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**Social choice and information: a note on the calculus
of mappings from utility spaces**

by

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Abstract

Social choice is studied in this paper as a mapping from information on utilities over states of the world to an ordering of those states of the world. The idea of using this type of information originates in the work of Sen and Roberts. This paper differs in that it uses theorems from analysis to derive its results in a straightforward manner. It also gives information on the way in which all states of the world, on any path through the set of states of the world, must be ordered.

Key words: social choice theory, information, analysis.

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1   Introduction

Social choice theory is concerned with aggregating individual reports on their desires over alternative states of the world in order to make a collective decision. What this involves, essentially, is squashing a lot of information living in a high dimension into a single dimension in order to rank the alternatives. Although choice theory is now well established, and there isn't much new to say, there is some value in looking at this squashing process from an unfamiliar angle. This is the aim of this paper.

The choice problem will be studied in this paper by treating the domain over which an ordering is to be made as the utility reports of individuals treated as an n dimensional Euclidean space with the reports of any one individual thought of as numbers on a line. This domain of information differs significantly from that in the tradition of Arrow which looks at choice over the ranking that individuals have for various states of the world. In this case the impossibility theorem shows that there is no aggregation procedure that can satisfy a small number of reasonable conditions such as: unrestricted domain of preferences; transitivity; selection of the unanimously preferred outcome; and no dictatorship. Looking at choice over the domain of utilities shifts the emphasis from impossibility theorems to questions about the existence, and characteristics of, the mappings given by choice functions under different information conditions. In addition we can use some familiar tools from calculus and analysis.

The specific purpose of the paper is to exploit these tools to develop some simple and constructive proofs of the conditions under which choice is possible and to study the character of the rankings produced by an acceptable choice function.

Sen and Roberts have also studied choice over the domain of utility reports and many of the results reported here are familiar from their work.[1] This paper derives its results in a more straightforward manner and also give information on the way in which all states of the world, on any path through the set of states of the world, must be ordered. This resolves the indeterminacy left by concepts like Pareto efficiency. This has the advantage of giving a better picture of the underlying spatial structure of choice than Arrow's more familiar combinatorical approach.[2] A further advantage of using with utility reports is that it allows questions to be asked about welfare implications of collective decisions.

I follow Sen in calling the mapping from the space of utility functions to an ordering a social welfare functional, or simply a welfare function.

I set out the main geometrical insights in section 2 and the results on the properties of the welfare functions

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[1]Sen ([?], 1111-12) gives an excellent summary and an extensive bibliography of this and other interpretations of the choice problem in the period up to 1986. The only notable work that is not extensively covered is the topological approach of Chichilnisky and Heal [?], [?], [?] and recent work by Saari [?] that explores the geometry of choice with ordinal information structures. See also [?]

[2]When the combinatoric proof of Arrow's theorem has been put on the board for undergraduates it is never clear what has been explained or what is happening to the space of reports under the choice mapping. For spatial intuition for ordinal data see the work by Saari [?].

that satisfy various conditions in section 3.

2 The social choice function and information equivalence.

The social choice and welfare functions are defined and, in Proposition 2, some a spatial representation is developed about the way in which the welfare function maps the n dimensional space of utility reports. This says, roughly, that the welfare function gives the same ranking to all points on hypersurfaces of dimension $n-1$ in this space. These surfaces cut across the space in a downward direction. Think of sheets of paper slicing through a three dimensional box, for example. It follows that the welfare function maps any path of utility reports, say a path something like a piece of string in the three dimensional box, according to the way in which it is sliced by these sheets of paper. See fig. 1 and fig. 2.

1. The social choice function.
The social choice function is a mapping from states of the world to an ordering of those states thought of as points on the real number line. A state of the world is written $s \in S$ and is any possible combination of allocations of goods, work, access to education and housing and whatever else might be subject to collective decision. There are n individuals. It is assumed that the set of states of the world forms a convex set and is restricted so that it is possible to associate with each individual i a continuous function $x_i : S \to \mathbb{R}$. This is not a completely innocent assumption since it essentially requires the existence of some universal, or all purpose, good that can be exchanged for all other goods in order to provide a metric on S and to make the idea of an arbitrarily small increase in s sensible. It is assumed that x_i increases with an increase in $|S|$ and that, if s is preferred by i to s' then $x_i(s) > x_i(s')$. It will be noted that the condition imposed on x does not give a unique value for x_i since, for any given x_i it can be met by any continuous bounded monotonic increasing transformation. Write the reports of the n individuals as the vector $x \in U \subset \mathbb{R}^n$ and the set of acceptable transformations of x be T where $T = \{f : U \to X \subset \mathbb{R}^n\}$. It follows that the space of utility reports is given by

$$S \xrightarrow{x} U \xrightarrow{T} X$$

where the range of the composite $T \circ x$ is understood to mean the range of $f(x)$ for all f in an acceptable T. It will be noted that x is not a homeomorphism since it is not one-one.

The choice function $\psi : S \to \mathbb{R}$ can be thought of as the composite function

$$\psi := \varphi \circ T \circ x \to \mathbb{R}$$

where the welfare function

$$\varphi : X \to \mathbb{R}$$

is a monotonic increasing function that maps information on utilities into an ordering.

It is assumed that X is a bounded subset of $\mathbb{R}^n$. This means that $x_i \leq K$ for each i for some K sufficiently large. It is also assumed that the welfare function is only required to process utilities with $x_i \geq \epsilon$ for each i in order to increase the space of welfare mappings that might be acceptable. These assumptions are only meant to tell us something about the information that a welfare function might reasonable be required to process. They are not necessarily a statements about the utilities that individuals might claim to have, although it is perfectly reasonable to assume that utilities would not, in fact, become infinite.[3] it is also necessary to require that $T \circ x$ is finite.

It is necessary to show that the choice function is decisive, in that it can pick a finite set of top ranked alternatives. Apart from the obvious condition that ψ is not constant this requires that it does not oscillate rapidly. This requires that each critical point for ψ is some distance from the next.

Proposition 1. [a]. *The function ψ has a maximum and a minimum on S*
[b]. *If ψ is differentiable critical points are isolated.*

Proof. [a]. Immediate from the facts that φ is continuous and X is a closed and bounded subset of $\mathbb{R}^n$.
[b]. Immediate from the Morse Theorem.
□

In what follows I will concentrate on the welfare function since it is the information contained in utility reports that is of most interest.

2. Conditions on the welfare function.
The welfare function is required to satisfy a set of reasonable conditions to ensure that it is not arbitrary. For the present purposes these will be taken to be the following

Ai. Continuity.
This says that the welfare function must map reports that are arbitrarily close together to rankings that are arbitrarily close together. This seems reasonable as a consistency requirement. It is not met by voting procedures like first past the post, for example, because the choices are discontinuous, and most of the criticism of these types of decision mechanisms can be boiled down to this fact. Segal ([**?**], p.575) has argued that collective choice should be discontinuous on the grounds that things like Medicaid are not continuous. This confuses convexity in the domain with continuity in the welfare function.

Aii. Pareto.
This says that if $x_i(s) \geq x_i(s')$ for all i, then $\varphi \circ x(s) \geq \varphi \circ x(s')$.

Aiii. Anonymity.
This says that the welfare function should not discriminate between individuals.[4]It imposes greater restrictions

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[3]It is only necessary to assume, for example, that $\frac{|x(s)|}{|s|'} \to 0$ as $|s|' \to \infty$ where $|x(s)|$ is the sup-norm.
[4]It is used in the topological approach, for example, but it is not used in Arrow type approaches

on the welfare function than non-dictatorship. Even though it is not used in all social choice theorems it seems fundamental It means that for s and s' states of the world and δ a permutation such that $\delta x(s) = x(s')$ then

$$\varphi \circ x(s) = \varphi \circ x(s')$$

Aiv. Differentiability.

This is required to keep the proofs simple and has already been used in the proof of Proposition 1. Since the set of differentiable functions is dense in the space of continuous functions on $\mathbb{R}^n$, any non-differentiable function can be approximated as closely as we wish by a differentiable function.[5]

3. Some general properties of the welfare function.

The mapping produced by the welfare function can be understood if we know which sets of points in the space of utility reports are given the same rank. These are given by $\varphi^{-1} := \{x : \varphi(x) = k\}$ for k some constant. See fig. 1 for an example in $\mathbb{R}^3$.
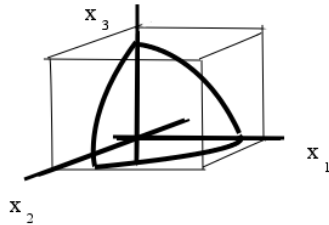

Figure 1. Example of the set $\{x : \varphi(x) = k\}$ in $\mathbb{R}^3$.

Proposition. 2. *The welfare function has sets $\varphi^{-1}(k)$ for each $k \in \mathbb{R}$ with the properties that:*

[a]. each set $\varphi^{-1}(k)$ is a manifold, or hypersurface, of dimension $n - 1$ in $\mathbb{R}^n$ for almost all points $k \in \mathbb{R}$

[b] there is only one set for each $k : \varphi(x) = k$ and each such set is intersected by the diagonal in the space $\mathbb{R}^n$ given by the ray σ through the points $f(0)$ and $(f(K), f(K), \ldots, f(K))$.

[c]. $\frac{\partial x_i}{\partial x_j} = -1$ in the direction x_j at any point $c \in \sigma$.

Proof. [a]. This follows immediately from Sard's theorem.

[b]. *Aii* says that the partial derivatives of the sets $x \in \varphi^{-1}(k)$ must be less than or equal to zero always and uniqueness follows from the continuity of φ. Intersection with σ follows from *Aiii*.

[c]. From the mean value theorem and the implicit function theorem $\frac{\partial x_i}{\partial x_j} = -1$ evaluated at points c_i, c_j and it is possible to use a squeezing argument from both sides of the diagonal to get $c_i = c_j$.

$\square$

In practical terms this proposition tells us that we can neglect points where $\varphi^{-1}(k)$ is not a manifold when thinking about the space of utility reports. This proposition is illustrated in fig. 2 where X is a circle, or

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[5]From the theory by Hilbert that the polynomials are dense in a cube [**?**], or by or approximating φ with an integral. Since the domain is $\mathbb{R}^n/0$ any first degree differentiable function is differentiable to any order.

closed loop, in $\mathbb{R}^2$. What is somewhat less obvious than this diagram suggests is that any welfare function from almost any loop in $\mathbb{R}^n$ that encloses a convex space will produce a mapping in $\mathbb{R}$ with the two critical values.[6]
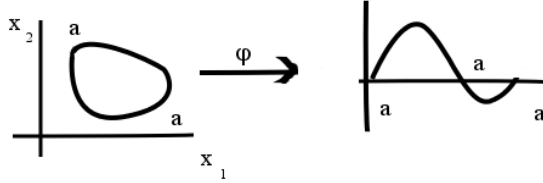

Figure 2. Example of $\varphi(\theta)$ for θ a closed loop.

In addition to satisfying $Ai - Aiii$ the choice function must always rank states of the world in the same order under all acceptable transformations. It is easy to see that this task becomes more difficult as the range of permissible transformations increases, which is the same as saying the information contained in a report decreases. If, for example, T were any possible monotonic transformation the choice function would be required to produce an ordering with only a small amount of information. From this perspective, the problem with the type of ordinal structure of reports used in Arrow's work is that the choice function is being asked to do too much with too little information. To make this more precise, a welfare function is said to be invariant under an acceptable transformation T, or to satisfy T if, for $\{x : \varphi(x) = k\}$ we have $\varphi \circ f(x) = \bar{k}$ for all $x \in \varphi^{-1}$ and $f \in T$ where k and $\bar{k}$ may be different.

One constructive approach to the problem is to accept that we often have access to information that is more finely grained than purely ordinal rankings. For example, we can get information on intensity of preferences from pressure group activity, membership of organizations, protests, focus groups and the like. This means that we can examine the welfare function over a smaller domain of transformations than that given by any positive monotonic function ([**?**], 1111).

4. Utility transformations.
The transformations of utility reports that will be considered are:[7]

Ti. Cardinal unit and level comparability.
This is given by $f(x) = (bx_1 + a_1, \ldots, bx_n + a_n)$ for $0 \leq a, b \leq r$ for r finite.

Tii. Ratio-scale non comparability.
This is given by $f(x) = (b_1 x_1, \ldots, b_n x_n)$ for $0 \leq b_i \leq r$.

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[6]More generally, any loop in R^n that only has two critical points. If a loop is entirely on a level surface it is mapped to a point.

[7]These are the most important for which a welfare function exists. A much larger set has been studied in the literature ([**?**], 1113).

5

Tiii. Ratio scale full comparability.

In this case $f(x) = bx$ for $0 < b \leq r$.

It is worthwhile noting the following result before beginning.

Proposition. 3. *The welfare function will satisfy any $T = \{f : f$ is any positive monotonic transformation of $X\}$ and $A_i - A_{ii}$ if and only if $\varphi \circ f(x) = \varphi(x_i)$ for some i.*

In this case the proposition restates Arrow's impossibility theorem. Note that this does not satisfy anonymity.

3   Properties of the welfare function.

The first proposition says that a welfare function will satisfy Ti if it gives reports the ranking they would be given under summation. In order to satisfy Tii the welfare function must rank reports according to their product and $Tiii$ is satisfied by a wider class of functions than Ti and Tii. Since the product welfare function gives a higher ranking to egalitarian distributions of utility than the additive function a tendency towards egalitarianism can be derived from requirements on information.

It is also asked whether a welfare function can satisfy a compensated Pareto outcome in which no-one can be made better off under side payments by a change in the state selected. It is shown that if φ satisfies Ti it satisfies this for every point on a level set. If it satisfies Tii it can satisfy this condition locally if it chooses an egalitarian outcome.

1. Propositions on transformations.

The invariance property for the welfare function can be rewritten using Proposition 2 and the chain rule to say that if $x \in \varphi^{-1}(k)$ it must be the case that

$$\left\langle \nabla\varphi, \frac{\partial f}{\partial x}dx \right\rangle = 0$$

and this is used to prove the following propositions.

Proposition. 4. *The welfare function satisfies $Ai - Aiv$ and Ti if and only if $\varphi(f) = \varphi(\sum f_i)$.*

Proof. (a). Only if. Suppose φ satisfies $Ai - Aiv$ and k is not a critical value for φ. Then

$$\left\langle \nabla\varphi, \frac{\partial f}{\partial x}dx \right\rangle = b(\frac{\partial \varphi}{\partial f_1}dx_1 + \ldots + \frac{\partial \varphi}{\partial f_n}dx_n) = 0$$

It follows that the term in brackets must be zero and all $\frac{\partial \varphi}{\partial f_i} \neq 0$ for at least one j. It is sufficient to consider $n = 2$. From the implicit function theorem we have $x_j = g(x_i)$ for some $x_j : \frac{\partial \varphi}{\partial f_j} \neq 0$. Hence

6

$$\frac{dg(x_i)}{dx_i} = -\frac{\partial\varphi/\partial f_i}{\partial\varphi/\partial f_j} \tag{1}$$

and the derivative cannot contain any f_i, f_j terms. It follows that φ is linear in f_i, f_j. From the symmetry imposed by Aii that $\varphi(f_1, f_2) = \varphi(f_1 + f_2)$ as required.

(b). If. Suppose $\varphi(\sum x_i) = k$. Then $\langle \nabla\varphi, dx_i \rangle = 0$. It follows that $\left\langle \nabla\varphi, \frac{\partial f}{\partial x}dx \right\rangle = b\langle \nabla\varphi, dx_i \rangle = 0$ as required.

$\square$

It will be noted that the set of permissible welfare functions includes all functions of the form $\varphi = (\sum f_i)^k$ and $\varphi = e^{\sum f_i}$. In this case the next result is a little surprising. See fig. 3. for an illustration in two dimensional space. Consider φ^1 and φ^2 equivalent if whenever $\varphi^1(x(s)) = \varphi^1(x(s'))$ then $\varphi^2(x(s)) = \varphi^2(x(s'))$. This means that φ^1 and φ^2 produce identical orderings.

Corollary of Proposition 4. *Every welfare function that satisfies $Ai - Aiv$ and Ti is equivalent to the additive welfare function $\varphi = \sum x_i$.*

Proof. Immediate from $\frac{dg(x_i)}{dx_i} = \frac{\partial\varphi/\partial x_i}{\partial\varphi/\partial x_j}$ and $\frac{\partial\varphi}{\partial x_i} = b\frac{\partial\varphi}{\partial \sum f_i}$. Hence $\frac{dg(x_i)}{dx_i} = -1$.

$\square$

Proposition. 5. *φ will satisfy $Ai - Aiv$ and Tii if and only if it is equivalent to the product welfare function $\Pi_i x_i$.*

Proof. (a). Only if. Suppose $\left\langle \nabla\varphi, \frac{\partial f}{\partial x}dx \right\rangle = 0$ for k some n on-critical value. As before it is only necessary to consider $n = 2$. Suppose φ is separable in f_i and f_j for all i, j so that Equation (**??**) gives

$$\frac{\partial\varphi}{\partial f_i} = \frac{b_i}{b_j}\frac{\partial\varphi}{\partial f_j}$$

which contradicts anonymity. From Aii the only permissable non-separable welfare functions must contain a product function $h(f_1 f_2)$ for h monotonically increasing.

(b). If. Suppose that we have $\varphi(\Pi x_i) = k$. Then $d\varphi = \frac{\partial\varphi}{\partial\Pi x_i}(\sum \Pi x_{-i}dx_i) = 0$ where Πx_{-i} is the product of all terms with x_i removed. This means that $\sum \Pi x_i dx_{-i} = 0$. Consider $\varphi(\Pi b_i x_i)$. In this case we have $d\varphi = (\Pi b_i)\frac{\partial\varphi}{\partial\Pi b_i x_i}(\sum \Pi x_i dx_{-i}) = 0$ as required.

$\square$

Information transformations in $Tiii$ are more restricted than those in Ti and Tii and can be satisfied by a wider set of welfare functions.

Proposition. 6. [a] *φ will satisfy $Ai - Aiv$ and $Tiii$ if and only if it satisfies Ti or is homogeneous of degree α for $\alpha > 0$.*
[b] *φ will satisfy $Ai - Aiii$ and $Tiii$ if and only if it satisfies $Ai - Aiv$ or if $\varphi(x) = \varphi\tilde{x}$ where $\tilde{x} = min$ or max $x_i \in x$.*

Proof. $[a]$. (a). Only if. Suppose $\langle \nabla \varphi \circ f(x), dx \rangle = 0$. As before it is only necessary to consider $n = 2$. Then the required condition is again given in Equation (**??**) and we also have

$$\frac{df_2}{df_1} - \frac{\partial \varphi / \partial f_1}{\partial \varphi / \partial f_2}$$

for $\varphi(f_1, f_2) = a$ where a is a constant. This means that $\frac{dx_2}{dx_1} = \frac{dkx_2}{dkx_1}$. This equality can be used to show that, either $\frac{dx_2}{dx_1}$ is constant and φ satisfies Ti or $\frac{dx_2}{dx_1} = h(k)$. From $Aiii$ it must be the case that $\frac{\partial \varphi / \partial f_1}{\partial \varphi / \partial f_2}$ is be symmetrical by interchanging f_1 and f_2 and hence the numerator cannot multiply the denominator by $h(k)$. This means that, either $\frac{\partial \varphi / \partial f_1}{\partial \varphi / \partial f_2} = c$ for c some constant and φ satisfies Ti, or from the linearity of the derivative, φ is homogeneous of degree α.

(b). If. Immediate from substitution.

$[b]$. Immediate since min or max x_i retains under $Tiii$. See ([**?**], 1116).

$\square$

A function is homogeneous of degree α if $\varphi(kx) = k^\alpha \varphi(x)$. Examples of welfare functions with this property are $\varphi = x_1 x_2 \ldots x_n$ or $\varphi = \sum (b_i x_i)^p$ for b_i a constant. In this last case, for $b_i = 1$ the level sets are the positive segment of the sphere in $\mathbb{R}^{n+1}$.

In $[b]$ the mapping is the lexi-min or lexi-max operator, or a positional dictator, which maps into the number associated with the lowest, or highest utility report. See fig. 4. To get some idea of what is happening note that, although it is not possible to have a person as a dictator under anonymity, it is possible to have a position as a dictator. It will be noted that the set $\varphi \in J$ that satisfy Tii and $[a]$ includes functions that are arbitrarily close to the lexi-min operator

2. Comparison of additive and product functions.

In order to see the difference in the orderings produced by the additive welfare function and the product welfare function consider the manifold M associated with $\varphi(\sum x_i) = k$. It is straightforward to show that $|x - \sigma|$ monotonically decreases $\varphi(x)$ for $x \in M$ monotonically increases under the mapping $\varphi(\Pi x)$. In other words for a set of points that are given the same ranking under the additive function the product mapping increases the ranking as they become more egalitarian.

3. A Utility compensated welfare functions.

Since we have some information on utilities it is reasonable to ask whether the welfare function can meet a compensated Pareto condition which says that it should not give states of the world s and s' equal ranking if a shift from s to s' makes one individual better off than it makes some individual worse off. It is obvious that a non-linear welfare function cannot meet this condition across the whole set of reports $x \in \varphi^{-1}(k)$. It is more interesting to ask, is there some state of the world such that no arbitrarily small change will make one individual better off than it makes another worse off by some small amount under any acceptable welfare function? In technical terms the question becomes, is there an s such that for $x(s) \in \varphi^{-1}(k)$ there is no $s' \in B(s, \epsilon)$

for ϵ arbitrarily small such that for $x(s') \in \varphi^{-1}(k)$ we have $|x(s) - x(s')| > \delta$ for some fixed $\delta > 0$? This gives us

Av. Local compensated Pareto principle.

Proposition. 7. *It is possible to satisfy Av for every φ that satisfies $Ti - Tiii$ and every fixed $\delta > 0$ if and only if for $x(s), x(s') \in \varphi^{-1}(k)$ and $s' \in B(s, \epsilon)$ we have $s : x(s) \in \sigma$.*

Proof. $[a]$. Immediate from the Corollary of Proposition 4.
□

In other words, if we wish to choose amongst states of the world with the same ranking, the states that satisfy the local compensation principle, it is necessary to choose those that give the same utility to every individual for reports on the scale $[0, K]$.

4   Conclusion.

This paper has analyzed choice problems in terms of the properties of a continuous welfare function defined on the space of utility reports. It has been shown that this gives us some insight in terms of the sets of points that are mapped into the same value.

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