Symmetric symplectic homotopy K3 surfaces

SYMMETRIC SYMPLECTIC HOMOTOPY $K3$ SURFACES

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Abstract. A study on the relation between the smooth structure of a symplectic homotopy $K3$ surface and its symplectic symmetries is initiated. A measurement of exoticness of a symplectic homotopy $K3$ surface is introduced, and the influence of an effective action of a $K3$ group via symplectic symmetries is investigated. It is shown that an effective action by various maximal symplectic $K3$ groups forces the corresponding homotopy $K3$ surface to be minimally exotic with respect to our measure. (However, the standard $K3$ is the only known example of such minimally exotic homotopy $K3$ surfaces.) The possible structure of a finite group of symplectic symmetries of a minimally exotic homotopy $K3$ surface is determined and future research directions are indicated.

1. Introduction

In the recent advances in topology and geometry of smooth 4-manifolds a very important role was played by one particular class of 4-manifolds, namely, the homotopy $K3$ surfaces. These manifolds have been used to test the flexibility of smooth and symplectic structures in comparison with the rigidity of holomorphic structures. To be more precise, let X be a homotopy $K3$ surface, namely, X is a closed, oriented smooth 4-manifold homeomorphic (as an oriented manifold) to the standard $K3$ surface. If such a manifold admits an orientation-compatible symplectic structure, then it is called a symplectic homotopy $K3$ surface. While the knot surgery of Fintushel and Stern (cf. [9]) allows construction of numerous examples of symplectic homotopy $K3$ surfaces, deep work of Taubes [26] gives very strong information about the smooth structures on such manifolds. For example, one can easily show that the set of Seiberg-Witten basic classes of X spans an isotropic sublattice L_X of $H^2(X;\mathbb{Z})$ (with respect to the cup product), so that its rank, denoted by r_X, must range from 0 to 3 (cf. Proposition 4.1). The rank r_X of the lattice L_X of the Seiberg-Witten basic classes gives a rough measurement of the exoticness of the smooth structure of X, with $r_X = 0$ being the minimally exotic and with $r_X = 3$ being the maximally exotic.

There are various known characterizations of the minimally exotic (i.e. $r_X = 0$) symplectic homotopy $K3$ surfaces X, which are all characteristics of the standard $K3$, namely:

- X has a trivial canonical class, i.e., $c_1(K_X) = 0$, cf. [26].
- X has a unique Seiberg-Witten basic class, cf. [21, 26].

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- X has the same Seiberg-Witten invariant of the standard $K3$, cf. [26].
- X is a simply-connected, minimal symplectic 4-manifold with zero Kodaira dimension, cf. [16].

Moreover, the standard $K3$ surface is the only known example of such a 4-manifold, and it has been a challenging problem as whether there is an exotic smooth structure with $r_X = 0$.

In [7] the authors studied the possible effect of a change of a smooth structure on the symmetry group of a closed, oriented 4-manifold. It was shown that for an infinite family of maximally exotic (i.e. $r_X = 3$) $K3$ surfaces, there are very significant constraints on the smooth symmetry groups of the manifolds. The current paper took a rather opposite viewpoint as one of its purposes is to investigate the implications of a (symplectic) group action for the smooth structure of a 4-manifold.

The interaction between smooth structures and symmetry groups of a manifold is one of the basic questions in the theory of differentiable transformation groups. In particular, the following classical theorem of differential geometry gives a characterization of the standard sphere $\mathbb{S}^n$ among all the homotopy n-spheres as having the largest degree of symmetry (cf. [13]).

Theorem (A Characterization of $\mathbb{S}^n$). *Let M^n be a closed, simply connected manifold of dimension n, and let G be a compact Lie group which acts smoothly and effectively on M^n. Then $\dim G \leq n(n+1)/2$, with equality if and only if M^n is diffeomorphic to $\mathbb{S}^n$.*

If X is a homotopy $K3$ surface then it is well known that a compact Lie group acting smoothly on X must be finite (cf. [2]). A finite group G is called a $K3$ group (resp. symplectic $K3$ group) if G can be realized as a subgroup of the automorphism group (resp. symplectic automorphism group) of a $K3$ surface. Finite automorphism groups of $K3$ surfaces were first systematically studied by Nikulin in [23]; in particular, he completely classified finite abelian groups of symplectic automorphisms. Subsequently, Mukai [22] determined all the symplectic $K3$ groups (see also [14, 27]). The following 11 groups are the maximal symplectic $K3$ groups:

$$L_2(7), A_6, S_5, M_{20}, F_{384}, A_{4,4}, T_{192}, H_{192}, N_{72}, M_9, T_{48}.$$

Motivated by the above characterization of the standard $\mathbb{S}^n$ we were led to the following:

Problem *Let X be a homotopy $K3$ surface supporting an effective action of a "large" $K3$ group via symplectic symmetries. What can be said about the smooth structure on X?*

Viewing the above maximal symplectic $K3$ groups as "large", our solution to this problem is contained in the following:

Theorem 1.0. *Let G be one of the following maximal symplectic $K3$ groups:*

$$L_2(7), A_6, M_{20}, A_{4,4}, T_{192}, T_{48},$$

and let X be a symplectic homotopy $K3$ surface. If X admits an effective G-action via symplectic symmetries, then X must be minimally exotic, i.e., $r_X = 0$.

Remarks It is possible to extend Theorem 1.0 to other $K3$ groups, or more generally, to give an upper bound on the exoticness r_X when X admits a "relatively large" symplectic symmetry group. However, we shall not pursue these extensions here as the detailed analysis depends very much on the structure of each individual group involved.

In fact, behind the proof of Theorem 1.0 a general method was devised in this paper which allows one to measure the effect of a symplectic finite group action on a homotopy $K3$ surface X in terms of its exoticness r_X. The basic idea of our method may be summarized as follows. Let a finite group G act on a homotopy $K3$ surface X via symplectic symmetries. Using the techniques developed in our previous work [6] and exploiting various features of the structure of G, one first determines the possible fixed point set of an arbitrary element $g \in G$, from which the trace $tr(g)$ of g on $H^*(X;\mathbb{Z})$ can be computed using the Lefschetz fixed point theorem. This leads to a calculation of

$$\dim(H^*(X;\mathbb{R}))^G = \frac{1}{|G|}\sum_{g\in G} tr(g).$$

On the other hand, there is an induced action of G on the lattice L_X of the Seiberg-Witten basic classes. The following basic inequality

$$\dim(L_X \otimes_{\mathbb{Z}} \mathbb{R})^G \leq \min(b_2^+(X/G), b_2^-(X/G)),$$

which follows from the fact that L_X is isotropic (cf. Proposition 4.1), plus the identity $\dim(H^*(X;\mathbb{R}))^G = 2 + b_2^+(X/G) + b_2^-(X/G)$ allows one to obtain information about $\dim(L_X \otimes_{\mathbb{Z}} \mathbb{R})^G$ and $r_X = \text{rank } L_X$.

For an illustration we consider the case where G is a nonabelian simple group. It is easily seen that in this case $b_2^+(X/G) = 3$ and $r_X = \dim(L_X \otimes_{\mathbb{Z}} \mathbb{R})^G$. The above basic inequality then becomes

$$r_X \leq \min(3, \dim(H^*(X;\mathbb{R}))^G - 5).$$

There are three nonabelian simple $K3$ groups: $L_2(7), A_5$ and A_6. For the case where $G = L_2(7)$ or A_6, we show in Section 2 that $\dim(H^*(X;\mathbb{R}))^G = 5$ (which is the same as that of a holomorphic G-action on a $K3$ surface), so that $r_X = 0$ as asserted in Theorem 1.0. For $G = A_5$, the fixed-point analysis only gives $\dim(H^*(X;\mathbb{R}))^G \leq 8$, cf. Lemma 2.5. (Note that even for a holomorphic A_5-action, one only gets $\dim(H^*(X;\mathbb{R}))^G = 6$.) Thus in the case of $G = A_5$, our method only gives $r_X \leq 3$, which does not yield any restriction on the exoticness r_X.

Our Theorem 1.0 naturally gives rise to the following question.

What can be said about a finite group G which can act effectively on a minimally exotic symplectic homotopy $K3$ surface via symplectic symmetries?

In the following theorem, we show that the symmetries of a minimally exotic symplectic homotopy $K3$ surface look very much like holomorphic automorphisms of the standard $K3$ surface; in particular, the symmetry groups are more or less $K3$ groups.

Theorem 1.1. *Let X be a minimally exotic symplectic homotopy $K3$ surface (i.e. $r_X = 0$) and let G be a finite group acting effectively on X via symplectic symmetries. Then there exists a short exact sequence of finite groups*

$$1 \to G_0 \to G \to G^0 \to 1,$$

where G^0 is cyclic and G_0 is a symplectic $K3$ group, such that G_0 is characterized as the maximal subgroup of G with the property $b_2^+(X/G_0) = 3$. Moreover, the induced action of G_0 on X has the same fixed point set structure as does a symplectic holomorphic action on the standard $K3$ by G_0.

Motivated by the above result we turn our attention to the problem of constructing symplectic finite group actions on *exotic* $K3$ surfaces. It seems that this line of research will require a developement of new techniques. Our next result could be viewed as a first, preliminary, step in this direction.

First of all, it is clear that the Fintushel-Stern knot surgery [9] can be suitably adapted for this purpose. More precisely, suppose a finite group G acts on the standard $K3$ surface preserving an elliptic fibration. Then under a certain condition (cf. Reamrk 4.3), one can perform knot surgery equivariantly to produce G-actions on exotic $K3$ surfaces. For example, every cyclic $K3$ group of prime order can act holomorphically on an elliptic $K3$ surface (cf. [25, 17]), and by a knot surgery one can easily show that such a group can act on an exotic $K3$ surface via symplectic symmetries. Concerning noncyclic $K3$ groups, the following theorem perhaps gives the most dramatic example of such a construction.

Theorem 1.2. *Let $G \equiv (\mathbb{Z}_2)^3$. There exists an infinite family of distinct maximally exotic (i.e. $r_X = 3$) symplectic homotopy $K3$ surfaces, such that each member of the exotic $K3$'s admits an effective G-action via symplectic symmetries. Moreover, the G-action is pseudofree and induces a trivial action on the lattice L_X of the Seiberg-Witten basic classes.*

The limitation of equivariant knot surgery is that the group G has to be a $K3$ group, and that it is difficult to construct group actions on homotopy $K3$ surfaces with a large exoticness (e.g., $r_X > 1$). In particular, the following questions seem to require techniques which go beyond the equivariant knot surgery:

Questions

(1) Are there any finite groups other than a $K3$ group which can act symplectically on a homotopy $K3$ surface?

(2) Are there any finite groups other than $(\mathbb{Z}_2)^3$ (or a subgroup of it) which can act symplectically on a homotopy $K3$ surface X with $r_X = 3$?

(3) Can $(\mathbb{Z}_2)^4$ act symplectically on an exotic $K3$ surface (i.e., with $r_X > 0$)?

We would like to point out that an earlier version of this paper was circulated under the title: *Symmetric Homotopy $K3$ Surfaces.*

The organization of the rest of the paper is as follows. The proofs of Theorem 1.0 and Theorem 1.1 are given in Sections 2 and 3 respectively. In Section 4 we show that the lattice L_X of Seiberg-Witten basic classes is isotropic and $r_X \leq 3$. The proof of Theorem 1.2 is also given in Section 4.

2. Proof of Theorem 1.0

Let (X, ω) be a symplectic homotopy $K3$ surface, and let G be a finite group which acts on X smoothly and effectively, preserving the symplectic structure ω. We pick an arbitrary ω-compatible, G-equivariant almost complex structure J on X, and we denote by g_J the associated Riemannian metric, i.e., $g_J(\cdot,\cdot) \equiv \omega(\cdot, J\cdot)$, which is also G-equivariant.

We derive some preliminary information about the G-action first.

Lemma 2.1. *Let G_0 be the maximal subgroup of G such that $b_2^+(X/G_0) = 3$. Then G/G_0 is cyclic. Moreover, the commutator $[G, G]$ is contained in G_0.*

Proof. Let H^+ be the space of g_J-self-dual harmonic 2-forms on X. Since the Riemannian metric g_J is G-equivariant, we see that H^+ is invariant under the action of G. Moreover, since $\omega \in H^+$ and G fixes ω, there is an induced action of G on the orthogonal complement $\langle\omega\rangle^\perp$ of ω in H^+. Note that $\dim H^+ = 3$, so that $\dim \langle\omega\rangle^\perp = 2$. We claim that the action of G on $\langle\omega\rangle^\perp$ is orientation-preserving (i.e. there are no reflections).

To see this, suppose there is a $g \in G$ such that the action of g on $\langle\omega\rangle^\perp$ is not orientation-preserving. This happens exactly when g fixes a 1-dimensional subspace, and it follows easily that in this case $b_2^+(X/\langle g\rangle) = 2$. On the other hand, $b_2^+(X/\langle g\rangle)$ must be odd. This is because for the symplectic 4-orbifold $X/\langle g\rangle$, the dimension of the Seiberg-Witten moduli space associated to the canonical $Spin^{\mathbb{C}}$ structure equals 0 (cf. [4], Appendix A). This gives rise to the equation

$$2 \cdot \text{index of Dirac operator} + (b_1(X/\langle g\rangle) - 1 - b_2^+(X/\langle g\rangle)) = 0.$$

It follows easily that $b_2^+(X/\langle g\rangle)$ is odd because $b_1(X/\langle g\rangle) = 0$.

With the preceding understood, we obtain an exact sequence of groups

$$1 \to G_0 \to G \to \mathbb{S}^1,$$

where the last homomorphism $G \to \mathbb{S}^1$ is induced from the action of G on $\langle\omega\rangle^\perp$. The lemma follows immediately from this.

□

For a symplectic $K3$ group G, the commutator $[G, G]$ and the quotient group (i.e. the abelianization) $G/[G, G]$ is determined in [27]. For the purpose of later discussions, the list of G where G is maximal is reproduced below.

- $G = L_2(7)$: $[G, G] = G$ and $G/[G, G] = 0$.
- $G = A_6$: $[G, G] = G$ and $G/[G, G] = 0$.
- $G = S_5$: $[G, G] = A_5$ and $G/[G, G] = \mathbb{Z}_2$.
- $G = M_{20} = 2^4 A_5$: $[G, G] = G$ and $G/[G, G] = 0$.
- $G = F_{384} = 4^2 S_4$: $[G, G] = 4^2 A_4$ and $G/[G, G] = \mathbb{Z}_2$.

- $G = A_{4,4} = 2^4 A_{3,3}$: $[G,G] = A_4^2$ and $G/[G,G] = \mathbb{Z}_2$.
- $G = T_{192} = (Q_8 * Q_8) \times_\phi S_3$: $[G,G] = (Q_8 * Q_8) \times_\phi \mathbb{Z}_3$ and $G/[G,G] = \mathbb{Z}_2$.
- $G = H_{192} = 2^4 D_{12}$: $[G,G] = 2^4\mathbb{Z}_3$ and $G/[G,G] = (\mathbb{Z}_2)^2$.
- $G = N_{72} = 3^2 D_8$: $[G,G] = A_{3,3}$ and $G/[G,G] = (\mathbb{Z}_2)^2$.
- $G = M_9 = 3^2 Q_8$: $[G,G] = A_{3,3}$ and $G/[G,G] = (\mathbb{Z}_2)^2$.
- $G = T_{48} = Q_8 \times_\phi S_3$: $[G,G] = T_{24} = Q_8 \times_\phi \mathbb{Z}_3$ and $G/[G,G] = \mathbb{Z}_2$.

The crucial step in the proof of Theorem 1.0 is to determine the possible fixed point set of an arbitrary element of G. This is done by combining the analysis in our previous work [6] with various G-index theorems, and by exploiting various specific features of the group G.

Here is the main technical input from [6] (cf. Lemma 3.1 in [6]). Since $b_2^+(X/G_0) = 3 \geq 2$, the canonical class $c_1(K_X)$ is represented by $\sum_i n_i C_i$, where $n_i \geq 1$ and $\{C_i\}$ is a finite set of J-holomorphic curves such that (i) $\cup_i C_i$ is invariant under the action of G_0, (ii) if $p \in X \setminus (\cup_i C_i)$ is fixed by an element $g \in G_0$, then the local representation of g at p must be contained in $SL_2(\mathbb{C})$. (In particular, p must be an isolated fixed point of g, and all the 2-dimensional components of the fixed point set $\mathrm{Fix}(g)$ are contained in $\cup_i C_i$.)

The fixed point set of an element of order 2 or 4 is determined in the following

Lemma 2.2. *(1) Let $g \in G$ be an involution. If $g \in G_0$, then Fix(g) consists of 8 isolated fixed points. If $g \in G \setminus G_0$, then Fix(g) is either empty or a disjoint union of embedded J-holomorphic curves $\{\Sigma_j\}$ such that $c_1(K_X) \cdot \Sigma_j = 0$ for each j.*

(2) Let $g \in G_0$ be an element of order 4. Then Fix(g) consists of 4 isolated fixed points, all with a local representation contained in $SL_2(\mathbb{C})$.

Proof. (1) Since X is simply-connected, the action of g can be lifted to the spin structure, where there are two cases: g is of even type, meaning that the order of the lifting is 2, and g is of odd type, meaning that the order of the lifting is 4. Moreover, g has only isolated fixed points in the case of an even type, and g is free or has only 2-dimensional fixed components in the case of an odd type (cf. [1]).

Suppose $g \in G_0$. Then $b_2^+(X/\langle g\rangle) = 3$, and by [3] g is of even type with 8 isolated fixed points. Now consider the case where $g \in G \setminus G_0$. In this case Fix(g) is either empty or is a disjoint union of embedded surfaces Σ_j (cf. [3]). Note that each Σ_j is J-holomorphic because we choose J to be G-equivariant.

We first show that $\sum_j c_1(K_X) \cdot \Sigma_j = 0$. To see this, suppose t is the dimension of the 1-eigenspace of g in $H^2(X;\mathbb{R})$. Then by the Lefschetz fixed point theorem and the G-signature theorem (cf. [12]), we obtain

$$\begin{cases} 2 + t - (22 - t) & = \quad \sum_j \chi(\Sigma_j) \\ 2(2-t) & = \quad -16 + \sum_j \frac{2^2-1}{3} \cdot \Sigma_j^2, \end{cases}$$

which gives $\sum_j (\chi(\Sigma_j) + \Sigma_j^2) = 0$. By the adjunction formula, we obtain

$$\sum_j c_1(K_X) \cdot \Sigma_j = \sum_j -(\chi(\Sigma_j) + \Sigma_j^2) = 0.$$

On the other hand, recall from [6] that $c_1(K_X)=\sum_i n_iC_i$ where $\{C_i\}$ is a finite set of J-holomorphic curves and $n_i\geq 1$. For any j, if $\Sigma_j\neq C_i$ for all i, then because of positivity of intersection of J-holomorphic curves, $c_1(K_X)\cdot\Sigma_j\geq 0$ with equality iff Σ_j is disjoint from $\cup_i C_i$. If $\Sigma_j=C_i$ for some i, then $c_1(K_X)\cdot\Sigma_j=c_1(K_X)\cdot C_i=0$ (cf. [6], Lemma 3.3). In any event we have $c_1(K_X)\cdot\Sigma_j\geq 0$, which implies $c_1(K_X)\cdot\Sigma_j=0$ because $\sum_j c_1(K_X)\cdot\Sigma_j=0$.

(2) Since $\mathrm{Fix}(g)\subset\mathrm{Fix}(g^2)$ and g^2 is an involution in G_0, we see immediately that g has only isolated fixed points, with local representations of either type $(1,1)$, $(3,3)$, or type $(1,3)$. We shall denote by s_+, s_- the number of fixed points of g of type $(1,3)$ and type $(1,1)$ or $(3,3)$ respectively. In order to determine s_+, s_-, we first compute with the Lefschetz fixed point theorem and the G-signature theorem. To this end, it is useful to observe that for the induced action of the involution g^2 on $H^2(X;\mathbb{R})$, the 1-eigenspace has dimension 14 and the (-1)-eigenspace has dimension 8. With this understood, if we denote by $t_\pm$ the dimension of the (± 1)-eigenspace of g in $H^2(X;\mathbb{R})$, then $t_+ + t_- = 14$. Now the Lefschetz fixed point theorem and the G-signature theorem (cf. [12]) give rise to the following system of equations

$$\begin{cases} 2+t_+-(14-t_+) & = & s_++s_- \\ 4(6-t_+) & = & -16+2s_++(-2)s_-, \end{cases}$$

where we use the assumption $g\in G_0$ so that $b_2^+(X/\langle g\rangle)=3$, and we use the fact that the signature defect at a fixed point of type $(1,3)$ and type $(1,1)$ or $(3,3)$ is $2,-2$ respectively, and the signature defect at a fixed point of g^2 is 0. (This follows by a direct calculation using the formulas in [12].) The solutions for s_+, s_- (note that $s_++s_-\leq 8$) are $s_+=4$ and $s_-=0,2$ or 4.

We proceed further by exploiting the fact that the action of g can also be lifted to the spin structure, and because g^2 is of even type, g is also of even type (i.e. a lifting of g to the spin structure is of order 4). Moreover, the induced lifting of g^2 to the spin structure is uniquely determined, i.e., it is independent of the different choices of liftings of g to the spin structure. With this understood, the computation of the "Spin-number" $Spin(g^2,X)$ plays a crucial role in the consideration which follows.

But first of all, a digression is needed in which we will recall a formula for the local contribution of a fixed point to the "Spin-number" (cf. Lemma 3.8 of [7]). Suppose h is an order p self-diffeomorphism ($p\geq 2$ and not necessarily prime) which is spin and almost complex. Then because of the h-equivariant almost complex structure, the h-equivariant spin structure corresponds to an h-equivariant complex line bundle L, such that at an isolated fixed point m of local representation type (a_m,b_m), the weight r_m of the representation of h on the fiber of L at m obeys $2r_m+a_m+b_m=0$ (mod p). Define $k(h,m)\equiv(2r_m+a_m+b_m)/p$. Then the local contribution of m to $Spin(h,X)$ is

$$I_m=(-1)^{k(h,m)+1}\cdot\frac{1}{4}\csc(\frac{a_m\pi}{p})\csc(\frac{b_m\pi}{p}).$$

End of digression.

We apply the above formula to the involution $h\equiv g^2$. For each fixed point m of g^2, $(a_m,b_m)=(1,1)$, so that the local contribution $I_m=\frac{1}{4}$ or $-\frac{1}{4}$, depending on whether

$r_m = 0$ or 1. Now suppose the g^2-index of the Dirac operator as a character is

$$\mathrm{index}_{g^2}\, D = d_0 + d_1\mathbb{C}_1$$

where $\mathbb{C}_k$ is the 1-dimensional weight-k representation. Then both d_0 and d_1 are even integers because of the quaternion structure. Since there are only 8 fixed points and each contributes $I_m = \frac{1}{4}$ or $-\frac{1}{4}$ to $Spin(g^2, X)$, it follows easily that $Spin(g^2, X) = d_0 - d_1$ only takes values of -2, 0, or 2. One can further eliminate the possibility of $Spin(g^2, X) = 0$ by observing that $d_0 + d_1 = -\mathrm{sign}(X)/8 = 2$ and that both d_0, d_1 are even. The crucial consequence of the fact that $Spin(g^2, X)$ equals either -2 or 2 is that the weight r_m of the representation of g^2 on the fiber of the complex line bundle L is independent of the fixed point m. This implies that for the element g, either $s_+ = 0$ or $s_- = 0$. Since $s_+ = 4$, s_- must be 0, and the lemma follows.

□

Next we discuss the fixed point set of an element of G_0 of an odd prime order. Unlike the cases we dealt with in Lemma 2.2, this requires analyzing the induced action on $\cup_i C_i$ in the way as we demonstrated in [6]. In particular, we shall rely on several specific results from Section 3 of [6]. We would like to point out that even though there is an additional assumption in [6] that the action is trivial on $H^2(X; \mathbb{R})$, this assumption is merely to ensure that each (-2)-sphere in $\cup_i C_i$ is invariant under the action.

We recall that the connected components of $\cup_i C_i$ may be divided into the following three types (cf. Section 3 in [6]):

(A) A single J-holomorphic curve of self-intersection 0 which is either an embedded torus, or a cusp sphere, or a nodal sphere.

(B) A union of two embedded (-2)-spheres intersecting at a single point with tangency of order 2.

(C) A union of embedded (-2)-spheres intersecting transversely.

Furthermore, a type (C) component may be conveniently represented by one of the graphs of type $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$ listed in Figure 1, where a vertex in a graph represents a (-2)-sphere and an edge connecting two vertices represents a transverse, positive intersection point of the two (-2)-spheres represented by the vertices.

With the preceding understood, let $g \in G_0$ be an element of order 3. Then $\mathrm{Fix}(g)$ may be divided into subsets (or groups) of the following four types.

(I) One fixed point with local representation in $SL_2(\mathbb{C})$.

(II) Three fixed points, all with local representation of type (k, k) for some $k \neq 0 \mod 3$.

(III) One fixed point of local representation type (k, k), $k \neq 0 \mod 3$, and one fixed spherical component of self-intersection -2.

(IV) One fixed toroidal component of self-intersection 0.

Moreover, a group of fixed points of type (III) comes only from a type (C) component of $\cup_i C_i$. For the sake of later arguments in this section, we shall give below a brief analysis of the action of g on a type (C) component of $\cup_i C_i$.

Let Λ be a type (C) component which is invariant under g. Then there is an induced action of g on the graph representing Λ. We consider first the case where Λ

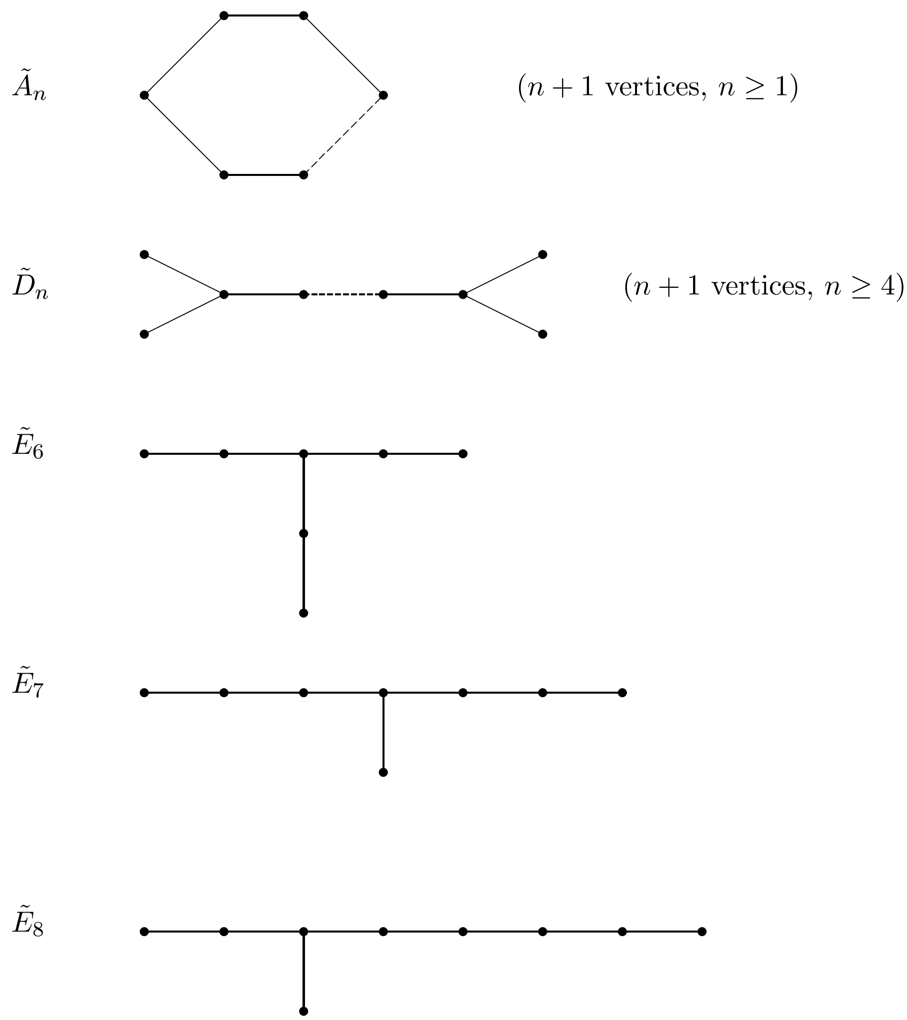

FIGURE 1.

is represented by a $\tilde{A}_n$ graph. Then the induced action of g on the graph is either a trivial action or a rotation. If the induced action is trivial, then the fixed points of g contained in Λ are either entirely of type (I) or consist of $(n+1)/3$ groups of type (III) fixed points (cf. Proposition 3.7 in [6]). (We note that by Lemma 3.6 in [6], Λ can not be a union of three (-2)-spheres intersecting transversely at a single point in this case.) If the induced action is a rotation, then either Λ contains no fixed points of g, or Λ is a union of three (-2)-spheres intersecting transversely at a single point, in which case the intersection point is the only fixed point of g contained in Λ and it is a type (I) fixed point.

Next we assume that Λ is represented by a $\tilde{E}_6$ graph. We claim that the induced action on the graph must be non-trivial. To see this, suppose the induced action of g on the graph is trivial. If we denote by C_0 the (-2)-sphere in Λ which is represented by the vertex of the graph that are adjacent to three other vertices (i.e., the central vertex), then C_0 must be fixed under the action of g (because a nontrivial cyclic group action on $\mathbb{S}^2$ has exactly 2 fixed points). Let C_1 be a (-2)-sphere in Λ intersecting with C_0 and C_2 be the (-2)-sphere intersecting with C_1. Then by Lemma 3.6 of [6], the rotation numbers at the two fixed points associated to C_1 must be $(0,1)$ and $(1,1)$, with $(0,1)$ being the rotation numbers at the intersection point of C_0 and C_1. (See Section 3 in [6] for a discussion on rotation numbers.) This implies, by Lemma 3.6 of [6] again, that the rotation numbers at the two fixed points of g associated to C_2 are $(1,1)$ and $(0,1)$, with $(1,1)$ being the rotation numbers at the intersection point of C_1 and C_2. It follows, since the rotation numbers at the other fixed point on C_2 are $(0,1)$, that C_2 must intersect with a 2-dimensional component of the fixed point set of g. But this is clearly a contradiction, hence our claim that the induced action of g on the graph must be non-trivial. With this understood, it is easily seen that Λ contains exactly two fixed points of g, and these two fixed points are on the (-2)-sphere C_0. Furthermore, it follows from Lemma 3.6 in [6] that these two fixed points are of type (I). By a similar argument, we show that Λ can not be represented by a $\tilde{E}_8$ graph, because a $\tilde{E}_8$ graph admits no non-trivial actions of g.

Suppose Λ is represented by a $\tilde{D}_n$ graph. Then the induced action on the graph must be trivial, and the fixed points of g contained in Λ consist of 1 group of type (II) fixed points and $(n-1)/3$ groups of type (III) fixed points. Suppose Λ is represented by a $\tilde{E}_7$ graph, then the induced action on the graph must be trivial and Λ gives rise to 3 groups of type (III) fixed points of g. (See Proposition 3.7 in [6].)

Finally, it is helpful to note that only a type $\tilde{A}_n$, $\tilde{D}_n$, or $\tilde{E}_7$ component of $\cup_i C_i$ can possibly contain a group of type (III) fixed points of g, and only a type $\tilde{D}_n$ component of $\cup_i C_i$ can contain a group of type (II) fixed points of g.

The fixed point set of an element of order 3 in G_0 is described in the following

Lemma 2.3. *Suppose $g \in G_0$ is an element of order 3. Let u, v and w be the number of groups of type (I), (II) and (III) fixed points of g respectively, and let $t = b_2(X/\langle g\rangle)$. Then*

1. *$2u + 3v = 12$, $w \leq 6$ and $t \geq 10$. Moreover, $t = 10$ iff $(u, v, w) = (6, 0, 0)$.*
2. *Suppose $w = 0$. If there exist 3 distinct involutions $h_1, h_2, h_3 \in G_0$ each of which commutes with g, then $(u, v) = (6, 0)$.*
3. *Suppose $w = 0$. If g is contained in a subgroup of G_0 which is isomorphic to T_{24}, then $(u, v) = (6, 0)$.*

Proof. (1) Note that a toroidal fixed component Y of g does not make any contribution in the Lefschetz fixed point theorem because $\chi(Y) = 0$, nor does it contribute in the G-signature theorem because $Y \cdot Y = 0$. Hence we shall ignore it in our calculations below.

Observe that $t = b_2(X/\langle g\rangle)$ is the dimension of the 1-eigenspace of g in $H^2(X;\mathbb{R})$, and that $t - (22-t)/2$ is the trace of g on $H^2(X;\mathbb{R})$. Hence the Lefschetz fixed point theorem and the G-signature theorem give rise to the following equations

$$\begin{cases} 2+t-(22-t)/2 & = & u+3v+3w \\ 3(6-t) & = & -16+\frac{2}{3}u-2v-6w \end{cases}$$

where we make use of $b_2^+(X/\langle g\rangle)=3$ and the fact that the total signature defect for a group of type (I), (II) and (III) fixed points is $\frac{2}{3}$, -2 and -6 respectively. (The claim concerning the total signature defect follows by a direct calculation using the formulas in [12].) The equation $2u+3v=12$ follows immediately, which has 3 solutions: $(u,v)=(6,0),(3,2)$, and $(0,4)$. The inequality $w\leq 6$ follows from $u+3v\geq 6$ and the fact that $t\leq b_2(X)=22$. It is also easy to check that $t\geq 10$, with $t=10$ iff $(u,v,w)=(6,0,0)$.

(2) Suppose $(u,v)=(0,4)$, in which case g has 12 isolated fixed points. From the analysis of a possible action of g on a type (C) component preceding Lemma 2.3, we see that only a component represented by a type $\tilde{D}_n$ graph can possibly contain a group of type (II) fixed points, and at the same time, there must be groups of type (III) fixed points. Since we assume that $w=0$, these 12 points can not be contained in type (C) components of $\cup_i C_i$. It follows by Proposition 3.7 of [6] that the 12 fixed points of g must be contained in 4 toroidal components of $\cup_i C_i$, where each toroidal component contains exactly 3 isolated fixed points.

By Lemma 2.2 each h_i has 8 isolated fixed points. Since g and h_i commute, there is an induced action of g on $\mathrm{Fix}(h_i)$, and it follows that g and h_i must have at least 2 common fixed points. This implies that one of the toroidal components containing the fixed points of g is invariant under h_i, and consequently, g and h_i generate an effective cyclic action of order 6 on that torus. Since an order-6 cyclic action on a torus is either free or has only 1 fixed point, we see that distinct common fixed points of g and h_i are contained in distinct toroidal components of $\cup_i C_i$. It follows easily that there are i,j with $i\neq j$ such that h_i and h_j leave one of the toroidal components invariant, because for each i, g and h_i have at least 2 common fixed points and there are exactly 4 toroidal components of $\cup_i C_i$ containing the fixed points of g. But this is easily seen a contradiction, as h_i acts freely on the set of common fixed points of g and h_j because $\mathrm{Fix}(h_i)\cap\mathrm{Fix}(h_j)=\emptyset$. The case where $(u,v)=(3,2)$ can be similarly eliminated. This proves that $(u,v)=(6,0)$.

(3) Note that $T_{24}=Q_8\times_\phi\mathbb{Z}_3$, where we may assume without loss of generality that the action of $\mathbb{Z}_3=\langle g\rangle$ on

$$Q_8=\{i,j,k|i^2=j^2=k^2=-1,\ ij=k,\ jk=i,\ ki=j\}$$

is given by $\phi(g)(i)=j$, $\phi(g)(j)=k$ and $\phi(g)(k)=i$.

By Lemma 2.2, it follows easily that Q_8 has either 2 or 4 isolated fixed points (see e.g. [27]). Since there is an induced action of g on the fixed point set of Q_8, we see immediately that T_{24} has at least 1 fixed point.

Suppose $(u,v)=(0,4)$. As we argued in (2) above, at least one of the 4 toroidal components must be invariant under T_{24} because it contains a fixed point of T_{24}. But this is impossible as there are no such T_{24}-actions on the torus (cf. [24]).

If $(u,v)=(3,2)$, then g and $-1\in Q_8$ must have 5 common fixed points. It follows as we argued in (2) above that each of the 2 toroidal components of $\cup_i C_i$ which contains the type (II) fixed points of g must be invariant under -1, with each containing exactly 1 common fixed point of g and -1. But on the other hand, by Proposition 3.7 in [6], each of the 2 toroidal components contains exactly 4 fixed points of -1, so that all of the fixed points of -1 are contained in there. This is a contradiction to the fact that the 3 type (I) fixed points of g, which are not contained in the 2 toroidal components, are also fixed under -1. Hence the case where $(u, v) = (3, 2)$ is also ruled out.

□

Proof of Theorem 1.0:

The general strategy goes as follows. For each of the 6 maximal symplectic $K3$ groups listed in Theorem 1.0, there is a subgroup $H \subset G_0$ such that for any symplectic holomorphic action of H on a $K3$ surface, one has $\mu(H) = 5$ where

$$\mu(H) \equiv \frac{1}{|H|} \sum_{g \in H} tr(g).$$

(See [22, 27] for the calculation of $\mu(H)$ for a symplectic automorphism group H of a $K3$ surface.) The main task in the proof of Theorem 1.0 is to show that any effective action of H on X via symplectic symmetries must have the same fixed point set as does a symplectic holomorphic action of H (except for possible toroidal fixed components). As a consequence this implies that

$$\dim(H^2(X; \mathbb{R}))^H = \dim(H^*(X; \mathbb{R}))^H - 2 = \mu(H) - 2 = 5 - 2 = 3.$$

On the other hand, $b_2^+(X/H) = 3$ because $H \subset G_0$, so that $H^2(X; \mathbb{R})^H$ must be positive-definite. It follows that $c_1(K_X) = 0$ because $c_1(K_X) \in H^2(X; \mathbb{R})^H$ and $c_1(K_X) \cdot c_1(K_X) = 0$, which is equivalent to X being minimally exotic, i.e., $r_X = 0$.

Case (1). $G = L_2(7)$. First note that $G_0 = G$, i.e., $b_2^+(X/G) = 3$. An element of G is of order 2, 3, 4, or 7. The following lemma describes the fixed point set of an order-7 element of G.

Lemma 2.4. *Let $g \in G = L_2(7)$ be any element of order 7. Then g has exactly 3 isolated fixed points, and is either pseudofree or has only toroidal fixed components.*

Proof. We first show that if a type (C) component of $\cup_i C_i$ contains a fixed point of g, then its local representation at the fixed point must lie in $SL_2(\mathbb{C})$. To this end, we recall that the normalizer of $\langle g \rangle$ in G is a maximal subgroup D of order 21 which is a semi-direct product of $\mathbb{Z}_7$ by $\mathbb{Z}_3$ (cf. [8]). Let Λ be a type (C) component which contains a fixed point of g. (Note that Λ is invariant under g.) If it is represented by a type $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$ graph, then since the (-2)-spheres in Λ generate a lattice in $H_2(X; \mathbb{Z})$ which contains a negative-definite sublattice of rank at least 4, the orbit of Λ under the action of G can have at most 4 components because of the constraint $b_2^-(X) = 19$. On the other hand, one can easily check that Λ is not invariant under $G = L_2(7)$, and since the index of the maximal subgroup D is 8, there are at least 8 components in the orbit of Λ, which is a contradiction. Suppose Λ is represented by a type $\tilde{A}_n$ graph. Then each (-2)-sphere in Λ is invariant under g (since Λ contains a fixed point of g). By Proposition 3.7 in [6], there are 3 possibilities: (i) $n = -1 \mod 7$, so that Λ contains at least seven (-2)-spheres, (ii) Λ only contains fixed points of g whose local representations lie in $SL_2(\mathbb{C})$, (iii) Λ is a union of three (-2)-spheres intersecting transversely at one single point. Note that case (i) can be eliminated by a similar argument as in the cases of type $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$ graphs. Case (iii) is ruled out as follows. Note that the maximal subgroup D can not act linearly and freely on $\mathbb{S}^3$, so that such a Λ can not be invariant under the action of D. Hence if such a Λ exists, there must be at least $3 \times 8 = 24$ components in the orbit of Λ under the action of G. But this is impossible because of the constraint $b_2^-(X) = 19$. This finishes the proof of our claim.

Secondly, we will show that there are no type (B) components which contain a fixed point of g. Suppose Λ is such a type (B) component. One can check easily that Λ can not be invariant under the action of D, so that there are at least 24 type (B) components in $\cup_i C_i$. But this contradicts the fact that $b_2^-(X) = 19$. Hence there are no type (B) components containing a fixed point of g.

Finally, suppose a type (A) component Λ of $\cup_i C_i$ contains a fixed point of g. Then by Proposition 3.7 in [6], Λ is either a fixed toroidal component, or Λ is a cusp sphere containing 2 fixed points of g of local representations of type $(2k, 3k)$, $(-k, 6k)$ for some $k \neq 0 \mod 7$ respectively.

With the preceding understood, we conclude that g has only fixed toroidal components, and that the isolated fixed points of g can be divided into groups of the following two types:

(1) One fixed point with local representation in $SL_2(\mathbb{C})$.
(2) Two fixed points with local representation of type $(2k, 3k)$, $(-k, 6k)$ for some $k \neq 0 \mod 7$ respectively.

Next we compute with the Lefschetz fixed point theorem and the G-signature theorem. Denote by t the dimension of the 1-eigenspace of g in $H^2(X; \mathbb{R})$ (note that $22 - t$ must be divisible by 6), and denote by u, v the number of groups of type (1), (2) isolated fixed points of g respectively. Then by the Lefschetz fixed point theorem and the G-signature theorem,

$$\begin{cases} 2 + t - (22 - t)/6 & = & u + 2v \\ 7(6 - t) & = & -16 + 10u - 8v, \end{cases}$$

where we make use of $b_2^+(X/\langle g \rangle) = 3$ and the fact that the total signature defect for a group of type (1), (2) fixed points of g is 10 and -8 respectively (cf. [6], Lemma 3.8). The solutions to the above system of equations are

$$(t, u, v) = (4, 3, 0), (10, 2, 4), (16, 1, 8), (22, 0, 12).$$

The cases where $(t, u, v) = (10, 2, 4)$ or $(16, 1, 8)$ can be ruled out as follows. The maximal subgroup D induces a $\mathbb{Z}_3$-action on the set of isolated fixed points of g, which must be free because D can not act freely and linearly on $\mathbb{S}^3$. This implies that the number of fixed points, which is $u + 2v$, must be divisible by 3.

In the case of $(t, u, v) = (22, 0, 12)$, g is homologically trivial. Since $G = L_2(7)$ is a simple group, it follows that the action of G is also homologically trivial. But, because G is nonabelian, this is impossible by McCooey's theorem in [19].

The only case left is $(t, u, v) = (4, 3, 0)$, which shows that g has exactly 3 isolated fixed points.

□

Next we consider the action of an element $g \in G$ of order 3. We claim that g has exactly 6 isolated fixed points, with possibly some fixed toroidal components. To see this, we note that there is an element $h \in G$ of order 7 such that $D = \langle g, h\rangle$ is a nonabelian subgroup of order 21, which is the normalizer of $\langle h\rangle$ (cf. [8]). From the proof of Lemma 2.4, we see that the dimension of the $\exp(\frac{2\pi ik}{7})$-eigenspace of h in $H^2(X;\mathbb{R})$ is $\frac{22-4}{6} = 3$ for each $1 \leq k \leq 6$. By examining the action of D on the $\exp(\frac{2\pi ik}{7})$-eigenspaces of h, $1 \leq k \leq 6$, one can check easily that the dimension of the 1-eigenspace of g in $H^2(X;\mathbb{R})$ is at most 10. By Lemma 2.3 (1), our claim follows.

Now with Lemma 2.2, which describes the number of fixed points of an element of order 2 or 4, we see that for any $g \in G$, the Lefschetz fixed point theorem implies that the trace $tr(g)$ is the same as that of a symplectic automorphism of order $|g|$ on a $K3$ surface. By Mukai [22], $\mu(G) = 5$ for a symplectic holomorphic $G = L_2(7)$ action. This implies that

$$\dim(H^*(X;\mathbb{R}))^G = \mu(G) \equiv \frac{1}{|G|}\sum_{g\in G} tr(g) = 5.$$

As we pointed out in the beginning of the proof of Theorem 1.0, this implies that $c_1(K_X) = 0$, and hence X is minimally exotic.

End of Case (1).

Case (2). $G = M_{20}$ or A_6. In this case we shall exploit the fact that there is a subgroup of G_0 which is isomorphic to either A_5 or A_6.

Lemma 2.5. *Suppose $H \subset G_0$ is a subgroup isomorphic to either A_5 or A_6. Let $g \in H$ be an element of odd order. Then g is either pseudofree or has only toroidal fixed components. Moreover, g has 4 isolated fixed points if $|g| = 5$, and g has either 6 or 12 isolated fixed points when $|g| = 3$.*

Proof. Suppose $g \in H$ is an element of order 5. Without loss of generality we may assume that $H \cong A_5$, because in the case of $H \cong A_6$, g is contained in an A_5-subgroup of H. With this understood, the maximal subgroup of H containing g is a dihedral group $D_{10} \subset H$ of index 6 (cf. [8]). One can similarly argue, as in the proof of Lemma 2.4, that if a type (C) component of $\cup_i C_i$ contains a fixed point of g, then it must be represented by a type $\tilde{A}_n$ graph and the fixed point is of local representation lying in $SL_2(\mathbb{C})$.

By Proposition 3.7 in [6], if a type (A) component Λ of $\cup_i C_i$ contains a fixed point of g, then Λ is either a fixed toroidal component, or Λ is a cusp or nodal sphere containing only fixed points of g of local representation lying in $SL_2(\mathbb{C})$. If a type (B) component Λ contains a fixed point of g, then Λ contains three fixed points of g, one with local representation of type $(k, 2k)$ and the other two of type $(-k, 4k)$ for some $k \neq 0 \mod 5$.

In conclusion, g has only toroidal fixed components and the isolated fixed points of g can be divided into groups of the following two types:

(1) One fixed point with local representation in $SL_2(\mathbb{C})$.
(2) Three fixed points, one with local representation of type $(k, 2k)$ and the other two of type $(-k, 4k)$ for some $k \neq 0 \mod 5$.

Denote by t the dimension of the 1-eigenspace of g in $H^2(X;\mathbb{R})$ (note that $22-t$ must be divisible by 4), and denote by u, v the number of groups of type (1), (2) isolated fixed points of g respectively. Then by the Lefschetz fixed point theorem and the G-signature theorem,

$$\begin{cases} 2+t-(22-t)/4 & = & u+3v \\ 5(6-t) & = & -16+4u-8v, \end{cases}$$

where we make use of $b_2^+(X/\langle g\rangle)=3$ and the fact that the total signature defect for a group of type (1), (2) fixed points is 4 and -8 respectively (cf. [6], Lemma 3.8). The solutions to the above system of equations are

$$(t,u,v)=(6,4,0),(10,3,2),(14,2,4),(18,1,6),(22,0,8).$$

The cases where $u=1$ or 3 can be eliminated as follows. There is an involution on the set of isolated fixed points of g induced by the action of D_{10}, which is free because D_{10} can not act freely and linearly on $\mathbb{S}^3$. Consequently, the number of isolated fixed points of g must be divisible by 2. To eliminate the case where $(t,u,v)=(14,2,4)$, note that in this case $\cup_i C_i$ has 4 type (B) components each of which contains a fixed point of g. Moreover, it is easy to see that each component is not invariant under the action of D_{10}. Since the index of $D_{10}\subset H$ is 6, there are at least $4\times 2\times 6=48$ type (B) components of $\cup_i C_i$, which contradicts $b_2^-(X)=19$. Finally, the case where $(t,u,v)=(22,0,8)$ is ruled out by McCooey's theorem [19] because H is simple and nonabelian. Hence g has 4 isolated fixed points when $|g|=5$.

Next suppose $g\in H$ is an element of order 3, where H is either A_5 or A_6. We claim that $\mathrm{Fix}(g)$ does not contain any group of type (III) fixed points (i.e., $w=0$ in Lemma 2.3). To see this, note first that there are no type (C) components of $\cup_i C_i$ which are represented by a graph of type $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$. The point is that such a component can not contain any fixed points of an order-5 element of H, hence can not be invariant under the action of an order-5 element. If such a component exists, then there are at least 5 such components in $\cup_i C_i$, and this contradicts $b_2^-(X)=19$. Hence if $\mathrm{Fix}(g)$ contains a group of type (III) fixed points, it must come from a type (C) component Λ which is represented by a type $\tilde{A}_n$ graph, where $n=-1 \mod 3$ (cf. Proposition 3.7 in [6]). Let $h\in H$ be an order-5 element. Since $g\neq hgh^{-1}$, it follows easily that h and g can not have a common isolated fixed point in Λ, which implies that either Λ is not invariant under h or h acts freely on Λ. In any event, the case where $n>2$ can be ruled out by using the fact $b_2^-(X)=19$. To eliminate the case where $n=2$, we note that there is a subgroup $K\subset H$ which is isomorphic to the symmetric group S_3 and contains $\langle g\rangle$ as a normal subgroup. Clearly K can not leave Λ invariant if it is represented by a $\tilde{A}_2$ graph, so that Λ must come in pairs. Again this is impossible by the fact that $b_2^-(X)=19$. Hence $\mathrm{Fix}(g)$ does not contain any group of type (III) fixed points. The action of K on $\mathrm{Fix}(g)$ also implies that u is even in Lemma 2.3 (because S_3 can not act freely and linearly on $\mathbb{S}^3$). Hence g has either 6 or 12 isolated fixed points when $|g|=3$.

Finally, note that g is either pseudofree or has only toroidal fixed components.

□

Let $G=M_{20}$. Since $[G,G]=G$, we see that $G_0=G$. We claim that for each $g\in G$ the trace $tr(g)$ on $H^*(X;\mathbb{R})$ is the same as that of a symplectic automorphism of order $|g|$ on a $K3$ surface. With this the proof of Theorem 1.0 proceeds identically as in the case of $L_2(7)$, as for $G=M_{20}$, $\mu(G)\equiv |G|^{-1}\sum_{g\in G} tr(g)=5$ is also true for a symplectic holomorphic action (cf. [22]). When $|g|\neq 3$ or 6, our claim follows readily from Lemma 2.2 and Lemma 2.5. For the case where $|g|=3$ or 6, we need to argue with some extra information about the structure of $G=M_{20}$.

According to Mukai [22], page 189, $M_{20}=2^4A_5$, where the action of A_5 on 2^4 is obtained by realizing 2^4 as the hypersurface $V=\{(a_i)|\sum_{i=1}^5 a_i=0\}\subset(\mathbb{Z}_2)^5$ with A_5 acting as permutations of the 5 coordinates. Clearly, for each element g of order 3 in A_5, there are 3 nonzero elements of V which are fixed under g. This gives 3 distinct involutions in G, each of which commutes with g. By Lemma 2.3 (2), g has 6 isolated fixed points. It also follows easily from the proof of Lemma 2.3 (2) that an order 6 element of G has 2 isolated fixed points, with possibly some fixed toroidal components. In conclusion, for an order 3 or 6 element $g\in G$, the trace $tr(g)$ on $H^*(X;\mathbb{R})$ is also the same as that of a symplectic automorphism on a $K3$ surface of the same order. This completes the proof for the case where $G=M_{20}$.

Let $G=A_6$. In this case, we also have $G_0=G$. As in the case of M_{20}, it suffices to show that for each $g\in G$ with $|g|=3$, there are 6 isolated fixed points. (Note that $\mu(G)=5$ is also true for a symplectic holomorphic A_6-action (cf. [22])). To this end, we recall the following fact about A_6: There are 2 conjugacy classes of elements of order 3 in A_6; the centralizer of each order 3 element in A_6 is isomorphic to $(\mathbb{Z}_3)^2$, hence has order 9. Now suppose an element g of order 3 in $G=A_6$ has, instead, 12 isolated fixed points. Then the conjugacy class of g will make an increase of $\frac{6}{9}=\frac{2}{3}$ to

$$\mu(G)\equiv \frac{1}{|G|}\sum_{g\in G} tr(g)$$

when compared with a holomorphic A_6-action. Since there are only two conjugacy classes of elements of order 3 in A_6, a nonzero increase to $\mu(G)$ is either $\frac{2}{3}$ or $\frac{4}{3}$, neither of which is integral. This shows that an element of order 3 in G must have 6 isolated fixed points, and the proof of Theorem 1.0 for the case of $G=A_6$ follows.

End of Case (2) where $G=M_{20}$ or A_6.

Case (3). $G=A_{4,4}$. Let $H\equiv[G,G]=A_4\times A_4$. Then since $[G,G]\subset G_0$, we have $b_2^+(X/H)=3$. Note that $\mu(H)=5$ for a symplectic automorphism group H of a $K3$ surface (cf. [27]). Hence by Lemma 2.2, it suffices to show that for each $g\in H$ of order 3, the trace $tr(g)$ on $H^*(X;\mathbb{R})$ is the same as that of a symplectic automorphism of order 3 on a $K3$ surface.

There are 4 conjugacy classes of order 3 elements in G, which are represented by $(g,1),(1,g),(g,g),(g,g^2)\in A_4\times A_4=H$ for some fixed element $g\in A_4$ of order 3.

Since the trace on $H^*(X;\mathbb{R})$ only depends on the conjugacy class in G, it suffices to examine these 4 elements of H.

We first show that there are no type (III) fixed points (i.e., $w=0$ in Lemma 2.3). Consider the case $(g,1)$ first. The normalizer of $\langle (g,1)\rangle$ in H is $\langle g\rangle\times A_4$ which has index 4. If Λ is a type (C) component of $\cup_i C_i$ which contains a group of type (III) fixed points of $(g,1)$, then the fact $b_2^-(X)=19$ immediately rules out the possibility that Λ is represented by a $\tilde{E}_7$ graph or a $\tilde{A}_n$ graph where $n\neq 2$. If Λ is represented by a $\tilde{D}_n$ graph or a $\tilde{A}_2$ graph, then one can check easily that the orbit of Λ under the normalizer $\langle g\rangle\times A_4$ has at least 3 components. This also contradicts $b_2^-(X)=19$, and hence there are no type (III) fixed points of $(g,1)$. The case of $(1,g)$ is completely parallel. For the case of (g,g) or (g,g^2), the normalizer of $\langle (g,g)\rangle$ or $\langle (g,g^2)\rangle$ in H is $\langle g\rangle\times\langle g\rangle$ which has index 16. It follows immediately from $b_2^-(X)=19$ that there are no type (III) fixed points.

Now by Lemma 2.3 (2), each of $(g,1)$ and $(1,g)$ has exactly 6 isolated fixed points. The case of (g,g) or (g,g^2) is more involved, which is addressed in the following

Lemma 2.6. *Suppose $c_1(K_X)\neq 0$. Then the number of isolated fixed points of (g,g) or (g,g^2) is even.*

Proof. We consider the case of (g,g) only. The case of (g,g^2) is completely parallel.

By Lemma 2.3, the number of isolated fixed points of (g,g) is either 6, 9 or 12. Suppose to the contrary that it is 9. A contradiction is derived as follows. Observe that there is an involution $h\in G\setminus H$ such that h and (g,g) generate a subgroup K of G, where K is isomorphic to S_3 and $\langle (g,g)\rangle$ is a normal subgroup of K. There is an induced action of K on $\mathrm{Fix}((g,g))$, which preserves the type of the fixed points. Since (g,g) has 3 type (I) fixed points, one of them, denoted by p, must be fixed by K. Note that $K\cong S_3$ can not have an isolated fixed point, hence $h\in G\setminus G_0$ and $\mathrm{Fix}(h)$ consists of a disjoint union of embedded J-holomorphic curves $\{\Sigma_j\}$ where $c_1(K_X)\cdot\Sigma_j=0$ for each j (cf. Lemma 2.2 (1)). It follows easily that there are fixed components $\Gamma_0,\Gamma_1,\Gamma_2$ of the three involutions h, $(g,g)h(g^{-1},g^{-1})$, $(g^2,g^2)h(g^{-2},g^{-2})$ of K respectively, which intersect transversely at p and have the same genus and self-intersection. We claim that $\Gamma_0,\Gamma_1,\Gamma_2$ are (-2)-spheres, and consequently $(\sum_{k=0}^2\Gamma_k)^2=0$. To see that each Γ_k is a (-2)-sphere, it suffices to show that $\Gamma_k^2<0$ because $c_1(K_X)\cdot\Gamma_k=0$. Suppose to the contrary that $\Gamma_k^2\geq 0$. Then $(\sum_{k=0}^2\Gamma_k)^2>0$, which we will show is impossible when $c_1(K_X)\neq 0$. To see this, note that all three classes $\sum_{k=0}^2\Gamma_k$, $c_1(K_X)$, and the symplectic structure ω are fixed under K. Since $b_2^+(X/K)=1$, we may write

$$\sum_{k=0}^{2}\Gamma_k=a_1\omega+\alpha_1,\quad c_1(K_X)=a_2\omega+\alpha_2$$

for some $a_1,a_2\in\mathbb{R}^+$ and $\alpha_1,\alpha_2\in H^2(X;\mathbb{R})$ such that $\alpha_i\cdot\omega=0$ and $\alpha_i^2\leq 0$ for $i=1,2$. Without loss of generality we assume that $\omega^2=1$. Then $(\sum_{k=0}^2\Gamma_k)^2>0$, $c_1(K_X)^2=0$, and $c_1(K_X)\cdot\sum_{k=0}^2\Sigma_k=0$ give rise to

$$a_1^2+\alpha_1^2>0,\ a_2^2+\alpha_2^2=0,\ \text{and } a_1a_2+\alpha_1\cdot\alpha_2=0.$$

We arrive at a contradiction to the triangle inequality

$$|\alpha_1 \cdot \alpha_2| = a_1 a_2 > (\alpha_1^2 \cdot \alpha_2^2)^{1/2}.$$

Hence $\Gamma_0, \Gamma_1, \Gamma_2$ are (-2)-spheres and $(\sum_{k=0}^2 \Gamma_k)^2 = 0$.

We claim that $\sum_{k=0}^2 \Gamma_k = \lambda c_1(K_X)$ for some $\lambda > 0$. To see this, let H^+ be the space of self-dual harmonic 2-forms. Then since $b_2^+(X/K) = 1$, the projections of the classes of $\sum_{k=0}^2 \Gamma_k$ and $c_1(K_X)$ into H^+ are linearly dependent. On the other hand, $\sum_{k=0}^2 \Gamma_k$ and $c_1(K_X)$ span an isotropic subspace because

$$(\sum_{k=0}^2 \Gamma_k)^2 = c_1(K_X)^2 = c_1(K_X) \cdot \sum_{k=0}^2 \Gamma_k = 0,$$

so that their projections into H^+ are injective. This proves the claim.

Now for each involution $h' \in H$, the set $h'(\cup_{k=0}^2 \Gamma_k)$ is disjoint from $\cup_{k=0}^2 \Gamma_k$ because of positivity of intersection of J-holomorphic curves and because

$$(h')^*(\sum_{k=0}^2 \Gamma_k) \cdot (\sum_{k=0}^2 \Gamma_k) = \lambda^2 (h')^* c_1(K_X) \cdot c_1(K_X) = \lambda^2 c_1(K_X)^2 = 0.$$

Since there are 15 distinct involutions in H, there must be 16 such configurations as $\cup_{k=0}^2 \Gamma_k$ which are mutually disjoint. This certainly contradicts $b_2^-(X) = 19$, and the lemma follows. □

If $c_1(K_X) = 0$, then X is already minimally exotic and we are done in this case. Suppose $c_1(K_X) \neq 0$, then with Lemma 2.6, we shall further argue that each of (g,g) or (g,g^2) must have 6 isolated fixed points. The reason is that if not, there will be an increase to $\mu(H) \equiv |H|^{-1} \sum_{g \in H} tr(g)$, in comparison with a symplectic automorphism group H of a $K3$ surface, of either $2 \times \frac{6}{9}$ or $4 \times \frac{6}{9}$, both of which are not integral. (The centralizer of (g,g) or (g,g^2) is $\langle g \rangle \times \langle g \rangle$ which has order 9, and (g,g), (g^2,g^2), and (g,g^2), (g^2,g) are not conjugate in H even though each pair of them are conjugate in G.) The proof for the case of $G = A_{4,4}$ is then completed.

End of Case (3) where $G = A_{4,4}$.

Case (4). $G = T_{192}$ or T_{48}. Set $H \equiv [G,G] \subset G_0$. Then in both cases, $\mu(H) = 5$ for a symplectic automorphism group H of a $K3$ surface (cf. [27]).

Let $G = T_{192}$. In this case $H = (Q_8 * Q_8) \times_\phi \mathbb{Z}_3$, where

$$Q_8 * Q_8 = Q_8 \times Q_8 / \langle (-1,-1) \rangle$$

is the central product of Q_8 with itself, and the action of $\mathbb{Z}_3$ on $Q_8 * Q_8$ is given by $\phi : x * y \mapsto \alpha^{-1}(x) * \alpha(y)$ for some fixed order-3 automorphism α of Q_8 (cf. [22]). The normalizer of $\mathbb{Z}_3$ in H is $\langle -1 \rangle \times \mathbb{Z}_3$, where $\langle -1 \rangle$ denotes the center of $Q_8 * Q_8$. It follows easily that for each $g \in \mathbb{Z}_3$, there are no type (III) fixed points of g because $b_2^-(X) = 19$ and the index of $\langle -1 \rangle \times \mathbb{Z}_3$ in H is 16. By Lemma 2.3 (3), each order-3 element of H has 6 isolated fixed points, with possibly some fixed toroidal components. Hence the case where $G = T_{192}$ follows.

Let $G = T_{48}$. Then H is isomorphic to $T_{24} = Q_8 \times_\phi \mathbb{Z}_3$. By Lemma 2.3 (3), one only needs to verify that for any nontrivial element $g \in \mathbb{Z}_3$, there are no groups of type (III) fixed points of g.

Suppose to the contrary that there is a group of type (III) fixed points, which is contained in a type (C) component Λ. We observe that the normalizer of $\mathbb{Z}_3$ in H is $\langle -1 \rangle \times \mathbb{Z}_3$ which has index 4. It follows immediately from $b_2^-(X) = 19$ that Λ is not represented by a $\tilde{E}_7$ graph, or a $\tilde{D}_n$ graph with $n > 4$, or a $\tilde{A}_n$ graph with $n > 2$. In fact Λ is not represented by a $\tilde{D}_4$ graph either. To see this, suppose Λ is of type $\tilde{D}_4$. If Λ is invariant under $-1 \in Q_8$, then the (-2)-sphere represented by the central vertex of Λ must contain 2 fixed points of -1. On the other hand, by Lemma 2.2, Q_8 has at least 1 fixed point (cf. e.g. [27]). Since $-1 \in Q_8$ has only 8 isolated fixed points, it follows that there must be an order-4 element of Q_8 which also fixes the central vertex of the $\tilde{D}_4$ graph. However, this would imply that the whole group Q_8 fixes the central vertex, and there is an induced effective action of Q_8 on $\mathbb{S}^2$, which is a contradiction. If Λ is not invariant under $-1 \in Q_8$, then the orbit of Λ under H has at least 8 components, contradicting $b_2^-(X) = 19$. Hence Λ is not represented by a $\tilde{D}_4$ graph.

It remains to eliminate the possibility that Λ is represented by a $\tilde{A}_2$ graph. Suppose this is the case. Then by the same argument as above, Λ can not be invariant under $-1 \in Q_8$, which means that Λ comes in pairs. Furthermore, the constraint $b_2^-(X) = 19$ allows for exactly two $\tilde{A}_2$ components, which give 2 groups of type (III) fixed points of g. To eliminate this possibility, we make use of the fact that there is an involution $h \in G \setminus H$, such that $hgh^{-1} = g^{-1}$. There is an induced action of h on the set of type (III) fixed points of g, where by replacing h with $(-1)h$, we may assume that h fixes the isolated fixed point in each of the 2 groups of type (III) fixed points. Since the local representation of g at the fixed point is of type $(1,1)$ or $(2,2)$, it follows that at the fixed point one has the commutativity relation $hg = gh$, which contradicts the fact that $hgh^{-1} = g^{-1}$. This finishes the proof that there are no groups of type (III) fixed points, and the case where $G = T_{48}$ follows.

3. Proof of Theorem 1.1

In the proof of Theorem 1.1, the determination of the structure and the action of the subgroup G_0 follows the strategy of Xiao [27]. However, it relies on the fundamental work of Taubes [26] to establish the necessary properties of the action of G in order to implement Xiao's strategy.

The first half of Theorem 1.1 is contained in the following

Proposition 3.1. *Let X be a minimally exotic symplectic homotopy $K3$ surface, and let G be a finite group acting on X effectively and symplectically. Then there exists a short exact sequence of finite groups*

$$1 \to G_0 \to G \to G^0 \to 1,$$

where G^0 is cyclic and G_0 is characterized as the maximal subgroup of G with property $b_2^+(X/G_0) = 3$. Moreover, for each $g \in G_0$ the action of g is pseudofree with local *representation at a fixed point contained in $SL_2(\mathbb{C})$, and the quotient orbifold X/G_0 can be smoothly resolved into a minimally exotic symplectic homotopy $K3$ surface.*

Proof. Let ω be a symplectic structure on X which is preserved under G, and we fix an ω-compatible, G-equivariant almost complex structure J on X. Let K_X be the canonical bundle with the choice of J, and let g_J be the associated Riemannian metric, both of which are G-equivariant.

Following Taubes [26], we consider the following family (parametrized by $r > 0$) of perturbed Seiberg-Witten equations

$$D_A\psi = 0 \text{ and } P_+F_A = \frac{1}{4}\tau(\psi \otimes \psi^*) + \mu,$$

where $\psi = \sqrt{r}(\alpha, \beta) \in \Gamma(K_X \oplus \mathbb{I})$, A is a $U(1)$-connection on K_X, and

$$\mu = -\frac{ir}{4}\omega + P_+F_{A_0}$$

for a canonical (up to gauge equivalence) connection A_0 on K_X^{-1}. According to [26], $c_1(K_X)$ is a Seiberg-Witten basic class, hence for any $r > 0$, there is a solution (ψ, A) with $\psi = \sqrt{r}(\alpha, \beta) \in \Gamma(K_X \oplus \mathbb{I})$. Moreover, as $r \to \infty$, the zero set $\alpha^{-1}(0) \subset X$ converges pointwise to a set of finitely many J-holomorphic curves with multiplicity, which represents the Poincaré dual of $c_1(K_X)$. Since X is minimally exotic by our assumption, $c_1(K_X) = 0$, and as a consequence $\alpha^{-1}(0)$ must be empty for sufficiently large $r > 0$. It follows that, when $r > 0$ is sufficiently large, there is a unique solution $(\sqrt{r}(\alpha_0, 0), A)$ (up to gauge equivalence) to the perturbed Seiberg-Witten equations, where $a_0 \equiv \frac{1}{2}(A - A_0)$ is a flat connection on K_X, $|\alpha_0| = 1$ and $\nabla_{a_0}\alpha_0 = 0$ (cf. Lemma 4.5 and the proof of Proposition 4.4 in Taubes [26]).

With the preceding understood, we note that since the family of perturbed Seiberg-Witten equations under consideration is G-equivariant (A_0 may be chosen such that $g^*A_0 = A_0$, $\forall g \in G$), the uniqueness of (α_0, a_0) up to gauge equivalence implies that for any $g \in G$, $g^*\alpha_0 = \phi(g)\alpha_0$ for some smooth circle-valued function $\phi(g) : X \to \mathbb{S}^1$. Since g is of a finite order, $\phi(g)$ must be a constant function because $\phi(g)^{|g|} = 1$. This gives rise to a homomorphism $\rho : G \to \mathbb{S}^1$ which is defined by $\rho : g \mapsto \phi(g) \in \mathbb{S}^1$. We define $G_0 \subset G$ to be the kernel of ρ and set $G^0 \equiv G/G_0$. Then clearly G^0 is cyclic. Moreover, if $g \in G$ has the property that $b_2^+(X/g) = 3$ then, as we argued in [6], the corresponding g-equivariant Seiberg-Witten invariant is nonzero, which implies $\phi(g) = 1$ and hence $g \in G_0$. Finally, we observe that for any $g \in G_0$, since α_0 is a nowhere vanishing section of K_X and $g^*\alpha_0 = \alpha_0$, g has at most isolated fixed points with a local representation contained in $SL_2(\mathbb{C})$.

It remains to show that the quotient orbifold X/G_0 can be smoothly resolved into a minimally exotic symplectic homotopy $K3$ surface. Note that this automatically implies $b_2^+(X/G_0) = 3$ as it equals the b_2^+ of the smooth resolution. In fact, in the next lemma we will prove an equivariant version of it. To finish the proof of the proposition, one simply uses the lemma with $H = K = G_0$.

∎

Consider a subgroup K of G which is contained in $G_0 = \ker\rho$ where $\rho : g \mapsto \phi(g)$, i.e., for any $g \in K$, $g^*\alpha_0 = \alpha_0$. Let H be a normal subgroup of K.

Lemma 3.2. *There exists a minimally exotic symplectic homotopy $K3$ surface X_H which is a smooth resolution of the orbifold X/H, such that K/H acts on X_H symplectically, extending the natural K/H-action on X/H under the resolution $X_H \to X/H$. Moreover, note that $b_2^+(X_H/(K/H)) = b_2^+(X/K) = b_2^+(X_K) = 3$.*

Proof. The construction of the smooth resolution of the symplectic orbifold X/H was given by McCarthy and Wolfson in [18]. We shall briefly review the procedure, indicating that it can be done equivariantly. In fact the construction is local, so we shall be focusing on a neighborhood of an isolated singular point of the orbifold, which by the equivariant Darboux' theorem is modeled on $\mathbb{C}^2/\Gamma$, where Γ is the isotropy group at the singular point which acts complex linearly on $\mathbb{C}^2$, and where the symplectic structure ω_0 on $\mathbb{C}^2/\Gamma$ is given by the standard one on $\mathbb{C}^2$, $\omega_{std} = i(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$.

Let U, V be the part of $\mathbb{C}^2/\Gamma$ which lies outside and inside of the unit ball over Γ respectively, and let $W = \partial U = \partial V$ which is the 3-manifold $\mathbb{S}^3/\Gamma$. Since V is an algebraic surface with an isolated singularity, there is a nonsingular, minimal projective resolution $\pi : Y \to V$. Note that Y is Kähler. We let τ be a Kähler form on Y. Then for any $\epsilon > 0$, $\omega_\epsilon \equiv \pi^*\omega_0 + \epsilon\tau$ is a Kähler form on Y. We shall show that for a sufficiently small $\epsilon > 0$, the two pieces (U, ω_0) and (Y, ω_ϵ) can be symplectically "glued" together, which gives a smooth resolution of $\mathbb{C}^2/\Gamma$ by a symplectic manifold.

To this end, we consider the contact structure ξ on W which is the distribution of complex lines in TW. Note that $\omega_0|_W = d\alpha$ for some contact form α such that $\xi = \ker \alpha$. On the other hand, since W is a rational homology 3-sphere, $\tau|_W = d\beta$ for a 1-form β, and hence $\omega_\epsilon|_W = d\alpha_\epsilon$ where $\alpha_\epsilon \equiv \alpha + \epsilon\beta$ is also a contact form when $\epsilon > 0$ is sufficiently small. By Moser's argument, there exists a self-diffeomorphism $\psi : W \to W$ such that $\psi^*\alpha_\epsilon = e^f \alpha$ for some smooth function $f : W \to \mathbb{R}$. Pick a constant $C > 0$ such that $f < C$ on W. Let $Z \subset (\mathbb{R} \times W, d(e^t\alpha))$ be the symplectic "cylinder" defined by

$$Z \equiv \{(t, x) | x \in W, f(x) - C \leq t \leq 0\}.$$

Then the smooth resolution of $\mathbb{C}^2/\Gamma$ by a symplectic manifold is given by

$$(X_{\epsilon,C}, \omega) \equiv (U, \omega_0) \cup (Z, d(e^t\alpha)) \cup (Y, e^{-C}\omega_\epsilon),$$

where the gluing between $\partial U = W$ and the component of ∂Z defined by $t = 0$ is by the identity map on W, and the gluing between the component of ∂Z defined by $t = f(x) - C$ and $\partial Y = W$ is by $(t, x) \mapsto \psi(x)$, where $\psi : W \to W$ is the self-diffeomorphism obtained above through Moser's argument. We leave it to the reader to follow through that if a finite group Γ' acts complex linearly on $\mathbb{C}^2/\Gamma$, then there is a corresponding symplectic Γ'-action on the smooth resolution $(X_{\epsilon,C}, \omega)$. (We remark that Moser's argument can be done equivariantly in the presence of a compact Lie group action; in particular, the self-diffeomorphism ψ of W can be made equivariant with respect to the Γ'-action on W, so that the gluing by $(t, x) \mapsto \psi(x)$ in the construction of $X_{\epsilon,C}$ is also equivariant.)

It remains to show that X_H is a minimally exotic symplectic homotopy $K3$ surface, and that $b_2^+(X_H/(K/H)) = 3$. The key step is the observation that X_H has a trivial canonical bundle. To see this, note that for any $g \in H$, since $g^*\alpha_0 = \alpha_0$, the nonzero section α_0 descends to a nonzero section $\hat{\alpha_0}$ of the canonical bundle of the symplectic orbifold X/H. With this understood it suffices to show that the canonical bundle of $(X_{\epsilon,C},\omega)$ is trivial, which is done by matching up the trivialization of the canonical bundle on the three pieces (U,ω_0), $(Z,d(e^t\alpha))$ and $(Y,e^{-C}\omega_\epsilon)$.

On (U,ω_0), the canonical bundle K_U is trivialized by $\hat{\alpha_0}$. On $(Z,d(e^t\alpha))$, the canonical bundle is the pull back of ξ^{-1}, the inverse line bundle of the contact structure ξ, via the projection $Z\to W$. Since $K_U|_W=\xi^{-1}$ and K_U is trivial, we see that K_Z is also trivial. Finally, K_Y is also trivial, because the symplectic form $e^{-C}\omega_\epsilon$ on Y is Kähler so that K_Y is simply given by the holomorphic canonical bundle. Since for each $g\in H$ the local representation at each fixed point of g is contained in $SL_2(\mathbb{C})$, the singularity of $\mathbb{C}^2/\Gamma$ is a Du Val singularity, and it is known that in this case Y has a trivial canonical bundle if it is taken minimal. Now since $H^1(W;\mathbb{Z})=0$ which parametrizes the homotopy classes of the non-zero sections of the trivial line bundle over W, one can matches up the trivialization of K_U, K_Z and K_Y to obtain a trivialization for the canonical bundle of $(X_{\epsilon,C},\omega)$.

As an immediate consequence, X_H is spin as $w_2(TX_H)=c_1(K_{X_H}) \pmod 2$ must vanish. By Rohlin's theorem, $\mathrm{sign}(X_H)$ is divisible by 16. Hence the intersection form on $H_2(X_H;\mathbb{Z})/Tor$ is given by $m\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\oplus 2k(\pm E_8)$, with $m=b_2^+(X_H)$ and $k=|\mathrm{sign}(X_H)|/16$. Now observe that the fundamental group of X_H is finite, which implies that $b_1(X_H)=0$. Hence

$$0=c_1^2(K_{X_H})=(2\chi+3\mathrm{sign})(X_H)=2(2+2m+16k)\pm 3\cdot 16k=0.$$

Since $m=b_2^+(X_H)=b_2^+(X/H)=1$ or 3 (cf. Lemma 2.1), the only solution for (m,k) from the above equation is $m=3$ and $k=1$, and moreover, $\mathrm{sign}(X_H)=-16$. This shows that X_H is a rational homology $K3$ surface. (Note that this conclusion also follows directly from Furuta's work on the $\frac{11}{8}$-conjecture, cf. [10].)

Next we show that $\pi_1(X_H)$ is trivial. Let $\widehat{X_H}$ be the universal cover of X_H, which is compact because $\pi_1(X_H)$ is finite. Then $\widehat{X_H}$ is a closed, simply-connected symplectic 4-manifold with trivial canonical bundle. It was shown by Morgan and Szabó [21] (compare also [16]) that the Betti numbers of $\widehat{X_H}$ satisfy

$$b_2^+(\widehat{X_H})=3 \text{ and } b_2^-(\widehat{X_H})=19.$$

With $b_2^+(X_H)=3$ and $b_2^-(X_H)=19$ it follows easily that $\pi_1(X_H)$ is trivial. This completes the proof that X_H is a minimally exotic symplectic homotopy $K3$ surface.

Finally, we observe that

$$b_2^+(X_H/(K/H))=b_2^+((X/H)/(K/H))=b_2^+(X/K)=b_2^+(X_K)=3.$$

□

Remark 3.3. The holomorphic version of Lemma 3.2 has been used in a fundamental way, first by Nikulin in [23] and then by Xiao in [27], to study finite symplectic automorphism groups of $K3$ surfaces. In particular, following the argument in Nikulin [23], one can show, with Lemma 3.2, that for any $g\in G_0$, the order $|g|\leq 8$ and the number of fixed points of g is the same as that of an order $|g|$ symplectic automorphism of a $K3$ surface. However, we would like to point out that this statement can also be proved directly, by a lengthy argument involving essentially the various G-index theorems. Even though we have no need to pursue it here, we would like to observe that $\mathrm{Fix}(g) \neq \emptyset$ directly implies that the smooth resolution X_H in Lemma 3.2 is simply-connected, without appealing to the result of Morgan and Szabó in [21] as we did in the proof of Lemma 3.2.

Now with Lemma 3.2 in place, we shall follow through the arguments of Xiao in [27] to complete the proof of Theorem 1.1 by showing that G_0 is a symplectic $K3$ group and that the action of G_0 on X has the same fixed point set structure as does a corresponding symplectic automorphism group of a $K3$ surface.

In Section 1 of Xiao [27], the only argument involving complex geometry is in the proof of Lemma 2 there. We shall give a pure algebraic topology proof of this result below. In order to state the lemma, we first need to introduce the necessary notations.

Let X be a minimally exotic symplectic homotopy $K3$ surface and let G be a finite group acting effectively on X via symplectic symmetries such that $b_2^+(X/G) = 3$. Then as we have shown, X/G is a symplectic orbifold of only Du Val singularities, which has a smooth resolution X_G as defined in Lemma 3.2. Let L' be the sublattice of $H_2(X_G;\mathbb{Z})$ generated by the (-2)-spheres in X_G which are sent to the singular points under $X_G \to X/G$, and let L be the smallest primitive sublattice of $H_2(X_G;\mathbb{Z})$ containing L'. Then the analog of Lemma 2 in Xiao [27] is contained in the following lemma.

Lemma 3.4. $L/L' \cong G/[G,G]$.

Proof. Let A be a regular neighborhood of the (-2)-spheres in X_G which are mapped to the singular points under $X_G \to X/G$, and let $B = X_G \setminus A$ be the complement of A. Then the long exact sequence associated to the pair (X_G, A) gives rise to

$$H_2(A;\mathbb{Z}) \xrightarrow{i_*} H_2(X_G;\mathbb{Z}) \xrightarrow{j_*} H^2(B;\mathbb{Z}) \to 0,$$

where we have used the excision and Poincaré duality to make the identification $H_2(X_G, A;\mathbb{Z}) \cong H_2(B, \partial B;\mathbb{Z}) \cong H^2(B;\mathbb{Z})$, and we have used the fact that A is simply-connected so that $H_1(A;\mathbb{Z}) = 0$. On the other hand, by the universal-coefficient theorem for cohomology, we have the short exact sequence

$$0 \to \mathrm{Ext}(H_1(B;\mathbb{Z}),\mathbb{Z}) \to H^2(B;\mathbb{Z}) \xrightarrow{h} \mathrm{Hom}(H_2(B;\mathbb{Z}),\mathbb{Z}) \to 0.$$

Now observe that for any element $x \in H_2(X_G;\mathbb{Z})$, $h \circ j_*(x) = 0$ if and only if the intersection product of x with any element $y \in H_2(B;\mathbb{Z})$ is zero, which is precisely if and only if $x \in L$. This gives a surjective homomorphism $j_* : L \to \mathrm{Ext}(H_1(B;\mathbb{Z}),\mathbb{Z})$ whose kernel is easily seen to be $L' = \mathrm{Im}(i_*)$. Hence $L/L' \cong \mathrm{Ext}(H_1(B;\mathbb{Z}),\mathbb{Z})$.

Finally, $\pi_1(B) = G$ so that $H_1(B;\mathbb{Z}) = G/[G,G]$ is a torsion group. This gives

$$L/L' \cong \mathrm{Ext}(H_1(B;\mathbb{Z}),\mathbb{Z}) \cong H_1(B;\mathbb{Z}) = G/[G,G].$$

□

In Section 2 of [27], Xiao formulated a set of criteria obtained from Section 1, and by a computer search a list of possibilities for a symplectic $K3$ group as well as the combinatorial types of the actions were generated. A few of the cases were further eliminated to reach the final list, where the arguments are those in [27] which precedes Lemma 5. We observe that this procedure can be used in the present situation verbatim. This finishes the proof of Theorem 1.1.

Remark 3.5. The holomorphic version of Theorem 1.1 is contained in Nikulin [23]. There it was also shown that the order of the cyclic group G^0 is bounded by 66 (which is a sharp bound). The proof of this result involves arguments in complex geometry which are not available in the present, symplectic category. However, we should point out that there are further informations contained in the proof of Proposition 3.1 which can be used to analyze G^0; in particular, it is very likely that $|G^0|$ has a universal upper bound. We shall not pursue this issue here, but wish to point out that because of the homological rigidity of symplectic symmetries of a minimally exotic symplectic homotopy $K3$ surface established in [6], the prime factors in $|G^0|$ are bounded by $b_2 = 22$.

4. The Lattice L_X and Proof of Theorem 1.2

Recall that the Seiberg-Witten invariant of a simply-connected, closed, oriented, smooth 4-manifold M with $b_2^+ \geq 2$ is a map

$$SW_M : \{\beta \in H^2(M;\mathbb{Z}) | \beta \equiv w_2(TM) \pmod 2\} \rightarrow \mathbb{Z}.$$

A class β is called a (Seiberg-Witten) basic class if $SW_M(\beta) \neq 0$. It is a fundamental fact that the set of basic classes is finite. Moreover, if β is a basic class, then so is $-\beta$ with

$$SW_M(-\beta) = (-1)^{(\chi+\text{sign})(M)/4} SW_M(\beta).$$

When M is symplectic, a fundamental result of Taubes says that the canonical class $c_1(K_X)$ associated to a symplectic structure is always a basic class. The Seiberg-Witten invariant SW_M is an invariant of the diffeomorphism class of M, whose sign depends on a choice of an orientation of

$$H^0(M;\mathbb{R}) \otimes \det H^{2,+}(M;\mathbb{R}).$$

In particular, the set of basic classes depends only on the diffeomorphism type of M. When M is a homotopy $K3$ surface, a theorem of Morgan and Szabó [21] says that $\beta = 0$ is always a basic class. Furthermore, when M is symplectic, work of Taubes [26] gives additional information about the Seiberg-Witten invariant, in particular, about the set of basic classes.

Let X be a symplectic homotopy $K3$ surface. We set

$$L_X \equiv \text{Span}(\beta \in H^2(X;\mathbb{Z}) | SW_X(\beta) \neq 0) \subset H^2(X;\mathbb{Z}),$$

and set $r_X \equiv \text{rank}(L_X)$. Let ω be any symplectic structure on X, and let K_X be the associated canonical bundle. Then Taubes [26] showed that $c_1(K_X) \in L_X$ and $0 \leq \beta \cdot [\omega] \leq c_1(K_X) \cdot [\omega]$ for any basic class β. In particular, $c_1(K_X) = 0$ iff $r_X = 0$.

Proposition 4.1. *Let X be a symplectic homotopy $K3$ surface. Then the lattice of basic classes L_X is isotropic, i.e., for any $x, y \in L_X$, the cup product of x and y is zero. As a consequence, the rank of L_X is bounded from above by* $\min(b_2^+, b_2^-) = 3$, *i.e., $r_X \leq 3$.*

Proof. Let ω be a symplectic structure of X, and let K_X be the canonical bundle. Since X is minimal, and $c_1^2(K_X) = 2\chi(X) + 3\mathrm{sign}(X) = 0$, a theorem of Taubes (cf. [26], Theorem 0.2 (5)) says that for any basic class β, $e_\beta \equiv \frac{1}{2}(c_1(K_X) + \beta) \in H^2(X;\mathbb{Z})$ is Poincaré dual to $\sum_i m_i T_i$, where $m_i > 0$ and $\{T_i\}$ is a finite set of disjoint, symplectically embedded tori with self-intersection 0.

To see L_X is isotropic, it suffices to show that for any basic classes β, β', the cup product $\beta \cdot \beta' = 0$, which follows from the generalized adjunction formula as follows. Suppose $e_\beta = \sum_i m_i T_i$ where $\{T_i\}$ is a finite set of disjoint, symplectically embedded tori with self-intersection 0. Then for any basic class β', we apply the generalized adjunction formula to T_i,

$$\mathrm{genus}(T_i) \geq 1 + \frac{1}{2}(|\beta' \cdot T_i| + T_i^2).$$

This implies, for each i, $\beta' \cdot T_i = 0$ because $\mathrm{genus}(T_i) = 1$ and $T_i^2 = 0$, and consequently, $e_\beta \cdot \beta' = 0$. In particular, since $c_1(K_X)$ is a basic class, we have $e_\beta \cdot c_1(K_X) = 0$, which implies that $\beta \cdot c_1(K_X) = 0$ for any basic class β. (This is because $e_\beta \equiv \frac{1}{2}(c_1(K_X) + \beta)$ and $c_1^2(K_X) = 0$.) Now we go back to $e_\beta \cdot \beta' = 0$, and conclude that

$$\beta \cdot \beta' = 2e_\beta \cdot \beta' - c_1(K_X) \cdot \beta' = 0.$$

Finally, we point out that $r_X \leq 3$ follows directly from the fact that the projection of L_X into $H^{2,+}(X;\mathbb{Z})$ is injective (because L_X is isotropic). $\square$

Remark 4.2. Suppose G is a finite group which acts on a symplectic homotopy $K3$ surface X via symplectic symmetries. Then there is an induced action of G on the set of basic classes, which can be extended to a linear action on the lattice L_X. Moreover, let ω be the symplectic structure which is preserved under the action of G, and let K_X be the associated canonical bundle. Then $c_1(K_X) \in L_X$ is fixed under the action of G, and since ω is also fixed, the function $\omega : L_X \to \mathbb{R}$ defined by $x \mapsto [\omega] \cdot x$ is G-invariant. On the other hand, since the action of G on $H^0(X;\mathbb{R}) \otimes \det H^{2,+}(X;\mathbb{R})$ is orientation-preserving (cf. Lemma 2.1), one has, for any basic class β,

$$SW_X(g \cdot \beta) = SW_X(\beta), \ \ \forall g \in G.$$

It is clear that the induced action of G on L_X may be exploited to relate the action of G on X and the underlying smooth structure of X.

Proof of Theorem 1.2

Construction of this type of exotic $K3$ surfaces is due to Fintushel and Stern, which is done by performing the knot surgery on three disjoint, homologically distinct, symplectically embedded tori in a Kummer surface (cf. [9], compare also [11]). Our observation here is that it can be done equivariantly. However, we would like to point out that the three tori (actually 12 tori divided into 3 groups) have to be chosen differently than in [9] and [11] (cf. Remark 4.3).

Consider the 4-torus $T^4 = (\mathbb{S}^1)^4$ with the involution ρ, which is defined in the angular coordinates by

$$\rho : (\theta_0, \theta_1, \theta_2, \theta_3) \mapsto (-\theta_0, -\theta_1, -\theta_2, -\theta_3), \text{ where } \theta_j \in \mathbb{R}/2\pi\mathbb{Z}.$$

There are 16 isolated fixed points $(\theta_0,\theta_1,\theta_2,\theta_3)$ where each θ_j takes values in $\{0,\pi\}$. A Kummer surface is a smooth 4-manifold which is obtained by replacing each of the singular points in the quotient $T^4/\langle\rho\rangle$ with an embedded (-2)-sphere. We denote the 4-manifold by X_0.

We shall give a more concrete description of X_0 below, where X_0 is also naturally endowed with a symplectic structure. Consider the symplectic form Ω on T^4, which is equivariant with respect to the involution ρ:

$$\Omega \equiv \sum_{(i,j,k)} (d\theta_0\wedge d\theta_i + d\theta_j\wedge d\theta_k)$$

where the sum is taken over $(i,j,k)=(1,2,3),(2,3,1)$ and $(3,1,2)$. This gives rise to a symplectic structure on the orbifold $T^4/\langle\rho\rangle$. One can further symplectically resolve the orbifold singularities to obtain a symplectic structure on X_0 as follows. By the equivariant Darboux' theorem, the symplectic structure is standard near each orbifold singularity. In particular, it is modeled on a neighborhood of the origin in $\mathbb{C}^2/\{\pm 1\}$ and admits a Hamiltonian $\mathbb{S}^1$-action with moment map $\mu:(w_1,w_2)\mapsto \frac{1}{4}(|w_1|^2+|w_2|^2)$, where w_1,w_2 are the standard coordinates on $\mathbb{C}^2$. Fix a sufficiently small $r>0$ and remove $\mu^{-1}([0,r))$ from $T^4/\langle\rho\rangle$ at each of its singular point. Then X_0 is diffeomorphic to the 4-manifold obtained by collapsing each orbit of the Hamiltonian $\mathbb{S}^1$-action on the boundaries $\mu^{-1}(r)$, which is naturally a symplectic 4-manifold (cf. [15]). We denote the symplectic structure on X_0 by ω_0.

Let $G=(\mathbb{Z}_2)^3$. We shall next describe a G-action on X_0 which preserves the symplectic structure ω_0. Consider first the following G-action on T^4:

$$a\cdot(\theta_0,\theta_1,\theta_2,\theta_3)=(\theta_0,\theta_1+\pi a_1,\theta_2+\pi a_2,\theta_3+\pi a_3)$$

where $a=(a_1,a_2,a_3)\in G$ with each $a_j\in\mathbb{Z}_2\equiv\mathbb{Z}/2\mathbb{Z}$. One can check easily that the above G-action commutes with the involution ρ, so that there is an induced G-action on the orbifold $T^4/\langle\rho\rangle$. Moreover, the G-action clearly preserves the symplectic form

$$\Omega \equiv \sum_{(i,j,k)} (d\theta_0\wedge d\theta_i + d\theta_j\wedge d\theta_k)$$

on T^4, hence it descends to a symplectic G-action on $T^4/\langle\rho\rangle$. From the description of (X_0,ω_0) given in the previous paragraph, it follows easily that there is an induced, symplectic G-action on (X_0,ω_0). (The key point here is that Lerman's symplectic cutting can be done equivariantly, cf. [15].) The G-action on X_0 is pseudofree; in fact, for any $0\neq a=(a_1,a_2,a_3)\in G$, a fixed point of a in X_0 is the image of a point in T^4 with angular coordinates $(\theta_0,\theta_1,\theta_2,\theta_3)$, where $\theta_0=0$ or π, and for $j=1,2,3$, $\theta_j=0$ or π if $a_j=0$ and $\theta_j=\pi/2$ or $3\pi/2$ if $a_j=1$. (Note that each $a\neq 0$ in G has 8 isolated fixed points.)

We shall next describe a set of 12 disjoint, symplectically embedded tori in (X_0,ω_0), which is invariant under the G-action. The 12 tori are divided into 3 groups, labeled naturally by $j=1,2,3$, where each group consists of 4 tori. The group G acts freely and transitively among the 4 tori in each group. For simplicity we shall only describe the group of tori indexed by $j=1$ in detail; the others are completely parallel.

Consider the projection π_1 from T^4 to T^2 where

$$\pi_1 : (\theta_0, \theta_1, \theta_2, \theta_3) \mapsto (\theta_2, \theta_3).$$

For any fixed $\delta_{12}, \delta_{13} \in \mathbb{R}/2\pi\mathbb{Z}$ other than 0, $\pi/2$, $3\pi/2$ and π, the 4 tori in T^4

$$T_{1,0} \equiv \pi_1^{-1}(\delta_{12}, \delta_{13}) \qquad T_{1,1} \equiv \pi_1^{-1}(\delta_{12}+\pi, \delta_{13})$$
$$T_{1,2} \equiv \pi_1^{-1}(\delta_{12}, \delta_{13}+\pi) \quad T_{1,3} \equiv \pi_1^{-1}(\delta_{12}+\pi, \delta_{13}+\pi)$$

are symplectically embedded with respect to the symplectic form

$$\Omega \equiv \sum_{(i,j,k)} (d\theta_0 \wedge d\theta_i + d\theta_j \wedge d\theta_k).$$

Moreover, they descend to 4 disjoint tori in $T^4/\langle\rho\rangle$, and if the distance between δ_{12}, δ_{13} to 0, $\pi/2$, $3\pi/2$ and π is sufficiently large, $T_{1,k}$, $0 \le k \le 3$, can be regarded as tori in X_0, which are disjoint and symplectically embedded. The union $\cup_k T_{1,k}$ is easily seen to be invariant under the action of G on X_0. Moreover, the action of G on $\cup_k T_{1,k}$ is transitive, and each $T_{1,k}$ is invariant under an involution of G, which acts on the torus freely via translations.

In the same vein, one can consider projections

$$\pi_2 : (\theta_0, \theta_1, \theta_2, \theta_3) \mapsto (\theta_1, \theta_3) \text{ and } \pi_3 : (\theta_0, \theta_1, \theta_2, \theta_3) \mapsto (\theta_1, \theta_2)$$

and choose $\delta_{21}, \delta_{23}, \delta_{31}, \delta_{32} \in \mathbb{R}/2\pi\mathbb{Z} \setminus \{0, \pi/2, \pi, 3\pi/2\}$ to obtain 8 other tori $T_{j,k}$, where $j = 2, 3$ and $0 \le k \le 3$. One can check easily that under further conditions:

$$\delta_{13} \ne \pm\delta_{23}, \pm(\delta_{23}+\pi),\ \delta_{12} \ne \pm\delta_{32}, \pm(\delta_{32}+\pi),\ \delta_{21} \ne \pm\delta_{31}, \pm(\delta_{31}+\pi)$$

the 12 tori $T_{j,k}$ in X_0 are disjoint.

The exotic $K3$ surfaces are constructed by performing the Fintushel-Stern knot surgery on each of the 12 tori $T_{j,k}$ in X_0 with a fibered knot. The key issue here is that the knot surgery needs to be performed equivariantly with respect to the G-action on X_0. To this end, we shall first give a brief review of the knot surgery from [9].

Let M be a simply-connected smooth 4-manifold with $b_2^+ > 1$, and let T be a c-embedded torus in M (cf. [9]) such that $\pi_1(M \setminus T) = 1$. Consider a knot K in $\mathbb{S}^3$, and let m denote a meridional circle to K. Let Y_K be the 3-manifold obtained by performing 0-framed surgery on K. Then m can also be viewed as a circle in Y_K. In $Y_K \times \mathbb{S}^1$ we have the smoothly embedded torus $T_m \equiv m \times \mathbb{S}^1$ of self-intersection 0. Since a neighborhood of m has a canonical framing in Y_K, a neighborhood of the torus T_m in $Y_K \times \mathbb{S}^1$ has a canonical identification with $T_m \times D^2$. With this understood, the knot surgery on T with knot K is the smooth 4-manifold M_K, which is the fiber sum

$$M_K \equiv M \#_{T=T_m} (Y_K \times \mathbb{S}^1) = [M \setminus (T \times D^2)] \cup [(Y_K \times \mathbb{S}^1) \setminus (T_m \times D^2)].$$

Here $T \times D^2$ is a tubular neighborhood of the torus T in M. The two pieces are glued together so as to preserve the homology class $[\mathrm{pt} \times \partial D^2]$. Note that the diffeomorphism type of the fiber sum is not uniquely determined in general, and the 4-manifold M_K is taken to be any manifold constructed in this fashion. A fundamental theorem of Fintushel and Stern states that M_K is naturally homeomorphic to M and the Seiberg-Witten invariants of the two manifolds are related by

$$sw_{M_K} = sw_M \cdot \Delta_K(t),$$

where sw_{M_K}, sw_M are certain Laurent polynomials defined from the Seiberg-Witten invariants of M_K and M respectively, and $\Delta_K(t)$ is the Alexander polynomial of K, with $t = \exp(2[T])$. See [9] for more details. We remark that when M is symplectic and T is symplectically embedded, M_K can be naturally made symplectic by choosing any fibered knot K. Note that when M is the standard $K3$ surface, one has $sw_M = 1$, so that M_K is an exotic $K3$ surface as long as the knot K has a nontrivial Alexander polynomial.

With the preceding understood, note that in our present situation, each of the 12 tori $T_{j,k}$ is invariant under an involution of G. Moreover, the action on the tubular neighborhood $T_{j,k} \times D^2$ projects to a trivial action on the D^2-factor. In order to do the knot surgery equivariantly, we shall consider the involution on $Y_K \times \mathbb{S}^1$ which is trivial on the Y_K-factor and is by translation on the $\mathbb{S}^1$-factor. Recall that the only requirement in the knot surgery is to preserve the homology class $[\mathrm{pt} \times \partial D^2]$ under the gluing. Since on the $Y_K \times \mathbb{S}^1$ side $\mathrm{pt} \times \partial D^2$ is given by a 0-framed copy of the knot K in Y_K and the involution on $Y_K \times \mathbb{S}^1$ is chosen to be trivial on the Y_K-factor, it follows easily that for any fixed fibered knot K, one can do the knot surgery simultaneously on each of the 12 tori $T_{j,k}$ with the knot K, such that the G-action on X_0 can be extended to a symplectic G-action on the resulting 4-manifold X_K. Moreover, X_K continues to be simply connected as repeated knot surgeries on parallel copies is equivalent to a single knot surgery using the connected sum of the knots (cf. Example 1.3 in [5]). Hence X_K is a symplectic homotopy $K3$ surface with

$$sw_{X_K} = \Delta_K(t_1)^4 \Delta_K(t_2)^4 \Delta_K(t_3)^4$$

where $t_j = \exp(2[T_{j,0}])$. Note that the three tori $T_{1,0}$, $T_{2,0}$ and $T_{3,0}$ are homologically linearly independent, so that X_K is maximally exotic, i.e., $r_X = 3$. By nature of construction, the G-action on X_K is clearly pseudofree and induces a trivial action on the lattice L_X of the Seiberg-Witten basic classes.

Finally, one obtains infinitely many distinct X_K by choosing K with distinct genus. □

Remark 4.3. We would like to explain why the tori in our construction have to be chosen differently than in [9] or [11], and point out that for the same reason our construction can not be extended to the symplectic $K3$ group $(\mathbb{Z}_2)^4$. The key point here is that one has to make sure that each $T_{j,k}$ can only be invariant under a cyclic subgroup of G. Otherwise, we will be forced to introduce a nontrivial cyclic action on the factor Y_K in $Y_K \times \mathbb{S}^1$. Of course, one way to obtain such a cyclic action on Y_K is to pick a cyclic action on $\mathbb{S}^3$ under which K is invariant, and then do the 0-framed surgery on K equivariantly. The problem is that the action on the tubular neighborhood $T_{j,k} \times D^2$ projects to a trivial action on the D^2-factor, and since under the knot surgery $\mathrm{pt} \times \partial D^2$ is glued to a 0-framed copy of K in Y_K, the action on $\mathbb{S}^3$ which we picked at the beginning has to fix the knot K. However, by the Smith conjecture [20] this is not possible unless K is the unknot. With this understood, we remark that with the choice of tori as in [9] or [11], one can only construct a $(\mathbb{Z}_2)^2$-action on a homotopy $K3$ surface with maximal exoticness. On the other hand, for the group $G = (\mathbb{Z}_2)^4$, our construction would not even yield an effective G-action on a homotopy $K3$ surface with nontrivial exoticness (i.e. $r_X > 0$).

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