu\nu}. \]  

(C.6)

The components of the Ricci Tensor are,

\[ R^{00} = -3 H', \quad R_{0i} = 0, \quad R_{ij} = (H' + 2H^2) \delta_{ij} + \left[ H' + 2H^2 + H \partial_\tau + \frac{1}{2} \partial_\tau \partial_\tau - \frac{1}{2} \Delta \right] h_{ij}. \]  

(C.7)

The Ricci scalar is,

\[ R = R^n_{\ mu} = \frac{6}{a^2} (H' + H^2). \]  

(C.8)

Plugging all of the above into Einstein’s field equation and applying the projection operator,

\[ \Pi^{l_{im}}_{ij} = \Pi^l_i \Pi^{m}_j - \frac{1}{2} \Pi_{ij} \Pi^{lm} \quad \Pi_{ij} = \delta_{ij} - \frac{\partial_i \partial_j}{\Delta}, \]  

(C.9)
to both the left and right hand sides we arrive at the sought after equation,

\[ h''_{ij} + 2 \frac{a'}{a} h'_{ij} - \Delta h_{ij} = \frac{2}{M_P^2} \Pi_{ij}^{ab} T_{ab}. \]  

(C.10)
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