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Pair Creation of Dilaton Black Holes

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Abstract

We consider dilaton gravity theories in four spacetime dimensions parametrised by a constant a, which controls the dilaton coupling, and construct new exact solutions. We first generalise the C-metric of Einstein-Maxwell theory ($a = 0$) to solutions corresponding to oppositely charged dilaton black holes undergoing uniform acceleration for general a. We next develop a solution generating technique which allows us to "embed" the dilaton C-metrics in magnetic dilaton Melvin backgrounds, thus generalising the Ernst metric of Einstein-Maxwell theory. By adjusting the parameters appropriately, it is possible to eliminate the nodal singularities of the dilaton C-metrics. For $a < 1$ (but not for $a \geq 1$), it is possible to further restrict the parameters so that the dilaton Ernst solutions have a smooth euclidean section with topology $S^2 \times S^2 - \{\text{pt}\}$, corresponding to instantons

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describing the pair production of dilaton black holes in a magnetic field. A different restriction on the parameters leads to smooth instantons for all values of a with topology $S^2 \times \mathbb{R}^2$.

1. Introduction

The idea that the topology of space might change in a quantum theory of gravity is an old one [1]. The "canonical" approach to quantum gravity, however, rules out the possibility from the start by taking the configuration space to be the space of three-geometries on a *fixed* three-manifold and the "covariant" approach assumes a fixed background *space-time*. Thus, the most natural framework for quantum gravity in which to investigate topology changing processes seems to be the sum-over-histories. In the sum-over-histories formulation a topology changing transition amplitude is given by a functional integral over four-metrics on four-manifolds (cobordisms) with boundaries which agree with the initial and final states. What conditions to place on the metrics summed over is a matter for some debate. One approach is to only sum over euclidean metrics [2]. Another proposal is to sum over almost everywhere lorentzian metrics, restricting the metrics to be causality preserving (i.e. no closed time-like curves), in which case the issue of the necessary singularities must be broached [3]. Although such functional integrals are ill-defined as yet, one can still do calculations by assuming that they can be well approximated by saddle point methods. An instanton, a euclidean solution that interpolates between the initial and final states of a classically forbidden transition, is a saddle point for both the "euclidean" and "lorentzian" functional integrals. We take the existence of an instanton as an indication that the transition has a finite rate and must be taken into consideration.

One such instanton in Einstein-Maxwell theory is the euclideanised Ernst metric [4] which is interpreted as describing the pair production of two magnetically charged Reissner-Nordstrom black holes in a Melvin magnetic universe [5,6]. This is the gravitational analogue of the Schwinger pair production of charged particles in a uniform electromagnetic field. In this process the topology of space changes from $\mathbb{R}^3$ to $S^2 \times S^1 - \{\mathrm{pt}\}$ corresponding to the formation of two oppositely charged black holes whose throats are connected by a handle. The calculation of the rate of this process leads to the observation that it is enhanced over the production rate of monopoles by a factor $e^{S_{bh}}$ where S_{bh} is the Hawking-Bekenstein entropy of the black holes [7]. This supports the notion that the entropy counts the number of "internal" states of the black hole.

Are similar processes described by instantons in other theories containing gravity? It is known, for example, that both the low energy limit of string theory [8,9] and 5-dimensional Kaluza-Klein theory [10,11,12] admit a family of charged black hole solutions. One may ask if instantons exist which describe their pair production. An action which includes all the above mentioned theories describes the interaction between a dilaton, a $U(1)$ gauge field and gravity and is given by

$$S = \int d^4x\sqrt{-g}\left[R - 2(\nabla\phi)^2 - e^{-2a\phi}F^2\right]. \tag{1.1}$$

For $a = 0$ this is just standard Einstein-Maxwell theory. For $a = 1$ it is a part of the action describing the low-energy dynamics of string theory, while for $a = \sqrt{3}$ it arises from 5-dimensional Kaluza-Klein theory. For each value of a, there exists a two parameter family of black hole solutions (which we shall briefly review in section 2) labelled by the mass m and the magnetic (or electric) charge q. That topology changing instantons for (1.1) exist, at least for $a < 1$, describing the pair creation of such black holes, will be one of the results of this paper.

It is important to note that we have used the "Einstein" metric to describe these theories. Metrics rescaled by a dilaton dependent factor are also of physical interest and may have different causal structures. For example, in string theory ($a = 1$) the "sigma-model" metric $g_\sigma = e^{2\phi}g$ is the metric that couples to the string degrees of freedom. If we consider the magnetically charged black holes in this theory, then for $m > \sqrt{2}q$ both the Einstein metric and the sigma model metric have a singularity cloaked by an event horizon. However, in the extremal limit, $m = \sqrt{2}q$, the Einstein metric has a naked singularity, whereas in the sigma model metric the singularity disappears from the space-time, down an infinitely long tube. In this limit the sigma model metric is geodesically complete and moreover the upper bound on the curvature can be made as small as one likes by choosing q large enough.

These properties of the sigma model metric are part of the motivation for using the $a = 1$ theory to further understand the issue of information loss in the scattering of matter with extremal black holes. The low-energy scattering of particles with such an extremal black hole, including the effects of back reaction on the metric, has been studied in [13,14,15]. One truncates to the s-wave sector of the theory and considers an effective two-dimensional theory defined in the throat region. Using semi-classical techniques, it has been argued that there may exist an infinite number of near degenerate states corresponding

to massless modes propagating down the throat. It was conjectured in [13,14,15] that these remnants or "cornucopions" are the end-points of Hawking evaporation. The infinite length of the throat allows for an arbitrarily large number of remnants, which can then store an arbitrarily large amount of information.

One objection to this scenario is that if an infinite number of such remnants exist, then we may expect them to each have a finite probability of being pair created. The infinite number of species would then lead to divergences in ordinary quantum field theory processes. A way around this objection was proposed in [16], where the rate of production of these remnants in a magnetic field was estimated using instanton methods. It was argued that the rate of pair production is not infinite, because the instanton would produce a pair of throats connected by a finite length handle. The finite length of the throat would then imply that only a finite number of remnants could be excited and that the total production rate would be finite. A shortcoming of the arguments in [16], however, was that no exact instanton solutions were constructed. Looking for such exact solutions was one of the motivations for the present work.

The plan of the rest of the paper is as follows. In section 2 we present the dilaton generalisations of the C-metric for arbitrary dilaton coupling a. These describe two oppositely charged dilaton black holes accelerating away from each other. We show that, just as in the Einstein-Maxwell C-metric, there exist nodal singularities in the metric which cannot be removed by any choice of period for the azimuthal coordinate. These can be thought of as providing the forces necessary to accelerate the black holes. In Einstein-Maxwell theory, string theory and Kaluza-Klein theory there are known transformations which generate new solutions starting from a known static, axisymmetric solution. In section 3, we show that such generating transformations exist for all a, and take flat space into dilaton magnetic Melvin universes. When applied to the C-metrics, these same transformations give dilaton generalisations of the Ernst solution; choosing the parameters appropriately, the magnetic field can provide exactly the right amount of acceleration to remove the nodal singularities. In section 4, we discuss the euclidean section of the dilaton Ernst solutions. To obtain a regular geometry, it is necessary that the Hawking temperatures of the black hole and acceleration horizons be equal. For $a < 1$, we find that it is possible to do this at non-zero temperature, and one obtains natural generalisations of the $q = m$ instantons discussed in [5,6] with topology $S^2 \times S^2 - \{\text{pt}\}$. These instantons describe the formation of a Wheeler wormhole on a spatial slice of a magnetic dilaton Melvin universe. For all values of a, it is possible to obtain a smooth euclidean section in the limit that the two horizons

have zero temperature. These instantons have topology $S^2 \times \mathbb{R}^2$. The physical interpretation of these instantons, however, is unclear. Section 5 is a summary and discussion of our results.

2. Dilaton C-metrics

2.1. Charged Black Holes in Dilaton Gravity

The equations of motion coming from the action (1.1) are given by

$$\begin{aligned} &\nabla_\mu(e^{-2a\phi}F^{\mu\nu}) = 0 \\ &\nabla^2\phi + \frac{a}{2}e^{-2a\phi}F^2 = 0 \\ &R_{\mu\nu} = 2\nabla_\mu\phi\nabla_\nu\phi + 2e^{-2a\phi}F_{\mu\rho}F_\nu^{\ \rho} - \frac{1}{2}g_{\mu\nu}e^{-2a\phi}F^2. \end{aligned} \tag{2.1}$$

These equations are invariant with respect to an electric-magnetic duality transformation, under which the metric is unchanged and the new field strength $\tilde{F}$ and dilaton $\tilde{\phi}$ are given by

$$\tilde{F}_{\mu\nu} = \tfrac{1}{2}e^{-2a\phi}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \qquad \tilde{\phi} = -\phi. \tag{2.2}$$

We will only consider the magnetically charged solutions below, but because of this duality our results also apply to the electric case.

For given a the equations of motion (2.1) admit a two parameter family of magnetically charged black hole solutions given by [1] [8][9]

$$\begin{aligned} &ds^2 = -\lambda^2 dt^2 + \lambda^{-2}dr^2 + R^2(d\theta^2 + \sin^2\theta d\varphi^2) \\ &e^{-2a\phi} = \left(1 - \frac{r_-}{r}\right)^{\frac{2a^2}{(1+a^2)}}, \qquad A_\varphi = q\cos\theta \\ &\lambda^2 = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^{\frac{(1-a^2)}{(1+a^2)}}, \qquad R^2 = r^2\left(1 - \frac{r_-}{r}\right)^{\frac{2a^2}{(1+a^2)}}. \end{aligned} \tag{2.3}$$

Assuming $r_+ > r_-$, then r_+ is the location of a black hole horizon. For $a = 0$, r_- is the location of the inner Cauchy horizon, however for $a > 0$ the surface $r = r_-$ is singular. The parameters r_+ and r_- are related to the ADM mass m and total charge q by

$$m = \frac{r_+}{2} + \left(\frac{1-a^2}{1+a^2}\right)\frac{r_-}{2}, \qquad q = \left(\frac{r_+r_-}{1+a^2}\right)^{\frac{1}{2}}. \tag{2.4}$$

[1] To obtain solutions where the dilaton asymptotically approaches an arbitrary constant ϕ_0, one can use the fact that the action is invariant under $\phi \to \phi + \phi_0$, $F \to e^{a\phi_0}F$ and the metric left unchanged. We will suppress ϕ_0 in the following.

The extremal limit occurs when $r_+ = r_-$.

Following [8] we introduce the "total metric", ds_T^2, defined via a conformal rescaling of the "Einstein" metric

$$ds_T^2 = e^{2\phi/a} ds^2. \tag{2.5}$$

For certain values of a this metric naturally appears in Kaluza-Klein theories [8]. For $a = 1$ this is just the sigma model metric that couples to the string degrees of freedom. For $a < 1$, in the extremal limit, the total metric is geodesically complete and the spatial sections have the form of two asymptotic regions joined by a wormhole, one region being flat, the other having a deficit solid angle. For $a = 1$ the geometry is that of an infinitely long throat.

2.2. Dilaton C-Metric

In Einstein-Maxwell theory the C-metric can be interpreted as the spacetime corresponding to two Reissner-Nordstrom black holes of opposite charge undergoing uniform acceleration [17]. The generalisation of this spacetime to dilaton gravity is given by

$$\begin{aligned}
ds^2 &= \frac{1}{A^2(x-y)^2}\left[F(x)\left\{G(y)dt^2 - G^{-1}(y)dy^2\right\} + F(y)\left\{G^{-1}(x)dx^2 + G(x)d\varphi^2\right\}\right] \\
e^{-2a\phi} &= \frac{F(y)}{F(x)}, \qquad A_\varphi = qx, \qquad F(\xi) = (1 + r_- A\xi)^{\frac{2a^2}{(1+a^2)}} \\
G(\xi) &= \bar{G}(\xi)(1 + r_- A\xi)^{\frac{(1-a^2)}{(1+a^2)}}, \qquad \bar{G}(\xi) = \left[1 - \xi^2(1 + r_+ A\xi)\right].
\end{aligned} \tag{2.6}$$

Note that the form of G as a product of two terms is quite similar to the form of λ in (2.3) and further, $\bar{G}$ is the cubic which appears in the uncharged C-metric in [17]. The parameters q, r_- and r_+ are related as in (2.4).

The metric (2.6) can be shown to give various known metrics in the appropriate limits. Setting $r_- = 0$ gives the uncharged C-metric (a vacuum solution and independent of a). Setting $a = 0$ gives the charged C-metric of Einstein-Maxwell theory, but in a slightly non-standard form: the function G is a quartic with a linear term. To compare with the form of the C-metric given in [17] one needs to change coordinates to obtain a quartic with no linear term. The appropriate transformations are discussed in [17].

In the limit of zero acceleration, the metric (2.6) reduces to the metric (2.3) for a single charged dilaton black hole. To see this, it is useful to use new coordinates given by

$$r = -\frac{1}{Ay}, \qquad T = A^{-1}t. \tag{2.7}$$

In these coordinates the metric (2.6) becomes

$$
\begin{aligned}
ds^2 = \frac{1}{(1+Arx)^2}\big[F(x)\left\{-H(r)dT^2 + H^{-1}(r)dr^2\right\} \\
+ R^2(r)\left\{G^{-1}(x)dx^2 + G(x)d\varphi^2\right\}\big] \\
H(r) = (1-\frac{r_+}{r} - A^2r^2)(1-\frac{r_-}{r})^{\frac{1-a^2}{(1+a^2)}},
\end{aligned}
\tag{2.8}
$$

where the function $R(r)$ is the same as that appearing in (2.3). Setting $A = 0$ and $x = \cos\theta$, we return to the metric (2.3) of the dilaton black holes.

The metric (2.6) has two Killing vectors, $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \varphi}$. For the range of parameters $r_+A < 2/(3\sqrt{3})$, the function $\bar{G}(\xi)$ has three real roots. Denote these in ascending order by ξ_2, ξ_3 and ξ_4 and define $\xi_1 \equiv -\frac{1}{r_-A}$. One can show $\xi_3 < \xi_4$ and we further restrict the parameters so that $\xi_1 < \xi_2$. The surface $y = \xi_1$ is singular for $a > 0$, as can be seen from the square of the field strength. This surface is analogous to the singular surface (the 'would be' inner horizon) of the dilaton black holes. The surface $y = \xi_2$ is the black hole horizon and the surface $y = \xi_3$ is the acceleration horizon; they are both Killing horizons for $\frac{\partial}{\partial t}$.

The coordinates (x, φ) in (2.6) are angular coordinates. To keep the signature of the metric fixed, the coordinate x is restricted to the range $\xi_3 \leq x \leq \xi_4$ in which $G(x)$ is positive. The norm of the Killing vector $\frac{\partial}{\partial \varphi}$ vanishes at $x = \xi_3$ and $x = \xi_4$, which correspond to the poles of spheres surrounding the black holes. The axis $x = \xi_3$ points along the symmetry axis towards spatial infinity. The axis $x = \xi_4$ points towards the other black hole[2]. Spatial infinity is reached by fixing t and letting both y and x approach ξ_3. Letting $y \to x$ for $x \neq \xi_3$ gives null or timelike infinity [18]. Since $y \to x$ is infinity, the range of the coordinate y is $-\infty < y < x$ for $a = 0$, $\xi_1 < y < x$ for $a > 0$.

2.3. Nodal Singularities

As is the case with the ordinary C-metric, it is not generally possible to choose the range of φ such that the metric (2.6) is regular at both $x = \xi_3$ and $x = \xi_4$. In order to see this, in a neighbourhood of each root define a new coordinate θ according to

$$
\theta = \int_{\xi_i}^{x} \frac{dx'}{\sqrt{G(x')}}. \tag{2.9}
$$

[2] The coordinates we are using only cover the region of spacetime where one of the black holes is.

The angular part of the metric near one the poles $x = \xi_i$, $i = 3, 4$ then has the form

$$dl^2 \approx \frac{F(y)}{A^2(\xi_i - y)^2}\left(d\theta^2 + \frac{1}{4}\lambda_i^2\theta^2 d\varphi^2\right), \tag{2.10}$$

where $\lambda_i = |G'(\xi_i)|$ and one can show that $\lambda_3 < \lambda_4$. Let the range of φ be $0 < \varphi \leq \alpha$, then the deficit angles at the two poles δ_3, δ_4 are given by

$$\delta_3 = 2\pi - \tfrac{1}{2}\alpha\lambda_3 \qquad \delta_4 = 2\pi - \tfrac{1}{2}\alpha\lambda_4 \tag{2.11}$$

We can remove the nodal singularity at $x = \xi_4$ by choosing $\alpha = 4\pi/\lambda_4$, but then there is a positive deficit angle running along the ξ_3 direction. This corresponds to the black holes being pulled by "cosmic strings" of positive mass per unit length $\mu = 1 - \lambda_3/\lambda_4$. Alternatively, we can choose $\alpha = 4\pi/\lambda_3$ to remove the nodal singularity at ξ_3. This means there is a negative deficit angle along the ξ_4 direction, which can be interpreted as the black holes being pushed apart by a "rod" of mass per unit length $\mu = 1 - \lambda_4/\lambda_3$ (which is negative). For a general choice of α, there will be nodal singularities on both sides. The mass per unit length of the outer singularity will always be greater than that on the inside.

There is a degenerate case when the metric is free from nodal singularities. Letting $r_+A = 2/(3\sqrt{3})$ the roots ξ_2 and ξ_3 of the cubic $\bar{G}$ become coincident. In this limit the range of x becomes $\xi_3 < x \leq \xi_4$ since the proper distance between ξ_3 and ξ_4 diverges. The point $x = \xi_3$ disappears from the (x, φ) section which is no longer compact but becomes topologically $\mathbb{R}^2$, the sphere gaining an infinitely long tail. One can eliminate the nodal singularity at $x = \xi_4$ by choosing $\alpha = 4\pi/\lambda_4$. It might seem that the acceleration and black hole horizons become coincident in this limit. This is not the case, however. The proper distance between the horizons (at fixed x and t) tends to a constant as $r_+A \to 2/(3\sqrt{3})$. This case will be discussed further in section 4.

3. Generating Dilaton Ernst

3.1. Generating new solutions

In the case of vanishing dilaton coupling ($a = 0$), Ernst [4] has shown that the nodal singularities can be removed by including a magnetic field of the proper strength running along the symmetry axis. The magnetic field provides the force necessary to accelerate the black holes. The magnetic field can be added to the C-metric via an Ehlers-Harrison type transformation [19], which takes an axisymmetric solution of the Einstein-Maxwell

equation into another such solution. The same transformation applied to flat spacetime produces Melvin's magnetic universe [20], which is the closest one can get to a constant magnetic field in general relativity. To follow the same path as Ernst, we first need to generalise the solution generating technique that he employed to dilaton gravity.

In the case of Kaluza-Klein theory ($a = \sqrt{3}$), this turns out to be quite simple. It is known that the charged black holes in Kaluza-Klein theory can be generated from the uncharged ones (*i.e.* Schwarzschild with an extra compact spatial dimension) by applying a coordinate transformation mixing the time and internal coordinates, a "boost", and then re-identifying the new internal coordinate [12]. Similarly, we can add magnetic field along the symmetry axis of an axisymmetric solution to Kaluza-Klein theory by doing a transformation mixing the internal and azimuthal coordinates, a "rotation". Applied to flat space, this transformation reproduces the dilaton Melvin solution given in [8] for $a = \sqrt{3}$. Together with the known form of the transformation without the dilaton field, the Kaluza-Klein case provides sufficient clues to guess the correct transformation for general dilaton coupling. In the string theory case ($a = 1$) this turns out to be one of the $O(1,2)$ transformations[3] that is known to act on the space of axisymmetric solutions [21].

Let $(g_{\mu\nu}, A_\mu, \phi)$ be an axisymmetric solution of (2.1). That is, all the fields are independent of the azimuthal coordinate φ. Let the other three coordinates be denoted by $\{x^i\}$. Suppose also that $A_i = g_{i\varphi} = 0$.[4] Then a new solution of (2.1) is given by

$$\begin{aligned} g'_{ij} &= \Lambda^{\frac{2}{1+a^2}} g_{ij}, \qquad g'_{\varphi\varphi} = \Lambda^{-\frac{2}{1+a^2}} g_{\varphi\varphi}, \\ e^{-2a\phi'} &= e^{-2a\phi} \Lambda^{\frac{2a^2}{1+a^2}}, \qquad A'_\varphi = -\frac{2}{(1+a^2)B\Lambda}(1 + \frac{(1+a^2)}{2} B A_\varphi), \\ \Lambda &= (1 + \frac{(1+a^2)}{2} B A_\varphi)^2 + \frac{(1+a^2)}{4} B^2 g_{\varphi\varphi} e^{2a\phi} \end{aligned} \tag{3.1}$$

The proof is presented in appendix A. This transformation generalises the "rotation"-type transformation of Kaluza-Klein theory to all a. Although we will not write the explicit transformations, it is possible to generalise the "boost"-type of transformation to all a, also.

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[3] To obtain the full $O(1,2)$ group one needs to include the antisymmetric tensor field in the action (1.1).

[4] We can relax this condition and construct new solutions only assuming axisymmetry. However, the transformations are somewhat more involved and will not be needed for our purposes.

3.2. Dilaton Melvin

Applying these transformations to flat space in cylindrical coordinates we obtain the dilaton Melvin solutions given by Gibbons and Maeda [8],

$$\begin{aligned} ds^2 &= \Lambda^{\frac{2}{1+a^2}}\left[-dt^2+d\rho^2+dz^2\right]+\Lambda^{-\frac{2}{1+a^2}}\rho^2 d\varphi^2 \\ e^{-2a\phi} &= \Lambda^{\frac{2a^2}{1+a^2}}, \qquad A_\varphi = -\frac{2}{(1+a^2)B\Lambda} \\ \Lambda &= 1+\frac{(1+a^2)}{4}B^2\rho^2 \end{aligned} \tag{3.2}$$

The parameter B gives the strength of the magnetic field on the axis via $B^2 = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}\big|_{\rho=0}$.

3.3. Dilaton Ernst

Applying the transformation (3.1) to the dilaton C-metric finally yields the dilaton Ernst solution.

$$\begin{aligned} ds^2 &= (x-y)^{-2}A^{-2}\Lambda^{\frac{2}{1+a^2}}\left[F(x)\left\{G(y)dt^2-G^{-1}(y)dy^2\right\}+F(y)G^{-1}(x)dx^2\right] \\ &\quad +(x-y)^{-2}A^{-2}\Lambda^{-\frac{2}{1+a^2}}F(y)G(x)d\varphi^2 \\ e^{-2a\phi} &= \Lambda^{\frac{2a^2}{1+a^2}}\frac{F(y)}{F(x)}, \qquad A_\varphi = -\frac{2}{(1+a^2)B\Lambda}(1+\frac{(1+a^2)}{2}Bqx) \\ \Lambda &= (1+\frac{(1+a^2)}{2}Bqx)^2+\frac{(1+a^2)B^2}{4A^2(x-y)^2}(1-x^2-r_+Ax^3)(1+r_-Ax) \end{aligned} \tag{3.3}$$

Defining $\mathcal{G}(y,x) = \Lambda^{-\frac{2}{1+a^2}}G(x)$ the nodal singularities of the C-metric will be removed if the period of ϕ is chosen to be $4\pi/|\partial_x\mathcal{G}|_{\xi_3}$ and we impose $|\partial_x\mathcal{G}|_{\xi_3} = |\partial_x\mathcal{G}|_{\xi_4}$. In the limit $r_\pm A \ll 1$, this constraint yields Newton's law

$$mA \approx Bq, \tag{3.4}$$

where m and q are identified in terms of r_+ and r_- according to (2.4).

Note that we can read off the dilaton Melvin metric in "accelerated" coordinates from (3.3) by setting $r_+ = r_- = 0$. The metric functions then reduce to

$$F = 1, \qquad G(x) = 1-x^2, \qquad \Lambda = 1+\frac{(1+a^2)B^2}{4A^2(x-y)^2}(1-x^2). \tag{3.5}$$

This form of the dilaton Melvin solution is useful for studying in what sense the dilaton Ernst solution approaches dilaton Melvin. This is discussed in detail in Appendix B. Here

we note that, if the value of the physical magnetic field parameter B_p is defined to be $\sqrt{\frac{1}{2}F_{\mu\nu}F^{\mu\nu}}$ on the axis as $y \to \xi_3$,

$$B_p = \frac{1}{2} \left. \frac{\partial_x G}{\Lambda^{\frac{3}{2}}} \right|_{x=\xi_3} B_e. \tag{3.6}$$

where B_e is the parameter that appears in (3.3). This value of B_p is also the amount of flux per unit area across a small area transverse to the axis in the limit $y \to \xi_3$ as shown in Appendix B. In the limit $r_\pm A \ll 1$ this reduces to $B_p = B_e$. Further, as discussed in Appendix B, we find coordinates in which the dilaton Ernst metric (3.3) approaches the dilaton Melvin metric (3.2) near the outer axis for $r = \frac{1}{x-y} \to \infty$.

4. Dilaton Instantons

The above solutions describe two dilaton black holes accelerating away from each other along the axis of a dilaton Melvin magnetic universe. Euclideanising (3.3) by setting $\tau = it$, we find that, just as in the $a = 0$ case, another condition must be imposed on the parameters in order to obtain a regular solution. The condition arises in order to eliminate conical singularities at both the black hole and acceleration horizons with a single choice of the period of τ. This is equivalent to demanding that the Hawking temperatures of the two horizons are equal.

In terms of the metric function $G(y)$ appearing in (3.3), the period of τ is taken to be $4\pi/|G'(\xi_2)|$ and the constraint is

$$|G'(\xi_2)| = |G'(\xi_3)|\,, \tag{4.1}$$

yielding

$$\left(\frac{\xi_2 - \xi_1}{\xi_3 - \xi_1}\right)^{\frac{1-a^2}{1+a^2}} (\xi_4 - \xi_2)(\xi_3 - \xi_2) = (\xi_4 - \xi_3)(\xi_3 - \xi_2). \tag{4.2}$$

Recall that we have restricted our parameters so that $\xi_4 > \xi_3 \geq \xi_2 > \xi_1$. For all values of a, (4.2) can be solved by demanding that the horizons have zero temperature i.e. $\xi_2 = \xi_3$. We will refer to these as type I instantons. For $a < 1$, the first factor on the left hand side of (4.2) is smaller than one, and there is also a second solution, which we will call type II instantons. For $a \geq 1$, the first factor on the left is greater than one, corresponding to the temperature of the black hole horizon always being greater than the temperature of the acceleration horizon, and there are no other solutions.

We first consider the type II solutions with $a < 1$. These generalise the regular euclidean metrics considered in the $a = 0$ case [5,6,7]; the condition (4.2) is the analogue of the $q = m$ condition on the parameters discussed in those papers. In the limit $r_{\pm}A << 1$, for $a < 1$ one can show that the condition (4.2) leads to $r_+ = r_-$. Since we have chosen the parameters so that there are no nodal singularities on the t, y and x, φ spheres, it is clear that the topology of these spacetimes is $S^2 \times S^2 - \{\text{pt}\}$ where the removed point is $x = y = \xi_3$.

This instanton is readily interpreted as a bounce: the surface defined by $\tau = 0$ and $\tau = \pi$ has topology $S^2 \times S^1 - \text{pt}$, which is that of a wormhole attached to a spatial slice of Melvin and is the zero momentum initial data for the lorentzian ernst solution. In addition the solution tends to the Melvin solution at euclidean infinity (see appendix B). The bounce describes the pair creation of a pair of oppositely charged dilaton black holes in a magnetic field which subsequently uniformly accelerate away from each other. From the metric we deduce that there is a horizon sitting inside the wormhole throat, located at a finite proper distance from the mouth.

Turning to the type I instantons, we note that this is again the case of two coincident roots, which has been discussed above in section 2.3 for $B = 0$. There it was pointed out that there are no nodal singularities even in the absence of a magnetic field, and that consequently there is no restriction on the value of B. The apparent coincidence of the two horizons is an artifact of a poor choice of coordinates and it can be shown that the proper distance between the horizons remains finite as the roots become coincident. Below, we exhibit a coordinate change, originally used by Ginsparg and Perry to study Schwarzschild-DeSitter instantons [22], which makes this explicit for the C-metrics.

In (2.6) let $r_+A = 2/(3\sqrt{3}) - \epsilon^2/\sqrt{3}$, so that the limit of coincident roots is $\epsilon \to 0$. Specifically, introducing $y_0 = -\sqrt{3}(1 + \frac{7}{6}\epsilon^2)$ the roots to order ϵ^2 are

$$\begin{aligned} \xi_{2,3} &= y_0 \mp \sqrt{3}\epsilon \\ \xi_4 &= \frac{\sqrt{3}}{2}(1 + \frac{1}{6}\epsilon^2). \end{aligned} \tag{4.3}$$

Writing the dilaton C-metric (2.6) in the coordinates

$$\chi = \cos^{-1}\left(\frac{1}{\sqrt{3}\epsilon}\,(y - y_0)\right), \quad \psi = \sqrt{3}\epsilon t \tag{4.4}$$

and taking the limit $\epsilon \to 0$, then gives

$$ds^2 = A^{-2}(x+\sqrt{3})^{-2}\Bigg[F(x)\left\{-F(-\sqrt{3})^{\frac{1-a^2}{2a^2}}\sin^2\chi d\psi^2 + F(-\sqrt{3})^{\frac{a^2-1}{2a^2}}d\chi^2\right\}$$
$$+ F(-\sqrt{3})\left\{G^{-1}(x)dx^2 + G(x)d\varphi^2\right\}\Bigg], \qquad A_\varphi = qx, \tag{4.5}$$
$$e^{-2a\phi} = \frac{F(-\sqrt{3})}{F(x)}, \qquad G(x) = -\frac{2}{3\sqrt{3}}(x+\sqrt{3})^2(x-\sqrt{3}/2)(1+r_-Ax)^{\frac{1-a^2}{1+a^2}}.$$

We may then apply the solution generating transformations (3.1) with arbitrary parameter B, though we will not do this explicitly here. Euclideanising (4.5) by setting $\Psi = i\psi$, we see that it is possible to eliminate the conical singularities at the north and south poles of the (Ψ, χ) section by making Ψ periodic with period $2\pi\left(1-\sqrt{3}r_-A\right)^{\frac{(a^2-1)}{(2a^2)}}$. Indeed, the (Ψ, χ) section is a round sphere and the topology of this solution is $S^2 \times \mathbb{R}^2$, in contrast to the type II instantons. It is not clear what, if any, the physical significance of these instantons may be.

5. Discussion

The instantons presented in section 4 suggest that topology changing processes can occur in dilaton gravity for $a < 1$. Specifically, the type II instantons describe the pair creation in a uniform magnetic field of an oppositely charged pair of $a < 1$ dilaton black holes. The rate of production of these black holes can be estimated in the semi-classical approximation by calculating the action [23].

It is interesting that the type II instantons exist only for $a < 1$. The interpretation of the type I instantons which exist for all a is unclear, especially since they exist for any value of the magnetic field. It therefore appears to be difficult to estimate the production rate of charged black holes in theories with $a \geq 1$ using semi-classical techniques.

Including additional matter fields may yield one way of modifying the type II solutions to obtain instantons for $a \geq 1$. In particular, in [24] it was argued that in the Einstein-Maxwell-Higgs theory which admits cosmic strings, euclidean solutions exist which correspond to a string world sheet wrapped around the horizon of a black hole. The effect of the string is to cut out a "wedge" from the (r, t) section of euclidean Schwarzschild, an effect which could be approximated in a vacuum theory by allowing a conical singularity at the horizon with a specified deficit. Similarly, for $a = 1$ say, cosmic strings could be added to

the model (1.1), in which case the type II instanton with a certain conical singularity in the (t, y) section could describe the (cosmic string induced) pair creation of $a = 1$ black holes.

Another interesting possibility is that the physics of black hole pair production for $a \geq 1$ is not so simply related to regular euclidean instantons. In reference [25] it was shown that the thermodynamic behavior of charged black holes with $a > 1$ differs from that given by the naive interpretation of their euclidean sections. For $0 \leq a < 1$ the temperature of the extremal black holes goes to zero and Hawking radiation is extinguished, as one would expect. For $a > 1$, however, the temperature of a black hole, given by the periodicity of its euclidean section, diverges as extremality is approached, a result that was regarded as puzzling [9]. This puzzle was partially resolved in [25], where it was shown that, for $a > 1$, the Hawking radiation is in fact shut off by infinite grey body factors. For $a = 1$ the temperature of the extremal black hole approaches a constant and the results of the analysis in [25] are inconclusive. In our case, demanding regularity of the euclidean section of the dilaton Ernst metric is equivalent to requiring that the black hole be in thermal equilibrium with the acceleration radiation. It is tempting to suspect that the inability to achieve this (for nonzero temperature) for $a \geq 1$ is somehow related to the physics uncovered in [25]. The study of black hole pair production with $a \geq 1$ would then require more subtle methods.

In the introduction, we summarised the cornucopion scenario for resolving the paradoxes associated with information loss in the scattering of particles from extremal $a = 1$ dilaton black holes. In this scenario it is important that the extremal black hole is non-singular, with an infinitely long throat leading to a second null infinity. Since the extremal black hole geometries (using the total metric (2.5)) for $0 < a < 1$ also have this property, it is natural to conjecture that these models all admit cornucopion type scenarios. Moreover, the instantons we have constructed above indicate that for $0 < a < 1$ there may not be a problematic infinite pair production of cornucopions in a magnetic field. Firstly, we expect the action of the instanton to be finite. Furthermore, since the created wormholes have finite length it would appear that if one included matter fields and calculated the one-loop determinants, one would not be including the infinite number of states living far down the static wormholes[5]. In conclusion, one expects the pair production rate of cornucopions to be finite since a cornucopion is not an elementary particle but is deeply interconnected

[5] We note however, that this logic has been questioned in ref. [26].

with the geometry of spacetime. As we have noted, we cannot say anything definite about the case $a = 1$. It would be interesting to understand the implications of our exact results for the approximate instantons presented in [16].

Note Added: We note that the dilaton Ernst solution (3.3) does not asymptote to the dilaton Melvin solution (3.2) although the metrics do match. A comparison of the gauge field and dilaton for (3.3) and (3.2) shows that the solutions are related by a constant shift in the dilaton and rescaling of the gauge field as in footnote 1. The details of this are given in [23].

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Appendix A.

Suppose that we have a solution to (2.1) that is axisymmetric, i.e. independent of the azimuthal coordinate φ, and further satisfies $A_i = g_{i\varphi} = 0$, where x^i are the other three coordinates. We prove that the transformations (3.1) generate a new solution by showing that the the transformations leave the action (1.1) invariant. We first rewrite the action in terms of the rescaled total metric (2.5) to obtain

$$S = \int d^4x \sqrt{-g_T} e^{-2\phi/a} \left[R_T + \left(\frac{6-2a^2}{a^2} \right) (\nabla\phi)^2 - e^{2\phi\frac{1-a^2}{a}} F^2 \right] \tag{A.1}$$

where indices are raised with the inverse of the total metric.

Introducing the definitions

$$\begin{aligned} {}^3g_{ij} &= g_{Tij}, \qquad\qquad V = g_{T\varphi\varphi} \\ \tilde{\phi} &= \phi - \frac{a}{4}\log V, \end{aligned} \tag{A.2}$$

we can recast the action into the form

$$S = \alpha \int d^3x \sqrt{-{}^3g} e^{-2\tilde{\phi}/a} \big[{}^3R + \frac{6-2a^2}{a^2} \partial_i \tilde{\phi} \partial^i \tilde{\phi} - \frac{a^2-1}{a} V^{-1} \partial_i \tilde{\phi} \partial^i V \\ - \frac{1+a^2}{8} V^{-2} \partial_i V \partial^i V - 2 e^{2\tilde{\phi} \frac{1-a^2}{a}} V^{-\frac{1+a^2}{2}} \partial_i A_\varphi \partial^i A_\varphi \big] \tag{A.3}$$

where we have carried out the integration over φ assuming its range is $0 \leq \varphi \leq \alpha$. The virtue of these definitions is that the transformations (3.1) now take the simple form

$$\begin{aligned} {}^3g'_{ij} &= {}^3g_{ij} \\ V' &= \Lambda^{\frac{-4}{1+a^2}} V \\ \tilde{\phi}' &= \tilde{\phi} \\ A'_\varphi &= -\frac{2}{(1+a^2)B\Lambda} (1 + \frac{(1+a^2)}{2} B A_\varphi) \end{aligned} \tag{A.4}$$

To complete the proof a straightforward calculation shows that the Lagrangian is invariant under these transformations.

Appendix B.

B.1. Determination of the magnetic flux

Here we will determine the value of the physical magnetic field parameter, B_p. The magnetic flux through a small area around the axis is given by

$$\begin{aligned} \Delta\text{flux} = \int da_{\hat{y}} B^{\hat{y}} &= \int d\varphi dx \partial_x A_\varphi \\ &\simeq \left. \frac{\partial A_\varphi}{\partial x} \right|_{x=\xi_3} \Delta\varphi \Delta x \end{aligned} \tag{B.1}$$

The flux per unit area in the limit $y \to \xi_3$ in dilaton Ernst is then given by

$$B_p = \left. \frac{\Delta\text{flux}}{\Delta\text{area}} \right|_{x=\xi_3} = \tfrac{1}{2} \left. \frac{\partial_x G}{\Lambda^{\frac{3}{2}}} \right|_{x=\xi_3} B_e. \tag{B.2}$$

and in the limit $r_\pm A \ll 1$ reduces to $B_p = B_e$. It is the same relation one gets from just matching $F_{\mu\nu} F^{\mu\nu}$ on the axis.

B.2. Matching the metric near the axis

We now show that the dilaton Ernst metric approaches the dilaton Melvin metric near the outer axis as $r \to \infty$. Note that for the euclidean section the outer axis, $x, y \to \xi_3$, is the only place where $r \to \infty$. We start with the dilaton Melvin metric expressed in accelerated coordinates, as discussed in section 3.3. There is then an acceleration parameter $\bar{A}$ at our disposal in the matching. Near the axis $G(x) \simeq \lambda_3(x - \xi_3)$, where $\lambda_3 = \partial_x G|_{x=\xi_3}$. Make the following coordinate transformation in the dilaton Ernst metric (3.3)

$$\bar{t} = \tfrac{1}{2}\lambda_3 t, \qquad \bar{y} = -\frac{y}{\xi_3}, \qquad \bar{\varphi} = \frac{\lambda_3 \varphi}{2\Lambda(\xi_3)^{\frac{2}{1+a^2}}}, \qquad \bar{\rho} = \left(\frac{2(\xi_3 - x)}{\xi_3}\right)^{\frac{1}{2}}. \tag{B.3}$$

Near the axis the dilaton Ernst metric then has the form

$$ds^2_{\text{ernst}} \simeq \frac{-2\xi_3 F(\xi_3)\Lambda(\xi_3)^{\frac{2}{1+a^2}}}{\lambda_3(1+\bar{y}^2)A^2\xi_3^2}\left[2(\bar{y}+1)d\bar{t}^2 - \frac{d\bar{y}^2}{2(1+\bar{y})} + \bar{\rho}^2 d\bar{\varphi}^2 + d\bar{\rho}^2\right] \tag{B.4}$$

We can make a similar coordinate transformation on the accelerated form of the dilaton Melvin metric near the axis to get

$$ds^2_{\text{melvin}} \simeq \frac{1}{\bar{A}^2(1+\bar{y})^2}\left[2(\bar{y}+1)d\bar{t}^2 - \frac{d\bar{y}^2}{2(1+\bar{y})} + \bar{\rho}^2 d\bar{\varphi}^2 + d\bar{\rho}^2\right] \tag{B.5}$$

These two are the same if we identify the acceleration of the Melvin coordinate system $\bar{A}$ according to

$$\frac{1}{\bar{A}^2} = -\frac{2}{A^2\lambda_3\xi_3}F(\xi_3)\Lambda(\xi_3)^{\frac{2}{1+a^2}} \tag{B.6}$$

Since the choice of $\bar{A}$ is just a choice of coordinates, this shows that the two metrics are the same near the axis.

We note that in the preceding calculation, we took the limit $x, y \to \xi_3$ in the manner

$$x - \xi_3 \to 0, \qquad y - \xi_3 \sim (x - \xi_3)^q, \qquad q < \tfrac{1}{2} \tag{B.7}$$

which lets $r^2 G(x) \to 0$ on the axis. Choosing $q > \frac{1}{2}$ gives an artificially singular slicing of the spacetime. Taking the limit for $q = \frac{1}{2}$ shifts the constant $\Lambda(\xi_3)$.

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