

1994

WEAKLY DAMPED MODES IN STAR-CLUSTERS AND GALAXIES

MD Weinberg

Follow this and additional works at: https://scholarworks.umass.edu/astro_faculty_pubs



Part of the [Astrophysics and Astronomy Commons](#)

Recommended Citation

Weinberg, MD, "WEAKLY DAMPED MODES IN STAR-CLUSTERS AND GALAXIES" (1994). *ASTROPHYSICAL JOURNAL*. 85.
[10.1086/173665](https://doi.org/10.1086/173665)

This Article is brought to you for free and open access by the Astronomy at ScholarWorks@UMass Amherst. It has been accepted for inclusion in Astronomy Department Faculty Publication Series by an authorized administrator of ScholarWorks@UMass Amherst. For more information, please contact scholarworks@library.umass.edu.

Weakly Damped Modes in Star Clusters and Galaxies

Martin D. Weinberg

Department of Physics and Astronomy

University of Massachusetts, Amherst, MA 01003

ABSTRACT

A perturber may excite a coherent mode in a star cluster or galaxy. If the stellar system is stable, it is commonly assumed that such a mode will be strongly damped and therefore of little practical consequence other than redistributing momentum and energy deposited by the perturber. This paper demonstrates that this assumption is false; weakly damped modes exist and may persist long enough to have observable consequences.

To do this, a method for investigating the dispersion relation for spherical stellar systems and for locating weakly damped modes in particular is developed and applied to King models of varying concentration. This leads to the following remarkable result: King models exhibit *very* weakly damped $m = 1$ modes over a wide range of concentration ($0.67 \leq c \leq 1.5$ have been examined). The predicted damping time is tens to hundreds of crossing times. This mode causes the peak density to shift from and slowly revolve about the initial center. The existence of the mode is supported by n-body simulation.

Higher order modes and possible astronomical consequences are discussed. Weakly damped modes, for example, may provide a natural explanation for observed discrepancies between density and kinematic centers in galaxies, the location of velocity cusps due to massive black holes, and $m = 1$ disturbances of disks embedded in massive halos. Gravitational shocking may excite the $m = 1$ mode in globular clusters, which could modify their subsequent evolution and displace the positions of exotic remnants.

Subject headings: galaxies: kinematics and dynamics, interactions, nuclei—
globular clusters: general

1. Introduction

A stellar system responds to a disturbance by both phase-mixing, an incoherent response, and by collective excitation of discrete modes, a coherent response. Since galaxy and star cluster models must be stable to be useful, *growing* modes¹ have been studied extensively both analytically in the form of stability criteria and using n-body simulation (see Binney & Tremaine 1987, hereafter BT, for a review). The most well-known of these is the bar-forming mode which significantly reorganizes its host galaxy. In general, *damped* modes will also be excited by a disturbance and transport momentum globally but their amplitude decays with time. If the decay time is short, these modes will not be perceptible. However, if a mode is weakly damped, it may persist for many dynamical times, even though the system is stable. For example, a galaxy “fly-by” could excite such a mode which might persist long after the encounter even if the system is not strictly unstable. Several authors, in particular Miller and Smith (Miller 1992, Miller & Smith 1993), have observed oscillatory modes in simulations. Mathur (1990) formally demonstrated the existence of oscillating solutions in both one-dimensional and spherical systems. In this paper, we explore the existence and implication of weakly damped modes in spherical stellar systems.

The existence of damped modes is not in question. Collective damping in a homogeneous plasma, *Landau damping*, is well-known. The simultaneous solution of the Vlasov equation and the Poisson equation yield a dielectric response function which describes the reaction of the plasma to an imposed disturbance. Even in the presence of no external disturbance, a stochastic space-charge field may give rise to spontaneous density fluctuations resulting in Landau modes. The zeros of the dielectric function are the conditions for collective solutions. This use of the response function is sometimes known as the *dispersion relation* (e.g. Ichimaru 1973) since it determines the frequency-wavelength relation for electromagnetic waves in a medium.² The relationship between the full modal spectrum and the initial value for the homogeneous plasma has been discussed by van Kampen (1955) and the same physical principles apply here.

Damped modes are difficult to study in arbitrary stellar systems for two reasons. First, the dispersion relation is analytically intractable in nearly all but the homogeneous

¹For our purposes, a mode is a well-defined pattern with a possibly complex frequency, ω . If $Im(\omega) > 0$, the initial equilibrium is unstable and the mode initially grows exponentially. If $Im(\omega) \leq 0$, the equilibrium is stable and the modes are oscillatory or damped.

²The analogy does not carry over the stellar dynamical problem but the term *dispersion relation* is still used.

cases. Second, although one can study damped modes in simulations, one must be able to discriminate the mode from numerical artifacts. In this paper, I will develop a numerical procedure for evaluating the dispersion relation for spherical stellar systems and locating damped modes (§2). Armed with a prediction for the shape and frequency of a damped mode, we may initialize and efficiently search for it in n-body simulations. Spherical models are considered because of the theoretical simplicity, because they represent a wide variety of astrophysical scenarios and to relate to the large body of literature on spherical stellar systems. In §3, the technique is applied to King models of varying concentration. King models are known to be stable, for example, by Antonov’s criterion (Antonov 1962, see also BT). Nonetheless, we will see that these models all have *very* weakly damped $l = m = 1$ modes, with damping times longer than ~ 20 half-mass crossing times. This is contrary to the widely expressed belief that a disturbance in a stable system will phase mix in several crossing times. As a check, we realize the $l = m = 1$ mode and follow its evolution in a set of n-body simulations. The results of the simulation (§4) show that the expected mode persists with no obvious decay. The paper concludes with a discussion of astronomical applications (§5).

2. Method

For collisionless systems, the dispersion relation describes the solution to an initial value problem as the result of a Laplace transform; each component has time-dependence of the form $\exp(-i\omega t)$, where the frequency ω may be complex. Although the dispersion relation for infinite homogeneous stellar systems has been derived (Ikeuchi et al. 1974, BT), most stellar systems are finite and inhomogeneous. In a stellar sphere in particular, orbits are quasiperiodic and the repetitive effect of a disturbance at the same point in an orbit’s phase can have a large effect (resonance). Periodicity may be introduced in a homogeneous medium by carving out a cube and connecting opposite faces; topologically, this is a 3-torus. The resulting so-called periodic cube has been used by several authors (Barnes et al. 1986, Weinberg 1993) to take the quasiperiodicity into account without sacrificing the theoretical simplicity of spatial homogeneity and will be used below as an example.

Inhomogeneous models of galaxies and clusters are considerably more difficult to treat analytically because the collisionless Boltzmann and Poisson equations are separable in different bases. By Jean’s theorem, the solution to the collisionless Boltzmann equation may be written conveniently in terms of its integrals of motion. In the spherical case, these are energy and angular momentum or radial and angular actions. Since the unperturbed

Hamiltonian is then independent of associated angle-like coordinate variables, a linear perturbation to this equilibrium may be written as a Fourier series in the coordinate angles. The Poisson equation is naturally expanded in a different basis: spherical harmonics. However, by expanding the self-consistent gravitational potential in an harmonic series which satisfies the Poisson equation by construction, the simultaneous linearized solution to the whole system may be reduced to a matrix equation. In particular, let a_j, b_j be two sets of expansion coefficients. Then the response b_j to a perturbation a_j is given by $b_i = M_{ij}(\omega)a_j$ where M_{ij} is the response matrix and describes the dynamical response of the stellar sphere to that part of perturbation with time-dependence $\exp(-i\omega t)$. A self-consistent response has $a_j = b_j$ (cf. Appendix B). This formalism, sometimes called the matrix method, has had varied applications beginning with Kalnajs (1977) who was interested in the modes of stellar disks. Polyachenko and Shukhman (1981) first adapted the method to the study a spherical system (see also Fridman and Polyachenko 1984, Appendix) and the method was later employed by both Palmer and Papaloizou (1987) in the study of the radial orbit instability and by Bertin and Pegoraro (1989) to study the instability of a family of models proposed by Bertin and Stiavelli (1984). Weinberg (1989) used the matrix formulation to study the response of a spherical galaxy to an encounter with a dwarf companion and Saha (1991) and Weinberg (1991, hereafter Paper I) investigated the stability of anisotropic galaxian models. Using this approach, the dispersion relation is the condition that the self-consistent response have a non-trivial solution: $\det[\delta_{ij} - M_{ij}(\omega)] = 0$. The general dispersion relation can be calculated for a spherical model with an arbitrary phase-space distribution function, although only isotropic distributions will be discussed here.

2.1. Analytic continuation of the dispersion relation

Just as in the case of the plasma dispersion relation, the derivation in Paper I explicitly shows that the dispersion relation as written (see Appendix B) is only valid in the upper-half ω plane. Causality in the initial value problem demands that various integrands be analytically continued to the lower-half plane. This is readily done in the plasma case (cf. Ichimaru 1973) where the natural coordinates for both the Boltzmann and Poisson equations are cartesian. This is problematic in the case of the sphere, whose dispersion relation is given by equations (B5), (B6), and (B7). Since equation (B5) contains the term $1/[\omega - \mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{I})]$, explicit analytic continuation requires analytic continuation of orbital frequencies and actions. To sidestep this daunting task, I propose to evaluate the dispersion relation in the upper-half plane and perform the analytic continuation numerically by

approximating it by a function which may be easily continued. Given that poles are expected because of the form of equation (B5), rational functions are a natural choice.

Before applying this to cases of interest, we would like some assurance that this procedure for analytic continuation will work reliably. To do this, I evaluated the dispersion relation for the periodic cube (eq. A1), both analytically and numerically. The simple form of the velocity integrals in equation (A2) allow explicit analytic continuation. Analytically generated contours for the fundamental harmonic in the dispersion relation, $|\mathcal{D}_{1,0,0}(\omega)|$, are shown in Figure 1. The zeros are at the centers of the concentric contours and the location of the zeros indicate the type of growth; if the zeros are in the lower (upper) half-plane, disturbances are damped (growing). The model depicted is stable. The most weakly damped mode has $Re(\omega) = 0$ and therefore damps in situ. The other modes both damp and propagate with group velocity $Re(\omega)$. As the stellar velocity dispersion is decreased, the solution with $Re(\omega) = 0$ moves toward the real axis from below and will become the Jeans instability for sufficiently small velocity dispersion. Modes in spherical systems, discussed below, have spatial distributions proportional to spherical harmonics and $Re(\omega)$ indicates a pattern speed rather than a group velocity.

I then evaluated $\mathcal{D}_{1,0,0}(\omega)$ at 20 points for $-40 < Re(\omega) < 40$ for each of $Im(\omega) = 0, 0.5, 1.0$ and found the corresponding rational function fit (cf. Appendix D.). Contours of the approximating rational function are shown in Figure 2. Comparing Figures 1 and 2 we see that the least damped modes are determined accurately. I checked the robustness of the method to the choice of evaluations by varying the number and placement of points included in the rational function fit. The three most weakly damped modes are accurately determined even if any two of the “rows” of constant $Im(\omega)$ are eliminated. The most weakly damped mode remains accurately determined even for 10 points along the real ω -axis from $-40 < Re(\omega) < 40$. The behavior of $\mathcal{D}_{1,0,0}$ near the real axis is dominated by the true weakly damped modes, allowing zeros to appear at $Im(\omega) \approx -28$ in Figure 2. This is an expected and generic feature; zeros on the lower-half plane will only be well-approximated by the rational function if they affect the \mathcal{D} on the upper-half plane. In general, we can not expect accurate determination of all features on the lower half-plane from a finite grid and, in particular, the upper half-plane will place little constraint on the rational function extrapolation on the lower half-plane if more weakly damped modes at similar values of $Re(\omega)$ exist.

2.2. Application to the dispersion relation for stellar spheres

In the next section, we investigate the low-order damped modes for isotropic King models with dimensionless central potential $W_0 \equiv (E_t - E_o)/\sigma^2 = 3, 5, 6, 7$ and concentration $c \equiv \log_{10}(r_t/r_c) = 0.67, 1.0, 1.3, 1.5$ respectively where r_c and r_t are the core and tidal radii (see King 1966 and BT for more details). Linear perturbation theory allows each harmonic order, $\propto Y_{lm}$, to be considered separately. However, because the sphere has no symmetry axis, all terms m are degenerate for a given l . In realizing specific modes below, we will choose $l = m$. Units are chosen so that $M = G = 1$ and the total gravitational potential $W = -1/4$. In most cases, we use a grid in the upper-half ω -plane with $Im(\omega) = 0.01, 0.05, 0.1$ and 10 to 40 evenly spaced points in $-1 < Re(\omega) < 1$. In some cases the domain is extended to $-5 < Re(\omega) < 5$ to check for “fast” pattern speeds but none were found in any cases explored here. The evaluations of $\mathcal{D}(\omega)$ (cf. eq. B7) are fit to a rational function and zeros in the lower-half plane are noted. The robustness of the zeros to the input grid is checked by successively truncating the grid in $|Re(\omega)|$, halving the resolution of the grid and leaving out “rows” in $Im(\omega)$. In all cases, the existence of the zeros and general topology of the function did not change although the position of particular zeros varied by as much as 10% in a few cases.

The elements of the response matrix $M_{ij}(\omega)$ (cf. eq. B5) require a two-dimensional integral and a two-dimensional sum in l_1 and l_2 . The sum is only two dimensional since $M_{ij}(\omega)$ is independent of $l_3 = m$ given l . The sum in l_1 must be truncated at a chosen value, $l_{1,max}$ and the sum in l_2 has $|l_2| < l$. Twenty point grids for the integral evaluation using rational functions were found to be adequate; large numbers of grid points ($\gtrsim 40$) cause significant truncation error (see App. D). The rational function technique gives surprisingly good results with a small number of integrand evaluations. Standard lower order integration techniques (e.g. Simpson’s rule) ultimately give better results but only after an order of magnitude more evaluations. The value of $l_{1,max}$ was successively increased to obtain convergence. The appropriate value of $l_{1,max}$ depends on l ; $l_{1,max}$ was chosen to be 6 and 10 for $l = 1$ and 2, respectively. Finally the order of the $n \times n$ matrix M_{ij} was varied to obtain convergence in $\mathcal{D}(\omega)$. The required order increases with $Re(\omega)$; $n = 30$ gave adequate convergence for $|Re(\omega)| \lesssim 1$. The resulting error in $\mathcal{D}(\omega)$ is estimated to be $\mathcal{O}(10^{-2})$.

3. Properties of damped modes

This section describes the application of the methods from §2 in determining the damped modes of King models with $W_0 = 3, 5, 6, 7$. The complex frequencies of the most

Table 1: Model parameters and modal frequencies

Model		Order	Frequency		Shift	Example
King W_0	c	l	$Re(\omega)$	$Im(\omega)$	$r_{shift}/r_c \epsilon$	Cluster name
3	0.67	1	0.010	−0.0016	0.41	NGC 6496
		2	0.10	−0.076		
5	1.0	1	0.034	−0.0011	0.40	NGC 288
		2	0.0	−0.11		
6	1.26	1	0.042	−0.0065	0.42	ω Cen
7	1.5	1	0.064	−0.025	0.54	M13
		2	0.0	−0.20		

weakly damped modes are shown in Table 1. To summarize, one finds the $l = 1$ modes are *very* weakly damped for all W_0 studied with the damping proportional to concentration. The existence of such modes imply that low-order excitations of stable galaxies and clusters may persist for many crossing times and scaled to physical units, perhaps as long as the age of the galaxy itself. Very weakly damped $l = 2$ modes were found for the $W_0 = 3, 5$ cases and the same mode is relatively more strongly damped for $W_0 = 7$. No weakly-damped $l = 3$ modes were found. A detailed discussion of these trends and an examination of the modal shape is given below.

3.1. Frequencies of lowest order modes

For comparison, Figure 3 plots the zeros of the dispersion relation for $l = 1$ damped modes in $W_0 = 3, 5, 6$ and 7 King models. For example, the $l = 1$ harmonic for the $W_0 = 5$ King model has a *very* weakly-damped mode at $\omega = (\pm 0.034, -0.0011)$. The ω -plane was searched for $-1 \leq Re(\omega), Im(\omega) \leq 1$ and no other modes were indicated. Changes in the input grid of $\mathcal{D}(\omega)$ change the location of $Im(\omega)$ for the modes by $\mathcal{O}(10^{-3})$. The errorbar in upper left corner of Figure 3 gives the estimated precision in ω . Both the $W_0 = 3$ and 5 have nearly zero damping within the resolution of the technique; the value $Im(\omega) = 0$ can not be ruled out for these cases. The analogous modes for the $W_0 = 6, 7$ models are clearly damped but still only weakly.

The $l = 2$ modes exhibit a similar trend: the rate of damping increases with concentration. For the $W_0 = 5, 7$ cases, the pattern speed is zero but it is nonzero for the lowest concentration model.

Grids for the $l = 3$ harmonic were constructed for models $W_0 = 3, 5, 7$ but no weakly damped modes with $|Re(\omega)| \lesssim 0.3$ and $Im(\omega) \gtrsim -0.3$ were found, suggesting that the $l = 3$ disturbances are moderately to strongly damped.

3.2. Shape of lowest order modes

The $l = 1$ weakly-damped mode for the $W_0 = 5$ King model discussed in §3.1. is shown in Figure 4. The procedure for finding the eigenfunction is discussed in Appendix C. The peak of the disturbance occurs at $\sim 75\%$ of the core radius of the background model. The half-mass radius is $r_{1/2} \approx 1.6$. At larger radii, the density contours remain concentric. However, the shape of an $m = 1$ mode depends on the expansion center; therefore one would observe this mode as a coherent $m = 1$ shift at large galactocentric radius by fixing the expansion center on the density center.

The $l = 1$ mode manifests itself as a shift of the central density. The shift is given by the linear relation $r_{shift}/r_c \approx 0.4\epsilon$, where ϵ is the ratio of perturbed density at peak relative to background and $\epsilon \lesssim 0.3$ to have positive density everywhere. As an example, the solid contours in Figure 5 show the combined background and perturbation with the near maximum amplitude of 30% at peak. The dotted contours show the unperturbed model at the same levels for comparison. The shifted center revolves about the original center at angular frequency $Re(\omega)$. The center-of-mass shift for other concentrations is given in Table 1. The value $r_{shift}/r_c\epsilon$ is similar for all concentrations.

The magnitude of the perturbed phase-space distribution averaged over angles is shown in Figure 6. The energies of perturbed orbits are small but do not include the most bound particles and the peak toward large κ^2 at constant E indicates that the perturbed orbits tend to be more circular on average.

The $l = m = 2$ mode (Fig. 7) appears as a bar which peaks at a fraction of the core radius, at roughly the same radial scale as the $l = 1$ mode. Figure 8 shows the mode at 30% amplitude combined with the $W_0 = 5$ King model background density. Since $Re(\omega) = 0$, the pattern will have a fixed orientation and the bar distortion will decay with time. The phase space perturbation is similar in extent to that shown in Figure 6 but with a larger contribution of eccentric orbits.

4. Simulation

Although the dispersion relation indicates the existence of a weakly damped $l = 1$ mode, can such a mode be excited at significant amplitude?

In partial answer, we may attempt to realize the mode in an n-body simulation. Success would also serve as an independent check of the linearized solution. Unfortunately, such a

simulation is fraught with serious systematic problems. First, most simulation schemes do not explicitly conserve momentum. Since an $l = 1$ perturbation shifts the density center, numerical artifacts may not be distinguishable from the mode itself. To minimize possible artifacts, we use a direct-summation force calculation which *does* conserve momentum. Second, minimization of two-body relaxation requires softening at, roughly, the mean interparticle spacing. However, the feature we wish to resolve is only a factor of a few larger than this scale for ~ 10000 bodies. Therefore the softening itself is likely to change the dispersion relation and this will require that the dependence on softening be investigated. Third, even with softening, relaxation significantly changes the background on a shorter timescale than the predicted damping, suggesting that relaxation in the simulation may affect the mode. Finally, although these difficulties could be addressed with larger n , the n^2 FLOP scaling of the direct summation scheme practically limited the simulations to $n = 10000$.³

Because the mode is computed in action-angle variables, the perturbed n-body phase space is easiest to realize in these variables. The details are described in Appendix E. The mode described in §3.2 is realized with a peak level of 20% of background. The softening is chosen to be the mean interparticle spacing at the half-mass radius. Because of both the imposed perturbation and modification of the interparticle force by softening, the initial conditions are not in equilibrium. The system appears to settle at $t = 25$ as indicated by a constant value of $-2T/W \sim 1$ (cf. Fig. 9). For comparison, the radial crossing time at the half-mass (≈ 1.6) is roughly 4 time units. The $-2T/W$ profile looks similar for initial conditions without an initial perturbation, suggesting that softening causes the poor initial equilibrium state.

The evolution of the introduced mode may be diagnosed using the same harmonic decomposition used to evaluate the dispersion relation (Appendix B). The original expansion center is preserved in the center-of-mass frame since the direct force scheme explicitly conserves momentum. The azimuthal phase of the $l = m = 1$ distortion for several of the higher-order radial wavefunctions are shown in Figure 10. The low-order radial terms are less sensitive to the mode which has a scale length of $\sim 1/10$ the system radius. The real part of the eigenfrequency is the pattern speed for an $m = 1$ distortion. The dashed line indicates the slope predicted by $Re(\omega)$. The agreement is quite good. However, the evolution due to relaxation removes sensitivity to changes in amplitude of the mode which is predicted to have a longer timescale.

³Computations were performed on a Sun Microsystems Sparc 10/41 workstation which required 3.4×10^2 CPU seconds per step.

The amplitude of a disturbance may be computed from the total gravitational potential energy in the disturbance. From Appendix B, it follows that this energy for a particular harmonic is $W_{lm} = -2\pi G \sum_{j=1}^{n_{max}} |a_j^{lm}|^2$ where a_j^{lm} are the expansion coefficients. For the current simulation, the energy per order l , $W_l \equiv \sum_m W_{lm}$, converges with increasing radial wavenumber n_{max} at fixed time, confirming that the disturbance is global. In addition, W_1 is roughly constant over the course the simulation. More precisely, W_1 does appear to grow slightly with time but this can not be distinguished from the effects of relaxation which causes the central potential to deepen. An additional simulation with a 7% rather than 20% amplitude perturbation exhibited similar behavior, both in phase and amplitude.

To understand the effects of softening, simulations with softening length of both half and double the half-mass interparticle spacing were performed with the same initial conditions. The same general trends were seen: the introduced $l = 1$ mode persisted with the expected pattern speed.

However, in all three runs, there were “bursts” of the $l = 2$ mode, which might be expected given its long damping time (cf. Table 1). The $l = 2$ power was larger for smaller softening, possibly suggesting that either or both relaxation and softening are changing the modal structure. In addition, for $t \gtrsim 30$, relaxation causes the central potential to deepen and at the same time, the relative contribution of the $l = 2$ disturbance grows and appears to saturate without damping, roughly at the level of the initial $l = 1$ perturbation. For the case with 7% perturbation initially, the $l = 1$ mode continues at the predicted pattern for $t \gtrsim 30$ even though the $l = 2$ distortion is evident. For the 20% case, the growth of the $l = 2$ distortion appears to significantly change the $l = 1$ mode. The importance of these effects (especially the possible implication for bar growth) remains to be investigated.

Finally, a set of simulations with no perturbation *a priori* were used to check the significance of the observed disturbance by comparing the gravitational potential energy in $l = 1$ mode. The stochastically excited $l = 1$ components contained $\lesssim 3\%$ of the energy in the introduced mode described above. Nevertheless, it is worth noting that a coherent pattern was seen in these control runs with the expected pattern speed even though the peak density contrast was only 1% to 3%.

5. Discussion and Summary

The numerical evaluation of the dispersion relation for King models predicts the existence of low-order weakly damped modes. In particular, the $l = 1$ ($m = 1$) mode is *very*

weakly damped with small pattern speed over a range of central concentrations. Similar results might be expected for other spherical systems.

This mode causes the density center to shift and rotate about the original center. Using the linear theory, the shift is given by the linear relation $r/r_c \approx 0.4\epsilon$, where ϵ is the ratio of perturbed density at peak relative to the background and must be $\lesssim 30\%$ of background at peak. This mode has been realized and observed in a simulation of a $W_0 = 5$ King model. A strong excitation of this mode results in a shift of the density center of roughly 10% of the core radius as predicted and such levels were maintained for the duration of the simulation. However, intrinsic limitations and features such as softening and relaxation modify the evolution of and limit temporal sensitivity to weakly damped modes. This, in part, explains both the reason they were not discovered by n-body simulation and the general belief that excitations of stable equilibria vanish in several crossing times.

Because the low-order damped modes are simple, they should result from general perturbations. For example, the differential acceleration due to a passing galaxy may cause a significant $m = 1$ perturbation which might mix quickly *except* for the weakly damped modes. A companion on an eccentric orbit may have similar effects. The induced distortion due to a companion on circular orbit will be dominated by the quadrupole ($l = 2$) moment; the dipole force is canceled by the barycentric motion (Weinberg 1989).

In the halo of a halo-dominated galaxy, such a mode could result in the appearance of an $m = 1$ distortion or position-shift of the luminous disk. Baldwin, Lynden-Bell and Sancisi (1980) have pointed out that a number of nearby galaxies have lopsided HI distributions relative to their optical centers and that a large fraction of spiral galaxies show this effect. Damped modes may be partly responsible for the ubiquity of such offsets, especially in the more moderate cases. Using the $W_0 = 5$ model as an example, the excitation of a 10% perturbation in a halo with core radius of 15 kpc by, say, a companion or “fly-by” would result a central density offset of ~ 600 pc which would persist long after the encounter. In the bulge or spheroid of a disk galaxy, the mode would shift the density center from the kinematic center as determined, say, by the large-scale HI rotation curve. Using the same example, for a bulge–spheroid with a 1 kpc core radius with $\epsilon = 0.01, 0.1$ the offset would be $\sim 4, 40$ pc. In addition, since the mode is slowly varying, one might expect a massive central black hole, $M \gtrsim 10^6 M_\odot$ to remain within the potential well of the shifted density center since the equipartition radius would be $r_c m_*/M_{bh}$ for mean stellar mass, m_* .

Similarly, a time-dependent external force or *gravitational shock* differentially accelerates globular clusters, possibly exciting weakly-damped modes. Clusters on eccentric orbits will suffer gravitational shocking due to the halo and bulge and all clusters with mean galactocentric radii within 20 kpc will suffer shocking due to the disk. In addition,

there are relaxation-driven internal momentum sources such as binary recoil which could in principle produce an $m = 1$ disturbance (Makino & Sugimoto 1987). Either way, an offset core may change the subsequent dynamical evolution by decreasing the thermal contact of the core and the population of eccentric halo orbits. This may also have implications for gravothermal oscillation. In addition, we may have to reexamine the interpretation of off-centered compact objects if any of the $m = 1$ excitations keep the density central offset, on average, from the geometric center. On the other hand, the physical picture is further complicated by two-body relaxation which may lead to more rapid damping of these modes. Also, the largest relative central density shifts and damping times occur for the lowest concentrations where we expect rates of relaxation and subsequent production of exotic objects to be low. Additional work will be necessary to understand these competing trends and the extent to which these modes may be excited in globular clusters.

I thank Susan Kleinmann and David van Blerkom for comments on the manuscript and Dave Chernoff and Ira Wasserman for discussion. This work was supported in part by NASA grant NAGW-2224.

REFERENCES

- Antonov, V. A. 1962, *Vestnik Leningrad Univ.*, 96, 19.
- Baldwin, J. E., Lynden-Bell, D., and Sancisi, R. 1980, *Mon. Not. R. Astron. Soc.*, 193, 313.
- Barnes, J., Goodman, J., and Hut, P. 1986, *Astrophys. J.*, 300, 112.
- Bertin, G. and Pegoraro, F. 1989, in T. D. Guyenne (ed.), *Plasma Astrophysics*, p. 329, ESA, ESA SP-285.
- Bertin, G. and Stiavelli, M. 1984, *Astron. Astrophys.*, 137, 26.
- Binney, J. and Tremaine, S. 1987, *Galactic Dynamics*, Princeton University Press, Princeton.
- Fridman, A. M. and Polyachenko 1984, *Physics of Gravitating Systems II*, Springer-Verlag, New York.
- Henrici, P. 1974, *Applied and Computational Complex Analysis*, Vol. I, John Wiley and Sons, New York.

- Ichimaru, S. 1973, *Basic Principles of Plasma Physics: A Statistical Approach*, W. A. Benjamin, Reading, Massachusetts.
- Ikeuchi, S., Nakamura, T., and Takahara, F. 1974, *Prog. Theor. Phys.*, 52.
- Kalnajs, A. J. 1977, *Astron. J.*, 212, 637.
- King, I. R. 1966, *Astron. J.*, 71, 64.
- Makino, J. and Sugimoto, D. 1987, *Publ. Astron. Soc. Jpn.*, 39, 589.
- Mathur, S. D. 1990, *Mon. Not. R. Astron. Soc.*, 243, 529.
- Miller, R. H. 1992, *Am. Sci.*, 80, 153.
- Miller, R. H. and Smith, B. F. 1993, *Galactic Oscillations*, preprint.
- Palmer, P. L. and Papaloizou, J. 1987, *Mon. Not. R. Astron. Soc.*, 224, 1043.
- Polyachenko, V. L. and Shukhman, I. 1981, *Sov. Astron.*, 25, 533.
- Press, W. H., Teukolsky, S. A., Vetterling, W. T., and Flannery, B. P. 1992, *Numerical Recipes in C*, Cambridge University Press, Cambridge.
- Saha, P. 1991, *Mon. Not. R. Astron. Soc.*, 248, 494.
- Stoer, J. and Bulirsch, R. 1980, *Introduction to Numerical Analysis*, Springer-Verlag, New York.
- Tremaine, S. and Weinberg, M. D. 1984, *Mon. Not. R. Astron. Soc.*, 209, 729.
- van Kampen, N. G. 1955, *Physica*, 21, 949.
- Weinberg, M. D. 1989, *Mon. Not. R. Astron. Soc.*, 239, 549.
- Weinberg, M. D. 1991, *Astrophys. J.*, 368, 66.
- Weinberg, M. D. 1993, *Astrophys. J.*, in press.

A. Dispersion relation for a stellar cube

As discussed in §2, the dispersion relation is the condition for simultaneous solution of the collisionless Boltzmann and Poisson equations in response to a perturbation. In the case of a homogeneous cube with periodic boundary conditions, the two equations are separable in the same coordinate system, cartesian, which leads to an analytic solution. Following Barnes et al. (1986), the dispersion relation may be written:

$$\mathcal{D}_{\mathbf{n}}(p) \equiv 1 + \frac{4\pi GM}{k^2} \tilde{\Delta}_{\mathbf{n}}, \quad (\text{A1})$$

where \mathbf{n} is a three vector of integers,

$$\tilde{\Delta}_{\mathbf{n}} \equiv i \int d^3v \frac{\mathbf{k} \cdot \partial f_o / \partial \mathbf{v}}{p + i\mathbf{k} \cdot \mathbf{v}}, \quad (\text{A2})$$

M is the total mass of stars in the cube with side length L , \mathbf{k} is the wave vector defined by $\mathbf{k} = 2\pi\mathbf{n}/L$, and f_o is the background phase-space distribution function. The eigenfunctions are proportional to $\exp(i\mathbf{k} \cdot \mathbf{x})$. Barnes et al. (1986) derive equation (A1) by explicitly introducing a vanishing small growth term. For an alternative derivation using the Laplace transform see Weinberg (1993).

B. Dispersion relation for the stellar sphere

Any disturbance in the sphere may be represented as a spherical harmonic expansion with an appropriate set of orthogonal radial wavefunctions. We choose a biorthogonal potential-density pair as described in Paper I. The pair (u_i^{lm}, d_i^{lm}) is constructed to satisfy Poisson's equation, $\nabla^2 u_i^{lm} = 4\pi G d_i^{lm}$, and to form a complete set of functions with the scalar the product

$$-\frac{1}{4\pi G} \int dr r^2 u_i^{lm*} d_j^{lm} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \quad (\text{B1})$$

The perturbed density and potential then have the following expansions:

$$\Phi_1 = \sum_{lm} Y_{lm}(\theta, \phi) \sum_j a_j^{lm}(t) u_j^{lm}(r), \quad (\text{B2})$$

$$\rho_1 = \sum_{lm} Y_{lm}(\theta, \phi) \sum_j a_j^{lm}(t) d_j^{lm}(r). \quad (\text{B3})$$

Paper I shows that the coupled system of collisionless Boltzmann and Poisson equations may be solved by reducing their simultaneous solution to a matrix equation using such

an expansion above. The solution may then be written as the non-trivial solution to the following linear equation

$$\tilde{a}_i^{lm} = M_{ij}(\omega)\tilde{a}_j^{lm}, \quad (\text{B4})$$

where summations over like indices are implied, the ‘~’ indicates a Laplace-transformed quantity,

$$M_{ij}(\omega) = \frac{(2\pi)^3}{4\pi G} \int \int \frac{dE dJ J}{\Omega_1(E, J)} \sum_{\mathbf{n}} \frac{2}{2l+1} \mathbf{n} \cdot \frac{\partial f_0}{\partial \mathbf{I}} \frac{1}{\omega - \mathbf{n} \cdot \boldsymbol{\Omega}} \times \\ |Y_{l,l_2}(\pi/2, 0)|^2 W_{l,l_2,l_3}^{l_1,i*}(\mathbf{I}) W_{l,l_2,l_3}^{l_1,j}(\mathbf{I}), \quad (\text{B5})$$

and

$$W_{l,l_2,l_3}^{l_1}(I_1, I_2) = \frac{1}{\pi} \int_0^\pi dw_1 \cos[l_1 w_1 - l_2(\psi - w_2)] u_j^{l_3}(r). \quad (\text{B6})$$

In the above equations, I_j are the actions, w_j are the angles, and $\mathbf{n} = (l_1, l_2, l_3)$ is a vector of integers. The actions were chosen by solution of the Hamiltonian-Jacobi equation; I_1 is the radial action, I_2 is the total angular momentum and I_3 is the z-projection of the angular momentum. The quantity $\psi - w_2$, the difference between the position angle of a star in its orbital plane and its mean azimuthal angle, depends only on w_1 , the angle describing the radial phase. The frequencies associated with the conjugate angles \mathbf{w} are defined by $\boldsymbol{\Omega} = \partial H / \partial \mathbf{I}$. Ω_1 is the radial frequency and reduces to the usual epicyclic frequency for nearly circular orbits. Ω_2 is the mean azimuthal frequency. $\Omega_3 = 0$ and the corresponding angle w_3 is the azimuth of the ascending node (see Paper I or Tremaine and Weinberg 1984, for details). It follows that $M_{ij}^*(\omega) = M_{ij}(-\omega^*)$ and this property is exploited in the computations described in §3 to reduce the number of numerical evaluations. Since most researchers quote phase-space distribution functions in energy and total angular momentum $f(E, J)$, I have transformed the integration in the definition of $\mathbf{M}(\omega)$ in equation (B5) to E and J variables. The factor $1/[\omega - \mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{I})]$ in the integrand of equation (B4) may cause discontinuities in the E and J integration. This is efficiently addressed using the rational function techniques discussed in Appendix D.

Equation (B4) only has a nontrivial solution if

$$\mathcal{D}(\omega) \equiv \det\{\mathbf{1} - \mathbf{M}(\omega)\} = 0. \quad (\text{B7})$$

As in plasma theory, we refer to the function $\mathcal{D}(\omega)$ as a dispersion relation. In general, ω will be complex. Recall that the coefficients \tilde{a}_j which appear in equation (B4) are Laplace transformed and describe the response of the system to a perturbation with a time-dependence of the form $\exp(-i\omega t)$. Because equation (B4) follows from a Laplace transform, it is only valid in its present form for ω in the upper-half complex plane. The dispersion relation must be analytically continued to $Im(\omega) < 0$ to find damped modes.

The response coefficients $\tilde{\mathbf{a}}^{lm}$ which solve equation (B4) for a particular eigenfrequency ω , describes a mode of the system.

Because a sphere has no unique symmetry axis, a mode can not depend on an arbitrary choice of coordinate axes. Since a rotation causes m components to mix according to the rotation matrices, the mode itself must depend on l alone. In particular, note that since the functions (u_i^{lm}, d_i^{lm}) may be chosen independent of m (e.g. the spherical Bessel function used by Fridman and Polyachenko 1984, and Weinberg 1989), equation (B5) and thus the dispersion relation is independent of m .

C. Eigenfunctions

The eigenfunction corresponding to the frequency at which $\mathcal{D}(\omega) = 0$ may be obtained by simultaneous solving the $n - 1$ independent equations implied by the $n \times n$ matrix given by equation (B4) for the expansion coefficients a_j . The density and potential perturbations follow directly using equations (B2) and (B3). Recall that the eigenvalue is found by rational function fit to a grid of $\mathcal{D}(\omega)$. The matrix elements at the eigenvalue are determined by rational function fit to the grid used to find the eigenvalue originally. The dispersion relation is not solved exactly by the new interpolated matrix $M_{ij}(\omega)$ and is iteratively refined to find a new solution to $\mathcal{D}(\omega) = 0$. Reassuringly the new eigenvalue is found within 5% in $|\omega|$ of the predicted value in all cases.

D. Rational function techniques

Rational functions are used for two tasks in the this paper. First, they are suitable estimators for the functional form of the dispersion relation. Using a discrete number of evaluations of the dispersion relation in the upper-half plane, the rational function may be trivially analytically continued to the lower-half plane and, in particular, zero locations may be straightforwardly determined. Second, rational functions are ideally suited to evaluating complex integrands with poles or near singularities. Since the integrands of equation (B5) have “vanishing” denominators—denominators of the form $1/[\omega - \mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{I})]$ —one expects that rational function to approximate these integrands rather well.

The general procedure is as follows. Suppose we are given n evaluations of a complex function $f(z_i)$. The standard techniques for rational function evaluation may be

straightforwardly extended to the complex domain. In particular, I adopted reciprocal differences along the main diagonal (Stoer & Bulirsch 1980) from which the coefficients of the numerator and denominator polynomials may be determined directly by recursion. Let the rational function be written as

$$R(z) = \frac{N(z)}{D(z)}, \quad (\text{D1})$$

where N and D are polynomials and let the order of N be n and the order of D be d . The reciprocal difference routine gives either $n = d$ or $n = d + 1$. Using synthetic division,

$$R(z) = Q(z) + \frac{\tilde{N}(z)}{D(z)} \equiv Q(z) + \tilde{R}(z) \quad (\text{D2})$$

where the order of Q and \tilde{N} will be 0 and $n - 1$ if $n = d$ or 1 and $n - 2$ if $n = d + 1$. Now since $\tilde{R}(\infty) = 0$, one can show that

$$\tilde{R}(z) = \sum_{i=1}^k S_i(z) \quad (\text{D3})$$

where the sum is over the k zeros in $D(z)$, z_i , and $S_i(z)$ is the principal part of $\tilde{R}(z)$ at z_i . Following Henrici (1974), we may write the principal as follows:

$$S_i(z) = \sum_{j=1}^{m_i} a_{i,j} (z - z_i)^{-j} \quad (\text{D4})$$

where m_i is the multiplicity of the zero at z_i .

An efficient method for computing the principle parts has been outlined by Henrici (1974); I will sketch his arguments here for completeness. Since the coefficients of \tilde{R} and D are available, the poles of D may be derived numerically using deflation and $\tilde{R}(z)$ may be written in the form

$$\tilde{R}(z) = \sum_{i=1}^k \left(\frac{1}{z - z_i} \right)^{m_i} \frac{n(z)}{d_i(z)}. \quad (\text{D5})$$

By construction $n(z)/d_i(z)$ is analytic at z_i and may be expanded in non-negative powers of $h \equiv z - z_i$ and therefore

$$\tilde{R}(z_i + h) = \sum_{n=0}^{\infty} c_{i,n} h^{n-m_i}. \quad (\text{D6})$$

Comparing equation (D6) with equations (D3) and (D4), it follows from the uniqueness of Laurent expansion that $a_{i,j} = c_{i,m_i-j}$ for $j = 1, \dots, m_i$. The desired values of $a_{i,j}$ may be then straightforwardly obtained from the explicit expression of $n(z)$ and $d_i(z)$.

This approach is remarkably useful since any rational function may be integrated exactly, limited only by truncation errors and the accuracy of the initial polynomial coefficients. Therefore, once an expression in the form of equation (D2) is obtained, the rational function R maybe be trivially integrated along any segment of the real line. In practice, I have found that m_i is rarely larger than 1 and the computational labor is usually much smaller than the statement of general method might suggest.

One unfortunate feature of the rational function scheme is that does not converge with increasing grid density due to truncation error. For example, imagine trying to approximate a lower order rational function with a very dense grid of points. The exact representation will have identical, and therefore canceling, roots in both the numerator and denominator. Since these will not cancel precisely in the numerical representation, a very dense grid may result in a noisy approximation. For zero location, incomplete cancellation is easily recognized. For integration, one must check the asymptotic behavior.

E. N-body realization in action-angle variables

The most common procedure for realizing an arbitrary but spherical phase-space distribution is the rejection method for psuedorandom variables (e.g. Press et al., 1992). The procedure is as follows. Let \mathcal{R} be a variate from the unit interval. If the phase-space volume is finite for the system, the radius may be computed from the cumulative mass distribution: $M(r) = \mathcal{R}M(r_{max})$. The gravitationally potential $\phi(r)$ then fixes the maximum velocity. A trial velocity may then be chosen uniformly from the sphere of radius $v_{max} = \sqrt{2[\phi(r_{max}) - \phi(r)]}$ from which the energy and angular momentum, E and J , may be chosen. Let $f_o(E, J)$ be the phase-space distribution function. The trial point is accepted if $f_o(E, J) / \sup |f_o(E, J)| > \mathcal{R}$. Otherwise the entire procedure is repeated.

The perturbed phase-space distribution, $f = f_o + \epsilon f_1$, is best realized in action-angle variables but otherwise the procedure is analogous. From Paper I (see also Appendix B), the perturbed mode may be written

$$f_1(\mathbf{I}, \mathbf{w}) = \sum_{\mathbf{n}} \mathbf{n} \cdot \frac{\partial f_o}{\partial \mathbf{I}} \frac{1}{\omega - \mathbf{n} \cdot \boldsymbol{\Omega}} e^{i(\mathbf{n} \cdot \mathbf{w} - \omega t)} \sum_j a_j^{lm} W_{l, l_2, m}^{l_1, j}(\mathbf{I}). \quad (\text{E1})$$

One chooses the three trial actions, or equivalently E, J, J_z , from an isotropic distribution, $f_{comp}(E)$, chosen to lie everywhere above f . The angle variables are chosen uniformly from the interval $[0, 2\pi]$. The orbit corresponding to the trial phase-space point is then specified in the background model. The phase point is accepted if $f(\mathbf{I}, \mathbf{w}) / f_{comp}(E) > \mathcal{R}$. This

procedure is efficient for a careful choice of f_{comp} . The quantity ϵ must be chosen to be sufficiently small that $f \geq 0$ everywhere.

Fig. 1.— Contours in magnitude of the analytically-generated dispersion relation for the stellar cube.

Fig. 2.— Same as in Fig. 1 but for the numerically-generated dispersion relation for the stellar cube.

Fig. 3.— Shows the location of the damped $l = 1$ modes for $W_0 = 3, 5, 6, 7$ King models (labeled by W_0). The estimated precision is indicated by the errorbar at the upper left.

Fig. 4.— Contours of the density perturbation for the $l = m = 1$ mode for the $W_0 = 5$ model on the x - y plane. Only the inner region of the model is shown; the tidal radius is $r_t = 8.72$. Overdensity (underdensity) is shown as a solid (dashed) line.

Fig. 5.— Effect of the density perturbation shown in Fig. 4 on the background model. The peak perturbation is 30% of the background. Note the shift of the central density peak. The core radius is at 0.81 in these units. The dotted contours show the unperturbed background at the same levels.

Fig. 6.— Absolute value of the phase-space perturbation for the $l = m = 1$ mode for the $W_0 = 5$ model as a function of energy E and $\kappa^2 = (J/J_{max})^2$.

Fig. 7.— Contours of the density perturbation for the $l = m = 2$ mode in the $W_0 = 5$ model on the x - y plane.

Fig. 8.— Effect of the density perturbation shown in Fig. 7 on the background model. The peak perturbation is 30% of the background.

Fig. 9.— The virial quantity $-2T/W$ (top panel) and the run of gravitational potential energy (bottom panel) shown as a function of time.

Fig. 10.— Position angle as a function of time for the $l = m = 1$ component in the n-body simulation described in §4. Three radial wavefunctions (orders $n = 8, 9, 10$) are shown separately (solid lines) along with the theoretical prediction from §3 (dashed line).