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# DEPARTMENT OF ECONOMICS

## Working Paper

**Contest Success Functions:  
Theory and Evidence**

by

Sung-Ha Hwang

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**UNIVERSITY OF MASSACHUSETTS  
AMHERST**

# Contest Success Functions: Theory and Evidence

Sung-Ha Hwang\*

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**Abstract** Contest success functions, which show how probabilities of winning depend on resources devoted to a conflict, have been widely used in the literature addressing appropriative activities (economics), international and civil wars (political science), and group conflict and selection (evolutionary biology). Two well-known forms of contest success functions predict contest outcomes from the difference between the resources of each side and from the ratio of resources. The analytical properties of a given conflict model, such as the existence of equilibrium, can be drastically changed simply by altering the form of the contest success function. Despite this problem, there is no consensus about which form is analytically better or empirically more plausible. In this paper we propose an integrated form of contest success functions, which has the ratio form and the difference form as limiting cases, and study the analytical properties of this function. We also estimate different contest success functions to see which form is more empirically probable, using data from battles fought in seventeenth-century Europe and during World War II.

**JEL Classification Numbers:** C70, D72, D74

**Keywords:** Conflicts; Contest Success Functions

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\*Department of Economics and Department of Mathematics and Statistics, University of Massachusetts at Amherst, email: [hwang@math.umass.edu](mailto:hwang@math.umass.edu), homepage: <http://people.umass.edu/sungha/>. I would like to thank Marjolein van Doorn-Strookman for the translation of the data from the German source, Samuel Bowles, Grace Chang, Seung-Yun Oh, Lynda Pickbourn, and Peter Skott for helpful comments and suggestions, the Behavioral Science Program in the Santa Fe Institute for the support. First draft: December 10, 2008.

# 1 Introduction

Traditionally scholars treated conflict as a pathological state requiring special treatment. However, in recent years, various theories of conflict have found important applications in and made contributions to fields such as economics, political science, and evolutionary biology. Economists have examined aspects and implications of appropriative activities, such as rent-seeking behaviors, and the trade-off between appropriation and production when property rights are not well-defined (Tullock, 1967, 1980; Hirshleifer, 1991; Grossman, 1994). Political scientists, focusing on political turmoil such as war, civil war, and demonstration, have scrutinized the cause of these conflict situations and their implications (Fearon, 1995; Collier and Hoeffler, 2001; Sambanis, 2004; Kalyvas et al., 2008). More importantly, early human lethal conflict is being recognized as a key factor in explaining human cooperation in evolutionary biology (Bowles, 2008; Choi and Bowles, 2007; Garcia and Bergh, 2008).

In these studies the technology of conflict is usually described by a function called the contest success function. A contest is “a game in which participants expend resources on arming so as to increase their probability of winning if conflict were to actually take place” (Garfinkel and Skaperdas, 2006, p.1) and contest success functions show how probabilities of winning depend on the resources devoted to conflict. Two well-known forms of contest success functions predict contest outcomes from the difference between the resources of each side and from the ratio of resources.

In spite of the frequent use of the two different forms of contest success functions, there is no agreement on which form better represents the technology of conflict. Jack Hirshleifer points out that the ratio form has the impractical

implication that a side investing zero effort loses everything as long as the opponents spend a small amount of resources (Hirshleifer, 1989, 1991). However, since the difference form does not admit the existence of an interior pure strategy Nash equilibrium in widely used conflict models, the ratio form is more commonly used.

In this paper we present an integrated form of contest success functions, which has the ratio form and the difference form as limiting cases, and study the analytical properties of this form. We also estimate different contest success functions using war data, which provide a natural candidate for a variable that measures effort or resources, namely the number of combatants.

To compare these two common functions we consider the following example. For concreteness we use the language of military combat, following Hirshleifer (1991). Suppose  $p$  is the winning probability of side 1 when two fighters of side 1 face one fighter of side 2 – a situation that we denote by  $(2, 1)$ . We ask the following question: when a thousand and one fighters of side 1 contend with a thousand fighters of side 2, namely  $(1001, 1000)$ , should we still assign the same value of  $p$  to the winning probability of side 1? Similarly, if the number of fighters of side 1 and side 2 are 2000 and 1000 respectively,  $(2000, 1000)$ , would  $p$  be the correct probability of side 1's winning? One may argue that because the importance of one more fighter becomes smaller as the total number of fighters grows, we should assign a probability less than  $p$  to  $(1001, 1000)$ . Regarding the case  $(2000, 1000)$ , one may think that the effectiveness of fighting ability may increase faster as fighter size increases, so side 1 can have a higher probability of winning in  $(2000, 1000)$  (see Lanchester, 1916).

The problem is that in analysis one necessarily chooses one specific form of

contest success functions, thus adopting one interpretation of these functions, even though we do not have a good answer to the above questions. The main purpose of this paper is to define a new contest success function which provides more flexibility in specification than the existing forms. Section 2 provides the derivation, which closely resembles that of a CES production function (Arrow et al., 1961). The probabilistic derivation, like McFadden's (1974) and Jia's (2008), is also provided. We examine the existence of a pure strategy interior Nash equilibrium. In section 3 we present the empirical estimation of various contest success functions using battle data of seventeenth-century Europe and World War II and section 4 concludes the paper.

## 2 Integrated Form

### 2.1 Derivation

Denoting the resources or fighting effort devoted to a contest by side 1 and side 2 by  $x_1$  and  $x_2$  respectively and winning probabilities of side 1 and side 2 by  $u(x_1, x_2)$  and  $v(x_1, x_2)$ , we have the difference form and the ratio form of contest success functions (Hirshleifer, 1989):

$$\begin{aligned} \text{Difference : } u^d(x_1, x_2) &= \frac{\exp(\kappa x_1)}{\exp(\kappa x_1) + \exp(\kappa x_2)} && \text{for } 0 \leq x_1, x_2 \\ v^d(x_1, x_2) &= \frac{\exp(\kappa x_2)}{\exp(\kappa x_1) + \exp(\kappa x_2)} && \text{for } 0 \leq x_1, x_2 \end{aligned}$$

$$\text{Ratio : } u^r(x_1, x_2) = \begin{cases} \frac{(x_1)^\kappa}{(x_1)^\kappa + (x_2)^\kappa} & \text{if } 0 < x_1 \text{ or } 0 < x_2 \\ \frac{1}{2} & \text{if } x_1 = 0 \text{ and } x_2 = 0 \end{cases}$$

$$v^r(x_1, x_2) = \begin{cases} \frac{(x_2)^\kappa}{(x_1)^\kappa + (x_2)^\kappa} & \text{if } 0 < x_1 \text{ or } 0 < x_2 \\ \frac{1}{2} & \text{if } x_1 = 0 \text{ and } x_2 = 0 \end{cases}$$

The superscript,  $d$  or  $r$ , indicates the difference or the ratio form. Clearly the difference form gives the winning probabilities based on the difference between resources,  $x_1 - x_2$ , since  $u^d(x_1, x_2) = \frac{1}{1 + \exp(-\kappa(x_1 - x_2))}$ , while the probability of winning in the ratio form depends only on the ratio,  $x_1/x_2$ , because  $u^r(x_1, x_2) = \frac{1}{1 + (x_2/x_1)^\kappa}$ . We also note that the ratio form of contest success functions is not continuous at  $(0, 0)$ , which accounts for the impossibility of  $(0, 0)$ 's being a Nash equilibrium in a conflict model. We will discuss this more precisely in section 2.2.

Note that in the example given in the introduction, the ratio of the increase in fighters of side 1 to the corresponding increase in side 2, necessary to keep the probability of winning constant, captures the degree of overvaluing (or undervaluing) the probability of winning. Specifically we compute

$$\text{case 1: ratio of increases in fighters} = \frac{1001 - 2}{1000 - 1} \approx 1 \quad (1)$$

$$\text{case 2: ratio of increases in fighters} = \frac{2000 - 2}{1000 - 1} \approx 2 \quad (2)$$

Motivated by this, we define a new rate which can serve as a measure comparing two forms of contest success functions and call this the marginal rate of augmentation (MRA). This measure shows the quantity of additional resources side 1 needs to augment its existing resources to keep the probability of winning constant against an increase in the other side's resources. More

precisely if we consider the level set of side 1's contest success function  $\bar{u} = u(x_1, x_2)$  and use the notation  $x_2 = x_2(x_1)$  such that  $\bar{u} = u(x_1, x_2(x_1))$ , MRA is the slope of  $x_2(x_1)$ :

$$\text{MRA} := \frac{dx_2}{dx_1} = -\frac{u_{x_1}}{u_{x_2}}$$

where  $u_{x_1} = \partial u / \partial x_1$ ,  $u_{x_2} = \partial u / \partial x_2$ . So if MRA is high more resources should be devoted to obtain the same probability of success. Using MRA we now define an elasticity of augmentation as follows:

$$\begin{aligned} & \text{elasticity of augmentation } (\rho) \\ &= \frac{\text{percentage increase in MRA}}{\text{percentage increase in relative size of contestants' resources}} \\ &= \frac{d \ln(-u_{x_1}/u_{x_2})}{d \ln(x_2/x_1)} \end{aligned}$$

The elasticity of augmentation is a normalized percentage increase in MRA and since  $u + v = 1$ , we can also write  $\rho = \frac{d \ln(v_{x_1}/u_{x_2})}{d \ln(x_2/x_1)}$ . When  $\rho$  is low, we expect that side 1 would need to augment its resources by a smaller amount to keep up the same probability of success. This may correspond to the situation described by the difference form. By contrast, a high  $\rho$  implies that side 1 should extend its resources by greater amounts to gain the same probability of success. This situation is possibly captured by the ratio form. By simple calculation we verify that for the difference form the elasticity of augmentation is 0, whereas for the ratio form the elasticity is 1.

The parameter  $\kappa$  in the two forms is a “mass effect parameter scaling the decisiveness of fighting effort disparities” (Hirshleifer, 1991) and measures the slope of the contest success function in an evenly balanced match – the contest where  $x_1 = x_2$ . By computing the marginal probabilities of winning in



an evenly balanced match,

$$\frac{\partial u^d}{\partial x_1} = \frac{\kappa}{4}, \quad \frac{\partial u^r}{\partial x_1} = \frac{\kappa}{4} \frac{\kappa}{x_1}$$

and we see that  $\partial u^d / \partial x_1 = \partial u^r / \partial x_1 = \frac{\kappa}{4}$  if  $x_1 = x_2 = 1$ . Since we would like to find an interpolation between the difference form and the ratio form with a constant elasticity of augmentation we require this probability to equal  $\frac{\kappa}{4}$  at  $x_1 = 1$  for a newly derived function. With these parameters, we have proposition 1:

**Proposition 1** *Suppose we have the following equations:*

$$\frac{d \ln(-u_{x_1}/u_{x_2})}{d \ln(x_2/x_1)} = \rho \quad \text{for } x_1, x_2 \geq 0 \quad (3)$$

$$u_{x_1}(x_1, x_2) = \frac{\kappa}{4} \quad \text{for } x_1 = x_2 = 1 \quad (4)$$

$$u(x_1, x_2; \rho) = \frac{f_\rho(x_1)}{f_\rho(x_1) + f_\rho(x_2)} \quad (5)$$

where  $0 \leq \rho$ ,  $\rho \neq 1$ ,  $\kappa > 0$  and  $f_\rho(0) > 0$ ,  $f_\rho$  is increasing and differentiable for  $x_1, x_2 \geq 0$ . Then equation (6) is a unique solution satisfying (3), (4), and (5)

$$u(x_1, x_2; \rho) = \frac{\exp\left(\kappa \frac{1}{1-\rho} x_1^{1-\rho}\right)}{\exp\left(\kappa \frac{1}{1-\rho} x_1^{1-\rho}\right) + \exp\left(\kappa \frac{1}{1-\rho} x_2^{1-\rho}\right)} \quad \text{for } 0 \leq \rho < 1 \quad (6)$$

Moreover we have

$$u(x_1, x_2; 0) = u^d(x_1, x_2) \quad \text{and} \quad u(x_1, x_2; \rho) \rightarrow u^r(x_1, x_2) \quad \text{as } \rho \rightarrow 1 \quad (7)$$

**Proof.** By rearranging (3) we obtain

$$(x_1)^\rho u_{x_1} + c(x_2)^\rho u_{x_2} = 0 \quad \text{for some } c \neq 0 \quad (8)$$

Using (5) and (8) we find

$$(x_1)^\rho f'_\rho(x_1) f_\rho(x_2) - c(x_2)^\rho f_\rho(x_1) f'_\rho(x_2) = 0 \quad \text{for } x_1, x_2 \geq 0 \quad (9)$$

By evaluating (9) at  $x_1 = x_2 > 0$  we conclude  $c = 1$ . We set  $x_2 = 1$  in (9) and find

$$\frac{f'_\rho(x_1)}{f_\rho(x_1)} = \frac{f'_\rho(1)}{f_\rho(1)} \frac{1}{x_1^\rho} \quad (10)$$

and using (4) and (5) we see that  $f'_\rho(1)/f_\rho(1) = \kappa$ . Then by solving (10) we obtain

$$u(x_1, x_2; \rho) = \frac{\exp(\kappa \frac{1}{1-\rho} x_1^{1-\rho})}{\exp(\kappa \frac{1}{1-\rho} x_1^{1-\rho}) + \exp(\kappa \frac{1}{1-\rho} x_2^{1-\rho})}$$

So we have  $u(x_1, x_2; 0) = u^d(x_1, x_2)$  and the fact that  $u(x_1, x_2; \rho) \rightarrow u^r(x_1, x_2)$  as  $\rho \rightarrow 1$  follows from an application of L'Hopital's rule. ■

We call  $u(x_1, x_2; \rho)$  in (6) an integrated form of contest success function and write  $u_\rho(x_1, x_2) := u(x_1, x_2; \rho)$ . According to proposition 1 an integrated contest success function equals the difference form when  $\rho = 0$  and approaches the ratio form as  $\rho \rightarrow 1$ . Skaperdas (1996) shows that a function of the form (5) satisfies the desirable axioms of contest success functions, where the desirable axioms include monotonicity, anonymity, and independence from irrelevant alternatives (See Skaperdas, 1996, pp.284-286). Hence by proposition 1 we also conclude that the integrated form is a unique function which satisfies the properties of (3, 4) and the desirable properties of a contest success function,

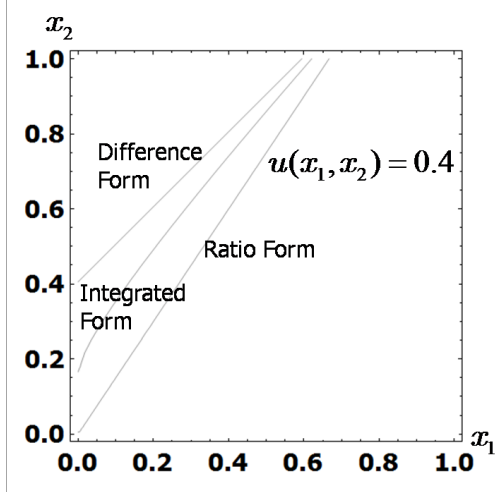


Figure 1: **Comparison of contest success functions.** Each line shows the combinations of  $x_1$  and  $x_2$  where side 1's probability of winning is 0.4. We use the values,  $\kappa = 1, \rho = 0.3$

and this provides an axiomatic characterization of the new integrated form. Figure 1 depicts the level sets of the integrated form, the difference form, and the ratio form. As we expect the integrated form describes the intermediate level of probabilities between the difference form and the ratio form.

Next we consider the probabilistic derivation of the integrated form. We write  $X \sim F(s)$  to indicate that the distribution of a random variable,  $X$ , is  $F(s)$ , and recall  $X$  follows Gumbel type (type I) extreme value distribution if  $X \sim \exp(-e^{-\kappa s})$  and Fréchet type (type II) extreme value distribution if  $X \sim \exp(-s^{-\kappa})$  for  $s \geq 0$ , where  $\kappa$  is a positive constant. We suppose the result of a contest depends on performance,  $h_i$ , and performance is in turn determined by  $x_i$  and a random factor  $\epsilon_i$ : i.e.  $h_i = h_i(x_i, \epsilon_i)$  (Garfinkel and Skaperdas, 2006).

If the specification of performance is in additive form,  $h_i^d = x_i + \epsilon_i$ , and  $\epsilon_i$  follows a Gumbel type distribution, the difference form of the contest success

function equals  $\Pr\{h_1^d > h_2^d\}$  (McFadden, 1974). When  $\epsilon_i$  follows a Fréchet type distribution and the specification of performance is in multiplicative form,  $h_i^r = x_i \epsilon_i$ , the ratio form of the contest success function can be derived from  $\Pr\{h_1^r > h_2^r\}$  (Jia, 2008). We set  $h_i(\rho) := \frac{x_i^{1-\rho}-1}{1-\rho} + \frac{\epsilon_i^{1-\rho}-1}{1-\rho}$  and easily see that  $\Pr\{h_1(0) > h_2(0)\} = \Pr\{h_1^d > h_2^d\}$  and  $\Pr\{h_1(1) > h_2(1)\} = \Pr\{h_1^r > h_2^r\}$ , where we use notations  $h_i(1) = \lim_{\rho \rightarrow 1} h_i(\rho)$ .

Since we have  $\Pr\{x_1 + \epsilon_1 < x_2 + \epsilon_2\} = \Pr\{\epsilon_1 - \epsilon_2 < x_2 - x_1\}$  for given  $x_1$  and  $x_2$ , McFadden (1974)'s result is obtained by showing that  $\epsilon_1 - \epsilon_2 \sim \frac{1}{1+e^{-s}}$  for  $\epsilon_i \sim \exp(-e^{-\kappa s})$ . Similarly because  $\Pr\{x_1 \epsilon_1 < x_2 \epsilon_2\} = \Pr\{\log \epsilon_1 - \log \epsilon_2 < \log x_2 - \log x_1\}$  holds, Jia's derivation is equivalent to showing that  $\log \epsilon_1 - \log \epsilon_2 \sim \frac{1}{1+e^{-s}}$  for  $\epsilon_i \sim \exp(-s^{-\kappa})$  for  $s \geq 0$ . Proposition 2 provides the generalization of these derivations. In the proposition we use the following definition of the rational power of real numbers,  $s^{\frac{n}{m}}$  : for  $m, n$  natural numbers

$$s^{\frac{n}{m}} := \begin{cases} (\sqrt[m]{s})^n & \text{if } s \geq 0 \\ -(\sqrt[m]{-s})^n & \text{if } s < 0 \end{cases} \quad (11)$$

where  $\sqrt[m]{s}$ , for  $s > 0$ , denotes a unique positive real number  $y$  such that  $y^m = s$ .

**Proposition 2** *Suppose that  $\rho = 1 - \frac{n}{m}$ ,  $m, n$  are natural numbers such that  $m > n$  and  $\epsilon_1, \epsilon_2 \sim F(s)$  i.i.d. and  $F(s) = \exp\left(-e^{-\kappa \frac{1}{1-\rho} s^{1-\rho}}\right)$  for  $-\infty < s < \infty$  and  $h_i(\rho) := \frac{x_i^{1-\rho}-1}{1-\rho} + \frac{\epsilon_i^{1-\rho}-1}{1-\rho}$ . Then*

$$\Pr\{h_1(\rho) > h_2(\rho)\} = \frac{\exp\left(\kappa \frac{1}{1-\rho} x_1^{1-\rho}\right)}{\exp\left(\kappa \frac{1}{1-\rho} x_1^{1-\rho}\right) + \exp\left(\kappa \frac{1}{1-\rho} x_2^{1-\rho}\right)}$$

**Proof.** From (11) we see that  $\frac{1}{1-\rho} s^{1-\rho}$  is continuous, increasing and  $\frac{1}{1-\rho} s^{1-\rho} \rightarrow -\infty$  as  $s \rightarrow -\infty$  and  $\frac{1}{1-\rho} s^{1-\rho} \rightarrow \infty$  as  $s \rightarrow \infty$ , so  $F(s)$  is indeed a

distribution function. Since  $\Pr\{h_1(\rho) > h_2(\rho)\} = \Pr\{\frac{1}{1-\rho}x_1^{1-\rho} - \frac{1}{1-\rho}x_2^{1-\rho} > \frac{1}{1-\rho}\epsilon_2^{1-\rho} - \frac{1}{1-\rho}\epsilon_1^{1-\rho}\}$ , in the view of McFadden (1974) (or lemma in the appendix) it is enough to show that  $\frac{1}{1-\rho}\epsilon_1^{1-\rho} \sim \exp(-e^{-\kappa s})$ . Again from (11) and the definition of  $F(s)$  we have

$$\begin{aligned} \Pr\{\frac{1}{1-\rho}\epsilon_1^{1-\rho} < s\} &= \Pr\{\epsilon_1 < \left(\frac{n}{m}s\right)^{\frac{m}{n}}\} \\ &= \exp(-e^{-\kappa s}) \end{aligned}$$

■

We note that the distribution function  $F(s)$  does not possess a continuous density since  $F(s)$  is not differentiable at 0. Moreover if  $\epsilon_i \sim F_i(s)$  independently and  $F_i(s) = \exp\left(-\gamma_i e^{-\kappa \frac{1}{1-\rho} s^{1-\rho}}\right)$ , from lemma 1 in the appendix we have

$$\Pr\{h_1 > h_2\} = \frac{\gamma_1 \exp\left(\kappa \frac{1}{1-\rho} x_1^{1-\rho}\right)}{\gamma_1 \exp\left(\kappa \frac{1}{1-\rho} x_1^{1-\rho}\right) + \gamma_2 \exp\left(\kappa \frac{1}{1-\rho} x_2^{1-\rho}\right)} \quad (12)$$

As  $\rho \rightarrow \infty$ , (12) approaches a generalized ratio form (Garfinkel and Skaperdas, 2006) and  $\gamma_1$  represents the relative fighting effectiveness of side 1 against side 2 (see Dupuy, 1987; Kalyvas et al., 2008).

## 2.2 Existence of Pure-Strategy Nash Equilibrium

Despite the fact that both the ratio form and the difference form have their respective analytical advantages, the ratio form is more commonly used since it admits an interior pure-strategy Nash equilibrium for frequently-used conflict models (Garfinkel and Skaperdas, 2006). We study the conditions for  $\rho$  which allows a pure strategy Nash equilibrium, using a simple conflict model (Hirshleifer, 1989; Garfinkel and Skaperdas, 2006). Assume side 1 and side 2

have resources  $x_1$  and  $x_2$ , where  $x_1, x_2 \in [0, \bar{x}]$  and  $\bar{x} \geq 1$ , and they compete for a prize of the value  $2\bar{x}$ , the sum of total available resources. The costs of competing are the resources devoted to the contest, so we write expected payoffs for side 1 and side 2:

$$\pi_1(x_1, x_2) = 2\bar{x} u_\rho(x_1, x_2) - x_1 \quad (13)$$

$$\pi_2(x_1, x_2) = 2\bar{x} v_\rho(x_1, x_2) - x_2 \quad (14)$$

where  $0 \leq \rho < 1$ . In the model with  $u_\rho$  being replaced by the ratio form  $u^r$  in (13) and (14),  $(x_1, x_2) = (0, 0)$  cannot be a Nash equilibrium since an arbitrary small increase in resources from 0 will raise the probability of winning from 0.5 to 1 and hence the marginal probability of winning at 0 is infinity (Hirshleifer, 1989). This is one of the main reasons why Hirshleifer criticizes the ratio form: peace is never observed as an equilibrium outcome (Hirshleifer, 1991, p.132).

We look for a symmetric interior pure-strategy Nash equilibrium. Denoting such an equilibrium by  $(x_1^*, x_2^*)$ , we find the first order condition for the best response of side 1,  $x_1^{BR}$ , given  $x_2$ :

$$2\kappa\bar{x} \frac{1}{(x_1^{BR})^\rho} u_\rho(x_1^{BR}, x_2)(1 - u_\rho(x_1^{BR}, x_2)) - 1 = 0 \quad (15)$$

At a symmetric equilibrium,  $u_\rho(x_1^{BR}, x_2) = \frac{1}{2}$ . Had we used the difference form instead of the integrated form or set  $\rho = 0$ , the left hand side of (15) would not have depended on  $x_1$ . Because of this an interior symmetric equilibrium generally fails to exist in the difference form, and this accounts for the more popular use of the ratio form in conflict models.

In the integrated form, if a symmetric equilibrium  $(x_1^*, x_2^*)$  exists, from (15)

$$x_1^* = x_2^* = \left(\frac{\kappa\bar{x}}{2}\right)^{\frac{1}{\rho}} \quad (16)$$

To simplify the analysis we assume  $\kappa\bar{x} < 2$  and  $\pi_*''(t) < 0$  for  $t \in [0, \bar{x}]$  where  $\pi_*(t) := 2\bar{x} u(t, x_2^*) - t$ . The first assumption,  $\kappa\bar{x} < 2$ , guarantees  $x_1^* = \left(\frac{\kappa\bar{x}}{2}\right)^{\frac{1}{\rho}} < \frac{\kappa\bar{x}}{2} < \bar{x}$  and the second one ensures that  $\pi_*(t)$  achieves a global maximum at  $x_1^*$ . With these two assumptions  $x_1^* = x_2^* = \left(\frac{\kappa\bar{x}}{2}\right)^{\frac{1}{\rho}}$  is indeed a unique symmetric Nash equilibrium and we view  $x_1^* = x_2^* = \left(\frac{\kappa\bar{x}}{2}\right)^{\frac{1}{\rho}}$  as a generalization of the solution in the case of the ratio form (For example, see equation (10) in Garfinkel and Skaperdas, 2006).

Moreover we verify that  $\lim_{\rho \rightarrow 0} x_1^* = 0$ ; in the limiting case approaching the difference form, an interior pure strategy Nash equilibrium converges to 0 and this shows one instance where there is no interior pure strategy Nash equilibrium in the difference form. We conclude that under reasonable conditions the integrated form of the contest success function allows an interior pure strategy Nash equilibria for all  $0 \leq \rho < 1$ . In figure 2 we present a numerical example of this analysis.

### 3 Empirical Evidence

Since we do not have an *a priori* answer as to which form of the contest success function is more plausible, we conduct an empirical analysis. As arms races and wars are the most important and obvious examples of conflictual contest (See Konrad, 2007, for various forms of conflict), we believe that the estimation of contest success functions using war data would provide mean-

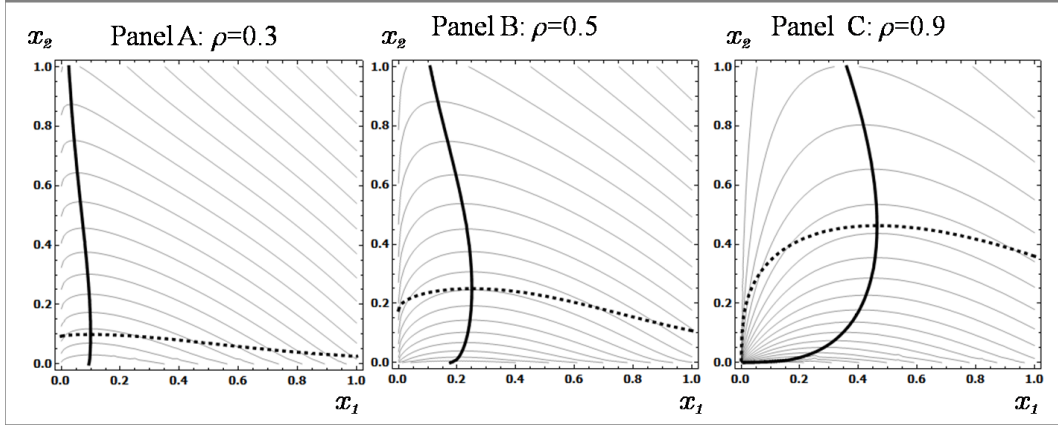


Figure 2: **Existence of an interior pure strategy Nash equilibrium.** We draw side 1's best response (thick line), side 2's best response (dashed line), side 1's indifference curves (thin line) in each panel. We use values,  $\kappa = 1$ ,  $\bar{x} = 1$ . From the shape of indifference curves we see that  $\pi''_*(t) < 0$  for all  $t$ .

ingful estimates of the parameters,  $\rho$  and  $\kappa$ .

### 3.1 Estimation Method

We use battle data from seventeenth century European wars in Bodart (1908, pp. 49-177) and from World War II in Dupuy (1987, pp. 293-295). Military combat, a violent, planned form of physical interaction between two hostile opponents, has a natural hierarchy: war, campaign, battle, engagement, and duel (Dupuy, 1987). Among these we consider two levels of military combat: war and battle. A war is an armed conflict or a state of belligerence usually lasting for months or years, while a battle involves combat between two armies with specific missions, normally lasting one or two days. The seventeenth century European wars data cover 315 battles with each battle corresponding to one observation in our data. Each observation has a record of the winner, the loser, and the total number of personnel in winning and



losing armies.

We observe that each battle gives two pieces of information: the winning probability of a winner and the losing probability of a loser. Because of this difficulty in interpretation, we consider two constructions of data sets from the original battle data. In the first construction – the case presented in the text – we associate each battle with either a winning event or a losing event depending on a random draw. Alternatively, we expand the original battle data such that each battle represents both winning and losing events to each battle, hence obtaining a new data set with 630 observations. In this case, presented in the appendix, we correct the standard errors by clustering battles. We provide descriptive statistics for European battles in the appendix.

Denoting the indicator of winning by  $y_i$  we use the following econometric model:

$$y_i \sim \text{Bernoulli}(\pi_i) \tag{17}$$

$$\text{(D)} \quad \pi_i = F(\kappa(x_{1i} - x_{2i}) + \beta_1 + \beta_2 D_{\text{army } i} + \beta_3 D_{\text{war } i})$$

$$\text{(R)} \quad \pi_i = F(\kappa(\ln x_{1i} - \ln x_{2i}) + \beta_1 + \beta_2 D_{\text{army } i} + \beta_3 D_{\text{war } i})$$

$$\text{(I)} \quad \pi_i = F(\eta(x_{1i}^{1-\rho} - x_{2i}^{1-\rho}) + \beta_1 + \beta_2 D_{\text{army } i} + \beta_3 D_{\text{war } i})$$

where  $F(s) = \frac{1}{1+e^{-s}}$ ,  $D_{\text{army}}$ ,  $D_{\text{war}}$  are dummy variables indicating the identity of armies and the kind of wars (see appendix).

The specifications of the models in equation (17) are the direct consequence of proposition 2 and the dummy variables control for combat effectiveness due to the identity of armies or the specificity of wars. Indeed using  $F(s)$  and

$\eta = \frac{\kappa}{1-\rho}$  we can write model **(I)** as

$$\pi_i = \frac{\exp^{\beta_1 + \beta_2 D_{army\ i} + \beta_3 D_{war\ i}} \exp(\kappa \frac{1}{1-\rho} x_{1i}^{1-\rho})}{\exp^{\beta_1 + \beta_2 D_{army\ i} + \beta_3 D_{war\ i}} \exp(\kappa \frac{1}{1-\rho} x_{1i}^{1-\rho}) + \exp(\frac{1}{1-\rho} x_{2i}^{1-\rho})}$$

so we can regard the part  $\exp^{\beta_1 + \beta_2 D_{army\ i} + \beta_3 D_{war\ i}}$  as a ratio of  $\gamma_i$ 's,  $\frac{\gamma_1}{\gamma_2}$ , in (12).

We also note that model **(D)** is a standard logit regression and model **(R)** is a logit regression with the data log-transformed. Model **(D)**, **(R)**, and **(I)** estimate the difference form, the ratio form, and the integrated form, respectively. We estimate each parameter using the maximum likelihood method, which is the standard method in estimating logit models. We could not estimate the difference form and the integrated form in the case of the World War II data since the data provides only the ratio of combat powers.

### 3.2 Estimation Results

In table 1 we note that all estimates of  $\kappa$  in the difference and the ratio forms are positive, which shows that one side's winning probability is an increasing function of that side's own effort. For the integrated form we can recover the implied  $\kappa = 2.28009$ , using the relation  $\kappa = (1 - \rho)\eta$ .

To compare  $\kappa$ 's in each model we compute one side's marginal probability of winning at an evenly balanced match when the number of combatants is half of its total available resources; i.e. if we denote the total available resources by  $\bar{x}$ , this marginal probability of winning is  $\frac{1}{4} \frac{\kappa}{(\bar{x}/2)^\rho}$ . Since the mean number of combatants in the seventeenth century European war data is 21,035 (see appendix), we can use this number as a proxy for  $\frac{\bar{x}}{2}$ . Table 2 shows the marginal probabilities of winning in an evenly balanced match.

	17C European War			World War II
	Difference	Ratio	Integrated	Ratio
$\kappa$	$1.98 \times 10^{-5}$ ( $9.32 \times 10^{-6}$ )	0.70377 (0.120365)		3.41982 (0.6776)
$\eta$			-18.19199 (6.4571)	
$\rho$			1.125335 (0.21571)	
Number of Observations	315	315	315	188
Percentage of Correctly Predicted	65.40	67.62	67.30	84.04
Log-likelihood Value	-200.8481	-188.23251	-188.0628	-70.6855

Table 1: **Estimation of contest success functions.** All estimates are significant at the 99% level. We use dummy variables of armies and wars in the European war estimation and dummy variables of armies indicating either Allied forces or German forces in the World War II data. Standard errors, in parentheses, are corrected for heteroskedasticity.

We may interpret these numbers as follows: in response to an increase of 10,000 in the number of combatants (from 21,035 original combatants), the difference form, the ratio form, and integrated form predict increases in winning probabilities by 4.95%, 8.36%, 7.78%, respectively. In the case of World War II, using the fact that the average strength of battles is around 14,000 (Dupuy, 1987, pp. 169) we compute a marginal probability of winning  $\times 10^4$  as 2.4472 (or 244.72%) which is much larger than those of the European wars. This fact suggests that the contest success function for World War II is more non-linear than the one for the seventeenth century European wars and “the tremendous advantage of being even just a little stronger than one’s

	D	R	I
marginal winning probability $\times 10^4$	0.0495	0.08364	0.077827

Table 2: **Estimates of the mass effect parameter**

opponent”, which is pointed out as one of the stylized facts of warfare by Hirshleifer (1991, p 131), only appears in World War II data.

Which form of contest success functions better describes battle? As we see in Table 1 and the appendix, estimates of  $\rho$  are close to 1. Our tentative conclusion would be that a contest success function close to the ratio form would best describe the probabilities of winning in battles. Of course the peculiarity of 17th century European wars or other possible data problems may have hindered the correct estimation of our model. This problem, if it exists, can be corrected by extending data sets to cover other kinds of wars.

## 4 Discussion

In the paper we have proposed an integrated contest success function which has the difference form and the ratio form as limiting cases. Also we have derived this new form axiomatically and provided a probabilistic derivation. These results provide a generalization of the existing results. In addition we have shown that the integrated form has desirable analytical properties which admit an interior pure strategy Nash equilibrium. Regarding the question of the empirical plausibility of contest success functions, a tentative conclusion is that the ratio form of contest success functions square with seventeenth century European wars.

Another way of interpreting the integrated contest success function is

through the transformation of variables. Since we do not know the exact unit of measurement for fighting effort or resources in various conflict situations, we may interpret the problem of choosing contest success functions as a problem of choosing a transformation method – the transformation of observed variables into variables with correct measurement. In this interpretation, as we have seen in the text, the difference form with the log transformation corresponds to the ratio form. More generally, the integrated form of contest success functions arises from the difference form with a transformation  $X_i = \frac{x_i - 1}{1 - \rho}$ .

## Appendix: Lemma and Tables

First we prove the lemma used in the text.

**Lemma 1** *Suppose that for  $i = 1, 2$ ,  $\epsilon_i \sim \exp(-\gamma_i e^{-\kappa s})$  independently where  $\gamma_i > 0, \kappa > 0$  and  $-\infty < s < \infty$ . Then  $\epsilon_1 - \epsilon_2 \sim \frac{\gamma_1}{\gamma_1 + \gamma_2 e^{-\kappa s}}$ .*

**Proof.** It is easy to check  $\Lambda(s) := \frac{\gamma_1}{\gamma_1 + \gamma_2 e^{-\kappa s}}$  is a distribution function. From the definition of  $\epsilon_i$ , we have  $\Pr\{\epsilon_2 \in ds\} = \exp(-\gamma_2 e^{-\kappa s}) \gamma_2 \exp(-\kappa s) \kappa ds$ . Hence from the definition of conditional probability and the independence between  $\epsilon_1$  and  $\epsilon_2$ , we have

$$\begin{aligned}
 \Pr\{\epsilon_1 < \epsilon_2 + x\} &= \int_{-\infty}^{\infty} \Pr\{\epsilon_1 < s + x\} \Pr\{\epsilon_2 \in ds\} \\
 &= \int_{-\infty}^{\infty} \exp(-\gamma_1 e^{-\kappa s - \kappa x}) \exp(-\gamma_2 e^{-\kappa s}) \gamma_2 \exp(-\kappa s) \kappa ds \\
 &= \int_0^{\infty} \exp(-t(\gamma_1 e^{-\kappa x} + \gamma_2)) \gamma_2 dt \\
 &= \frac{\gamma_2}{\gamma_1 e^{-\kappa x} + \gamma_2} \tag{18}
 \end{aligned}$$

We use the change of variable,  $t = \exp(-\kappa s)$ , in the third line and the asserted claim follows from (18) ■

We provide tables containing descriptive statistics and alternative estimations.

War	Number of Battles
War of the Spanish Succession	108
Thirty Years' War	64
Austro-Turkish War	34
Great Northern War	29
Dutch War	19
War of the League of Augsburg	18
Other wars	43

**Table A1: 17th century European wars.** Other wars include wars with less than ten battles. These are the Turkish War with Venice and Austria, English Civil War, Hungarian-Turkish War, Polish-Turkish War, Second English Civil War, The Fronde, War of the Quadruple Alliance, Polish-Swedish War, Spanish-Portuguese War, Swedish-Danish War, The First Northern War, War of Devolution, Chamber of Reunion, English Scottish War, Franco-Spanish War, Moldavian Campaign, Monmoth's Rebellion, Polish Insurgency, and Turkish-Venian War. The classification of war is based on Dupuy and Dupuy (1986) and Palmer and Colton (1984).

Statistics	Number of Personnel
Number of observations	630
Mean	21035
Maximum	260000
Minimum	1000
Standard Deviation	24047
Median	15000

**Table A2: Descriptive statistics for number of personnel involved in 17th century European war.**

Statistics	Combat Power Ratio
Number of observations	188
Mean	1.4332
Maximum	7.54
Minimum	0.1326
Standard Deviation	1.30212
Median	1

**Table A3: Descriptive statistics for combat power ratio in World War II data.**

	17C European War		
	Difference	Ratio	Integrated
$\kappa$	$2.24 \times 10^{-5}$ ( $9.30 \times 10^{-6}$ )	0.803 (0.133)	
$\eta$			-20.478 (5.159)
$\rho$			1.117 (0.214)
Number of Observations	630	630	630
Percentage of Correctly Predicted	70.63	73.97	73.73
Log-likelihood Value	-382.443	-354.87	-354.552

**Table A4: Alternative data set.** Each battle represents both a winning event and a losing event. The standard errors, in parentheses, are corrected for heteroskedasticity and clustering.



	Model 1	Model 2	Model 3	Model 4
$\eta$	-2505.666 (226.199)	-19.123 (7.736)	-20.099 (19.510)	-49.866 (93.992)
Implied $\kappa$	0.749194	2.52424	3.739	13.9193
$\rho$	1.0003 (0.0000299)	1.132 (0.204)	1.186 (0.242)	1.279 (0.340)
Number of Observation	308	315	315	184
Log-likelihood Value	-182.557	-185.612	-192.636	-95.279

**Table A5: Alternative estimation.** Model 1: Excludes observations with armies of size greater than 100,000; Model 2: Some observations indicate that the battle took place in a garrison. We use the dummy variable when the observation has this indication; Model 3: Excludes dummy variables for wars; Model 4: Includes only battles among eight major armies: French, Imperial, Swedish, Spanish, Turkish, English, Dutch, Russian. The standard errors, in parentheses, are corrected for heteroskedasticity and clustering.

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