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Vortex–soliton complexes in coupled nonlinear Schrödinger equations with unequal dispersion coefficients

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We consider a two-component, two-dimensional nonlinear Schrödinger system with unequal dispersion coefficients and self-defocusing nonlinearities. In this setting, a natural waveform with a nonvanishing background in one component is a vortex, which induces an effective potential well in the second component. We show that the potential well may support not only the fundamental bound state, which forms a vortex–bright (VB) soliton, but also multi-ring excited radial state complexes for suitable ranges of values of the dispersion coefficient of the second component. We systematically explore the existence, stability, and nonlinear dynamics of these states. The complexes involving the excited radial states are weakly unstable, with a growth rate depending on the dispersion of the second component. Their evolution in the case examples considered leads to transformation of the multi-ring complexes into stable VB soliton ones with the fundamental state in the second component.

I. INTRODUCTION

Multi-component nonlinear Schrödinger (NLS) systems emerge in a variety of optical [1] and atomic physics [2, 3] contexts. In the former setting, they model, in particular, the interaction of waves with different carrier wavelengths [4, 5], while in atomic Bose-Einstein condensates (BECs) they apply [in the form of coupled Gross-Pitaevskii (GP) equations] to spinor (or pseudo-spinor) systems, which represent mixed condensates formed by different hyperfine states of the same atom species [6–8], as well as to heteronuclear mixtures composed by different atom species (see, e.g., Ref. [9]).

In such multi-component settings, when the nonlinearity is self-defocusing (self-repulsive), a prototypical example of a one-dimensional self-trapped structure is given by dark-bright (DB) solitons. These are ubiquitous in two-component systems with the self- and cross-repulsion (alias self- and cross-phase-modulation, SPM and XPM, respectively) represented by cubic terms. Since long ago, the DB solitons have drawn much attention in nonlinear optics [10–16], including their realization in pioneering experiments of [17, 18]. More recently, the remarkable control available in the pristine experimental settings of atomic BECs in ultracold gases, such as $^87\text{Rb}$ and $^{23}\text{Na}$, with a multitude of available co-trapped hyperfine states, as well as in heteronuclear mixtures, such as $^{87}\text{Rb}-^{13}\text{K}$ [9], has opened a new gateway to the realization of DB solitons. Indeed, these structures were created in a multitude of state-of-the-art experiments either controllably, or spontaneously, and their pairwise interactions, as well as interactions with external potentials, were studied [19–23]. Related $SO(2)$ rotated DB soliton states, in the form of dark-dark solitons, were also experimentally produced [24, 25].

The formation of the DB solitons is based on the fact that dark solitons are fundamental modes in one-component, one-dimensional (1D) self-defocusing media. These modes, when created in one component of a two-component system, induce an effective potential well in the other component. This potential well features an isolated fundamental bound state, i.e., the ground state (GS), which represents the bright component of the DB soliton complex. The knowledge of the explicit form of the dark soliton enables one to explore the induced potential well, which is generically of the Pöschl-Teller type [26]. It is possible to demonstrate, as done recently [27], that, if the dispersion coefficient in the second component is different from its counterpart in the first component, not only the ground state, but also excited ones can be trapped by the potential well in the second component. When the DB soliton states emerge at their bifurcation point, they have an infinitesimal amplitude of the bright component in the effective potential induced by the action of the SPM term. However, they can be readily continued numerically to finite values of the amplitude. Heteronuclear BEC mixtures with different atomic masses of the components provide a straightforward realization of the coupled GP equations with different dispersion coefficients (inverse atomic masses). In addition, spin-orbit coupled BECs [28] offer the same possibility, for states that coexist in the upper- and lower-energy bands of the linear...
spectrum \[29\]. Finally, in terms of optics, a similar realization is provided by the copropagation of two beams carried by widely different wavelengths in a self-defocusing medium.

It is natural and interesting to extend the concept of DB soliton states to higher dimensions. In particular, the fact that the component carrying delocalized patterns induces an effective trapping potential in the other component, remains valid in this case. In the 2D setting, delocalized states are well-known stable vortices supported by a nonvanishing background \[30\] (vortices were studied in multi-component systems too \[31\]). A vortex in one component generates an effective axisymmetric potential well in the other component of the two-component system, which may trap a bright 2D solitary wave, producing a complex that was given different names in different communities, including a vortex-bright (VB) soliton \[32\], a half-quantum vortex \[33\], a filled-core vortex \[34\], as well as a baby Skyrmion \[35\]. Similar stable two-component modes are “semi-vortices” in the free 2D space with the attractive SPM and XPM terms, which are made stable by the spin-orbit coupling \[36\]; they are composed of a bright vortex soliton in one component, and a bright fundamental one in the other.

It is important to note that the vortex-bright soliton complexes in the self-repulsive setting of a mixture of internal states of \(^{87}\)Rb atoms were created experimentally in the early work of \[34\]. Subsequently, their stability \[32, 37\] and dynamics \[32, 35\] have been examined theoretically. It was shown that these states feature intriguing interactions that decay with the distance \(r\) between them as \(1/r^3\) \[33\]. Pairs of VB soliton complexes can form bound states in atomic BECs, as shown in detail in Ref. \[38\].

Our objective in the present work is to consider VB soliton complexes in the system featuring repulsive SPM and XPM interactions, and different dispersion coefficients of the two components (i.e., different atomic masses in the respective coupled GP equations, or different propagation constants in the coupled NLS equations for optical beams). We aim to generate a broad set of novel families of excited complexes, with the vortex in the first component potentially trapping not only the fundamental bright solitons, but also excited radial states in the second component, represented by confined multi-ring shaped waveforms. We demonstrate that such complexes are possible in the two-component NLS/GP system. The fundamental state among them, the VB soliton complex, is generically found to be stable. On the other hand, the complexes whose bright component is represented by the excited ring-shaped modes are found to be unstable. However, varying the dispersion coefficient of the second component, we conclude that it is possible to render the corresponding instability very weak, and the associated structures very long-lived. Lastly, we showcase basic scenarios of the instability development, inferring that the unstable multi-ring states are typically transformed into the stable VB fundamental ones.

The presentation of the paper is structured as follows. The model is introduced in Section II. In Sec. III, we discuss the computational analysis of the model, presenting both the numerical methods and results. Finally, in Sec. IV, we summarize our findings and mention possible directions for future studies.

### II. THE MODEL AND ANALYTICAL CONSIDERATIONS

Motivated by the above-mentioned realizations in BECs and nonlinear optics, we consider the coupled defocusing GP/NLS system in \((2+1)\) dimensions (two spatial and one temporal). In the scaled form, the relevant system reads:

\[
\begin{align*}
    i\partial_t \Phi_- &= -\frac{D_-}{2} \nabla^2 \Phi_- + \gamma \left( g_1 |\Phi_-|^2 + \sigma_{12} |\Phi_+|^2 \right) \Phi_- + V(x, y)\Phi_- , \\
    i\partial_t \Phi_+ &= -\frac{D_+}{2} \nabla^2 \Phi_+ + \gamma \left( \sigma_{12} |\Phi_-|^2 + g_2 |\Phi_+|^2 \right) \Phi_+ + V(x, y)\Phi_+ ,
\end{align*}
\]

where \(\nabla^2 = \partial_x^2 + \partial_y^2\) is the Laplacian in 2D, while various parameters stand for the dispersion coefficients \(D_\pm\), nonlinearity strength \(\gamma\), and SPM and XPM interaction coefficients \(g_j\) (\(j = 1, 2\)) and \(\sigma_{12}\), respectively. The model may, in principle, also include the external harmonic-oscillator trapping potential \(V(x, y)\), as is commonly the case in atomic BECs, although in what follows we will focus on the case where such a potential is absent i.e., \(V(x, y) = 0\).

The fields \(\Phi_-\) and \(\Phi_+\) will carry the delocalized vortex and bright-soliton components, respectively. From now on, we focus on the basic case of equal interaction coefficients, \(g_{1,2} = \sigma_{12} = 1\), and use rescaling to fix \(D_- = \gamma = 1\), while \(D_+ \equiv D \geq 0\) is the relative dispersion coefficient in the second component.

Notice that in the case of the binary heteronuclear BECs, coefficient \(D\) is determined by the two atomic masses, \(D = m_- / m_+\), while in the case of the spin-orbit coupled BECs, is given by the ratio of the group-velocity dispersion coefficients, as found by the corresponding dispersion relation of the two-component branches \[29\]. On the other hand, in the optics model, time \(t\) is replaced by the propagation coordinate, \(z\), in the corresponding bulk waveguide \[1\], and \(D\) is determined by the carrier wavelengths of the two beams, \(D = \Lambda_+ / \Lambda_-\).

Stationary solutions to Eqs. (1)-(2) with chemical potentials \(\mu_\pm\) (or propagation constants \(-\mu_\pm\), in terms of the optical beams) are looked for as \(\Phi_\pm(x, y, t) = \phi_\pm(x, y) \exp(-i\mu_\pm t)\), reducing Eqs. (1)-(2) to the coupled system of
stationary equations:
\begin{align}
\mu_- \phi_- &= -\frac{1}{2} \nabla^2 \phi_- + \left( |\phi_-|^2 + |\phi_+|^2 \right) \phi_- , \quad (3) \\
\mu_+ \phi_+ &= -\frac{D}{2} \nabla^2 \phi_+ + \left( |\phi_-|^2 + |\phi_+|^2 \right) \phi_+. \quad (4)
\end{align}

Further, we introduce the following ansätze for the stationary fields, with real radial functions \( f_\pm(r) \):
\begin{align}
\phi_-(r, \theta) &= f_-(r) e^{iS\theta}, \\
\phi_+(r, \theta) &= f_+(r) e^{in\theta}.
\end{align}

Here \( r \) is the radial distance, \( \theta \) is the polar angle, and the integer topological charges of the delocalized vortex and bright solitons are \( S \) and \( n \), respectively. Thus, Eqs. (3)-(4) reduce to the radial equations:
\begin{align}
\nabla^2_r f_- - \frac{S^2 f_-}{r^2} - 2 \left( f_-^2 + f_+^2 - \mu_- \right) f_- &= 0, \quad (7) \\
\nabla^2_r f_+ - \frac{n^2 f_+}{r^2} - \frac{2}{D} \left( f_-^2 + f_+^2 - \mu_+ \right) f_+ &= 0, \quad (8)
\end{align}
with
\[ \nabla^2_r = d^2/dr^2 + r^{-1}d/dr \]
being the radial part of the Laplace operator.

As indicated above, our fundamental premise, similar to that adopted in the study of the 1D setting of \([27]\), is that the dark mode (dark soliton in 1D, and vortex in this present case) of the defocusing NLS equation induces an effective potential (via the XPM interaction) in the other component, which in turn gives rise to trapping of the bright-soliton state in it. Thus, Eq. (3) simplifies to the single-component equation,
\[ \nabla^2_r f_- - \frac{S^2 f_-}{r^2} - 2 \left( f_-^2 - \mu_- \right) f_- = 0, \quad (9) \]
in the absence of bright component \( f_+ \). Equation (9) was solved numerically via a fixed-point (Newton-Raphson) iteration (see, Sec. III below for details). Suitable approximate solutions for the vortical waveform are known too (see, e.g. Ref. [39]), and they may be useful as initial guesses for the iterative process described numerically below. From here on, we assume that this iterative process converges to a radial solution for the vortex. This is different from the 1D case, where the dark soliton is available in the commonly known analytical form, and the Pöschl-Teller potential \([27]\) that it formulates in the other component is thus analytically tractable \([27]\). In the 2D system presented in this work, the analysis was completed numerically.

Thus, the resulting vortex profile \( f_- \) (or \( \phi_- \), for given \( S \)) of Eq. (5) plays the role of the background for the weak component \( f_+ \) (or \( \phi_+ \), for given \( n \)). As follows from Eq. (9), the amplitude of the background for the vortex is \( f_- (r = \infty) = \sqrt{\mu_-} \) which (upon rescaling) is set to be \( \mu_- = 1 \), in our numerical computations below. Then, when the solution for the component \( f_+ \) bifurcates from its linear limit (with \( f_+ \to 0 \)), it can be obtained by solving the linear eigenvalue problem
\[ \mathcal{L} f_+ = \mu_+ f_+, \quad (10) \]
for known \( f_- \), where \( \mathcal{L} = - (D/2) \left( \nabla^2_r - n^2/r^2 \right) + f_-^2 \) is a linear operator and \( (\mu_+, f_+) \) is the eigenvalue-eigenvector pair.

It is natural to expect that nonlinear solutions to Eqs. (7)-(8) and (3)-(4), corresponding to the ground and excited states in the linear limit for component \( f_+ \) emerge (bifurcate) at some critical values of \( D \) with the corresponding eigenvalues \( \mu_+ \) of the linear problem based on Eq. (10). These values are found below by performing numerical continuations over the aforementioned parameters.

### III. NUMERICAL ANALYSIS

#### A. Computational methods

In this section, numerical results are presented for the coupled GP/NLS system \([11-22]\). Our analysis addresses the existence, stability, and dynamical evolution of the nonlinear modes under consideration. As concerns the existence and stability, a parametric continuation is performed in the chemical potential \( \mu_+ \) of the bright component for given values of the relative dispersion coefficient \( D \). The corresponding states are thus identified along with their stability
complex eigenvalue quartets. By the Newton-Raphson code has been corroborated upon using a collocation method [40] for solving boundary value problems.

Having identified the stationary states, we turn to the study of their stability, using linearized equations for small perturbations (see Appendix A). In particular, motivated by the decomposition described in [41], an eigenvalue problem [cf. Eq. (A3)] is derived [at order $O(\varepsilon)$] and solved numerically for each azimuthal mode $m$. As a result, the full spectrum is obtained by putting together the spectra for different integer values of $m$. This was done for $m \geq 0$ only ($m = 0, 1, 2, 3, 4$ and $5$ in this work), since the sets of the eigenvalues for $\pm m$ (with $m > 0$) are complex conjugates. The corresponding steady state is classified as a stable one if none of the eigenvalues $\lambda = \lambda_r + i \lambda_i$ has $\lambda_r \neq 0$, given the Hamiltonian nature of our system. Note that two types of instabilities can be thus identified: i) exponential, characterized by a pair of real eigenvalues with $\lambda_i = 0$, and ii) oscillatory instabilities, characterized by complex eigenvalue quartets.

Finally, the results for the spectral stability, obtained from the solution of the eigenvalue problem [see, Eq. (A3)] were corroborated by means of direct simulations of the coupled GP/NLS system [11]-[12]. To do so, a parallel version

FIG. 1: (Color online) Vortex profiles for various values of the topological charge, $S$, as functions of the radial distance, $r$. spectra. When the solutions are predicted to be stable, this is verified by means of direct simulations. For unstable states, the simulations aim to reveal the eventual states into which they are spontaneously transformed.

In our numerical computations, a 1D uniform spatial grid is employed along the radial direction, consisting of $N$ points $r_j = j \delta r$, with $j = 1, \ldots, N$ and lattice spacing $\delta r = 0.05$. The origin is located at $j = 0$, whereas the domain cut-off ($r_{\text{max}}$) is set at $j = N + 1$ (from now on, we fix $r_{\text{max}} = 50$). In this way, both fields $f_{\pm}(r)$ are replaced by their discrete counterparts on the spatial grid, $f_{j, \pm} = f_{\pm}(r_j)$. Then, the radial Laplacian $\nabla_r^2$ in Eqs. (7)-(8), (9) and (10) as well as in Eqs. (A4), (A5), (A9) and (A11) in Appendix A is replaced by second-order central-finite-difference formulas for the first and second derivatives. To secure a well-posed problem, we employ the boundary conditions (BCs) $f_-(r = 0) = 0$ and $df_-/dr(r = \infty) = 0$ for the vortex soliton component, and $df_+/dr(r = 0) = 0$ and $f_+(r = \infty) = 0$ for the bright-soliton one. In particular, the zero-derivative (Neumann) BCs are incorporated into the internal discretization scheme using the first-order backward and forward difference formulas, respectively. Essentially, the zero-derivative (no-flux) BCs are enforced by requiring $f_{N+1,-} = f_{N,-}$ and $f_{0,+} = f_{1,+}$, whereas $f_{0,-} = f_{N+1,+} = 0$, per the corresponding homogeneous Dirichlet BCs.

Our starting point is Eq. (9) for the radial profile $f_-$ of the vortex component. From now on, we fix the vorticities of the vortex and bright-soliton components to be $S = 1$ and $n = 0$, respectively, given that our emphasis is on VB soliton complexes. We solve Eq. (9) by means of the standard Newton-Raphson method, which converges to vortex profile as long as a sufficiently good initial guess is used. To this end, an example of an input to the solver which ensures both the convergence and compliance with error-tolerance criteria is $f_-(r) = \tanh r$. Examples of the radial profile $f_-$ with various values of vorticity $S \geq 1$, satisfying the dependence $f_-(r \to 0) \sim r^S$, are depicted in Fig. 1.

Subsequently, with the background field $f_-$ at hand, we solve the eigenvalue problem (10) numerically, to obtain the corresponding bright component $f_+(r)$ along with the associated chemical potential, $\mu_+$. Our study focuses on the bound states of a given order (the ground state, first excited state, and so on) and value of $D$. Specifically, we determine values of $\mu_+$ corresponding to one of the lowest eigenstates (the ground state corresponds to lowest $\mu_+$, the first excited state is related to the second lowest eigenvalue, and so on) and the corresponding bright eigenfunction $f_+$ is obtained afterwards. This way, the fully nonlinear self-Trapped states of system (7)-(8) can be obtained with the help of the Newton-Raphson scheme, the seed for the respective iterations consisting of the vortex radial profile $f_-(r)$ together with the eigenvalue-eigenvector pair $(\mu_+, f_+)$. In addition, we trace the stationary solutions, for a given value of dispersion coefficient $D$, by performing a numerical continuation with respect to the chemical potential $\mu_+$, by dint of the sequential continuation method, i.e., using the solution for given $\mu_+$, found by the solver, as the seed for the next continuation step. We are thus able to numerically determine not only the range of dispersion coefficient $D$, but also the range of chemical potential $\mu_+$ for each case of interest. The validity of the stationary solutions produced by the Newton-Raphson code has been corroborated upon using a collocation method [40] for solving boundary value problems.

Having identified the stationary states, we turn to the study of their stability, using linearized equations for small perturbations (see Appendix A). In particular, motivated by the decomposition described in [41], an eigenvalue problem [cf. Eq. (A3)] is derived [at order $O(\varepsilon)$] and solved numerically for each azimuthal mode $m$. As a result, the full spectrum is obtained by putting together the spectra for different integer values of $m$. This was done for $m \geq 0$ only ($m = 0, 1, 2, 3, 4$ and $5$ in this work), since the sets of the eigenvalues for $\pm m$ (with $m > 0$) are complex conjugates. The corresponding steady state is classified as a stable one if none of the eigenvalues $\lambda = \lambda_r + i \lambda_i$ has $\lambda_r \neq 0$, given the Hamiltonian nature of our system. Note that two types of instabilities can be thus identified: i) exponential, characterized by a pair of real eigenvalues with $\lambda_i = 0$, and ii) oscillatory instabilities, characterized by complex eigenvalue quartets.

Finally, the results for the spectral stability, obtained from the solution of the eigenvalue problem [see, Eq. (A3)] were corroborated by means of direct simulations of the coupled GP/NLS system [11]-[12]. To do so, a parallel version
(using OpenMP) of the standard fourth-order Runge-Kutta method (RK4) with a fixed time-step of $\delta t = 10^{-4}$ was employed. The simulations were initialized at $t = 0$ using the available stationary solutions. To obtain the latter, we employed the Newton-Raphson method in a two-dimensional rectangular domain for system (3)-(4), using the NITSOL package [42]. The 2D uniform spatial grid was built of $N_x \times N_y$ grid points with $N_x = 251$ and resolution $\delta x = \delta y = 0.08$. Both fields $\phi_{\pm}(x, y)$ were replaced by their discrete counterparts on the 2D spatial grid, i.e., $\phi_{i,j,\pm} = \phi_{\pm}(x_i, y_j)$, with $i = 1, \ldots, N_x$ and $j = 1, \ldots, N_y$. Then, the Laplacian on the rectangular grid is replaced by the second-order central-finite-difference formula. As mentioned above, the Neumann BCs for both fields at edges of the grid were replaced by the first-order forward and backward difference formulas.

Two possible initializations of the direct simulations can be distinguished. On the one hand, we initialized the dynamics in the presence of small (uniformly distributed) random perturbations with amplitude $\varepsilon = 10^{-3}$, added to presumably stable stationary states. An alternative approach was to initialize the evolution using the linearization ansatz $\mathbf{A}\mathbf{1}-\mathbf{A}\mathbf{2}$ for unstable solutions, with $\varepsilon = 10^{-3}$, eigenvector $\mathbf{V}$ and given $m$ corresponding to a (complex) eigenvalue being responsible for the instability. The latter approach helps to stimulate the onset of the expected instability and observe the ensuing dynamics.

**B. Numerical results**

We start by considering the most fundamental state, namely the VB soliton one in Fig. [2]. In this case, as well as in all the other cases that we have considered herein, we find the VB structure (shown, e.g., in the left panel of Fig. [2]) to be stable, as evidenced by the absence of eigenvalues with nonzero real part in the right panel of Fig. [2].

The first excited state is shown in Fig. [3]. The profile displayed in the top left panel demonstrates that the vortex retains its general structure; yet, it develops a modified spatial profile due to the presence of a more complex spatial pattern in the bright component, with a local density ($|f_+|^2$) maximum at the center and another one at the periphery, separated by a zero-crossing, which has the form of a dark ring. The typical linearization spectrum shown in the top right panel illustrates the presence of complex instabilities. It is relevant to stress that, both in this case and in those considered below, the instabilities are associated with quartets of complex eigenvalues (even when their imaginary parts are so small that the eigenvalues may appear as real ones in the figure). The detailed spectra shown in the middle and bottom panels of the figure (cf. Figs. [3(c)]-[3(g)]) make it clear that the lowest perturbation eigenmodes are always prone to instability, especially the ones with $m = 0$ and $m = 1$. For larger values of $D$, higher eigenmodes may become unstable too, and their instabilities may eventually (i.e., for large $\mu_+$) even dominate the respective growth rate. It is also relevant to stress that the oscillatory pattern, featured, especially, by the $m = 0$ mode is associated with the presence of gaps in the spectrum (for our finite-domain computation), which allow the relevant eigenmode to periodically restabilize, before it collides with another one and destabilizes anew. Similar features for other “dark” patterns have long been known (see, in particular, Ref. [43]), and are absent in the infinite-domain limit, where the relevant eigenvalue follows the envelope of the respective mode “trajectory”.

Similar findings were obtained for the second and third excited states in the bright component, see Figs. [4] and [5] respectively. The former state features a triple local density maximum in the bright component, with these maxima separated by two dark rings; the latter state has four local maxima, separated by three dark rings, as shown in the respective top left panels of the figures. One can also observe in the corresponding top right panels, which showcase typical examples of the spectral plane, $(\lambda_r, \lambda_i)$, of eigenvalues $\lambda = \lambda_r + i\lambda_i$, that the number of unstable modes is getting progressively larger with the increase of the order of the state. Some additional relevant observations regarding these figures are as follows. In Fig. [4] we observe that, for sufficiently large dispersion coefficient $D$, eigenvalues of higher-order perturbation modes, including ones for $m = 3$ (and even $m = 4$ and 5) grow fast with $\mu_+$, so that they

![FIG. 2: (Color online) Steady-state profiles (a) of the vortex and bright soliton components (black and blue lines, respectively), and the eigenvalue spectrum (b) corresponding to a ground state with $D = 0.6$ and $\mu_+ = 0.95$.](image)
FIG. 3: (Color online) Bound states and continuation results corresponding to the first excited state in the bright component. Top row: (a) Stationary profiles of the vortex and bright components (the black and blue lines, respectively). (b) The corresponding eigenvalue spectrum for \( D = 0.1 \) and \( \mu_+ = 0.97 \). Middle and bottom rows: The largest real part of the eigenvalues as a function of \( \mu_+ \) at various fixed values of \( D \): (c) \( D = 0.5 \), (d) \( D = 0.4 \), (e) \( D = 0.3 \), (f) \( D = 0.2 \), and (g) \( D = 0.1 \).

play a dominant role in the resulting dynamics, making it different from that in more typical cases of \( m = 0 \) and \( m = 1 \). As for the waveform in Fig. 5 on the other hand, it is relevant to point out that the internal structure of the ring state has a conspicuous feedback effect on the spatial profile of the vortex. In this case, the vortex features a nearly non-monotonic profile. Here, too, higher perturbation eigenmodes, including most notably \( m = 3 \), but also, in some cases, \( m = 2 \), \( m = 4 \), etc., may result in the largest growth rate of the instability.

Having examined the spectral stability of the different states, we now turn to direct simulations to study the evolution of the presented states. First, in Fig. 6, we confirm that the evolution of the fundamental VB soliton branch (where the bright component is the GS of the vortex-induced potential) does not exhibit any instability in long simulations (up to \( t = 2000 \)), even though the solution is initially perturbed.

Nevertheless, the situation is different for the excited states. This is observed, in particular, in the evolution of the structure with the bright component represented by the first excited state displayed in Fig. 7 for the second and third excited states in the bright component the same is shown in Figs. 8 and 9 respectively. In the case of the first excited state, we see in Fig. 7 that the shape becomes elongated, resulting in the breakup of the dark density ring embedded into the bright component. As a result, the bright component gradually transforms into the GS (see, e.g., the right panel in the figure).

In the case of the second excited state shown in Fig. 8 the instability breaks the two dark rings embedded into the bright component. As a result, more “mass” (or optical power) migrates from the outside rings towards the mode’s core, being trapped by the potential well induced by the vortex from the mate component. In this case, the vortex structure is only weakly affected by the instability of the bright component. Eventually (see the panel on the right side of the figure), the bright waveform develops an intense blob at the center, having shed off considerable
amount of radiation. Thus, this solution approaches the GS in the bright component too, as a result of the instability development.

Finally, the waveform featuring the third excited state (including a triple dark ring) in the bright component exhibits a fairly complex evolution, as seen in Fig. 9. The rings get distorted, as is shown in the second column of the figure—the outer nodal line is no longer a ring, while the middle one has already been broken up. In the third column, the outer and middle dark-ring patterns are severely distorted, resulting, eventually (in the right column), in the transfer of the norm of the bright component towards the center, although surrounded by a complex pattern involving multiple nodal structures. Thus, one again sees a tendency for the accumulation of the norm of the bright component towards the center, which suggests the rearrangement of the mode into the GS. Here (as well as in the case of the bright component shaped as the first excited state), the vortex component suffers a more significant feedback from the instability development in the bright one (resulting in complex patterns in the dark component as well). Nevertheless, the central core of the vortex remains intact, thus keeping a stable effective potential trapping the bright waveform.

**IV. CONCLUSION**

We have considered the 2D, two-component GP/NLS system with the self-defocusing nonlinearity, in which a delocalized vortex in one component induces an effective trapping potential in the other (bright) component. The system models heteronuclear BEC mixtures, as well as spin-orbit coupled BECs, and, on the other hand, the copropagation of optical beams carried by different wavelengths. Depending on the relative dispersion parameter of this second component, the effective potential can trap not only the GS (ground state) in the bright component, but also the first,
second and even third excited radial states. This results in complexes with multi-ring solitons in the bright component, which produce a feedback on the vortex in the first component. Among these complexes, the VB (vortex-bright) one, with the GS in the bright component, has been identified as spectrally stable and has been confirmed to generically be dynamically robust. On the other hand, the complexes with the bright component represented by the excited states are unstable, although the instability growth rate is broadly tunable, and may be made quite small, by means of the variation of the relative dispersion parameter of the bright component. The stability and instability, predicted by the analysis of small perturbations, were corroborated by direct simulations. In particular, the unstable complexes with the excited bright component have been shown in the case examples considered herein to spontaneously rearrange into VB modes with the GS in the bright part.

This work paves the way for explorations of related systems. Some of the possibilities are as follows. First, we actually considered only the bright component with zero vorticity, \( n = 0 \) in Eq. (6) (i.e., without the angular momentum). The existence and stability of complexes with a vortical bright component, \( n \neq 0 \), is a very relevant generalization. In particular, the stability may be quite different for the same shape of the vortical bright mode with opposite signs of the vorticity, \( n = \pm 1 \), while \( S = +1 \) is fixed in the first component, cf. the stability of two-component trapped modes with the \textit{hidden vorticity} studied in Ref. [44]. Further, in this work we restrict the considerations to 2D settings, while recent work [45] has shown that 3D vortical structures are capable of trapping bright states. Another possibility is to systematically consider the states reported here in the presence of the external harmonic-oscillator trapping potential, which is very commonly present in experiments with atomic BECs. Finally, the present analysis is restricted to axially symmetric states trapped by the vortex-induced effective potential. It is also interesting to check if an azimuthally modulated state (\textit{azimuthon} [46]) may be produced in the present setup. Some of these extensions are presently under consideration, and will be reported elsewhere.
FIG. 6: (Color online) The evolution of densities $|\Phi_-(x, t)|^2$ and $|\Phi_+(x, t)|^2$ (the top and bottom rows), displayed at different instants of time: $t = 0$ (left panels), $t = 1000$ (middle panels), and $t = 2000$ (right panels), for perturbed complexes with the bright component in the form of the ground state, at $D = 0.6$ and $\mu_+ = 0.95$.

FIG. 7: (Color online) The same as Fig. 6 but for the complexes with the bright component in the form of the first excited state. Top and bottom rows display densities $|\Phi_-(x, t)|^2$ and $|\Phi_+(x, t)|^2$, respectively, at $t = 0$ (left panels), $t = 80$ (middle panels), and $t = 160$ (right panels) for $D = 0.1$, $\mu_+ = 0.97$ and $m = 2$. 
FIG. 8: (Color online) Same as Fig. but for the complexes with the bright component in the form of the second excited state. Top and bottom rows display densities $|\Phi_-(x, t)|^2$ and $|\Phi_+(x, t)|^2$, respectively, at $t = 0$ (left panels), $t = 210$ (middle panels), and $t = 430$ (right panels), for $D = 0.05$, $\mu_+ = 0.95$ and $m = 2$.

FIG. 9: (Color online) Same as Fig. but for the complexes with the bright component represented by the third excited state. Top and bottom rows display densities $|\Phi_-(x, t)|^2$ and $|\Phi_+(x, t)|^2$, respectively, at $t = 0$ (panels (a) and (e)), $t = 130$ (panels (b) and (f)), $t = 180$ (panels (c) and (g)), and $t = 230$ (panels (d) and (h)), for $D = 0.04$, $\mu_+ = 0.9825$ and $m = 4$.

Appendix A: Stability analysis

We start with the perturbation ansatz around stationary solutions, expressed in terms of the polar coordinates

$$
\Phi_-(r, \theta, t) = e^{-i\mu t} e^{iS\theta} \left\{ f_0 + \varepsilon \sum_{|m|=0}^{\infty} \left[ a_m(r)e^{im\theta} + b_m^*(r)e^{-im\theta} \right] \right\},
$$

(A1)
\[ \Phi_+ (r, \theta, t) = e^{-i \mu + t} e^{i \theta} \left\{ f_+^0 + \varepsilon \sum_{|m|=0}^\infty \left[ c_m(r) e^{\lambda t} e^{i m \theta} + d_m^*(r) e^{\lambda^* t} e^{-i m \theta} \right] \right\}, \] (A2)

where \( \lambda \) is the (complex) eigenvalue, \( \varepsilon \) is an infinitesimal amplitude of the perturbation, and the asterisk stands for complex conjugate. We insert Eqs. (A1)–(A2) into Eqs. (1)–(2) and thus obtain, at order \( \varepsilon \), an eigenvalue problem in the following matrix form:

\[ \lambda \begin{pmatrix} a_m \\ b_m \\ c_m \\ d_m \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ -A^*_{12} & A_{22} & -A^*_{14} & -A^*_{13} \\ A^*_{13} & A_{14} & A_{33} & A_{34} \\ -A^*_{14} & -A^*_{13} & -A^*_{34} & A_{44} \end{pmatrix} \begin{pmatrix} a_m \\ b_m \\ c_m \\ d_m \end{pmatrix}, \] (A3)

with eigenfrequencies \( \tilde{\lambda} = i \lambda \), eigenvectors \( \mathbf{\nu} = (a_m, b_m, c_m, d_m)^T \), and matrix elements given by

\[
\begin{align*}
A_{11} &= -\frac{D_2^2}{2} \left[ \nabla_r^2 - \frac{(S + m)^2}{r^2} \right] + \gamma \left[ 2 g_1 |f_-^0|^2 + \sigma_{12} |f_+^0|^2 \right] + V - \mu_-, \\
A_{12} &= \gamma g_1 \left( f_-^0 \right)^2, \\
A_{13} &= \gamma \sigma_{12} f_0^+ \left( f_-^0 \right)^* \\
A_{14} &= \gamma \sigma_{12} f_0^- \left( f_+^0 \right)^* \\
A_{22} &= -\frac{D_2^2}{2} \left[ \nabla_r^2 - \frac{(S - m)^2}{r^2} \right] - \gamma \left[ 2 g_1 |f_-^0|^2 + \sigma_{12} |f_+^0|^2 \right] - (V - \mu_-), \\
A_{33} &= -\frac{D_2^2}{2} \left[ \nabla_r^2 - \frac{(n + m)^2}{r^2} \right] + \gamma \left[ \sigma_{12} |f_-^0|^2 + 2 g_2 |f_+^0|^2 \right] + V - \mu_+, \\
A_{34} &= \gamma g_2 \left( f_-^0 \right)^2, \\
A_{44} &= -\frac{D_2^2}{2} \left[ \nabla_r^2 - \frac{(n - m)^2}{r^2} \right] - \gamma \left[ \sigma_{12} |f_-^0|^2 + 2 g_2 |f_+^0|^2 \right] - (V - \mu_+).
\end{align*}
\]

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