University of Massachusetts - Amherst
ScholarWorks@UMass Amherst

Mathematics and Statistics Department Faculty Publication Series

Mathematics and Statistics

2010

CELLS IN COXETER GROUPS I

M Belolipetsky

PE Gunnells
University of Massachusetts - Amherst, gunnells@math.umass.edu

Follow this and additional works at: http://scholarworks.umass.edu/math_faculty_pubs

Part of the Physical Sciences and Mathematics Commons

Belolipetsky, M and Gunnells, PE, "CELLS IN COXETER GROUPS I" (2010). *Mathematics and Statistics Department Faculty Publication Series.* Paper 1179.
http://scholarworks.umass.edu/math_faculty_pubs/1179

This Article is brought to you for free and open access by the Mathematics and Statistics at ScholarWorks@UMass Amherst. It has been accepted for inclusion in Mathematics and Statistics Department Faculty Publication Series by an authorized administrator of ScholarWorks@UMass Amherst. For more information, please contact scholarworks@library.umass.edu.

arXiv:1012.0489v1 [math.RT] 2 Dec 2010

CELLS IN COXETER GROUPS I

MIKHAIL V. BELOLIPETSKY AND PAUL E. GUNNELLS

1. Introduction

In their seminal paper [KL1] Kazhdan and Lusztig introduced the notion of cells. These are equivalence classes in a Coxeter group W and corresponding Hecke algebra $\mathcal{H}$ that can be defined combinatorially and that have deep connections with representation theory. Since then there has been considerable interest in cells and related topics: we can mention the famous papers by Lusztig on cells in affine Weyl groups, research of J.-Y. Shi on the combinatorics of cells, the work of Bezrukavnikov and Ostrik on geometry of unipotent conjugacy classes of simple complex algebraic groups among the numerous other contributions. See [G] for references. Even a short survey of related results would take us far beyond the scope of this paper.

The purpose of this article is to shed new light on the combinatorial structure of cells in infinite Coxeter groups. Our main focus is the set $\mathcal{D}$ of distinguished involutions in W, which was introduced by Lusztig in [L2]. We conjecture that the set $\mathcal{D}$ of an arbitrary Coxeter group has a simple recursive structure and can be enumerated algorithmically starting from the distinguished involutions of finite Coxeter groups. Moreover, to each element of $\mathcal{D}$ we assign an explicitly defined set of equivalence relations on W that altogether conjecturally determine the partition of W into left (right) cells. We are able to prove these conjectures only in a special case, but even from these partial results we can deduce some interesting corollaries. For example, we show that many non-affine infinite Coxeter groups contain infinitely many one-sided cells. This was known before only for a special class of right-angled Coxeter groups [Bel] and several other hyperbolic examples [Bed].

In a forthcoming article [BG2] we will present the experimental support for the conjectures. In particular, we will show that the conjectures hold for infinite affine groups of small rank (see also [BG1] for the affine groups of rank 3) and for large subsets of hyperbolic triangle groups. Some of the experimental results were first presented in [G]. The results of the current article and [BG2] were announced in [BG1].

The paper is organized as follows. In §2 we introduce our main conjectures and show that if valid, these conjectures indeed determine the partition of W into cells

Belolipetsky partially supported by EPSRC grant EP/F022662/1.
Gunnells partially supported by NSF grant DMS 08–01214.

(see Theorem 2.7). Section 3 provides a proof of the conjectures in a special case when reduced expressions of the elements of W satisfy certain restrictions. Here we make extensive use of some unpublished results of Lusztig and Springer. In §4 we present some corollaries and applications of the results.

Acknowledgements. We would like to thank W. Casselman, J. Humphreys, N. Libedinsky and D. Rumynin for helpful discussions.

2. Conjectures

Let (W, S) be a Coxeter system. We will usually denote general elements of W by v, w, x, y, z, and simple reflections from S by s, t. For $x, y \in W$, by $z = x.y$ we mean that $z = xy$ and $l(z) = l(x) + l(y)$, where $l\colon W \to \mathbb{Z}$ is the length function in (W, S).

Throughout the paper we use the terminology of [KL1, L1, L2]. Thus, let $\mathcal{H}$ denote the Hecke algebra of W over the ring $\mathcal{A} = \mathbb{Z}[q^{1/2}, q^{-1/2}]$ of Laurent polynomials in $q^{1/2}$. Along with the standard basis $(T_w)_{w\in W}$ of $\mathcal{H}$ we have the basis $(C_w)_{w\in W}$ of [KL1], where $C_w = \sum_{y\le w}(-1)^{l(w)-l(y)}q^{l(w)/2-l(y)}P_{y,w}(q^{-1})T_y$ and

$$P_{y,w} = \mu(y, w)q^{\frac{1}{2}(l(w)-l(y)-1)} + \text{lower degree terms}$$

are the so-called *Kazhdan-Lusztig polynomials* introduced in [KL1]. Considering the multiplication of the basis elements in $\mathcal{H}$ we see that there exist $h_{x,y,z} \in \mathcal{A}$ such that $C_xC_y = \sum_z h_{x,y,z}C_z$. The value of $a(z)$ is defined to be the smallest integer such that $q^{-\frac{a(z)}{2}}h_{x,y,z} \in \mathbb{Z}[q^{-\frac{1}{2}}]$ for all $x, y \in W$, or to be infinity if such an integer does not exist. If the function a on W takes only finite values (which is conjecturally true for any group W), then for every $x, y, z \in W$ we can write

$$h_{x,y,z} = \gamma_{x,y,z}q^{\frac{a(z)}{2}} + \delta_{x,y,z}q^{\frac{a(z)-1}{2}} + \text{lower degree terms.}$$

This formula defines the constants $\gamma_{x,y,z}$ and $\delta_{x,y,z}$ that we will need later.

Using the polynomials $P_{y,w}$ one can define preorders $\le_L$, $\le_R$, $\le_{LR}$ and the associated equivalence relations $\sim_L$, $\sim_R$, $\sim_{LR}$ on W [KL1]. The equivalence classes for $\sim_L$ (respectively $\sim_R$, $\sim_{LR}$) are called *left cells* (resp. *right cells*, *two-sided cells*) in W. Every result about left cells translates to right cells and vice versa by the duality, so in our considerations we will usually mention only one of the two.

Let $\mathcal{D}_i = \{z \in W \mid l(z) - a(z) - 2\delta(z) = i\}$, where $l(z)$ is the length of z in (W, S), $\delta(z)$ is the degree of the polynomial $P_{e,z}$, so $P_{e,z} = \pi(z)q^{\delta(z)} + \text{lower degree terms}$, and the function $a(z)$ is defined as above. The set $\mathcal{D} = \mathcal{D}_0$ is the set of *distinguished involutions* of W, which was introduced in [L2, §1.3].

Our goal is to detect an inductive structure inside $\mathcal{D}$ and to describe an explicit relationship between the elements of $\mathcal{D}$ and equivalence relations on W that determines its partition into cells. To this end let us formulate two conjectures.

We first introduce some more notations. Let $w \in W$. Denote by $\mathcal{Z}(w)$ the set of all $v \in W$ such that $w = x.v.y$ for some $x, y \in W$ and $v \in W_I$ for some $I \subset S$ with W_I finite. We call $v \in \mathcal{Z}(w)$ *maximal in* w and write $v \in \mathcal{M}(w)$, if it is not a proper subword of any other $v' \in \mathcal{Z}(w)$ such that $w = x'.v'.y'$ with $x' \leq x$ and $y' \leq y$. Let $\mathcal{Z} = \mathcal{Z}(W)$ be the union of $\mathcal{Z}(w)$ over all $w \in W$, $\mathcal{D}_f := \mathcal{D} \cap \mathcal{Z}$ be the set of distinguished involutions of the finite standard parabolic subgroups of W and $\mathcal{D}_f^\bullet = \mathcal{D}_f \smallsetminus (S \cup \{1\})$. We will call $w = x.v.y$ *rigid at* v if (i) $v \in \mathcal{D}_f$, (ii) v is maximal in w, and (iii) for every reduced expression $w = x'.v'.y'$ with $a(v') \geq a(v)$, we have $l(x) = l(x')$ and $l(y) = l(y')$. This notion of *combinatorial rigidity* for the elements of W is essential for our conjectures and results. We refer to [BG1, §4] for some comments about its meaning.

Conjecture 2.1. *("distinguished involutions") Let $v = x.v_1.x^{-1} \in \mathcal{D}$ with $v_1 \in \mathcal{D}_f^\bullet$ and $a(v) = a(v_1)$, and let $v' = s.v.s$ with $s \in S$. Then if sxv_1 is rigid at v_1, we have $v' \in \mathcal{D}$.*

Conjecture 2.2. *("basic equivalences") Let $w = y.v_0$ with $v_0 \in \mathcal{M}(w)$.*

(a) *Let $u = x.v_1.x^{-1} \in \mathcal{D}$ satisfies $a(u) \leq a(v_0)$ and $w' = wu$ is reduced and has $a(w') = a(w)$. Then there exists v_{01} such that $v_0 = v_0'.v_{01}$, $v_{01}'xv_1$ is rigid at v_1 for every v_{01}' such that $v_0 = v_0''.v_{01}'$ and $l(v_{01}') = l(v_{01})$, the right descent set $\mathcal{R}(w'v_{01}^{-1}) \subsetneq \mathcal{R}(w)$, and $\mu(w, w'v_{01}^{-1}) \neq 0$, which implies $w \sim_R w'v_{01}^{-1} \sim_R w'$.*

(b) *Let $w'' = w.v_1$ with $v_1 \in \mathcal{D}_f$ not maximal in w'' and $a(w'') = a(v_0)$. Then we can write $w = y.v_{01}.v_{02}.v_{03}$ so that $v_{03}.v_1 \in \mathcal{M}(w'')$, $\mathcal{R}(w''v_{02}^{-1}) \neq \mathcal{R}(w)$, and $\mu(w, w''v_{02}^{-1}) \neq 0$. So again $w \sim_R w''v_{02}^{-1} \sim_R w''$.*

In practice it is usually easy to find the required 'endings' v_{01} and v_{02} as in the conjecture. This can be seen, in particular, in the statement of Theorem 3.2 and in results from [BG2]. At the same time some examples of [BG2] show that there are cases, such as for instance affine $\tilde{\mathrm{F}}_4$, which require a special attention.

In order to be able to apply these conjectures we will need to recall some other conjectures from the theory. The first is a variant of a conjecture of Lusztig about the function a (cf. [L3, §13.12]):

Conjecture 2.3. *("the function a") For every $w \in W$, $a(w) = a'(w)$ where $a'(w) = \max_{v \in \mathcal{M}(w)} a(v)$.*

One of the immediate corollaries of this conjecture is that there exists a constant $N \geq 0$, which depends only on (W, S), such that for every $w \in W$, $a(w) \leq N$. The groups whose a-function satisfies this property are called *bounded*. For affine and some hyperbolic Coxeter groups the boundedness can be verified directly [L1, Bel].

Remark 2.4. There is a small gap in the proof of Theorem 4.2 in [Bel] which, however, is easy to fix: One has to replace the corresponding part of line 3 on page

332 there by "in which $i_{j+1} = i_j + 1$, for $j = 1, \dots, p-1$, and $s_{i_j} \in \mathcal{L}(s_{i_1-1} \dots s_1 y)$, for $j = 1, \dots, p$" and everywhere after in this paragraph replace y by $y' = s_{i_1-1} \dots s_1 y$. We would like to thank Nanhua Xi for pointing out this issue.

Finally, we recall the well known positivity conjecture (see e.g. [L1, §3]).

Conjecture 2.5. *("positivity") For all $x, y, z \in W$, the coefficients of Kazhdan-Lusztig polynomials $P_{x,y}(q)$ and polynomials $h_{x,y,z}(q^{1/2}, q^{-1/2})$ are positive integers.*

Positivity of the coefficients of Kazhdan-Lusztig polynomials is well known for finite and affine Weyl groups where it is proved using the relation between Kazhdan-Lusztig polynomials and singularities of Schubert varieties [KL2, L1]. Later on this result was extended to all crystallographic Coxeter groups using similar geometric ideas (cf. [Ku, Theorem 12.2.9]). This, however, can hardly be generalized to non-crystallographic cases. Another approach to positivity based on categorification of the Hecke algebra was suggested by Soergel in [So]. We refer to [Li] for some recent results in this direction.

Lemma 2.6. *Assume Conjectures 2.1, 2.2, and 2.3 hold. Let $w = x_1.v_1.x_2.v_2$ with $v_i \in \mathcal{M}(w)$, $a(v_1) \geq a(v_2) > a(x_2)$ and $a(x_2 v_2) = a(v_2)$. Then $w \sim_R x_1 v_1$.*

Proof. We are going to use induction by $a(v_2)$. First note that if $a(v_2) = 1$, then $v_2 = s \in S$, $x_2 = e$ and $\mathcal{R}(w) \not\supset \mathcal{R}(x_1 v_1)$ (by the maximality of v_1 in w), so $w = x_1 v_1 s \sim_R x_1 v_1$ follows from the definition of $\sim_R$.

If $a(v_2) > 1$ and $v_2 \notin \mathcal{D}_f$, we can replace it by some v_2' so that $w \sim_R x_1.v_1.x_2.v_2'$ and $v_2' \in \mathcal{D}_f$. In order to do so first find a distinguished involution v_2' of the finite parabolic subgroup W_I containing v_2 which is right equivalent to v_2 in W_I (its existence and uniqueness is proved in [L2]). Then using the maximality of v_2 in w and the known properties of the relation $\sim_R$ we can lift $v_2 \sim_R v_2'$ in W_I to $x_1.v_1.x_2.v_2 \sim_R x_1.v_1.x_2.v_2'$ in W.

Now, assume that $a(v_2) > 1$, $v_2 \in \mathcal{D}_f$ and $x_2 v_2$ is rigid at v_2. Then $u = x_2 v_2 x_2^{-1}$ (and also $x_1 v_1 x_2 v_2 x_2^{-1}$) is reduced, indeed, if it is not then there is $s \in \mathcal{R}(x_2)$ such that $s v_2 s = v_2$, which contradicts the rigidity of $x_2 v_2$ at v_2. Rigidity of $x_2 v_2$ at v_2 enables us to apply Conjecture 2.1 $l(x_2)$ times starting from v_2 to show that $u \in \mathcal{D}$. By Conjecture 2.3, it follows that $a(u) = a(v_2)$ and $a(w x_2^{-1}) = a(x_1 v_1)$. We are now in a position to use Conjecture 2.2(a), which gives $x_1 v_1 \sim_R x_1 v_1 x_2 v_2 x_2^{-1}$. As $a(x_2) < a(v_2)$, we can apply the inductions hypothesis together with Conjecture 2.2(a) to show that in turn $x_1 v_1 x_2 v_2 x_2^{-1} \sim_R x_1 v_1 x_2 v_2$, which finished the proof for this case.

It remains to consider the case when $x_2 v_2$ is not rigid at v_2. It implies that we can write $x_2 v_2 = x.v.y$ with $v \in \mathcal{M}(xvy)$, where $a(v) = a(v_2)$ and $l(y) > 0$. We can choose such an expression with $l(y)$ maximal and $a(y) < a(v)$. Then the induction hypothesis applies to show that $x_2 v_2 \sim_R xv$ and $w \sim_R x_1 v_1 x v$. Thus we can replace

x_2v_2 with xv and repeat the procedure if rigidity fails at $v_2 = v$. As each iteration reduces lengths of the elements, we eventually reach the case when x_2v_2 is rigid at v_2, which allows us to use the previous argument. □

Theorem 2.7. *If Conjectures 2.1, 2.2, 2.3 and 2.5 are satisfied for an infinite Coxeter group W, then the following are true:*

(1) *The set $\mathcal{D}$ of distinguished involutions consists of the union of $v \in \mathcal{D}_f$ and the elements of W obtained from them using Conjecture 2.1.*
(2) *The relations described in Conjecture 2.2 determine the partition of W into right cells, i.e. $x \sim_R y$ in W if and only if there exists a sequence $x = x_0$, x_1, $\dots$, $x_n = y$ in W such that $\{x_{i-1}, x_i\} = \{v, v'\}$ as in 2.2 for every $i = 1, \dots, n$.*
(3) *The relations described in Conjecture 2.2 together with its $\sim_L$-analogue determine the partition of W into two-sided cells.*

Proof. By [L2], Conjectures 2.3 and 2.5 imply that each right cell of W contains a uniquely defined distinguished involution. To prove assertions (1) and (2) we thus need to show that for any $w \in W$ we can find a sequence as in (2) such that $x_0 = w$ and $x_n = d$ where $d = d(w)$ satisfies the conditions of Conjecture 2.1.

So let $w \in W$ be an arbitrary element. If $\#\mathcal{R}(w) = 1$ then $w \sim_R w' = ws$ such that $l(ws) = l(w) - 1$. Repeating this procedure we come to $w \sim_R w_1$ with either $\#\mathcal{R}(w_1) > 1$ or $w_1 = s \in S$. In the second case $w_1 \in \mathcal{D}$ and we are done. Thus we can assume that $\#\mathcal{R}(w_1) > 1$ and $w_1 = x.v$ with a maximal v. If $a(x) \geq a(v)$, then using Conjecture 2.3 we can write $w_1 = x_1.v_1.x_2.v_2$ as in Lemma 2.6, which implies $w_1 \sim_R x_1.v_1$. Repeating the procedure if necessary, we come to $w_1 \sim_R w_2 = y.u$ with $\#\mathcal{R}(w_2) > 1$, $u \in \mathcal{M}(w_2)$ and $a(y) < a(u) = a(w_2)$. As in the proof of Lemma 2.6 we can assume here that $u \in \mathcal{D}_f$. If w_2 is rigid at u, we can apply Conjecture 2.1 $l(y)$ times to show that $y.u.y^{-1} \in \mathcal{D}$. Otherwise we can shift u to the left using Conjecture 2.2(b), and repeating this procedure if necessary we eventually reduce w_2 to a right-equivalent element of the same form which is rigid at u.

Thus in all the cases we show that $w \sim_R y.u$ such that $d = y.u.y^{-1} \in \mathcal{D}$. It remains to prove that $d \sim_R y.u$ which can be done applying the same procedure as in the previous paragraph to d instead of w and using the fact that $y.u$ is rigid at u and satisfies $a(y) < a(u)$ by the construction. This completes the proof of (1) and (2).

To show (3) we just need to recall the definition of the two-sided equivalence relation and then refer to the previous argument together with its analogue for the left cells. □

Let us consider two examples which demonstrate the necessity of the main conditions imposed in the conjectures.

Example 2.8. Let W be an affine group of type $\tilde{\mathrm{A}}_4$ with extended Dynkin diagram labelled as in Figure 1.

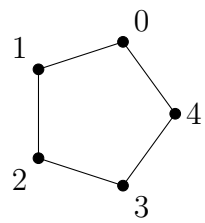

FIGURE 1. Dynkin diagram for $\tilde{\mathrm{A}}_4$.

The element $v_1 = s_4 s_0 s_4 s_2$ is the longest element of the finite standard parabolic subgroup W_I with $I = \{s_0, s_2, s_4\}$, so $v_1 \in \mathcal{D}_f$. We can check by direct computation that $s_1\, s_4 s_0 s_4 s_2\, s_1$, $s_3 s_1\, s_4 s_0 s_4 s_2\, s_1 s_3$, and $s_2 s_3 s_1\, s_4 s_0 s_4 s_2\, s_1 s_3 s_2$ are in $\mathcal{D}$, which agrees with Conjecture 2.1. However, the same computation shows that $s_0 s_2 s_3 s_1\, s_4 s_0 s_4 s_2\, s_1 s_3 s_2 s_0$ is not in $\mathcal{D}$! The only possible reason for this in view of the conjecture is that $s_0 s_2 s_3 s_1\, s_4 s_0 s_4 s_2$ might not be rigid at v_1, and, indeed, we can check that

$$s_0 s_2 s_3 s_1\, s_4 s_0 s_4 s_2 = s_2 s_0 s_3 s_1\, s_0 s_4 s_0 s_2 = s_2\, s_3 s_0 s_1 s_0\, s_4 s_0 s_2,$$

where $s_3 s_0 s_1 s_0 \in \mathcal{D}_f$ and $a(s_3 s_0 s_1 s_0) = a(v_1)$.

Example 2.9. It is easy to check that if W_I is a finite standard parabolic subgroup of W then its longest element w_0 is always in $\mathcal{D}_f$. The statements of our conjectures would simplify quite a bit if all involutions in $\mathcal{D}_f$ had this form. However, this is not always the case. For example, let W be a Weyl group of type D_4 with Dynkin diagram as in Figure 2.

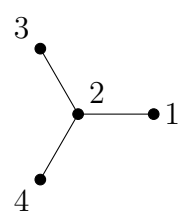

FIGURE 2. Dynkin diagram for D_4.

Let $v = s_2 s_4 s_1 s_3 s_2 s_1 s_3 s_2 s_4 s_2 s_1$. Then $a(v) = 7$ (see [J] or [BG2]) and a direct computation shows that $\delta(v) = 2$, so we have $l(v) - a(v) - 2\delta(v) = 11 - 7 - 4 = 0$, and thus $v \in \mathcal{D}$. At the same time, it is clear that v is not the longest element in W or in any of its standard parabolic subgroups.

3. Results

Throughout this section we assume the positivity conjecture and boundedness of the function a on W. These assumptions can be slightly relaxed in some cases but can not be completely removed. The positivity and boundness conjectures are widely believed to be true for any Coxeter group and are proved in a number important special cases, we refer to the previous section for a related discussion.

Theorem 3.1. *Let $v = x.v_1.x^{-1} \in \mathcal{D}$ with $v_1 \in \mathcal{D}_f^\bullet$, $a(v) = a(vs)$ and $\mathcal{L}(vs) \smallsetminus \mathcal{R}(vs) \neq \emptyset$; and let $v' = s.v.s$. Then if v' is rigid at v_1, we have $v' \in \mathcal{D}$.*

Proof. The argument naturally splits into two steps:

(a) show that $\delta(vs)$, which is $\deg(P_{e,vs})$, equals $\delta(v)$;
(b) show that $\delta(svs) = \delta(vs) + 1$.

The first step is a corollary of some results from the correspondence of Lusztig and Springer [LS]. Let us briefly recall the argument (see also [Xi, §1.4]).

We have $\mu(v, vs) = 1$ (by [KL1, 2.3.f]) and $a(v) = a(vs)$ (by the assumption), so by [L2, 1.9], $v \sim_R vs$. As $\mathcal{L}(vs) \neq \mathcal{R}(vs)$, clearly, $vs \not\sim_R sv = (vs)^{-1}$. By Springer's formula $\mu(v, vs)$, which equals $\mu(vs, v)$, can be written

$$\mu(v, vs) = \sum_{v' \in \mathcal{D}} \delta_{sv,v,v'} + \sum_{f \in \mathcal{D}_1} \gamma_{sv,v,f} \pi(f).$$

By [L2, Thm. 1.8], we have $\gamma_{sv,v,f} = \gamma_{f,sv,v}$. If $\gamma_{f,sv,v} \neq 0$, then [L2, Prop. 1.4(a)] implies that $f = sv$ (and $\gamma_{f,sv,v} = 1$). Thus in this case sv and also vs are in $\mathcal{D}_1$.

Assume now that vs is not in $\mathcal{D}_1$. Then by the previous argument all $\gamma_{sv,v,f} = 0$ and hence $\mu(v, vs) = \sum_{v' \in \mathcal{D}} \delta_{x^{-1},v,v'}$. But if $\delta_{x^{-1},v,v'} \neq 0$, then by a result of Springer we have $sv \sim_R v'$ and $v \sim_L v'$. Since each left cell contains only one distinguished involution ([L2, Thm. 1.10]) we must have $v = v'$. Hence we get $sv \sim_R v$. But $\mathcal{L}(sv) \neq \mathcal{L}(v)$ and we come to a contradiction with [KL1, Prop. 2.4(ii)].

Therefore, we must have $vs \in \mathcal{D}_1$ which means

$$\delta(vs) = \frac{1}{2}(l(vs) - a(vs) - 1) = \frac{1}{2}(l(v) - a(v) - 1) = \delta(v).$$

We proceed with the second step.

By [KL1, 2.2.c], we have

$$P_{e,svs} = qP_{s,vs} + P_{e,vs} - \sum_{\substack{z \prec vs \\ sz < z}} \mu(z, vs) q_z^{-1/2} q_{vs}^{1/2} q^{1/2} P_{e,z} = qP_{s,vs} + P_{e,vs} - \Sigma;$$

and by [KL1, 2.3.g], $P_{s,vs} = P_{e,vs}$, thus

$$P_{e,svs} = (q+1)P_{e,vs} - \Sigma.$$

As $z \prec vs$ and $s \in \mathcal{L}(z) \smallsetminus \mathcal{L}(vs)$, we have $z \leq_L vs$ for any z in Σ. By [L2, 1.5(c)], this implies $a(z) \geq a(vs) = a(v)$. Now, by [L2, 1.3(a)], we obtain

$$\delta(z) \leq \frac{1}{2}(l(z) - a(z)) \leq \frac{1}{2}(l(z) - a(v)).$$

This inequality enables us to bound the degree d_1 of the summands in Σ:

$$d_1 \leq \frac{1}{2}(-l(z) + l(vs) + 1 + l(z) - a(z)) \leq \frac{1}{2}(2\delta(v) + a(v) + 1 + 1 - a(v)) = \delta(v) + 1,$$

thus by Step (a), $d_1 \leq \delta(vs) + 1$. Note that the leading term of $(q+1)P_{e,vs}$ may cancel only if $d_1 = \delta(vs) + 1$, in which case we obtain

$$a(z) = a(v) \quad \text{and} \quad \delta(z) = \frac{1}{2}(l(z) - a(z)).$$

Hence any such z has to be a distinguished involution. Moreover, as $a(z) = a(vs)$ and $z \leq_L vs$, by [L2, 1.9(b)] we have $z \sim_L vs$. Hence we have $\mathcal{L}(z) = \mathcal{R}(z) = \mathcal{R}(vs)$. Now $z \prec vs$ and $\mathcal{L}(vs) \smallsetminus \mathcal{R}(vs) \neq \emptyset$ together with [KL1, 2.3.e] imply $vs = tz$ for some $t \in S$. Hence we have $vs = x.v_1.x^{-1}.s = tz = tz^{-1} = x_1.v_1.x_2$ with $l(x_1) = l(x) + 2$ and $l(x_2) = l(x) - 1$. This contradicts the rigidity of vs (and also svs) at v_1.

We showed that such z does not exist. Therefore the degree of $P_{e,svs}$ is equal to the degree of $(q+1)P_{e,vs}$, and recalling again the result of Step (a), we have

$$\delta(svs) = \delta(vs) + 1 = \delta(v) + 1.$$

On the other hand, $l(svs) = l(v) + 2$ and $a(svs) \geq a(vs) = a(v)$, so $l(svs) - a(svs) - 2\delta(svs) \leq 0$ but for any $w \in W$, $l(w) - a(w) - 2\delta(w) \geq 0$ (see [L2, 1.3(a)]). We thus showed that $a(svs) = a(v)$, $l(svs) - a(svs) - 2\delta(svs) = 0$ and $svs \in \mathcal{D}$. □

We see that with the extra condition $\mathcal{L}(vs) \smallsetminus \mathcal{R}(vs) \neq \emptyset$ and the assumption that v' is rigid at v_1, Conjecture 2.1 is true. The extra condition is used only in Step (b) of the proof. The rigidity assumption here obviously implies the rigidity of sxv_1 at v_1 (as in Conjecture 2.1). The converse is probably also true but we do not know how to show it in a general setting.

The proof of Theorem 3.1 together with the following argument allows us to establish also a special case of Conjecture 2.2:

Theorem 3.2. *Let $w = x.v_0 = t_n \dots t_1.s_l \dots s_1$ with $t_i, s_i \in S$, $v_0 = s_l \dots s_1 \in \mathcal{D}_f$ is the longest element of a standard finite parabolic subgroup of W which is maximal in w and $a(w) = a(v_0)$; $u = y.u_0.y^{-1} \in \mathcal{D}$ with $u_0 \in \mathcal{D}_f$ such that $a(u) = a(u_0) = l$; and $w' = w.u.v_{01}$ with $v_{01} = s_1 \dots s_{l-1}$ has $a(w') = a(w)$ and $\mathcal{R}(w') \subsetneq \mathcal{R}(w)$.*

Assume that

(1) *For any $v_j = t_j \dots t_1 v_0 t_1 \dots t_j$, $j = 0, \dots, n-1$ and $t = t_{j+1}$ or $t = t_{j-1}$ if $t_{j-1} \notin \mathcal{R}(v_j)$, we have $a(v_j t) = a(v_j)$, $\mathcal{L}(v_j t) \smallsetminus \mathcal{R}(v_j t) \neq \emptyset$ and $tv_j t$ is rigid at v_0.*

(2) *For any* $u_j = s_{j-1}\dots s_1 u s_1 \dots s_{j-1}$, $j = 1, \dots, l-1$ *with* $u_1 = u$, *we have* $a(u_j s_j) = a(u_j)$, $\mathcal{L}(u_j s_j) \smallsetminus \mathcal{R}(u_j s_j) \neq \emptyset$ *and* $s_j u_j s_j$ *is rigid at* u_0*;*

Then $\mu(w, w') \neq 0$ *and* $w \sim_R w'$.

We note that conditions (1) and (2) are imposed in order to apply Theorem 3.1. If we would be able to prove Conjecture 2.1 in its full generality, the assumptions of the theorem would immediately simplify.

Proof. Let us first consider the case when $x = e$, so $w = v_0$. As $v_0 = s_l \dots s_1$ is the longest element of a standard parabolic subgroup, for every i, $s_i \in \mathcal{L}(v_0) = \mathcal{R}(v_0)$. Thus by [KL1, 2.3.g],

$$P_{v_0,\ v_0 u v_{01}} = P_{e,\ v_0 u v_{01}}.$$

Theorem 3.1 applied to each u_j, $j = 1, \dots, l-1$ as in (2) implies that $u_{l-1} = v_{01}^{-1} u v_{01} \in \mathcal{D}$, moreover, the argument of Step (a) of the proof shows that $\deg(P_{e,\ v_0 u v_{01}}) = \deg(P_{e,\ v_{01}^{-1} u v_{01}})$. So

$$\deg(P_{v_0,\ v_0 u v_{01}}) = \frac{1}{2}(2l(v_{01}) + l(u) - a(u)) = \frac{1}{2}(a(u) + l(u) - 2) = \frac{1}{2}(l(u) + l(v_{01}) - 1).$$

Hence $\mu(w, w') \neq 0$ (in fact, we have $\mu = 1$). Now, as $\mathcal{R}(w') \subsetneq \mathcal{R}(w)$, we have $w \leq_R w'$ by the definition of the preorder $\leq_R$ (see [KL1]). The opposite inequality $w' \leq_R w$ is easy to show using induction by $l(w') - l(w)$ and relations of the form $w_i s \leq_R w_i$ which follow from the definition. Thus we obtain that $w \sim_R w'$.

Now let $w = x.v_0$ with $x = t_n \dots t_1$ nontrivial. In order to prove the theorem for this case we use induction on the length of x. The base case $x = e$ has already been considered. Assume that the theorem is proven for all $w_i = t_i \dots t_1.v_0$, $i = 0, \dots, n-1$ and corresponding w_i'. We need to show that then it follows for $w_n = s.xv_0$ and $w_n' = s.xv_0 u v_{01}$ with $s = t_n$, in particular, the assumption (1) is satisfied for s.

By [KL1, 2.2.c], we have

$$\begin{aligned} P_{sxv_0, sxv_0 u v_{01}} &= P_{w_{n-1}, w_{n-1}'} + q P_{w_n, w_{n-1}'} - \sum_{\substack{w_n \leq z \prec w_{n-1}' \\ sz < z}} \mu(z, w_{n-1}') q_z^{-1/2} q_{w_{n-1}'}^{1/2} q^{1/2} P_{w_n, z} \\ &= P_{w_{n-1}, w_{n-1}'} + q P_{w_n, w_{n-1}'} - \Sigma. \end{aligned}$$

By the induction hypothesis, $\mu(w_{n-1}, w_{n-1}') \neq 0$. It follows that a summand of Σ can have the same degree as $P_{w_{n-1}, w_{n-1}'}$ only if it corresponds to z with $w_n \prec z$.

Thus we have to consider $z \in W$ such that

$$sxv_0 \prec z \prec xv_0 u v_{01}.$$

By assumption (1), there exists $t \in S$ such that $t \in \mathcal{L}(xv_0 x^{-1} s) \smallsetminus \mathcal{R}(xv_0 x^{-1} s) = \mathcal{L}(xv_0) \smallsetminus \mathcal{L}(sxv_0)$. It then belongs to $\mathcal{L}(xv_0 u v_{01}) \smallsetminus \mathcal{L}(sxv_0)$. We consider two possible cases.

(1) First assume $t \notin \mathcal{L}(z)$. Then by [KL1, 2.3.e] we have $tz = w'_{n-1} = tx'v_0uv_{01}$ and thus $z = w'_{n-2}$. We have $z = w'_{n-2} \sim_R w_{n-2}$ by the induction hypothesis (assuming $n \geq 2$), which is then easily seen to be right equivalent to $x'v_0x'^{-1}$ using the definition of $\sim_R$. Theorem 3.1 then implies $x'v_0x'^{-1} \in \mathcal{D}$. On the other hand, by a similar argument the relations $sxv_0 \prec z$, $\mathcal{R}(z) \subsetneq \mathcal{R}(sxv_0)$ and $a(z) = a(sxv_0)$ imply $z \sim_R sxv_0 \sim_R sxv_0x^{-1}s \in \mathcal{D}$. We thus see that both $x'v_0x'^{-1} \sim_R sxv_0x^{-1}s$ are in $\mathcal{D}$ and are not equal to each other, which is a contradiction.

The case $n = 1$ has to be considered separately. We get $tz = v_0uv_{01}$, which implies $\mathcal{L}(z) \subset \{s_1, \ldots, s_l\} = \mathcal{L}(v_0)$ but $s \notin \mathcal{L}(v_0)$ and thus we arrive at a contradiction again.

(2) Now assume $t \in \mathcal{L}(z)$. As $t \notin \mathcal{L}(sxv_0)$, by [KL1, 2.3.e] we have $z = t.sxv_0$. A similar argument to that given above shows that $z \sim_R xv_0uv_{01} \sim_R xv_0$, which gives rise to two different but equivalent distinguished involutions — a contradiction.

It follows that such z does not exist. Hence $\deg(P_{w_n,w'_n}) = \deg(P_{w_{n-1},w'_{n-1}})$, $\mu(w_n, w'_n) \neq 0$ and $w_n \sim_R w'_n$. This finishes the proof of the theorem. □

4. Applications

4.1. **Groups of type** (n)**.** Let W be a Coxeter group whose Coxeter matrix non-diagonal entries $m_{i,j} = m(s_i, s_j)$, $i \neq j$ are either n or infinity. Thus the only non-trivial relations in W are those of the form $(s_is_j)^n = e$ with $s_i, s_j \in S$. We say that such Coxeter groups are *of type* (n).

Coxeter groups of type (n) include all dihedral groups, the affine Weyl groups $\tilde{\mathrm{A}}_1$ and $\tilde{\mathrm{A}}_2$, and infinitely many hyperbolic Coxeter groups. We claim that if some elements of an infinite group of type (n) satisfy the conditions of Conjectures 2.1 and 2.2, then they also satisfy the conditions of Theorems 3.1 and 3.2. This follows easily from the combinatorics of the relations in W and the fact that the only finite standard parabolic subgroups of W of rank > 1 have the form W_I with $I = \{s_i, s_j\}$, and all such subgroups are isomorphic to each other. Conjecture 2.3 for groups of type (n) can be verified using Tits' elementary M-operations similar to the proof of Theorem 4.2 in [Bel]. Although the positivity conjecture is not known in general for such groups there are partial results in its direction (see §2 for a short discussion).

Let us consider some concrete examples.

Example 4.1. Let $W = \langle s_1, s_2, s_3 \mid s_1^2,\ s_2^2,\ s_3^2,\ (s_1s_2)^3,\ (s_2s_3)^3,\ (s_3s_1)^3\rangle$ be the affine Weyl group $\tilde{\mathrm{A}}_2$, which is of type (3). Then $\mathcal{D}_f = \{s_1,\ s_2,\ s_3,\ s_1s_2s_1,\ s_2s_3s_2,\ s_3s_1s_3\}$ and $\mathcal{D}_f^\bullet = \{s_1s_2s_1,\ s_2s_3s_2,\ s_3s_1s_3\}$. Let $v_1 = s_1s_2s_1 \in \mathcal{D}_f^\bullet$. By Conjecture 2.1 (or here Theorem 3.1) it gives rise to the distinguished involution $s_3\, s_1s_2s_1\, s_3$ in W. If we

try to continue the process, we see that both reduced elements $s_2s_3\,s_1s_2s_1\,s_3s_2$ and $s_1s_3\,s_1s_2s_1\,s_3s_1$ are not rigid at v_1. Thus the inductive construction terminates here. The same can be done for the other two elements in $\mathcal{D}_f^\bullet$. We obtain that in total there are $6+3$ distinguished involutions in W. Thus there are 9 non-trivial left cells in W which agrees with the result of [L1, §11]. The partition of W into cells can be recovered using Conjecture 2.2 which in this case is equivalent to Theorem 3.2. It is a simple exercise to check that the resulting partition coincides with the one described by Lusztig (loc. cit.).

Example 4.2. Let $W = P_n = \langle s_1, s_2, \dots, s_n \mid s_i^2 = 1,\ (s_is_{i+1})^2 = 1,\ i = 1, \dots, n\rangle$ (for the sake of convenience we assume $s_{n+1} = s_1$). Assume $n \geq 5$. Then W can be realized as a group generated by reflections in the hyperbolic plane about the sides of a right-angled n-gon. This is a special class of right-angled Coxeter groups studied in detail in [Bel] and is a group of type (2). We have $\mathcal{D}_f = \{s_i, s_is_{i+1} \mid i = 1, \dots, n\}$ and $\mathcal{D}_f^\bullet = \{s_is_{i+1} \mid i = 1, \dots, n\}$. By Theorem 3.1, the elements of the form $t_1 \dots t_ks_is_{i+1}t_k \dots t_1$ with $m(t_j, t_{j+1}) = \infty$ $(j = 1, \dots, k-1)$ and $m(t_k, s_i) = m(t_k, s_{i+1}) = \infty$ are in $\mathcal{D}$. These are, in fact, the only elements of W that satisfy the conditions of the theorem, and thus from the discussion above we conclude that the elements of this form give us the whole set $\mathcal{D}$ of the distinguished involutions. By Theorem 3.2 we obtain that together with the obvious right equivalences W also admits the equivalences of the form $xs_is_{i+1} \sim_R xs_is_{i+1}t_1 \dots t_ks_js_{j+1}t_k \dots t_1s_i$. Theorem 2.7 now shows that these equivalences are sufficient for describing the right cells of W.

The resulting description of the right cells and distinguished involutions in W agrees with the one which is given in [Bel], so we obtain an alternative proof for the main results there.

4.2. **Groups with infinitely many one-sided cells.** Consider Coxeter groups W with the following property:

(*) There exist $s, t_1, t_2 \in S$ such that $m(s, t_1) = m(s, t_2) = \infty$ and $m(t_1, t_2)$ is finite.

We claim that, assuming the positivity and boundness conjectures, (*) implies that W has infinitely many one-sided cells. Indeed, let v_0 be the longest element in the standard parabolic subgroup generated by t_1 and t_2. Then all elements with reduced expressions of the form $t_is \dots t_jsv_0st_j \dots st_i$ $(i, j = 1$ or $2)$ satisfy the conditions of Theorem 3.1, and hence are in $\mathcal{D}$. As under our assumptions each one-sided cell of W contains a uniquely defined distinguished involution ([L2]), the claim follows.

Let us put this result in a general perspective. The coexistence of infinite and finite exponents in the Coxeter matrix of W implies that W contains a non-abelian free subgroup (in our case it is the subgroup generated by, for example, t_1t_2s and t_1st_2). A group is called *large* if a subgroup of finite index in it has a non-abelian free quotient. In [MV] it was shown that any non-affine infinite indecomposable Coxeter

group of finite rank is large. Any large group is SQ-universal, and hence contains a non-abelian free subgroup. Thus an indecomposable infinite Coxeter group is either affine or contains a non-abelian free subgroup. By the work of Lusztig [L2], any affine Coxeter group has only finitely many cells. Our result here gives a support to the *conjecture* that all other indecomposable infinite Coxeter groups except those whose all exponents are infinite should have infinitely many one-sided cells.

Let us point out that there are large groups for which one cannot produce infinitely many distinguished involution using only Theorem 3.1. For a simple example consider the Hurwitz triangle group $(2,3,7)$ with presentation $\Gamma = \langle s_1, s_2, s_3 \mid (s_1s_3)^2 = (s_1s_2)^3 = (s_2s_3)^7 \rangle$. It is easy to check that applying Theorem 3.1 while starting from the distinguished involutions in the standard parabolic subgroups of Γ we can only get finitely many elements of $\mathcal{D}$, while Conjecture 2.1, if true, implies that it should be possible to continue the process to infinity. In [BG2] we present an experimental conformation of the infiniteness of the number of one-sided cells as well as our main conjectures for this and some other groups.

References

[Bed] R. Bédard, Left V-cells for hyperbolic Coxeter groups, *Comm. Algebra*, **17** (1989), 2971–2997.

[Bel] M. Belolipetsky, Cells and representations of right-angled Coxeter groups, *Selecta Math., N. S.* **10** (2004), 325–339.

[BG1] M. Belolipetsky, P. E. Gunnells, Kazhdan-Lusztig cells in infinite Coxeter groups, Proceedings of the Workshop on 'Algebra, Combinatorics and Dynamics', *to appear*.

[BG2] M. Belolipetsky, P. E. Gunnells, Cells in Coxeter groups, II, in preparation.

[G] P. E. Gunnells, Cells in Coxeter groups, *Notices of the AMS* **53** (2006), 528–535.

[J] Du Jie, Cells in the Affine Weyl groups of type $\tilde{\mathrm{D}}_4$, *J. Algebra*, **128** (1990), 384–404.

[KL1] D. Kazhdan, G. Lusztig, Representations of Coxeter groups and Hecke algebras, *Invent. Math.* **53** (1979), 165–184.

[KL2] D. Kazhdan, G. Lusztig, Schubert varieties and Poicaré duality, in *Proc. Sympos. Pure Math.*, XXXVI, Amer. Math. Soc., Providence, RI, 1980, pp. 185–203.

[Ku] S. Kumar, *Kac-Moody Groups, their Flag Varieties and Representation Theory*, Progr. Math., **204**, Birkhäuser, Boston, 2002.

[Li] N. Libedinsky, New bases of some Hecke algebras via Soergel bimodules, *preprint* arXiv:0907.0031v1 [math.RT].

[L1] G. Lusztig, Cells in affine Weyl groups, *Algebraic Groups and Related Topics,* Adv. Stud. Pure Math., **40**, Kinokuniya and North-Holland, Amsterdam, 1985, pp. 255–287.

[L2] G. Lusztig, Cells in affine Weyl groups, II, *J. Algebra,* **109** (1987), 536–548.

[L3] G. Luzstig, *Hecke algebras with unequal parameters,* CRM Monographs Ser., **18**, Amer. Math. Soc., 2003.

[LS] G. Lusztig, T. A. Springer, Correspondence, 1987.

[MV] G. A. Margulis, È. B. Vinberg, Some linear groups virtually having a free quotient, *J. Lie Theory,* **10** (2000), 171–180.

[So] W. Soergel, Kazhdan-Lusztig polynomials and indecomposable bimodules over polynomial rings, *Journal of the Inst. of Math. Jussieu*, **6** (2007), 501–525.

[Xi] N. Xi, On the characterization of the set $\mathcal{D}_1$ of the affine Weyl group of type $\tilde{A}_n$, *Representation theory of algebraic groups and quantum groups,* Adv. Stud. Pure Math., **40**, Math. Soc. Japan, Tokyo, 2004, pp. 483–490.

Department of Mathematical Sciences, Durham University, South Rd, Durham DH1 3LE, UK, and Institute of Mathematics, Koptyuga 4, 630090 Novosibirsk, RUSSIA

E-mail address: mikhail.belolipetsky@durham.ac.uk

Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003, USA

E-mail address: gunnells@math.umass.edu