Comment on “Hausdorff Dimension of Critical Fluctuations in Abelian Gauge Theories”

N Prokof’ev  
University of Massachusetts - Amherst, prokofev@physics.umass.edu

B Svistunov  
University of Massachusetts - Amherst, svistunov@physics.umass.edu

Follow this and additional works at: http://scholarworks.umass.edu/physics_faculty_pubs

Part of the Physics Commons

Recommended Citation

Physics Review Letters. 1122.  
http://scholarworks.umass.edu/physics_faculty_pubs/1122

This Article is brought to you for free and open access by the Physics at ScholarWorks@UMass Amherst. It has been accepted for inclusion in Physics Department Faculty Publication Series by an authorized administrator of ScholarWorks@UMass Amherst. For more information, please contact scholarworks@library.umass.edu.
Comment on “Hausdorff Dimension of Critical Fluctuations in Abelian Gauge Theories”

In their Letter [1], Hove, Mo, and Sudbo derive a simple connection between the anomalous scaling dimension, \( \eta \), of the U(1) universality class order parameter, \( \phi(x) \), and the Hausdorff dimension, \( D_H \), of critical loops:

\[
\eta + D_H = 2 .
\]  (1)

In the loop representation, the correlator \( G(r) = \langle \phi(r)\phi^*(0) \rangle \propto r^{-(d-2+\eta)} \) describes the distribution of the end-points in open loops. For definiteness, one may think of the high-temperature-expansion loops for the lattice \( |\phi|^4 \)-model.

The analysis of Ref. [1] might seem absolutely compelling, being just a translation of the hyperscaling hypothesis into the loop language: At the critical point there should be about one loop of diameter \( r \) per volume element \( r^d \). Nevertheless, given the result \( \eta = 0.0380(4) \) of Ref. [2], the relation (1) is in strong contradiction with the value \( D_H = 1.7655(20) \) which we obtained for the 3D \( |\phi|^4 \)-model with suppressed leading corrections to scaling (and also—with a bit less accuracy—for the standard bond-current model [3], and its special version with excluded loop overlaps and self-crossings). The simulations were done with the Worm algorithm [4].

The hidden flaw in the treatment of Ref. [1] is as follows. When introducing the self-similar expression

\[
P(r,N) \propto N^{-\rho} F(r/N^\Delta), \quad \Delta = 1/D_H \]  (2)

for the probability to find the ends of an open loop of length \( N \) being distance \( r \) away from each other, which is then used to establish the connection between the open and closed loops, the authors take for granted that \( F(0) \) is finite. While looking innocent, this is an arbitrary assumption, since the self-similar form (2) is valid only for \( r \gg a \), where \( a \) is a microscopic cutoff (e.g., the lattice period). Strictly speaking, a closed loop of length \( N \) corresponds to \( F(a/N^\Delta) \) rather than to \( F(0) \), and one has to work with the generic asymptotic form

\[
F(x) \propto x^\theta \quad \text{at} \quad x \ll 1,
\]  (3)

with some exponent \( \theta \). With Eq. (3), the hyperscaling argument yields \( \rho = (d-\theta)/D_H \), and from \( G(r) \propto \int dNP(r,N) \) one then obtains

\[
\eta + D_H = 2 - \theta .
\]  (4)

Using high-precision data for \( \eta \) and \( D_H \) mentioned above, we find \( \theta = 0.1965(20) \).

It is instructive to explicitly verify Eq. (1) by simulating \( P(r,N) \). In Fig. 1 we present results of such a simulation for the \( |\phi|^4 \)-model. We plot the value of \( P(r,N)N^{d\Delta} \) as a function of \( r \) for three different values of \( N \). In view of the self-similarity of \( P(r,N) \), the qualitative difference between the cases of \( \theta \neq 0 \) and \( \theta = 0 \) is readily seen. In the former case, curves for different values of \( N \) should merge for \( r/N^\Delta \ll 1 \)—and they do in Fig. 1. In the latter case, as \( r \to 0 \) one should see a fan of curves with essentially different slopes and a common origin at \( r = 0 \).

![FIG. 1: Distribution of open loops over radii for three different values of N. The Worm algorithm simulation for the loop representation (high-temperature expansion) of the 3D lattice \( |\phi|^4 \)-model with \( L = 192^3 \) sites at the special critical point with suppressed leading corrections to scaling.](image.png)

One important implication of Eq. (1) in the absence of additional relation between \( D_H \), \( \eta \) and \( \theta \), is that the anomalous scaling dimension can not be deduced from simulations of closed loops which determine \( D_H \) only.

We are grateful to A. Kuklov, Z. Tesanović, and G. Williams for numerous discussions. This work was supported by the National Science Foundation under Grant DMR-0071767.

Nikolay Prokof’ev and Boris Svistunov,
Department of Physics, University of Massachusetts, Amherst, MA 01003.

PACS numbers: 05.70.Jk, 64.60.Ak, 05.70.Fh