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THE SINGULAR SUPPORTS OF IC SHEAVES ON QUASIMAPS’ SPACES ARE IRREDUCIBLE

MICHAEL FINKELBERG, ALEXANDER KUZNETSOV, AND IVAN MIRKOVIĆ

1. Introduction

1.1. Let $C$ be a smooth projective curve of genus 0. Let $\mathcal{B}$ be the variety of complete flags in an $n$-dimensional vector space $V$. Given an $(n-1)$-tuple $\alpha \in \mathbb{N}[I]$ of positive integers one can consider the space $Q_\alpha$ of algebraic maps of degree $\alpha$ from $C$ to $\mathcal{B}$. This space is noncompact. Some remarkable compactifications $Q^D_\alpha$ (Quasimaps), $Q^L_\alpha$ (Quasiflags) of $Q_\alpha$ were constructed by Drinfeld and Laumon respectively. In [Ku] it was proved that the natural map $\pi : Q^L_\alpha \to Q^D_\alpha$ is a small resolution of singularities. The aim of the present note is to study the singular support of the Goresky-MacPherson sheaf $IC_\alpha$ on the Quasimaps’ space $Q^D_\alpha$.

Namely, we prove that this singular support $SS(IC_\alpha)$ is irreducible. The proof is based on the factorization property of Quasimaps’ space and on the detailed analysis of Laumon’s resolution $\pi : Q^L_\alpha \to Q^D_\alpha$.

We are grateful to P.Schapira for the illuminating correspondence.

This note is a sequel to [Ku] and [FK]. In fact, the local geometry of $Q^D_\alpha$ was the subject of [Ku]; the global geometry of $Q^D_\alpha$ was the subject of [FK], while the microlocal geometry of $Q^D_\alpha$ is the subject of the present work. We will freely refer the reader to [Ku] and [FK].

2. Reductions of the main theorem

2.1. Notations.

2.1.1. We choose a basis $\{v_1, \ldots, v_n\}$ in $V$. This choice defines a Cartan subgroup $H \subset G = SL(V) = SL_n$ of matrices diagonal with respect to this basis, and a Borel subgroup $B \subset G$ of matrices upper triangular with respect to this basis. We have $B = G/B$.

Let $I = \{1, \ldots, n-1\}$ be the set of simple coroots of $G = SL_n$. Let $R^+$ denote the set of positive coroots, and let $2p = \sum_{\theta \in R^+} \theta$. For $\alpha = \sum a_i i \in \mathbb{N}[I]$ we set $|\alpha| := \sum a_i$. Let $X$ be the lattice of weights of $G, H$. Let $X^+ \subset X$ be the set of dominant (with respect to $B$) weights. For $\lambda \in X^+$ let $V_\lambda$ denote the irreducible representation of $G$ with the highest weight $\lambda$.

Recall the notations of [Ku] concerning Kostant’s partition function. For $\gamma \in \mathbb{N}[I]$ a Kostant partition of $\gamma$ is a decomposition of $\gamma$ into a sum of positive coroots with multiplicities. The set of Kostant partitions of $\gamma$ is denoted by $\mathcal{K}(\gamma)$.

There is a natural bijection between the set of pairs $1 \leq q \leq p \leq n-1$ and $R^+$, namely, $(p,q)$ corresponds to $i_q + i_{q+1} + \ldots + i_p$. Thus a Kostant partition $\kappa$ is given by a collection of nonnegative integers $(\kappa_{p,q})$, $1 \leq q \leq p \leq n-1$. Following loc. cit. (9) we define a collection $\mu(\kappa)$ as follows: $\mu_{p,q} = \sum_{r \leq q \leq p} \kappa_{r,s}$.

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Recall that for $\gamma \in \mathbb{N}[I]$ we denote by $\Gamma(\gamma)$ the set of all partitions of $\gamma$, i.e. multisubsets (subsets with multiplicities) $\Gamma = \{\{\gamma_1, \ldots, \gamma_k\}\}$ of $\mathbb{N}[I]$ with $\sum_{r=1}^{k} \gamma_r = \gamma$, $\gamma_r > 0$ (see e.g. [Ku], 1.3).

The configuration space of colored effective divisors of multidegree $\gamma$ (the set of colors is $I$) is denoted by $C(\gamma)$. The diagonal stratification $C(\gamma) = \bigcup_{\Gamma \in \Gamma(\gamma)} C_\Gamma$ was introduced e.g. in loc. cit. Recall that for $\Gamma = \{\{\gamma_1, \ldots, \gamma_k\}\}$ we have $\dim C_\Gamma = k$.

2.1.2. For the definition of Laumon’s Quasiflags’ space $Q^L_\alpha$ the reader may consult [La] 4.2, or [Ku] 1.4. It is the space of complete flags of locally free subsheaves $0 \subset E_1 \subset \cdots \subset E_{n-1} \subset V \otimes O_C =: V$ such that $\text{rank}(E_k) = k$, and $\text{deg}(E_k) = -a_k$.

It is known to be a smooth projective variety of dimension $2|\alpha| + \dim B$.

2.1.3. For the definition of Drinfeld’s Quasimaps’ space $Q^D_\alpha$ the reader may consult [Ku] 1.2. It is the space of collections of invertible subsheaves $L_\lambda \subset V_\lambda \otimes O_C$ for each dominant weight $\lambda \in X^+$ satisfying Plücker relations, and such that $\text{deg} L_\lambda = -(\lambda, \alpha)$.

It is known to be a (singular, in general) projective variety of dimension $2|\alpha| + \dim B$.

The open subspace $Q_\alpha \subset Q^D_\alpha$ of genuine maps is formed by the collections of line subbundles (as opposed to invertible subsheaves) $L_\lambda \subset V_\lambda \otimes O_C$. In fact, it is an open stratum of the stratification by the type of degeneration of $Q^D_\alpha$ introduced in [Ku] 1.3:

$$Q^D_\alpha = \bigsqcup_{\Gamma \in \Gamma(\alpha - \beta)} D^{\beta, \Gamma}$$

We have $D_{\alpha, \emptyset} = Q_\alpha$, and $D^{\beta, \Gamma} = Q_\beta \times C_{\Gamma}^{\alpha - \beta}$ (see loc. cit. 1.3.5).

The space $Q^D_\alpha$ is naturally embedded into the product of projective spaces

$$\mathbb{P}_\alpha = \prod_{1 \leq p \leq n-1} \mathbb{P}(\text{Hom}(O_C(-\langle \omega_p, \alpha \rangle), V_{\omega_p} \otimes O_C))$$

and is closed in it (see loc. cit. 1.2.5). Here $\omega_p$ stands for the fundamental weight dual to the coroot $i_p$.

The fundamental representation $V_{\omega_p}$ equals $\Lambda_i$.

2.2. We will study the characteristic cycle of the Goresky-MacPherson perverse sheaf (or the corresponding regular holonomic $D$-module) $IC_\alpha$ on $Q^D_\alpha$. As $Q^D_\alpha$ is embedded into the smooth space $\mathbb{P}_\alpha$, we will view this characteristic cycle $SS(IC_\alpha)$ as a Lagrangian cycle in the cotangent bundle $T^*\mathbb{P}_\alpha$. A priori we have the following equality:

$$SS(IC_\alpha) = T^*\mathbb{P}_\alpha + \sum_{\beta < \alpha} m_{\alpha, \beta} T^*D^{\beta, \Gamma}_{\alpha, \beta} \mathbb{P}_\alpha,$$

closures of conormal bundles with multiplicities.

Theorem. $SS(IC_\alpha) = T^*\mathbb{P}_\alpha$ is irreducible.

In the following subsections we will reduce the Theorem to a statement about geometry of Laumon’s resolution.
2.3. We fix a coordinate $z$ on $C$ identifying it with the standard $\mathbb{P}^1$. We denote by $Q^\infty_\alpha \subset Q^D_\alpha$ the open subspace formed by quasimaps which are genuine maps in a neighbourhood of the point $\infty \in C$. In other words, $(L_\lambda \subset \mathcal{V}_\lambda \otimes \mathcal{O}_C)_{\lambda \in X^+} \in Q^\infty_\alpha$ iff for each $\lambda$ the invertible subsheaf $L_\lambda \subset \mathcal{V}_\lambda \otimes \mathcal{O}_C$ is a line subbundle in some neighbourhood of $\infty \in C$.

Evidently, $Q^\infty_\alpha$ intersects all the strata $D_{\beta, \Gamma}$. Thus it suffices to prove the irreducibility of the singular support of Goresky-MacPherson sheaf of $Q^\infty_\alpha$.

There is a well-defined map of evaluation at $\infty \in C$:

$$\Upsilon_\alpha : Q^\infty_\alpha \to B$$

It is compatible with the stratification of $Q^\infty_\alpha$ and realizes $Q^\infty_\alpha$ as a (stratified) fibre bundle over $B$. In effect, $G$ acts naturally both on $Q^\infty_\alpha$ (preserving stratification) and on $B$; the map $\Upsilon_\alpha$ is equivariant, and $B$ is homogeneous. We denote the fiber $\Upsilon_\alpha^{-1}(B)$ over the point $B \in B$ by $Z_\alpha$.

It inherits the stratification

$$Z_\alpha = \bigsqcup_{\beta \leq \alpha} Z^\beta, \Gamma$$

from $Q^\infty_\alpha$ and $Q^D_\alpha$. It is just the transversal intersection of the fiber $\Upsilon_\alpha^{-1}(B)$ with the stratification of $Q^\infty_\alpha$. As in [K] 1.3.5 we have $Z^\beta, \Gamma \sim \to Z_\beta \times (C - \infty)^{r - \beta}$.

Hence it suffices to prove the irreducibility of the singular support $SS(IC(Z_\alpha))$ of Goresky-MacPherson sheaf $IC(Z_\alpha)$ of $Z_\alpha$.

2.4. Factorization. The Theorem 6.3 of [FM] admits the following immediate Corollary. Let

$$(\phi_\beta, \gamma_1 x_1, \ldots, \gamma_k x_k) = \phi_\alpha \in Z_\beta \times (C - \infty)^{r - \beta} = Z^\beta, \Gamma \subset Z_\alpha.$$ Consider also the points

$$(\phi_\beta, \gamma_1 x_1) = \phi_\gamma \in Z_0 \times (C - \infty)^{r - \beta} = Z^0, \Gamma \subset Z_\gamma, 1 \leq r \leq k.$$ Proposition. There is an analytic open neighbourhood $U_\alpha$ (resp. $U_\beta$, resp. $U_\gamma$, $1 \leq r \leq k$) of $\phi_\alpha$ (resp. $\phi_\beta$, resp. $\phi_\gamma$, $1 \leq r \leq k$) in $Z_\alpha$ (resp. $Z_\beta$, resp. $Z_\gamma$, $1 \leq r \leq k$) such that

$$U_\alpha \sim \to U_\beta \times \prod_{1 \leq r \leq k} U_{\gamma_r}$$

Recall the nonnegative integers $m^\beta, \Gamma_\alpha$ introduced in 2.2. The Proposition implies the following Corollary.

Corollary. $m^\beta, \Gamma_\alpha = \prod_{1 \leq r \leq k} m^{0, \Gamma \gamma_r}$.

Thus to prove that all the multiplicities $m^\beta, \Gamma_\alpha$ vanish, it suffices to check the vanishing of $m^{0, \Gamma \gamma_\alpha}$ for arbitrary $\gamma > 0$.

2.5. It remains to prove that the conormal bundle $T^{\ast}_{\mathcal{B}^{0, \Gamma \gamma_\alpha}} \mathbb{P}_\alpha$ to the closed stratum of $Q_\gamma$ enters the singular support $SS(IC_\alpha)$ with multiplicity 0. To this end we choose a point $(B, \gamma_0) = \phi \in B \times C = Q_0 \times C^{\gamma_0} = D^{\gamma_0, \Gamma \gamma_0} \subset Q_\gamma \subset \mathbb{P}_\gamma$. We also choose a sufficiently generic meromorphic function $f$ on $\mathbb{P}_\gamma$ regular around $\phi$ and vanishing on $D^{0, \Gamma \gamma_0}$.

According to the Proposition 8.6.4 of [KS], the multiplicity in question is 0 iff $\Phi_f(IC_\gamma)_{\phi} = 0$, i.e. the stalk of vanishing cycles sheaf at the point $\phi$ vanishes.

To compute the stalk of vanishing cycles sheaf we use the following argument, borrowed from [BFL] §1. As $\pi : Q^L_\gamma \to Q^D_\gamma$ is a small resolution of singularities, up to a shift, $IC_\alpha = \pi_* Q^L_\gamma$. By the proper base change, $\Phi_f \pi_* Q^L_\gamma = \pi_* \Phi_f Q^D_\gamma$. So it suffices to check that $\Phi_f Q^D_\gamma|_{\pi^{-1}(\phi)} = 0$. 


Let us denote the differential of the function \( f \) at the point \( \phi \) by \( \xi \) so that \( (\phi, \xi) \in T^*_\mathbb{F}_{\phi} \mathbb{P} \). Then the support of \( \Phi f \mathbb{Q}_{\phi} \mathbb{Q}^{\pi-1}(\phi) \) is a priori contained in the microlocal fiber over \( (\phi, \xi) \) which we define presently.

2.5.1. Definition. Let \( \varpi : A \to B \) be a map of smooth varieties. For \( a \in A \) let \( d_a^s \varpi : T^*_{\mathbb{P}(a)} B \to T^*_a A \) denote the codifferential, and let \( (b, \eta) \) be a point in \( T^* B \). Then the microlocal fiber of \( \varpi \) over \( (b, \eta) \) is defined to be the set of points \( a \in \varpi^{-1}(b) \) such that \( d^s_a \varpi(\eta) = 0 \).

2.5.2. Thus we have reduced the Theorem 2.2 to the following Proposition.

Proposition. For a sufficiently generic \( \xi \) such that \( (\phi, \xi) \in T^*_\mathbb{F}_{\phi} \mathbb{P} \), the microlocal fiber of Laumon’s resolution \( \pi \) over \( (\phi, \xi) \) is empty. Equivalently, the cone \( \cup_{E_\bullet \in \pi^{-1}(\phi)} \ker(d_{E_\bullet}^s \pi) \) is a proper subvariety of the fiber of \( T^*_\mathbb{F}_{\phi} \mathbb{P} \) at \( \phi \).

2.6. Piecification of a simple fiber. The fiber \( \pi^{-1}(\phi) \) was called the simple fiber in Ku §2. It was proved in loc. cit. 2.3.3 that \( \pi^{-1}(\phi) \) is a disjoint union of (pseudo)affine spaces \( \mathcal{S}(\mu(\kappa)) \) where \( \kappa \) runs through the set \( \mathcal{A}(\gamma) \) of Kostant partitions of \( \gamma \) (for the notation \( \mu(\kappa) \) see 2.1.1 or Ku (9)). Another way to parametrize these pseudoaffine pieces was introduced in FK 2.11. Let us recall it here.

We define nonnegative integers \( c_p, 1 \leq p \leq n - 1 \), so that \( \gamma = \sum_{p=1}^{n-1} c_p p \).

2.6.1. Definition. \( \mathcal{D}(\gamma) \) is the set of collections of nonnegative integers \( (d_{p,q})_{1 \leq q \leq p \leq n-1} \) such that

a) For any \( 1 \leq q \leq p \leq r \leq n - 1 \) we have \( d_{r,q} \leq d_{p,q} \);

b) For any \( 1 \leq p \leq n - 1 \) we have \( \sum_{q=1}^{p} d_{p,q} = c_p \).

2.6.2. Lemma. The correspondence \( \kappa = (\kappa_{p,q})_{1 \leq q \leq p \leq n-1} \mapsto (d_{p,q} := \sum_{r=p}^{n-1} \kappa_{r,q})_{1 \leq q \leq p \leq n-1} \) defines a bijection between \( \mathcal{A}(\gamma) \) and \( \mathcal{D}(\gamma) \). □

2.6.3. Using the above Lemma we can rewrite the parametrization of the pseudoaffine pieces of the simple fiber as follows:

\[ \pi^{-1}(\phi) = \bigcup_{\mathfrak{d} \in \mathcal{D}(\gamma)} \mathcal{S}(\mathfrak{d}) \]

In these terms the dimension formula of Ku 2.3.3 reads as follows: for \( \mathfrak{d} = (d_{p,q})_{1 \leq q \leq p \leq n-1} \) we have \( \dim \mathcal{S}(\mathfrak{d}) = \sum_{1 \leq q \leq p \leq n-1} d_{p,q} \).

Note also that \( \sum_{1 \leq q \leq p \leq n-1} d_{p,q} = \sum_{1 \leq p \leq n-1} c_p = |\gamma| \).

2.7. Proposition. For arbitrary \( \mathfrak{d} = (d_{p,q})_{1 \leq q \leq p \leq n-1} \in \mathcal{D}(\gamma) \) and arbitrary quasiflag \( E_\bullet \in \mathcal{S}(\mathfrak{d}) \subset \pi^{-1}(\phi) \) we have \( \dim \ker(d_{E_\bullet}^s \pi) \leq \sum_{1 \leq p \leq n-1} d_{p,q} + \sum_{1 \leq q \leq p \leq n-1} d_{p,q} - 1 \).

This Proposition implies the Proposition 2.5.2 straightforwardly. In effect, \( \text{codim} \ker(d_{E_\bullet}^s \pi) = \dim \mathbb{Q}^{\pi-1}(\phi) - \dim \ker(d_{E_\bullet}^s \pi) > 2|\gamma| + \dim \mathcal{B} - \sum_{1 \leq p \leq n-1} d_{p,p} - \sum_{1 \leq q \leq p \leq n-1} d_{p,q} + 1 = \dim \mathcal{B} + 1 + \sum_{1 \leq q \leq p \leq n-1} d_{p,q} \).

Hence the subspace \( \ker(d_{E_\bullet}^s \pi) \subset T^*_\mathbb{F}_{\phi} \mathbb{P} \) has codimension greater than \( \dim \mathcal{B} + 1 + \sum_{1 \leq q \leq p \leq n-1} d_{p,q} \).

Recall that \( \dim \mathbb{Q}^{\pi-1}(\phi) = \dim \mathcal{B} + 1 \). Hence the codimension of \( \ker(d_{E_\bullet}^s \pi) \cap T^*_\mathbb{F}_{\phi} \mathbb{P} \) in the fiber of \( T^*_\mathbb{F}_{\phi} \mathbb{P} \) at \( \phi \) is greater than \( \sum_{1 \leq q \leq p \leq n-1} d_{p,q} = \dim \mathcal{S}(\mathfrak{d}) \). Hence the cone \( \cup_{E_\bullet \in \mathcal{S}(\mathfrak{d})} \ker(d_{E_\bullet}^s \pi) \) is a proper subvariety of the fiber of \( T^*_\mathbb{F}_{\phi} \mathbb{P} \) at \( \phi \).
The union of these proper subvarieties over \( \mathfrak{d} \in \mathcal{D}(\gamma) \) is again a proper subvariety of the fiber of \( T^*_{\mathcal{O}_\gamma,\{\gamma\}} \mathbb{P}_\gamma \) at \( \phi \) which concludes the proof of the Proposition 2.5.2.

2.8. Fixed points. It remains to prove the Proposition 2.7. To this end recall that the Cartan group \( H \) acts on \( V \) and hence on \( Q^L_\alpha \). The group \( \mathbb{C}^* \) of dilations of \( C = \mathbb{P}^1 \) preserving 0 and \( \infty \) also acts on \( Q^L_\alpha \) commuting with the action of \( H \). Hence we obtain the action of a torus \( T := H \times \mathbb{C}^* \) on \( Q^L_\alpha \).

It preserves the simple fiber \( \pi^{-1}(\phi) \) and its pseudoaffine pieces \( \mathfrak{S}(\mathfrak{d}) \), \( \mathfrak{d} \in \mathcal{D}(\gamma) \), for evident reasons. It was proved in [FK] 2.12 that each piece \( \mathfrak{S}(\mathfrak{d}) \), \( \mathfrak{d} = (d_{p,q})_{1 \leq q \leq p \leq n-1} \) contains exactly one \( T \)-fixed point \( \delta(\mathfrak{d}) = (E_1, \ldots, E_{n-1}) \). Here

\[
\begin{align*}
E_1 &= E_{1,1} \\
E_2 &= E_{2,1} \oplus E_{2,2} \\
&\vdots \\
E_{n-1} &= E_{n-1,1} \oplus E_{n-1,2} \oplus \ldots \oplus E_{n-1,n-1}
\end{align*}
\]

and \( E_{p,q} = \mathcal{O}(-d_{p,q}) \subset \mathcal{O}_{V_q} \subset V = V \otimes \mathcal{O}_C \) with quotient sheaf \( \mathcal{O}/\mathcal{O}(-d_{p,q}) \) concentrated at 0 \( \in C \).

2.8.1. Now the \( T \)-action contracts \( \mathfrak{S}(\mathfrak{d}) \) to \( \delta(\mathfrak{d}) \). Since the map \( \pi \) is \( T \)-equivariant, and the dimension of \( \text{Ker}(d_{E,\pi}) \) is lower semicontinuous, the Proposition 2.7 follows from the next one.

**Key Proposition.** For arbitrary \( \mathfrak{d} = (d_{p,q})_{1 \leq q \leq p \leq n-1} \in \mathcal{D}(\gamma) \) (\( \gamma \neq 0 \)) we have \( \dim \text{Ker}(d_{\delta(\mathfrak{d}),\pi}) < \sum_{1 \leq p \leq n-1} d_{p,p} + \sum_{1 \leq q \leq p \leq n-1} d_{p,q} - 1 \).

The proof will be given in the next section.

2.8.2. Remark. In general, the pieces \( \mathfrak{S}(\mathfrak{d}) \) of the simple fiber are not equisingular, i.e. \( \dim \text{Ker}(d_{E,\pi}) \) is not constant along a piece. The simplest example occurs for \( G = SL_3 \), \( \gamma = 2i_1 + 2i_2 \). Then the simple fiber is a singular 2-dimensional quadric. Its singular point is the fixed point of the 1-dimensional piece \( \mathfrak{S}(\mathfrak{d}) \) where \( d_{1,1} = 2, d_{2,1} = d_{2,2} = 1 \). At this point we have \( \dim \text{Ker}(d_{\delta(\mathfrak{d}),\pi}) = 3 \) while at the other points in this piece we have \( \dim \text{Ker}(d_{E,\pi}) = 2 \).

3. The proof of the Key Proposition

3.1. **Tangent spaces.** Let \( \Omega \) be the following quiver: \( \Omega = 1 \rightarrow 2 \rightarrow \ldots \rightarrow n - 1 \). Thus the set of vertices coincides with \( I \). A quasiflag \( (E_1 \leftarrow E_2 \leftarrow \ldots \leftarrow E_{n-1} \subset V) \in \mathcal{Q}^L_\bullet \) may be viewed as a representation of \( \Omega \) in the category of coherent sheaves on \( C \). If we denote the quotient sheaf \( V/E_p \) by \( Q_p \), \( 1 \leq p \leq n-1 \), we have another representation of \( \Omega \) in coherent sheaves on \( C \), namely,

\[
Q_\bullet := (Q_1 \rightarrow Q_2 \rightarrow \ldots \rightarrow Q_{n-1})
\]

3.1.1. Exercise. \( T_{E_\bullet} \mathcal{Q}_\gamma^L = \text{Hom}_\Omega(E_\bullet, Q_\bullet) \) where \( \text{Hom}_\Omega(?, ?) \) stands for the morphisms in the category of representations of \( \Omega \) in coherent sheaves on \( C \).

3.1.2. Consider a point \( L_\bullet = (L_1, \ldots, L_{n-1}) \in \mathbb{P}_\gamma \). Here \( L_p \subset V_{\omega_p} \otimes \mathcal{O}_C \) is an invertible subsheaf, the image of morphism \( \mathcal{O}_C(-\langle \omega_p, \gamma \rangle) \rightarrow V_{\omega_p} \otimes \mathcal{O}_C \).

**Exercise.** \( T_{L_\bullet} \mathbb{P}_\gamma = \prod_{p=1}^{n-1} \text{Hom}(L_p, V_{\omega_p} \otimes \mathcal{O}_C/L_p) \).
3.1.3. Recall that for $E_\bullet \in Q^L_\gamma$ we have $\pi(E_\bullet) = \mathcal{L}_\bullet \in \mathbb{P}_\gamma$ where $\mathcal{L}_p = \Lambda^p E_p$ for $1 \leq p \leq n - 1$.

**Exercise.** For $h_\bullet = (h_1, \ldots, h_{n-1}) \in T_E, Q^L_\gamma$ we have $d_{E_\bullet}(h_\bullet) = (\Lambda^1 h_1, \Lambda^2 h_2, \ldots, \Lambda^{n-1} h_{n-1}) \in T_{E_\bullet} \mathbb{P}_\gamma$.

3.2. From now on we fix $\gamma > 0$, $\mathcal{D} \in D(\gamma)$, $\delta(\mathcal{D}) = E_\bullet$. To unburden the notations we will denote the tangent space $T_{E_\bullet} Q^L_\gamma$ by $T$. Since $Q^L_\gamma$ is a smooth $(2|\gamma| + \dim B)$-dimensional variety it suffices to find a subspace $N \subset T$ of dimension
\[
\sum_{1 \leq p \leq n-1} d_{p,p} - \sum_{1 \leq q \leq p \leq n-1} d_{p,q} + 1 = \sum_{1 \leq q < p \leq n-1} (d_{p,q} + 1) + 1
\]
such that $d_{E_\bullet} \pi|_N$ is injective.

3.3. Let $N_0 = \bigoplus_{n-1 \geq p > q \geq 1} \Hom(O(-d_{p,q}), O)$. We have $\dim N_0 = \sum_{n-1 \geq p > q \geq 1} (d_{p,q} + 1)$.

Recall that we have canonically $T = \Hom(E_\bullet, Q_\bullet)$, where
\[
Q_p = \mathcal{V}/E_p = \left( \bigoplus_{q=1}^p \left( \frac{O}{O(-d_{p,q})} \right) v_q \right) \oplus \left( \bigoplus_{q=p+1}^n O v_q \right).
\]

3.4. Let us define a map $\nu_0 : N_0 \to T$ assigning to an element $(f_{p,q}) \in N_0$ a morphism $\nu_0(f_{p,q}) := F \in \Hom(E_\bullet, Q_\bullet)$ of graded coherent sheaves, where $F|_{E_{p,q}} = \bigoplus_{r=p+1}^n F^r_{p,q}$, and
\[
\begin{align*}
F^r_{p,q} : E_{p,q} \rightarrow O v_r \subset Q_p \quad &\text{is defined as the composition} \quad E_{p,q} \subset E_{r,q} = O(-d_{r,q}) \rightarrow O v_r \\
\end{align*}
\]

3.5. **Lemma.** The map $F : E_\bullet \rightarrow Q_\bullet$ is a morphism of representations of the quiver $\Omega$.

**Proof.** We need to check the commutativity of the following diagram
\[
\begin{array}{ccc}
E_p & \longrightarrow & E'_{p'} \\
F \downarrow & & F \downarrow \\
Q_p & \longrightarrow & Q'_{p'}
\end{array}
\]
Since $E_p$ and $Q'_{p'}$ are canonically decomposed into the direct sum it suffices to note that for any $q \leq p \leq p' < r$ the following diagram
\[
\begin{array}{ccc}
E_{p,q} & \longrightarrow & E'_{p',q} \longrightarrow E_{r,q} \\
F^r_{p,q} \downarrow & & F^r_{p',q} \downarrow & & F_{r,q} \downarrow \\
O v_r & \longrightarrow & O v_r & \longrightarrow & O v_r
\end{array}
\]
commutes and for any $q \leq p < r \leq p'$ the following diagram
\[
\begin{array}{ccc}
E_{p,q} & \longrightarrow & E_{r,q} \longrightarrow E'_{p',q} \\
F^r_{p,q} \downarrow & & f_{r,q} \downarrow & & 0 \downarrow \\
O v_r & \longrightarrow & O v_r & \longrightarrow & \left( \frac{O}{O(-d_{r,q})} \right) v_r
\end{array}
\]
commutes as well. \[\square\]
3.6. Let $N_1 = \mathbb{C}$. Let $p_0 = \min\{1 \leq p \leq n-1 \mid d_{p,p} > 0\}$ and pick a non-zero element $f \in \text{Hom}(\mathcal{O}(-d_{p,p}), \mathbb{C}_{p,-d_{p,p}})$. Define the map $\nu_1 : N_1 \to T$ by assigning to $1 \in N_1$ the element $\tilde{f} \in \text{Hom}_{\mathcal{O}}(E_\bullet, Q_\bullet)$ defined on $E_{p,p}$ as the composition

$$E_{p,p} = \mathcal{O}(-d_{p,p}) \overset{1}{\longrightarrow} \frac{\mathcal{O}}{\mathcal{O}(-d_{p,p})} v_p \subset Q_p$$

and with all other components equal to zero.

3.7. Let $\mathcal{M}(r,d;\mathcal{V})$ denote the space of rank $r$ and degree $d$ subsheaves in $\mathcal{V}$.

3.7.1. Let $E \subset \mathcal{V}$ be a rank $k$ and degree $d$ subsheaf in the vector bundle $\mathcal{V}$. Let $\mathcal{V}/E = T \oplus \mathcal{F}$ be a decomposition of the quotient sheaf into the sum of the torsion $T$ and a locally free sheaf $\mathcal{F}$. Consider the map $\det : \mathcal{M}(r,d;\mathcal{V}) \to \mathcal{M}(1,d;\Lambda^k \mathcal{V})$ sending $E$ to $\Lambda^k E$. Then the restriction of its differential $d_E \det : T_E \mathcal{M}(r,d;\mathcal{V}) = \text{Hom}(\mathcal{E}/E, \mathcal{V}/E) \to \text{Hom}(\Lambda^k E, \Lambda^k \mathcal{V}/\Lambda^k E) = T_{\Lambda^k E} \mathcal{M}(1,d;\Lambda^k \mathcal{V})$ to the subspace $\text{Hom}(E, \mathcal{F}) \subset \text{Hom}(E, \mathcal{V}/E)$ factors as $\text{Hom}(\Lambda^k E, \Lambda^{k-1} E \otimes \mathcal{F}) \subset \text{Hom}(\Lambda^k E, \Lambda^k \mathcal{V}/\Lambda^k E)$. Therefore it is injective.

3.7.2. Let $E = \mathcal{O}^{\oplus (r-1)} \oplus \mathcal{O}^{\oplus (d)}$ be a subsheaf in $\mathcal{V} = \mathcal{O}^{\oplus m}$. Then the restriction of differential $d_E \det$ to the subspace $\text{Hom}(E, T) \subset \text{Hom}(E, \mathcal{V}/E)$ is injective.

This immediately follows from the following fact. Let $\bar{E} = \mathcal{O}^{\oplus r} \subset \mathcal{V}$ be the normalization of $E$ in $\mathcal{V}$, that is, the maximal vector subbundle $\bar{E} \subset \mathcal{V}$ such that $\bar{E}/E$ is torsion. Then $T = \bar{E}/E = \Lambda^k \bar{E}/\Lambda^k E \subset \Lambda^k \mathcal{V}/\Lambda^k E$.

3.7.3. Clearly, the subsheaves $\Lambda^{k-1} E \otimes \mathcal{F} \subset \Lambda^k \mathcal{V}/\Lambda^k E$ and $T \subset \Lambda^k \mathcal{V}/\Lambda^k E$ do not intersect.

3.7.4. It follows from [3.7.1] [3.7.2] and [3.7.3] that the composition $d_{E,p} \pi \circ (\nu_0 \oplus \nu_1) : N_0 \oplus N_1 \to T_{\pi(E)} \mathbb{P}^r$ is injective, hence $N := (\nu_0 \oplus \nu_1)(N_0 \oplus N_1) \subset T_{\pi(E)} \mathbb{P}^r$ enjoys the desired property. Namely, $d_{E,p} \pi|_N$ is injective, and dim $N = \sum_{1 \leq q \leq n-1} (d_{p,q} + 1) + 1$.

This completes the proof of the Key Proposition [2.8.1] along with the Main Theorem [2.2] $\square$

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