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Abstract. Assuming suitable convergence properties for the Gromov-Witten potential of a Calabi-Yau manifold X, one may construct a polarized variation of Hodge structure over the complexified Kähler cone of X. In this paper we show that, in the case of fourfolds, there is a correspondence between "quantum potentials" and polarized variations of Hodge structures that degenerate to a maximally unipotent boundary point. Under this correspondence, the WDVV equations are seen to be equivalent to the Griffiths' trasversality property of a variation of Hodge structure.

1. Introduction

The Gromov-Witten potential of a Calabi-Yau manifold is a generating function for some of its enumerative data. It may be written as $\Phi^{GW} = \Phi_0 + \Phi_\hbar$, where Φ_0 is determined by the cup product structure in cohomology. For a quintic hypersurface $X \subset P^4$, the quantum potential $\Phi_\hbar$ is holomorphic in a neighborhood of $0 \in \mathbb{C}$; the coefficients of its expansion

$$\Phi_\hbar(q) \ := \ \sum_{d=1}^{\infty} \langle I_{0,0,d} \rangle \cdot q^d,$$

are the Gromov-Witten invariants $\langle I_{0,0,d} \rangle$ which encode information about the number of rational curves of degree d in X. This potential gives rise to a flat connection on the trivial bundle over the punctured disk Δ^* with fiber $V = \oplus_p H^{p,p}(X)$. This flat bundle is shown to underlie a polarized variation of Hodge structure whose degeneration at the origin is, in an appropriate sense, maximal. The Mirror Theorem in the context of [**4**], [**17**], [**13**], [**10**, Theorem 11.1.1] asserts that this is the variation of Hodge structure arising from a family of mirror Calabi-Yau threefolds and, therefore, that the Gromov-Witten potential may be computed from the period map of this family, written with respect to a canonical coordinate at "infinity". This leads to the effective computation (and prediction) of the number of rational curves, of a given degree, in a quintic threefold.

The Mirror Theorem, in the sense sketched above, has a conjectural generalization for toric Calabi-Yau threefolds [**10**, Conjecture 8.6.10] but the situation in the higher dimensional case is considerably murkier. Still, one may write a formal potential in terms of axiomatically defined Gromov-Witten invariants and, assuming suitable convergence properties, construct from it an abstract polarized variation of Hodge structure. The third derivatives of Φ^{GW} may be used to define a quantum

product on $H^*(X)$ whose associativity is equivalent to a system of partial differential equations satisfied by Φ^{GW}. These are the so-called WDVV equations, after E. Witten, R. Dijkgraaf, H. Verlinde, and E. Verlinde.

The purpose of this paper is to further explore the relationship between quantum potentials and variations of Hodge structure. The weight-three case has been extensively considered in [**10, 21**]; here we consider the case of structures of weight four.

In §2, after recalling some basic notions of Hodge theory, we describe the behavior of variations at "infinity" in terms of two ingredients: a nilpotent orbit and a holomorphic function Γ with values in a graded nilpotent Lie algebra. Nilpotent orbits may be characterized in terms of polarized mixed Hodge structures; in the particular case when these split over $\mathbb{R}$, their structure mimics that of the cohomology of a Kähler manifold under multiplication by elements in the Kähler cone. For a Calabi-Yau fourfold X, this action —together with the intersection form— characterizes the cup product in $\oplus_p H^{p,p}(X)$. The holomorphic function Γ, in turn, is completely determined by one of its components, Γ_{-1}, relative to the Lie algebra grading. Moreover, it must satisfy the differential equation (2.14) involving the monodromy of the variation. This is the content of Theorem 2.7 which generalizes a result of P. Deligne [**11**, Theorem 11] for a case when the variation of Hodge structure is also a variation of mixed Hodge structure. Throughout this section, and indeed this whole paper, we restrict ourselves to real variations of Hodge structure without any reference to integral structures.

The asymptotic data associated with a variation of Hodge structure depends on the choice of local coordinates. In §3 we describe how the nilpotent orbit and Γ behave under a change of coordinates and show that in certain cases there are canonical choices of coordinates. This happens, for example, in the cases of interest in mirror symmetry and we recover, in this manner, Deligne's Hodge theoretic description of these canonical coordinates.

In §4 we begin our discussion of quantum products by concentrating on their "constant" part. We define the notion of polarized, graded Frobenius algebras and show that they give rise to nilpotent orbits which are maximally degenerate in an appropriate sense. Moreover, in the weight-four case this correspondence may be reversed.

Finally, in §5 we define a notion of quantum potentials in polarized, graded Frobenius algebras of weight four that abstracts the main properties of the Gromov-Witten potential for Calabi-Yau fourfolds. Following the arguments of [**10**, §8.5.4] we construct a variation of Hodge structure associated with such a potential and explicitly describe its asymptotic data. We show, in particular, that the function Γ is completely determined by the potential in a manner that transforms the WDVV equations into the horizontality equation (2.14). This is done in Theorem 5.3 which establishes an equivalence between such potentials and variations with certain prescribed limiting behavior. For the case of weight three a similar theorem is due to G. Pearlstein [**21**, Theorem 7.20].

A distinguishing property of the weight 3 and 4 cases is that the quantum product on the (p,p)-cohomology is determined by the $H^{1,1}$-module structure. In the case of constant products, this statement is the content of Proposition 4.4. For general weights the PVHS on the complexified Kähler cone determines only the structure of the (p,p)-cohomology as a $\operatorname{Sym} H^{1,1}$-module, relative to the small

quantum product. At the abstract level, one can prove that there is a correspondence between germs of PVHS of weight k at a maximally unipotent boundary point whose limiting mixed Hodge structure is Hodge-Tate and flat, real families of polarized graded Frobenius $\operatorname{Sym} H^{1,1}$-modules over $\oplus_{p=0}^{k} H^{p,p}$. Details will appear elsewhere.

Barannikov ([**1**], [**2**] and [**3**]) has shown that for projective complete intersections, the PVHS constructed from the Gromov-Witten potential is of geometric origin and coincides with the variation arising from the mirror family. He has introduced, moreover, the notion of semi-infinite variations of Hodge structure. These more general variations are shown to correspond to solutions of the WDVV equations.

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2. Variations at Infinity

We begin by reviewing some basic results about the asymptotic behavior of variations of Hodge structure. We refer to [**8, 14, 15, 22**] for details and proofs.

Let M be a connected complex manifold, a (real) *variation of Hodge structure* (VHS) $\mathcal{V}$ over M consists of a holomorphic vector bundle $\mathcal{V} \to M$, endowed with a flat connection ∇, a flat real form $\mathcal{V}_{\mathbb{R}}$, and a finite decreasing filtration $\mathcal{F}$ of $\mathcal{V}$ by holomorphic subbundles —the *Hodge filtration*— satisfying

$$\nabla \mathcal{F}^p \subset \Omega^1_M \otimes \mathcal{F}^{p-1} \quad \text{(Griffiths' horizontality) and} \tag{2.1}$$

$$\mathcal{V} = \mathcal{F}^p \oplus \overline{\mathcal{F}}^{k-p+1} \tag{2.2}$$

for some integer k —the *weight* of the variation— and where $\overline{\mathcal{F}}$ denotes conjugation relative to $\mathcal{V}_{\mathbb{R}}$. As a C^∞-bundle, $\mathcal{V}$ may then be written as a direct sum

$$\mathcal{V} = \bigoplus_{p+q=k} \mathcal{V}^{p,q}\,, \qquad \mathcal{V}^{p,q} = \mathcal{F}^p \cap \overline{\mathcal{F}}^q\,; \tag{2.3}$$

the integers $h^{p,q} = \dim \mathcal{V}^{p,q}$ are the *Hodge numbers*. A *polarization* of the VHS is a flat non-degenerate bilinear form $\mathcal{S}$ on $\mathcal{V}$, defined over $\mathbb{R}$, of parity $(-1)^k$, whose associated flat Hermitian form $\mathcal{S}^h(\,.\,,\,.\,) = i^{-k}\,\mathcal{S}(\,.\,,\bar{\ .})$ is such that the decomposition (2.3) is $\mathcal{S}^h$-orthogonal and $(-1)^p\mathcal{S}^h$ is positive definite on $\mathcal{V}^{p,k-p}$.

Specialization to a fiber defines the notion of *polarized Hodge structure* on a $\mathbb{C}$-vector space V. We will denote by D [**14**] the *classifying space* of all polarized Hodge structures of given weight and Hodge numbers on a fixed vector space V, endowed with a fixed real structure $V_{\mathbb{R}}$ and the polarizing form S. Its Zariski closure $\check{D}$ in the appropriate variety of flags consists of all filtrations F in V, with $\dim F^p = \sum_{r\geq p} h^{r,k-r}$, satisfying $S(F^p, F^{k-p+1}) = 0$. The complex Lie group $G_{\mathbb{C}} = \operatorname{Aut}(V,S)$ acts transitively on $\check{D}$ —therefore $\check{D}$ is smooth— and the group of real points $G_{\mathbb{R}}$ has D as an open dense orbit. We denote by $\mathfrak{g} \subset \mathfrak{gl}(V)$, the Lie algebra of $G_{\mathbb{C}}$, and by $\mathfrak{g}_{\mathbb{R}} \subset \mathfrak{g}$ that of $G_{\mathbb{R}}$. The choice of a base point $F \in \check{D}$ defines a filtration in $\mathfrak{g}$

$$F^a\mathfrak{g} = \{\, T \in \mathfrak{g} \ :\ T\,F^p \subset F^{p+a} \,\}\,.$$

The Lie algebra of the isotropy subgroup $B \subset G_{\mathbb{C}}$ at F is $F^0\mathfrak{g}$ and $F^{-1}\mathfrak{g}/F^0\mathfrak{g}$ is an $\mathrm{Ad}(B)$-invariant subspace of $\mathfrak{g}/F^0\mathfrak{g}$. The corresponding $G_{\mathbb{C}}$-invariant subbundle of the holomorphic tangent bundle of $\check{D}$ is the *horizontal tangent bundle*. A polarized VHS over a manifold M determines —via parallel translation to a typical fiber V— a holomorphic map $\Phi\colon M \to D/\Gamma$ where Γ is the monodromy group (Griffiths' period map). By definition, it has *horizontal* local liftings into D, *i.e.*, its differentials take values on the horizontal tangent bundle.

Example 2.1. Let X be an n-dimensional, smooth projective variety, $\omega \in H^{1,1}(X)$ a Kähler class. For any $k = 0, \dots, 2n$, the Hodge decomposition (see [**16**])

$$H^k(X,\mathbb{C}) \;=\; \bigoplus_{p+q=k} H^{p,q}(X); \quad \overline{H^{p,q}} = H^{q,p}$$

determines a Hodge structure of weight k by

$$F^p \;:=\; \bigoplus_{a\geq p} H^{a,k-a}.$$

Its restriction to the *primitive cohomology*

$$H_0^{n-\ell}(X,\mathbb{C}) \;:=\; \{\alpha \in H^{n-\ell}(X,\mathbb{C}) \,:\, \omega^{\ell+1}\cup\alpha = 0\}\,,\ \ell \geq 0,$$

is polarized by the form $Q_\ell(\alpha,\beta) = Q(\alpha, \beta\cup\omega^\ell)$, $\alpha,\beta \in H_0^{n-\ell}(X,\mathbb{C})$, and where Q denotes the signed intersection form given, for $\alpha \in H^k(X,\mathbb{C})$, $\alpha' \in H^{k'}(X,\mathbb{C})$ by:

$$Q(\alpha,\alpha') \;:=\; (-1)^{k(k-1)/2} \int_X \alpha\cup\alpha'. \tag{2.4}$$

A family $\mathcal{X} \to M$ of smooth projective varieties gives rise to a polarized variation of Hodge structure $\mathcal{V} \to M$, where $\mathcal{V}_m \cong H_0^k(\mathcal{X}_m,\mathbb{C})$, $m \in M$.

Our main concern is the asymptotic behavior of Φ near the boundary of M, with respect to some compactification $\overline{M}$ where $\overline{M} - M$ is a divisor with normal crossings (the divisor at "infinity"). Such compactifications exist, for instance, if M is quasiprojective. Near a boundary point $p \in \overline{M} - M$ we can choose an open set W such that $W \cap M \simeq (\Delta^*)^r \times \Delta^m$ and then consider the local period map

$$\Phi : (\Delta^*)^r \times \Delta^m \to D/\Gamma. \tag{2.5}$$

We shall also denote by Φ its lifting to the universal covering $U^r \times \Delta^m$, where U denotes the upper-half plane. We denote by $z = (z_j)$, $t = (t_l)$ and $s = (s_j)$ the coordinates on U^r, Δ^m and $(\Delta^*)^r$ respectively. By definition, we have $s_j = e^{2\pi i z_j}$.

According to Schmid's Nilpotent Orbit Theorem [**22**], the singularities of Φ at the origin are, at worst, logarithmic; this is essentially equivalent to the regularity of the connection ∇. More precisely, assuming quasi-unipotency —this is automatic in the geometric case— and after passing, if necessary, to a finite cover of $(\Delta^*)^r$, there exist commuting nilpotent elements[1] $N_1, \dots, N_r \in \mathfrak{g}_{\mathbb{R}}$, with $N^{k+1} = 0$ and such that

$$\Phi(s,t) \;=\; \exp\left(\sum_{j=1}^r \frac{\log s_j}{2\pi i} N_j\right) \cdot \Psi(s,t)\,, \tag{2.6}$$

where $\Psi\colon \Delta^{r+m} \to \check{D}$ is holomorphic.

[1] Our sign convention is consistent with [**11, 22**] but opposite to that in [**10, 20**].

We will refer to $N_1, \dots, N_r$ as the *local monodromy logarithms* and to $F_0 := \Psi(0)$ as the *limiting Hodge filtration*. They combine to define a *nilpotent orbit* $\{N_1, \dots, N_r; F_0\}$. We recall:

Definition 2.2. With notation as above, $\{N_1, \dots, N_r; F_0\}$ is called a nilpotent orbit if the map

$$\theta(z) = \exp(\sum_{j=1}^{r} z_j N_j) \cdot F_0$$

is horizontal and there exists $\alpha \in \mathbb{R}$ such that $\theta(z) \in D$ for $\mathrm{Im}(z_j) > \alpha$.

Theorem 2.3 below gives an algebraic characterization of nilpotent orbits which will play a central role in the sequel.

We point out that the local monodromy is topological in nature, while the limiting Hodge filtration depends on the choice of coordinates s_j. To see this, we consider, for simplicity, the case $m = 0$. A change of coordinates compatible with the divisor structure must be, after relabeling if necessary, of the form $(s'_1, \dots, s'_r) = (s_1 f_1(s), \dots, s_r f_r(s))$ where f_j are holomorphic around $0 \in \Delta^r$ and $f_j(0) \neq 0$. We then have from (2.6),

$$\begin{aligned} \Psi'(s') &= \exp(-\frac{1}{2\pi i} \sum_{j=1}^{r} \log(s'_j) N_j) \cdot \Phi(s') \\ &= \exp(-\frac{1}{2\pi i} \sum_{j=1}^{r} \log(f_j) N_j) \exp(-\frac{1}{2\pi i} \sum_{j=1}^{r} \log(s_j) N_j) \cdot \Phi(s) \\ &= \exp(-\frac{1}{2\pi i} \sum_{j=1}^{r} \log(f_j) N_j) \cdot \Psi(s), \end{aligned} \tag{2.7}$$

and, letting $s \to 0$

$$F'_0 = \exp(-\frac{1}{2\pi i} \sum_{j} \log(f_j(0)) N_j) \cdot F_0. \tag{2.8}$$

These constructions may also be understood in terms of Deligne's canonical extension [**12**]. Let $\mathcal{V} \to (\Delta^*)^r \times \Delta^m$ be the local system underlying a polarized VHS and pick a base point (s_0, t_0). Given $v \in V := \mathcal{V}_{(s_0,t_0)}$, let $v^\flat$ denote the multivalued flat section of $\mathcal{V}$ defined by v. Then

$$\tilde{v}(s,t) \ := \ \exp\left(\sum \frac{\log s_j}{2\pi i} N_j\right) \cdot v^\flat(s,t) \tag{2.9}$$

is a global section of $\mathcal{V}$. The canonical extension $\overline{\mathcal{V}} \to \Delta^{r+m}$ is characterized by its being trivialized by sections of the form (2.9). The Nilpotent Orbit Theorem then implies that the Hodge bundles $\mathcal{F}^p$ extend to holomorphic subbundles $\overline{\mathcal{F}}^p \subset \overline{\mathcal{V}}$. Writing the Hodge bundles in terms of a basis of sections of the form (2.9) yields the holomorphic map Ψ. Its constant part —corresponding to the nilpotent orbit— defines a polarized VHS as well.

A nilpotent linear transformation $N \in \mathfrak{gl}(V_\mathbb{R})$ defines an increasing filtration, the *weight filtration*, $W(N)$ of V, defined over $\mathbb{R}$ and uniquely characterized by requiring that $N(W_l(N)) \subset W_{l-2}(N)$ and that $N^l : \mathrm{Gr}_l^{W(N)} \to \mathrm{Gr}_{-l}^{W(N)}$ be an isomorphism. It follows from [**7**, Theorem 3.3] that if $N_1, \dots, N_r$ are local monodromy logarithms arising from a polarized VHS then the weight filtration $W(\sum \lambda_j N_j)$,

$\lambda_j \in \mathbb{R}_{>0}$, is independent of the choice of $\lambda_1, \dots, \lambda_r$ and, therefore, is associated with the positive real cone $\mathcal{C} \subset \mathfrak{g}_{\mathbb{R}}$ spanned by $N_1, \dots, N_r$.

A *mixed Hodge Structure* (MHS) on V consists of a pair of filtrations of V, (W, F), W defined over $\mathbb{R}$ and increasing, F decreasing, such that F induces a Hodge structure of weight k on Gr_k^W for each k. Equivalently, a MHS on V is a bigrading

$$V \;=\; \bigoplus I^{p,q}$$

satisfying $I^{p,q} \equiv \overline{I^{q,p}} \mod (\oplus_{a<p,b<q} I^{a,b})$ (see [**9**, Theorem 2.13]). Given such a bigrading we define: $W_l = \oplus_{p+q\le l} I^{p,q}$, $F^a = \oplus_{p\ge a} I^{p,q}$. A MHS is said to *split* over $\mathbb{R}$ if $I^{p,q} = \overline{I^{q,p}}$; in that case the subspaces $V_l = \oplus_{p+q=l} I^{p,q}$ define a real grading of W. A map $T \in \mathfrak{gl}(V)$ such that $T(I^{p,q}) \subset I^{p+a,q+b}$ is called a morphism of bidegree (a, b).

A *polarized MHS* (PMHS) [**7**, (2.4)] of weight k on $V_{\mathbb{R}}$ consists of a MHS (W, F) on V, a $(-1,-1)$ morphism $N \in \mathfrak{g}_{\mathbb{R}}$, and a nondegenerate bilinear form Q such that

1. $N^{k+1} = 0$,
2. $W = W(N)[-k]$, where $W[-k]_j = W_{j-k}$,
3. $Q(F^a, F^{k-a+1}) = 0$ and,
4. the Hodge structure of weight $k+l$ induced by F on $\ker(N^{l+1} : \mathrm{Gr}_{k+l}^W \to \mathrm{Gr}_{k-l-2}^W)$ is polarized by $Q(\cdot, N^l\cdot)$.

Theorem 2.3. *Given a nilpotent orbit $\theta(z) = \exp(\sum_{j=1}^r z_j N_j)\cdot F$, the pair $(W(\mathcal{C}), F)$ defines a MHS polarized by every $N \in \mathcal{C}$. Conversely, given commuting nilpotent elements $\{N_1, \dots, N_r\} \in \mathfrak{g}_{\mathbb{R}}$ with the property that the weight filtration $W(\sum \lambda_j N_j)$, $\lambda_j \in \mathbb{R}_{>0}$, is independent of the choice of $\lambda_1, \dots, \lambda_r$, if $F \in \check{D}$ is such that $(W(\mathcal{C}), F)$ is polarized by every element $N \in \mathcal{C}$, then the map $\theta(z) = \exp(\sum_{j=1}^r z_j N_j)\cdot F$ is a nilpotent orbit. Moreover, if $(W(\mathcal{C}), F)$ splits over $\mathbb{R}$, then $\theta(z) \in D$ for $\mathrm{Im}(z_j) > 0$.*

The first part of Theorem 2.3 was proved by Schmid [**22**, Theorem 6.16] as a consequence of his SL_2-orbit theorem. The converse is Proposition 4.66 in [**9**]. The final assertion is a consequence of [**7**, Proposition 2.18].

Example 2.4. Let X be an n-dimensional, smooth projective variety. Let $V = H^*(X, \mathbb{C})$, $V_{\mathbb{R}} = H^*(X, \mathbb{R})$. The bigrading $I^{p,q} := H^{n-p,n-q}(X)$ defines a MHS on V which splits over $\mathbb{R}$. The weight and Hodge filtrations are then

$$W_l \;=\; \bigoplus_{d\ge 2n-l} H^d(X, \mathbb{C}), \quad F^p \;=\; \bigoplus_{s} \bigoplus_{r\le n-p} H^{r,s}(X).$$

Given a Kähler class $\omega \in H^{1,1}(X, \mathbb{R}) := H^{1,1}(X) \cap H^2(X, \mathbb{R})$, let $L_\omega \in \mathfrak{gl}(V_{\mathbb{R}})$ denote multiplication by ω. Note that L_ω is an infinitesimal automorphism of the form (2.4) and is a $(-1,-1)$ morphism of (W, F). Moreover, the Hard Lefschetz Theorem and the Riemann bilinear relations are equivalent to the assertion that L_ω polarizes (W, F). Let $\mathcal{K} \subset H^{1,1}(X, \mathbb{R})$ denote the Kähler cone and

$$\mathcal{K}_{\mathbb{C}} := H^{1,1}(X, \mathbb{R}) \oplus i\mathcal{K} \subset H^2(X, \mathbb{C})$$

the complexified Kähler cone. It then follows from Theorem 2.3 that for every $\xi \in \mathcal{K}_{\mathbb{C}}$, the filtration $\exp(L_\xi)\cdot F$ is a Hodge structure of weight n on V polarized by Q. The map $\xi \in \mathcal{K}_{\mathbb{C}} \mapsto \exp(L_\xi)\cdot F$ is the period map (in fact, the nilpotent

orbit) of a variation of Hodge structure over $\mathcal{K}_{\mathbb{C}}$. Note that we can restrict the above construction to $V = \oplus_p H^{p,p}(X)$; this is the case of interest in mirror symmetry.

Remark 2.5. The notion of nilpotent orbit is closely related to that of Lefschetz modules introduced by Looijenga and Lunts [**18**]. Indeed, it follows from [**18**, Proposition 1.6] that if (W, F) is a MHS, polarized by every N in a cone $\mathcal{C}$, and $\mathcal{A}$ denotes the linear span of $\mathcal{C}$ in $\mathfrak{g}_{\mathbb{R}}$, then V is a Lefschetz module of $\mathcal{A}$.

Let now Φ be as in (2.5), $\{N_1, \dots, N_r; F_0\}$ the associated nilpotent orbit and $(W(\mathcal{C}), F_0)$ the *limiting mixed Hodge structure*. The bigrading $I^{*,*}$ of V defined by $(W(\mathcal{C}), F_0)$ defines a bigrading $I^{*,*}\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ associated with the MHS $(W(\mathcal{C})\mathfrak{g}, F_0\mathfrak{g})$. Set

$$\mathfrak{p}_a \;:=\; \bigoplus_q I^{a,q}\mathfrak{g} \quad \text{and} \quad \mathfrak{g}_- \;:=\; \bigoplus_{a\le -1} \mathfrak{p}_a. \tag{2.10}$$

The nilpotent subalgebra $\mathfrak{g}_-$ is a complement of the stabilizer subalgebra at F_0. Hence $(\mathfrak{g}_-, X \mapsto \exp(X)\cdot F_0)$ provides a local model for the $G_{\mathbb{C}}$-homogeneous space $\check{D}$ near F_0 and we can rewrite (2.6) as:

$$\Phi(s,t) \;=\; \exp\left(\sum_{j=1}^r \frac{\log s_j}{2\pi i} N_j\right)\; \exp\Gamma(s,t)\cdot F_0\,, \tag{2.11}$$

where $\Gamma\colon \Delta^{r+m} \to \mathfrak{g}_-$ is holomorphic and $\Gamma(0) = 0$. The lifting of Φ to $U^r \times \Delta^m$ may then be expressed as:

$$\Phi(z,t) = \exp X(z,t)\cdot F_0$$

with $X : U^r \times \Delta^m \to \mathfrak{g}_-$ holomorphic. Setting $E(z,t) := \exp X(z,t)$, the horizontality of Φ is then expressed by:

$$E^{-1}\,dE = dX_{-1} \in \mathfrak{p}_{-1} \otimes T^*(U^r \times \Delta^m). \tag{2.12}$$

The following explicit description of period mappings near infinity is proved in [**8**, Theorem 2.8].

Theorem 2.6. *Let $\{N_1, \dots, N_r; F_0\}$ be a nilpotent orbit and $\Gamma : \Delta^r \times \Delta^m \to \mathfrak{g}_-$ be holomorphic, such that $\Gamma(0,0) = 0$. If the map*

$$\Phi(z,t) = \exp(\sum z_j N_j)\;\; \exp(\Gamma(s,t))\cdot F_0 \tag{2.13}$$

is horizontal (i.e., (2.12) is satisfied), then $\Phi(z,t)$ is a period mapping.

Given a period mapping as in (2.13), let $\Gamma_{-1}(s,t)$ denote the $\mathfrak{p}_{-1}$-component of Γ. Then,

$$X_{-1}(z,t) \;=\; \sum_{j=1}^r z_j N_j + \Gamma_{-1}(s,t)\,,\ \text{with } s_j = e^{2\pi i z_j},$$

and it follows from (2.12) that

$$dX_{-1} \wedge dX_{-1} = 0 \tag{2.14}$$

The following theorem shows that this equation characterizes period mappings with a given nilpotent orbit.

THEOREM 2.7. *Let $R : \Delta^r \times \Delta^m \to \mathfrak{p}_{-1}$ be a holomorphic map with $R(0) = 0$. Let $X_{-1}(z,t) = \sum_{j=1}^r z_j N_j + R(s,t)$, $s_j = e^{2\pi i z_j}$, and suppose that the differential equation (2.14) holds. Then, there exists a unique period mapping (2.11) defined in a neighborhood of the origin in Δ^{r+m} and such that $\Gamma_{-1} = R$.*

PROOF. To prove uniqueness, we begin by observing that if Φ and Φ' are period mappings with the same associated nilpotent orbit and $\Gamma_{-1}(s,t) = \Gamma'_{-1}(s,t)$ then, for any $v \in F^p$, we may consider the sections $\nu(s,t) = E(s,t) \cdot v^\flat(s,t)$ and $\nu'(s,t) = E'(s,t) \cdot v^\flat(s,t)$ of the canonical extension $\overline{\mathcal{V}}$. Clearly, $\nu(s,t) \in \mathcal{F}^p_{(s,t)}$ and $\nu'(s,t) \in \mathcal{F}'^p_{(s,t)}$. On the other hand, since $\Gamma_{-1}(s,t) = \Gamma'_{-1}(s,t)$, it follows that $E_{-1}(s,t) = E'_{-1}(s,t)$ and, consequently, $\nu(s,t) - \nu'(s,t)$ is a ∇-flat section which extends to the origin and takes the value zero there. Hence, $\nu(s,t) - \nu'(s,t)$ is identically zero and $\mathcal{F}^p_{(s,t)} = (\mathcal{F}'_{(s,t)})^p$ for all values of (s,t).

To complete the proof of the Theorem it remains to show the existence of a period mapping with given nilpotent orbit and $\Gamma_{-1}(s,t) = R(s,t)$. This amounts to finding a solution to the differential equation (2.12) with

$$X_{-1}(z,t) \;=\; \sum_{j=1}^r z_j N_j + R(s,t)\,, \text{ with } s_j = e^{2\pi i z_j},$$

assuming that the integrability condition (2.14) is satisfied. Set $G(s,t) = \exp \Gamma(s,t)$ and $\Theta = d(\sum_{j=1}^r z_j N_j)$. Then (2.12) may be rewritten as

$$dG = [G, \Theta] + G dG_{-1} \text{ with } G(0,0) = \mathbb{I}, \tag{2.15}$$

where $\mathbb{I}$ denotes the identity, while the condition (2.14) takes the form:

$$dG_{-1} \wedge \Theta + \Theta \wedge dG_{-1} + dG_{-1} \wedge dG_{-1} = 0. \tag{2.16}$$

By considering the $\mathfrak{p}_{-l}$-graded components of (2.15) we obtain a sequence of equations:

$$dG_{-l} = [G_{-l+1}, \Theta] + G_{-l+1} dG_{-1}, \quad G_{-l}(0,0) = 0, \quad l \geq 2. \tag{2.17}$$

Assume inductively that, for $l \geq 2$, we have constructed G_{-l+1} satisfying (2.17) and such that

$$dG_{-l+1} \wedge \Theta + \Theta \wedge dG_{-l+1} + dG_{-l+1} \wedge dG_{-1} = 0.$$

Then, the initial value problem

$$dG_{-l} = [G_{-l+1}, \Theta] + G_{-l+1} dG_{-1}, \quad G_{-l}(0,0) = 0,$$

has a solution which verifies

$$\begin{aligned}
&dG_{-l} \wedge \Theta + \Theta \wedge dG_{-l} + dG_{-l} \wedge dG_{-1} = \\
&\quad = \; [G_{-l+1}, \Theta] \wedge \Theta + G_{-l+1} dG_{-1} \wedge \Theta + \Theta \wedge [G_{-l+1}, \Theta] + \Theta \wedge G_{-l+1} dG_{-1} + \\
&\qquad\quad + [G_{-l+1}, \Theta] \wedge dG_{-1} + G_{-l+1} dG_{-1} \wedge dG_{-1} \\
&\quad = \; -\Theta \wedge G_{-l+1}\Theta + G_{-l+1} dG_{-1} \wedge \Theta + \Theta \wedge G_{-l+1}\Theta + \Theta \wedge G_{-l+1} dG_{-1} + \\
&\qquad\quad + G_{-l+1}\Theta \wedge dG_{-1} + G_{-l+1} dG_{-1} \wedge dG_{-1} - \Theta \wedge G_{-l+1} dG_{-1} \\
&\quad = \; G_{-l+1}(dG_{-1} \wedge \Theta + \Theta \wedge dG_{-1} + dG_{-1} \wedge dG_{-1}) = 0.
\end{aligned}$$

Thus we may, inductively, construct a solution of (2.16). Theorem 2.6 now implies that the map

$$\Phi(z,t) = \exp(\sum_{j=1}^{r} z_j N_j)\ G(s,t)\cdot F_0$$

is the desired period map. □

Remark 2.8. The uniqueness part of the argument is contained in Lemmas 2.8 and 2.9 of [**6**], while the existence proof is contained in the unpublished manuscript [**5**]. A particular case of Theorem 2.7 is given in [**11**, Theorem 11]; a generalization to the case of variations of MHS appears in [**21**].

3. Canonical Coordinates

The asymptotic data of a polarized variation of Hodge structure over an open set $W \simeq (\Delta^*)^r \times \Delta^m$ depends on the choice of coordinates on the base. We have already observed that the local monodromy logarithms N_j are independent of coordinates and have shown in (2.8) how the limiting Hodge filtration changes under a coordinate transformation. Here we will discuss the dependence of the holomorphic function $\Gamma\colon (\Delta^*)^r \times \Delta^m \to \mathfrak{p}_{-1}$ and show that, in special cases, there is a natural choice of coordinates. This choice will be seen to agree with that appearing in the mirror symmetry setup and which has already been given a Hodge-theoretic interpretation by Deligne [**11**]. These canonical coordinates may also be interpreted, in the case of families of Calabi-Yau threefolds as the coordinates where the Picard-Fuchs equations take on a certain particularly simple form ([**10**, Prop. 5.6.1]). To simplify the discussion we will restrict our discussion to the case $m = 0$. The general case, which follows easily, will not be needed in the sequel.

Since we are required to preserve the divisor structure at the boundary, we want to study the behavior of the asymptotic data under coordinate changes of the form

$$s'_j = s_j f_j(s) \tag{3.1}$$

where the functions f_j are holomorphic in a neighborhood of $0 \in \Delta^r$ and $f_j(0) \neq 0$.

Given a PVHS over $(\Delta^*)^r$ and a choice of local coordinates $(s_1, \ldots, s_r)$ around 0, we write the associated period map as in (2.11):

$$\Phi(s) \ = \ \exp(\sum_{j=1}^{r} \frac{\log s_j}{2\pi i} N_j)\exp(\Gamma(s))\cdot F_0$$

Given another system of coordinates $s' = (s'_1, \ldots, s'_r)$ as in (3.1), let F'_0 and Γ' denote the corresponding asymptotic data. By (2.8), $F'_0 = \mathcal{M}\cdot F_0$, where

$$\mathcal{M} \ := \ \exp(-\frac{1}{2\pi i}\sum_{j=1}^{r} \log f_j(0)\, N_j).$$

Proposition 3.1. *Under a coordinate change as in (3.1):*

$$\mathcal{M}^{-1}\exp(\Gamma'(s'))\mathcal{M} \ = \ \exp\left(-\frac{1}{2\pi i}\sum_{j=1}^{r}\log\frac{f_j(s)}{f_j(0)}\, N_j\right)\exp(\Gamma(s)). \tag{3.2}$$

Proof. Let W denote the filtration $W(N)[-k]$, where N is an arbitrary element in the cone $\mathcal{C}$ positively spanned by $N_1, \dots, N_r$. Note that $\mathcal{M}$ leaves W invariant. Moreover, since the monodromy logarithms are $(-1,-1)$-morphisms of the mixed Hodge structure (W, F_0) it follows easily that

$$I^{a,b}(W, F_0') \;=\; \mathcal{M} \cdot I^{a,b}(W, F_0),$$

where $I^{*,*}(W, F_0)$ denotes the canonical bigrading of the MHS. This implies that the associated bigrading of the Lie algebra $\mathfrak{g}$ and, in particular, that the subalgebra $\mathfrak{g}_-$ defined in (2.10) are independent of the choice of coordinates.

According to (2.7), $\Psi'(s') \;=\; \exp\left(\frac{1}{2\pi i}\sum_j \log f_j(s) N_j\right) \cdot \Psi(s)$, therefore

$$\exp\Gamma'(s')\,\mathcal{M}\cdot F_0 \;=\; \exp\left(\frac{1}{2\pi i}\sum_{j=1}^{r} \log f_j(s)\, N_j\right)\, \exp\Gamma(s)\cdot F_0. \tag{3.3}$$

This identity, in turn, implies (3.2) since the group elements in both sides of (3.3) lie in $\exp(\mathfrak{g}_-)$. □

Corollary 3.2. *With the same notation of Proposition 3.1,*

$$\mathcal{M}^{-1}\Gamma'_{-1}\mathcal{M} \;=\; -\frac{1}{2\pi i}\sum_j \log\frac{f_j(s)}{f_j(0)}\, N_j + \Gamma_{-1}. \tag{3.4}$$

Proof. This follows considering the $\mathfrak{p}_{-1}$-component in (3.2), given the observation that this subspace is invariant under coordinate changes. □

Up to rescaling, we may assume that our coordinate change (3.1) satisfies $f_j(0) = 1$, $j = 1\dots, r$. Such changes will be called *simple*. In this case $\mathcal{M} = \mathbb{I}$, $F_0' = F_0$, and the transformation (3.4) is just a translation in the direction of the nilpotent elements N_j. Thus, whenever the subspace spanned by $N_1, \dots, N_r$ has a natural complement in $\mathfrak{p}_{-1}$ we will be able to choose coordinates, unique up to scaling, such that Γ_{-1} takes values in that complement. This is the situation in the variations of Hodge structure studied in mirror symmetry. In this context, one analyzes the behavior of PVHS near some special boundary points. They come under the name of "large radius limit points" (see [**10**, §6.2.1]) or "maximally unipotent boundary points" (see [**10**, §5.2]). For our purposes, we have

Definition 3.3. Given a PVHS of weight k over $(\Delta^*)^r$ whose monodromy is unipotent, we say that $0 \in \Delta^r$ is a *maximally unipotent boundary point* if

1. $\dim I^{k,k} = 1$, $\dim I^{k-1,k-1} = r$ and $\dim I^{k,k-1} = \dim I^{k-2,k} = 0$, where $I^{a,b}$ is the bigrading associated to the limiting MHS and,
2. $\mathrm{Span}_{\mathbb{C}}(N_1(I^{k,k}), \dots, N_r(I^{k,k})) = I^{k-1,k-1}$, where N_j are the monodromy logarithms of the variation.

Under these conditions, we may identify $\mathrm{Span}_{\mathbb{C}}(N_1, \dots, N_r) \cong \mathrm{Hom}(I^{1,1}, I^{0,0})$. Hence, denoting by $\rho\colon \mathfrak{p}_{-1} \to \mathrm{Hom}(I^{1,1}, I^{0,0})$ the restriction map, the subspace $K = \ker(\rho)$ is a canonical complement of $\mathrm{Span}_{\mathbb{C}}(N_1, \dots, N_r)$ in $\mathfrak{p}_{-1}$.

Note that both, the notion of maximally unipotent boundary point and the complement K are independent of the choice of basis.

Definition 3.4. Let $\mathcal{V} \to (\Delta^*)^r$ be a PVHS having the origin as a maximally unipotent boundary point. A system of local coordinates $(q_1, \dots, q_r)$ is called *canonical* if the associated holomorphic function Γ_{-1} takes values in K.

PROPOSITION 3.5. *Let $\mathcal{V} \to (\Delta^*)^r$ be a PVHS having the origin as a maximally unipotent boundary point. Then there exists, up to scaling, a unique system of canonical coordinates.*

PROOF. Let $s = (s_1, \ldots, s_r)$ be an arbitrary system of coordinates around 0. We can write

$$\rho(\Gamma(s)) \;=\; \sum_{j=1}^{r} \gamma_j(s)\, N_j,$$

where $\gamma_j(s)$ are holomorphic in a neighborhood of $0 \in \Delta^r$ and $\gamma_j(0) = 0$. The transformation formula (3.4) now implies that the coordinate system

$$q_j \;:=\; s_j \exp(2\pi i \gamma_j(s)) \tag{3.5}$$

is canonical. Moreover, that same formula shows that it is unique up to scaling. □

In [**11**], Deligne observed that a variation of Hodge structure whose limiting MHS is of Hodge-Tate type defines, together with the monodromy weight filtration, a variation of mixed Hodge structure. In this context, the holomorphic functions $q_j(s)$ defined by (3.5) constitute part of the extension data of this family of mixed Hodge structures. He shows, moreover, that they agree with the special coordinates studied in [**4**], [**20**], and [**19**] for families of Calabi-Yau manifolds in the vicinity of a maximally unipotent boundary point. We sketch this argument for the sake of completeness.

Given a coordinate system $s = (s_1, \ldots, s_r)$, let (W, F_0) be the limiting MHS of weight k and choose $e^0 \in I^{0,0}$. Let $e^k \in I^{k,k}$ be such that $Q(e^0, e^k) = (-1)^k$. Since the origin is a maximally unipotent boundary point, there exists a basis $e_1^1, \ldots, e_r^1$ of $I^{1,1}$ such that $N_j(e_l^1) = \delta_{jl} e^0$. We can define a (multi-valued) holomorphic section of $\mathcal{F}^k$ by

$$\omega(s) \;:=\; \exp\Big(\sum_{j=1}^{r} \frac{\log s_j}{2\pi i} N_j\Big) \exp(\Gamma(s)) \cdot e^k\,.$$

In the geometric setting of a family of varieties X_s the coefficients

$$h_0(s) \;:=\; -Q(e^0, \omega(s)) \quad \text{and} \quad h_j(s) \;:=\; Q(e_j^1, \omega(s))$$

may be interpreted as integrals $\int_\alpha \omega(s)$ over appropriate cycles $\alpha \in H_k(X_{s_0})$ on the typical fiber. Clearly, our assumptions imply that $h_0(s) = (-1)^{k+1}$ and

$$\begin{aligned} h_j(s) \;&=\; Q\Big(e_j^1, \Big(\sum_l \frac{\log s_l}{2\pi i} N_l + \Gamma_{-1}(s)\Big) \cdot e^k\Big) \\ &=\; -\frac{\log s_j}{2\pi i} Q(e^0, e^k) - Q(\Gamma_{-1}(s) \cdot e_j^1, e^k) \\ &=\; -\frac{\log s_j}{2\pi i} Q(e^0, e^k) - Q\Big(\sum_l \gamma_l(s) N_l e_j^1, e^k\Big) \\ &=\; (-1)^{k+1}\Big(\frac{\log s_j}{2\pi i} + \gamma_j(s)\Big). \end{aligned}$$

Therefore, $q_j(s) = \exp(2\pi i h_j(s)/h_0(t))$, which agrees with Morrison's geometric description of the canonical coordinates in [**19**, §2].

4. Graded Frobenius Algebras and Potentials

In this section we will abstract the basic properties of the cup product in the cohomology subalgebra $\oplus_p H^{p,p}(X)$ for a smooth projective variety X and show that this product structure may be encoded in a single homogeneous polynomial of degree 3.

We recall that $(V, *, e_0, \mathcal{B})$ is called a Frobenius algebra if $(V, *)$ is an associative, commutative $\mathbb{C}$-algebra with unit e_0, and $\mathcal{B}$ is a nondegenerate symmetric bilinear form such that $\mathcal{B}(v_1 * v_2, v_3) = \mathcal{B}(v_1, v_2 * v_3)$. The algebra is said to be real if V has a real structure $V_{\mathbb{R}}$, $e_0 \in V_{\mathbb{R}}$ and both $*$ and $\mathcal{B}$ are defined over $\mathbb{R}$. Throughout this paper we will be interested in *graded*, real Frobenius algebras of weight k. By this we mean that V has an even grading

$$V \;=\; \bigoplus_{p=0}^{k} V_{2p},$$

defined over $\mathbb{R}$, and such that:

1. $V_0 \;\cong\; \mathbb{C}$,
2. $(V, *)$ is a graded algebra,
3. $\mathcal{B}(V_{2p}, V_{2q}) \;=\; 0$ if $p + q \neq k$.

The product structure on a Frobenius algebra $(V, *, e_0, \mathcal{B})$ may be encoded in the trilinear function $\tilde{\phi}_0 : V \times V \times V \to \mathbb{C}$

$$\tilde{\phi}_0(v_1, v_2, v_3) \;:=\; \mathcal{B}(e_0, v_1 * v_2 * v_3),$$

or, after choosing a graded basis $\{e_0, \dots, e_m\}$ of V, in the associated cubic form:

$$\phi_0(z_0, \dots, z_m) \;:=\; \frac{1}{6}\tilde{\phi}_0(\gamma, \gamma, \gamma) \;=\; \frac{1}{6}\mathcal{B}(e_0, \gamma^3), \quad \gamma \;=\; \sum_{a=0}^{m} z_a e_a. \tag{4.1}$$

Let $\{e_a^{\mathcal{B}} \;:=\; \sum_b h_{ba} e_b\}$ denote the $\mathcal{B}$-dual basis of $\{e_a\}$.

THEOREM 4.1. *The cubic form ϕ_0 defined by (4.1) is (weighted) homogeneous of degree k with respect to the grading defined by $\deg z_a = \deg e_a$, and satisfies the algebraic relation:*

$$\sum_{d,f} \frac{\partial^3 \phi_0}{\partial z_a \partial z_b \partial z_d} h_{df} \frac{\partial^3 \phi_0}{\partial z_f \partial z_c \partial z_g} \;=\; \sum_{d,f} \frac{\partial^3 \phi_0}{\partial z_b \partial z_c \partial z_d} h_{df} \frac{\partial^3 \phi_0}{\partial z_a \partial z_f \partial z_g}. \tag{4.2}$$

The product $*$ and the bilinear form $\mathcal{B}$ may be expressed in terms of ϕ_0 by

$$\mathcal{B}(e_a, e_b) \;=\; \frac{\partial^3 \phi_0}{\partial z_0 \partial z_a \partial z_b} \tag{4.3}$$

$$e_a * e_b \;=\; \sum_c \frac{\partial^3 \phi_0}{\partial z_a \partial z_b \partial z_c} e_c^{\mathcal{B}}\,. \tag{4.4}$$

Conversely, let V be an evenly graded vector space, $V_0 \cong \mathbb{C}$, $\{e_0, \dots, e_m\}$ a graded basis of V, and $\phi_0 \in \mathbb{C}[z_0, \dots, z_m]$ a cubic form, homogeneous of degree k relative to the grading $\deg z_a = \deg e_a$, satisfying (4.2). Then, if the bilinear symmetric form defined by (4.3) is non-degenerate, the product (4.4) turns $(V, *, e_0, \mathcal{B})$ into a graded Frobenius algebra of weight k.

Proof. We note, first of all, that the quasi-homogeneity of ϕ_0 follows from the assumption that $*$ is a graded product. Moreover, (4.3) is a consequence of the fact that e_0 is the unit for $*$, while (4.2) is the associativity condition for $*$. On the other hand, (4.1) and the fact that $(V, *, e_0, \mathcal{B})$ is Frobenius imply that

$$\frac{\partial^3 \phi_0}{\partial z_a \partial z_b \partial z_c} \;=\; \mathcal{B}(e_0, e_a * e_b * e_c) \;=\; \mathcal{B}(e_a * e_b, e_c)$$

and (4.4) follows.

The converse is immediate since (4.4) defines a commutative structure whose associativity follows from (4.2); the quasi-homogeneity assumption implies that the product is graded; e_0 is the unit because of (4.3). The compatibility between $\mathcal{B}$ and $*$ comes from $\mathcal{B}(e_a * e_b, e_c) = \frac{\partial^3 \phi_0}{\partial z_a \partial z_b \partial z_c} = \mathcal{B}(e_a, e_b * e_c)$. □

Example 4.2. Let $(V, *, e_0, \mathcal{B})$ be a graded real Frobenius algebra of weight four. Setting $\dim V_2 = r$ and $\dim V_4 = s$, we have $\dim V = 2r + s + 2 := m + 1$. We choose a basis $\{T_0, \dots, T_m\}$ of V as follows: $T_0 = e_0 \in V_0$ is the multiplicative unit, $\{T_1, \dots, T_r\}$ is a real basis of V_2, $\{T_{r+1}, \dots, T_{r+s}\}$ is a basis of V_4 such that $B(T_{r+a}, T_{r+b}) = \delta_{a,b}$. Finally, $\{T_{r+s+1}, \dots, T_{m-1}\}$ and T_m are chosen as the $\mathcal{B}$-duals of $\{T_1, \dots, T_r\}$ and T_0 in V_6 and V_8, respectively. We will say that such a basis is *adapted* to the graded Frobenius structure.

It now follows from (4.1), that with respect to such a basis, the polynomial ϕ_0 is given by

(4.5)
$$\phi_0(z) = \frac{1}{2} z_0^2 z_m + z_0 \sum_{j=1}^{r} z_j z_{r+s+j} + \frac{1}{2} z_0 \sum_{a=1}^{s} z_{r+a}^2 + \frac{1}{2} \sum_{a=1}^{s} z_{r+a} P^a(z_1, \dots, z_r),$$

where $P^a(z_1, \dots, z_r) = \sum_{j,k=1}^{r} P^a_{jk} z_j z_k$ are homogeneous polynomials of degree 2 determined by:

$$P^a_{jk} = \mathcal{B}(T_{r+a}, T_j * T_k). \tag{4.6}$$

Specializing to the case when $V = \oplus_{p=0}^{4} H^{p,p}(X)$ for a smooth, projective fourfold X, endowed with the cup product and the intersection form

$$\mathcal{B}(\alpha, \beta) \;:=\; \int_X \alpha \cup \beta\,,$$

we obtain

$$\tilde{\phi}_0^{cup}(\omega_1, \omega_2, \omega_3) = \int_X \omega_1 \cup \omega_2 \cup \omega_3.$$

In order to complete the analogy with the structure deduced from the cup product on the cohomology of a smooth projective variety we need to require that the Lefschetz Theorems be satisfied. Given $w \in V_2$, let $L_w \colon V \to V$, denote the multiplication operator $L_w(v) = w * v$. The fact that $*$ is associative and commutative implies that the operators L_w, $w \in V_2$, commute, while the assumption that $*$ is graded implies that L_w is nilpotent and $L_w^{k+1} = 0$. Given a basis $w_1, \dots, w_r$ of V_2, we can view V as a module over the polynomial ring $\mathfrak{U} := \mathbb{C}[L_{w_1}, \dots, L_{w_r}]$. Clearly, if $V = \mathfrak{U} \cdot e_0$, then the product structure $*$ may be deduced from the action of $\mathfrak{U}$. This conclusion is also true for $k = 3$ or $k = 4$ without further conditions, as can be checked explicitly.

If V underlies a real, graded Frobenius algebra of weight k, we can define on V a mixed Hodge structure of Hodge-Tate type by $I^{p,q}=0$ if $p\neq q$, and $I^{p,p}=V_{2(k-p)}$. If we set

$$Q(v_a,v_b)=(-1)^a\mathcal{B}(v_a,v_b) \text{ if } v_a\in V_{2a}, \tag{4.7}$$

then Q has parity $(-1)^k$ and for $w\in V_2\cap V_{\mathbb{R}}$, the operator L_w is an infinitesimal automorphism of Q and a $(-1,-1)$ morphism of the MHS. We also observe, that in the geometric case —such as in Example 4.2— this construction agrees with that in Example 2.4.

Definition 4.3. An element $w\in V_2\cap V_{\mathbb{R}}$ is said to *polarize* $(V,*,e_0,\mathcal{B})$ if $(I^{*,*},Q,L_w)$ is a polarized MHS. A real, graded Frobenius algebra is said to be polarized if it contains a polarizing element. In this case, the cubic form ϕ_0 is called a *classical potential*.

By Theorem 2.3, a polarizing element w determines a one dimensional nilpotent orbit $(W(L_w),F)$, where $W_l(L_w)=\oplus_{2j\geq 2k-l}V_{2j}$ is the weight filtration of L_w and $F^p=\oplus_{j\leq k-p}V_{2j}$.

Given a polarizing element w, the set of polarizing elements is an open cone in $V_2\cap V_{\mathbb{R}}$. Then, it is possible to choose a basis $w_1,\dots,w_r$ of $V_2\cap V_{\mathbb{R}}$ spanning a simplicial cone $\mathcal{C}$ contained in the closure of the polarizing cone and with $w\in\mathcal{C}$. Such a choice of a basis of V_2 will be called a *framing* of the polarized Frobenius algebra.

Since the weight filtration is constant over all the elements L_w for $w\in\mathcal{C}$, it follows from Theorem 2.3 that $(W(\mathcal{C}),F^*)$ is a nilpotent orbit. Hence, we can define a polarized VHS on $(\Delta^*)^r$ whose period mapping is given by

$$\theta(q_1,\dots,q_r) \;=\; \exp(\sum_{j=1}^r z_jL_{w_j})\cdot F\;;\quad q_j=e^{2\pi i z_j}\,. \tag{4.8}$$

Note that the origin is a maximally unipotent boundary point in the sense of Definition 3.3.

We conclude this section showing that in the weight-four case, maximally unipotent, Hodge-Tate, nilpotent orbits yield graded Frobenius algebras. In the weight-three case this is done in [**11**, Example 14].

Proposition 4.4. *Let $(N_1,\dots,N_r;F)$ be a weight-four nilpotent orbit, polarized by Q, whose limiting MHS is Hodge-Tate. Suppose that* $\dim I^{4,4}=1$ *and choose a non-zero element $e_0\in I^{4,4}\cap V_{\mathbb{R}}$; let $e_0^*\in I^{0,0}\cap V_{\mathbb{R}}$ be such that $Q(e_0,e_0^*)=1$. Assume, moreover, that $\{N_1(e_0),\dots,N_r(e_0)\}$ are a basis of $I^{3,3}$. Let $\mathcal{B}$ be obtained from Q as in (4.7). Then, there exists a unique product $*$ on V with unit e_0 such that*

$$N_j(e_0)*v \;:=\; N_j(v)\,,\ \textit{for } v\in V,\ j=1,\dots,r \tag{4.9}$$

$$v_1*v_2 \;:=\; \mathcal{B}(v_1,v_2)\,e_0^*\,,\ \textit{for } v_1,v_2\in I^{2,2} \tag{4.10}$$

Furthermore, $(V,*,e_0,\mathcal{B})$ is a graded, polarized, real Frobenius algebra.

Proof. It is clear that (4.9) and (4.10) define a graded product whose unit is e_0. Commutativity follows immediately from the symmetry of Q and the commutativity of the operators N_j.

There are two non-trivial cases to check in order to prove the associativity of the product. When all three factors lie in $I^{3,3}$ this follows, again, from the commutativity of the operators N_j. On the other hand, given $v \in I^{2,2}$:

$$\begin{aligned}(N_j(e_0) * N_k(e_0)) * v &= \mathcal{B}(N_j(e_0) * N_k(e_0), v)\, e_0^* = \mathcal{B}(N_j(N_k(e_0)), v)\, e_0^* \\ &= \mathcal{B}(N_k(N_j(e_0)), v)\, e_0^* = \mathcal{B}(e_0, N_j(N_k(v)))\, e_0^* \\ &= \mathcal{B}(e_0, N_j(e_0) * (N_k(e_0) * v))\, e_0^* \\ &= N_j(e_0) * (N_k(e_0) * v).\end{aligned}$$

Thus, $(V, *, e_0)$ is a graded, commutative, associative algebra with unit e_0. It is straightforward to check that $\mathcal{B}$ is compatible with the product. □

5. Quantum Products

By a quantum product we will mean a suitable deformation of the (constant) product on a graded, polarized, real Frobenius algebra. The weight-three case has been extensively studied in the context of mirror symmetry for Calabi-Yau threefolds ([**10**, Chapter 8], [**21**]). Here we will restrict our attention to the $k = 4$ case. In order to motivate our definitions, we recall the construction of the Gromov-Witten potential in the case of Calabi-Yau fourfolds; we refer to [**10**, Ch. 7 and 8] for proofs and details.

As in Example 4.2, let X be a Calabi-Yau fourfold, and consider the graded, polarized real Frobenius algebra $V = \oplus_{p=0}^{4} H^{p,p}(X)$, endowed with the cup product and the intersection form $\mathcal{B}$. We choose a basis $\{T_0, \dots, T_m\}$ as in the example with the added assumption that $\{T_1, \dots, T_r\}$ be a $\mathbb{Z}$-basis of $H^{1,1}(X, \mathbb{Z})$ lying in the closure of the Kähler cone.

Following [**10**, § 8.2], we define the Gromov-Witten potential as

$$\phi(z) = \phi^{GW}(z) = \sum_n \sum_{\beta \in H_2(X,\mathbb{Z})} \frac{1}{n!} \langle I_{0,n,\beta} \rangle (\gamma^n) q^\beta \tag{5.1}$$

where $\gamma = \sum_{j=0}^{m} z_j T_j$ and $\langle I_{0,n,\beta} \rangle$ is the Gromov-Witten invariant [**10**, (7.11)]. The term q^β may be interpreted as a formal power or, given a class ω in the complexified Kähler cone, as $q^\beta := \exp(2\pi i \int_\beta \omega)$.

The term corresponding to $\beta = 0$ in (5.1) yields the classical potential $\phi_0^{cup}(z) = (1/6) \int_X \gamma^3$; moreover, if we set $\delta = \sum_{j=1}^{r} z_j T_j$ and $\epsilon = \gamma - \delta - z_0 T_0$ and apply the Divisor Axiom (see [**10**, §8.3.1]) we may rewrite (5.1) as

$$\phi^{GW}(z) = \phi_0^{cup}(z) + \sum_n \sum_{\beta \in H_2(X,\mathbb{Z}) - \{0\}} \frac{1}{n!} \langle I_{0,n,\beta} \rangle (\epsilon^n) \exp(\int_\beta \delta) q^\beta.$$

Now, the homogeneity properties of the Gromov-Witten potential allow us to further simplify this expression in case X is a Calabi-Yau fourfold

$$\phi^{GW}(z) = \phi_0^{cup}(z) + \sum_{a=1}^{s} \sum_{\beta \in H_2(X,\mathbb{Z}) - \{0\}} \langle I_{0,1,\beta} \rangle (T_{r+a}) z_{r+a} e^{2\pi i \sum_{j=1}^{r} z_j \int_\beta T_j} q^\beta.$$

Note that the above series depends linearly on $z_{r+1}, \dots, z_{r+s}$ while the variables $z_1, \dots, z_r$, appear only in exponential form. Hence, we can write

$$\phi^{GW}(z) = \phi_0^{cup}(z) + \sum_{a=1}^{s} z_{r+a} \phi_h^a(z_1, \dots, z_r).$$

with $\phi_h^a(z_1,\dots,z_r) = \Psi^a(e^{2\pi i z_1},\dots,e^{2\pi i z_r})$. It follows from the Effectivity Axiom (see [**10**, § 7.3]) that $\Psi^a(0)=0$.

This construction motivates the following definition of an abstract potential function for graded, polarized, real Frobenius algebras of weight four.

Definition 5.1. Let $(V, *_0, e_0, \mathcal{B})$ be a graded, polarized, real Frobenius algebra of weight four and let $\{T_0,\dots,T_m\}$ be an adapted basis as in Example 4.2. Assume, moreover, that $T_1,\dots,T_r$ are a framing of V. A *potential* on $(V, *_0, e_0, \mathcal{B})$ is a function

$$\phi(z) = \phi_0(z) + \phi_\hbar(z)\ ; \quad \phi_\hbar(z) = \sum_{a=1}^{s} z_{r+a}\,\phi_h^a(z_1,\dots,z_r)\,, \tag{5.2}$$

where $\phi_0(z)$ is the classical potential associated with $(V, *_0, e_0, \mathcal{B})$, $\phi_h^a(z_1,\dots,z_r) = \psi^a(q_1,\dots,q_r)$, $q_j = \exp(2\pi i z_j)$, and $\psi^a(q)$ are holomorphic functions in a neighborhood of the origin in $\mathbb{C}^r$ such that $\psi^a(0)=0$. We will refer to $\phi_\hbar(z)$ as the *quantum* part of the potential.

Given a potential ϕ we define a *quantum* product on V by

$$T_a * T_b \ = \ \sum_{c=0}^{m} \frac{\partial^3\phi}{\partial z_a \partial z_b \partial z_c} T_c^{\mathcal{B}}\,, \tag{5.3}$$

where $\{T_0^{\mathcal{B}},\dots,T_m^{\mathcal{B}}\}$ denotes the $\mathcal{B}$-dual basis. Clearly, $*$ is commutative and $e_0 = T_0$ is still a unit. The quantum product is associative if and only if the potential satisfies the WDVV equations:

$$\sum_{a=1}^{s} \frac{\partial^3\phi}{\partial z_i \partial z_j \partial z_{r+a}}\ \frac{\partial^3\phi}{\partial z_k \partial z_l \partial z_{r+a}} \ = \ \sum_{a=1}^{s} \frac{\partial^3\phi}{\partial z_k \partial z_j \partial z_{r+a}}\ \frac{\partial^3\phi}{\partial z_i \partial z_l \partial z_{r+a}},$$

with i,j,k,l running from 1 to r. In view of (5.2) and (4.5) these equations are equivalent to (4.2) and

$$\begin{aligned} \sum_{a=1}^{s}\Big(P_{ij}^a \frac{\partial^2\phi_h^a}{\partial z_k \partial z_l} + P_{kl}^a \frac{\partial^2\phi_h^a}{\partial z_i \partial z_j} + \frac{\partial^2\phi_h^a}{\partial z_i \partial z_j}\frac{\partial^2\phi_h^a}{\partial z_k \partial z_l}\Big) &= \\ = \sum_{a=1}^{s}\Big(P_{il}^a \frac{\partial^2\phi_h^a}{\partial z_j \partial z_k} + P_{jk}^a \frac{\partial^2\phi_h^a}{\partial z_i \partial z_l} + \frac{\partial^2\phi_h^a}{\partial z_i \partial z_l}\frac{\partial^2\phi_h^a}{\partial z_j \partial z_k}\Big), \end{aligned} \tag{5.4}$$

for all $i,j,k,l = 1,\dots,r$, and where P_{ij}^a denotes the coefficients (4.6).

Remark 5.2. For a classical potential the WDVV equations reduce to the algebraic relation (4.2). The Gromov-Witten potential (5.1) satisfies the WDVV equations (see [**10**, Theorem 8.2.4]).

We can now state and prove the main theorem of this section.

Theorem 5.3. *There is a one-to-one correspondence between*

- *Associative quantum products on a framed Frobenius algebra of weight four.*
- *Germs of polarized variations of Hodge structure of weight four for which the origin $0 \in \mathbb{C}^r$ is a maximally unipotent boundary point, and whose limiting mixed Hodge structure is of Hodge-Tate type.*

This correspondence, which depends on the choice of an element corresponding to the unit, identifies the classical potential with the nilpotent orbit of the PVHS while the quantum part of the potential is equivalent to the holomorphic function Γ *defined by (2.11) relative to a canonical basis.*

Proof. Let $(V, *_0, \mathcal{B}, e_0)$ be a graded, polarized, real Frobenius algebra of weight four. Let $T_0, \dots, T_m$ be an adapted basis such that $T_1, \dots, T_r$ is a framing of V_2. Let $\mathcal{H} \subset V_2$ be the tube domain

$$\mathcal{H} \ := \ \{ \sum_{j=1}^{r} z_j \, T_j \ ; \ \mathrm{Im}(z_j) > 0 \}$$

We view $\mathcal{H}$ as the universal covering of $(\Delta^*)^r$ via the map $(z_1, \dots, z_r) \mapsto (q_1, \dots, q_r)$, $q_j = \exp(2\pi i z_j)$. Let $\mathcal{V}$ denote the trivial bundle over $(\Delta^*)^r$ with fiber V and $\mathcal{F}^p$ the trivial subbundle with fiber $\sum_{a \leq 8-2p} V_a$.

Given a potential ϕ and elements $w \in V_2$, $v \in V$, the quantum product $w * v$ may be thought of as a V-valued function on V. Let $w *_s v$ denote its restriction to $\mathcal{H}$. It follows easily from (5.2) and (4.5) that $w *_s v$ depends only on $q_1, \dots, q_r$ and, therefore, it descends to a V-valued function on $(\Delta^*)^r$, *i.e.* a section of $\mathcal{V}$. This allows us to define a connection ∇ on $\mathcal{V}$ by

$$\nabla_{\frac{\partial}{\partial q_j}} v \ := \ \frac{1}{2\pi i q_j} (T_j *_s v) \, ,$$

where in the left-hand side v represents the constant section defined by $v \in V$. As shown in [**10**, Proposition 8.5.2] the WDVV equations for the potential ϕ imply that ∇ is flat.

We can compute explicitly the connection forms relative to the constant frame $\{T_0, \dots, T_m\}$. We have, for $j, l = 1, \dots, r$ and $a = 1, \dots, s$:

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial q_j}} T_0 &= \frac{1}{2\pi i q_j} T_j \\
\nabla_{\frac{\partial}{\partial q_j}} T_l &= \frac{1}{2\pi i} \sum_{b=1}^{s} \left(\frac{P^b_{jl}}{q_j} + 2\pi i \frac{\partial}{\partial q_j} (2\pi i q_l \frac{\partial \psi^b}{\partial q_l}) \right) T_{r+b} \\
\nabla_{\frac{\partial}{\partial q_j}} T_{r+a} &= \frac{1}{2\pi i} \sum_{k=1}^{r} \left(\frac{P^a_{jk}}{q_j} + 2\pi i \frac{\partial}{\partial q_j} (2\pi i q_k \frac{\partial \psi^a}{\partial q_k}) \right) T_{r+s+k} \\
\nabla_{\frac{\partial}{\partial q_j}} T_{r+s+l} &= \frac{1}{2\pi i q_j} \delta_{jl} T_m \\
\nabla_{\frac{\partial}{\partial q_j}} T_m &= 0,
\end{aligned} \tag{5.5}$$

where the coefficients P^a_{jk} are defined as in (4.6). Note that these equations imply that the bundles $\mathcal{F}^p$ satisfy the horizontality condition (2.1). Moreover, suppose we define a bilinear form Q on V as in (4.7) and extend it trivially to a form $\mathcal{Q}$ on $\mathcal{V}$, then it is straightforward to check that

$$\mathcal{Q}(\nabla_{\frac{\partial}{\partial q_j}} T_a, T_b) + \mathcal{Q}(T_a, \nabla_{\frac{\partial}{\partial q_j}} T_b) = 0$$

for all $j = 1, \dots, r$ and all $a, b = 0, \dots, m$. Hence the form $\mathcal{Q}$ is ∇-flat.

We may also deduce from (5.5) that ∇ has a simple pole at the origin with residues

$$\mathrm{Res}_{q_j=0}(\nabla) \;=\; \frac{1}{2\pi i}\, L^0_{T_j} \;;\quad j=1,\dots,r,$$

where $L^0_{T_j}$ denotes multiplication by T_j relative to the constant product $*_0$. It then follows from [**12**, Théorème II.1.17] that, written in terms of the (multivalued) ∇-flat basis $T_0^\flat,\dots,T_m^\flat$ the matrix of the monodromy logarithms N_j coincides with the matrix of $2\pi i\,\mathrm{Res}_{q_j=0}(\nabla)$ in the constant basis $T_0,\dots,T_m$, *i.e.* with $L^0_{T_j}$. Because the operators $L^0_{T_j}$ are real, so is the monodromy $\exp(N_j)$ and therefore we can define a flat real structure $\mathcal{V}_\mathbb{R}$ on $\mathcal{V}$.

Since $T_1,\dots,T_r$ are a framing of the polarized Frobenius algebra $(V,*_0,\mathcal{B},e_0)$, it follows from (4.8) that the map

$$\theta(q_1,\dots,q_r) \;=\; \exp(\sum_{j=1}^r z_jN_j)\cdot F \;;\quad z_j=e^{2\pi i q_j}\,.$$

is the period map of a VHS (a nilpotent orbit) in the bundle $(\mathcal{V},\mathcal{V}_\mathbb{R},\nabla,\mathcal{Q})$. Since the bundles $\mathcal{F}^p$ are already known to satisfy (2.1), we can apply Theorem 2.6 to conclude that they define a polarized VHS on $(\mathcal{V},\mathcal{V}_\mathbb{R},\nabla,\mathcal{Q})$.

In order to complete the asymptotic description of the PVHS defined by $\mathcal{F}$ on $\mathcal{V}$, we need to compute the holomorphic function $\Gamma\colon\Delta^r\to\mathfrak{g}_-$. Because of Theorem 2.7, it suffices to determine the component Γ_{-1}. Moreover, it follows from Proposition 3.5 that we may choose canonical coordinates $(q_1,\dots,q_r)$ on Δ^r, so that, in terms of the basis $T_0,\dots,T_m$, $\Gamma(q)$ has the form:

$$\Gamma(q)=\left(\begin{array}{c|c|c|c|c}
 & & & & \\ \hline
0 & & & & \\ \hline
-D^t(q) & C^t(q) & & & \\ \hline
* & * & C(q) & & \\ \hline
* & * & D(q) & 0 &
\end{array}\right). \tag{5.6}$$

Thus, $\Gamma(q)$ is completely determined by the $r\times s$-matrix $C(q)$.

On the other hand, as noted in §2, $\Psi(q)=\exp\Gamma(q)\cdot F_0$ is the expression of the Hodge bundles $\mathcal{F}^p$ in terms of the canonical sections (2.9):

$$\tilde T(z) \;=\; \exp(\sum_{j=1}^r z_jN_j)\cdot T^\flat \tag{5.7}$$

The matrix $\exp(-\Gamma(q))$, in the basis $T_0,\dots,T_m$, is the matrix expressing the canonical sections $\tilde T_0,\dots,\tilde T_m$ in terms of the constant frame. Therefore

$$\tilde T_{r+a}(q) \;=\; T_{r+a}-\sum_{k=1}^r C_{ka}(q)T_{r+s+k}-D_a(q)T_m$$

and it suffices to compute $\tilde T_{r+a}(q)$ to determine C (and D).

It is straightforward to show, using the formulae (5.5), that

$$\begin{aligned} T^\flat_m &= T_m \\ T^\flat_{r+s+l} &= T_{r+s+l} - z_l T_m \\ T^\flat_{r+a} &= T_{r+a} - \sum_{l=1}^{r} \frac{\partial(P^a + \phi^a_h)}{\partial z_l} T_{r+s+l} + (P^a + \phi^a_h) T_m \end{aligned}$$

Hence, to obtain $\tilde{T}_{r+a}(q)$ it suffices to apply (5.7), together with the fact that the matrix of N_j in the basis $\{T^\flat_p\}$ coincides with that of $*_0$-multiplication by T_j relative to $\{T_p\}$. Thus, $N_j(T^\flat_m) = 0$ and

$$N_j(T^\flat_{r+s+l}) \;=\; \delta_{jl}\, T^\flat_m\,; \quad N_j(T^\flat_{r+a}) \;=\; \sum_{k=1}^{r} P^a_{jk}\, T^\flat_{r+s+k}\,.$$

This, together with the fact that P^a is a homogeneous polynomial of degree 2, implies that

$$\begin{aligned} (\sum_{j=1}^{r} z_j N_j) T^\flat_{r+a} &= \sum_{j,l=1}^{r} z_j \frac{\partial^2 P^a}{\partial z_j \partial z_l} T^\flat_{r+s+l} = \sum_{l=1}^{r} \frac{\partial P^a}{\partial z_l} T^\flat_{r+s+l} \\ &= \sum_{l=1}^{r} \frac{\partial P^a}{\partial z_l}(T_{r+s+l} - z_l T_m) = \sum_{l=1}^{r} \frac{\partial P^a}{\partial z_l} T_{r+s+l} - 2P^a T_m \end{aligned}$$

and

$$\frac{1}{2}(\sum_{j=1}^{r} z_j N_j)^2 T^\flat_{r+a} = \frac{1}{2}(\sum_{j=1}^{r} z_j N_j) \sum_{l=1}^{r} \frac{\partial P^a}{\partial z_l} T^\flat_{r+s+l} = \frac{1}{2} \sum_{j=1}^{r} z_j \frac{\partial P^a}{\partial z_j} T^\flat_m = P^a T_m.$$

Hence

$$\begin{aligned} \tilde{T}_{r+a} \;&=\; T^\flat_{r+a} + (\sum_{j=1}^{r} z_j N_j) T^\flat_{r+a} + \frac{1}{2}(\sum_{j=1}^{r} z_j N_j)^2 T^\flat_{r+a} \\ &=\; T^\flat_{r+a} + \sum_{l=1}^{r} \frac{\partial P^a}{\partial z_l} T_{r+s+l} - 2P^a T_m + P^a T_m \\ &=\; T_{r+a} - \sum_{l=1}^{r} \frac{\partial \phi^a_h}{\partial z_l} T_{r+s+l} + \phi^a_h T_m. \end{aligned}$$

Thus,

$$C_{ka} \;=\; \frac{\partial \phi^a_h}{\partial z_k} \quad \text{and} \quad D_a \;=\; -\phi^a_h\,.$$

Conversely, suppose now that $(\mathcal{V}, \mathcal{V}_{\mathbb{R}}, \mathcal{F}, \nabla, \mathcal{Q})$ is a polarized VHS of weight four over $(\Delta^*)^r$, that the origin is a maximally unipotent boundary point, and that the limiting MHS is of Hodge-Tate type. Let $\{N_1, \ldots, N_r; F\}$ denote the associated nilpotent orbit and set $V_{8-2p} := I^{p,p}$. It follows from Proposition 4.4 that we can define a product $*_0$, and a bilinear form $\mathcal{B}$ —as in (4.7)— turning $(V, *_0, \mathcal{B})$ into a polarized, real, graded Frobenius algebra with unit $e_0 \in V_0$. This structure is

determined by the choice of unit and the fact that, relative to an adapted basis $\{T_0, \ldots, T_m\}$ as in Example 4.2,

$$N_j(T_{r+a}) \;=\; \sum_{k=1}^{r} \mathcal{B}(T_j *_0 T_{r+a}, T_k)\, T_{r+s+k} \;=\; \sum_{k=1}^{r} P^a_{jk}\, T_{r+s+k}, \tag{5.8}$$

$j = 1, \ldots, r$, $a = 1, \ldots, s$, and P^a_{jk} are the coefficients (4.6) of the associated classical potential ϕ_0.

We have already noted that in canonical coordinates, the holomorphic function Γ associated with the PVHS takes on the special form (5.6). Moreover, since $\Gamma(q)$ satisfies the differential equation (2.14), we have from (2.17) that

$$dG_{-2} = \Gamma_{-1}\Theta - \Theta\Gamma_{-1} + \Gamma_{-1} d\Gamma_{-1}$$

for $\Theta = d(\sum_{j=1}^{r} z_j N_j)$. Consequently,

$$d(D_a) = -\sum_{k=1}^{r} C_{ka}\, dz_k. \tag{5.9}$$

We now define a potential on V by

$$\phi(z) \;:=\; \phi_0(z) - \sum_{a=1}^{s} z_{r+a} D_a(q)\,. \tag{5.10}$$

Since we already know that the classical potential ϕ_0 satisfies (4.2), the WDVV equations for Φ reduce to the equations (5.4). But this is a consequence of the integrability condition (2.14); indeed, note that given (5.8), if we let $\Xi = (\xi_{ka})$ be the $r \times s$-matrix of one forms

$$\xi_{ka} \;=\; \sum_{j=1}^{r} (P^a_{jk} + \frac{\partial C_{ka}}{\partial z_j})\, dz_j,$$

the equation (2.14) reduces to

$$\Xi \wedge \Xi^t \;=\; 0 \tag{5.11}$$

which, in view of (5.9) and (5.10), is easily seen to be equivalent to (5.4). □

Remark 5.4. Note that (5.11) expresses the WDVV equations in a very compact form. Also, note that even though the quantum product is defined in terms of third derivatives of ϕ, one recovers the full potential ϕ from the PVHS. Since (5.3) only allows for an ambiguity which is, at most, quadratic in z, the quantum part $\phi_\hbar$ is uniquely determined.

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