PERVERSE SHEAVES ON LOOP GRASSMANNIANS AND LANGLANDS DUALITY

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1. Introduction.

In this paper we outline a proof of a geometric version of the Satake isomorphism. Namely, given a connected, complex algebraic reductive group G we show that the tensor category of representations of the dual group $\check{G}$ is naturally equivalent to a certain category of perverse sheaves on the loop Grassmannian of G. The tensor category structure on this category of perverse sheaves is given via a convolution product.

The above result is not new. It has been announced by Ginsburg in [G] and some of the arguments in section 5 of this paper are borrowed from [G]. However, at crucial points our proof differs from Ginsburg's. First, we use a more "natural" commutativity constraint for the convolution product. This commutativity constraint, explained in section 3, is due to Drinfeld and was explained to us by Beilinson. Secondly, in section 4, we give a direct geometric proof that the global cohomology functor is exact and decompose this cohomology functor into a direct sum of weights (Theorem 4.3). We completely avoid the use of the decomposition theorem of [BBD] which makes our techniques applicable to perverse sheaves with coefficients over arbitrary commutative rings.

This note includes sketches of (some of the) proofs. The details, as well as the generalization of the results from $\mathbb{C}$-representations to representations over arbitrary fields and commutative rings will appear elsewhere.

2. The Convolution Product.

Let G be a connected, complex algebraic reductive group. Denote by $\mathcal{O} = \mathbb{C}[[t]]$ the ring of formal power series in one variable and by $\mathcal{K} = \mathbb{C}((t))$ its fraction field, the field of formal Laurent series. The loop Grassmannian, as a set, is defined as $\mathcal{G} = G(\mathcal{K})/G(\mathcal{O})$, where, as usual, $G(\mathcal{K})$ and $G(\mathcal{O})$ denote the sets of the $\mathcal{K}$-valued and the $\mathcal{O}$-valued points of G respectively. The sets $G(\mathcal{K})$, $G(\mathcal{O})$, and $\mathcal{G}$ have an algebraic structure as $\mathbb{C}$-spaces. The space $G(\mathcal{O})$ is a group scheme over $\mathbb{C}$ but the spaces $G(\mathcal{K})$ and $\mathcal{G}$ are only ind-schemes[1]. To see that $G(\mathcal{K})$ is an ind-scheme, one embeds G in $SL_N(\mathbb{C})$. The filtration by order of pole in $SL_N(\mathcal{K})$ induces a filtration

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[1]By an ind-scheme we mean an ind-scheme in a strict sense, i.e., an inductive system of schemes where all maps are closed embeddings.

of $G(\mathcal{K})$ which exhibits $G(\mathcal{K})$ as an inductive limit of schemes. The filtration above is invariant under the (right) action of $G(\mathcal{O})$ on $G(\mathcal{K})$ and thus, after taking the quotient of $G(\mathcal{K})$ by $G(\mathcal{O})$ one gets a filtration of $\mathcal{G}$ which exhibits it as a union of finite dimensional projective schemes. Furthermore, the morphism $\pi : G(\mathcal{K}) \to \mathcal{G}$ is locally trivial in the Zariski topology, i.e., there exists a Zariski open subset $U \subset \mathcal{G}$ such that $\pi^{-1}(U) \cong U \times G(\mathcal{O})$ and π restricted to $U \times G(\mathcal{O})$ is simply projection to the first factor. For details see for example [BL1,LS].

The group scheme $G(\mathcal{O})$ acts on $\mathcal{G}$ with finite dimensional orbits. In order to describe the orbit structure, let us fix a maximal torus $T \subset G$. We write W for the Weyl group and $X_*(T)$ for the coweights $\mathrm{Hom}(\mathbb{C}^*, T)$. Then the $G(\mathcal{O})$-orbits on $\mathcal{G}$ are parametrized by the W-orbits in $X_*(T)$, and given $\lambda \in X_*(T)$ the $G(\mathcal{O})$-orbit associated to it is $\mathcal{G}_\lambda = G(\mathcal{O}) \cdot \lambda \subset \mathcal{G}$, where we have identified $X_*(T)$ as a subset of $G(\mathcal{K})$.

Let $\Bbbk$ be a field of characteristic zero, which we fix for the rest of the paper. All sheaves that we encounter in this paper will be sheaves in the classical topology. We denote by $\mathrm{P}_{G(\mathcal{O})}(\mathcal{G}, \Bbbk)$ the category of $G(\mathcal{O})$-equivariant perverse $\Bbbk$-sheaves on $\mathcal{G}$ with finite dimensional support and by $\mathrm{P}_{\mathcal{S}}(\mathcal{G}, \Bbbk)$ the category of perverse $\Bbbk$-sheaves on $\mathcal{G}$ which are constructible with respect to the orbit stratification $\mathcal{S}$ of $\mathcal{G}$ and which have finite dimensional support. We use the notational conventions of [BBD] for perverse sheaves, in particular, in order for the constant sheaf on a $G(\mathcal{O})$-orbit $\mathcal{G}_\lambda$ to be perverse it has to be placed in degree $-\dim \mathcal{G}_\lambda$.

Proposition 2.1. *The forgetful functor* $\mathrm{P}_{G(\mathcal{O})}(\mathcal{G}, \Bbbk) \to \mathrm{P}_{\mathcal{S}}(\mathcal{G}, \Bbbk)$ *is an equivalence of categories.*

We will now put a tensor category structure on $\mathrm{P}_{G(\mathcal{O})}(\mathcal{G}, \Bbbk)$ via the convolution product. Consider the following diagram of maps (of sets)

$$\mathcal{G} \times \mathcal{G} \xleftarrow{p} G(\mathcal{K}) \times \mathcal{G} \xrightarrow{q} G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{G} \xrightarrow{m} \mathcal{G}\,. \tag{2.2}$$

Here $G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{G}$ denotes the quotient of $G(\mathcal{K}) \times \mathcal{G}$ by $G(\mathcal{O})$ where the action is given on the $G(\mathcal{K})$-factor via right multiplication by an inverse and on the $\mathcal{G}$-factor by left multiplication. The p and q are projection maps and m is the multiplication map. All other terms in (2.2) have been given a structure of an ind-scheme except $G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{G}$. The description of this structure is easier in the global context of section 3 where it is a special case of a more general construction and thus we omit the details here. We define the convolution product $A_1 * A_2$ of $A_1, A_2 \in \mathrm{P}_{G(\mathcal{O})}(\mathcal{G}, \Bbbk)$ by the formula

$$A_1 * A_2 \;=\; Rm_* \tilde{A} \qquad \text{where} \;\; q^* \tilde{A} = p^*(A_1 \boxtimes A_2)\,. \tag{2.3}$$

To make sense of this definition we first use the fact that p and q are locally trivial in the Zariski topology. This guarantees the existence of $\tilde{A} \in \mathrm{P}_{G(\mathcal{O})}(G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{G}, \Bbbk)$. To see the local triviality of q one can use the same arguments as for example in [BL1,LS], and as was pointed out above, the local triviality of p is proved in those references. It remains to show that $Rm_* \tilde{A} \in \mathrm{P}_{G(\mathcal{O})}(\mathcal{G}, \Bbbk)$. To that end we introduce the notion of a stratified semi-small map.

Let us consider two complex stratified spaces $(Y, \mathcal{T})$ and $(X, \mathcal{S})$ and a map $f : Y \to X$. We assume that the two stratifications are locally trivial with connected strata and that f is a stratified with respect to the stratifications $\mathcal{T}$ and $\mathcal{S}$, i.e.,

that for any $T \in \mathcal{T}$ the image $f(T)$ is a union of strata in $\mathcal{S}$ and for any $S \in \mathcal{S}$ the map $f|f^{-1}(S) : f^{-1}(S) \to S$ is locally trivial in the stratified sense. We say that f is a stratified semi-small map if

$$
\begin{aligned}
&a) \text{ for any } T \in \mathcal{T} \text{ the map } f|\bar{T} \text{ is proper} \\
&b) \text{ for any } T \in \mathcal{T} \text{ and any } S \in \mathcal{S} \text{ such that } S \subset f(\bar{T}) \text{ we have} \\
&\quad \dim(f^{-1}(x) \cap \bar{T}) \leq \frac{1}{2}(\dim f(\bar{T}) - \dim S) \\
&\quad \text{for any (and thus all) } x \in S\,.
\end{aligned} \tag{2.4}
$$

Next the notion of a small stratified map. We say that f is a small stratified map if there exists a (non-trivial) open stratified subset W of Y such that

$$
\begin{aligned}
&a) \text{ for any } T \in \mathcal{T} \text{ the map } f|\bar{T} \text{ is proper} \\
&b) \text{ the map } f|W : W \to f(W) \text{ is proper and has finite fibers} \\
&c) \text{ for any } T \in \mathcal{T},\, T \subset W, \text{ and any } S \in \mathcal{S} \text{ such that } S \subset f(\bar{T}) - f(T) \\
&\quad \text{we have } \ \dim(f^{-1}(x) \cap \bar{T}) \leq \frac{1}{2}(\dim f(\bar{T}) - \dim S) \\
&\quad \text{for any (and thus all) } x \in S\,.
\end{aligned} \tag{2.5}
$$

The result below follows directly from dimension counting:

Lemma 2.6. *If f is a semismall stratified map then $Rf_*A \in \mathrm{P}_{\mathcal{S}}(X, \Bbbk)$ for all $A \in \mathrm{P}_{\mathcal{T}}(Y, \Bbbk)$. If f is a small stratified map then, with any W as above, and any $A \in \mathrm{P}_{\mathcal{T}}(W, \Bbbk)$, we have $Rf_*j_{!*}A = \tilde{j}_{!*}f_*A$, where $j : W \hookrightarrow Y$ and $\tilde{j} : f(W) \hookrightarrow X$ denote the two inclusions.*

We apply the above considerations, in the semismall case, to our situation. We take $Y = G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{G}$ and choose $\mathcal{T}$ to be the stratification whose strata are $p^{-1}(\mathcal{G}_\lambda) \times_{G(\mathcal{O})} \mathcal{G}_\mu$, for $\lambda, \mu \in X_*(T)$. We also let $X = \mathcal{G}$, $\mathcal{S}$ the stratification by $G(\mathcal{O})$-orbits, and choose $f = m$. To conclude the constructions of the convolution product on $\mathrm{P}_{G(\mathcal{O})}(\mathcal{G}, \Bbbk)$ it suffices to note that the sheaf $\tilde{A}$ is constructible with respect to the stratification $\mathcal{T}$ and appeal to the following

Theorem 2.7. *The multiplication map $G(\mathcal{K}) \times_{G(\mathcal{O})} \mathcal{G} \xrightarrow{m} \mathcal{G}$ is a stratified semi-small map with respect to the stratifications above.*

For an outline of proof, see the appendix.

One can define the convolution product of three sheaves completely analogously to (2.3). This gives an associativity constraint for the convolution product thus giving $\mathrm{P}_{G(\mathcal{O})}(\mathcal{G}, \Bbbk)$ the structure of an associative tensor category. In the next section we construct a commutativity constraint for the convolution product.

3. The Commutativity Constraint.

In order to construct the commutativity constraint we will need to consider the convolution product in the global situation. Let X be a smooth curve over the

complex numbers. Let $x \in X$ be a closed point and denote by $\mathcal{O}_x$ the completion of the local ring at x and by $\mathcal{K}_x$ its fraction field. Then the Grassmannian $\mathcal{G}_x = G(\mathcal{K}_x)/G(\mathcal{O}_x)$ represents the following functor from $\mathbb{C}$-algebras to sets :

$$R \mapsto \{\mathcal{F} \text{ a } G\text{-torsor on } X_R,\ \nu : G \times X_R^* \to \mathcal{F}|X_R^* \text{ a trivialization on } X_R^*\ \}\,. \tag{3.1}$$

Here the pairs $(\mathcal{F}, \nu)$ are to be taken up to isomorphism, $X_R = X \times \mathrm{Spec}(R)$, and $X_R^* = (X - \{x\}) \times \mathrm{Spec}(R)$. For details see for example [BL1,BL2,LS]. We now globalize this construction and at the same time form the Grassmannian at several points on the curve. Denote the n fold product by $X^n = X \times \cdots \times X$ and consider the functor

$$R \mapsto \left\{ \begin{array}{l} (x_1, \ldots, x_n) \in X^n(R), \quad \mathcal{F} \text{ a } G\text{-torsor on } X_R\,, \\ \nu_{(x_1,\ldots,x_n)} \text{ a trivialization of } \mathcal{F} \text{ on } X_R - \cup x_i \end{array} \right\}. \tag{3.2}$$

Here we think of the points $x_i : \mathrm{Spec}(R) \to X$ as subschemes of X_R by taking their graphs. One sees that the functor in (3.2) is represented by an ind-scheme $\mathcal{G}_X^{(n)}$. Of course $\mathcal{G}_X^{(n)}$ is an ind-scheme over X^n and its fiber over the point $(x_1, \ldots, x_n)$ is simply $\prod_{i=1}^k \mathcal{G}_{y_i}$, where $\{y_1, \ldots, y_k\} = \{x_1, \ldots, x_n\}$, with all the y_i distinct. We write $\mathcal{G}_X^{(1)} = \mathcal{G}_X$.

We will now extend the diagram of maps (2.2), which was used to define the convolution product, to the global situation, i.e., to a diagram of ind-schemes over X^2:

$$\mathcal{G}_X \times \mathcal{G}_X \xleftarrow{p} \widetilde{\mathcal{G}_X \times \mathcal{G}_X} \xrightarrow{q} \mathcal{G}_X \tilde{\times} \mathcal{G}_X \xrightarrow{m} \mathcal{G}_X^{(2)}\,. \tag{3.3}$$

Roughly, the diagram starts with a pair of torsors, each trivialized off one point. One chooses a trivialization of the first torsor near the second point, and uses it to glue the torsors.

More precisely, $\widetilde{\mathcal{G}_X \times \mathcal{G}_X}$ denotes the ind-scheme representing the functor

$$R \mapsto \left\{ \begin{array}{l} (x_1, x_2) \in X^2(R);\ \mathcal{F}_1, \mathcal{F}_2\ G\text{-torsors on } X_R;\ \nu_i \text{ a trivialization of} \\ \mathcal{F}_i \text{ on } X_R - x_i, \text{ for } i = 1, 2;\ \ \mu_1 \text{ a trivialization of } \mathcal{F}_1 \text{ on } \widehat{(X_R)}_{x_2} \end{array} \right\}, \tag{3.4}$$

where $\widehat{(X_R)}_{x_2}$ denotes the formal neighborhood of x_2 in X_R. The "twisted product" $\mathcal{G}_X \tilde{\times} \mathcal{G}_X$ is the ind-scheme representing the functor

$$R \mapsto \left\{ \begin{array}{l} (x_1, x_2) \in X^2(R);\ \mathcal{F}_1, \mathcal{F}\ G\text{-torsors on } X_R;\ \nu_1 \text{ a trivialization} \\ \text{of } \mathcal{F}_1 \text{ on } X_R - x_1;\ \eta : \mathcal{F}_1|(X_R - x_2) \xrightarrow{\simeq} \mathcal{F}|(X_R - x_2) \end{array} \right\}. \tag{3.5}$$

It remains to describe the morphisms p, q, and m in (3.3). Because all the spaces in (3.3) are ind-schemes over X^2, and all the functors involve the choice of the same $(x_1, x_2) \in X^2(R)$ we omit it in the formulas below. The morphism p simply forgets the choice of μ_1, the morphism q is given by the natural transformation

$$(\mathcal{F}_1, \nu_1, \mu_1; \mathcal{F}_2, \nu_2) \mapsto (\mathcal{F}_1, \nu_1, \mathcal{F}, \eta), \tag{3.6}$$

where $\mathcal{F}$ is the G-torsor gotten by gluing $\mathcal{F}_1$ on $X_R - x_2$ and $\mathcal{F}_2$ on $\widehat{(X_R)}_{x_2}$ using the isomorphism induced by $\nu_2 \circ \mu_1^{-1}$ between $\mathcal{F}_1$ and $\mathcal{F}_2$ on $(X_R - x_2) \cap \widehat{(X_R)}_{x_2}$. The morphism m is given by the natural transformation

$$(\mathcal{F}_1, \nu_1, \mathcal{F}, \eta) \mapsto (\mathcal{F}, \nu)\,, \tag{3.7}$$

where $\nu = (\eta \circ \nu_1)|(X_R - x_1 - x_2)$.

Next, the global analog of $G(\mathcal{O})$ is the group-scheme $G_X^{(n)}(\mathcal{O})$ which represents the functor

$$R \mapsto \left\{ \begin{array}{l} (x_1, \ldots, x_n) \in X^n(R), \quad \mathcal{F} \text{ the trivial } G\text{-torsor on } X_R\,, \\ \mu_{(x_1,\ldots,x_n)} \text{ a trivialization of } \mathcal{F} \text{ on } \widehat{(X_R)}_{(x_1 \cup \cdots \cup x_n)} \end{array} \right\}. \tag{3.8}$$

Just as in section 2 we define the convolution product of $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{P}_{G_X(\mathcal{O})}(\mathcal{G}_X, \Bbbk)$ by the formula

$$\mathcal{B}_1 \underset{X}{*} \mathcal{B}_2 \;=\; Rm_* \tilde{\mathcal{B}} \qquad \text{where} \;\; q^* \tilde{\mathcal{B}} = p^*(\mathcal{B}_1 \boxtimes \mathcal{B}_2)\,. \tag{3.9}$$

Precisely as in section 2, the sheaf $\tilde{\mathcal{B}}$ exists because q is locally, even in the Zariski topology, a product. Furthermore, the map m is a stratified small map – regardless of the stratification on X. To see this, let us denote by $\Delta \subset X^2$ the diagonal and set $U = X^2 - \Delta$. Then we can take, in definition (2.5), as W the locus of points lying over U. That m is small now follows as m is an isomorphism over U and over points of Δ the map m coincides with its analogue in section 2 which is semi-small by theorem 2.7.

Let us now, for simplicity, choose $X = \mathbb{A}^1$. Then the choice of a global coordinate on $\mathbb{A}^1$, trivializes $\mathcal{G}_X$ over X; let us write $\rho : \mathcal{G}_X \to \mathcal{G}$ for the projection. Let us denote $\rho^0 = \rho^*[1] : \mathcal{P}_{G(\mathcal{O})}(\mathcal{G}, \Bbbk) \to \mathcal{P}_{G_X(\mathcal{O})}(\mathcal{G}_X, \Bbbk)$. By restricting $\mathcal{G}_X^{(2)}$ to the diagonal $\Delta \cong X$ and to U, and observing that these restrictions are isomorphic to $\mathcal{G}_X$ and to $(\mathcal{G}_X \times \mathcal{G}_X)|U$ respectively, we get the following diagram

$$\begin{array}{ccccc} \mathcal{G}_X & \xrightarrow{\;i\;} & \mathcal{G}_X^{(2)} & \xleftarrow{\;j\;} & (\mathcal{G}_X \times \mathcal{G}_X)|U \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & X^2 & \longleftarrow & U \end{array}\,. \tag{3.10}$$

Lemma 3.11. *For $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{P}_{G(\mathcal{O})}(\mathcal{G}, \Bbbk)$ we have*

$$\begin{array}{ll} a) & \rho^0 \mathcal{A}_1 \underset{X}{*} \rho^0 \mathcal{A}_2 \;\cong\; j_{!*}\big((\rho^0 \mathcal{A}_1 \boxtimes \rho^0 \mathcal{A}_2)|U\big) \\ b) & \rho^0(\mathcal{A}_1 * \mathcal{A}_2) \;\cong\; i^0(\rho^0 \mathcal{A}_1 \underset{X}{*} \rho^0 \mathcal{A}_2)\,. \end{array}$$

Part a) of the lemma follows from smallness of m and lemma 2.6.

Lemma 3.11 gives us the following sequence of isomorphisms:

$$\begin{aligned} \rho^0(\mathcal{A}_1 * \mathcal{A}_2) &\cong i^0 j_{!*}\big((\rho^0 \mathcal{A}_1 \boxtimes \rho^0 \mathcal{A}_2)|U\big) \\ &\cong i^* j_{!*}((\rho^0 \mathcal{A}_2 \boxtimes \rho^0 \mathcal{A}_1)|U) \cong \rho^0(\mathcal{A}_2 * \mathcal{A}_1)\,. \end{aligned} \tag{3.12}$$

Specializing this isomorphism to (any) point on the diagonal yields a functorial isomorphism between $\mathcal{A}_1 * \mathcal{A}_2$ and $\mathcal{A}_2 * \mathcal{A}_1$. We take this isomorphism as our commutativity constraint.

Remark 3.13. The construction of the commutativity constraint can be carried out in a more elegant way as follows. We first observe that the image of the embedding $\rho^0 = \rho^*[1] : \mathcal{P}_{G(\mathcal{O})}(\mathcal{G}, \Bbbk) \to \mathcal{P}_{G_X(\mathcal{O})}(\mathcal{G}_X, \Bbbk)$ consists precisely of objects in $\mathcal{P}_{G_X(\mathcal{O})}(\mathcal{G}_X, \Bbbk)$ which are "constant" along X. This subcategory of $\mathcal{P}_{G_X(\mathcal{O})}(\mathcal{G}_X, \Bbbk)$ coincides with $\mathcal{P}_{\tilde{G}_X(\mathcal{O})}(\mathcal{G}_X, \Bbbk)$, where $\tilde{G}_X(\mathcal{O})$ denotes the semi direct product of $G_X(\mathcal{O})$ and the groupoid which consists of pairs of points $(x, y) \in X \times X$ together with an isomorphism between the formal neighborhood of x and the formal neighborhood of y. Now $\rho^0 = \rho^*[1] : \mathcal{P}_{G(\mathcal{O})}(\mathcal{G}, \Bbbk) \to \mathcal{P}_{\tilde{G}_X(\mathcal{O})}(\mathcal{G}_X, \Bbbk)$ is an equivalence whose inverse is $i^0 = i^*[-1]$, where $i : \mathcal{G}_x \hookrightarrow \mathcal{G}_X$ is the inclusion. If X is an arbitrary smooth curve then the functor $i^0 : \mathcal{P}_{\tilde{G}_X(\mathcal{O})}(\mathcal{G}_X, \Bbbk) \to \mathcal{P}_{G(\mathcal{O})}(\mathcal{G}, \Bbbk)$ still has meaning and is an equivalence of categories. It is clear that the convolution product (3.9) gives us a convolution product on the category $\mathcal{P}_{\tilde{G}_X(\mathcal{O})}(\mathcal{G}_X, \Bbbk)$. Thus, we can give the construction of the commutativity constraint in terms of $\mathcal{P}_{\tilde{G}_X(\mathcal{O})}(\mathcal{G}_X, \Bbbk)$ and i^0 without specializing to $X = \mathbb{A}^1$ and choosing a global coordinate.

4. The Fiber Functor.

Let $\mathrm{Vec}_\Bbbk$ denote the category of finite dimensional vector spaces over $\Bbbk$. Let us consider the global cohomology functor $\mathbb{H}^* : \mathrm{P}_{G(\mathcal{O})}(\mathcal{G}, \Bbbk) \to \mathrm{Vec}_\Bbbk$, where we ignore the grading on global cohomology.

Proposition 4.1. *The functor $\mathbb{H}^* : \mathrm{P}_{G(\mathcal{O})}(\mathcal{G}, \Bbbk) \to \mathrm{Vec}_\Bbbk$ is a tensor functor.*

Let r denote the map $r : \mathcal{G}_X^{(2)} \to X^2$. That $\mathbb{H}^*$ is tensor functor follows immediately from

$$\begin{aligned} &\text{a)}\quad Rr_*(\rho^0(A_1) *_X \rho^0(A_2))|U \ \text{ is the constant sheaf } \ \mathbb{H}^*(A_1) \otimes \mathbb{H}^*(A_2)\,. \\ &\text{b)}\quad Rr_*(\rho^0(A_1) *_X \rho^0(A_2))|\Delta = \rho^0(\mathbb{H}^*(A_1 * A_2)) \\ &\text{c)}\quad Rr_*(\rho^0(A_1) *_X \rho^0(A_2)) \ \text{ is a constant sheaf} \end{aligned} \tag{4.2}$$

The claims a) and b) follow from lemma 3.11. It remains to note that, in the notation of formula (3.9), the sheaf $R(r \circ m)_* \tilde{B}$ is constant; this implies c).

We now come to the main technical result of this paper. In order to state it we will fix some further notation. We choose a Borel subgroup $B \subset G$ which contains the maximal torus T. This, of course, determines a choice of positive roots. Let N denote the unipotent radical of B. As usual, we denote by ρ half the sum of positive roots of G. For any $\nu \in X_*(T)$ we write $\mathrm{ht}(\nu)$ for the height of ν with respect to ρ. The $N(\mathcal{K})$-orbits on $\mathcal{G}$ are parametrized by $X_*(T)$; to each $\nu \in X_*(T) = \mathrm{Hom}(\mathbb{C}^*, T)$ we associate the $N(\mathcal{K})$-orbit $S_\nu =_{\mathrm{def}} N(\mathcal{K}) \cdot \nu$. Note that these orbits are neither of finite dimension nor of finite codimension.

Theorem 4.3. *a) For all $\mathcal{A} \in \mathrm{P}_{G(\mathcal{O})}(\mathcal{G}, \Bbbk)$ we have*

$$\mathrm{H}_c^k(S_\nu, \mathcal{A}) = 0 \quad \textit{if} \quad k \neq 2\,\mathrm{ht}(\nu)\,.$$

In particular, the functors $\mathrm{H}_c^{2\,\mathrm{ht}(\nu)}(S_\nu,\ ) : \mathrm{P}_{G(\mathcal{O})}(\mathcal{G},\Bbbk) \to \mathrm{Vec}_\Bbbk$ *are exact.*

b) We have a natural equivalence of functors

$$\mathbb{H}^* \ \cong \bigoplus_{\nu\in X_*(T)} \mathrm{H}_c^{2\,\mathrm{ht}(\nu)}(S_\nu,\ )\ :\ \mathrm{P}_{G(\mathcal{O})}(\mathcal{G},\Bbbk) \to \mathrm{Vec}_\Bbbk$$

This result immediately gives the following consequence:

Corollary 4.4. *The global cohomology functor* $\mathbb{H}^* : \mathrm{P}_{G(\mathcal{O})}(\mathcal{G},\Bbbk) \to \mathrm{Vec}_\Bbbk$ *is exact.*

Here is a brief outline of the proof of theorem 4.3. Let us consider unipotent radical $\bar{N}$ of the borel $\bar{B}$ opposite to B. The $\bar{N}(\mathcal{K})$-orbits on $\mathcal{G}$ are parametrized by $X_*(T)$: to each $\nu \in X_*(T)$ we associate the orbit $T_\nu = \bar{N}(\mathcal{K})\cdot\nu$. Recall that the $G(\mathcal{O})$-orbits are parametrized by $X_*(T)/W$. The orbits S_ν and T_ν intersect the orbits $\mathcal{G}_\lambda$ as follows:

$$\begin{aligned} &\text{a)}\ \ \dim(S_\nu\cap\mathcal{G}_\lambda) \ = \ \mathrm{ht}(\nu+\lambda) \qquad \text{if } \lambda \text{ is chosen dominant}\\ &\text{b)}\ \ \dim(T_\nu\cap\mathcal{G}_\lambda) \ = \ -\,\mathrm{ht}(\nu+\lambda) \qquad \text{if } \lambda \text{ is chosen anti-dominant}\\ &\text{c)}\ \ \text{the intersections in a) and b) are of pure dimension}. \end{aligned} \tag{4.5}$$

In proving estimates a) and b) we use the fact that the boundary ∂S_ν is given by one equation in the closure $\bar{S}_\nu$. For the idea behind the proof of c), see the appendix. From the dimension estimates (4.5a,b) above we conclude immediately that

$$\begin{aligned} \mathrm{H}_c^k(S_\nu,\mathcal{A}) \ &= \ 0 \quad \text{if } k > 2\,\mathrm{ht}(\nu)\\ \mathrm{H}_{T_\nu}^k(\mathcal{G},\mathcal{A}) \ &= \ 0 \quad \text{if } k < 2\,\mathrm{ht}(\nu)\,. \end{aligned} \tag{4.6}$$

Theorem 4.3 follows immediately from (4.6) and the following statement:

$$\mathrm{H}_c^k(S_\nu,\mathcal{A}) \ = \mathrm{H}_{T_\nu}^k(\mathcal{G},\mathcal{A}) \qquad \text{for all } k\,. \tag{4.7}$$

To see (4.7) we use the fact that $N(\mathcal{K})$-orbits and $\bar{N}(\mathcal{K})$-orbits are in general position with respect to each other.

Remark 4.8. The decomposition of functors in theorem 4.3b is independent of the choice of N. In the case of N and its opposite unipotent subgroup $\bar{N}$ the corresponding decompositions are explicitly related by $\mathrm{H}_{S_\nu}^k(\mathcal{G},\mathcal{A}) \cong \mathrm{H}_{T_{w_0\cdot\nu}}^k(\mathcal{G},\mathcal{A})$, where w_0 is the longest element in the Weyl group. From this, and (4.7), we conclude that we could state theorem 4.3 replacing the functors $\mathrm{H}_c^{2\,\mathrm{ht}(\nu)}(S_\nu,\ )$ by the equivalent set of functors $\mathrm{H}_{S_\nu}^{2\,\mathrm{ht}(\nu)}(\mathcal{G},\ )$, where $\mathrm{H}_c^{2\,\mathrm{ht}(\nu)}(S_\nu,\ ) \cong \mathrm{H}_{S_{w_0\cdot\nu}}^{-2\,\mathrm{ht}(\nu)}(\mathcal{G},\ )$.

Remark 4.9. The decomposition of $\mathcal{G}_\lambda$ into $N(\mathcal{K})$-orbits and $\bar{N}(\mathcal{K})$-orbits is an example of a perverse cell complex. Perverse cell complexes are the analogues of CW-complexes for computing cohomology of perverse sheaves instead of the ordinary cohomology. In the case at hand we are in the situation analogous to the one for CW-complexes where the dimensions of all cells are of the same parity. We will develop the general theory of perverse cell complexes elsewhere.

5. The dual group.

We will now apply Tannakian formalism as in [DM] to $\mathrm{P}_{G(\mathcal{O})}(\mathcal{G},\Bbbk)$ and the functor $\mathbb{H}^*$. In sections 2 and 3 we have given a tensor product structure on the category $\mathrm{P}_{G(\mathcal{O})}(\mathcal{G},\Bbbk)$ via convolution and we have given functorial associativity and commutativity constraints for this tensor product. To see that $\mathrm{P}_{G(\mathcal{O})}(\mathcal{G},\Bbbk)$ is a rigid tensor category, we still must exhibit the identity object and construct duals. The identity object is given by the sky scraper sheaf supported on the point $1\cdot G(\mathcal{O})\in\mathcal{G}$ whose stalk is $\Bbbk$. The dual $A^\vee$ of a sheaf $A\in\mathrm{P}_{G(\mathcal{O})}(\mathcal{G},\Bbbk)$ is given as follows. Consider the following sequence of maps

$$\mathcal{G}\xleftarrow{\pi} G(\mathcal{K})\xrightarrow{i} G(\mathcal{K})\xrightarrow{\pi}\mathcal{G}\,, \tag{5.1}$$

where i is the inversion on $G(\mathcal{K})$, i.e., $i(g)=g^{-1}$. We define an equivalence

$$\iota:\mathrm{P}_{G(\mathcal{O})}(\mathcal{G},\Bbbk)\to\mathrm{P}_{G(\mathcal{O})}(\mathcal{G},\Bbbk)\text{ by }\iota(A)=\pi_*\tilde{A}\text{ where }i^*\tilde{A}=\pi^*A\,.$$

Then the dual $A^\vee$ is given by $A^\vee=\iota(\mathbb{D}A)$, where $\mathbb{D}$ denotes the Verdier dual.

In 4.1 we showed that the tensor product gets taken to the ordinary tensor product in $\mathrm{Vec}_\Bbbk$ by the functor $\mathbb{H}^*$. Furthermore, the associativity and the commutativity constraints on $\mathrm{P}_{G(\mathcal{O})}(\mathcal{G},\Bbbk)$ get mapped to the standard ones on $\mathrm{Vec}_\Bbbk$ by $\mathbb{H}^*$. Corollary 4.4 says that $\mathbb{H}^*$ is exact and from this it is not hard to deduce that it is also faithful. Thus, we have verified that $\mathrm{P}_{G(\mathcal{O})}(\mathcal{G},\Bbbk)$ together with $\mathbb{H}^*$ constitutes a neutral Tannakian category and by [DM, theorem 2.11] we conclude:

Proposition 5.1. *There exists an affine group scheme $\check{G}$ such that the tensor category $\mathrm{P}_{G(\mathcal{O})}(\mathcal{G},\Bbbk)$ is equivalent to the (tensor category) of representations of $\check{G}$. This equivalence is given via the fiber functor $\mathbb{H}^*$.*

We claim:

Proposition 5.2. *The affine groups scheme $\check{G}$ is isomorphic to the Langlands dual of G.*

To see this, one may argue as follows. First of all, it is not difficult to see that $\check{G}$ is connected. By theorem 4.3b) we conclude that the dual torus $\check{T}$ of T is contained in $\check{G}$ and then one shows, as in [G], that the torus $\check{T}$ is maximal. If $\check{G}$ had a unipotent radical, then the category of $\mathrm{P}_{G(\mathcal{O})}(\mathcal{G},\Bbbk)$ would have certain non-trivial self extensions of objects and this can easily be ruled out (this argument is due to Soergel). As one can express the root datum of a reductive group in terms of its irreducible representations one concludes, following [G], that $\check{G}$ is the dual group of G.

A few remarks are in order. Because $\check{G}$ is reductive, one concludes immediately that $\mathrm{P}_{G(\mathcal{O})}(\mathcal{G},\Bbbk)$ is semisimple. One can also see directly that $\mathrm{P}_{G(\mathcal{O})}(\mathcal{G},\Bbbk)\cong\mathrm{P}_{\mathcal{S}}(\mathcal{G},\Bbbk)$ is semisimple, for example from [Lu, theorem 11c].

Let us make the statements of propositions 5.1 and 5.2 more concrete. Let $\lambda\in X_*(T)/W=X^*(\check{T})/W$. To λ we can associate an irreducible representation V_λ of the Langlands dual group $\check{G}$ on one hand, and a $G(\mathcal{O})$-orbit $\mathcal{G}_\lambda$, and thus an irreducible perverse sheaf $\mathcal{V}_\lambda=j_{!*}\Bbbk_\lambda[\dim\mathcal{G}_\lambda]$, $j:\mathcal{G}_\lambda\hookrightarrow\mathcal{G}$, on the other. Under the equivalence of proposition 5.1 the sheaf $\mathcal{V}_\lambda$ and the representation V_λ correspond to each other. Furthermore, the representation space of V_λ gets identified with

the global cohomology of $\mathcal{V}_\lambda$, i.e., $V_\lambda = \mathbb{H}^*(\mathcal{G}, \mathcal{V}_\lambda)$. This interpretation gives a canonical basis for $\mathcal{V}_\lambda$ as follows. From theorem 4.3, the fact that $j_{!*}\Bbbk_\lambda[\dim \mathcal{G}_\lambda] = {}^p j_! \Bbbk_\lambda[\dim \mathcal{G}_\lambda]$, and (4.5c) we conclude:

$$\begin{aligned} \mathbb{H}^k(\mathcal{G}, \mathcal{V}_\lambda) \;&=\; \bigoplus_{\substack{\nu \in X_*(T) \\ k = 2\,\mathrm{ht}(\nu)}} \mathrm{H}_c^{2\,\mathrm{ht}(\nu)}(S_\nu, \mathcal{V}_\lambda) \;= \\ \bigoplus_{\substack{\nu \in X_*(T) \\ k = 2\,\mathrm{ht}(\nu)}} \mathrm{H}_c^{2\,\mathrm{ht}(\lambda+\nu)}(S_\nu \cap \mathcal{G}_\lambda, \Bbbk) \;&=\; \bigoplus_{\substack{\nu \in X_*(T) \\ k = 2\,\mathrm{ht}(\nu)}} \Bbbk[\mathrm{Irr}(S_\nu \cap \mathcal{G}_\lambda)]\,. \end{aligned} \tag{5.3}$$

Here $\Bbbk[\mathrm{Irr}(S_\nu \cap \mathcal{G}_\lambda)]$ denotes the vector space spanned by the irreducible components of $S_\nu \cap \mathcal{G}_\lambda$. Thus we get

$$V_\lambda \;=\; \mathbb{H}^*(\mathcal{G}, \mathcal{V}_\lambda) \;=\; \bigoplus_{\nu \in X_*(T)} \Bbbk[\mathrm{Irr}(S_\nu \cap \mathcal{G}_\lambda)]\,. \tag{5.4}$$

Note that the results above imply that the cohomology group $\mathbb{H}^*(\mathcal{G}, \mathcal{V}_\lambda)$ is generated by algebraic cycles.

6. Appendix.

In this appendix we outline the proofs of theorem 2.7 and the statement (4.5c). Theorem 2.7 follows from the estimate:

$$\dim[m^{-1}S_\nu \;\cap\; p^{-1}(\mathcal{G}_\lambda) \underset{G(\mathcal{O})}{\times} \mathcal{G}_\mu] \le \mathrm{ht}(\lambda + \mu + \nu) \tag{6.1}$$

for coweights $\lambda, \mu, \nu \in X_*(T)$ such that λ and μ are dominant and $\nu \in \overline{\mathcal{G}_{\lambda+\mu}}$. The statement (6.1) can be proved exactly the same way as the estimates (4.5a,b). We first directly verify (6.1) in the two cases when $\nu = \lambda + \mu$ is dominant and when $\nu = w_0(\lambda + \mu)$ is antidominant (w_0 the longest element in the Weyl group). Then we use the fact that the boundary ∂S_ν is given by one equation in the closure $\bar{S}_\nu$.

The proof of the estimate (4.5c) is more involved as we use a Poisson structure on the ind-variety $\mathcal{G}$. We choose an invariant non-degenerate bilinear form χ on $\mathfrak{g}$ and define an invariant non-degenerate form $(\ ,\ )$ on $\mathfrak{g}_\mathcal{K}$ by the formula $(x, y) = \mathrm{Res}\ \chi(x, y)$ for $x, y \in \mathfrak{g}_\mathcal{K}$. The pair $(\mathfrak{g}_\mathcal{K},\ (-,-))$, has a Manin decomposition $(\mathfrak{g}_\mathcal{K})_+ = \mathfrak{g}_\mathcal{O}$ and $(\mathfrak{g}_\mathcal{K})_- = \mathfrak{g}_{z^{-1}\mathbb{C}[z^{-1}]}$, see, for example, [Dr]. This formally defines a Poisson structure on the ind-group $G(\mathcal{K})$ which descends to a Poisson structure on $\mathcal{G} = G(\mathcal{K})/G(\mathcal{O})$. We have:

Lemma 6.2. *(a) The symplectic leaves in $\mathcal{G}$ are the intersections of $G(\mathcal{O})$-orbits and the orbits of the negative congruence subgroup $K_- =_{\mathrm{def}} G(z^{-1}\mathbb{C}[z^{-1}])$.*

(b) The $N(\mathcal{K})$-orbits are coisotropic subvarieties of the Grassmannian $\mathcal{G}$.

For a coweight $\nu \in X_*(T)$ we write $\mathcal{G}^\nu = K_- \cdot \nu \subseteq \mathcal{G}$ for its orbit under the negative congruence subgroup K_-. When the coweight ν is antidominant the intersection $S_\nu \;\cap\; \mathcal{G}_\lambda$ is a Lagrangian subvariety of the symplectic leaf $\mathcal{G}^\nu \cap \mathcal{G}_\lambda$. This

implies (4.5c) in the antidominant case. To deduce (4.5c) for general $\nu \in X_*(T)$ we use the factorization

$$S_\nu \cap \mathcal{G}_\lambda \cong (S_\nu \cap \mathcal{G}^\nu \cap \mathcal{G}_\lambda) \times (S_\nu \cap \mathcal{G}_\nu)$$

and observe that the first factor $S_\nu \cap \mathcal{G}^\nu \cap \mathcal{G}_\lambda$ is a Lagrangian subvariety of the symplectic leaf $\mathcal{G}^\nu \cap \mathcal{G}_\lambda$.

Remark 6.4. Using the same techniques one can also prove a stronger form of estimate (6.1). Namely, that the variety $m^{-1}S_\nu \cap p^{-1}(\mathcal{G}_\lambda) \times_{G(\mathcal{O})} \mathcal{G}_\mu$ is of pure dimension $\mathrm{ht}(\lambda + \mu + \nu)$.

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