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A General Measure for Output-Variable Input Demand Elasticities

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In recent years the development of new production function forms has given impetus to empirical work of measuring input demand and substitution elasticities in a variety of industries. The so-called "flexible" functional forms have given us a much richer set of tools to investigate these relationships, as compared to the familiar Cobb-Douglas and constant-elasticity-of-substitution functions. The majority of researchers have reported their results in terms of input parameters estimated under the assumption of fixed output. While this is appropriate for some questions, we argue in the next section that output-variable measures often will be more useful for the problem at hand. In the third section, we derive a general expression for the output variable price elasticity of input demand, of which the well-known expression of Allen is a special case. Last, we discuss this measure in the context of several specific functional forms.

The Question of Output Variability

Although a bewildering number and variety of input substitution and demand parameters have been put forth by researchers, perhaps the most widely used measure is the simple price elasticity of input demand:

$$\epsilon_{ij} = \frac{\partial \ln X_i}{\partial \ln P_j} \bigg| \; Q, P_k \text{ for all } k \neq j,$$

where X_i is the quantity demanded of the ith input, P_j the price of the jth input, and Q refers to total output. To be consistent with demand theory, this is the elasticity whose sign should determine whether an input pair are substitutes or complements. It also may be expressed in a slightly different form: $A_{ij} = \epsilon_{ij}/S_j$, where S_j is the share of the jth input in total cost. A_{ij} is the Allen-Uzawa partial elasticity of "substitution," so called despite the fact that it is simply a normalized price elasticity of demand.

The elasticity ϵ_{ij}, calculated under the assumption that output is held constant, can indicate the characteristics of particular production surfaces that influence policy direction, i.e., the matter of substitutability and complementarity among inputs. Combinations of ϵ_{ij} also can be used to construct higher-order elasticities of substitution to study the curvature properties of production surfaces.

For matters of public policy, however, the assumption of constant output is often a disadvantage. There we are usually concerned with measuring the consequence of particular actions; for example, the effects of given subsidies on capital, of limits on land inputs, or of increases in energy prices. To be complete, we must take into account both input substitutions along given isoquants and the effects of output changes on input demand. What are needed in this case are measures of total elasticity:

$$\eta_{ij} = \frac{\partial \ln X_i}{\partial \ln P_j} \bigg| \; P_k \text{ for all } k \neq j.$$

In this case, quantities of all inputs as well as output are allowed to adjust to input price changes.

The expression that results, equation (4) below, is similar to the Slutsky equation of consumer demand theory. Thus, the general measure is somewhat analogous to the difference between ordinary and compensated demand curves in consumer behavior. The ϵ_{ij} above are analogous to elasticities on the compensated demand curve; whereas, for predicting real-world changes in consumption, we wish to know the elasticities of the ordinary demand curves. For most goods it will not make much difference which measure is used because consumers normally will spread their incomes over a large number of goods. On the production side, however, this is not the case. Most production functions contain only three or four measured inputs. For many of these inputs, therefore, the difference between elasticities with and without an output effect could be considerable.

One output-variable input demand elasticity already is available. In the case of constant returns to scale (CRTS) in production but a downward sloping output demand function, changes in output are produced when the cost function shifts, the extent of the output change being related to the price elasticity of output demand. This total input demand elasticity was provided by Allen (p. 508). Output effects also can be produced, even with a constant output price, if the production function is non-

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CRTS. In this case, shifts in a sloping cost (supply) curve over a horizontal demand curve produce changes in output. Of course output effects could result from both a downsloping output demand function and non-CRTS in production. In the next section we derive a general expression for the output-variable elasticity of substitution.

A General Expression

To derive a general expression for η_{ij}, i.e., one that permits both nonconstant returns and nonconstant output price, we make use of the following relations: production function $Q = f(X)$, dual minimum cost function $C = C(P,Q)$, and demand function $Q = \phi(R)$, where X and P are n-tuples of input quantities and prices, respectively, Q is output, and R is output price. Market clearing requires that marginal cost equal output price, or

$$C_q(P,Q) = \phi^{-1}(Q). \tag{1}$$

We use subscripts to denote partial differentiation with respect to that variable. By differentiating (1) totally, setting $dP = 0$ for all but the jth factor price and rearranging, we get:

$$\frac{\partial Q}{\partial P_J} = \eta \quad \frac{C_{qj}Q}{(R - C_{qq}Q\eta)},$$

where $\eta = \partial \ln Q/\partial \ln R$, the price elasticity of demand for output.

According to Shepard's lemma, $\partial C/\partial P_i = C_i = X_i(P,Q)$, the cost-minimizing demand curve for input i. Differentiating this demand curve with respect to the jth factor price gives

$$\frac{\partial X_i}{\partial P_j} = C_{ij} + C_{iq}\frac{\partial Q}{\partial P_j}. \tag{3}$$

Using (2) and substituting appropriately gives[1]

$$\eta_{ij} = \left.\frac{\partial \ln X_i}{\partial \ln P_j}\right|_{P_k,\ k \neq j} = S_j(A_{ij} + \eta\psi) = \epsilon_{ij} + S_j\psi\eta,$$

where $S_j = P_jX_j/C$, the share of the jth input in total cost; $A_{ij} = CC_{ij}/C_iC_j$, the Allen-Uzawa elasticity of "substitution" expressed in terms of the cost function, and

$$\psi = \frac{C_{iq}C_{jq}Q^2}{C_iC_j(1 - R^{-1}C_{qq}Q\eta)}.$$

Suppose we have a downward sloping demand curve: $0 < |\eta| < \infty$, and CRTS. In this case, marginal cost is constant, or $C_{qq} = 0$, giving

$$\psi = \frac{C_{iq}C_{jq}Q^2}{C_iC_j} = 1$$

since CRTS implies $C_i = QC_{iq}\ \forall\ i$. This gives the expression derived by Allen (p. 508):

$$\eta_{ij} = S_j(A_{ij} + \eta). \tag{4}$$

There are two cases where the expression gives constant-output elasticities, either CRTS ($C_{qq} \rightarrow \infty$) or a perfectly vertical demand curve ($\eta = 0$).

The case that has not been considered before is that characterized by $\eta \rightarrow -\infty$ and $C_{qq} > 0$, where the output effect is produced by the shifting of a sloping supply curve over a horizontal demand curve. In this case,

$$\lim_{\eta \rightarrow -\infty} (\eta\psi) = -A_{ij}\frac{C_{iq}C_{jq}}{C_{ij}C_{qq}},$$

giving

$$\eta_{ij} = S_jA_{ij}\left(1 - \frac{C_{iq}C_{jq}}{C_{ij}C_{qq}}\right). \tag{5}$$

Special Cases

It is of interest to consider the case of an output-price-constant, quantity-variable elasticity, equation (5), in the case of specific production function forms. Several recent studies (Sidhu and Baanante; Yotopoulos, Lau and Lin) have used a non-CRTS Cobb-Douglas function: $\ln Q = \Sigma_i\alpha_i \ln X_i$, with, of course, $A_{ij} = 1$, and $\Sigma\alpha_i = \mu < 1$.[2] In this case, making appropriate substitutions into (5), and recognizing that $\alpha_j = S_j/\mu$,

$$\eta_{ij} = -\frac{\alpha_j}{1 - \mu},$$

a result that was derived originally by Lau and Yotopoulos. Note that, as long as decreasing returns to scale pertain (i.e., $\mu < 1$), all inputs will be judged "complements" ($\eta_{ij} < 0$) despite the fact that $A_{ij} = 1\ \forall\ i, j$. This is another manifestation of the Cobb-Douglas inflexibility.

In the case of a multiple-input CES function,

$$Q = (\Sigma_i\alpha_iX_i^{-\beta})^{-\frac{\mu}{\beta}},$$

with a cost function of

$$C = Q^{\frac{1}{\mu}}\left(\Sigma_i\alpha_i^{\frac{1}{1+\beta}}P_i^{\frac{\beta}{1+\beta}}\right)^{\frac{1+\beta}{\beta}},$$

we have

$$\eta_{ij} = S_j\sigma\left(1 - \frac{1}{\mu}\right),$$

[1] This step makes use of the zero-profit condition, $C = RQ$, and of the symmetry conditions, $C_{ij} = C_{ji}$ and $C_{qi} = C_{iq}$.

[2] The non-CRTS functions in these studies involved subsets of inputs from overall functions that are CRTS. Suppose, on the other hand, we have an overall function that is non-CRTS but has a subset of inputs that are CRTS (when all inputs not in the subset are held constant). Then the elasticities of equation (4) apply only to this subset of inputs and are constructed from shares, S_j, Allen partial elasticities, A_{ij}, and "output" elasticities that refer to this subset of inputs. Of course, to analyze an overall production function in which some inputs are fixed and some variable would require a different analysis than that presented above.

where $\sigma = \dfrac{\beta}{1 + \beta}$. Finally, suppose we have a translog cost function:

$$\begin{aligned} \ln C = {} & \ln\alpha_0 + \Sigma_i\alpha_i \ln P_i + \alpha_q \ln Q \\ & + 1/2\Sigma_i\Sigma_j\gamma_{ij}\ln P_i \ln P_j + \\ & \Sigma_i\gamma_{iq}\ln Q \ln P_i + 1/2\gamma_{qq} (\ln Q)^2, \\ & i, j = 1, \ldots, n. \end{aligned}$$

In this case,

$$\eta_{ij} = S_j A_{ij}\left[1 - \frac{(\gamma_{iq} + S_i S_q)(\gamma_{jq} + S_j S_q)}{(\gamma_{ij} + S_i S_j)(\gamma_{qq} + S_q^2)}\right],$$

where

$$\begin{aligned} S_i &= \alpha_i + \Sigma_k\gamma_{ik}\ln P_k + \gamma_{iq} \ln Q, \text{ and} \\ S_q &= \alpha_q + \Sigma_k\gamma_{kq} \ln P_k + \gamma_{qq} \ln Q. \end{aligned}$$

If factors i and j are substitutes, then $(\gamma_{ij} + S_i S_j) > 0$. Furthermore, positive marginal cost, monotonicity, and decreasing returns to scale imply that $S_q > 0$, $S_i, S_j > 0$, and $\gamma_{qq} > 0$, respectively. Homotheticity requires that $\gamma_{iq} = 0 \;\forall\; i$. In this case, the output variable elasticity will be less than the output constant elasticity (i.e., $\eta_{ij} < S_j A_{ij}$). It need not be always negative, however, as is the case with the *CD* function. If the function is sufficiently nonhomothetic, the "output" effect could lead to $\eta_{ij} > S_j A_{ij}$. If total output decreases as the supply function shifts up (assuming decreasing returns to scale), nonhomotheticity of the right type and magnitude (say γ_{iq}, strongly negative while γ_{jq}, close to zero) could give larger output-variable elasticities than output-constant elasticities. In this case the "warping" of the isoquants is strong enough to offset the impact of the change in output.

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