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Isoperimetric inequality and area growth of surfaces with bounded mean curvature

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**ISOPERIMETRIC INEQUALITY AND AREA GROWTH
OF SURFACES WITH BOUNDED MEAN CURVATURE**

A Dissertation Presented

by

DECHANG CHEN

Submitted to the Graduate School of the
University of Massachusetts Amherst in partial fulfillment
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2014

Mathematics & Statistics

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ABSTRACT

ISOPERIMETRIC INEQUALITY AND AREA GROWTH OF SURFACES WITH BOUNDED MEAN CURVATURE

MAY 2014

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Ph.D., UNIVERSITY OF MASSACHUSETTS AMHERST

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In this thesis, we give a lower bound on the areas of small geodesic balls in an immersed hypersurface M contained in a Riemannian manifold N . This lower bound depends only on an upper bound for the absolute mean curvature function of M , an upper bound of the absolute sectional curvature of N and a lower bound for the injectivity radius of N . As a consequence, we prove that if M is a noncompact complete surface of bounded absolute mean curvature in Riemannian manifold N with positive injectivity radius and bounded absolute sectional curvature, then the area of geodesic balls of M must grow at least linearly in terms of their radius. In particular, this result implies the classical result of Yau that a complete minimal hypersurface in \mathbb{R}^n must have infinite area. We also attain partial results on the conjecture: *If M is a compact immersed surface in hyperbolic 3-space \mathbb{H}^3 , and the absolute mean curvature function of M is bounded from above by 1, then $\text{Area}(M) \leq \frac{(\text{Length}(\partial M))^2}{4\pi}$.*

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INTRODUCTION

The thesis includes two parts; the first part deals with the volume growth of a hypersurface M with bounded absolute mean curvature in a Riemannian manifold in terms of the radius r of intrinsic geodesic balls $B_M(p, r)$ centered at a point p in M . The second part of the thesis concerns the existence of isoperimetric inequalities in \mathbb{H}^3 for compact surfaces with boundary and absolute mean curvature function bounded from above by 1.

In Chapter 2, we give lower volume growth estimates for geodesic balls in a complete noncompact hypersurface M with bounded mean curvature in a complete n -manifold N with bounded sectional curvature and positive injectivity radius, where M is allowed to have compact boundary. More precisely, as a consequence of Theorem 0.0.1 below (see Theorem 2.3.6) such an M has at least linear volume growth with respect to the distance function to its boundary; see Corollary 0.0.2 for this consequence. This result generalizes an earlier theorem of Yau [18] which states that a complete, noncompact minimal hypersurface of \mathbb{R}^{n+1} has infinite volume (also see [2] and [18] for some related results).

Our first main result can be stated as follows:

Theorem 0.0.1. *Let H_0, I_0, S_0 be positive numbers. Suppose that M is a complete oriented hypersurface with boundary in an $(n + 1)$ -manifold (N, g) such that*

- *the absolute mean curvature function of M is at most H_0 ,*
- *the injectivity radius of N is at least I_0 ,*
- *the absolute sectional curvature of N is at most S_0 .*

Then there exist constants $c = c(n, H_0, I_0, S_0)$, $\sigma = \sigma(n, H_0, I_0, S_0)$ such that for any point $p \in M$ of distance at least σ from ∂M , and for $r \in (0, \sigma)$, the volume of the intrinsic Riemannian ball $B_M(p, r)$ is greater than or equal to cr^n .

Corollary 0.0.2. *Under the hypothesis of Theorem 0.0.1, for any point $p \in M$ and for any $R \in [\sigma, \text{dist}(p, \partial M)]$, then $\text{Vol}(B_M(p, R)) \geq CR$ for some constant C depending on σ, H_0, I_0, S_0 .*

In the case the dimension of M is 2, we have the following result, where ε can be taken to be σ in an appropriate application of Theorem 0.0.1.

Corollary 0.0.3. *Let M be a complete surface with compact boundary in a 3-manifold N with bounded sectional curvature and positive injective radius. Suppose for some $H_0 \geq 0$, the absolute mean curvature function H_M of M satisfies $|H_M| \leq H_0$ and the boundary ∂M has at most m boundary components with total length D . Then for any point p in M such that there exists a point $q \in M$ with $R_q = d_M(p, q) > 2m\varepsilon + \frac{D}{2}$ (ε is small enough), then for any $r \in [2m\varepsilon + \frac{D}{2}, R_q]$, the area of the intrinsic Riemannian ball satisfies $\text{Area}(B_M(p, r)) \geq C\varepsilon(r - 2m\varepsilon - \frac{D}{2})$ for some constant C .*

As a consequence of the above corollary, we conclude that certain noncompact hypersurfaces in certain Riemannian manifolds have infinite area. We can also obtain an isoperimetric inequality given in the next theorem.

We first make the following definitions.

Definition 0.0.4. The *diameter* of a compact Riemannian manifold M is defined as

$$\text{Diameter}(M) := \sup_{p, q \in M} d(p, q) \in (0, \infty).$$

Definition 0.0.5. The *radius* of compact Riemannian manifold with boundary is defined as

$$\text{Radius}(M) := \sup_{p \in M} d(p, \partial M) \in (0, \infty).$$

Theorem 0.0.6 (Isoperimetric Inequality 1). *Suppose X is a Riemannian manifold without boundary that satisfies the following isoperimetric inequality: Given L_0, H_0 , there exists A_0 such that for any compact immersed surface Σ in X with mean curvature function satisfying $|H_\Sigma| \leq H_0$, and with the length L of its boundary satisfying $L \leq L_0$,*

$$\text{Area}(\Sigma) \leq A_0 L.$$

Then there exists a C such that for any compact hypersurface Σ with at most one boundary component, mean curvature function $|H_\Sigma| \leq H_0$ and boundary length $L \leq L_0$, then

$$\text{Diameter}(\Sigma) + \text{Radius}(\Sigma) + \text{Area}(\Sigma) \leq CL.$$

In Chapter 3, we study the existence of isoperimetric inequalities in \mathbb{H}^3 for certain compact surfaces. The classical isoperimetric inequality in \mathbb{R}^2 states that $4\pi \text{Area}(D) \leq \text{Length}(\partial D)^2$ holds for any compact planar domain D , where the equality is attained precisely when ∂D is a circle. The inequality is conjectured to hold for compact minimal surfaces in \mathbb{R}^n , where it is known to hold when the minimal surface has at most two boundary curves, see [10].

In another direction, one may try to extend the classical isoperimetric inequality to more general submanifolds with variable mean curvature vector in a Riemannian manifold. W. Allard [1] gave an isoperimetric inequality for submanifolds in \mathbb{R}^n which involves the mean curvature term:

$$\text{Vol}(M)^{m-1} \leq c(m)(\text{Vol}(\partial M) + \int_M |H|^m). \quad (1)$$

Then D. Hoffman and J. Spruck [7] generalized this result to submanifolds in a Riemannian manifold.

For the next discussion we need the following definition; recall that a metric Lie group is just a Lie group with a left invariant metric, see Definition 1.6.3.

Definition 0.0.7. For a 3-dimensional metric Lie group X , the *critical mean curvature* $H(X)$ of X is defined to be

$$H(X) = \inf_{\Sigma \in \mathcal{A}} \max_{\Sigma} |H_{\Sigma}|,$$

where \mathcal{A} is the collection of all compact, immersed orientable surfaces in X and H_{Σ} stands for the mean curvature function of Σ .

Meeks, Mira, Perez and Ros proved that if X is a noncompact, simply connected, 3-dimensional metric Lie group, then the critical mean curvature of X is equal to twice the Cheeger constant of X ; see [11] for details.

Conjecture 0.0.8 (Meeks, Mira, Perez and Ros). *Let X be a metric Lie group diffeomorphic to \mathbb{R}^3 . Given $L > 0$, there exists a $C_L > 0$ such that for any compact immersed surface Σ in X with one boundary component of length at most L and absolute mean curvature function satisfying $|H_X| \leq H(X)$, where $H(X)$ is the critical curvature of X , then*

$$\text{Area}(\Sigma) \leq C_L \cdot \text{Length}(\partial\Sigma).$$

In this thesis, we give some results on isoperimetric inequalities for compact surfaces immersed in \mathbb{H}^3 , which has critical curvature 1; note that \mathbb{H}^3 is an example of a metric Lie group where the group is the orientation preserving similarities of \mathbb{R}^2 .

We prove the following linear isoperimetric inequality for certain surfaces in \mathbb{H}^3 .

Theorem 0.0.9. *Let Σ be a compact surface with boundary in \mathbb{H}^3 . Suppose $|H_{\Sigma}| \leq 1 - \varepsilon$, where $\varepsilon \in (0, 1]$. There exists a constant $C(\varepsilon)$ such that*

$$\text{Area}(\Sigma) \leq C(\varepsilon) \cdot \text{Length}(\partial\Sigma).$$

If we assume that a surface Σ in \mathbb{H}^3 is contained in a bounded domain, then we obtain the next result.

Theorem 0.0.10. *Let Σ be a compact surface in a bounded domain R in \mathbb{H}^3 with compact boundary. Suppose $|H_\Sigma| \leq 1$. Then there exists some constant C_R such that*

$$\text{Area}(\Sigma) \leq C_R \cdot \text{Length}(\partial\Sigma).$$

Using the observation that compact surfaces with one boundary component of length at most L_0 and immersed in \mathbb{H}^3 are contained in a bounded geodesic ball of radius at most $L_0/2$, we can obtain the following result.

Corollary 0.0.11. *Suppose that Σ is a compact surface with one boundary component. Suppose $\text{Length}(\partial\Sigma) \leq L_0$ and $|H_\Sigma| \leq 1$. Then we have*

$$\text{Area}(\Sigma) \leq \frac{e^{L_0}}{2} \text{Length}(\partial\Sigma).$$

Furthermore, we conjecture:

Conjecture 0.0.12. *Let Σ be an immersed compact surface in \mathbb{H}^3 with absolute mean curvature function $|H_\Sigma| \leq 1$ and boundary of length at most $L > 0$, then*

$$\text{Area}(\Sigma) \leq \frac{e^{D(L)}}{2} \text{Length}(\partial\Sigma)$$

for some constant $D(L)$.

A conjecture on the sharp isoperimetric inequality problem in \mathbb{H}^3 can be stated below which holds for disks in \mathbb{H}^3 .

Conjecture 0.0.13. *Let Σ be a compact immersed surface with boundary of length at most $L > 0$ and absolute mean curvature function $|H_\Sigma| \leq 1$, then*

$$\text{Area}(\Sigma) \leq \frac{1}{4\pi} (\text{Length}(\partial\Sigma))^2.$$

Moreover, if one has equality in the above formula, then Σ is a round disk in a horosphere in \mathbb{H}^3 .

CHAPTER 1

FOUNDATIONAL MATERIAL

1.1 Riemannian manifold

Definition 1.1.1. A *Riemannian metric* g on a differentiable manifold M is a family of inner products $\{g_p\}$ on the tangent spaces T_pM which depend smoothly on the point p ; (M, g) is called *Riemannian manifold*.

In the local coordinates $x = (x^1, x^2, \dots, x^n)$ of M , the metric is represented by a symmetric positive definite matrix

$$(g_{ij}(x))_{i,j=1,\dots,n}.$$

The *inner product* of two vectors $v = v^i \frac{\partial}{\partial x^i}, w = w^j \frac{\partial}{\partial x^j}$ in the tangent space T_pM is

$$\langle v, w \rangle = g_{ij}(x(p))v^i w^j.$$

Then the length of v is given by

$$\|v\| = \langle v, v \rangle^{\frac{1}{2}}.$$

Suppose $\gamma: [a, b] \rightarrow M$ is a smooth curve, then the *length* of γ is defined as

$$L(\gamma) = \int_a^b \left\| \frac{d\gamma}{dt} \right\| dt.$$

The distance between two points p, q can be defined as

$$d(p, q) := \inf_{\gamma \in \mathcal{C}} \{L(\gamma)\},$$

where \mathcal{C} is the set of all piecewise smooth curves connecting p, q .

The *diameter* of M is defined as

$$\text{Diameter}(M) := \sup_{p, q \in M} d(p, q) \in (0, \infty].$$

The *radius* of a compact Riemannian manifold with boundary is defined as

$$\text{Radius}(M) := \sup_{p \in M} d(p, \partial M) \in (0, \infty).$$

If there is a constant c such that for any $t \in [a, b]$ there is a neighborhood $U \subset [a, b]$ of t such that for any $t_1, t_2 \in U$, we have

$$d(\gamma(t_1), \gamma(t_2)) = c|t_1 - t_2|,$$

then γ is called a *geodesic*. The geodesic is often equipped with a natural parametrization, i.e., in the above identity $c = 1$, and the parametrization is unit speed:

$$d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|, \text{ for } t_1, t_2 \in U.$$

Geodesics joining p and q are not necessarily unique and may not exist for some cases. If the Riemannian manifold is compact, there always exists at least one geodesic with length $d(p, q)$ connecting points p and q . Without compactness, this result need not be true. For example, the two points on the punctured plane $\mathbb{R}^2 \setminus \{0\}$ that are symmetric about the origin have no geodesic joining them.

For any point p in M and for any vector v in T_pM there exists a unique geodesic $\gamma: I \rightarrow M$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$, where I is a maximal open interval in \mathbb{R} containing 0. We denote this geodesic γ by $c_v(t)$.

Definition 1.1.2. Let $V_p = \{v \in T_pM : c_v \text{ is defined on } [0, 1]\}$. The mapping $\exp_p: V_p \rightarrow M$ defined by $\exp_p(v) = c_v(1)$ is called the *exponential map* of M at p .

Definition 1.1.3. Let M be a Riemannian manifold. If $\exp_p(v)$ is well defined on T_pM at any point p of M , then M is *geodesically complete*.

In Riemannian manifolds, the properties of geodesic completeness and of metric completeness are equivalent to each other according to Hopf-Rinow Theorem [8]. So in Riemannian manifolds, we can use complete for all cases.

Note that a Riemannian manifold M is compact if and only if it is complete and has finite diameter.

Definition 1.1.4. Let M be a Riemannian manifold and let $p \in M$. The *injectivity radius function* of M at the point p is

$$i(p) := \sup\{\rho : \exp_p \text{ is a diffeomorphism on } B_\rho(0) \subset T_pM\},$$

where $B_\rho(0)$ is the ball in T_pM centered at the origin 0 of radius ρ . The *injectivity radius of M* is

$$i(M) := \inf_{p \in M} i(p).$$

1.2 Riemannian connection

To do calculus on manifolds, we need to define a connection or covariant derivative on the manifold.

Definition 1.2.1. Suppose M is a smooth manifold, and $\Gamma(TM)$ is the collection of all smooth tangent vector fields on M . Then an *affine connection* on M is a map

$$\begin{aligned}\nabla: \Gamma(TM) \times \Gamma(TM) &\rightarrow \Gamma(TM), \\ (X, Y) &\mapsto \nabla_X Y,\end{aligned}$$

such that for any $X, Y, Z \in \Gamma(TM)$ and any smooth functions $f, g \in C^\infty(M)$, it satisfies

- $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$,
- $\nabla_X(fY + gZ) = (Xf)Y + f\nabla_X Y + (Xg)Z + g\nabla_X Z$,

where Xf denotes the pointwise derivative of f in the direction of X and $\nabla_X Y$ is called the *covariant derivative* of Y in the direction of X .

Suppose (U, p) is a local coordinate chart of M , and e_i is a basis of local vector fields on U , then we have

$$\nabla_{e_j} e_k = \Gamma_{jk}^i e_i,$$

where Γ_{ij}^k are called the connection coefficients or Christoffel symbols, and these functions determine the connection on M .

Definition 1.2.2. Suppose (M, g) is a Riemannian manifold. A *Riemannian connection* ∇ is an affine connection satisfying two more conditions:

- $\nabla_X Y - \nabla_Y X = [X, Y]$, where $[X, Y]$ is the Lie bracket of the vector fields X, Y ;
- $X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product for g .

There exists a unique Riemannian connection, which is also called *Levi-Civita connection*, and the term “covariant derivative” is often referred to as the *Levi-Civita connection* in the theory of Riemannian manifolds. From the given conditions of the

Levi-Civita connection, we can find the Christoffel symbols of Levi-Civita connection which can be expressed in terms of g ,

$$\Gamma_{jk}^i = \frac{1}{2}g^{il} \left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right).$$

1.3 Sectional curvature

Let (M, g) be a n -dimensional Riemannian manifold with Levi-Civita connection ∇ . Suppose $X, Y, Z \in \Gamma(TM)$, where $\Gamma(TM)$ is the collection of all smooth vector fields on M . The Riemann curvature tensor of the Levi-Civita connection ∇ is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

where $[,]$ is the Lie bracket. In local coordinates $x = (x^1, x^2, \dots, x^n)$, we define R_{lij}^k by the formula

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l} = R_{lij}^k \frac{\partial}{\partial x^k},$$

and

$$R_{klij} = \left\langle R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l}, \frac{\partial}{\partial x^k} \right\rangle.$$

Definition 1.3.1. The *sectional curvature* of the plane spanned by the tangent vectors $X, Y \in T_x M$ of the Riemannian manifold M at x is

$$K(X, Y) := \frac{\langle R(X, Y)Y, X \rangle}{\langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2}.$$

If $\{e_i\}$ is a orthonormal basis of $T_x M$, then the sectional curvature of the plane spanned by e_i, e_j is R_{ijij} .

If M is a two-dimensional surface, then the sectional curvature is simply the Gaussian curvature. In the special case that M is Euclidean n -space, the sectional curvature of M is identically zero.

1.4 Mean curvature

Consider (M, g) as an oriented n -dimensional isometric immersed submanifold of $(n+p)$ -dimensional Riemannian manifold (N, \tilde{g}) . Suppose $X, Y \in \Gamma(TM)$, we can extend X, Y to the local vector fields \tilde{X}, \tilde{Y} on N . Let $\tilde{\nabla}$ denote the Levi-Civita connection on N . We can simply denote the $\tilde{\nabla}_{\tilde{X}}\tilde{Y}|_M$ as $\tilde{\nabla}_X Y$. It can be decomposed as

$$\tilde{\nabla}_X Y = \nabla_X Y + B(X, Y),$$

where $\nabla_X Y \in \Gamma(TM)$, $B(X, Y) \in \Gamma(TM)^\perp$. The defined map

$$B: \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM)^\perp$$

is called the *second fundamental form* of M in N . It can be represented as

$$B(X, Y) = \sum_{\alpha=n+1}^{n+p} h^\alpha(X, Y)\xi_\alpha,$$

where ξ_α is a local orthonormal basis of $\Gamma(TM)^\perp$. Because we have

$$\langle \tilde{\nabla}_X Y, \xi_\alpha \rangle + \langle Y, \tilde{\nabla}_X \xi_\alpha \rangle = \langle Y, \xi_\alpha \rangle = 0,$$

by definition of second fundamental form, we get $h^\alpha(X, Y) = -\langle Y, \tilde{\nabla}_X \xi_\alpha \rangle$.

In a local coordinate neighborhood U of a point on N , suppose $\{e_1, \dots, e_{n+p}\}$ is a orthonormal basis of $\Gamma(TU)$, such that ordered set $\{e_1, \dots, e_n\} \in \Gamma(TM)$ corresponds to the orientation of M , $\{e_{n+1}, \dots, e_{n+p}\} \in \Gamma(TM)^\perp$, and let w^1, \dots, w^{n+p} be the dual basis. Then B can be represented in local coordinates as

$$B = h_{ij}^\alpha w^i \otimes w^j \otimes e_\alpha.$$

The norm of the B

$$\|B\| = \sqrt{\langle B, B \rangle} = \sqrt{\sum_{\alpha, i, j} (h_{ij}^\alpha)^2},$$

is called the norm of the second fundamental form.

Definition 1.4.1. The *mean curvature vector* of M is

$$\mathbf{H} = \frac{1}{n} \text{trace}(B) = \frac{1}{n} \sum_i B(e_i, e_i) = \sum_\alpha \left(\frac{1}{n} \sum_i h_{ii}^\alpha \right) e_\alpha.$$

The norm of the mean curvature vector field is called the *mean curvature function* of M . By the definition, we have

$$\mathbf{H} = \sum_\alpha \left(\frac{1}{n} \sum_i h_{ii}^\alpha \right) e_\alpha = -\frac{1}{n} \sum_\alpha \left(\sum_i \langle e_i, \tilde{\nabla}_{e_i} e_\alpha \rangle \right) e_\alpha.$$

Definition 1.4.2. The mean curvature function H_M of M is

$$H_M = \|\mathbf{H}\| = \sqrt{\langle \mathbf{H}, \mathbf{H} \rangle} = \frac{1}{n} \sqrt{\sum_\alpha \left(\sum_i h_{ii}^\alpha \right)^2}.$$

If M^n is a hypersurface of N^{n+1} , then the second fundamental form of M^n is

$$B(X, Y) = h(X, Y)\xi,$$

where ξ is a unit normal vector field of M^n . In local coordinates, we can represent the second fundamental form as

$$h = h_{ij} w^i \otimes w^j.$$

The mean curvature function is simply equal to

$$H = \frac{1}{n} \sum_{i=1}^n h_{ii}.$$

1.5 Divergence of a vector field

Let (M, g) be an n -dimensional isometrically immersed submanifold of $(n + p)$ -dimensional Riemannian manifold (N, \tilde{g}) . Given $X: M \rightarrow \Gamma(TN)$, a vector field along M , let $\tilde{\nabla}X: \Gamma(TM) \rightarrow \Gamma(TN)$ be defined as the map $Y \rightarrow \tilde{\nabla}_Y X$, where we consider X to be extended to a vector field in N denoted by the same letter. (see Section 3 of [7])

Definition 1.5.1. The *divergence* of X on M is $\mathbf{div}(X) = \text{trace}_M \tilde{\nabla}X$ which is the trace of $\tilde{\nabla}X$ on $\Gamma(TM)$.

Let $\{e_1, e_2, \dots, e_n\}$ denote the orthonormal basis of the smooth vector field on M , and then we have

$$\mathbf{div}(X) = \sum_{i=1}^n \langle \tilde{\nabla}_{e_i} X, e_i \rangle.$$

By the definition of mean curvature vector, we can write the mean curvature vector as

$$\mathbf{H} = -\frac{1}{n} \sum_{\alpha} (\mathbf{div}(e_{\alpha})) e_{\alpha}.$$

In particular, if M is a hypersurface of N , we can obtain

$$\mathbf{div}(\xi) = -nH_M,$$

where ξ is the unit normal vector field to M and H_M is the mean curvature function of M .

The divergence of X in the ambient space N is given by the definition $\mathbf{DIV}(X) = \text{trace}_N \tilde{\nabla}X$. If $\{e_{n+1}, \dots, e_{n+p}\}$ is a local orthonormal basis of the normal bundle to M , then we have

$$\mathbf{DIV}(X) = \sum_{i=1}^{n+p} \langle \tilde{\nabla}_{e_i} X, e_i \rangle.$$

So we have

$$\mathbf{DIV}(X) = \mathbf{div}(X) + \sum_{i=n+1}^{n+p} \langle \tilde{\nabla}_{e_i} X, e_i \rangle.$$

In the case that M is a hypersurface of N , because of

$$\langle \tilde{\nabla}_\xi \xi, \xi \rangle = 0,$$

we have

$$\mathbf{DIV}(\xi) = \mathbf{div}(\xi) = -nH_M.$$

Next, let's introduce the divergence theorem.

Theorem 1.5.2. *Suppose M is a compact Riemannian manifold with boundary ∂M .*

For any $X \in \Gamma(TM)$, we have

$$\int_M \mathbf{div}(X) = - \int_{\partial M} \langle X, \eta \rangle,$$

where η is inward pointing co-normal to ∂M .

See [15] for more discussion on what follows. When M is a submanifold of N , it is interesting to compute $\int_M \mathbf{div}(X)$ in case the condition $X \in \Gamma(TM)$ is dropped. Let X be a smooth vector field on N . We firstly decompose X into tangential and normal parts on M :

$$X = X^\top + X^\perp,$$

where locally $X^\perp = \sum_\alpha \langle X, e_\alpha \rangle e_\alpha$. Then we have

$$\mathbf{div}(X^\perp) = \sum_\alpha \langle X, e_\alpha \rangle \mathbf{div}(e_\alpha) = -n \langle X, \mathbf{H} \rangle. \quad (1.1)$$

Applying the divergence theorem to X^\top , we can get

$$\int_M \mathbf{div}(X^\top) = - \int_{\partial M} \langle X^\top, \eta \rangle.$$

Therefore we have

$$\int_M \mathbf{div}(X) = - \int_{\partial M} \langle X^\top, \eta \rangle - n \int_M \langle X, \mathbf{H} \rangle.$$

We also can use \mathbf{div} to define the Laplace operator Δ on M by

$$\Delta f = \mathbf{div}(\nabla^M f),$$

where $\nabla^M f$ is the gradient of the function f on M .

1.6 Lie groups and homogeneous 3-manifolds

Definition 1.6.1. A Riemannian n -manifold X is *homogeneous* if the isometry group $I(X)$ of X acts transitively on X .

Definition 1.6.2. A *Lie Group* X is a smooth manifold whose group operation $(g_1, g_2) \mapsto g_1 g_2$ from $G \times G$ to G is smooth and the inverse mapping $I: G \rightarrow G$ given by $I(x) = x^{-1}$ is also smooth.

Given an element $x \in X$, let $l_x: X \rightarrow X$, $l_x(y) = xy$, denote the left translation by x . If the metric g on X is invariant under the left translation, then we say that the metric is left invariant.

Definition 1.6.3. A *metric Lie group* X is a Lie group equipped with a left invariant metric.

Let X denote a simply connected, homogeneous 3-manifold. If X is not isomorphic to the Riemannian manifold $\mathbb{S}^2 \times \mathbb{R}$, then X is isometric to a metric Lie group; see [?] for a proof.

Definition 1.6.4. For a simply connected, homogeneous 3-manifold X , the *critical mean curvature* $H(X)$ of X is defined to be

$$H(X) = \inf_{\Sigma \in \mathcal{A}} \max_{\Sigma} |H_{\Sigma}|,$$

where \mathcal{A} is the collection of all compact, immersed orientable surfaces in X and H_{Σ} stands for the mean curvature function of Σ .

1.7 Hyperbolic 3-space \mathbb{H}^3

Hyperbolic 3-space is a homogeneous 3-manifold X with critical curvature equals $H(X) = 1$ whose isometry group is 6 dimensional, which we denote by \mathbb{H}^3 . In general, hyperbolic $(n + 1)$ -space \mathbb{H}^{n+1} can be seen as being a Lie group isomorphic to the group of similarities of \mathbb{R}^n , for any $(\mathbf{a}, b) \in \mathbb{H}^{n+1}$ using upper halfspace model $\{(\mathbf{a}, b) \mid \mathbf{a} \in \mathbb{R}^n, b > 0\}$, we have

$$\begin{aligned} \phi_{(\mathbf{a}, b)}: \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ x &\mapsto bx + \mathbf{a}. \end{aligned}$$

In the upper halfspace model $\mathbb{R}_+^{n+1} = \{(x, y) \mid x \in \mathbb{R}^n, y > 0\}$, the metric for \mathbb{R}_+^{n+1} isometric to \mathbb{H}^{n+1} is described by

$$ds^2 = \frac{dx_1^2 + \dots + dx_n^2 + dy^2}{y^2}.$$

Another model for \mathbb{H}^{n+1} is the open unit ball \mathbf{B}^{n+1} centered at the origin $\mathbf{0}$ in \mathbb{R}^{n+1} with conformal metric

$$ds^2 = \frac{4(dx_1^2 + \dots + dx_{n+1}^2)}{(1 - r^2)^2}.$$

It is straightforward to verify that the following maps are mutually inverse isometries:

$$F: \mathbf{B}^{n+1} \rightarrow \mathbb{R}_+^{n+1}$$

$$(x, y) \mapsto \frac{(2x, -|x|^2 - y^2 + 1)}{(1 - y)^2 + |x|^2}$$

and

$$G: \mathbb{R}_+^{n+1} \rightarrow \mathbf{B}^{n+1}$$

$$(x, y) \mapsto \frac{(2x, |x|^2 + y^2 - 1)}{(1 + y)^2 + |x|^2}.$$

Any hypersurface of \mathbb{H}^{n+1} on which the isometries of \mathbb{H}^{n+1} act transitively must have constant mean curvature. Hence any horizontal hyperplane $y = y_0$ in the upper halfspace model is a constant mean curvature hypersurface. Also the spheres tangent to the boundary of the upper halfspace also have constant mean curvature. Actually, we always can find an isometric group action to transform tangent spheres to horizontal hyperplanes.

We next show the mean curvature of the horizontal hyperplanes is constant 1. In the upper halfspace model, we take the basis $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y}\}_{i=1, \dots, n}$ for the smooth vector fields on \mathbb{H}^{n+1} . Obviously $\{y \frac{\partial}{\partial x_i}, y \frac{\partial}{\partial y}\}_{i=1, \dots, n}$ is an orthonormal basis of the smooth vector fields. Here $e_{n+1} = y_0 \frac{\partial}{\partial y}$ is the unit normal vector field on hyperplane $y = y_0$, and $\{e_i = y_0 \frac{\partial}{\partial x_i}\}_{i=1, \dots, n}$ is an orthonormal basis of the vector fields tangent to the hyperplane. Let $\tilde{\nabla}$ denote the Levi-Civita connection on \mathbb{H}^{n+1} . Then the mean curvature of the hyperplane $y = y_0$ is

$$H = \frac{1}{n} \sum_{i=1}^n h(e_i, e_i) = -\frac{1}{n} \sum_{i=1}^n g(e_i, \tilde{\nabla}_{e_i} e_{n+1}).$$

If $\{\Gamma_{ij}^k\}$ is the Christoffel Symbol in basis $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y}\}_{i=1, \dots, n}$, then we have $\Gamma_{i(n+1)}^i = -\frac{1}{y}$ for all i and 0 otherwise. Therefore,

$$\tilde{\nabla}_{e_i} e_{n+1} = \tilde{\nabla}_{y_0 \frac{\partial}{\partial x_i}} y_0 \frac{\partial}{\partial y} = y_0^2 \tilde{\nabla}_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial y} = -y_0 \frac{\partial}{\partial x_i} = -e_i.$$

Hence, the mean curvature of the hyperplane $y = y_0$ is $H = 1$.

In the ball model, both the horizontal hyperplanes and the tangent spheres in the upper halfspace model appear in the ball model as Euclidean spheres tangent to $\partial\mathbf{B}^{n+1}$. We remark that all the horizontal hyperplanes in the halfspace model correspond to the spheres tangent to the same point in ball model, similarly the tangent spheres at a particular point in the upper halfspace model correspond to spheres in the ball model tangent to some particular point in $\partial\mathbf{B}^{n+1}$. We call these tangent spheres in ball model *horospheres*, and we just showed that all horospheres have constant mean curvature 1. The horospheres tangent at the same point in $\partial\mathbf{B}^{n+1}$ form a codimension-one foliation of \mathbb{H}^{n+1} , according to the following definitions.

Definition 1.7.1. A smooth *codimension-one lamination* of a Riemannian n -manifold X is the union of a collection of pairwise disjoint, connected, injectively immersed surfaces, with a certain local product structure. More precisely, it is a pair $(\mathcal{L}, \mathcal{A})$ satisfying:

1. \mathcal{L} is a closed subset of X ;
2. $\mathcal{A} = \{\varphi_\beta: \mathbb{D} \times (0, 1) \rightarrow U_\beta\}_\beta$ is an atlas of smooth coordinate charts of X (here \mathbb{D} is the open unit disk in \mathbb{R}^{n-1} , $(0, 1)$ is the open unit interval in \mathbb{R} and U_β is an open subset of X).
3. For each β , there exists a closed subset C_β of $(0, 1)$ such that $\varphi_\beta^{-1}(U_\beta \cap \mathcal{L}) = \mathbb{D} \times C_\beta$.

We will simply denote laminations by \mathcal{L} , omitting the charts φ_β in \mathcal{A} unless explicitly necessary. A smooth lamination \mathcal{L} is said to be a smooth *foliation* of X if $\mathcal{L} = X$. Every lamination \mathcal{L} decomposes into a collection of disjoint connected

topological hypersurfaces (locally given by $\varphi_\beta(\mathbb{D} \times \{t\})$, $t \in C_\beta$, with the notation above), called the *leaves* of \mathcal{L} . Note that if $\Delta \subset \mathcal{L}$ is any collection of leaves of \mathcal{L} , then the closure of the union of these leaves has the structure of a lamination within \mathcal{L} , which we will call a *sublamination*.

A smooth codimension-one lamination \mathcal{L} of X is said to be a *CMC lamination* if each of its leaves has constant mean curvature (possibly varying from leaf to leaf). Given $H \in \mathbb{R}$, an *H-lamination* of X is a CMC lamination all whose leaves have the same mean curvature H . If $H = 0$, the H -lamination is called a *minimal lamination*.

The horosphere foliation (Figure 1.1) is an H -foliation with constant mean curvature 1 by our previous calculations. A typical *CMC*-foliation in hyperbolic space punctured at the origin is the foliation of geodesic spheres centered at origin in the ball model (Figure 1.2). Let S_R denote the geodesic sphere of hyperbolic radius R centered at origin. Then we have

$$R = \int_0^r \frac{2}{1-r^2} dr = \ln\left(\frac{1+r}{1-r}\right).$$

Equivalently, we have $r = \tanh(R/2)$. In the polar coordinates, we have the form of metric

$$g = dr^2 + \sinh^2(r)d\theta^2,$$

where $d\theta^2$ is the standard round metric. Hence, the area of the boundary sphere is proportional to $\sinh^2(R)$. The first variation of area gives S_R has constant mean curvature $H = \coth(R)$; see Appendix B for these calculations.

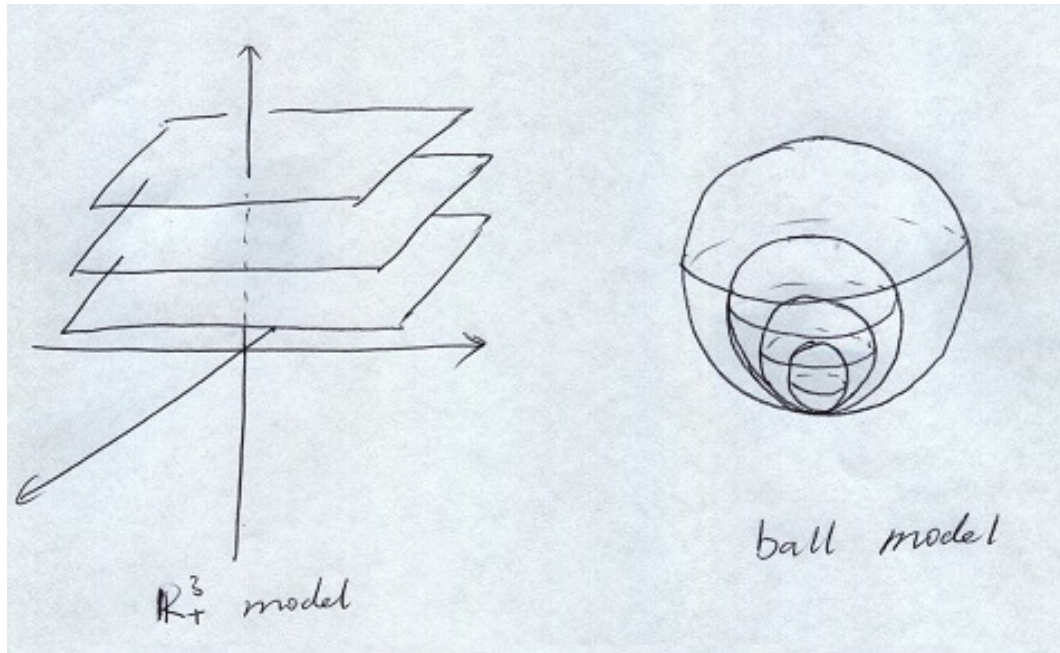


Figure 1.1. Horosphere foliation of \mathbb{H}^3 in upper half space and ball models

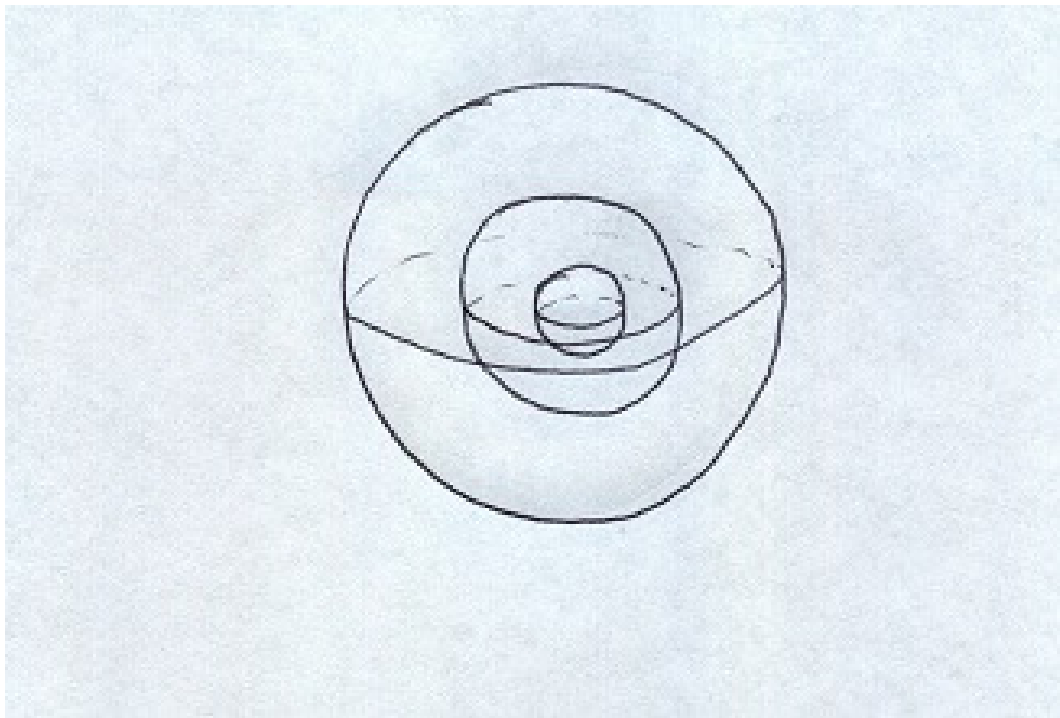


Figure 1.2. Geodesic sphere foliation of \mathbb{H}^3 in ball model

CHAPTER 2

AREA GROWTH OF HYPERSURFACES

In this chapter, we will show the uniform lower bound for the local area of a surface with bounded mean curvature. We will give estimates for areas of both extrinsic and intrinsic balls. As a consequence, we show that the area of the surface grows at least linearly. Before we give the proof of the main theorems, we introduce the coarea formula which is useful in the proof of the theorems.

2.1 Coarea formula

The coarea formula can be considered as a formula that expresses the integral of a function on a open set in terms of the integral of the integrals over level sets of some other function. The simplest version of coarea formula can be stated as Fubini Theorem

Theorem 2.1.1 (Fubini Theorem). *Suppose ϕ is an integrable function on \mathbb{R}^{n+k} , then we have*

$$\int_{\mathbb{R}^{n+k}} \phi(x^1, \dots, x^{n+k}) dx^1 \dots dx^{n+k} = \int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^n} \phi(x^1, \dots, x^{n+k}) dx^1 \dots dx^n \right) dx^{n+1} \dots dx^{n+k}.$$

Let F to be a submersion from \mathbb{R}^{n+k} to \mathbb{R}^k ,

$$\begin{aligned} F: \mathbb{R}^{n+k} &\rightarrow \mathbb{R}^k \\ (x, y) &\rightarrow y. \end{aligned}$$

Then we can reformulate Fubini theorem as

$$\int_{\mathbb{R}^{n+k}} \phi(x, y) dv_{n+k}(x, y) = \int_{\mathbb{R}^k} \int_{F^{-1}(y)} \phi(x, y) dV_n(x) dV_k(y)$$

In the general case, we consider X and Y as C^1 Riemannian manifolds of dimension $n + k$ and k equipped with metrics g_X and g_Y . Suppose $F: X \rightarrow Y$ is a C^1 function with surjective differential $D_p F: T_p X \rightarrow T_{F(p)} Y$.

To state the coarea formula, we need to define the jacobian of F . Let's consider X and Y as open subset of \mathbb{R}^{n+k} and \mathbb{R}^k . Choose (x, y) and y as the coordinate systems of X and Y . We have

$$dV_X = \rho_X dx^1 \dots dx^n dy^1 \dots dy^k,$$

$$dV_Y = \rho_Y dy^1 \dots dy^k,$$

$$dV_{F^{-1}(q)} = \rho_F dx^1 \dots dx^n,$$

where $dV_{F^{-1}(q)}$ denotes the volume density on $F^{-1}(q)$ induced by the restriction of g_X on $F^{-1}(q)$. The jacobian of F is defined as

$$J_F = \frac{\rho_Y \rho_F}{\rho_X}.$$

The general definition of jacobian can be defined via partitions of unity and the implicit function theorem. Then the general coarea formula can be stated as follows (see the proof in [13])

Theorem 2.1.2. *Let X and Y as C^1 Riemannian manifold of dimension $n + k$ and k equipped with metric g_X and g_Y . Suppose ϕ is a measurable function on X respect to the measure defined by dV_X . Then we have*

$$\int_X J_F(p) \phi(p) dV_X(p) = \int_Y \left(\int_{F^{-1}(q)} \phi(p) dV_{F^{-1}(q)}(p) \right) dV_Y(q).$$

Corollary 2.1.3. *Let X be a C^1 Riemannian manifold equipped with metric g_X . Suppose $F: X \rightarrow \mathbb{R}$ is a C^1 regular function, and $\phi: X \rightarrow \mathbb{R}$ is a measurable function. Then we have*

$$\int_X |\nabla F(p)| \phi(p) dV_X(p) = \int_{\mathbb{R}} \left(\int_{F=t} \phi(p) dV_{F^{-1}(t)}(p) \right) dt.$$

In particular, if $\phi(p) = \frac{1}{|\nabla F(p)|}$, we have

$$\text{Vol}(X) = \int_{\mathbb{R}} \left(\int_{F=t} \frac{1}{|\nabla F(p)|} dV_{F^{-1}(t)}(p) \right) dt$$

Example 2.1.4. The volume of a ball in Euclidean space \mathbb{R}^{n+1} can be described as the integral of the volume of the level set of the distance function. In Euclidean space, the distance to origin is $f(x) = |x|$. The gradient of f is $\nabla f = \frac{x}{|x|}$, and hence $|\nabla f| = 1$. Let B denote the unit ball.

$$\text{Vol}(B) = \int_0^1 \text{Vol}(\partial B_s) ds.$$

Example 2.1.5. We also can use the coarea formula to calculate the volume of unit sphere $\text{Vol}(\partial B)$. Consider the unit sphere embedded in \mathbb{R}^{n+1} ,

$$S^n = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n x_i^2 = 1\}.$$

Let f be the coordinate function x_0 , $f(p) = x_0(p) = t$. It is easy to verify this function is regular on S^n . Clearly, the level set $\{f = t\}$ is a $(n-1)$ -sphere with radius $(1-t^2)^{1/2}$. Let $\tilde{\nabla}$ be the gradient of the ambient Euclidean space, and ∇ be the gradient on unit sphere. Then we have

$$\nabla f(p) = (\tilde{\nabla} f(p))^\top = (\partial_{x_0})^\top,$$

which is the projection of $\tilde{\nabla}f(p) = \partial_{x_0}$ on the tangent plane $T_p S^n$. Hence we have,

$$|\nabla f(p)| = |\partial_{x_0}|(1 - x_0^2)^{1/2} = (1 - t^2)^{1/2}.$$

Applying coarea formula, we get

$$\begin{aligned} \text{Vol}(S^n) &= \int_{-1}^1 \left(\int_{f=t} \frac{1}{|\nabla f(p)|} dV_{f^{-1}(t)}(p) \right) dt \\ &= \int_{-1}^1 ((1 - t^2)^{-1/2} \text{Vol}(f = t)) dt \\ &= \int_{-1}^1 ((1 - t^2)^{(n-2)/2} \text{Vol}(S^{n-1})) dt \\ &= \text{Vol}(S^{n-1}) \int_{-1}^1 (1 - t^2)^{(n-2)/2} dt \end{aligned}$$

We can get the volume of the unit sphere for any dimension using this recursive formula.

2.2 Monotonicity formula

In this section, we will state the Monotonicity formula of volume for minimal submanifolds in \mathbb{R}^n , (see [4]).

Theorem 2.2.1 (Monotonicity Formula). *Let $M^k \subset \mathbb{R}^n$ be a minimal submanifold. Given $x_0 \in \mathbb{R}^n$, let $V(s) = \text{Vol}(B_s(x_0) \cap M^k)$, where $B_s(x_0)$ is the ball centered at x_0 in \mathbb{R}^n . Then for all $0 < s < t$,*

$$\frac{V(t)}{t^k} - \frac{V(s)}{s^k} = \int_{(B_t(x_0) \setminus B_s(x_0)) \cap M^k} \frac{|(x - x_0)^N|}{|x - x_0|^{k+2}}.$$

The proof of the theorem is based on the coarea formula (see [5] for a proof). Defining the function $\Theta_{x_0}(s)$ as

$$\Theta_{x_0}(s) = \frac{\text{Vol}(B_s(x_0) \cap M^k)}{\text{Vol}(B_s \subset \mathbb{R}^k)},$$

we have the following corollary from the theorem.

Corollary 2.2.2. *Let $M^k \subset \mathbb{R}^n$ be a minimal submanifold and $x_0 \in \mathbb{R}^n$. Then the function $\Theta_{x_0}(s)$ is a nondecreasing function of s . Moreover, $\Theta_{x_0}(s) \geq 1$ if $x_0 \in M^k$, $\Theta_{x_0}(s) = 1$ if and only if M^k is a part of some k -dimensional plane in \mathbb{R}^n .*

Since $\Theta_{x_0}(s)$ is nondecreasing, we can define the density at x_0 as

$$\Theta_{x_0} = \lim_{s \rightarrow 0} \Theta_{x_0}(s).$$

If $x_0 \in M^k$, then we have $\Theta_{x_0} \geq 1$ by the corollary.

Corollary 2.2.3. *Let $M^k \subset \mathbb{R}^n$ be a minimal submanifold and $V(s) = \text{Vol}(B_s(x_0) \cap M^k)$, then $V(s) \geq w(k)s^k$ when $x_0 \in M$, where $w(k)$ is the unit volume of k -dimensional ball.*

For constant mean curvature submanifolds, we can establish a similar monotonicity formula. In 1989, Korevaar, Kusner, and Solomon [9] proved the “monotonicity of the area growth” of a constant mean curvature surface. The proof of the following theorem can be found in [9].

Theorem 2.2.4. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be a properly embedded hypersurface of constant mean curvature H ($H > 0$), and $A(r) = \text{Area}(B_r(p) \cap \Sigma)$ ($p \in \mathbb{R}^{n+1}$). Then we have the inequality*

$$\frac{d}{dr} \left(\frac{A(r)}{r^n} \right) \geq -(n+1)w(n+1)H,$$

and for $0 < r < 1$,

$$r^{-n}A(r) \leq |A(1)| + (n+1)w(n+1)H(1-r),$$

where $w(n+1)$ is unit volume of $(n+1)$ -dimensional ball.

Next, we will show another version of monotonicity formula for a constant mean curvature hypersurface. More generally, we will consider the hypersurface with bounded absolute mean curvature in \mathbb{R}^{n+1} .

Theorem 2.2.5. *Let $M \subset \mathbb{R}^{n+1}$ be a hypersurface with bounded mean curvature $|H| \leq H_0$. Given $x_0 \in \mathbb{R}^{n+1}$, let $V(s) = \text{Vol}(B_s(x_0) \cap M)$, where $B_s(x_0)$ is the ball centered at x_0 in \mathbb{R}^{n+1} . Then we have the following inequalities,*

$$\frac{d}{ds} \left(\frac{e^{H_0 s} V(s)}{s^n} \right) \geq \frac{e^{H_0 s}}{s^{n+1}} \int_{\partial B_s(x_0) \cap M} \frac{|(x-x_0)^N|^2}{|(x-x_0)^T|},$$

$$\frac{e^{H_0 t} V(t)}{t^n} - \frac{e^{H_0 s} V(s)}{s^n} \geq \int_{(B_t(x_0) \setminus B_s(x_0)) \cap M} e^{H_0 |x-x_0|} \frac{|(x-x_0)^N|^2}{|x-x_0|^{n+2}}.$$

Proof. Without loss of generality, we can pick $x_0 = 0$. Let the function d be the extrinsic distance to x_0 on M , $d(x) = |x|$. Let Δ be the Laplace operator on M , ∇ be the gradient operator on M , and N be the unit normal vector on M . Then we have $\nabla d = x^\top / |x|$, where x^\top is the tangent part of x to M , and

$$\Delta d^2(x) = \Delta \langle x, x \rangle = 2 \langle \nabla x, \nabla x \rangle + 2 \langle \Delta x, x \rangle = 2n + 2H \langle N, x \rangle.$$

Because of $|\langle N, x \rangle| \leq |N| \cdot |x| = |x|$, we can get

$$\Delta d^2(x) \geq 2n - 2H_0 |x|.$$

By divergence theorem,

$$\int_{B_s(x_0) \cap M} \Delta d^2 = \int_{\partial B_s(x_0) \cap M} \langle \nabla d^2, \eta \rangle = 2 \int_{\partial B_s(x_0) \cap M} |x^T|.$$

Integrating the right term of the above inequality,

$$\int_{B_s(x_0) \cap M} (2n - 2H_0|x|) = 2nV(s) - 2H_0 \int_{B_s(x_0) \cap M} |x| \geq 2nV(s) - 2H_0sV(s).$$

By the above formulas, we have

$$H_0sV(s) - nV(s) \geq - \int_{\partial B_s(x_0) \cap M} |x^T|.$$

The coarea formula gives

$$V(s) = \int_{B_s(x_0) \cap M} |\nabla d|^{-1} |\nabla d| = \int_0^s \int_{\partial B_t(x_0) \cap M} |\nabla d|^{-1} dt = \int_0^s \int_{\partial B_t(x_0) \cap M} \frac{|x|}{|x^T|} dt,$$

hence we have

$$\frac{dV(s)}{ds} = \int_{\partial B_s(x_0) \cap M} \frac{|x|}{|x^T|} ds.$$

Then we have

$$\begin{aligned} \frac{d}{ds} \left(\frac{e^{H_0s} V(s)}{s^n} \right) &= \frac{H_0 e^{H_0s} V(s)}{s^n} - \frac{n e^{H_0s} V(s)}{s^{n+1}} + \frac{e^{H_0s}}{s^n} \frac{dV(s)}{ds} \\ &= \frac{e^{H_0s}}{s^{n+1}} \left(H_0sV(s) - nV(s) + s \frac{dV(s)}{ds} \right) \\ &\geq \frac{e^{H_0s}}{s^{n+1}} \left(- \int_{\partial B_s(x_0) \cap M} |x^T| + s \int_{\partial B_s(x_0) \cap M} \frac{|x|}{|x^T|} ds \right) \\ &\geq \frac{e^{H_0s}}{s^{n+1}} \left(- \int_{\partial B_s(x_0) \cap M} |x^T| + \int_{\partial B_s(x_0) \cap M} \frac{|x|^2}{|x^T|} ds \right) \\ &= \frac{e^{H_0s}}{s^{n+1}} \int_{\partial B_s(x_0) \cap M} \left(\frac{|x|^2}{|x^T|} - |x^T| \right) \\ &= \frac{e^{H_0s}}{s^{n+1}} \int_{\partial B_s(x_0) \cap M} \frac{|x^N|^2}{|x^T|}. \end{aligned}$$

Hence we have the first inequality in the statement of Theorem 2.2.5.

Next rewrite the inequality as

$$\frac{d}{ds} \left(\frac{e^{H_0 s} V(s)}{s^n} \right) \geq \int_{\partial B_s(x_0) \cap M} \frac{e^{H_0 |x|} |x^N|^2}{|x|^{n+1} |x^T|} = \int_{\partial B_s(x_0) \cap M} \frac{e^{H_0 |x|} |x^N|^2}{|x|^{n+2}} |\nabla d|^{-1},$$

and then integrating above inequality and applying coarea formula once again, gives the second inequality. \square

Corollary 2.2.6. *Suppose $M \subset \mathbb{R}^{n+1}$ is a hypersurface with bounded mean curvature $|H| \leq H_0$, then $e^{H_0 s} s^{-n} V(s)$ is nondecreasing in s . In particular, $V(s) \geq e^{-H_0 s} w(n) s^n$ when $x_0 \in M$. If s satisfies $0 < s < R$ ($R < \text{dist}(x_0, \partial M)$), we have $V(s) \geq C s^n$, where $C = e^{-H_0 R} w(n)$.*

Proof. By the previous theorem, $e^{H_0 s} s^{-n} V(s)$ is nondecreasing in s . When $x_0 \in M$, we have

$$\lim_{s \rightarrow 0} \frac{e^{H_0 s} V(s)}{s^n} = w(n),$$

and hence we conclude

$$\frac{e^{H_0 s} V(s)}{s^n} \geq w(n).$$

Equivalently, we have $V(s) \geq e^{-H_0 s} w(n) s^n$. For $0 < s < R$, we have $e^{-H_0 s} > e^{-H_0 R}$, so

$$V(s) \geq e^{-H_0 R} w(n) s^n.$$

\square

The above results are for hypersurfaces in Euclidean spaces, but they should be true for submanifolds of higher codimension. In fact, W. Allard ([1]) showed similar results for general dimensional varifolds in Euclidean spaces.

2.3 Area growth of surfaces

In Corollary 2.2.6, we proved a monotonicity result for the volume of the intersection of extrinsic ball and the surface. In fact, later we will describe similar results

for the volume of intrinsic balls. Similarly, we have the following result for minimal surfaces in Euclidean spaces.

Theorem 2.3.1 (Classical Monotonicity Theorem). *Let M be a complete minimal hypersurface in \mathbb{R}^{n+1} , and $B(s)$ is a geodesic ball in M with radius s , then $\text{Vol}(B(s)) \geq w(n)s^n$.*

We can get the next result directly by applying the above theorem. (See [2] for details.)

Corollary 2.3.2. *If M is a complete minimal hypersurface in \mathbb{R}^{n+1} , then every end of M has infinite volume.*

Proof. In fact, we will show that for any compact set $K \subset M$, every component of $M \setminus K$ has infinite volume. Let E be a component of $M \setminus K$. If E has finite volume, choose R big enough such that

$$w(n)R^n > \text{Vol}(E).$$

Let p be a point in E such that the intrinsic distance $r(p, \partial E) \geq R$ to the boundary of E , and let $B(R)$ be the geodesic ball of E centered at p . By the Theorem 2.3.1, then

$$\text{Vol}(E) \geq \text{Vol}(B(R)) \geq w(n)R^n > \text{Vol}(E),$$

a contradiction. □

In the case M is a minimal submanifold in a complete simply connected manifold N with non-positive curvature, we can obtain a lower estimate of the volume of a geodesic ball of M as follows.

Theorem 2.3.3 (Yau [18]). *If M is a minimal n -manifold in a complete simply connected manifold N with non-positive sectional curvature. $B(s)$ is a geodesic ball in M , then $s^{-n}\text{Vol}(B(s))$ is non-decreasing. Moreover*

$$\text{Vol}(B(s)) \geq w(n)s^n.$$

A consequence of the above theorem is that complete minimal submanifold of a simply connected manifold with non-positive sectional curvature has infinite volume ([18]).

In the case that the hypersurface is not necessary minimal, Cheung and Leung [3] proved the following result

Theorem 2.3.4. *If M is a n -dimensional complete noncompact submanifold with bounded mean curvature in \mathbb{R}^n or \mathbb{H}^n , then the rate of volume growth of M is at least linear, that is, for any $p \in M$ and sufficiently large $R > 0$,*

$$\text{Vol}(B_p(R)) \geq CR$$

for some constant $C > 0$, where $B_p(s)$ denotes the geodesic ball centered at p with radius R .

Similarly, a direct consequence of the above theorem is that a submanifold satisfying the hypotheses of the above theorem has infinite volume. Moreover, recall from the introduction the isoperimetric inequality formula 1 for any compact n -submanifold M with compact boundary in a Riemannian manifold N

$$\text{Vol}(M)^{n-1} \leq c(n) \left(\text{Vol}(\partial M) + \int_M |H| \right)^n.$$

If one applies the above isoperimetric inequality to the geodesic ball $B_p(R)$, then one immediately gets

$$c(n)^{\frac{1}{n}}(CR)^{\frac{n-1}{n}} \leq c(n)^{\frac{1}{n}}\text{Vol}(B_p(R))^{\frac{n-1}{n}} \leq \text{Vol}(\partial(B_p(R))) + \int_{B_p(R)} |H|,$$

which implies that the term $\text{Vol}(\partial(B_p(R))) + \int_{B_p(R)} |H|$ has a lower bound $c(n)^{\frac{1}{n}}(CR)^{\frac{n-1}{n}}$ for sufficiently large R .

For geodesic balls with small radius in hypersurface M embedded in Euclidean space, we can obtain a lower bound for the volume as follows

Theorem 2.3.5. *Suppose that M is a complete oriented hypersurface with compact boundary in \mathbb{R}^{n+1} . If the absolute mean curvature of M is bounded with $|H_M| \leq H_0$, then the volume of the intrinsic Riemannian ball $B_M(p, r)$ is at least Cr^n for $0 < r < R < \text{dist}(p, \partial M)$, where constant C depends on H_0 and R .*

Proof. We let d be the distance function of \mathbb{R}^{n+1} , and r be the intrinsic distance function of M . Without loss of generality, we can choose $p = 0$. And we will write the distance functions as $d(x)$, $r(x)$ if the base point is 0. Let Δ denote the laplace operator on M .

Suppose x is the vector/position function of M .

$$\Delta d^2(x) = \Delta \langle x, x \rangle = 2\langle \nabla x, \nabla x \rangle + 2\langle \Delta x, x \rangle = 2n + 2H\langle N, x \rangle,$$

where H is the mean curvature function on M , N is the normal vector.

$$|\langle N, x \rangle| \leq \|N\| \cdot \|x\| = d(x) \leq r(x).$$

Since $|H| \leq H_0$, then

$$\Delta d^2(x) \geq 2n - 2H_0r.$$

Let $B(s)$ be the geodesic ball of M of radius s centered at 0. Integrating the above inequality, we obtain

$$\int_{B(s)} \Delta d^2(x) dV \geq \int_{B(s)} (2n-2H_0r) dV \geq \int_{B(s)} (2n-2H_0s) dV = (2n-2H_0s)\text{Vol}(B(s)).$$

Applying Stoke's theorem, we obtain:

$$\int_{B(s)} \Delta d^2(x) dV = \int_{\partial B(s)} \langle \nabla d^2, \eta \rangle d\Sigma = \int_{\partial B(s)} 2d \frac{\partial d}{\partial r} d\Sigma \leq \int_{\partial B(s)} 2s d\Sigma = 2s\text{Vol}(\partial B(s)).$$

Thus we have

$$(2n - 2H_0s)\text{Vol}(B(s)) \leq 2s\text{Vol}(\partial B(s)).$$

In any manifold, we have

$$\text{Vol}(\partial B(s)) = \frac{\partial}{\partial r} \Big|_{r=s} \text{Vol}(B(r)).$$

We thus obtain

$$s \frac{\partial}{\partial r} \Big|_{r=s} \text{Vol}(B(s)) - (n - H_0s)\text{Vol}(B(s)) \geq 0.$$

$$\frac{\partial(s^{-n}\text{Vol}(B(s)))}{\partial s} = s^{-n-1}(s \frac{\partial}{\partial r} \Big|_{r=s} \text{Vol}(B(r)) - n\text{Vol}(B(s))) \geq -H_0s^{-n}\text{Vol}(B(s))$$

$$s^{-n}\text{Vol}(B(s)) \geq \omega(n)e^{-H_0s}.$$

Letting $C = \omega(n)e^{-H_0R}$, we have

$$\text{Vol}(B(s)) \geq Cs^n.$$

□

We just showed the local intrinsic geodesic balls in hypersurfaces of fixed bounded mean curvature in Euclidean spaces of the same dimension have volume uniformly bounded from below in terms of their radius. In fact, we can generalize the result to the hypersurfaces in ambient Riemannian manifolds.

Theorem 2.3.6. *Let H_0, I_0, S_0 be positive numbers. Suppose that M is a complete oriented hypersurface with boundary in an $(n + 1)$ -manifold N such that*

- *the absolute mean curvature function of M is at most H_0 ,*
- *the injectivity radius of N is at least I_0 ,*
- *the absolute sectional curvature of N is at most S_0 .*

Then there exist constants $c = c(n, H_0, I_0, S_0)$, $\sigma = \sigma(n, H_0, I_0, S_0)$ such that for any point $p \in M$ of distance at least σ from ∂M , and for $r \in (0, \sigma)$, the volume of the intrinsic Riemannian ball $B_M(p, r) \geq cr^n$.

First let's fix some notations and recall some definitions. Let M, N be the Riemannian manifolds in the theorem, and we consider M as a submanifold of N . Let p be a base point of M , and let U_M be a small neighborhood of p in M . For any point $q \in U_M$, we choose a normal coordinates $(u^1, u^2, u^3, \dots, u^n)$ at q , then at q we have

$$\left\langle \frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\beta} \right\rangle = \delta_{\alpha\beta},$$

$$\nabla_{\frac{\partial}{\partial u^\alpha}}^M \frac{\partial}{\partial u^\beta} = 0$$

for $\alpha, \beta = 1, \dots, n$, here ∇^M is the Riemannian connection of M .

In order to express the distance function d in N explicitly, we also choose normal coordinates $(x^1, x^2, x^3, \dots, x^{n+1})$ at p in a neighborhood U_N of N . Assume the coordinates for $q \in U_N$ is $x = (x^1, x^2, x^3, \dots, x^{n+1})$, then we have

$$d(p, q) = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2 + \dots + (x^{n+1})^2}.$$

At p we have

$$\begin{aligned} \left\langle \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta} \right\rangle &= \delta_{\alpha\beta}, \\ \nabla_{\frac{\partial}{\partial x^\alpha}}^N \frac{\partial}{\partial x^\beta} &= 0 \end{aligned}$$

for $\alpha, \beta = 1, \dots, n+1$; here ∇^N is the Riemannian connection of N . Furthermore, the Riemannian metric at q is given by the formulas below; see [14]:

$$g_{\alpha\beta}(q) = \delta_{\alpha\beta} - \frac{1}{3} x^i x^j R_{\alpha i \beta j} + o(d^2).$$

Since $\{\frac{\partial}{\partial u^\alpha}\}_{\alpha=1, \dots, n}$ are orthonormal at q (we are now using the normal coordinates around q),

$$e_\alpha = x_* \left(\frac{\partial}{\partial u^\alpha} \right) = \frac{\partial x^i}{\partial u^\alpha} \frac{\partial}{\partial x^i}$$

are orthonormal at q . Assume v is a unit normal vector to $T_q M$, then the mean curvature function of M at q is

$$H_M = \sum_{\alpha=1}^n \langle \nabla_{e_\alpha}^N e_\alpha, v \rangle.$$

Because M is isometrically immersed in N , for all X and $Y \in \Gamma(TM)$, (we consider $\Gamma(TM)$ as a subspace of $\Gamma(TN)$)

$$\nabla_X^M Y = (\nabla_X^N Y)^\top,$$

where \top is the orthogonal projection from $\Gamma(TN)$ to $\Gamma(TM)$.

So at q ,

$$\begin{aligned}
H_M v &= \sum_{\alpha=1}^n (\nabla_{e_\alpha}^N e_\alpha)^\perp = \sum_{\alpha=1}^n \nabla_{e_\alpha}^N e_\alpha - \sum_{\alpha=1}^n (\nabla_{e_\alpha}^N e_\alpha)^\top \\
&= \sum_{\alpha=1}^n \nabla_{e_\alpha}^N e_\alpha - \sum_{\alpha=1}^n \nabla_{\frac{\partial}{\partial u^\alpha}}^M \frac{\partial}{\partial u^\alpha} \\
&= \sum_{\alpha=1}^n \nabla_{e_\alpha}^N e_\alpha \\
&= \sum_{\alpha=1}^n \nabla_{\frac{\partial x^i}{\partial u^\alpha} \frac{\partial}{\partial x^i}}^N \frac{\partial x^j}{\partial u^\alpha} \frac{\partial}{\partial x^j} \\
&= \sum_{\alpha=1}^n \left(\frac{\partial^2 x^j}{(\partial u^\alpha)^2} \frac{\partial}{\partial x^j} + \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^k}{\partial u^\alpha} \Gamma_{ik}^j \frac{\partial}{\partial x^j} \right)
\end{aligned}$$

Proof of Theorem 2.3.6. We already defined the distance function $d(\cdot, \cdot)$ of N . Let $r(\cdot, \cdot)$ be the distance function of M with respect to the induced metric. We fix a base point p and will write $d(q), r(q)$ to mean the related distances from q to p .

$$\begin{aligned}
\Delta_M d^2(x) &= \sum_{\alpha=1}^n \frac{\partial^2}{\partial u^\alpha{}^2} \left(\sum_{i=1}^{n+1} (x^i)^2 \right) \\
&= 2 \sum_{\alpha,i} \left(\frac{\partial x^i}{u^\alpha} \right)^2 + 2 \sum_{\alpha,i} x^i \frac{\partial^2 x^i}{(\partial u^\alpha)^2}
\end{aligned}$$

Without of loss of generality, we can assume the coordinates of q are $(x^1, 0, 0, \dots, 0)$.

Then at q ,

$$\Delta_M d^2(x) = 2 \sum_{\alpha,i} \left(\frac{\partial x^i}{\partial u^\alpha} \right)^2 + 2x^1 \sum_{\alpha} \frac{\partial^2 x^1}{(\partial u^\alpha)^2}. \quad (2.1)$$

Also,

$$\begin{aligned}
d^2(q) &= (x^1)^2, \\
g_{ij}(q) &= \delta_{ij} - \frac{1}{3} d^2(q) R_{i1j1} + o(d^2).
\end{aligned}$$

Assume the absolute sectional curvature of N is bounded by S_0 , and we can assume

$|R_{ikjk}| < S_0$. So,

$$\delta_{ij} - \frac{S_0}{3} d^2(q) + o(d^2) \leq g_{ij}(q) \leq \delta_{ij} + \frac{S_0}{3} d^2(q) + o(d^2).$$

Since $\langle \frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\alpha} \rangle = 1$, for all $\alpha = 1 \dots n$,

$$\begin{aligned}
1 &= \left\langle \frac{\partial x^i}{\partial u^\alpha} \frac{\partial}{\partial x^i}, \frac{\partial x^i}{\partial u^\alpha} \frac{\partial}{\partial x^i} \right\rangle = g_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\alpha} \\
&= \sum_i g_{ij} \left(\frac{\partial x^i}{\partial u^\alpha} \right)^2 + \sum_{i \neq j} g_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\alpha} \\
&\leq (1 + 1/3S_0d^2 + o(d^2)) \sum_i \left(\frac{\partial x^i}{\partial u^\alpha} \right)^2 \\
&\quad + 1/2(1/3S_0d^2 + o(d^2)) \sum_{i \neq j} \left(\left(\frac{\partial x^i}{\partial u^\alpha} \right)^2 + \left(\frac{\partial x^j}{\partial u^\alpha} \right)^2 \right) \\
&= (1 + 1/3S_0d^2 + n/3S_0d^2 + o(d^2)) \sum_i \left(\frac{\partial x^i}{\partial u^\alpha} \right)^2 \\
&= \left(1 + \frac{n+1}{3}S_0d^2 + o(d^2) \right) \sum_i \left(\frac{\partial x^i}{\partial u^\alpha} \right)^2.
\end{aligned}$$

Similarly,

$$1 \geq \left(1 - \frac{n+1}{3}S_0d^2 + o(d^2) \right) \sum_i \left(\frac{\partial x^i}{\partial u^\alpha} \right)^2.$$

By Taylor series, we have

$$\sum_i \left(\frac{\partial x^i}{\partial u^\alpha} \right)^2 \geq \frac{1}{1 + \frac{n+1}{3}S_0d^2 + o(d^2)} = 1 - \frac{n+1}{3}S_0d^2 + o(d^2),$$

and then we obtain

$$2 \sum_{\alpha, i} \left(\frac{\partial x^i}{u^\alpha} \right)^2 \geq 2n \left(1 - \frac{n+1}{3}S_0d^2 + o(d^2) \right).$$

Next we estimate the second term of $\Delta_M d^2$ in equation 2.1: we have shown that

$$H_M v = \sum_{\alpha=1}^n \left(\frac{\partial^2 x^j}{(\partial u^\alpha)^2} \frac{\partial}{\partial x^j} + \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^k}{\partial u^\alpha} \Gamma_{ik}^j \frac{\partial}{\partial x^j} \right),$$

where v is the unit normal vector field to M . Let v^1 denote the first component of v , $|v^1| \leq M_1$ for some constant M_1 because v is a unit normal vector field. For some M_2 , $|\Gamma_{ik}^l| \leq M_2$ in a neighborhood of p . Then we have

$$\begin{aligned}
2x^1 \sum_{\alpha} \frac{\partial^2 x^1}{(\partial u^{\alpha})^2} &= 2H_M x^1 v^1 - 2x^1 \sum_{\alpha} \Gamma_{ik}^1 \frac{\partial x^i}{\partial u^{\alpha}} \frac{\partial x^k}{\partial u^{\alpha}} \\
&\geq -2M_1 H_0 d - 2(n+1)M_2 d \sum_{i,\alpha} \left(\frac{\partial x^i}{\partial u^{\alpha}}\right)^2 \\
&\geq -2M_1 H_0 d - \frac{2n(n+1)M_2}{1 - \frac{n+1}{3}S_0 d^2 + o(d^2)} d \\
&\geq -\widetilde{M}_1 H_0 d - \widetilde{M}_2 d,
\end{aligned}$$

where $\widetilde{M}_1 = 2M_1 > 0$ and $\widetilde{M}_2 = \frac{2n(n+1)M_2}{1 - \frac{n+1}{3}S_0 d^2 + o(d^2)} > 0$.

Hence we have:

$$\begin{aligned}
\Delta_M d^2 &\geq 2n\left(1 - \frac{n+1}{3}S_0 d^2 + o(d^2)\right) - \widetilde{M}_1 H_0 d - \widetilde{M}_2 d \\
&\geq 2n - Qd,
\end{aligned}$$

where $Q = \widetilde{M}_1 H_0 + \widetilde{M}_2$.

Let $B(s)$ be the geodesic ball in M of radius s centered at p .

$$\int_{B(s)} \Delta_M d^2 dV \geq \int_{B(s)} (2n - Qd) dV \geq \int_{B(s)} (2n - Qs) dV = (2n - Qs)\text{Vol}(B(s)). \quad (2.2)$$

Applying Stoke's theorem

$$\int_{B(s)} \Delta_M d^2 dV = \int_{\partial B(s)} 2d \frac{\partial d}{\partial r} d\Sigma \leq \int_{\partial B(s)} 2s d\Sigma = 2s\text{Vol}(\partial B(s)), \quad (2.3)$$

By the inequalities 2.2 and 2.3, we obtain:

$$(2n - Qs)\text{Vol}(B(s)) \leq 2s\text{Vol}(\partial B(s)).$$

In any manifold,

$$\text{Vol}(\partial B(s)) = \frac{\partial}{\partial r} \Big|_{r=s} \text{Vol}(B(r)),$$

and we thus obtain

$$s \frac{\partial}{\partial r} \Big|_{r=s} \text{Vol}(B(r)) - (n - \frac{Q}{2}s) \text{Vol}(B(s)) \geq 0,$$

which implies $e^{\frac{Q}{2}s} s^{-n} \text{Vol}(B(s))$ is nondecreasing. Therefore

$$\frac{e^{\frac{Q}{2}s} \text{Vol}(B(s))}{s^n} \geq \lim_{s \rightarrow 0} \frac{e^{\frac{Q}{2}s} \text{Vol}(B(s))}{s^n} = \omega(n).$$

Hence, for $0 < r < \sigma$, where σ is chosen to be small enough such that the above inequalities are true, we have

$$\text{Vol}(B(r)) \geq cr^n,$$

where $c = \omega(n)e^{-\frac{Q\sigma}{2}}$. □

S. Gadgil and H. Seshadri ([6]) show a similar result to one given in Theorem 2.3.6 for two-dimensional surfaces in Riemannian manifolds based on the isoperimetric inequality. These results can be generalized to the case of any dimensional submanifold; see Appendix A for details.

Based on the local area estimate given in Theorem 2.3.6, we claim the volume of the hypersurface grows linearly respect to the geodesic radius.

Corollary 2.3.7. *Under the hypothesis of Theorem 2.3.6, for any point $p \in M$ and for any $R \in [\sigma, \text{dist}(p, \partial M)]$, then $\text{Vol}(B_M(p, R)) \geq CR$ for some constant C depending on σ, H_0, I_0, S_0 .*

Proof. Along a geodesic with least distance joining the point p to the boundary, we can cover the geodesic by geodesic balls of radius σ without overlapping. So we can find at least $\lceil \frac{R-\sigma}{2\sigma} \rceil + 1$ geodesic balls such that the balls do not intersect with the

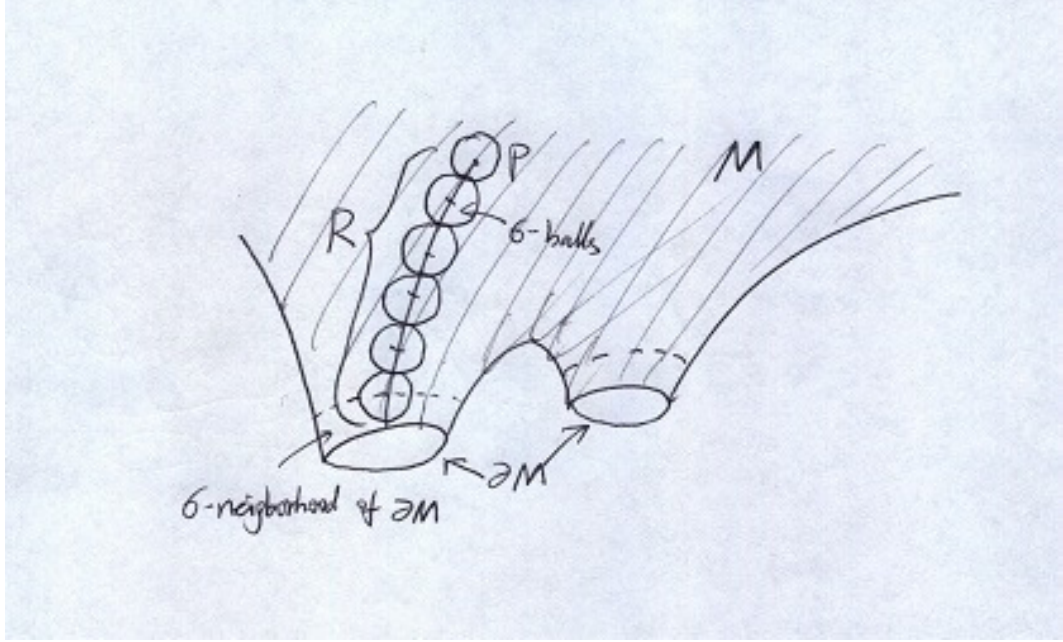


Figure 2.1. Linear area growth of geodesic balls

boundary of surface (Figure 2.1). For each geodesic ball, we have the volume of the geodesic ball is at least $c\sigma^n$. Sum up the area of these geodesic balls, we can get the area of the geodesic ball of radius R is at least $c\sigma^n([\frac{R-\sigma}{2\sigma}] + 1) \geq c\sigma^{n-1}R/2$.

□

For a 2-dimensional surface embedded in a 3-manifold, we have following result.

Theorem 2.3.8. *Let M be a complete surface with nonempty compact boundary in 3-manifold N with bounded sectional curvature, positive lower bound of injective radius. Suppose the absolute mean curvature of M satisfies $|H_M| \leq H_0$ and ∂M has at most m boundary components with total length D . Then for any point p in M such that there exists a point $q \in M$ with $R_q = d_M(p, q) > 2m\varepsilon + \frac{D}{2}$ (ε is small enough) and for any $r \in [2m\varepsilon + \frac{D}{2}, R_q]$, the area of the intrinsic Riemannian ball satisfies $\text{Area}(B_M(p, r)) \geq C\varepsilon(r - 2m\varepsilon - \frac{D}{2})$ for some constant C .*

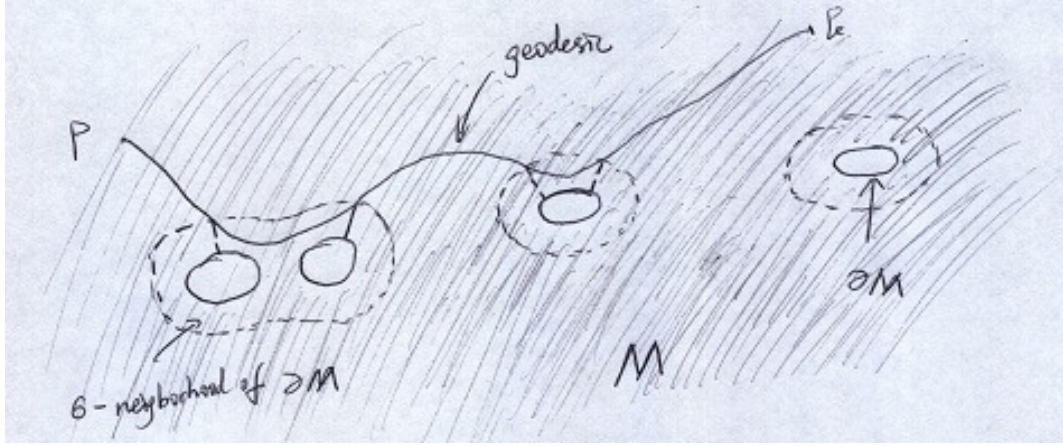


Figure 2.2. Geodesic entering the σ -neighborhood of ∂M

Proof. By the triangle inequality, we know the length of the geodesic in the ε -neighborhood of ∂M is at most $2m\varepsilon + \frac{D}{2}$ (Figure 2.2). Along the rest of the geodesic, we may cover it by geodesic balls of radius ε without overlapping. We can find at least $\lceil \frac{r-2m\varepsilon-\frac{D}{2}}{2\varepsilon} \rceil + 1$ geodesic balls of radius ε on the geodesic that do not intersect ∂M (Figure 2.3). From Theorem 2.3.6, the area of the geodesic ball of radius ε is at least $C\varepsilon^2$ for the constant C given there. Then we have the area of $B_M(p, r)$ can be estimated:

$$\text{Area}(B_M(p, r)) \geq \left(\left\lceil \frac{r-2m\varepsilon-\frac{D}{2}}{2\varepsilon} \right\rceil + 1 \right) \cdot C\varepsilon^2 \geq C\varepsilon \left(r - 2m\varepsilon - \frac{D}{2} \right).$$

□

As we know the area of surface with bounded mean curvature grows at least linearly, it is easy to prove that any end of such a surface should have infinite area. We may also use this fact to study some problems related to the isoperimetric problem.

Theorem 2.3.9. *Suppose X is a Riemannian manifold without boundary and satisfies the following isoperimetric inequality: Given L_0, H_0 , there exists A_0 such that for any compact surface Σ with $|H_\Sigma| \leq H_0$, and with the length of its boundary $L \leq L_0$,*

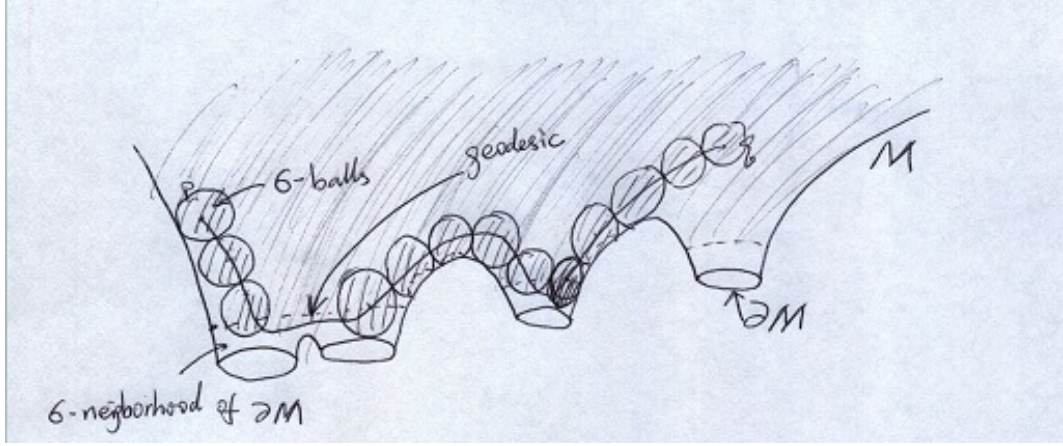


Figure 2.3. Illustration of linear area growth

$$\text{Area}(\Sigma) \leq A_0 L.$$

Then given L_0, H_0 , there exists a $C = C(H_0, L_0)$ such that for any compact hypersurface Σ with at most one boundary component, $|H_\Sigma| \leq H_0$ and bounded boundary length $L \leq L_0$,

$$\text{Diameter}(\Sigma) + \text{Radius}(\Sigma) + \text{Area}(\Sigma) \leq CL.$$

Proof. First, let's prove $\text{Radius}(\Sigma) \leq C_R \cdot L$ for some constant C_R . If it is not true, then for any constant C_R , we can find a surface Σ with $\text{Length}(\Sigma_n) \leq L_0$, $|H_{\Sigma_n}| \leq H_0$ such that

$$\text{Radius}(\Sigma) > C_R \cdot L.$$

By Corollary 2.3.7, we can find a constant c such that $\text{Area}(\Sigma) \geq c \cdot \text{Radius}(\Sigma)$. Hence, we have

$$\text{Area}(\Sigma) > c \cdot C_R \cdot L.$$

Choose a sequence C_R goes to infinity, then we get contradiction to $\text{Area}(\Sigma) \leq A_0 L$.

Then, we prove $\text{Diameter}(\Sigma) \leq C_D L$ for some constant C_D . Pick any two points p and q on Σ . Because $\text{Radius}(\Sigma) \leq C_R \cdot L$, then we have $\text{dist}(p, \partial\Sigma) \leq C_R \cdot L$ and $\text{dist}(q, \partial\Sigma) \leq C_R \cdot L$. Hence

$$\text{dist}(p, q) \leq \text{dist}(p, \partial\Sigma) + \text{dist}(q, \partial\Sigma) + \frac{L}{2} \leq C_R \cdot L + C_R \cdot L + \frac{L}{2} = C_D L,$$

where $C_D = (2C_R + \frac{1}{2})$.

Therefore, we have

$$\text{Diameter}(\Sigma) + \text{Radius}(\Sigma) + \text{Area}(\Sigma) \leq (C_R + C_D + A_0)L.$$

□

CHAPTER 3

ISOPERIMETRIC INEQUALITY IN \mathbb{H}^3

3.1 Classical isoperimetric inequalities

The classical isoperimetric inequality can be stated as following theorem

Theorem 3.1.1. *Let $D \subset \mathbb{R}^2$ be a planar region bounded by the simple closed curve C with length L . Then*

$$4\pi \text{Area}(D) \leq L^2,$$

with equality if and only if C is a circle.

More generally, the isoperimetric inequality can be extended to subregions in higher dimensional Euclidean spaces.

Theorem 3.1.2. *For any bounded region $D \subset \mathbb{R}^m$, we have*

$$m^m w(m) \text{Vol}(D)^{m-1} \leq \text{Vol}(\partial D)^m,$$

where $w(m)$ is the volume of unit ball in \mathbb{R}^m , and equality holds if and only if D is a ball.

Conjecture 3.1.3. *Any compact m -dimensional minimal submanifold M of \mathbb{R}^n satisfies the above inequality with equality if and only if M is an m -dimensional ball.*

Some partial results have been proved for $m = 2$. L. Simon showed the isoperimetric inequality that $2\pi \text{Area}(\Sigma) \leq \text{Length}(\partial\Sigma)^2$ for any minimal surface $\Sigma \subset \mathbb{R}^n$. And then A. Stone improved the result to $2\sqrt{2}\pi \text{Area}(\Sigma) \leq \text{Length}(\partial\Sigma)^2$ [16]. Moreover,

for $m = 2$ we have the next conjecture: which is known to hold when the number of boundary curves of the compact minimal surface is at most 2; see the paper [10] by P. Li, R. Schoen and S. T. Yau for this last result.

Conjecture 3.1.4. *Any compact minimal surface $\Sigma \subset \mathbb{R}^n$ satisfies:*

$$4\pi \text{Area}(\Sigma) \leq \text{Length}(\partial\Sigma)^2,$$

and equality holds if and only if Σ is a disk.

For an m -dimensional compact submanifold $M \subset \mathbb{R}^n$, the following inequality is given by W. Allard [1], J. Michael and L. Simon [12]:

$$\text{Vol}(M)^{m-1} \leq c(m)(\text{Vol}(\partial M) + \int_M |H|^m),$$

where $c(m)$ is a constant which depends on m and H is the mean curvature vector of M in \mathbb{R}^n . Then D. Hoffman and J. Spruck generalized these results to obtain the next theorem; see Theorem 2.2 in [7]:

Theorem 3.1.5 (Hoffman-Spruck). *Let M be a compact m -submanifold with boundary ∂M immersed in Riemannian n -manifold N with sectional curvature satisfies $K \leq b^2$, where b is either positive or pure imaginary. Then*

$$\text{Vol}(M)^{m-1} \leq c(m)(\text{Vol}(\partial M) + \int_M |H|^m),$$

provided for some $a \in \mathbb{R}$,

$$b^2(1 - a)^{-2/m}(\omega(m)^{-1}\text{Vol}(M))^{2/m} \leq 1,$$

and the injectivity radius I_0 of N satisfies

$$2\rho_0 \leq I_0,$$

where $\rho_0 = b^{-1} \sin^{-1} b(1 - a)^{-1/m} (\omega(m)^{-1} \text{Vol}(M))^{1/m}$ for b is real, and $\rho_0 = (1 - a)^{-1/m} (\omega(m)^{-1} \text{Vol}(M))^{1/m}$ for b is imaginary.

The following is a corollary of the above theorem.

Corollary 3.1.6. *Let M be a compact m -submanifold with boundary in a Riemannian n -manifold (N, g) with the same hypothesis as Theorem 3.1.5. Then there exists a constant $v_0 = v_0(n, I_0, K)$ such that either $\text{Vol}(M) \geq v_0$ or*

$$\text{Vol}(M)^{m-1} \leq \beta(\text{Vol}(\partial M) + \int_M |H|^m).$$

3.2 Isoperimetric inequalities in \mathbb{H}^3

In this chapter, we will show some isoperimetric inequalities in the case the ambient space $X = \mathbb{H}^3$. Before we give the statement of the theorem, we prove a lemma which is important in the proofs of theorems stated later in this chapter.

Lemma 3.2.1. *Suppose Σ is a immersed hypersurface in \mathbb{H}^3 , $\{e_1, e_2\}$ is an orthonormal basis of tangent vector field on Σ and e_3 is a unit normal vector field to Σ . Let \mathcal{F} be either a family of all geodesic spheres centered at a point of \mathbb{H}^3 or a horosphere foliation of \mathbb{H}^3 . Let $N_{\mathcal{F}}$ be the unit normal field to \mathcal{F} so that with respect to the induced orientation on the leaves, the leaves have positive mean curvature. Then we have for all $p \in \Sigma$,*

$$\langle \nabla_{e_3} N_{\mathcal{F}}, e_3 \rangle(p) = -H_{\mathcal{F}} \cdot \|e_3^{\top}\|^2(p),$$

where e_3^{\top} is the projection of e_3 to the tangent plane to the leaf of \mathcal{F} at p , $H_{\mathcal{F}}$ is the mean curvature function of the foliation.

Proof. Decompose e_3 to its tangent and normal parts to the leaves of \mathcal{F} as $e_3 = e_3^{\top} + e_3^{\perp}$ (Figure 3.1), then where it makes sense, we have

$$\nabla_{e_3} N_{\mathcal{F}} = \nabla_{e_3^{\top} + e_3^{\perp}} N_{\mathcal{F}} = \nabla_{e_3^{\top}} N_{\mathcal{F}} + \nabla_{e_3^{\perp}} N_{\mathcal{F}} = \nabla_{e_3^{\top}} N_{\mathcal{F}},$$

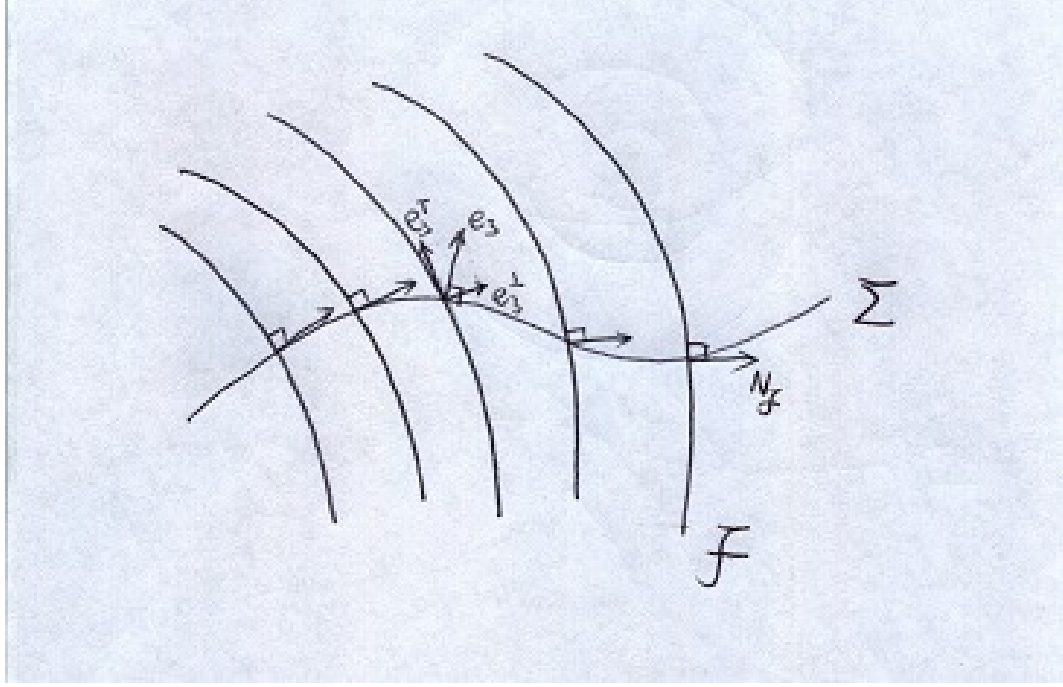


Figure 3.1. Normal vector field on Σ and normal vector field to foliation

because $\nabla_{e_3^\perp} N_{\mathcal{F}} = 0$. Since $\langle \nabla_{e_3^\perp} N_{\mathcal{F}}, e_3^\perp \rangle = 0$, we have

$$\langle \nabla_{e_3} N_{\mathcal{F}}, e_3 \rangle = \langle \nabla_{e_3^\perp} N_{\mathcal{F}}, e_3 \rangle = \langle \nabla_{e_3^\perp} N_{\mathcal{F}}, e_3^\top + e_3^\perp \rangle = \langle \nabla_{e_3^\perp} N_{\mathcal{F}}, e_3^\top \rangle.$$

Defining $\xi = \frac{e_3^\top}{\|e_3^\top\|}$, we have

$$\langle \nabla_{e_3^\perp} N_{\mathcal{F}}, e_3^\top \rangle = \|e_3^\top\|^2 \langle \nabla_{\xi} N_{\mathcal{F}}, \xi \rangle.$$

Note $-\langle \nabla_{\xi} N_{\mathcal{F}}, \xi \rangle$ equals the principal curvature of the leaf of \mathcal{F} in the direction ξ , and since horospheres and geodesic spheres have constant second fundamental forms,

$$\langle \nabla_{\xi} N_{\mathcal{F}}, \xi \rangle = -H_{\mathcal{F}}.$$

Therefore we have

$$\langle \nabla_{e_3} N_{\mathcal{F}}, e_3 \rangle = -H_{\mathcal{F}} \cdot \|e_3^\top\|^2.$$

□

We next give a linear isoperimetric inequality for surfaces with absolute mean curvature bounded by some number less than the critical mean curvature 1 of the ambient space \mathbb{H}^3 .

Theorem 3.2.2. *Let Σ be a compact surface with boundary in \mathbb{H}^3 and suppose $|H_\Sigma| \leq 1 - \varepsilon$, where $\varepsilon \in (0, 1]$. There exists a constant $C(\varepsilon)$ such that*

$$\text{Area}(\Sigma) \leq C(\varepsilon) \cdot \text{Length}(\partial\Sigma).$$

Proof. In what follows, we let \mathbf{H} denote the mean curvature vector field of Σ .

Let \mathcal{F} be the horosphere foliation of \mathbb{H}^3 , and $N_{\mathcal{F}}$ be the unit normal vector field of \mathcal{F} such that the mean curvature of the leaves is 1. Let \mathbf{DIV} be the divergence on \mathbb{H}^3 and \mathbf{div} be the divergence on Σ , then we have

$$\mathbf{DIV}N_{\mathcal{F}} = \mathbf{div}N_{\mathcal{F}} + \langle \nabla_{e_3}N_{\mathcal{F}}, e_3 \rangle, \quad (3.1)$$

where e_3 is a unit normal vector field of Σ . By the lemma, we have $\langle \nabla_{e_3}N_{\mathcal{F}}, e_3 \rangle = -\|e_3^\top\|^2$. If we denote the angle between $N_{\mathcal{F}}$ and e_3 by θ , then we have $\langle \nabla_{e_3}N_{\mathcal{F}}, e_3 \rangle = -\sin^2\theta$.

Using $\langle \nabla_{e_3}N_{\mathcal{F}}, e_3 \rangle = -\sin^2\theta$ and integrating the above equality 3.1 over Σ gives us

$$\int_{\Sigma} \mathbf{DIV}N_{\mathcal{F}} = \int_{\Sigma} \mathbf{div}N_{\mathcal{F}} - \int_{\Sigma} \sin^2\theta.$$

Since we have

$$\int_{\Sigma} \mathbf{div}N_{\mathcal{F}} = - \int_{\partial\Sigma} \langle N_{\mathcal{F}}^\top, \eta \rangle - 2 \int_{\Sigma} \langle N_{\mathcal{F}}, \mathbf{H} \rangle,$$

where η is the inward conormal to the boundary of surface, we obtain

$$\int_{\Sigma} \mathbf{DIV}N_{\mathcal{F}} = - \int_{\partial\Sigma} \langle N_{\mathcal{F}}^\top, \eta \rangle - 2 \int_{\Sigma} \langle N_{\mathcal{F}}, \mathbf{H} \rangle - \int_{\Sigma} \sin^2\theta. \quad (3.2)$$

As we know

$$\mathbf{DIV} N_{\mathcal{F}} = -2H_{\text{horosphere}} = -2,$$

and

$$\langle N_{\mathcal{F}}, \mathbf{H} \rangle = H_{\Sigma} \cos \theta.$$

Substituting the above terms into equation 3.2, we get the following equality

$$\int_{\Sigma} (-2 + 2H_{\Sigma} \cos \theta + \sin^2 \theta) = - \int_{\partial\Sigma} \langle N_{\mathcal{F}}^{\top}, \eta \rangle.$$

By straightforward computation, we obtain

$$-2 + 2H_{\Sigma} \cos \theta + \sin^2 \theta = -1 + H_{\Sigma}^2 - (\cos \theta + H_{\Sigma})^2 \leq -1 + H_{\Sigma}^2 < 0.$$

Taking absolute value of the previous integral, we have

$$\left| \int_{\Sigma} (-2 + 2H_{\Sigma} \cos \theta + \sin^2 \theta) \right| \geq (1 - H_{\Sigma}^2) \text{Area}(\Sigma).$$

Since

$$|\langle N_{\mathcal{F}}^{\top}, \eta \rangle| \leq 1,$$

we have

$$\left| - \int_{\partial\Sigma} \langle N_{\mathcal{F}}^{\top}, \eta \rangle \right| \leq \text{Length}(\partial\Sigma).$$

Hence, we have

$$(1 - H_{\Sigma}^2) \text{Area}(\Sigma) \leq \text{Length}(\partial\Sigma),$$

or equivalently,

$$\text{Area}(\Sigma) \leq C(\varepsilon) \cdot \text{Length}(\partial\Sigma),$$

where $C(\varepsilon) = \frac{1}{2\varepsilon - \varepsilon^2}$.

□

If we assume that $|H| \leq 1$ instead of $|H| \leq 1 - \varepsilon$, then the isoperimetric inequality in the previous theorem is still true for surfaces in a bounded domain of \mathbb{H}^3 where the constant depends on the domain.

Theorem 3.2.3. *Let Σ be a compact surface in a bounded domain \mathcal{R} in \mathbb{H}^3 with compact boundary and suppose $|H_\Sigma| \leq 1$. Then there exists some constant $C_{\mathcal{R}}$ such that*

$$\text{Area}(\Sigma) \leq C_{\mathcal{R}} \cdot \text{Length}(\partial\Sigma).$$

Proof. In what follows, we let \mathbf{H} denote the mean curvature vector field of Σ .

Consider a family \mathcal{F} of geodesic spheres S_r of radius r centered at a fixed point in \mathbb{H}^3 , and let $N_{\mathcal{F}}$ be the unit normal vector field of \mathcal{F} such that the mean curvature is positive. As mentioned previously, the mean curvature of S_r is $H_r = \coth(r)$. Then we have

$$\mathbf{DIV} N_{\mathcal{F}} = \mathbf{div} N_{\mathcal{F}} + \langle \nabla_{e_3} N_{\mathcal{F}}, e_3 \rangle,$$

where e_3 is a unit normal vector field of Σ . We have $\langle \nabla_{e_3} N_{\mathcal{F}}, e_3 \rangle = -H_r \sin^2 \theta$, and integrating the equality over Σ gives us

$$\int_{\Sigma} \mathbf{DIV} N_{\mathcal{F}} = \int_{\Sigma} \mathbf{div} N_{\mathcal{F}} - \int_{\Sigma} H_r \sin^2 \theta.$$

By divergence theorem, we have

$$\int_{\Sigma} \mathbf{div}(N_{\mathcal{F}}) = - \int_{\partial\Sigma} \langle N_{\mathcal{F}}^{\top}, \eta \rangle - 2 \int_{\Sigma} \langle N_{\mathcal{F}}, \mathbf{H} \rangle.$$

Because

$$\langle N_{\mathcal{F}}, \mathbf{H} \rangle = H_{\Sigma} \cos \theta,$$

and

$$\mathbf{DIV} N_{\mathcal{F}} = -2H_r.$$

We obtain the equality

$$\int_{\Sigma} (-2H_r + H_r \sin^2 \theta + 2H_{\Sigma} \cos \theta) = - \int_{\partial\Sigma} \langle N_{\mathcal{F}}^{\top}, \eta \rangle.$$

By straightforward computation and since $H_r > 1$,

$$-2H_r + H_r \sin^2 \theta + 2H_{\Sigma} \cos \theta = \frac{H_{\Sigma}^2 - H_r^2}{H_r} - H_r \left(\cos \theta - \frac{H_{\Sigma}}{H_r} \right)^2 \leq \frac{H_{\Sigma}^2 - H_r^2}{H_r}.$$

Hence we must have

$$\int_{\Sigma} \frac{H_r^2 - H_{\Sigma}^2}{H_r} \leq \int_{\partial\Sigma} \langle N_{\mathcal{F}}^{\top}, \eta \rangle.$$

We will next use the estimate

$$H_r = \coth(r) = \frac{e^r + e^{-r}}{e^r - e^{-r}} = 1 + \frac{2}{e^{2r} - 1} > 1 + e^{-2r} > 1.$$

We have

$$\frac{H_r^2 - H_{\Sigma}^2}{H_r} = \frac{(H_r + |H_{\Sigma}|)(H_r - |H_{\Sigma}|)}{H_r} > H_r - |H_{\Sigma}| > 1 + e^{-2r} - 1 = e^{-2r}.$$

Because the surface is in a bounded domain \mathcal{R} , we can find a ball B_{r_0} such that

$\mathcal{R} \subset B_{r_0}$. Then we have

$$\frac{H_r^2 - H_{\Sigma}^2}{H_r} > e^{-2r_0}$$

on surface Σ . Moreover we have

$$|\langle N_{\mathcal{F}}^{\top}, \eta \rangle| \leq 1.$$

Therefore

$$e^{-2r_0} \text{Area}(\Sigma) \leq \text{Length}(\Sigma),$$

or equivalently,

$$\text{Area}(\Sigma) \leq C_{\mathcal{R}} \cdot \text{Length}(\partial\Sigma),$$

where constant $C_{\mathcal{R}} = \frac{e^{2r_0}}{2}$ depends on the radius of geodesic ball containing bounded domain \mathcal{R} . □

From the proof of the above theorem, we get the following conclusion if the bounded domain is a geodesic ball.

Corollary 3.2.4. *Let Σ be a compact surface in a geodesic ball with radius r in \mathbb{H}^3 with compact boundary and suppose $|H_{\Sigma}| \leq 1$. Then*

$$\text{Area}(\Sigma) \leq \frac{e^{2r}}{2} \cdot \text{Length}(\partial\Sigma).$$

As a corollary of Theorem 3.2.3, we will show that we have isoperimetric inequality for certain compact surfaces Σ immersed in hyperbolic space \mathbb{H}^3 with only one boundary component and $\text{Length}(\partial\Sigma) \leq L_0$.

Theorem 3.2.5 (Maximum principle for CMC surfaces). *Assume Σ_1 and Σ_2 are two surfaces with constant mean curvatures H_1, H_2 tangent at point p having the constant mean curvature vectors oriented at the same direction. Suppose Σ_1 lies on the positive constant mean curvature vector side of Σ_2 , then $H_1 > H_2$. Furthermore, if $H_1 = H_2$, then Σ_1 must coincide with Σ_2 .*

By the above maximum principle, we next show a compact surface immersed in hyperbolic space \mathbb{H}^3 with only one boundary component is contained a bounded domain in \mathbb{H}^3 .

Lemma 3.2.6. *Suppose Σ is a compact surface with one compact boundary and suppose $|H_{\Sigma}| \leq 1$. Then the surface is contained in the geodesic ball of radius $R \leq \frac{L}{2}$, where $L = \text{Length}(\partial\Sigma)$.*

Proof. First, fix any point $p \in \partial\Sigma$ and note that $\partial\Sigma \subset B_{L/2}(p)$, where $B_{L/2}(p)$ is the extrinsic ball in \mathbb{H}^3 centered at p of radius $L/2$. If the surface is not contained in $B_{L/2}(p)$, we can expand the radius of the geodesic ball until the boundary sphere S_r with radius r is tangent to the surface a last time. For any geodesic sphere S_r , we know the mean curvature is $H_r = \coth(r) \geq 1$. Since the mean curvature function of surface satisfies $|H_\Sigma| \leq 1$, we obtain a contradiction to the maximum principle at the point where S_r is furthest from p . \square

Corollary 3.2.7. *Suppose Σ is a compact surface with one boundary component and $|H_\Sigma| \leq 1$. Then we have*

$$\text{Area}(\Sigma) \leq \frac{e^L}{2}L,$$

where $L = \text{Length}(\partial\Sigma)$.

Proof. After an isometric translation of the surface Σ , we can find an ambient geodesic ball with radius $R \leq \frac{L}{2}$ containing Σ described in the statement of Lemma 3.2.6. Then we apply the Corollary 3.2.4 to prove the above estimate. \square

Furthermore, we have the following result for surfaces described in Corollary 3.2.7 with bounded boundary length.

Corollary 3.2.8. *Let Σ be a compact surface with one compact boundary component. Suppose $\text{Length}(\partial\Sigma) \leq L_0$ and $|H_\Sigma| \leq 1$. Then we have*

$$\text{Area}(\Sigma) \leq \frac{e^{L_0}}{2}\text{Length}(\partial\Sigma).$$

Proof. After an isometric translation of the surface Σ , we can assume that Σ with the properties in the corollary is contained in the geodesic ball with radius $R = \frac{L_0}{2}$ centered at some point of the boundary of Σ . Applying Corollary 3.2.4 gives the result. \square

We next state several conjectures about the diameter of compact surfaces immersed in \mathbb{H}^3 with bounded boundary length.

Conjecture 3.2.9. *Every immersed compact surface Σ in \mathbb{H}^3 with absolute mean curvature function $|H_\Sigma| \leq 1$ and boundary of length at most $L > 0$ has diameter less than some constant $D(L)$ depending L .*

Conjecture 3.2.10. *Let Σ be an immersed compact surface in \mathbb{H}^3 with absolute mean curvature function $|H_\Sigma| \leq 1$ and boundary of length at most $L > 0$, then*

$$\text{Area}(\Sigma) \leq \frac{e^{D(L)}}{2} \text{Length}(\Sigma)$$

for some constant $D(L)$.

Moreover, we have the following general conjecture:

Conjecture 3.2.11. *Let Σ be a immersed compact surface in \mathbb{H}^3 with absolute mean curvature function $|H_\Sigma| \leq 1$, then we have*

$$\text{Area}(\Sigma) \leq C(\text{Length}(\partial\Sigma)) \cdot \text{Length}(\partial\Sigma).$$

A conjecture on the sharp isoperimetric inequality problem in \mathbb{H}^3 can be stated below. We remark that the next conjecture holds for disks in \mathbb{H}^3 by the following reasoning; by the Gauss equation: $-1 = K(T_p\mathbb{H}^3) = K_\Sigma(p) - \det(B)(p)$, a surface Σ of absolute mean curvature at most one in \mathbb{H}^3 has non-positive Gaussian curvature, and thus the isoperimetric inequality below holds for disks by a classical result of A. Weil [17].

Conjecture 3.2.12. *Let Σ be a compact immersed surface with boundary and absolute mean curvature function $|H_\Sigma| \leq 1$, then*

$$\text{Area}(\Sigma) \leq \frac{1}{4\pi} (\text{Length}(\partial\Sigma))^2.$$

Moreover, if one has equality in the above formula, then Σ is a round disk in a horosphere in \mathbb{H}^3 .

APPENDIX A

LOWER AREA BOUNDS OF SMALL BALLS

Theorem A.0.13. *Let M be a compact m -submanifold with mean curvature $|H| \leq H_0$ in a Riemannian n -manifold (N, g) with the same hypothesis in Theorem 3.1.5. Then there exist $c = c(n, I_0, H_0, K)$ and $\sigma = \sigma(n, I_0, H_0, K)$ such that the volume of any geodesic ball of radius $s \leq \sigma$ in M with the induced metric satisfies*

$$\text{Vol}(B(p, s)) \geq cs^m.$$

Proof. The boundary of intrinsic geodesic ball $B(p, s)$ is piecewise smooth when r is smaller than injectivity radius. We will apply the isoperimetric inequality in Corollary 3.1.6 to the geodesic ball $B(p, r)$, $0 < r \leq s$.

If $\text{Vol}(B(p, r)) \geq v_0$ for some $r < s$, then $\text{Vol}(B(p, s)) > \text{Vol}(B(p, r)) \geq v_0$. Hence $\text{Vol}(B(p, s)) \geq s^m$ if $s \leq \sqrt[m]{v_0}$.

So we can suppose that for every $r \leq s$, we have the isoperimetric inequality

$$\text{Vol}(B(p, r))^{m-1} \leq \beta \left(\text{Vol}(\partial B(p, r)) + \int_{B(p, r)} |H| \right)^m.$$

It follows from co-area formula that $\text{Vol}(B(p, r)) = \int_0^r \text{Vol}(\partial B(p, t)) dt$. We then have

$$\begin{aligned} \frac{d}{dr} \text{Vol}(B(p, r)) &= \text{Vol}(\partial B(p, r)) \\ &\geq \text{Vol}(B(p, r)) \\ &\geq \beta^{-1} \text{Vol}(B(p, r))^{\frac{m-1}{m}} - \int_{B(p, r)} |H| dV_M \\ &\geq \beta^{-1} \text{Vol}(B(p, r))^{\frac{m-1}{m}} - H_0 \text{Vol}(B(p, r)) \end{aligned}$$

We can assume $\beta^{-1}\text{Vol}(B(p, r))^{\frac{m-1}{m}} > 2H_0\text{Vol}(B(p, r))$ for all $0 < r \leq s$. If not, we have

$$\text{Vol}(B(p, r)) \geq \left(\frac{1}{2\beta H_0}\right)^m \geq s^m$$

for $s \leq 2\beta H_0$.

Because of $\beta^{-1}\text{Vol}(B(p, r))^{\frac{m-1}{m}} > 2H_0\text{Vol}(B(p, r))$, we obtain

$$\frac{d}{dr}\text{Vol}(B(p, r)) > (2\beta)^{-1}\text{Vol}(B(p, r))^{\frac{m-1}{m}}.$$

Hence we have

$$\frac{d}{dr}\text{Vol}(B(p, r))^{\frac{1}{m}} = \frac{1}{m}\text{Vol}(B(p, r))^{\frac{1}{m}-1}\frac{d}{dr}\text{Vol}(B(p, r)) > \frac{1}{2m\beta}.$$

Integrating the above inequality gives us

$$\text{Vol}(B(p, s)) > \left(\frac{s}{2m\beta}\right)^m.$$

Therefore, for all $s \leq \sigma$ for some constant $\sigma = \sigma(n, I_0, H_0, K)$, we have

$$\text{Vol}(B(p, s)) \geq cs^m.$$

□

APPENDIX B

THE FIRST VARIATION OF AREA

Let $F: \Sigma \times (\varepsilon, \varepsilon) \rightarrow N$ be a variation of Σ with compact support and fixed boundary. Then F is the identity outside a compact set, $F(x, 0) = x$, and $F(x, t) = x$ for all $x \in \partial\Sigma$. The vector field $F_t = F_*\left(\frac{\partial}{\partial t}\right)$ restricted to Σ is called the *variational vector field* of the variation F . Then we want to compute the first variation of area of the one-parameter family of surfaces arising from the variation F of Σ . Let x_i be local coordinates on Σ . Set

$$g_{ij}(t) = g(F_{x_i}, F_{x_j}).$$

Without loss of generality, we can choose a local coordinate system on Σ such that at x it is orthonormal, that is $g_{ij}(0) = \delta_{ij}$. Hence, the area formula is

$$\text{Vol}(F(\Sigma, t)) = \int_{\Sigma} \sqrt{\det(g_{ij}(t))}.$$

Differentiating the above formula gives

$$\left. \frac{d}{dt} \right|_{t=0} \text{Vol}(F(\Sigma, t)) = \int_{\Sigma} \left. \frac{d}{dt} \right|_{t=0} \sqrt{\det(g_{ij}(t))}.$$

Because $\nabla_{F_t} F_{x_i} - \nabla_{F_{x_i}} F_t = [F_t, F_{x_i}] = 0$, we get at x ,

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \sqrt{\det(g_{ij}(t))} &= \frac{1}{2} \sum_{i=1}^k \left. \frac{d}{dt} \right|_{t=0} g_{ii}(t) = \sum_{i=1}^k g(\nabla_{F_t} F_{x_i}, F_{x_i}) \\ &= \sum_{i=1}^k g(\nabla_{F_{x_i}} F_t, F_{x_i}) = \mathbf{div} F_t. \end{aligned}$$

We have seen that (see equation 1.1)

$$\mathbf{div}F_t = \mathbf{div}F_t^\perp + \mathbf{div}F_t^\top = -kg(F_t, \mathbf{H}) + \mathbf{div}F_t^\top,$$

where \mathbf{H} is the mean curvature vector of Σ at x . Since F is identity outside a compact set, we then have $\int \mathbf{div}F_t^\top = 0$ by Stoke's theorem. Therefore, we have the first variation formula:

$$\left. \frac{d}{dt} \right|_{t=0} \text{Vol}(F(\Sigma, t)) = -k \int_{\Sigma} g(F_t, \mathbf{H}) = \int_{\Sigma} \mathbf{div}F_t.$$

For the geodesic balls centered at 0 in hyperbolic 3-space, in the polar coordinates of the ball model, the variational vector field can be taken as the unit vector field to the foliation by geodesic spheres; hence we have $|F_r| = 1$. The mean curvature of boundary of a geodesic ball is constant since it is the orbit of a subgroup of the group of isometries fixing 0. So from the first variation formula and the fact that the volume of geodesic spheres of radius r in \mathbb{H}^3 is proportional to $\sinh(r)$, we have

$$kH \sinh^k(r) = \frac{d}{dr} \sinh^k(r) = k \sinh^k(r) \coth(r).$$

Therefore, the mean curvature of the boundary of the geodesic ball with radius r is $H = \coth(r)$.

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