2007

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UMass Amherst Computer Science Department Technical Report 2006-40

First version: August 1, 2006

This version: August 22, 2006

Abstract

We study the interaction among users of unstructured file sharing applications, who compete for available network resources (link bandwidth or capacity) by opening multiple connections on multiple paths so as to accelerate data transfer. We model this interaction with an *unstructured file sharing game*. Users are players and their strategies are the numbers of sessions on available paths. We consider a general bandwidth sharing framework proposed by Kelly [1] and Mo and Walrand [2], with TCP as a special case. Furthermore, we incorporate the Tit-for-Tat strategy (adopted by BitTorrent [3] networks) into the unstructured file sharing game to model the competition in which a connection can be set up only when both users find this connection beneficial. We refer to this as an *overlay formation game*. We prove the existence of Nash equilibrium in several variants of both games, and quantify the losses of efficiency of Nash equilibria. We find that the loss of efficiency due to selfish behavior is still unbounded even when the Tit-for-Tat strategy is believed to prevent selfish behavior.

I. Introduction

Recently peer-to-peer applications (e.g., BitTorrent [3], Kazaa, eDonkey, and Gnutella [4]) have become very popular. They can be major contributors of the Internet traffic. For example, Sprint's IP Monitoring Project [5] shows that in April 2003, $20 - 40\%$ of total bytes corresponded to peer-to-peer traffic on one backbone link. CacheLogic [6] estimates that peer-to-peer generated 60% of all US Internet traffic at the end of 2004.

We refer to the networks for these peer-to-peer applications as unstructured file sharing overlay networks. These networks are overlay networks since users forward or relay traffic for each other. These networks are also *unstructured* because there are no well-defined network topologies, and users are not under the control of some central entity. For comparison, Resilient Overlay Network [7] is a *structured* overlay network. Given the increasingly large share of Internet traffic from unstructured file sharing networks, it is important to understand the behavior and performance of such networks, and such a fundamental understanding will certainly help ISPs and aid in the design of future Internet architecture.

In this paper, we investigate the strategic behavior of self-interested peers/users of such unstructured file sharing overlay networks. Our work differs from previous works on peer-to-peer applications, whose focus are on file searching and replication [8], and topology discovery [9]. Specifically, our investigations are from two different angles.

First, we study the interaction among users of unstructured file sharing applications, who compete for available network resources (link bandwidth or capacity) by opening multiple connections or sessions on multiple paths so as to accelerate data transfer. We introduce an *unstructured file sharing game* to model this interaction. In this game, users are players and their strategies are the numbers of sessions on available paths. The data rate allocated to connections are determined by the network. The mechanism of rate allocation considered by us is a general bandwidth sharing framework proposed by Kelly [1] with TCP networks as special cases [10][2]. Our focus is on TCP networks in which all connections/sessions are TCP connections. The unstructured file sharing game generalizes the *TCP connection game* introduced in [11] where the competition for a single bottleneck link capacity is investigated.

Second, we incorporate the Tit-for-Tat strategy into the unstructured file sharing game. This strategy is widely known and built into BitTorrent [3] networks. With this strategy, peers set up a connection between themselves only when they both find it beneficial. We model this interaction scenario as an *overlay formation game*. In order to make our model tractable, we restrict users to open either zero or one connection to another peer.

In both games, users are interested in maximizing their benefits, a combination of some utility function and the cost associated with maintaining data transfer sessions. We assume that utility functions are increasing and concave functions of the data throughput in bits per second. Throughput is defined as the successful packet delivery rate. The cost incurred to users includes memory cost and CPU cost. As in [11], we consider a cost that is proportional to the total number of connections opened by a user. We also consider another type of cost which is proportional to a user's packet sending rate.

We are interested in the following questions. First, does there exist a stable network state (i.e., Nash equilibrium (NE) [12]) in both games? If so, what is the system performance at a NE? Specifically, we are interested in the loss of efficiency of a NE and the price of anarchy [13] of NE(s). The loss of efficiency of a NE is defined as the ratio of the optimal system performance over the system performance at the NE, and the worst loss of efficiency is referred to as the price of anarchy [13]. These metrics capture how bad the competition can be among self-interested TCP users. Here we focus on pure strategy NE.

We make the following contributions.

First, we give a formal formulation of unstructured file sharing game, and show by examples that multiple NEs exist on general network topologies. We then focus on parallel link networks and star networks, which are used to model peer-to-peer applications (similar topologies were also studied in [14][15]). We prove the existence of

NE of unstructured file sharing games on both networks, and find that, if users are not resource constrained, the efficiency loss of NEs can be unbounded (i.e., price of anarchy is arbitrarily large). Fortunately, if there are resource constraints for users, the efficiency loss is upper bounded. We also demonstrate the stability of NE in best-response dynamics in several variants of the game.

Second, we model the Tit-for-Tat strategy in unstructured file sharing networks by an overlay formation game. We show analytically the existence of equilibrium overlay networks and that the loss of efficiency can be arbitrarily large. Tit-for-Tat is believed to prevent selfish behavior. However, our results show that the loss of efficiency due to selfish behavior can still be unbounded.

The rest of this paper is organized as follows. Related work is presented in Section II. The problem formulation for unstructured file sharing game is given in Section III. In Sections IV and V, we focus on unstructured file sharing game on a parallel link network and star network. We address the overlay formation game in Section VI. Conclusions are given in Section VII.

II. Related Work

Johari *et al* [16] study a congestion game where users of a congested resource anticipate the effect of their actions on the price of the resource. In [16] users compete for each link independently from other links in the network. But this independence characteristic is not true for our model, because if a user opens a connection on a path, then all links of this path must carry this connection. [17] and [11] study the interactions among selfish TCP users competing for a single bottleneck link. The unstructured file sharing game in this paper can be thought of as a generalized version of the game in [11].

[18][19] propose multi-path congestion controllers by which users can coordinate the data transfer sessions on several different paths to improve data throughput. A multi-path congestion controller chooses rates at which to send data on all of the paths available to it. In our models, all sessions controlled by a single user are independent congestion controllers. [14] studies how Tit-for-Tat affects selfish peers who are able to set their uploading bandwidth. Our work differs from [14] in that we assume that a user can benefit by changing the number of connections to open. The analytical framework for our overlay formation game is in [20].

III. Unstructured File Sharing Game

A. Formulations

Consider a network consisting of J links, numbered $1, ..., J$. Link j has a capacity given by $C_j > 0$; we let $\mathbf{C} = (C_1, C_2, ..., C_J)$ denote the vector of capacities. A set of users $\{1, ..., R\}$ share this network. We assume that there exists a set of paths through the network, numbered $1, ..., P$. By an abuse of notation, we will use J, N, P to also denote the sets of links, users, and paths, respectively. Each path $p \in P$ uses a subset of the set of links J; if

link j is used by path p, we will denote this by writing $j \in p$. Each user $r \in \mathbf{R}$ has a collection of paths available through the network; if path p serves user r, we will denote this by writing $p \in r$.

Each user can open a number of concurrent connections n_{rp} on each path p with $p \in r$. This defines a strategy vector for user r as $\mathbf{n}_r = (n_{rp})$ with $p \in P$ and $p \in r$. Then a composite strategy vector of all users is given by $\mathbf{n} = (\mathbf{n}_1, ..., \mathbf{n}_R)$. For a given $\mathbf{n}$, a certain rate allocation mechanism allocates a traffic rate y_p to each connection on path p. We will discuss rate allocation mechanisms in the following section. For now, we simply state that, $\forall p \in P$, y_p is a function of $\mathbf{n}$. We use vector $\mathbf{y} = (y_p, p \in P)$ to represent a rate allocation on all paths.

The total date rate or throughput G_r obtained by a user r is: $G_r(\mathbf{n}_r) = \sum_{p \in r} n_{rp} y_p$, where n_{rp} is the number of connections opened by user r on path p. As y_p ($\forall p \in P$) is a function of $\mathbf{n}$, the throughput of user r is a function of the number of connections of all users, namely, $G_r = f(\mathbf{n})$. Any feasible rate allocation $\mathbf{y}$ must satisfy the capacity constraint: $\sum_{r \in \mathbf{R}} \sum_{p: j \in p} n_{rp} y_p \leq C_j, j \in J$.

We assume that user r receives a utility $U_r(G_r)$ when obtaining throughput G_r. We assume that U_r is a continuous, concave, and non-decreasing function of G_r, with domain $G_r \geq 0$. A user r has some cost $\Phi_r(\mathbf{n}_r)$ associated with opened connections. We assume that this cost is proportional to the total number of connections opened by this user on all its available paths: $\Phi_r(\mathbf{n}_r) = \beta \sum_{p \in r} n_{rp}$. Note that $\beta \in [0, 1]$, and it is interpreted as the aggressiveness coefficient. Smaller β corresponds to more powerful computation resources. This type of cost is also considered in [11]. In general, we can assume that Φ_r is a continuous, convex, and non-decreasing function of $\mathbf{n}_r$. The payoff or benefit of a user r is a linear combination of utility U_r and cost Φ_r, defined as:

$$B_r(\mathbf{n}_r) = U_r(\mathbf{n}_r) - \Phi_r(\mathbf{n}_r). \tag{1}$$

B. Rate Allocation Mechanism

We assume that the network allocates data rates to connections based on the α-bandwidth allocation scheme [10][1][2]:

$$\text{maximize}_{\mathbf{y}} \quad \sum_p w_p n_p^{\alpha} \frac{(y_p n_p)^{(1-\alpha)}}{1-\alpha} \tag{2}$$

$$\text{subject to} \quad \sum_{r \in \mathbf{R}} \sum_{p: j \in p} n_{rp} y_p \leq C_j, j \in J \tag{3}$$

$$n_p = \sum_{r: p \in r} n_{rp} \forall p \in P. \tag{4}$$

where w_p is the weight of path p. n_p is the number of connections or sessions on path p. Different values of α give different rate allocations. For example, as $\alpha \to \infty$, this allocation mechanism corresponds to Max-Min fairness. Rate allocation in a TCP network is well approximated with $\alpha = 2$ and $w_p = 1/(RTT_p)^2$. Here, RTT_p is the Round Trip Time (RTT) of path p.

In a single link case and where all paths have the same RTT, this α-bandwidth allocation is simplified to a *simple rate allocation mechanism*. That is, for a link shared by n flows with the same RTT, each flow or connection gets

an equal share of the bandwidth of the link, namely,

$$y = C/n. \tag{5}$$

Thus if a user r has n_r flows, then its throughput G_r is:

$$G_r(\mathbf{n}_r) = \begin{cases} Cn_r / \sum_{w \in \mathbf{R}} n_w, & \text{if } n_r > 0 \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

Remarks. Note that this *simple rate allocation mechanism* cannot be extended to a network setting. Specifically, after we calculate the rate allocated to each user on each link according to (5), we cannot simply say that the allocated rate on a path can be given by $y_{rp} = \min_{j \in p} y_{rj}, \forall r \in \mathbf{R}$. An illustrative example is given in Appendix I.

Note that the authors of [16] can use this rate allocation mechanism because in their case, users compete for each link independently from other links. However, in our case, links can not be treated independently, as all links of a path must carry the connections opened on this path. As shown in the following section, this requirement makes the throughput of a user neither a concave nor convex in the numbers of connections opened by this user. Thus, it is difficult to apply the existing game-theoretic results (which requires concavity of utility functions) to the unstructured file sharing game on general network topology. Thus, in this paper we focus on two specific networks: parallel links and a star.

C. Unstructured File Sharing Game

Based on the previous formulations, we now introduce an *unstructured file sharing game*. In this game, each user r tries to maximize its aggregate benefit B_r by adjusting $\mathbf{n}_r$, its number of connections on its available paths. Namely, a user r tries to solve the following optimization problem:

$$\max_{\mathbf{n}_r} \quad B_r(\mathbf{n}_r, \mathbf{y}^*(\mathbf{n}_r)) \tag{7}$$

$$\text{s.t.} \quad n_{r_p} \in [0, n_{r_p}^{max}], \quad \forall r_p \in P_r \tag{8}$$

$$\mathbf{y}^* = \text{argmax}_{\mathbf{y}} \quad \sum_p w_p n_p^{\alpha} \frac{(y_p n_p)^{(1-\alpha)}}{1-\alpha} \tag{9}$$

$$\text{s.t.} \quad \sum_{r \in \mathbf{R}} \sum_{p: j \in p} n_{r_p} y_p \leq C_j, j \in J$$

$$n_p = \sum_{r: p \in r} n_{r_p}, \forall p \in P$$

The decision variables of user r is given by vector $\mathbf{n}_r$. The set of available paths of user r is represented by P_r. (9) indicates that the throughput of each connection on a path is the solution of the optimization problem defined in (2). If $\alpha = 2$ and $w_p = 1/(RTT_p)^2$ and the network is a single bottleneck link, this game becomes the TCP connection game [11].

For a general network, we cannot obtain an explicit form of function $B_r(\mathbf{n}_r)$ because there is no closed form solution for the rate allocation problem (9). However, as shown later, we can obtain an explicit form of $B_r(\mathbf{n}_r)$ for some specific networks such as grid network, parallel link, and star network.

In fact, (7) is a Bi-level Programming problem which in general is NP-hard [21]. In this paper, we do not try to obtain a general solution for (7) for each user. Instead, we focus on some special network topologies for which there exist analytically tractable and closed form solutions to (9), and for these networks, we investigate the existence of Nash equilibrium.

Let $\mathbf{n}_r^*$ represent the solution to user r's optimization problem defined above. Formally, we have:

$$\mathbf{n}_r^* = \text{argmax}_{\mathbf{n}_r} B_r(\mathbf{n}).$$

A Nash equilibrium (NE) is defined as a composite strategy profile or a vector of connections of all users, and no user can gain by unilaterally deviating from it. We denote a Nash equilibrium by: $\mathbf{n}^* = (\mathbf{n_1}^*, \mathbf{n_2}^*, ..., \mathbf{n_R}^*)$.

The NE of this game represents the stable network state of the interaction among all users. The network performance at a NE is described by the loss of efficiency, defined as:

$$L_{eff} = B_{max}/B_{ne} \tag{10}$$

where B_{ne} is the total benefit of all users when the network is at a NE, and B_{max} is the maximum benefit. The worst efficiency loss is also known as the *price of anarchy* [13].

Remarks. It is not necessarily true that the throughput $G_r(\mathbf{n}_r)$ is an increasing function of $\mathbf{n}_r$. For example, in the network shown in Figure 1, user r has three paths: p_1, p_2 and p_3. p_1 contains two links j_1 and j_2 with capacity C. p_2 contains link j_1 and p_2 contains link j_2. According to the simple rate allocation mechanism introduced before, if $\mathbf{n}_r = (0, 1, 1)$, then $G_r(\mathbf{n}_r) = 2C$. However, if user r increases its number of connections on path p_1 from zero to one, then $G_r(\mathbf{n}_r) = 3C$

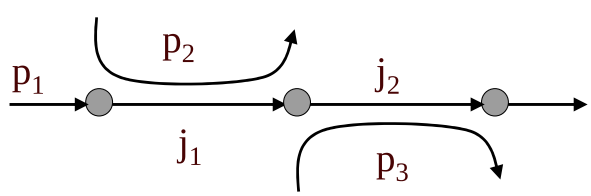

Fig. 1. A case where the throughput of user r is not increasing in $\mathbf{n}_r$.

One interesting special case is that a user can only choose either zero or one connection on a given available path. That is, (8) can be described as $n_{r_p} \in \{0, 1\}, \forall r_p \in P_r$. In this case, each user only has finite number of strategies. This variant of the game is a finite game. According to [22], this game admits a mixed strategy NE. This NE is related to randomly choosing of connections to other peers in BitTorrent applications [3]. This is an interesting future research topic.

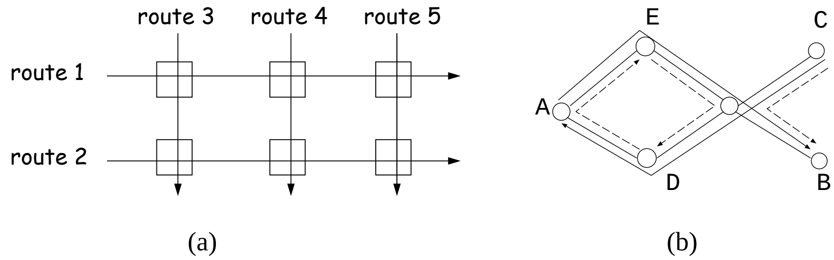

Fig. 2. (a) is a grid network where squares represent links. (b) is an instance of (a). $A \to E \to B$ and $C \to D \to A$ correspond to route 1 and 2 in (a). $D \to A \to E, E \to F \to D, C \to F \to B$ correspond to routes $3, 4, 5$.

D. Existence of Multiple Nash Equilibria in Grid Network

In this section, we use a simple example to illustrate the unstructured file sharing game and possible NEs. The network topology in this example is a so called grid network introduced in [10], shown in Figure 2.(a). A possible instance of this grid network is called "fish" network, shown in Figure 2.(b).

A closed form rate allocation based on the α-bandwidth sharing mechanism for such a grid network is given in [10]. Specifically, if there are K horizontal routes and L vertical routes, then the total throughput on horizontal path p is given by

$$n_p y_p = \frac{(\sum_{k=1}^{K} \frac{1}{RTT_k} n_k^{\alpha})^{1/\alpha}}{(\sum_{k=1}^{K} \frac{1}{RTT_k} n_k^{\alpha})^{1/\alpha} + (\sum_{l=1}^{L} \frac{1}{RTT_l} n_l^{\alpha})^{1/\alpha}} \tag{11}$$

where n_p denotes the number of flows on horizontal path p. y_p is the throughput of a single flow on path p.

In the following, we discuss two variants of the game by considering two users playing the game on the grid network. User 1 uses route 1 and user 2 uses route 2. Suppose $\alpha = 2$ in (11), which corresponds to TCP. Suppose that all vertical and horizontal routes have RTT of 50ms, and there are 10 background flows on all vertical routes.

Benefit includes throughput only. When both users are only concerned with total throughput and have no resource limitations, we have identified the following case where there is a unique NE, at which both players open their maximal allowable number of connections.

There are two users. User 1 uses the upper horizontal route and user 2 uses the lower horizontal route. Suppose that user 2 opens 100 connections. In Figure 3, we plot the throughput of user 1 as a function of its number of connections on its single available path. We find that the throughput of user 1 is neither a concave nor convex function of its number of connections on its single available path. This suggests that the current results on the existence of Nash equilibrium cannot be applied here because these results require the concavity of the utility function [12][23].

However, note that the throughput of user 1 is an increasing function of n_1, which can be verified by checking its first-order derivative. Similarly, we can also show that user 2's throughput is also an increasing function of its number of connections. Therefore, if both users play the unstructured file sharing game, there is a unique NE. Furthermore, at the NE both players opens their maximal allowable number of connections.

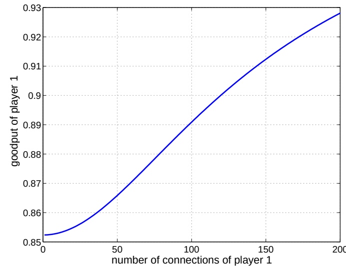

Fig. 3. Total data rate G of user 1 as a function of the number of connections on its path, when user 2 has 100 connections.

Benefit includes both throughput and cost. In this variant of the game, not only is that B_r neither a concave nor a convex function of its number of connections $\mathbf{n}_r$, but B_r is not always increasing in $\mathbf{n}_r$.

For example, suppose $\beta = 0.0005$ in the cost function $\Phi(\mathbf{n}_r)$. We plot in Figures 4 and 5 the benefit B of user 1 as a function of its number of connections on its single available route, given that the number of connections of user 2 is 50 and 100 respectively. Note that, depending on the number of connections opened by user 2, the benefit of user 1 can be either an increasing or a decreasing function of $\mathbf{n}_1$.

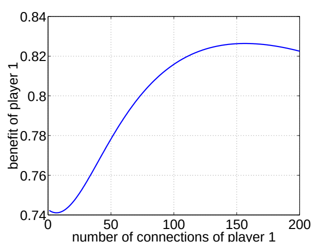

Fig. 4. Benefit of user 1 as a function of the number of connections when user 2 has 50 connection.

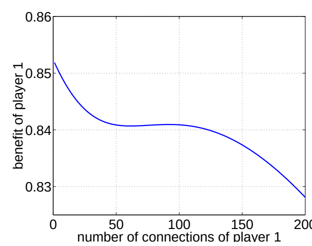

Fig. 5. Benefit of user 1 as a function of the number of connections when user 2 has 100 connection.

We define the best response $\mathbf{n}_r^*$ of player r as the solution of r's optimization problem given fixed strategies of all other players $\mathbf{n}_{-r}$. In Figure 6, we plot the best response curves of both players. Note that there are three intersecting points. An intersecting point is a NE because at that point, each user's response is the best response to the other user's strategy. Thus, there are three NE in this game. For comparison, in the single link TCP connection game [11], there is only one unique NE when the cost is proportional to the number of connections.

It is also interesting to note that these two players do not share any common link (Figure 2), so, their interaction arises because they share links with other common sessions.

This simple example indicates that the interaction among multiple users on a general network topology can be much more complex than the single link TCP connection game. The existence and uniqueness of Nash equilibrium can depend on network topologies and the utility functions adopted by users.

In the following, we focus on two special networks: a parallel link network and a star network. Both can be used

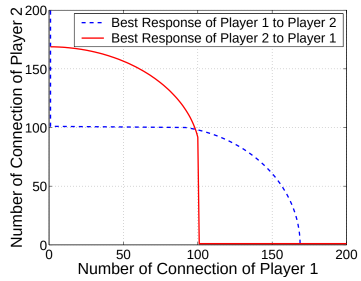

Fig. 6. Best response curves of both player 1 and player 2.

to model peer-to-peer networks.

IV. Parallel Link Network

In this section, we investigate an unstructured file sharing game on a parallel-link network where all users share a common source and a common destination node interconnected by a number of parallel links. Parallel-link networks can be used as simple models for unstructured file sharing. For example, in eDonkey networks [4], a peer can download a file from multiple other peers providing this file. There are possibly many peers simultaneously downloading the same file, and they can be thought of as associated with a common destination node. Each of the file-providing peers can be thought of as a "link" or "path" connecting the common destination node with a common super virtual file-providing source node. Those downloading peers compete for these parallel links/paths for bandwidth. This scenario can be approximated by a parallel link network.

In this section, we first show the existence of stable network states (NEs) on a parallel-link network. We then present the results on the efficiency loss of NE and the stability of NE in the best-response dynamics.

A. Nash equilibrium

Suppose that there are L links and R users. By an abuse of notation, we will use L and R to denote the set of links and the set of users respectively. An example of a parallel link network is shown in Figure 7. The throughput G_{rj} obtained by user r on link j is given by the simple rate allocation mechanism introduced in the previous section: $G_{rj}(n_{rj}) = C_j n_{rj}/RTT_{rj}/(\sum_{k=1}^{R} n_{kj}/RTT_{kj})$, where RTT_{rj} is the Round Trip Time of user r on link/path j, C_j is the capacity of link j, and n_{rj} is the number of connections of user r on link j. The strategy of user r is a vector of the number of connections on its available paths or links: $\mathbf{n}_r = (n_{r1}, ..., n_{rL})$ and $n_{rj} \in (0, n_r^{max}], \forall j \in L$. n_r^{max} is the maximum allowable number of connections for user r. Note that this game is a continuous kernel game [12] as we assume that a user's strategy is a real-valued vector.

In this section, we only consider the case where $U_r(\mathbf{n}_r) = G_r(\mathbf{n}_r)$. The benefit or payoff obtained by user r is: $B_r(\mathbf{n}_r) = G_r(\mathbf{n}_r) - \Phi_r(\mathbf{n}_r)$.

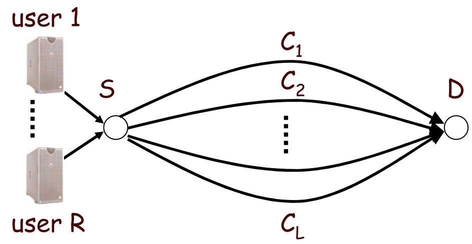

Fig. 7. A parallel-link network topology.

We consider two scenarios: an unconstrained game and a constrained game. In an unconstrained game, there is no upper limit on the total number of connections a user can open. In a constrained game, each user must choose a certain total number of connections[1]. We have shown the existence of a unique NE in both constrained and unconstrained games.

B. Unconstrained Game

In an unconstrained game, users essentially play an independent game on each distinct path/link. Since a NE exists and is unique on a single link game [11], we know that a NE also exists and is unique on this parallel link network. This is summarized in the following theorem.

Theorem 1: There exists a unique interior-point NE in an unstructured file sharing game on a parallel link network.

Social Benefit at Nash equilibrium. As shown in [11], a single bottleneck link TCP connection game admits a symmetric NE when users have the same Round Trip Times (RTT) and their benefit function includes throughput and a cost proportional to the number of connections. This result can be extended to our uncontrained game. That is, when all users have the same RTTs, the unique NE is symmetric, in the sense that all users have the same number of connections at the NE.

Solving the optimization problem for a user r, we can get the vector of connections of user r at the symmetric NE as:

$$n_{rj}^* = (R-1)C_j/(R^2\beta).$$

Then, user r's benefit at the NE is

$$B_r^* = \sum_{j=1}^{L} C_j/R - \sum_{j=1}^{L}(R-1)C_j/R^2.$$

Therefore, the total social benefit of the NE is

$$B_{ne} = \sum_{j=1}^{L} C_j/R.$$

[1]This is motivated by BitTorrent [3] where each peer always has 5 active connections open to 5 different other peers.

Note that B_{ne} is not related to the cost of users. It is simply a function of the total network capacity and the number of users. As the number of users increases, the total social benefit of the NE goes to zero.

Reaction functions. The reaction function of a user r is defined as the best response of user r as a function (if it exists) of the total number of connections of all other users. A response of user r is $\mathbf{n_r} = (n_{r1}, n_{r2}, ..., n_{rL})$. Since in an unconstrained game users essentially play an independent game on each individual link, we can solve for a user's best response on each link separately. Specifically, for any link j, we have

$$\bar{n}_{rj} = \text{argmax}_{n_{rj} \in (0,\infty)} B_{rj}(\sum_{k \neq r}^{R} n_{kj}). \tag{12}$$

For convenience, let n_{-rj} denote $\sum_{k\neq r}^{R} n_{kj}$. It is easy to show that

$$\bar{n}_{rj} = f(n_{-rj}) = -n_{-rj} + \sqrt{C_j n_{-rj}/\beta}. \tag{13}$$

$\bar{n}_{rj}$ is a continuous function of n_{-rj}. We note that in order to guarantee that the best response of user r is an interior point of its strategy space, we must have

$$\bar{n}_{rj} > 0 \quad \text{or} \quad n_{-rj} < C_j/\beta. \tag{14}$$

As shown in Section III-D, we can use reaction functions to identify NEs by checking the intersecting point(s) of the reaction function (best response) curves of all players. We can also use reaction functions to investigate the best-response dynamics of the game playing process, as discussed later.

Stability of NE in Best-response Dynamics. Suppose that users interact with each other using best-response in a discrete time process, a so called *best-response dynamics* [12][11]. This process proceeds in discrete time steps or rounds, and only one randomly chosen user makes a move at each round. Whenever a user makes a move, it calculates its best response to other users' numbers of connections which are determined in previous steps. That is, the user who makes a move solves its optimization problem to maximize its benefit. If all users' strategies converge to or stabilize at some point $\mathbf{n}_s$ as time goes to infinity, then $\mathbf{n}_s$ is a NE, and it is *globally stable*. Regarding an unstructured file sharing game on a parallel link network, we have the following stability result.

Theorem 2: The unique NE is globally stable in the two-player version of the unstructured file sharing game on parallel link network when both players use best-response to play the game.

Proof: We want to show that the best response of a user is a concave function of the other player's number of connections. In the unconstrained game, users actually play independent games on different links. For a given user r, the best response function or reaction function on link j is given by (13), and re-stated as follows:

$$\bar{n}_{r,j} = -n_{-r,j} + \sqrt{C_j n_{-r,j}/\beta},$$

where $n_{-r,j}$ is the number of connections of all other users. It can be shown that

$$\partial^2 \bar{n}_{r,j} / \partial n_{-r,j}^2 = (-1/4)\sqrt{C_j/\beta} \cdot n_{-r,j}^{-3/2} \le 0.$$

Thus, the reaction function of user r is a concave function of number of connections of other users. Then, from [24], we know that in a two-player version of the game, Nash equilibrium is globally stable. ■

Efficiency loss of Nash equilibrium

First note that the maximal system benefit is the solution of a straightforward optimization problem. The system benefit can be represented as:

$$B = \sum_{r=1}^{R} B_r = \sum_{r=1}^{R}\sum_{j=1}^{L} G_{rj} - \beta \sum_{r=1}^{R}\sum_{j=1}^{L} n_{rj}. \tag{15}$$

We find that the maximal value of B is

$$B_{max} = \sum_{j=1}^{L} C_j - \beta N_{min}. \tag{16}$$

Consider a homogeneous network where all links have the same capacity. Then we have $B_{max} = LC - \beta L$, as we need at least one connection for each link in order to get the bandwidth of each link. The efficiency loss of a NE is given by

$$L_{eff} = \frac{B_{max}}{B_{ne}} = \frac{LC - \beta L}{LC/R}. \tag{17}$$

This result essentially suggests that the efficiency loss of the unique NE is bounded. However, if L, C are fixed, and let $R \to \infty$, then $L_{eff} \to \infty$. This suggests that the system performance at NE can degrade arbitrarily if the number of users becomes large.

Socially Responsible Users

Note that we can think of users as data senders in the game discussed above. Let the packet loss rate associated with each link/path j be p_j. Suppose that the packet sending rate of a TCP connection of user r on path/link j is T_{rj}. The throughput of this connection is given by $G_{rj} = T_{rj}(1 - p_j)$. Not all packets coming to bottleneck link j are delivered. The network resources before link j are partially wasted because that they carry data at a higher rate than the actual delivery rate of link j. Therefore we can think of this extra traffic as a cost to the network and that is proportional to the packet sending rate T_{rj}. A user is considered as socially responsible if his/her benefit function includes this cost term. That is, we have $B_r(\mathbf{n}_r) = G_{rj} - \gamma \sum_{i=1}^{L} n_{rj} T_{rj}$, where $\gamma \in (0,1)$. Based on [11], we can show that there exists a pure strategy unique NE because users actually play a game on each link independently from other links. It also follows that the loss of efficiency of the NE is bounded as the unique NE is an interior point in the strategy space. Note that the definition of loss of efficiency in unstructured file sharing game is different from that of the single bottleneck link TCP connection game. The latter is defined as the ratio of total sending rate from all users at NE over the minimum total sending rate. The latter is the efficiency loss from network's point of view, whereas the former is from user's point of view.

C. Constrained Game

Consider another model where the total number of connections that are allowed to open by a user is fixed. Formally, for any user r, we have $\sum_j^L x_{rj} = n_r$, where n_r is the required total number of connections.

We refer to this game as a *constrained game*. As summarized in the following theorem, this game admits a unique symmetric Nash equilibrium.

Please see Appendix II for the proof of this theorem.

Theorem 3: There exists a unique interior-point symmetric Nash equilibrium in a constrained unstructured file sharing game in parallel-link network.

Remarks. It can be true that there are *asymmetric* NE. For example, suppose that there are two users and two links with the same capacity, and each user is constrained to use two and only two connections. Then one NE is that user 1 opens its two connections on link 1 and user 2 opens its two connections on link 2, or a NE could be that user 1 opens its two connections on link 2, and user 2 opens its two connections on link 1.

An Illustrative Example for the existence and stability of NE. We use a simple example to illustrate the Nash equilibrium proved in Theorem 3 . There are three users: A, B, and C. There are two paths (or two links) in a parallel link topology. Suppose that the capacity of link 1 is $C_1 = 25$Mbps and the capacity of link 2 is $C_2 = 100$Mbps. Suppose that each user has to open 20 connections. As proved in Theorem 3, at Nash equilibrium, each user will open 4 and 16 connections on link 1 and 2 respectively, because $n_1^*/n_2^* = C_1/C_2 = 1/4$. That is, at Nash equilibrium we have, $\mathbf{n_A^*} = \mathbf{n_B^*} = \mathbf{n_C^*} = (4, 16)$.

Suppose users interact with each other using *best-response dynamics* [12][11]. If all users' strategies converge to or stabilize at some points as time goes by, then the stablized numbers of connections are the Nash equilibrium strategies for all users. As shown in Figure 8, the best-response dynamics indeed converges to a stable point which corresponds to the Nash equilibrium obtained from the previous analysis.

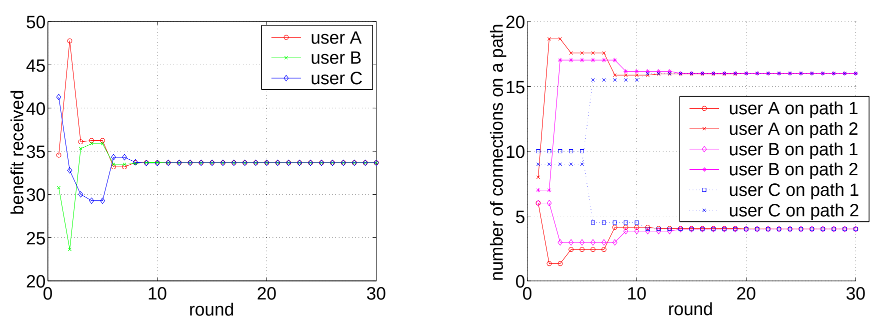

Fig. 8. An example of the best-response dynamics on two parallel links. This dynamic process converges to Nash equilibrium. The left figure shows the benefit of three users. The right figure shows the numbers of connections.

Loss of Efficiency. Given the constraint that the total number of connections of user r should be equal to n_r, the maximal value of (15) is given by $B_{max} = \sum_{j=1}^L C_j - \beta \sum_{r=1}^R n_r$. The system optimal performance is exactly the

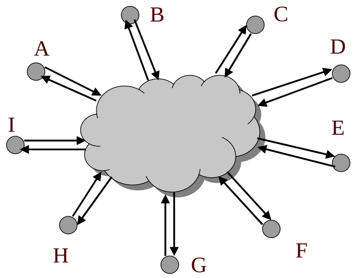

Fig. 9. An example of star network.

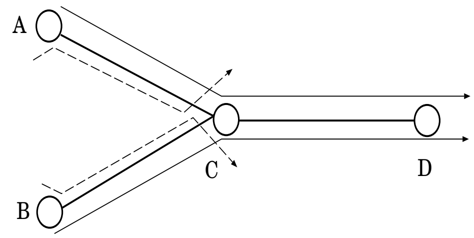

Fig. 10. A three node topology.

same as the system performance at Nash equilibrium. Then, the Nash equilibrium has no efficiency loss, that is, $L_{eff} = 1$.

V. Star Network

In this section, we use a star network to approximately model a peer-to-peer file sharing overlay network, and investigate the unstructured file sharing game on such a star network. Figure 9 presents one such example.

In the star network, we assume that a user has two asymmetric access links to the Internet: one downstream link and one upstream link. This assumption is supported in a measurement study in [25], where it is found that most users in current peer-to-peer networks use cable modem or ADSL to get connected to the Internet. Usually the downstream link has higher capacity than the upstream link [25].

A user r uses its downstream link to get data from other peers. The downstream link of user r is a "private" link in the sense that this link is only used by user r itself. On the other hand, the upstream link of user r is shared by all other peers or users who are downloading files from user r. We can think of the upstream link of user r as a "public" link from the point of view of user r.

In addition, similar to [14][15], we assume that in a peer-to-peer file sharing network, bottlenecks can occur at access links, not in the core Internet. This assumption is a reasonable approximation of the current peer-to-peer file sharing networks such as Gnutella and BitTorrent, where usually the data throughput is limited by the "last mile" (cable or ADSL or modem) of a connection. Thus, in the star network shown in Figure 9, the Internet cloud can be represented simply as a central node.

In the following, we first prove the existence of NE in unstructured file sharing game on a star network. We then use examples to illustrate the best response dynamics of this game playing process, and finally we present our results on the loss of efficiency of NE.

A. Nash Equilibrium

Recall that the benefit of user r is given by (1). In the following, we first present a lemma (Lemma 1) and later use it to prove that a utility function[2] $U_r(G_r(\mathbf{n_r}))$ is a non-decreasing, continuous, and concave function of user r's number of connections $\mathbf{n_r} = (n_{r1}, ..., n_{rP_r})$, where P_r represents the set of available paths of user r and the number of paths as well. Since we assume that cost $\Phi_r(\mathbf{n}_r)$ is an increasing and convex function of $\mathbf{n}_r$, it then follows that the benefit B_r is a non-decreasing, continuous, and concave function of $\mathbf{n_r}$.

Lemma 1 is introduced for the simple network in Figure 10, where a user r has two paths ($A \to C \to D$ and $B \to C \to D$) to transfer data to destination node D. Both paths share a common link CD. Suppose that the number of connections user r opens on path $A \to C \to D$ is n_{p1}, and on path $B \to C \to D$ is n_{p2}. Then we have $\mathbf{n}_r = (n_{p1}, n_{p2})$.

We assume that link CD is a private link of user r, i.e., no other users use this link. This private link corresponds to the downstream link of user r in a star network. On the other hand, links AC and BC are shared by user r and other users. AC and BC correspond to two public links of user r in a star network.

Recall that throughput G_r obtained by user r is a function of $\mathbf{n}_r$. Lemma 1 shows that G_r is a concave function of $\mathbf{n}_r$.

Lemma 1: Throughput G_r of user r in Figure 10 is a concave function of $\mathbf{n}_r = (n_{p1}, n_{p2})$.

Proof: The strategy vector of user r is $\mathbf{n}_r = (n_{p1}, n_{p2})$. Let $z = \frac{n_{p1}C_1}{n_{p1}+n_{-r1}} + \frac{n_{p2}C_2}{n_{p2}+n_{-r2}}$.

Then, the throughput obtained by user r is

$$G(n_{p1}, n_{p2}) = \begin{cases} C_3 & \text{, if } z \geq C_3 \\ z & \text{, if } z \leq C_3 \end{cases} \tag{18}$$

First, we note that this function is continuous and increasing. Second, this function has two parts, with each part being a concave function. Now we want to show that this function is a concave function of $\mathbf{n}_r$ *everywhere* in its domain.

Take any two points $\mathbf{n^1}$ and $\mathbf{n^2}$. Without loss of generality, we assume that $\mathbf{n^1}$ satisfies $z \leq C_3$ and that $\mathbf{n^2}$ satisfies $z \geq C_3$, as shown in Figure 11. We would like to show that

$$G(\delta\mathbf{n^1} + (1-\delta)\mathbf{n^2}) \geq \delta G(\mathbf{n^1}) + (1-\delta)G(\mathbf{n^2}), \delta \in [0, 1].$$

If we connect points $\mathbf{n^1}$ and $\mathbf{n^2}$ with a line, then this line intersects with the boundary of region $z \geq C_3$ at point $\mathbf{n^0}$. Then we have,

$$\begin{aligned} G(\delta\mathbf{n^1} + (1-\delta)\mathbf{n^2}) &\geq G(\delta\mathbf{n^1} + (1-\delta)\mathbf{n^0}) && (19) \\ &\geq \delta G(\mathbf{n^1}) + (1-\delta)G(\mathbf{n^0}) && (20) \\ &= \delta G(\mathbf{n^1}) + (1-\delta)G(\mathbf{n^2}) && (21) \end{aligned}$$

[2] $U_r(x)$ is assumed to be continuous, nondecreasing, and concave.

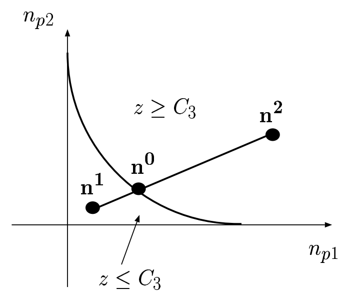

Fig. 11. The domain of G_r, the throughput of user r, can be divided into two regions. One region is $z \geq C_3$, and the other region is $z \leq C_3$.

(19) is true because that $G(\mathbf{x})$ is an increasing function of $\mathbf{x}$, and $\delta\mathbf{n^1} + (1-\delta)\mathbf{n^2} \geq \delta\mathbf{n^1} + (1-\delta)\mathbf{n^0}$. (20) is true because function G is a concave function in region $z \leq C_3$. (21) is true because function G is a continuous function.

■

An illustrative example. In Figure 10, suppose we choose 6bps as capacities for links $A \rightarrow C$ and $B \rightarrow C$ and 2bps for link $C \rightarrow D$. User r wants to open some number of connections on paths $A \rightarrow C \rightarrow D$ (path 1) and $B \rightarrow C \rightarrow D$ (path 2) to transfer data from A and B to destination node D. The numbers of connections or sessions from other users on links AC and BC are 100. We vary the numbers of connections from user r on path 1 and 2, and then compute the throughput received by user r. As shown in Figure 12, we see that user r's throughput is indeed a concave function.

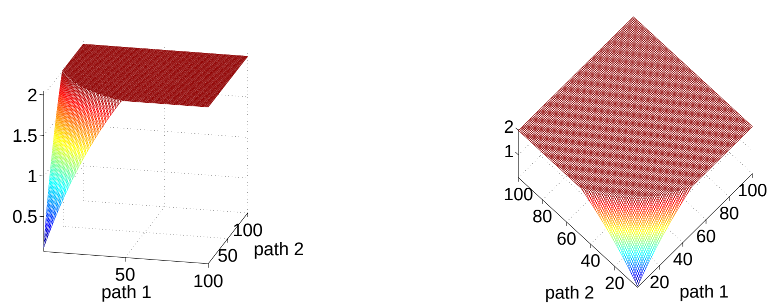

Fig. 12. Throughput G_r of user r as a function of the number of connections on both paths. The left figure is a side view. The right figure is a top view.

Consider the network in Figure 13, a generalized version of the network in Figure 10. In Figure 13, there are M (multiple) paths along which user r can get data from the sender. All paths share a common link BA. A strategy vector of user r is $\mathbf{n}_r = (n_{r1}, n_{r2}, ..., n_{rM})$ with $M \geq 2$. We can extend the result in Lemma 1 to show that a user r's throughput is also a concave function of $\mathbf{n}_r$. This is summarized in Lemma 2.

Lemma 2: Suppose that user r has M ($M \geq 2$) paths in the network shown in Figure 13, then the throughput

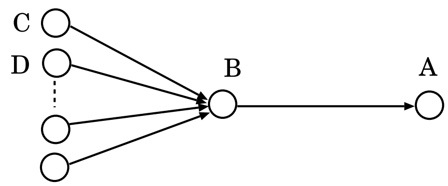

Fig. 13. A network where a user has multiple paths (or peers) to get data.

of user r is a concave function of its strategy vector $\mathbf{n}_r = (n_{r1}, n_{r2}, ..., n_{rM})$.

Based on Lemma 2, we can show in the following theorem the existence of NE on a star network. One example of such star network is shown in Figure 9.

Theorem 4: There exists a Nash equilibrium of unstructured overlay game on a star network (shown in Figure 9).

Proof: According to Lemma 2, each user r's total throughput is a concave function of the vector of number of connections $\mathbf{n_r}$. Then it is easy to show that user r's benefit or payoff function B_r is a concave function of $\mathbf{n_r}$. In addition, B_r is continuous in $\mathbf{n}$. Thus we have a multi-player *concave* game. Based on the result in [23], we conclude that Nash equilibrium exists in this game. ∎

An illustrative example. We use a simple star network shown in Figure 14 to illustrate the existence of NE proved in Theorem 4. On this star network, there are 6 links AD, DA, BD, DB, CD, and DC. The capacities of all links are $C_{AD} = 10, C_{DA} = 20, C_{BD} = 30, C_{DB} = 40, C_{CD} = 50$, and $C_{DC} = 60$. There are three users associated with nodes A, B and C respectively. For convenience, we refer to the user at node A as user A. Note that each user has two download paths with each path consisting of two links. For example, user A has two download paths $B \to D \to A$ and $C \to D \to A$. For any given download path, one link is shared with other users, and the other link is a private link. For example, for user A, path $B \to D \to A$ has two links: BD and DA. Link BD is a link shared with user C. Link DA is a private link of user A, which is shared by both of its paths $B \to D \to A$ and $C \to D \to A$.

User A's strategy is a vector of number of connections on two available paths, i.e., $\mathbf{n}_A = (n_{BA}, n_{CA})$. Similarly, strategies of user B and C are: $\mathbf{n}_B = (n_{AB}, n_{CB})$ and $\mathbf{n}_C = (n_{AC}, n_{BC})$.

Consider the unstructured file sharing game played by users A, B, and C. Each user tries to maximize its benefit B_r ($r = A, B, C$). We use best response dynamics to demonstrate the existence of a NE in this game. At the first step, each user opens a random number of connections on two available paths. In the following steps, only one player is randomly chosen to compute its best response at each step. As shown in Figure 14, the best response dynamics converges to a NE, which can be verified by checking the optimality of benefits of all three users.

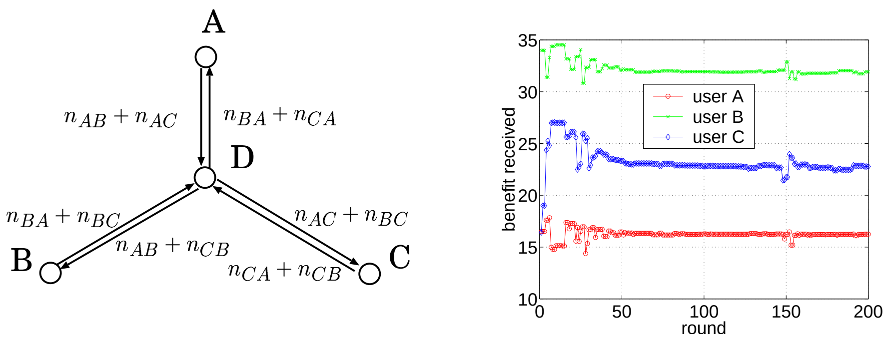

Fig. 14. The left figure shows a simple star topology with three users A, B, and C. The right figure shows the best response dynamics. All three users' benefits converge to the Nash equilibrium.

B. Loss of Efficiency

Consider the case where all downstream links have higher capacity than upstream links and users are homogeneous. We can show that in this case, the loss of efficiency of any NE in the game is bounded. However, if users are aggressive in the sense that their benefit functions do not contain cost terms, then a unique NE is a point where all users open their maximum allowable number of connections. Clearly, the loss of efficiency of the NE is unbounded if users can open arbitrarily large numbers of connections. In order to show these results, we need to do a simple transformation as described below.

In the star topology shown in Figure 9, if all users' private downstream links have much higher capacities than the upstream links of those other peers, then this game can be thought of a variant of the game on a parallel link network. For example, we can transform the simple star network in the left sub-figure of Figure 14 into Figure 15. Center node D in Figure 14 is decomposed into six interconnected virtual nodes $D_{Ad}, D_{Au}, D_{Bd}, D_{Bu}, D_{Cd}, D_{Cu}$. Links between these six virtual nodes have infinite capacity. Node A is decomposed into nodes A_{down} and A_{up}. Link $D_{Ad}A_{down}$ represents the ... n link of node A. Other links have similar interp...

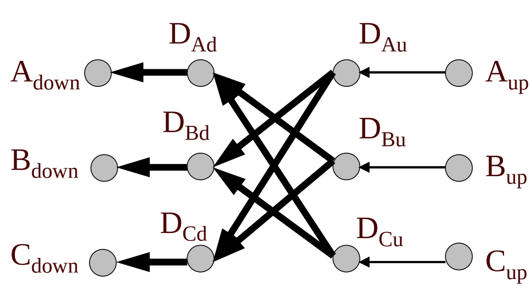

Fig. 15. Transformation of star network into equivalent parallel link network.

Based on the transformation illustrated in Figure 15, the result for the loss of efficiency at NE on a parallel link network can be applied to a star network. That is, the loss of efficiency at NE of the unstructured file sharing game

can be arbitrarily large if the number of users becomes large in this special case.

We also consider another special case where users are aggressive in the sense that users do not have cost constraint and only care about their throughputs [11]. That is, user's benefit function is represented as: $B_r(\mathbf{n_r}) = G_r(\mathbf{n_r})$. In this special case, there exist a unique Nash equilibrium where all users open their maximum allowable number of connections, and the price of anarchy can be unbounded when users can open arbitrary large number of connections.

Network Resource Utilization. Suppose that all downstream links have higher capacities than upstream links. Then the capacities of all upstream links will be fully utilized at the NE. This is a good situation in terms of the network resource utilization because the total throughput can be supported by the network is just the aggregate capacity of these upstream links. Note that this is not always true for general network topologies, which is demonstrated in an example in Appendix III. A similar example is given in [26].

VI. Overlay Formation Game

In this section, we introduce an *overlay formation game* to study the Tit-for-Tat strategy adopted by BitTorrent (BT) [3], one of the most popular peer-to-peer applications.

As before, we assume that the physical network is a star network where each peer is attached to a physical node, and the center node models the Internet, and peers connect to the center node via access links. However, unlike the last section, here we assume that bottlenecks only occur at upstream access links. As before, we assume that peers always have demands that can be satisfied by each other, and that connections are always allowed.

A connection between a pair of peers can be thought of as a virtual link. Through setting up connections between themselves, peers form an overlay network, in which each node represents a unique peer, and virtual links are connections between peers. A peer i can get a share of the upload bandwidth (BW) of peer j through the connection (or virtual link) between i and j. In the mean time, other peers may want to get some share of peer j's upload BW by setting up connections with j. The upload BW of j is equally shared among all connections with other peers. Note that a peer may want to get BW shares of all other peers' upload BW and want to maximize its received total BW. If all peers behave this way, we have a game among peers, and any stable point of this game is an overlay network consisting of a set of virtual links among peer nodes. We call this game an *overlay formation game*.

We can think of the *overlay formation game* as a variant of an *unstructured file sharing game* with two major unique characteristics: 1) two peers set up a connection between themselves only when they both find it beneficial; 2) there can only be zero or one connection between a pair of peers. The first characteristic captures the reciprocation feature of the so called *Tit-for-Tat strategy* in BitTorrent (BT) protocol [3]. According to Tit-for-Tat strategy each peer uploads to the n_u peers (the default value is 4) from which it can download at the highest rate, i.e., its best uploaders.

The Tit-for-Tat strategy is generally considered robust. To the best of our knowledge, the only analytical support for this belief is in [14]. The authors of [14] study how Tit-for-Tat can affect selfish peers who are able to set their upload bandwidth in a BT network. Under several assumptions, they show that there is a *good* NE at which each peer uploads at the maximum rate. Note that in [14], for a given peer, the total number of other peers to set up a connection with is fixed. However, we observe that BT clients can change the number of connections to open in order to gain advantage or to improve their performance. We illustrate this observation in the following example.

An Illustrative Example. Consider 10 peers divided into two groups. Five peers have physical upload bandwidth $C_1 = 3$ and the other five have bandwidth $C_2 = 2$. Suppose that the default number of connections is $n_u = 3$. According to [14], peers would use all their upload bandwidth and would create the overlay shown in Figure 16, where big circles and small circle respectively represent high-bandwidth and low bandwidth peers. Note that the peers do not receive the same download rate, even if they belong to the same group. Four high-bandwidth peers receive a download rate of 3 ($= 3C_1/3$), while the peer connecting the two groups (peer **S** in the figure) receives only $8/3$ ($= 2C_1/3 + C_2/3$). Similarly four low-bandwidth peers receive rates of 2, while the other receives $7/3$. According to [14] the formed overlay network is stable in the sense that no peer wants to change a link (or reduces its uploading rate).

Let us now remove the constraint on the number of connections. For example, peer **S** decides to increase its number of connections to 5. If all other peers keep $n_u = 3$, the new equilibrium is presented in Figure 17. Note that peer **S** improves its performance, because its download rate increases from $8/3$ to $10/3$ ($= 5C_2/3$).

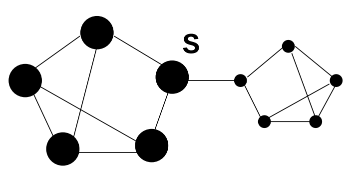

Fig. 16. Regular Graph.

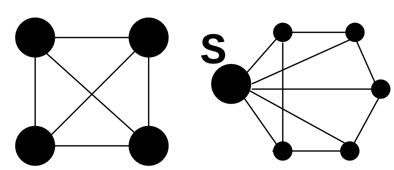

Fig. 17. Peers can change numbers of connections.

This example shows that peers can benefit by changing their numbers of connections. This is formally supported by a result in Section VI-B.1 regarding a homogeneous network where all peers have the same capacity. In the rest of this section, we first formally introduce the overlay formation game in which peers act selfishly as player **S**. We then study the network equilibria arising in this game and quantify the loss of efficiency using the analytical framework of network formation games [20].

A. Model of Overlay Formation Game

We formally introduce the overlay formation game in this section. Assumptions are detailed in the previous section. We refer to peers as players and to connections as links. As before, let $\mathbf{R} = \{1, 2, \cdots, R\}$ denote the set of players. The strategy of a player i is the set of intended connections player i wants to establish, which is

denoted by $s_i = \{s_{i,j} | \, j \in \mathbf{R} \backslash \{i\}\}$, where $s_{i,j} = 1$ means that player i intends to create a link (open a connection) with player j and $s_{i,j} = 0$ means that player i does not intend to create such a link. With the *Tit-for-Tat strategy*, both players have to agree in order to create a link, hence a link between players i and j is formed if and only if $s_{i,j} = s_{j,i} = 1$. A strategy profile $s = \{s_1, s_2, \cdots, s_R\}$ therefore induces a network $g(s) = \{g_{i,j}, i, j \in \mathbf{R}\}$, where $g_{i,j} = 1$ denotes the existence of link (i, j) and $g_{i,j} = 0$ denotes the absence of link (i, j). Given a network g, we use $g + g_{i,j}$ or $g - g_{i,j}$ to denote the network obtained by adding or severing the link (i, j). We also let $N_i(g) = \{j \in \mathbf{R} : j \neq i, g_{i,j} = 1\}$ be the set of player i's neighbors in graph g, and let $n_i(g) = |N_i(g)|$. A network is symmetric if $n_i(g) = n, \forall i \in \mathbf{R}$, i.e. all players have the same number of connections, also known as a regular graph.

The payoff or benefit of player i is given by its download rate minus the cost of opening connections: $B_i = G_i - \Phi_i(n_i) = \sum_{j \in N_i(g)} C_j/n_j - \Phi_i(n_i)$. As before, we assume that Φ_i is a convex function of n_i. The marginal benefit for player i to open a new connection with player j is:

$$\begin{aligned} b_i(n_i(g), n_j(g)) &= B_i(g + g_{i,j}) - B_i(g) \\ &= \frac{C_j}{n_j(g) + 1} - \Phi_i(n_i(g) + 1) + \Phi_i(n_i(g)). \end{aligned}$$

A connection between two players can be set up only when both of them find this connection beneficial. This coordination requirement makes the concept of Nash equilibrium (NE) *partially inadequate*. To address this issue, the idea of NE has been supplemented with the requirement of pairwise stability [27], described below.

Definition 1: A network g is a *pairwise equilibrium network* (PEN) if the following conditions hold: 1) there is a NE strategy profile which supports g; 2) for $g_{i,j} = 0$, $B_i(g + g_{i,j}) > B_i(g) \Rightarrow B_j(g + g_{i,j}) < B_j(g)$.

B. Equilibria in Homogeneous Networks

In this section we consider homogeneous networks in which all peers have the same upload capacity and payoff function.

1) Overlay Network Characterization: Based on the previous assumptions, our game is the local spillovers game with strategic substitutes properties studied in [28]. Some of the following results (Theorems 5, 6 and 8) can be derived from [28]. Please see Appendix IV for details.

Theorem 5: If the number of players is even, a symmetric PEN always exists. Specifically, if $b(0, 0) \leq 0$, the empty network is a PEN; if $b(r-2, r-2) \geq 0$, the complete network is a PEN; if $b(k, k) \leq 0 \leq b(k-1, k-1)$, the regular graph with degree k is a PEN. When the previous inequalities are strict, the degree of the PEN is unique.

Remarks. *First*, note that for a set of R players or nodes, if R is even, we can expect a PEN to be a symmetric or regular graph of any possible degree from 0 to $R - 1$; this is not true when the number of players is odd. *Second*, this theorem states that the degree of a PEN can be determined by considering only the marginal benefit $b(k, k)$

for a pair of nodes with the same number of connections k, and in particular this degree is the smallest value k that makes $b(k,k)$ negative. *Third*, the symmetric network at equilibrium is not necessarily connected. Figure 18 shows two possible equilibria with $r = 8$ players and degree $k = 2$. *Finally*, even when a symmetric network can arise from player interaction according to Theorem 5, the degree of the network is in general different from the default value used in current BitTorrent implementation ($n_u = 4$). This means that the symmetric network created by compliant peers in BitTorrent networks is not in general a PEN for our overlay formation game.

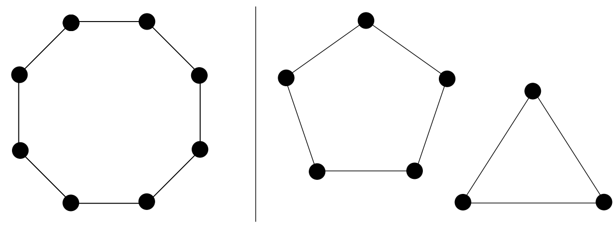

Fig. 18. Different Pairwise Symmetric Equilibria.

Besides symmetric PENs discussed in the above, we have the following theorem addressing asymmetric PENs.

Theorem 6: There can be at most one player or node not connected to any other players in a PEN and the rest of the network is a symmetric network of a unique degree. In asymmetric networks with a single component, if two players with the same number of connections k (i.e. two nodes with the same degree k) are connected to each other, then any two players with fewer number of links than k (or two nodes with lower degrees than k) must be mutually connected.

Remarks. *First*, this property rules out two or more isolated players and interlinked stars with two or more central players, but does allow a star to arise in equilibrium[3]. Note that for file sharing purposes, an overlay with a star topology is very inefficient: the operation falls back to the server-client paradigm with the center of the star acting as the server. *Second*, in some cases symmetric and asymmetric networks can be pairwise equilibria for a given set of link capacities and cost functions (see [28] for examples).

The following theorem (not derived from [28]) shows some other restrictions as regards asymmetric networks when the marginal benefit for player i to open a connection with player j only depends on the number of connections of players i and j (as in our case). This new result rules out also star topologies. Please see Appendix V for a detailed proof.

Theorem 7: In a scenario where a unique degree -h- is possible for the symmetric PENs, there can be at most h

[3] An interlinked star network has a maximally connected group and a minimally connected group of players. In addition, the maximally connected players are connected to all players while the minimally connected group has links only with the players in the maximally connected set.

players with degree smaller than h. Say l the number of players with degree smaller than h, there can be at most $(h-l)l$ players with degree bigger than h, each of them with degree at most $h+l$. If the cost function is linear then there are no players with degree bigger than h.

Remark. Note that the degree of symmetric PENs h depends only on the cost function $\Phi()$ and the capacity C, and is independent from the number of players R. Hence the *distance* between a PEN and a symmetric PEN is bounded and becomes less significant as the number of players R increases. Formally:

$$\lim_{R\to\infty} \frac{1}{R} E\left\{\sum_{i=1}^{R} |n_i(g_{PEN}) - h|\right\} = 0.$$

Similarly the average payoff per player in a PEN converges to that of a symmetric PEN.

The following result shows that players having more connections gain higher payoffs than other players, supporting the example introduced at the beginning of this section.

Theorem 8: Let g be a pairwise equilibrium network in which $n_i(g) < n_j(g)$. If $\forall u \in N_i(g)$, $\exists v \in N_j(g)$ s.t. $n_u = n_v$, then $B_i(g) < B_j(g)$.

Note that if player i's neighborhood is included in player j's neighborhood ($N_i \subset N_j$), the condition, "$\forall u \in N_i(g)$, $\exists v \in N_j(g),$ s.t. $n_u = n_v$", is satisfied.

2) Loss of Efficiency of Symmetric Equilibria: In our game, given the number of players, the number of possible overlays players can create is finite. Hence there is one network g_{opt} with the highest total payoff $\sum_{i\in\mathbf{R}} B_i(g_{opt})$. We define the efficiency loss of a PEN g as the ratio of the highest total payoff over the total payoff of the PEN:

$$L_{eff}(r, C, \Phi) = \frac{\sum_{i\in\mathbf{R}} B_i(g_{opt})}{\sum_{i\in\mathbf{R}} B_i(g)}.$$

We note that L_{eff} depends in general on the number of players, and the upload capacities and cost functions of those players. The following theorem states that L_{eff} is unbounded even for the class of linear connection cost functions ($\Phi(n) = \alpha n$). Therefore, the price of anarchy (the worst efficiency loss of all NEs) is infinite[4]. Please see Appendix VI for a detailed proof.

Theorem 9: For the class of linear connection cost functions, the loss of efficiency is unbounded. In particular, given an even number of players and an upload capacity C, $\forall M \in \mathbb{R}, \exists \alpha^* \in \mathbb{R}^+$ s.t. $L_{eff}(r, C, \Phi^*) > M$, where $\Phi^*(n) = \alpha^* n$.

C. Dynamic Models

We investigate in this section how peers can dynamically reach a PEN. Here we consider linear costs ($\Phi(n_i) = \alpha n_i$). We consider the following dynamic discrete-time process. Starting from an empty network, at each time a player pair (i,j) is randomly chosen. Link (i, j) is created (or kept) if both players find it beneficial. An existing

[4]This is different from what happens for selfish routing, where the price of anarchy is finite, and independent from the network topology for networks in which edge latency does not depend in a highly nonlinear fashion on the edge congestion [29].

link is removed if at least one of the two players of that link does not find it useful. We are going to show that this dynamic process always reaches a PEN.

Let us introduce some terminology according to [20]. A network g' is *adjacent* to a network g if $g' = g + g_{i,j}$ or $g' = g - g_{i,j}$ for some pair (i, j). A network g' *defeats* another network g if either $g' = g - g_{i,j}$ and $B_i(g') > B_i(g)$, or if $g' = g + g_{i,j}$ with $B_i(g') \geq B_i(g)$ and $B_j(g') \geq B_j(g)$ with at least one inequality holding strictly. A network game exhibits *no indifference* if for any two adjacent networks, one defeats the other.

According to this terminology in the dynamic process we described above, the current network is altered if and only if the addition or deletion of a link would defeat the current network. The process leads to an *improving path*, i.e. a sequence of networks $g_1, g_2, ..., g_K$ where each network g_k is defeated by the subsequent (adjacent) network g_{k+1}. There are two kind of improving paths: those exhibiting cycles (which have infinite length) and those terminating with a PEN (called *stable state*). The following lemma (a theorem in [30]) characterizes when there are no cycles and pairwise stable networks exist.

Lemma 3: Given G the set of all the possible networks g, if there exists a real valued function $w : G \to \mathbb{R}$ such that "g' defeats g" if and only if "$w(g') > w(g)$ and g' and g are adjacent", then there are no cycles. Conversely, if the network game exhibits no indifference, then there are no cycles only if there exists a function $w : G \to \mathbb{R}$ such that "g' defeats g" if and only if "$w(g') > w(g)$ and g' and g are adjacent".

Based on this lemma, we have the following result.

Theorem 10: If the connection cost function is a linear function $\Phi(n) = \alpha n$, the dynamic process introduced at the beginning of Section VI-C always reaches a PEN.

Sketch of the proof. If $h \in \{0, 1, \cdots, R-1\}$ is the degree of a symmetric equilibrium according to Theorem 5 and $b(h, h) < 0$ for $h \neq R-1$, the following function $w : G \to \mathbb{R}$:

$$w(g) = -\sum_{i=1}^{R} f(n_i),$$

where

$$f(n_i) = \begin{cases} h - n_i & \text{if } h \geq n_i \\ R(n_i - h) & \text{otherwise} \end{cases}$$

satisfies the relation in Lemma 3 for our overlay formation game, hence the dynamic process always reaches a PEN. If $h \neq R-1$ and $b(h, h) = 0$, then in a PEN there can be also nodes with degree $h+1$ (as well as nodes with degrees $0, 1, \cdots, h$), in this case the following function can be considered:

$$f(n_i) = \begin{cases} h - n_i & \text{if } h \geq n_i \\ R(n_i - (h+1)) & \text{otherwise} \end{cases}$$

The details of the proof are in Appendix VIII.

Simulations. We present some simulation results. We considered a number of players ranging from 100 to 10000 and $\alpha = 0.245$, for which the degree of a symmetric PEN is 4. For each setting we simulated 5000 runs of the above dynamic process. Each run terminates with a PEN. We denote the average degree for this PEN over all players as d_{avg}.

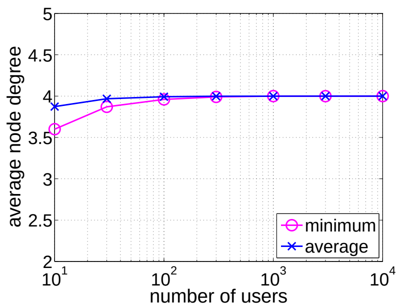

Fig. 19. Average node degree.

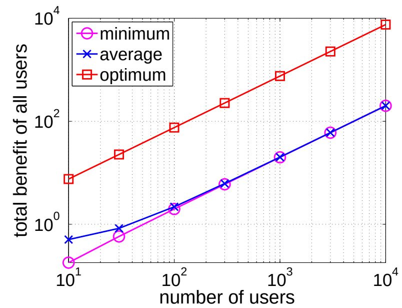

Fig. 20. Total benefit.

Figure 19 shows the minimum and the mean of d_{avg} over all the runs. We see that as R increases both the mean and the minimum converge to 4. This result confirms Theorem 7: as R increases the PENs *converge* to a symmetric one.

In Figure 20, the mean and the minimum of the total benefit are compared with the highest total benefit, which can be directly evaluated from the results in Appendix VII. This figure shows also the convergence of the payoffs of all PENs to the payoff of the symmetric PEN when R increases.

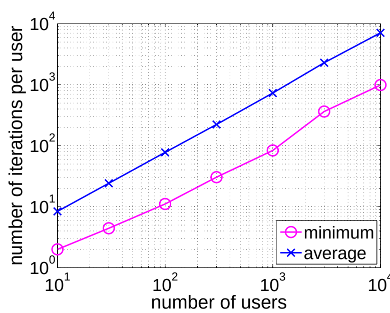

Fig. 21. Number of iterations per peer.

In addition, we present the number of iterations per peer in Figure 21. We observe that the average number of iterations to reach a PEN is of the order of R^2 and hence the number of iteration per peer is of the order of R. Let us consider this number of iterations in the context of BitTorrent (BT) [3]. Each peer in a BT network tries to replace an existing connection with a new, better connection every 10 seconds. All peers do such replacement simultaneously, unlike the sequential replacement in our simulations. So R^2 iterations in our simulations corresponds to $10R$ seconds in a BT network. For a population of 100 peers, the time needed to reach a PEN is of the order of

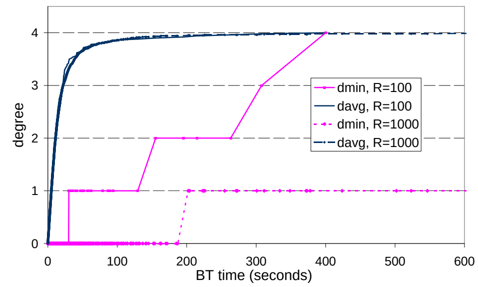

Fig. 22. Convergence to the PEN: the degree.

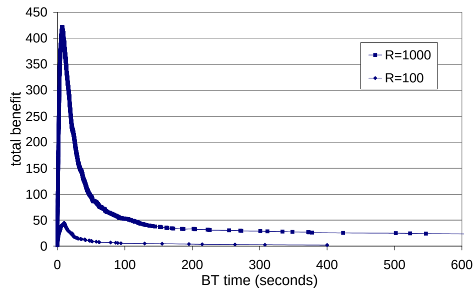

Fig. 23. Convergence to the PEN: the benefit.

about 17 minutes, which is typically faster than the average time between changes in the population of peers (due to arrivals or departures). Figure 22 shows how the average and minimum degrees change during two simulation runs respectively for $R = 100$ and for $R = 1000$. The initial values are equal to 0 and converge to 4. The time scale represents time in a BT network; namely, R iterations are represented by 10s. We can observe that: 1) with this time scale the evolution of the average degree seems independent from the number of players; 2) the network converges quite rapidly to the PEN. In particular, the average degree reaches 3.8, i.e. 95% of the final value, after less than 80 seconds in both cases, or, equivalently, after less than 800 iterations for $R = 100$ and less than 8000 for $R = 1000$.

Finally Figure 23 shows the time evolution of the process as regards the total benefit. We can note that for both runs, as the process begins the total benefit grows because of the high benefit of the initial connections, while it falls down to the expected value when the network approaches the equilibrium.

VII. Conclusions

Motivated by unstructured file sharing networks such as BitTorrent [3], we introduced an unstructured file sharing game and an overlay formation game to model the interaction among self-interested users who can open multiple connections on multiple paths to accelerate data transfer. Users are modelled as players, and each user adjusts its numbers of connections on its available paths to maximize its benefit.

We demonstrated by examples that there exist multiple stable network states, so called Nash equilibria (NE), in the unstructured file sharing game on general networks. We further restrict our attention to parallel link networks and star networks which are used to model unstructured file sharing networks. We proved the existence of NE in several variants of the game on both networks. We found that the loss of efficiency of NE can be arbitrarily large if users have no cost constraints. However, when there are cost constraints, the loss of efficiency is bounded. In addition, we proved the global stability of NE in some variants of the game. Furthermore, we studied the Tit-for-Tat strategy (built in BitTorrent [3]) through an overlay formation game. We proved the existence of equilibrium overlays, and demonstrated the convergence of the dynamical game-playing process. Although the general belief is that the Tit-for-Tat can prevent selfish behavior, we showed that it can still lead to an unbounded loss of efficiency.

ACKNOWLEDGMENT

The authors would like to thank Jim Kurose, Arun Venkataramani, and Chun Zhang for their helps. This research has been supported in part by NSF under grant awards ANI-0085848, CNS-0519998, CNS-0519922, and EIA-0080119. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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APPENDIX I

AN EXAMPLE FOR THE SIMPLE RATE ALLOCATION MECHANISM

This example is to show that the simple rate allocation mechanism in Section III-B cannot be extended to a general network.

Suppose that there are two paths p_1 and p_2 which belong to two user $r1$ and $r2$ respectively. These two paths share a single common link l. Let user $r1$ open n_{1p_1} number of connections on p_1 and user 2 open n_{2p_2} number of connections on p_2.

Suppose that

$$\forall m \in p_1, m \neq l, C_m n_{1p_1} / \sum_{w \in \mathbf{R}} n_{wm} > C_l n_{1p_1} / (n_{1p_1} + n_{2p_2}).$$

Here, n_{wm} represents the number of connections opened by user w on link m.

If we conclude that

$$y_{1p_1} = C_l n_{1p_1} / (n_{1p_1} + n_{2p_2}),$$

then we might be wrong. The reason is as follows.

It is possible that there is a link k on p_2 satisfying

$$k \in p_2, k \neq l, C_k n_{2p_2} / \sum_{w \in \mathbf{R}} n_{wk} \leq C_j n_{2p_2} / \sum_{w \in \mathbf{R}} n_{wj}, \forall j \in p_2,$$

then user $r2$'s obtained rate is

$$C_k n_{2p_2} / \sum_{w \in \mathbf{R}} n_{wk},$$

and the actual allocated rate of user $r2$ on link l is $C_k n_{2p_2} / \sum_{w \in \mathbf{R}} n_{wk}$.

If we have

$$\begin{aligned} C_m n_{1p_1} / \sum_{w \in \mathbf{R}} n_{wm} &> C_l - C_k n_{2p_2} / \sum_{w \in \mathbf{R}} n_{wk} \\ &> C_l n_{1p_1} / (n_{1p_1} + n_{2p_2}) \end{aligned}$$

then, the actual rate obtained by user 1 is

$$C_l - C_k n_{2p_2} / \sum_{w \in \mathbf{R}} n_{wk},$$

not

$$C_l n_{1p_1} / (n_{1p_1} + n_{2p_2}).$$

Appendix II

Proof of Theorem 3

Consider the Lagrangian of the constrained optimization problem of any given user r.

$$L(\mathbf{n}_r) = B_r(\mathbf{n}_r) + \lambda(n_r - \sum_{j=1}^{L} n_{rj}).$$

The optimal solution can be obtained by solving the following equations.

$$\partial L / \partial n_{rj} = 0, \forall j \quad (22)$$

$$\partial L / \partial \lambda = 0 \quad (23)$$

That is,

$$\partial L / \partial n_{rj} = \frac{\sum_{k \neq r}^{R} n_{kj}}{(n_{rj} + \sum_{k \neq r}^{R} n_{kj})^2} C_j - \beta - \lambda = 0 \quad (24)$$

$$\partial L / \partial \lambda = n_r - \sum_{j=1}^{L} n_{rj} = 0 \quad (25)$$

We consider a symmetric Nash equilibrium where all users have the same number of connections on each path/link. Then, we get

$$C_i / n_{ri}^* = C_j / n_{rj}^*, \forall i, j$$

Combined with $\sum_{j=1}^{L} n_{rj}^* = n_r$, we can get the vector of number of flows at Nash equilibrium. Specifically, for a given user r, its number of connections at link j at Nash equilibrium is given by $n_{rj}^* = n_r C_j / \sum_{k=1}^{L} C_k$.

APPENDIX III

AN EXAMPLE FOR UNDER-UTILIZED NETWORK RESOURCES DUE TO SELFISH BEHAVIOR OF USERS.

In the triangle network shown in Figure 24, consider that all links are bi-directional and all links have the same capacity C. We have $C_{AB} = C_{BA} = C_{AC} = C_{CA} = C_{BC} = C_{CB} = C$. There are six users:

- User AB wants to transfer data from node A to node B.
- User BA wants to transfer data from node B to node A.
- User BC wants to transfer data from node B to node C.
- User CB wants to transfer data from node C to node B.
- User AC wants to transfer data from node A to node C.
- User CA wants to transfer data from node C to node A.

Consider that each user has two paths to transfer data and it can only open at most one connection on each path. For clarity, in Figure 24, we only show connections opened by user AB and user BA. Assume that all users try to maximize its total throughput, then at the NE, every user opens one connection on each of its two paths. Each user gets a total throughput of $2C/3(= C/3 + C/3)$. Then, the total throughput from all six users is $4C$. However, the total capacity provided by the network is $6C$. Thus, the network resource is not fully utilized in this example. A similar example is given in [26].

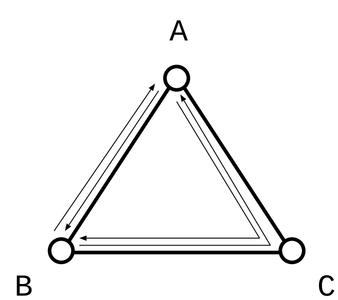

Fig. 24. Triangle Network.

APPENDIX IV

PROOFS OF THEOREMS 5, 6 AND 8

The theorems are derived from results in [28], in particular from Proposition 4.4, from the remarks about the characterization of asymmetric equilibria after Proposition 4.4 and from Proposition 4.5. In this section we just show that we can apply those results to our problem.

In section 4 of [28] the authors define a *local spillovers game* as a network game where aggregate gross payoff of player i can be written as:

$$\pi_i(g) = \Psi_1(\eta_i(g)) + \sum_{j\in N_i(g)} \Psi_2(\eta_j(g)) + \sum_{j\notin N_i(g)} \Psi_3(\eta_j(g)),$$

where $\eta_i(g)$ is the number of links of player i in graph g ($n_i(g)$ according to our notation).

We can recognize the same benefit function of our overlay formation game where:

$\Psi_1(n_i)$ is the cost of opening n_i connections ($-\Phi(n_i)$),

$\Psi_2(n_j)$ is the downloading rate user i receives from a peer with n_j connections (C/n_j),

$\Psi_3()$ is identically null.

In our game $\Phi()$ is a convex function and the downloading rate is a decreasing function of n_j. Hence according to the terminology of [28] our aggregate payoff function "satisfies the local spillovers property, concavity in own links ($\Psi_1()$ is concave) and strategic substitutability ($\Psi_2(k+1) - \Psi_3(k+1) < \Psi_2(k) - \Psi_3(k+1)$)". In this case Proposition 4.4 about symmetric equilibria, remarks about the characterization of asymmetric equilibria after Proposition 4.4 and Proposition 4.5 about payoff distribution hold.

APPENDIX V
PROOF OF THEOREM 7

The result does not depend on the specific form of the payoff function we considered, it holds when the aggregate payoff function satisfies the local spillover property, concavity in own links and strategic substitutability and the marginal benefit depends only on the number of connections of players i and j, i.e., when $\Psi_3()$ is identically null.

If h is the degree of symmetric PENs, $b(h-1,h-1) > 0$ and $b(h,h) < 0$. The marginal benefit $b(n_i,n_j) = \Psi_1(n_i+1) - \Psi_1(n_i) + \Psi_2(n_j+1)$ is a decreasing function of n_i (because $\Psi_1()$ is concave) and of n_j (because of strategic substitutability). Hence $b(u,v) \geq b(h-1,h-1) > 0$ for $u,v < h$. As a consequence, given a PEN, all the players with degree smaller than h are mutually connected and their number is at most h (otherwise their degree would be at least h).

Say l the number of players with degree smaller than h. They can have at most $h-1$ connections and they are mutually connected, i.e. they have $l-1$ connections with other players with less than h connections. Hence, they can have at most $(h-1)-(l-1) = h-l$ connections with players which have at least h connections. Therefore the number of players with more than h connections is bounded by $(h-l)l$ ($< h^2$).

If the cost function is linear all the nodes have degree at most h. In fact in this case $b(n_i,n_j)$ depends only on n_j ($b(n_i,n_j) = C/(n_j+1) - \alpha = b(n_j)$). If $n_j \geq h$, $b(n_j) \geq b(h) < 0$, no player would create a connection with a player that has already h connections or more.

The remark after Theorem 7 follows from the fact that in a PEN the number of nodes with degree different from h is bounded by $h + (h-l)l \leq h + h^2$ and that for each of these nodes the difference between its degree and h is

bounded by h and by $h+l<2h$, respectively for players with less than h connections and for players with more than h connections[5].

APPENDIX VI
PROOF OF THEOREM 9

Without loss of generality we can assume that $C=1$.

Given $R>2$ players and a symmetric equilibria g with degree $k \in \{2, ..., R-2\}$, let us consider the network $\tilde{g}$ where all the players have degree equal to one[6]. It holds:

$$\begin{aligned}
L_{eff} &= \frac{\sum_{i\in\mathbf{R}} B_i(g_{opt})}{\sum_{i\in\mathbf{R}} B_i(g)} \geq \\
&\geq \frac{\sum_{i\in\mathbf{R}} B_i(\tilde{g})}{\sum_{i\in\mathbf{R}} B_i(g)} = \\
&= \frac{\sum_{i=1}^{R} 1-\Phi(1)}{\sum_{i=1}^{R}\left(\sum_{j\in N_i}^{R}\frac{1}{k}\right)-\Phi(k)} = \\
&= \frac{R(1-\Phi(1))}{R(1-\Phi(k))} = \frac{1-\Phi(1)}{1-\Phi(k)}
\end{aligned}$$

If the connection cost $\Phi(n)$ is a linear function ($\Phi(n)=\alpha n$), in order to support an equilibrium with degree k we can consider α such that $1/(k+1)<\alpha<1/k$ (Theorem 5). We can choose $\alpha = 1/k(1-\epsilon)$ with $\epsilon>0$. In this case

$$\begin{aligned}
L_{eff} &= \frac{1-\Phi(1)}{1-\Phi(k)} = \\
&= \frac{1-\alpha}{1-\alpha k} = \\
&= \frac{1-(1-\epsilon)/k}{\epsilon},
\end{aligned}$$

and the loss of efficiency is clearly unbounded.

APPENDIX VII
OPTIMAL NETWORKS

In this section we characterize the networks with the highest global payoff for our overlay formation game. We observe that in the homogenous scenario the global payoff $B_S=\sum_{i\in\mathbf{R}} B_i$ does not change when we permute the players, because we are simply changing their labels. Without loss of generality we consider $\Phi(0)=0$

Theorem 11: If $C\leq\Phi(1)$, the empty network is an optimal network. If $C\geq\Phi(1)$ and R is even a symmetric network with degree one is an optimal network. If $C\geq\Phi(1)$, R is odd and $C\leq\Phi(2)$ a network with all the nodes with degree one but one with degree zero is an optimal network. If $C\geq\Phi(1)$, R is odd and $C\geq\Phi(2)$ a network

[5] Here we want to show the existence of bounds independent from R, not to determine tight bounds.

[6] In Appendix VII we prove that network $\tilde{g}$ is a network with the optimal payoff, but this is not necessary for this proof.

with all the nodes with degree one but one with degree two is an optimal network. When the previous inequalities are strict the optimal networks differ only for a permutation of the players.

First we note that the global payoff B_S can be expressed as follows:

$$\begin{aligned} B_S(g) &= \sum_{i \in \mathbf{R}} B_i(g) = \\ &= \sum_{i \in \mathbf{R}} \sum_{j \in N_i} \left(\frac{C}{n_j(g)} - \Phi(n_i(g)) \right) = \\ &= \sum_{\substack{i \in \mathbf{R}: \\ n_i(g) \geq 1}} C - \sum_{i \in \mathbf{R}} \Phi(n_i(g)) = \\ &= \sum_{\substack{i \in \mathbf{R}: \\ n_i(g) \geq 1}} \left(C - \Phi(n_i(g)) \right). \end{aligned} \tag{26}$$

If $C \leq \Phi(1)$ then the empty (network g_0) is an optimal network, in fact for any network g:

$$\begin{aligned} B_S(g) &= \sum_{\substack{i \in \mathbf{R}: \\ n_i(g) \geq 1}} \left(C - \Phi(n_i(g)) \right) \leq \\ &\leq \sum_{\substack{i \in \mathbf{R}: \\ n_i(g) \geq 1}} \left(C - \Phi(1) \right) \leq \\ &\leq 0 = B_S(g_0). \end{aligned}$$

Similarly, if $C \geq \Phi(1)$ and R is even then a symmetric network with degree one (g_1) is an optimal network, in fact for any network g:

$$\begin{aligned} B_S(g) &= \sum_{\substack{i \in \mathbf{R}: \\ n_i(g) \geq 1}} \left(C - \Phi(n_i) \right) \leq \\ &\leq \sum_{\substack{i \in \mathbf{R}: \\ n_i \geq 1}} \left(C - \Phi(1) \right) \leq \\ &\leq \sum_{i \in \mathbf{R}} \left(C - \Phi(1) \right) = \\ &= B_S(g_1). \end{aligned}$$

Let us consider now $C \geq \Phi(1)$ and R odd. Given a network g, there are two possibilities: 1) all the players have at least a connection, or 2) there is at least one player without connections.

In case 1) there is at least one player -say player l- with two or more connections. Let consider the network g_{12} where that player has two connections and all the other players have only one. It holds:

$$\begin{aligned} B_S(g) &= \sum_{\substack{i\in\mathbf{R}:\\ n_i(g)\geq 1}} (C-\Phi(n_i)) \leq \\ &\leq \sum_{\substack{i\in\mathbf{R}-\{l\}:\\ n_i(g)\geq 1}} (C-\Phi(1)) + (C-\Phi(2)) \leq \\ &\leq \sum_{i\in\mathbf{R}-\{l\}} (C-\Phi(1)) + (C-\Phi(2)) = \\ &= B_S(g_{12}). \end{aligned}$$

In case 2) there is at least one player -say player m- without any connection. Let consider the network g_{10} where that player has no connection and all the other players have only one. It holds:

$$\begin{aligned} B_S(g) &= \sum_{\substack{i\in\mathbf{R}:\\ n_i(g)\geq 1}} (C-\Phi(n_i)) = \\ &= \sum_{\substack{i\in\mathbf{R}-\{m\}:\\ n_i(g)\geq 1}} (C-\Phi(n_i)) \leq \\ &\leq \sum_{\substack{i\in\mathbf{R}-\{m\}:\\ n_i(g)\geq 1}} (C-\Phi(1)) \leq \\ &\leq \sum_{i\in\mathbf{R}-\{m\}} (C-\Phi(1)) = \\ &= B_S(g_{10}). \end{aligned}$$

One out of g_{12} and g_{10} is an optimal network. The two networks differ only for the connections of three nodes. The difference of their payoffs is:

$$\begin{aligned} B(g_{12}) &- B(g_{10}) = \\ &= \left(2\left(\frac{C}{2}-\Phi(1)\right)+2C-\Phi(2)\right)+ \\ & -2\Big(C-\Phi(1)\Big) = \\ &= C-\Phi(2). \end{aligned}$$

Hence if $C\geq\Phi(2)$ g_{12} is an optimal network, while if $C\leq\Phi(2)$ g_{10} is an optimal network.

If the inequalities in the hypothesis are strict, then all the optimal networks differ only for a permutation of the players. For example let us consider $C>\Phi(1)$ and R even. Given a symmetric network with degree one g_1, all the other symmetric networks with degree one have clearly the same payoff because of Eq. (26) and can be obtained by permutation of the players in g_1. Let us consider another network g which cannot be obtained by a permutation of players in g_1, g differs from g_1 at least for the degree of one player, say it l. Player l has no connection or has more than one. If player l is not connected in g, hence its contribution to $B_S(g)$ is null, if it is connected

$C - \Phi(n_l) < C - \Phi(1)$ because $n_l > 1$. In both cases its contribution to $B_S(g)$ is smaller than its contribution in g_1, and it follows:

$$B_S(g) < B_S(g_1),$$

hence g cannot be an optimal network. Similar reasoning leads to the result for the other cases.

APPENDIX VIII
PROOF OF THEOREM 10

We prove the result for $h \in \{1, 2, \cdots, R-2\}$ and $b(h,h) < 0$, the other cases ($b(0,0) < 0$, $h = R-1$, $h \in \{0, 1, \cdots R-2\}$and $b(h) = 0$) can be carried on similarly. In this case $b(h-1,h-1)$¿0.

We need only to check that the function $w : G \to \mathbf{R}$:

$$w(g) = -\sum_{i=1}^{R} f(n_i),$$

where

$$f(n_i) = \begin{cases} h - n_i & \text{if } h \geq n_i \\ R(n_i - h) & \text{otherwise} \end{cases}$$

satisfies the relation in Lemma 3 for our overlay formation game.

Part I: "g' defeats g" $\Rightarrow$ "$w(g') > w(g)$ and g' and g are adjacent"

Clearly g' and g are adjacent by definition of defeat.

If g' defeats g, then either 1) $g' = g - g_{i,j}$ and $B_i(g') > B_i(g)$ or 2) $g' = g + g_{i,j}$ and $B_i(g') > B_i(g)$ and $B_j(g') \geq B_j(g)$. Note that $B_i(g - g_{i,j}) > B_i(g) \Leftrightarrow b(n_i(g'), n_j(g')) = b(n_i(g) - 1, n_j(g) - 1) < 0 \Leftrightarrow n_j(g) > h$. Hence in case 1) $n_j(g) > h$ and

$$\begin{aligned} w(g') - w(g) &= -f(n_i(g) - 1) - f(n_j(g) - 1) + \\ &\quad + f(n_i(g)) + f(n_j(g)) = \\ &= (f(n_i(g)) - f(n_i(g) - 1)) + \\ &\quad + (f(n_j(g)) - f(n_j(g) - 1)) = \\ &= (f(n_i(g)) - f(n_i(g) - 1)) + R. \end{aligned}$$

If $n_i > h$, $f(n_i(g)) - f(n_i(g) - 1) = R$, otherwise $f(n_i(g)) - f(n_i(g) - 1) \geq -1$. In both cases:

$$w(g') - w(g) > 0.$$

The marginal benefit $b(n_i, n_j)$ is only a function of n_j and it decreases as n_j increases (see comments in Appendix V). We are considering $b(h,h) = b(h) < 0$ and $b(h-1,h-1) = b(h-1) > 0$, hence $b(k)$ is always different from zero, and $B_j(g') \geq B_i(g) \Leftarrow B_j(g') > B_i(g)$. Hence if $g' = g + g_{i,j}$ defeats g then $B_i(g') > B_i(g)$

and $B_j(g') > B_j(g)$. Note also that $B_i(g + g_{i,j}) > B_i(g) \Leftrightarrow b(n_i(g), n_j(g)) > 0 \Leftrightarrow n_j(g) < h$. Hence in case 2) $n_j(g) < h$ and $n_i(g) < h$. It follows:

$$\begin{aligned} w(g') - w(g) &= -f(n_i(g)+1) - f(n_j(g)+1) + \\ &\quad +f(n_i(g)) + f(n_j(g)) = \\ &= (f(n_i(g)) - f(n_i(g)+1)) + \\ &\quad +(f(n_j(g)) - f(n_j(g)+1)) = \\ &= 2 > 0. \end{aligned}$$

Part II: "$w(g') > w(g)$ and g' and g are adjacent" $\Rightarrow$ "g' defeats g"

If $g' = g + g_{i,j}$,

$$\begin{aligned} w(g') - w(g) &= -f(n_i(g)+1) - f(n_j(g)+1) + \\ &\quad +f(n_i(g)) + f(n_j(g)) = \\ &= (f(n_i(g)) - f(n_i(g)+1)) + \\ &\quad +(f(n_j(g)) - f(n_j(g)+1)), \end{aligned}$$

and $w(g') - w(g)$ can be positive only if $n_i(g) < h$ and $n_j(g) < h$. In this case $b(n_i(g), n_j(g)) > 0$ and $b(n_j(g), n_i(g)) > 0$ and g' defeats g.

If $g' = g - g_{i,j}$,

$$\begin{aligned} w(g') - w(g) &= -f(n_i(g)-1) - f(n_j(g)-1) + \\ &\quad +f(n_i(g)) + f(n_j(g)) = \\ &= (f(n_i(g)) - f(n_i(g)-1)) + \\ &\quad +(f(n_j(g)) - f(n_j(g)-1)), \end{aligned}$$

and $w(g') - w(g)$ can be positive only if $n_i(g) > h$ or $n_j(g) > h$. In this case $b(n_i(g'), n_j(g')) < 0$ or $b(n_j(g'), n_i(g')) < 0$, hence g' defeats g.