1996

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February 1996

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Abstract

Efficient formulae of Ricci tensor for an arbitrary diagonal metric are presented.

Introduction

Calulation of the Ricci tensor is often a cumbersome task. In this note useful formulae of the Ricci tensor are presented in equations (1) and (2) for the case of the diagonal metric tensor. Application of the formulae in computing the Ricci tensor of n-sphere is also presented.

Derivation

The sign conventions and notation of Wald [1] will be followed. Latin indices are part of abstract index notation and Greek indices denote basis components. The Riemann curvature tensor $R_{abc}{}^{d}$ is defined by $\nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c = R_{abc}{}^{d} \omega_d$. The Ricci tensor in a coordinate basis is $R_{\lambda\nu} = \partial_\rho \Gamma^\rho_{\lambda\nu} - \partial_\lambda \Gamma^\rho_{\nu\rho} - \Gamma^\sigma_{\lambda\rho}\Gamma^\rho_{\sigma\nu} + \Gamma^\rho_{\lambda\nu}\Gamma^\sigma_{\rho\sigma}$ where the Christoffel symbols are $\Gamma^\rho_{\lambda\nu} = g^{\rho\sigma}\left(\partial_\lambda g_{\nu\sigma} + \partial_\nu g_{\lambda\sigma} - \partial_\sigma g_{\lambda\nu}\right)/2$. *For the rest of this note we assume that* $g_{\rho\nu}$ *is diagonal unless otherwise indicated.* Also to avoid having to write down "$\mu \neq \nu$" repeatedly we will *always* take $\mu \neq \nu$ and no sum will be assumed on repeated μ and ν.

We first note that $\Gamma^\rho_{\mu\nu} = 0$ if $\mu \neq \rho \neq \nu$. It means that at least two indices of Γ must be the same for it to be nonvanishing. Also $\ln|g| = \sum_{\sigma=1}^{n} \ln|g_{\sigma\sigma}|$ where g and n are determinant of the metric and dimension of manifold, respectively. Then it is easy to show that $2\Gamma^\mu_{\sigma\mu} = \partial_\sigma \ln|g_{\mu\mu}|$ for all σ and that $2\Gamma^\nu_{\mu\mu} = -g^{\nu\nu}\partial_\nu g_{\mu\mu}$. (Again $\mu \neq \nu$ and no sum on μ and ν.) These two equations give

$$4\Gamma^\nu_{\mu\mu}\Gamma^\mu_{\nu\nu} = g^{\nu\nu}\left(\partial_\nu g_{\mu\mu}\right) g^{\mu\mu}\left(\partial_\mu g_{\nu\nu}\right) = \left(\partial_\mu \ln|g_{\nu\nu}|\right)\left(\partial_\mu \ln|g_{\nu\nu}|\right) = 4\Gamma^\mu_{\mu\nu}\Gamma^\nu_{\mu\nu}$$

Using above facts, we get, with each sum having upper limit n and lower limit 1,

$$\begin{aligned} R_{\mu\nu} =& \partial_\mu \Gamma^\mu_{\mu\nu} + \partial_\nu \Gamma^\nu_{\mu\nu} - (1/2)\partial_\mu\partial_\nu \ln|g| - \Gamma^\mu_{\mu\rho}\Gamma^\rho_{\mu\nu} - \Gamma^\nu_{\mu\rho}\Gamma^\rho_{\nu\nu} - \sum_{\sigma\neq\mu,\nu} \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\sigma\nu} \\ &+ (1/2)\Gamma^\mu_{\mu\nu}\partial_\mu \ln|g| + (1/2)\Gamma^\nu_{\mu\nu}\partial_\nu \ln|g| \\ =& (1/2)\partial_\mu\partial_\nu \ln|\frac{g_{\mu\mu}g_{\nu\nu}}{g}| - \Gamma^\mu_{\mu\mu}\Gamma^\mu_{\mu\nu} - \Gamma^\mu_{\mu\nu}\Gamma^\nu_{\mu\nu} - \Gamma^\nu_{\mu\mu}\Gamma^\mu_{\nu\nu} - \Gamma^\nu_{\nu\mu}\Gamma^\nu_{\nu\nu} - \sum_{\sigma\neq\mu,\nu} \Gamma^\sigma_{\sigma\mu}\Gamma^\sigma_{\sigma\nu} \\ &+ \Gamma^\mu_{\mu\nu}\partial_\mu \ln\sqrt{|g_{\mu\mu}|} + \Gamma^\mu_{\mu\nu}\partial_\mu \ln\sqrt{|g_{\nu\nu}|} + \Gamma^\nu_{\mu\nu}\partial_\nu \ln\sqrt{|g_{\mu\mu}|} + \Gamma^\nu_{\mu\nu}\partial_\nu \ln\sqrt{|g_{\nu\nu}|} \\ &+ (1/2)\Gamma^\mu_{\mu\nu}\partial_\mu \ln|\frac{g}{g_{\mu\mu}g_{\nu\nu}}| + (1/2)\Gamma^\nu_{\mu\nu}\partial_\nu \ln|\frac{g}{g_{\mu\mu}g_{\nu\nu}}| \end{aligned}$$

$$
\begin{aligned}
=&(1/2)\partial_\mu\partial_\nu \ln|\frac{g_{\mu\mu}g_{\nu\nu}}{g}| - \Gamma^\mu_{\mu\mu}\Gamma^\mu_{\mu\nu} - \Gamma^\mu_{\mu\nu}\Gamma^\nu_{\mu\nu} - \Gamma^\nu_{\mu\nu}\Gamma^\mu_{\mu\nu} - \Gamma^\nu_{\nu\mu}\Gamma^\nu_{\nu\nu} - \sum_{\sigma\neq\mu,\nu}\Gamma^\sigma_{\sigma\mu}\Gamma^\sigma_{\sigma\nu} \\
&+ \Gamma^\mu_{\mu\mu}\Gamma^\mu_{\mu\nu} + \Gamma^\mu_{\mu\nu}\Gamma^\nu_{\mu\nu} + \Gamma^\nu_{\mu\nu}\Gamma^\mu_{\mu\nu} + \Gamma^\nu_{\nu\mu}\Gamma^\nu_{\nu\nu} \\
&+ (1/2)\Gamma^\mu_{\mu\nu}\partial_\mu \ln|\frac{g}{g_{\mu\mu}g_{\nu\nu}}| + (1/2)\Gamma^\nu_{\mu\nu}\partial_\nu \ln|\frac{g}{g_{\mu\mu}g_{\nu\nu}}| \\
=&(1/2)\partial_\mu\partial_\nu \ln|\frac{g_{\mu\mu}g_{\nu\nu}}{g}| - \sum_{\sigma\neq\mu,\nu}\Gamma^\sigma_{\sigma\mu}\Gamma^\sigma_{\sigma\nu} + \frac{1}{2}\Gamma^\mu_{\mu\nu}\partial_\mu \ln|\frac{g}{g_{\mu\mu}g_{\nu\nu}}| + \frac{1}{2}\Gamma^\nu_{\mu\nu}\partial_\nu \ln|\frac{g}{g_{\mu\mu}g_{\nu\nu}}|
\end{aligned}
$$

or (remember $\mu \neq \nu$)

$$
4R_{\mu\nu} = (\partial_\mu \ln|g_{\nu\nu}| - \partial_\mu)\,\partial_\nu \ln|\frac{g}{g_{\mu\mu}g_{\nu\nu}}| + (\mu \leftrightarrow \nu) - \sum_{\sigma\neq\mu,\nu}\partial_\mu \ln|g_{\sigma\sigma}|\partial_\nu \ln|g_{\sigma\sigma}| \qquad (1)
$$

where $(\mu \leftrightarrow \nu)$ stands for preceding terms with μ and ν interchanged.* A number of useful facts follow from the above formula. First of all $R_{\mu\nu} = 0$ *if* x_μ *is an ignorable coordinate* because ∂_μ is in each term of the formula. If there are m ignorable coordinates then maximum number of nonzero off-diagonal components of the Ricci tensor is $(n-m)(n-m-1)/2$. Thus a sufficient condition for the Ricci tensor to be diagonal is that the (diagonal) metric be a function only of one coordinate. The Robertson-Walker metric with flat spatial sections, $ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)$, satisfies this condition and its Ricci tensor is consequently diagonal. If the (diagonal) metric is a function only of two coordinates, say x_μ and x_ν, then all off diagonal components of the Ricci tensor except $R_{\mu\nu}$ are zero. For example, for Schwartzschild metric $ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$ only $R_{\theta r}$ needs to be explicitly calculated, all other being zero. It is easily checked using equation (1) that $R_{\theta r}$ is also zero. Symmetry arguments can also be given† as to why the Ricci tensor is diagonal in this case. Secondly *second derivative terms are identically zero whenever each component of the metric is a product of functions of one single coordinate* i.e. if, for all μ, $g_{\mu\mu} = \prod_{l=1}^n f^l_\mu(x_l)$ where some of f^l_μ may be one. A final trivial corollary

---

* Alternative forms of equation (1) are
$4R_{\mu\nu} = \sum_{\sigma\neq\mu,\nu}\left[(\partial_\mu \ln|g_{\nu\nu}| - \partial_\mu)\,\partial_\nu \ln|g_{\sigma\sigma}| + (\mu \leftrightarrow \nu) - \partial_\mu \ln|g_{\sigma\sigma}|\partial_\nu \ln|g_{\sigma\sigma}|\right]$ and
$8R_{\mu\nu} = \sum_{\sigma\neq\mu,\nu}\left[\left(\partial_\mu \ln\frac{g^2_{\nu\nu}}{|g_{\sigma\sigma}|} - 2\partial_\mu\right)\partial_\nu \ln|g_{\sigma\sigma}| + (\mu \leftrightarrow \nu)\right]$

† Page 178 of Weinberg [2].

of equation (1) is that the Ricci tensor is diagonal in 2-dimensions. This fact also follows trivially from the fact that in 2-dimensions, the Ricci tensor is the metric tensor (not necessarily diagonal) up to a factor of a scalar function. When $n = 3$, it is easy to find an example of a diagonal metric which results in a non-diagonal Ricci tensor. For example, $ds^2 = dx^2 + xdy^2 + ydz^2$ gives $4R_{xy} = 1/xy$.

We now calculate the diagonal components of Ricci tensor. We have

$$
\begin{aligned}
R_{\mu\mu} =& \partial_\sigma \Gamma^\sigma_{\mu\mu} - \partial^2_\mu \ln\sqrt{|g|} - \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\mu\mu}\partial_\rho \ln\sqrt{|g|} \\
=& \partial^2_\mu \ln\sqrt{|g_{\mu\mu}|} + \sum_{\sigma\neq\mu}\partial_\sigma\Gamma^\sigma_{\mu\mu} - \partial^2_\mu \ln\sqrt{|g|} - \Gamma^\mu_{\mu\rho}\Gamma^\rho_{\mu\mu} \\
& - \sum_{\sigma\neq\mu}\Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\mu\sigma} + \Gamma^\mu_{\mu\mu}\partial_\mu \ln\sqrt{|g|} + \sum_{\sigma\neq\mu}\Gamma^\sigma_{\mu\mu}\partial_\sigma \ln\sqrt{|g|} \\
=& (1/2)\partial^2_\mu \ln|\frac{g_{\mu\mu}}{g}| + \sum_{\sigma\neq\mu}\partial_\sigma\Gamma^\sigma_{\mu\mu} - \left(\Gamma^\mu_{\mu\mu}\right)^2 - \sum_{\sigma\neq\mu}\Gamma^\mu_{\mu\sigma}\Gamma^\sigma_{\mu\mu} - \sum_{\sigma\neq\mu}\Gamma^\sigma_{\mu\mu}\Gamma^\mu_{\sigma\mu} \\
& - \sum_{\sigma\neq\mu}\Gamma^\sigma_{\sigma\mu}\Gamma^\sigma_{\sigma\mu} + \Gamma^\mu_{\mu\mu}\partial_\mu \ln\sqrt{|g_{\mu\mu}|} + \Gamma^\mu_{\mu\mu}\partial_\mu \ln\sqrt{|\frac{g}{g_{\mu\mu}}|} + \sum_{\sigma\neq\mu}\Gamma^\sigma_{\mu\mu}\partial_\sigma \ln\sqrt{|g|} \\
=& (1/4)\left(\partial_\mu \ln|g_{\mu\mu}| - 2\partial_\mu\right)\partial_\mu \ln|\frac{g}{g_{\mu\mu}}| + \sum_{\sigma\neq\mu}\Big[\partial_\sigma\Gamma^\sigma_{\mu\mu} - 2\Gamma^\sigma_{\mu\mu}\Gamma^\mu_{\mu\sigma} - \left(\Gamma^\sigma_{\sigma\mu}\right)^2 \\
& + \Gamma^\sigma_{\mu\mu}\partial_\sigma \ln\sqrt{|g|}\Big] \\
=& (1/4)\left(\partial_\mu \ln|g_{\mu\mu}| - 2\partial_\mu\right)\partial_\mu \ln|\frac{g}{g_{\mu\mu}}| + \sum_{\sigma\neq\mu}\Big[-(1/2)\partial_\sigma\left(g^{\sigma\sigma}\partial_\sigma g_{\mu\mu}\right) \\
& + g^{\sigma\sigma}\left(\partial_\sigma g_{\mu\mu}\right)\partial_\sigma \ln\sqrt{|g_{\mu\mu}|} - \left(\partial_\mu \ln\sqrt{|g_{\sigma\sigma}|}\right)^2 - (g^{\sigma\sigma}/2)(\partial_\sigma g_{\mu\mu})\partial_\sigma \ln\sqrt{|g|}\Big] \\
=& \frac{1}{4}\left(\partial_\mu \ln|g_{\mu\mu}| - 2\partial_\mu\right)\partial_\mu \ln|\frac{g}{g_{\mu\mu}}| - \frac{1}{4}\sum_{\sigma\neq\mu}\Big[2\partial_\sigma\left(g^{\sigma\sigma}\partial_\sigma g_{\mu\mu}\right) + \left(\partial_\mu \ln|g_{\sigma\sigma}|\right)^2 \\
& + g^{\sigma\sigma}\left(\partial_\sigma g_{\mu\mu}\right)\partial_\sigma \ln\frac{|g|}{g^2_{\mu\mu}}\Big]
\end{aligned}
$$

or

$$
4R_{\mu\mu} = \left(\partial_\mu \ln|g_{\mu\mu}| - 2\partial_\mu\right)\partial_\mu \ln|\frac{g}{g_{\mu\mu}}| - \sum_{\sigma\neq\mu}\Big[\left(\partial_\mu \ln|g_{\sigma\sigma}|\right)^2 + \left(\partial_\sigma \ln\frac{|g|}{g^2_{\mu\mu}} + 2\partial_\sigma\right)g^{\sigma\sigma}\partial_\sigma g_{\mu\mu}\Big] \tag{2}
$$

where $2\partial_\sigma$ acts everything on its right. The efficiency of this formula is obvious for Schwartzschild metric given earlier. To compute R_{tt} we note that all terms with ∂_t are zero.

Thus $4R_{tt} = -\sum_{\sigma\neq t}\left(\partial_\sigma \ln\frac{|g|}{g_{tt}^2} + 2\partial_\sigma\right)(g^{\sigma\sigma}\partial_\sigma g_{tt})$. Because $g_{tt} = -B(r)$ only the $\sigma = r$ term survives and we get, after very little computation, $R_{tt} = \frac{B'}{Ar} - \frac{B'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{B''}{2A}$ where prime indicates differentiation with respect to r.‡ Other diagonal components are just as easy to compute.

Application to n-sphere

To illustrate the use of above formulae we compute Ricci tensor of n-sphere which in spherical coordinates has diagonal metric $g_{11} = 1$ and $g_{\mu+1,\mu+1} = g_{\mu\mu}\sin^2 x_\mu$ for $1 \leq \mu \leq n-1$. First note that $\ln|g_{\mu\mu}| = \sum_{\nu=1}^{\mu-1} 2\ln|\sin x_\nu|$ and thus $\partial_\rho g_{\mu\mu} = 0$ if $n \geq \rho \geq \mu \geq 1$. These facts will be used repeatedly in what follows. We first compute the off diagonal components of Ricci tensor. According to the general argument given earlier second derivative terms will be zero. Let $\mu > \nu$. Then equation (1) becomes

$$\begin{aligned} 4R_{\mu\nu} =& \sum_{\sigma>\mu}\left[(\partial_\nu \ln|g_{\mu\mu}|)\,\partial_\mu \ln|g_{\sigma\sigma}| - (\partial_\mu \ln|g_{\sigma\sigma}|)\,\partial_\nu \ln|g_{\sigma\sigma}|\right] \\ =& 4\sum_{\sigma>\mu}(\cot x_\nu \cot x_\mu - \cot x_\mu \cot x_\nu) = 0 \end{aligned}$$

We next compute the diagonal components. We have, from equation (2),

$$\begin{aligned} 4R_{\mu\mu} =& -\sum_{\sigma>\mu}\left[2\partial_\mu^2 \ln|g_{\sigma\sigma}| + (\partial_\mu \ln|g_{\sigma\sigma}|)^2\right] - \sum_{\sigma<\mu}\left(\partial_\sigma \ln\frac{|g|}{g_{\mu\mu}^2} + 2\partial_\sigma\right) g^{\sigma\sigma}\partial_\sigma g_{\mu\mu} \\ =& \sum_{\sigma>\mu}\left(-2\times 2\partial_\mu \cot x_\mu - 4\cot^2 x_\mu\right) - \left(\partial_{\mu-1}\ln\frac{|g|}{g_{\mu\mu}^2} + 2\partial_{\mu-1}\right) g^{\mu-1,\mu-1}\partial_{\mu-1}g_{\mu\mu} \\ & - \sum_{\sigma<\mu-1}\left(\partial_\sigma \ln\frac{|g|}{g_{\mu-1,\mu-1}^2} - 4\partial_\sigma \ln|\sin x_{\mu-1}| + 2\partial_\sigma\right) g^{\sigma\sigma}\partial_\sigma\left(g_{\mu-1,\mu-1}\sin^2 x_{\mu-1}\right) \\ =& \sum_{\sigma>\mu}\left(\frac{4}{\sin^2 x_\mu} - 4\frac{\cos^2 x_\mu}{\sin^2 x_\mu}\right) \\ & - \left[\Big(\sum_{\sigma>\mu-1}\partial_{\mu-1}\ln|g_{\sigma\sigma}|\Big) - 2\partial_{\mu-1}\ln|g_{\mu\mu}| + 2\partial_{\mu-1}\right] g^{\mu-1,\mu-1}\partial_{\mu-1}g_{\mu\mu} \\ & - \sin^2 x_{\mu-1}\sum_{\sigma<\mu-1}\left(\partial_\sigma \ln\frac{|g|}{g_{\mu-1,\mu-1}^2} + 2\partial_\sigma\right) g^{\sigma\sigma}\partial_\sigma g_{\mu-1,\mu-1} \end{aligned}$$

‡ Cf. page 178 of Weinberg and note that his $R_{abc}{}^{d}$ is minus of Wald's. Also see problem 6.2 of [1].

$$
\begin{aligned}
=&4(n-\mu) - \Big[\big(2\cot x_{\mu-1}\sum_{\sigma>\mu-1}\big) - 4\cot x_{\mu-1} + 2\partial_{\mu-1}\Big] g^{\mu-1,\mu-1} g_{\mu-1,\mu-1}\sin 2x_{\mu-1} \\
&\quad - \sin^2 x_{\mu-1}\sum_{\sigma<\mu-1}\Big(\partial_\sigma \ln\frac{|g|}{g^2_{\mu-1,\mu-1}} + 2\partial_\sigma\Big) g^{\sigma\sigma}\partial_\sigma g_{\mu-1,\mu-1} \\
=&4(n-\mu) - [2\cot x_{\mu-1}(n+1-\mu) - 4\cot x_{\mu-1}]\sin 2x_{\mu-1} \;\; - 2\times 2\cos 2x_{\mu-1} \\
&\quad - \sin^2 x_{\mu-1}\sum_{\sigma<\mu-1}\big(\partial_\sigma \ln\frac{|g|}{g^2_{\mu-1,\mu-1}} + 2\partial_\sigma\big) g^{\sigma\sigma}\partial_\sigma g_{\mu-1,\mu-1} \\
=&4(n-\mu) - 4\big[\cos^2 x_{\mu-1}(n-\mu+1) - 2\cos^2 x_{\mu-1} + \cos^2 x_{\mu-1} - \sin^2 x_{\mu-1}\big] \\
&\quad - \sin^2 x_{\mu-1}\sum_{\sigma<\mu-1}\big(\partial_\sigma \ln\frac{|g|}{g^2_{\mu-1,\mu-1}} + 2\partial_\sigma\big) g^{\sigma\sigma}\partial_\sigma g_{\mu-1,\mu-1} \\
=&4\left[n-\mu-\cos^2 x_{\mu-1}(n-\mu+1) + 1\right] \\
&\quad - \sin^2 x_{\mu-1}\sum_{\sigma<\mu-1}\big(\partial_\sigma \ln\frac{|g|}{g^2_{\mu-1,\mu-1}} + 2\partial_\sigma\big) g^{\sigma\sigma}\partial_\sigma g_{\mu-1,\mu-1} \\
=&\sin^2 x_{\mu-1}\Big[4(n-\mu+1) - \sum_{\sigma<\mu-1}\Big(\partial_\sigma \ln\frac{|g|}{g^2_{\mu-1,\mu-1}} + 2\partial_\sigma\Big) g^{\sigma\sigma}\partial_\sigma g_{\mu-1,\mu-1}\Big]
\end{aligned}
$$

Thus

$$
4R_{\mu+1,\mu+1} = \sin^2 x_\mu \left[4(n-\mu) - \sum_{\sigma<\mu}\left(\partial_\sigma \ln\frac{|g|}{g^2_{\mu\mu}} + 2\partial_\sigma\right) g^{\sigma\sigma}\partial_\sigma g_{\mu\mu}\right]
$$

We now rewrite the expression inside the square brackets. Break up the sum into $\sigma = \mu-1$ and $\sigma < \mu - 1$ and use $g_{\mu\mu} = g_{\mu-1,\mu-1}\sin^2 x_{\mu-1}$. Then

$$
\begin{aligned}
[\quad] =&4(n-\mu) - \big(\partial_{\mu-1}\ln\frac{|g|}{g^2_{\mu\mu}} + 2\partial_{\mu-1}\big) g^{\mu-1,\mu-1}\partial_{\mu-1} g_{\mu\mu} \\
&\quad - \sum_{\sigma<\mu-1}\Big(\partial_\sigma \ln\frac{|g|}{g^2_{\mu-1,\mu-1}} - 4\partial_\sigma \ln|\sin x_{\mu-1}| + 2\partial_\sigma\Big) g^{\sigma\sigma}\partial_\sigma g_{\mu\mu} \\
=&4(n-\mu) - \Big(\partial_{\mu-1}\ln|g| - 2\partial_{\mu-1}\ln|g_{\mu\mu}| + 2\partial_{\mu-1}\Big) g^{\mu-1,\mu-1} g_{\mu-1,\mu-1}\sin 2x_{\mu-1} \\
&\quad - \sum_{\sigma<\mu-1}\Big(\partial_\sigma \ln\frac{|g|}{g^2_{\mu-1,\mu-1}} + 2\partial_\sigma\Big) g^{\sigma\sigma}\partial_\sigma\left(g_{\mu-1,\mu-1}\sin^2 x_{\mu-1}\right) \\
=&4(n-\mu) - \Big[\big(\sum_{\sigma>\mu-1}\partial_{\mu-1}\ln|g_{\sigma\sigma}|\big) - 2\times 2\cot x_{\mu-1} + 2\partial_{\mu-1}\Big]\sin 2x_{\mu-1} \\
&\quad - \sin^2 x_{\mu-1}\sum_{\sigma<\mu-1}\left(\partial_\sigma \ln\frac{|g|}{g^2_{\mu-1,\mu-1}} + 2\partial_\sigma\right) g^{\sigma\sigma}\partial_\sigma g_{\mu-1,\mu-1}
\end{aligned}
$$

$$
\begin{aligned}
=&4(n-\mu)-[2\cot x_{\mu-1}(n-\mu+1)-4\cot x_{\mu-1}]\sin 2x_{\mu-1}-4\cos 2x_{\mu-1}\\
&\quad-\sin^2 x_{\mu-1}\sum_{\sigma<\mu-1}\left(\partial_\sigma \ln\frac{|g|}{g^2_{\mu-1,\mu-1}}+2\partial_\sigma\right)g^{\sigma\sigma}\partial_\sigma g_{\mu-1,\mu-1}\\
=&4(n-\mu)-4\Big[\cos^2 x_{\mu-1}(n-\mu+1)-2\cos^2 x_{\mu-1}+\cos^2 x_{\mu-1}-\sin^2 x_{\mu-1}\Big]\\
&\quad-\sin^2 x_{\mu-1}\sum_{\sigma<\mu-1}\left(\partial_\sigma \ln\frac{|g|}{g^2_{\mu-1,\mu-1}}+2\partial_\sigma\right)g^{\sigma\sigma}\partial_\sigma g_{\mu-1,\mu-1}\\
=&4\Big[n-\mu-\cos^2 x_{\mu-1}(n+1-\mu)+\sin^2 x_{\mu-1}+\cos^2 x_{\mu-1}\Big]\\
&\quad-\sin^2 x_{\mu-1}\sum_{\sigma<\mu-1}\Big(\partial_\sigma \ln\frac{|g|}{g^2_{\mu-1,\mu-1}}+2\partial_\sigma\Big)g^{\sigma\sigma}\partial_\sigma g_{\mu-1,\mu-1}\\
=&4(n+1-\mu)\sin^2 x_{\mu-1}-\sin^2 x_{\mu-1}\sum_{\sigma<\mu-1}\Big(\partial_\sigma \ln\frac{|g|}{g^2_{\mu-1,\mu-1}}+2\partial_\sigma\Big)g^{\sigma\sigma}\partial_\sigma g_{\mu-1,\mu-1}\\
=&4R_{\mu\mu}
\end{aligned}
$$

Next

$4R_{11}=-\sum_{\sigma>1}\left[2\partial_1^2\ln|g_{\sigma\sigma}|+(\partial_\mu \ln|g_{\sigma\sigma}|)^2\right]=4\left(\frac{1}{\sin^2 x_1}-\cot^2 x_1\right)\sum_{\sigma>1}=4(n-1)$

All preceding calculations amount to proving that $R_{\rho\mu}=(n-1)g_{\rho\mu}$. Let $\{v_a^\rho\}$ be the dual basis of the tangent space where ρ labels a particular basis vector. Then

$$R_{ab}=R_{\rho\sigma}v_a^\rho v_b^\sigma=(n-1)g_{\rho\sigma}v_a^\rho v_b^\sigma=(n-1)g_{ab}$$

This fact about n-sphere is also provable using the machinery of symmetric spaces. See chapter 13 of [2].

Equations (1) and (2) were derived when attempt was made to establish any connection between diagonality of $R_{\nu\sigma}$ and that of $g_{\sigma\nu}$. Finally we remark that Dingle [3] has listed values of T_ν^σ for the case of 4-dimensional diagonal metric.

Acknowledgements

I would like to thank Professor Jennie Traschen for much encouragement and discussion and for useful comments on the manuscript.

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