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Abstract

This paper presents an intertemporal model of pre-industrial economies defined with leisure preference to study the condition of the *emergence* and *persistence* of exploitation as unequal exchange of labor. We show that pure workers are exploited in any finite periods if there is positive real profit rate, even though labor allocation among agents tends to be equalized in the limit regardless of the saving behaviors. The so-called Fundamental Marxian Theorem and Profit-Exploitation Correspondence Principle are generalized in the intertemporal setting with exploitation in the whole life, and the Class-Exploitation Correspondence Principle is established with exploitation within period.

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1 Introduction

In Marxian economics, exploitation associated with class structure is an enduring characteristic of capitalism. It has been shown that one can find an appropriate definition of exploitation that is both logically coherent and empirically meaningful (Veneziani & Yoshihara, 2015, 2017b; Yoshihara, 2010). However, the research on mechanisms generating persistent exploitation is still ongoing (e.g., Galanis, Veneziani, & Yoshihara, 2019; Kaneko & Yoshihara, 2019; Skillman, 2017; Veneziani, 2013).

In this paper, we focus on the dynamics of exploitation in pre-industrial capitalist economies within an intertemporal framework, and establish the relationship among exploitation, positive profits and class status. The main contribution of this paper is showing (i) the *persistence of exploitation* at equilibrium (reproducible solution) in the sense that pure workers are exploited in any finite periods if there is positive real profits and (ii) the *asymptotically egalitarian allocation of labor time* at equilibrium — each agent tends to supply to same labor time in the infinite limit — regardless of the their saving behavior.

In our model of economies with linear technology, producers are assumed to choose labor supply and plan of investments on both production and speculation to minimize lifelong labor time, or equivalently maximize utility representing the so-called leisure preference — once subsistence good is secured agents prefer to enjoy as much leisure time as possible. This simple model of subsistence economy is not just a first step in theory before studying a more general accumulating economy, but also a reasonable approximation of the pre-industrial capitalism before the new time-discipline was imposed by the eighteenth century (e.g., Cunningham, 1980, 2014; Thompson, 1967).

Moreover, we assume production takes one period of time. Outputs generated in this period is used as inputs in the next, and therefore the sell of outputs in period t and the purchase of inputs in $t+1$ are just two sides of the same coin and thus should have the same prices. Then for the production in period t, the prices of inputs (same as the prices of outputs in the previous period) could be different from that of outputs (same as the prices of inputs in the next period). For this reason, we relax the conventional assumption of stationary prices, which would have substantial effects on the asymptotic behavior of the prices (see Lemma 4.2).

We first generalize Roemer (1980)'s equilibrium concept of reproducible solution (RS) to the intertemporal setting. The concept of RS is a Marxian refinement of general competitive equilibrium with reproducibility, from which we derive aggregate properties of the production activity and characterizations of the price system.

To reproduce the economy as a whole, each agent should have the subsistence consumption bundle, which implies that the total subsistence goods should be produced as net outputs and thus every commodity must be produced, which has several consequences as follows: (i) The individual constraints must be binding in order to be consistent with the aggregate properties — each agent should use up all the stock of materials she owns in production and the total income must just meet the expense since they are so in the social level; (ii) There must be a uniform profit rate — otherwise no one would invest on the sector with low profit rate; (iii) by applying the same argument to the trade-off between production and speculation, we have the non-negative real profit, i.e., the profit rate must be large sufficient to induce the agents to invest the stock of materials on production rather than carrying them to the next period. Moreover, by considering the intertemporal trade-offs of allocation labor time along the life, we establish the Euler equation for the minimization problem, i.e., $1/w_t = (1+\pi_t)/w_{t+1}$ — the additional amount of labor to earn one more dollar should equal to the amount of labor saved thanks to the return from the additional dollar.

With the help of these characteristics of prices system, we derive the lifelong labor time as a function of initial endowment and prices irregardless of the saving behaviors. While in each period, the agent should supply sufficient labor to earn revenue that could cover the gap between total expenditure and the property income. Since both the expenditure and property income within period are influenced by the saving behavior, we could have different reproducible solutions with the same prices and total labor time but different allocations of labor time along the life.

We then define exploitation as unequal exchange of labor (UEL) in the intertemporal RS by generalizing the standard Okishio (1963)–Morishima (1973) formula. At an RS, an agent is said to be exploited if she receives less labor than she supplies. To elaborate this idea in the intertemporal setting, we consider the exploitative status within period t (WP_t) and that in the whole life (WL), since they may not be identical with each other when saving is allowed. Then the first question is: Which definition of exploitation is appropriate to discuss its relationship with class status and profits?

This is answered by Theorem 1, the first main result in this paper, in which we show (i) the equivalence of WL exploitation and the possibility of positive real profits in economies with inegalitarian endowments, the fundamental Marxian theorem (**FMT**); (ii) the profit-exploitation correspondence principle (**PECP**) that each propertyless workers are WL exploited if and only if there is possibility of positive real profits; and (iii) the correspondence of WP_t exploitation and class status in the sense that pure workers are WP_t exploited while capitalists are WP_t exploiting, the so-called class-exploitation correspondence principle (**CECP**). Therefore, when discuss the relationship between exploitation and profits, the WL exploitation is appropriate since it takes account of the historical factors (e.g., Wolff, 1999, p. 117), while WP_t exploitation is proper with respect to the relation with class structure, as both of them vary with the saving behavior. Moreover, we show that for the interior RS (RS without saving), all the three principles mentioned above are established with WP_t exploitation.

Finally, we tackle the main issue of this paper — the long-run behavior of exploitation — by first providing a formal definition of persistence. Exploitation is persistent if it exists in any finite periods, which should be distinguished from the limiting behavior described by asymptotics. Persistent exploitation could be consistent with the asymptotically egalitarian allocation of labor time, even the latter could be interpreted as exploitation disappears in the infinite limit. This distinction would help us clarify the implications of Theorem 3, the second main result in our paper, stating that any RS is asymptotically egalitarian — allocation of labor time among agents would tend to be equalized in the limit.

This result, which is observed in existing literature with interior RS (e.g., Kaneko & Yoshihara, 2019; Veneziani, 2007), is established with RS in general without any restriction on saving behavior for the first time. It may be interpreted as an impossibility theorem since it implies that exploitation disappears at the limit. However, such a limiting behavior could be consistent with the persistence of exploitation in the sense that exploitation exists in any finite period. Indeed, we show that pure workers are exploited all the time except in the limit if there is possibility of positive profits (Proposition 4).

We argue that persistence of exploitation is sufficient to capture the Marxian idea of exploitation as an enduring characteristics of capitalism, and asymptotically egalitarian labor allocation or the disappearance of exploitation in the limit is also consistent with the Marxian view of capitalism as one stage in human history that is neither sustainable nor permanent. Therefore, we might ask too much in the search of the mechanism that generates RS without asymptotically egalitarian labor allocation. Nevertheless, the result that labor allocation tends to be equalized does provide us some information about the long-run behavior of exploitation. It implies that after some point the degree of exploitation can be arbitrary small, if we measure it by the difference between labor supplied and labor received. In other words, exploitation, if exists, persists but tends to be smaller and smaller in the long-run, if we agree on this particular measure of the size of exploitation.

Related literature. This paper is part of the ongoing research project on the persistence of exploitation. After Roemer (1982a, 1982b)'s seminal work on modern theory of exploitation, critics have touched the issue of persistence by arguing that capital accumulation would drive profit to zero when Roemer (1982b)'s static model is run for many periods, however, without explicitly analyzing intertemporal decisions (Devine & Dymski, 1991; Hahnel, 2006; Skillman, 1995). Veneziani (2007), for the first time, discusses this issue in an intertemporal general equilibrium framework, which is the closest model to ours in this paper.

Veneziani (2007, 2013) presents a model of subsistence economy with linear technology but mainly focus on the reproducible solutions without saving. They show that labor time tends to be egalitarian in the limit, which is generalized to reproducible solutions without any restrictions on the saving behavior in this paper (see Theorem 3).

Moreover, Veneziani (2007, 2013) assume stationary prices, i.e., the price of inputs is identical with that of outputs at the same period, which leads to the fact that profit rate converges to zero. In this paper, we assume that inputs and outputs are transacted in the beginning and at the ending of the period respectively and thus could have different prices. Therefore, we observe a different asymptotic behavior of the prices — there exists an RS such that WP_t exploitation disappears in the limit while real profit rate does not vanish there.

This feature brought by the assumption of non-stationary prices is shared with Kaneko and Yoshihara (2019)'s model of international trade in pre-industrial world economies, which, however, differs from our model in two aspects. First, they focus on the interior reproducible solution as Veneziani (2007, 2013). Second, there is no international labor market in their model. In this sense, our model can also be seen as a model of international trade with labor market and savings, a generalization of Kaneko and Yoshihara (2019).

All of the papers mentioned above show only the asymptotically egalitarian labor allocation in the interior RS, while in this paper we distinguish two different types of long-run behaviors for the first time and establish the persistence of exploitation in the sense that pure workers are exploited in any finite periods. The RS without asymptotically egalitarian labor allocation is observed in Galanis et al. (2019), another current paper that is close to ours, which, however, "crucially relies on a strictly positive rate of time preference" (Galanis et al., 2019, p. 41).

Theorem 1 shows the equivalence of positive real profit and exploitation in the whole life, and the correspondence of exploitation within-period and class status, and thus relates this paper to the literature on the relationship among exploitation, profit and class structure (Morishima, 1974; Roemer, 1982a; Veneziani & Yoshihara, 2015; Yoshihara & Kaneko, 2016). Also, our discussion on the use of WL exploitation and WP_t exploitation is relevant to the long-lasting efforts to find an appropriate definition of exploitation (Ferguson & Steiner, 2018; Yoshihara, 2017; Yoshihara & Veneziani, 2018).

2 Reproducible Solution

Economic Environment. Suppose that there are N identical producers living for T periods, denoted by $\mathcal{N}$ with the genetic element ν, requiring for subsistence a vector $b \in \mathbb{R}^n_+$ in each period. They have the same knowledge of a certain Leontief input–output technology (A, L) with the following assumption[1].

Assumption 1. The nonnegative matrix A is productive, indecomposable and labor is indispensable. That is, for the matrix $A \geq 0$,

(i) $\exists x \geq 0$, $x - Ax \geq 0$;

(ii) $\forall i, j, \exists t, (A)^t_{ij} > 0$; and

(iii) $L > 0$.

Let the initial endowment be $\Omega = (\omega^1_1, \ldots, \omega^N_1)$ and denote the economy environment by $\mathcal{E}(\Omega) = \langle \mathcal{N}, A, L, b, \Omega \rangle$. If the total endowment is just sufficient to reproduce, i.e.,

$$\sum_{\nu \in \mathcal{N}} \omega^\nu_1 = \overline{\omega} \equiv A(I - A)^{-1}(Nb)$$

denote the initial endowment by $\overline{\Omega}$ and thus the economy $\mathcal{E}(\overline{\Omega})$.

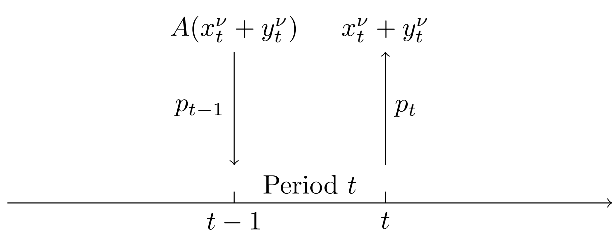

Figure 1: The time structure of the economy.

Time Structure. We assume that (i) production takes time, (ii) wage is paid *post factum*, and (iii) non-stationary price, i.e., prices of inputs and that of outputs may differ. Specifically, as shown in Figure 1, define period t be the time interval starting from time $t-1$ to time t, i.e., $[t-1, t]$. Let x^ν_t be the activity level of self-operated production,

[1] For $a, b \in \mathbb{R}$, we use $\geq$ and $\leq$ to denote the weak inequality. For $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n)$, $a \geqq b$ means $a_i \geq b_i$ for all i; $a \geq b$ means $a \geqq b$ and $a \neq b$; $a > b$ means $a_i > b_i$ for all i.

and y_t^ν the activity level that ν hires others to operate during period t. First, we assume that it takes one period to produce. In other words, means of production $A(x_t + y_t)$ is inputted at time $t-1$, the beginning of period t, and outputs $x_t + y_t$ will be generated at time t, the end of period t. Second, wages w_t are paid at time t, though labor is inputted at time $t-1$. Finally, let p_t be the prices at time t, then the prices of outputs p_t could be different from that of inputs p_{t-1}.

The first assumption that production takes time is a conventional assumption in classical economics. It is one of the key assumptions that bring about positive profit rate in equilibrium with constant-return-to-scale technology, in contrast with zero profit in neoclassical model (e.g., Roemer, 1981, pp.81-86).

Secondly, the assumption that wage is paid *post factum* is usually adopted in Sraffian tradition following Sraffa (1960), while in Marxian tradition, wage is typically assumed to be advanced at the beginning of the period. However, the choice between these two assumptions has no essential impact on the results because it is equivalent for the agent to pay w at the beginning or $(1+\pi)w$ and the end where π is the profit rate (e.g., Bidard, 2004, p.39; Abraham-Frois & Berrebi, 1997, p. 55).

Finally, although the assumption of stationary prices, i.e., prices of inputs and that of outputs are the same, is conventional in relevant literature (e.g. Galanis et al., 2019; Roemer, 1982a; Veneziani, 2007), we do not adopt here for two reasons. First, it is natural to have non-stationary prices in the intertemporal setting. Outputs generated in this period is used as inputs in the next and thus the sell of outputs in period t and the purchase of inputs in $t+1$ are just two sides of the same coin and thus should have the same prices. Then for the production in period t, the prices of inputs (same as the prices of outputs in the previous period) could be different from that of outputs (same as the prices of inputs in the next period). Second, allowing non-stationary prices would make a substantial difference in the possible trajectory of the prices as shown in Lemma 4.2 and Kaneko and Yoshihara (2019).

The Programming Problem. Let the price vector be $(p, w) = \{p_{t-1}, w_t\}_{t=1}^T$, and the agent ν's lifetime plan $\xi^\nu = (x^\nu, y^\nu, z^\nu, \delta^\nu, \omega^\nu)$ where $x^\nu = \{x_t^\nu\}_{t=1}^T$ is the lifetime plan of activity levels operated by herself, $y^\nu = \{y_t^\nu\}_{t=1}^T$ the plan of activity levels that ν hires others to operate, $z^\nu = \{z_t^\nu\}_{t=1}^T$ the plan of labor supply, $\delta^\nu = \{\delta_t^\nu\}_{t=1}^T$ the plan of speculation which is purchased at the beginning and sold at the end of each period, and $\omega^\nu = \{\omega_t^\nu\}_{t=2}^{T+1}$ the plan of endowments. Let $\Lambda_t^\nu = Lx_t^\nu + z_t^\nu$ be ν's total labor time at period t. The agent is assumed to choose ξ^ν to solve the following minimization programming problem (MP),

$$
\begin{aligned}
\min \quad & \Lambda^\nu = \sum_{t=1}^{T} \Lambda_t^\nu \\
\text{s.t.} \quad & p_t(x_t^\nu + \delta_t^\nu) + (p_t - w_t L)y_t^\nu + w_t z_t^\nu \geqq p_t b + p_t \omega_{t+1}^\nu \\
& p_{t-1}A(x_t^\nu + y_t^\nu) + p_{t-1}\delta_t^\nu \leq p_{t-1}\omega_t^\nu \\
& Lx_t^\nu + z_t^\nu \leq 1 \\
& \omega_{T+1}^\nu \geqq \omega_1^\nu \\
& x_t^\nu, y_t^\nu, \omega_t^\nu \geqq \mathbf{0}, z_t^\nu \geq 0
\end{aligned}
\tag{MP}
$$

In other words, the agent ν is assumed to minimize the undiscounted labor time subject to the constraint that (1) total revenue is sufficient to purchase the consumption goods and the stock of materials for the next period; (2) the investment plan in both production and speculation must be affordable in each period; (3) total labor performed in each period cannot go beyond the physical limit which is normalize to one; and (4) she should pass the resources to the next generation at least as much as she inherited.

Two things to be noted about the objective function. First, to minimize labor time is equivalent to maximize utility representing the so-called *leisure preference*, i.e.,

$$u_t^\nu(c, \Lambda_t^\nu) = 1 - \Lambda_t^\nu$$

where $c \geqq b$ is the consumption vector, and $\Lambda_t^\nu \in [0, 1]$ the labor supply. That is, the agent prefers to enjoy as much leisure time as possible after securing the subsistence goods b. With the leisure preference, the economy is sometimes referred to the "subsistence economy"(Roemer, 1982a; Veneziani, 2007), or following Kaneko and Yoshihara (2019) "the pre-industrial economy" since such preference was ubiquitous in the pre-industrial society by eighteenth-century, in which well-paid male workers took time off when they could (e.g., Cunningham, 1980, 2014).

> The idea of a regular working week is further challenged by the issue that so exercised contemporary commentators, the habit for workers to work only for as many hours or days as were necessary to keep them at a standard of living to which they were accustomed. They worked the full day and the full week only when wages were low and prices high. Rather than responding to the incentive of high wages by working harder and longer, they did exactly the opposite. In short, they were addicted to idleness. (Cunningham, 2014, p. 41)

Moreover, the total labor time is aggregated without discount factor, with which it is easy to have positive real profit rate and thus persistent exploitation as shown in Galanis et al. (2019). However, for our purpose to examine the persistent exploitation as a objective feature of capitalism from the viewpoint of historical materialism, it is not sufficient to show persistent exploitation brought from subjective discount factor. Therefore, it is appropriate to assume no discount factor here.

Denote the set of solutions by $\mathcal{A}^\nu(p, w)$. Then $\mathcal{A}^\nu(p, w)$ is not a singleton since there are multiple solutions to this problem in general. First, it is equivalent for the agent to operate herself on the one hand and to hire others to operate and supply her labor on the market on the other hand. Indeed, for any $\xi^\nu = (x^\nu, y^\nu, z^\nu, \delta^\nu, \omega^\nu) \in \mathcal{A}^\nu(p, w)$, we have

$$\bar{\xi}^\nu = (0, x^\nu + y^\nu, z^\nu + Lx^\nu, \delta^\nu, \omega^\nu) \in \mathcal{A}^\nu(p, w) \tag{1}$$

where $x^\nu + y^\nu = \{x_t^\nu + y_t^\nu\}_{t=1}^T$, $z^\nu + Lx^\nu = \{z_t^\nu + Lx_t^\nu\}_{t=1}^T$. Second, the agent could choose to seize the day or to work hard and save for the future. Actually, the agent could have different optimal saving paths associate with the equilibrium prices, as we can see later (Lemma 3.4).

Reproducible Solution. Let $x_t = \sum_\nu x_t^\nu$ and similar notation with y_t, z_t, δ_t and ω_t, we define the reproducible solution (RS) as follows.

Definition 1. A price vectors (p, w) and the associated lifetime plan $\Xi = (\xi^1, \ldots, \xi^N)$ is a *reproducible solution* (RS) for the economy $\mathcal{E}(\Omega)$ if:

(i) $\xi^\nu \in \mathcal{A}^\nu(p, w), \forall \nu$

(ii) $x_t + y_t + \delta_t \geqq Nb + \omega_{t+1}, \forall t$

(iii) $A(x_t + y_t) + \delta_t \leqq \omega_t, \forall t$

(iv) $Ly_t = z_t, \forall t$

(v) $\omega_{T+1} \geqq \omega_1$.

Condition (i) means that the economy activities must be consistent with the labor time minimization, (ii) is the condition of reproducibility of the economy as a whole, (iii) means that the investments in both production and speculation should not exceed the endowment at aggregate level, (iv) is the equilibrium condition in labor market, and finally (v) means that the physical endowment for the next generation should not be less than the initial endowment in this generation, which is implied by the constraint of the minimization program.

In this paper, we discuss the persistent exploitation in RS in general without any *ad hoc* assumption on the saving behavior. However, we also consider the *interior* RS defined below as a special case, since it is the main focus of the existing literature(e.g., Kaneko & Yoshihara, 2019; Veneziani, 2007).

Definition 2. An interior RS (IRS) for $\mathcal{E}(\Omega)$ is an RS such that

$$p_t A(x_t^\nu + y_t^\nu) + p_t \delta_t^\nu = p_t \omega_{t+1}^\nu, \forall \nu, \forall t$$

In other words, if the wealth at the end of the period, $p_t \omega_{t+1}^\nu$, is just sufficient to maintain the investments (both in production and speculation), $A(x_t^\nu + y_t^\nu) + \delta_t^\nu$, we say there is no saving and called such a reproducible solution an interior RS.

In the rest of this section, we provide some properties of the reproducible solutions. For example, it is easy to see that if an RS exists, then there are multiple RSs. Indeed, for any RS $\{(p, w), \Xi\}$, we have $\{(p, w), \overline{\Xi}\}$ is an RS, where $\overline{\Xi} = (\overline{\xi}^1, \ldots, \overline{\xi}^N)$ and $\overline{\xi}^\nu$ is defined by (1).

To avoid uninteresting technicalities, we focus on those solutions with wealth maximization (WM) such that if the agent ν supply zero labor at equilibrium she could choose the path that maximizes the wealth, stated as an assumption below[2].

Assumption 2 (WM)**.** Let $\{(p, w), \Xi)$ be an RS for $\mathcal{E}(\overline{\Omega})$. If there exists $\nu \in \mathcal{N}$ such that $\Lambda^\nu = 0$, then ν chooses y^ν and δ^ν to maximize wealth $p_T \omega_{T+1}^\nu$.

Moreover, from now on we will only consider the economy $\mathcal{E}(\overline{\Omega})$ with initial endowment $\overline{\omega}$. Recall that $\overline{\omega} = A(I - A)^{-1}(Nb)$ is the minimal endowment to produce Nb as net output. Therefore it is impossible to accumulate for the economy as a whole as shown in Lemma 2.1 below. In other words, this assumption implies 'simple reproduction', which is consistent with the behavior assumption on labor time minimization

[2]This assumption plays a similar role as the "nonbenevolent capitalist" assumption (NBC) in Roemer (1982a, 1988), Veneziani (2007). It is not strong in our study of persistent exploitation, since when T is large no one can supply zero labor at RS, as we can see later in Lemma 3.4.

or the leisure preference. Indeed, given any initial endowment greater than $\overline{\omega}$, it can be shown that the social stock of material inputs would reduce to $\overline{\omega}$ in the long-run. Therefore, we focus on $\mathcal{E}(\overline{\Omega})$ without loss of generality.

Lemma 2.1. *Let $\{(p,w),\Xi\}$ be an RS for $\mathcal{E}(\overline{\Omega})$. Then $\omega_t = \overline{\omega}$ for any t. Furthermore, $\omega^\nu_{T+1} = \omega^\nu_1$ for all $\nu \in \mathcal{N}$.*

Proof. See Appendix A. □

By fixing the end points ω^ν_{T+1}, Lemma 2.1 provides the *transversality conditions* of the optimal problem (MP)[3]. Moreover, with the fixed aggregate stock of material inputs $\overline{\omega}$ in each period, it can be shown that the aggregate production activity $x_t + y_t$ to generate Nb as net outputs must use up all the stock of material inputs $\overline{\omega}$ in production and therefore no speculation, $\delta_t = 0$, as summarized in the following proposition.

Proposition 1. *Let $\{(p,w),\Xi\}$ be an RS for $\mathcal{E}(\overline{\Omega})$. Define the value vector by $v = L(I-A)^{-1}$. Then*

(1) $\delta_t = 0, \forall t$

(2) $x_t + y_t = Nb + \overline{\omega}, \forall t$

(3) $A(x_t + y_t) = \overline{\omega}, \forall t$

(4) $\sum_{\nu=1}^N \Lambda^\nu_t = vNb, \forall t$

(5) $\sum_{\nu=1}^N \Lambda^\nu = vTNb.$

Proof. By Lemma 2.1, we have $\omega_t = \overline{\omega}, \forall t$. Together with (ii) and (iii) in Definition 1, we have

$$x_t + y_t + \delta_t \geqq Nb + \overline{\omega}$$
$$A(x_t + y_t) + \delta_t \leqq \overline{\omega}$$

then $x_t + y_t \geqq (I-A)^{-1}Nb$ and thus $A(x_t + y_t) \geqq \overline{\omega}$, which establishes (1) and (3). Suppose (2) does not hold, then we have

$$A(x_t + y_t) > A(Nb + \overline{\omega}) = \overline{\omega}$$

contradicted with (3). Therefore, (2) holds. Then, we have $x_t + y_t = Nb + \overline{\omega} = (I - A)^{-1}Nb$. Then

$$\sum_\nu \Lambda^\nu_t = \sum_\nu (Lx_t + z_t) = \sum_\nu L(x_t + y_t) = L(I-A)^{-1}Nb = vNb$$

For (5), by $\Lambda^\nu = \sum_{t=1}^T \Lambda^\nu_t$, we have

$$\sum_{\nu=1}^N \Lambda^\nu = \sum_{t=1}^T \sum_\nu \Lambda^\nu_t = vTNb$$

□

[3] The Euler equations, which provides information on the price system, will be derived in Lemma 3.3 after we establish the uniform rate of profit.

Proposition 1 gives the aggregate properties of RS guaranteed by the restrictions imposed by reproduction of the economy as a whole. From (2) we can see that all commodities are produced at RS. Then together with the equilibrium condition in the labor market, we can conclude that wage and price must be positive. Moreover, to be consistent with these aggregate properties, the individual capital and reproducible constraints must be binding.

Corollary 2.1. *Let $\{(p,w),\Xi\}$ be an RS for $\mathcal{E}(\overline{\Omega})$ with WM. Then $w_t > 0$ and $p_t > 0$.*

Proof. See Appendix A. □

Corollary 2.2. *Let $\{(p,w),\Xi\}$ be an RS for $\mathcal{E}(\overline{\Omega})$ with WM. Then*

(1) *The capital constraint is binding, i.e.,*

$$p_{t-1}A(x_t^\nu + y_t^\nu) = p_{t-1}\omega_t^\nu, \forall\nu, \forall t$$

(2) *The reproducibility constraint is binding, i.e.,*

$$p_t x_t^\nu + (p_t - w_t L)y_t^\nu + w_t z_t^\nu = p_t b + p_t \omega_{t+1}^\nu, \forall\nu, \forall t$$

Proof. Both are established by the (2) and (3) of Proposition 1 and the positivity of price from Corollary 2.1. □

3 Exploitation

Exploitation as Unequal Exchange of Labor. We define exploitation as the unequal exchange of labor (henceforth, UEL exploitation), i.e., an agent is said to be exploited if and only if the amount of labor she receives is less than that she contributes in production. In our setting with linear technology, the amount of labor one receives in each period via consumption goods b is well-defined as the labor value, vb, and thus the amount of labor one receives in the whole life is vTb. Then we have the following definition.

Definition 3. Let $\{(p,w),\Xi\}$ be an RS for $\mathcal{E}(\overline{\Omega})$. We say in the whole life (WL)

$$\nu \text{ is exploited} \Leftrightarrow \Lambda^\nu > vTb, \quad \nu \text{ is exploiting} \Leftrightarrow \Lambda^\nu < vTb$$

and within period t (WP_t) ,

$$\nu \text{ is exploited} \Leftrightarrow \Lambda_t^\nu > vb, \quad \nu \text{ is exploiting} \Leftrightarrow \Lambda_t^\nu < vb$$

Definition 3 generalizes the standard Okishio (1963)–Morishima (1973) form of UEL exploitation in subsistence economies with linear technology to the intertemporal framework. As mentioned in Section 1, the concept of UEL exploitation is well generalized in a broad class of economies (Veneziani & Yoshihara, 2015; Yoshihara, 2010). Although there are many alternative forms of UEL exploitation in general economies, as discussed in Veneziani and Yoshihara (2015, 2017b), Yoshihara (2010, 2017), Yoshihara and Veneziani (2018), all forms are reduced to Definition 3 in subsistence economies with linear technology. Therefore, our definition here is appropriate regardless of what is the best form in general.

Moreover, the concept of UEL exploitation is of great normative relevance as discussed in Section 1. In this paper, we will focus on its implications with respect to profits and class structure in capitalism. Before that, we shall first discuss what positive profit means in our setting with non-stationary prices, and characterizes the prices system at RS.

Possibility of Positive Real Profits. Using the fact that all commodities are produced at RS, the following lemma first establishes the uniform rate of profit.

Lemma 3.1. *Let $\{(p, w), \Xi\}$ be an RS for $\mathcal{E}(\overline{\Omega})$. Assume WM. Then for any t there exist π_t such that*

$$p_t = (1 + \pi_t)p_{t-1}A + w_t L \tag{2}$$

Proof. See Appendix A. □

Moreover, to be consistent with no speculations at RS, $\delta_t = 0$, as shown in Proposition 1, the profit rate should be large sufficient to induce the agents to invest on production rather than speculations. Therefore, the real profit rate should be nonnegative, i.e.,

$$(1 + \pi_t)p_{t-1} \geqq p_t, \forall t \tag{3}$$

otherwise the agent would speculate to save labor. However, since we assume non-stationary prices, the nominal rate of profit π_t can be negative. Indeed, in the case with deflation, $p_t < p_{t-1}$, even with some negative profit rate the agent would prefer productive investments to speculations. Nevertheless, the nominal profit rate has a lower bound since prices are positive by Corollary 2.1. The reasoning above is presented formally in the following lemma which characterizes the relationship between prices and profit rates at RS and defines the *possibility of positive real profits* (PPRP).

Lemma 3.2. *Let $\{(p, w), \Xi\}$ be an RS for $\mathcal{E}(\overline{\Omega})$ with WM. Then*

(i) $(1 + \pi_t)p_{t-1} \geqq p_t$ *and* $p_t \geqq w_t v, \forall t$.

(ii) $(1 + \pi_t)p_{t-1} \geq p_t \Rightarrow p_t > w_t v$. *In this case, we say there is a* possibility of positive real profits *(PPRP).*

(iii) $\pi_t + 1 > 0, \forall t$.

Proof. See Appendix A. □

So far we derive the uniform profit rate (2) and nonnegative real profit rate (3) by considering the agent's investment decisions within each period. The choice among alternative sectors helps to establish the uniform rate of profit (Lemma 3.1), and the trade off between productive and speculative investments leads to the nonnegative real profit rate (Lemma 3.2). Both are common features shared with the static models (e.g., Roemer, 1982a). Contrarily, equation (4) in the following lemma is established by considering the agent's intertemporal trade-offs in the allocation of labor along the life.

Lemma 3.3. *Let* $\{(p,w),\Xi\}$ *be an RS for* $\mathcal{E}(\overline{\Omega})$ *with WM. Then*

$$w_{t+1} = (1+\pi_{t+1})w_t, \forall t \tag{4}$$

Proof. See Appendix A. □

Equation (4) is the Euler equations for MP, derived from the standard perturbation method. By rearranging, we have

$$\frac{1}{w_t} = \frac{1+\pi_{t+1}}{w_{t+1}}, \forall t \tag{5}$$

To see the intuition, consider a deviation whereby the producer works more in period t to earn one more dollar, which she invests and earns a return $(1+\pi_{t+1})$ and hence can work less next period. To be optimal at RS, this marginal deviation should lead to the same labor time. Indeed, the left hand side of equation (5) is the amount of extra labor to earn one more dollar in period t, while the right hand side is the amount of labor saved in next period thanks to the return $(1+\pi_{t+1})$.

Note that this specific information provided by Lemma 3.3 on the sequence of wages is the product of labor minimization. Because of the leisure preference, wages are involved in the intertemporal trade-offs as in Veneziani and Yoshihara (2017a, Lemma 2, p. 453) and Kaneko and Yoshihara (2019), contrasted with the monotonic preference over consumption goods where wages do not appear in the Euler equations (See e.g. Lemma C.1 in Appendix C and Galanis et al., 2019, Lemma 3, p. 37).

By Lemma 3.1 and 3.3, the price system (p,w) for any possible reproducible solutions must satisfies (2) and (4), from which (p,w) can be determined recursively by the initial prices (p_0,w_1) and the sequence of nominal profit rate $\{\pi_t\}$. Moreover, with these characteristics of price system at RS, the following proposition shows that if there is possibility of positive real profit at the beginning then it is so in every period and vice versa.

Proposition 2. *Let* $\{(p,w),\Xi\}$ *be any RS for* $\mathcal{E}(\overline{\Omega})$ *with WM. Then*

$$(1+\pi_1)p_0 - p_1 \geq 0 \Leftrightarrow (1+\pi_t)p_{t-1} - p_t \geq 0 \quad \forall t \tag{6}$$

Proof. See Appendix A. □

Proposition 2 shows that whether we have PPRP at each period is determined by the initial prices p_0, π_1 and w_1, since

$$(1+\pi_1)p_0 - p_1 \geq 0 \Leftrightarrow (1+\pi_1)p_0(I-A) - w_1 L \geq 0.$$

In other words, it is impossible to have PPRP in some periods but no PPRP at others. Therefore, we can say there is PPRP or not at an RS with no ambiguity.

Exploitation, PPRP and Class. One of the main purposes of this paper is to examine the relationship between exploitation, capitalist profits and class status in the intertemporal framework with non-stationary prices and no *ad hoc* assumption on savings. Specifically, we will focus on the following principles that are well-established in static framework.

First, as an indicator of the overall economy, UEL exploitation is equivalent to positive profit rate in inegalitarian capitalist economy. This is the so-called Fundamental Marxian Theorem (**FMT**) originally developed by Okishio (1963), capturing the Marxian perception of capitalism as an exploitative system. Second, exploitation status corresponds to the class membership in the sense that pure workers are exploited while capitalists are exploiters, as summarized in the Class-Exploitation Correspondence Principle (**CECP**) proposed by Roemer (1982b). The third principle is the so-called Profit-Exploitation Correspondence Principle (**PECP**) proposed by Veneziani and Yoshihara (2015). It captures the idea that in any economic equilibrium, positive profits imply that each propertyless worker is exploited and vice versa. This principle differs in two aspects from **FMT**: (i) with positive total profits, **PECP** requires each propertyless worker is exploited while **FMT** just implies that the working class is exploited as a whole; (ii) zero profit implies no exploitation based on **FMT** but not so according to **PECP**[4].

There have been a lot of discussions on the robustness of these three principles in general economic environments, and all except **PECP** are problematic (Yoshihara, 2017). However, all of the three principles are well-established with linear technology in static framework[5]. In this section, we will examine them in the intertemporal framework with non-stationary prices and no *ad hoc* assumption on saving behaviors.

From the individual binding constraints in Corollary 2.2, we can see that within each period the agent should work to earn sufficient wage revenue to cover the gap between the total expenditure $p_t b + p_t \omega_{t+1}^{\nu}$ and the property income $(1+\pi_t)p_{t-1}\omega_t^{\nu}$. Then using the characterizations of the prices system, we have the labor time as shown in the following lemma.

Lemma 3.4. *Let $\{(p,w),\Xi\}$ be an RS for $\mathcal{E}(\overline{\Omega})$ with WM. Then*

(1) *The labor time within period t is*

$$\Lambda_t^{\nu} = \frac{p_t b + p_t \omega_{t+1}^{\nu} - (1+\pi_t)p_{t-1}\omega_t^{\nu}}{w_t}, \forall t, \forall \nu$$

(2) *The labor time in the whole life*

$$\Lambda^{\nu} = \sum_{t=1}^{T} \frac{p_t b}{w_t} - \frac{(1+\pi_1)p_0\omega_1^{\nu}}{w_1} + \frac{p_T \omega_{T+1}^{\nu}}{w_T}$$

Proof. See Appendix A. □

Note that $\omega_{T+1}^{\nu} = \omega_1^{\nu}$ by Lemma 2.1 therefore the total labor time Λ^{ν} is invariant with respect to the saving behaviors, i.e., the choice of $\{\omega_t^{\nu}\}_{t=1}^{T}$. Moreover, in the economy $\mathcal{E}(\overline{\Omega})$, RSs with the different prices system could have different distributions of total labor time $(\Lambda^{\nu})_{\nu\in\mathcal{N}}$, even with the same initial distribution of physical endowments $\overline{\Omega}$. This is because different price systems could lead to different distributions of wealth. Finally, compare the RSs with the same prices (p,w) and thus the same total labor time, the agent's saving behavior would influence the labor time within each period. In other

[4] See Veneziani and Yoshihara (2015, 2017b), Yoshihara (2017) for detailed discussions.

[5] In recent literature, these principles are also generalized into various intertemporal settings without saving (Veneziani, 2007; Yoshihara & Kaneko, 2016).

words, Λ^ν is fixed once the price system (p, w) and the initial endowment ω_1^ν are given, while the choice among different paths of ω_t^ν determines her intertemporal allocation of labor time $\{\Lambda_t^\nu\}_{t=1}^T$.

Theorem 1. *Let $\{(p, w), \Xi\}$ be an RS for $\mathcal{E}(\overline{\Omega})$ with WM. Denote the set of propertyless agent at period t by $\mathcal{W}_t \equiv \{\nu \in \mathcal{N} \mid \omega_t^\nu = 0\}$. Then*

(i) $\forall \nu \in \mathcal{W}_1,\ (1+\pi_1)p_0 \geq p_1 \Rightarrow \Lambda^\nu > vTb.$

(ii) $\forall \nu \in \mathcal{W}_t,\ (1+\pi_t)p_{t-1} \geq p_t \Rightarrow \Lambda_t^\nu > vb, \forall t.$

(iii) $\forall \nu \in \mathcal{N},\ (1+\pi_1)p_0 = p_1 \Rightarrow \Lambda^\nu = vTb$

(iv) *If $\{(p, w), \Xi\}$ is an IRS, then* $(1+\pi_t)p_{t-1} = p_t \Rightarrow \Lambda_t^\nu = vb, \forall t, \forall \nu$

Proof. See Appendix A.

□

Theorem 1 is one of the main results in our paper. First, (i) and (iii) together establish the **FMT** with respect to WL exploitation. That is, WL exploitation exists if and only if PPRP, for the economy with $\mathcal{W}_1 \neq \emptyset$. Indeed, we have **PECP** as well — each $\nu \in \mathcal{W}_1$ is WL exploited if and only if PPRP. Second, in terms of WP_t exploitation, we have both **FMT** and **PECP** for IRS from (ii) and (iv). Finally, (ii) means that pure workers are WP_t exploited and from the expression of labor time in Lemma 3.4 it is easy to see that any pure capitalist must be exploiter within period t. Therefore, we have **CECP** for any RS in terms of WP_t exploitation. The above discussions are summarized in Table 1.

Table 1: Exploitation, profit and class..

	FMT	**CECP**	**PECP**
Exploitation WP_t	IRS	✓	IRS
Exploitation WL	✓		✓

Note that for RS in general (i) **FMT** and **PECP** do not hold with respect to WP_t exploitation since some producer may be exploited WP_t even when there is no PPRP — one can work more in period t to save in the next if there is another one does the opposite; (ii) **CECP** fails in terms of WL exploitation because class membership may change along the life without changing the WL exploitation status — a WL exploited agent would be a capitalist in some period if she saves in the previous periods and there is other agent doing the opposite. Indeed, Example 1 shows the existence of such RS with particular saving behaviors as counterexamples with the help of Theorem 2 below.

Example 1. For the economy $\mathcal{E}(\overline{\Omega})$ with $\mathcal{W}_1 \neq \emptyset$, if there is an RS $\{(p, w), \Xi\}$ with no PPRP, i.e., $p_0(1+\pi_1) = p_1$, then there exists an RS with the same price (p, w) such that exploitation exists within period t for some t.

Therefore, we can conclude that

(1) for RS in general, it is appropriate to use the definition of WL exploitation to discuss the relationship between exploitation and positive profits, and WP_t exploitation to discuss the relationship between exploitation status and class membership;

(2) for IRS in particular, WP_t exploitation is meaningful for the discussions on the relationship among exploitation, profits and class, as established in the existing literature (e.g. Veneziani, 2007; Yoshihara & Kaneko, 2016).

Existence of RS. Given any sequence of profit rate $\pi = \{\pi_t\}_{t=1}^{T}$ such that $\pi_t + 1 > 0, \forall t$. For any initial prices (p_0, w_1), let $\mathbf{p}(p_0, w_1, \pi)$ be the sequence of prices obtained from

$$w_t = (1 + \pi_t)w_{t-1} \tag{7a}$$
$$p_t = (1 + \pi_t)p_{t-1}A + w_t L \tag{7b}$$

Let

$$\Upsilon = \{(Au^\nu)_{\nu\in\mathcal{N}} \mid u^\nu \geqq 0, \sum_\nu u^\nu = (I - A)^{-1}(Nb)\}$$

and the set of initial endowment be

$$\mathbf{\Delta}(p_0, w_1, \pi) \equiv \{(\omega_1^\nu)_{\nu\in\mathcal{N}} \in \Upsilon \mid 0 \le \Psi(\omega_1^\nu) \le T, \forall\nu\}$$

where

$$\Psi(\omega_1^\nu) = \sum_{t=1}^{T} \frac{p_t b}{w_t} - \frac{(1 + \pi_1)p_0\omega_1^\nu}{w_1} + \frac{p_T\omega_1^\nu}{w_T} \tag{8}$$

In the economy $\mathcal{E}(\Omega)$ for any $\Omega \in \mathbf{\Delta}(p_0, w_1, \pi)$, denote the set of labor supply of agent ν by

$$Z^\nu(p_0, w_1, \pi, \Omega) = \{z^\nu = \{z_t^\nu\}_{t=1,\ldots,T} \mid \sum_t z_t^\nu = \Psi(\omega_1^\nu), 0 \le z_t^\nu \le 1\}$$

and let

$$Z(p_0, w_1, \pi, \Omega) \equiv \prod_{\nu\in\mathcal{N}} Z^\nu(p_0, w_1, \pi, \Omega).$$

For any $(z^\nu) \in Z(p_0, w_1, \pi, \Omega)$, is there any RS with the path of labor time $\Lambda_t^\nu = z_t^\nu$? The following proposition gives necessary and sufficient conditions. For any $(z^\nu) \in Z$, define wealth by

$$W_{t+1}^\nu = w_t z_t^\nu + (1 + \pi_t)W_t^\nu - p_t b, \quad t = 1, \cdots, T - 1, \forall\nu \tag{9}$$

for all $\nu \in \mathcal{N}$.

Theorem 2. *Given (p_0, w_1, π) with $p_0 \geqq p_0 A + \frac{w_1}{1+\pi_1}L$, the economy $\mathcal{E}(\Omega)$ for any $\Omega \in \mathbf{\Delta}(p_0, w_1, \pi)$ has an RS with labor time $(z^\nu) \in Z(p_0, w_1, \pi, \Omega)$ if and only if*

(1) $W_t^\nu \ge 0, \forall t, \forall\nu$

(2) $\sum_{\nu\in\mathcal{N}} W_t^\nu = p_{t-1}\overline{\omega}$, $\forall t$

where $W_1^\nu = p_0\omega_1^\nu$ and W_t^ν defined by (9).

Proof. (⇒) If the eoconomy $\mathcal{E}(\Omega)$ has an RS with $\Lambda_t^\nu = z_t^\nu$, then by Lemma 3.4, we have $W_t^\nu = p_{t-1}\omega_t^\nu$ and thus (1) and (2) hold.

(⇐) If (1) and (2) hold, then define the profile $\mathbf{\Xi}(p_0, w_1, \pi, \Omega)$ with $z^\nu \in Z^\nu$ by

$$x_t^\nu = \delta_t^\nu = \mathbf{0}, \quad \forall t \forall \nu \tag{10a}$$

$$\omega_{T+1}^\nu = \omega_1^\nu \text{ and } \omega_t^\nu = \frac{W_t^\nu}{p_{t-1}\overline{\omega}}\overline{\omega}, \quad t = 2, \cdots, T \tag{10b}$$

$$y_1^\nu = u \text{ and } y_t^\nu = \frac{W_t^\nu}{p_{t-1}\overline{\omega}}(I - A)^{-1}(Nb), \quad t = 2, \cdots, T \tag{10c}$$

where u is the vector assoicated with $\Omega \in \Upsilon$ by the definition of Υ. Then it can be checked that $\{\mathbf{p}(p_0, w_1, \pi), \mathbf{\Xi}(p_0, w_1, \pi, \Omega)\}$ satisfies all the conditions in Defintion 1 (RS). Indeed, (ii) (iii) and (v) are straightforward since $y_t = (I - A)^{-1}(Nb)$ and $\omega_t = \overline{\omega}$ by (10). For (i), it is obvious that the profile satisfies all the constraints in the minimization program (MP), then by Lemma 3.4, the lifelong labor time $\Psi(\omega_1^\nu)$ equal to the minimum labor time, thus we have $\xi^\nu(p_0, w_1, \pi, \Omega) \in \mathcal{A}^\nu(\mathbf{p}(p_0, w_1, \pi))$ for all ν. Condition (iv) is obtained by the fact that $z_t = v(Nb)$, which is nontrivial and shown in the second part of Lemma 3.5 below. □

Lemma 3.5. (1) *Given $\Psi(\omega_1^\nu)$ defined in (8), we have*

$$\sum_{\nu \in \mathcal{N}} \Psi(\omega_1^\nu) = v(TNb)$$

(2) *For any $(z^\nu) \in Z(p_0, w_1, \pi, \Omega)$, if the conditions in Theorem 2 hold, then we have*

$$\sum_{\nu \in \mathcal{N}} z_t^\nu = v(Nb), \quad \forall t$$

Proof. See Appendix A. □

Theorem 2 gives the conditions for a path of labor allocation $(z^\nu) \in Z(p_0, w_1\pi, \Omega)$ to be realized in the RS. Then the existence of RS can be shown by constructing a path of labor allocation satisfies these conditions. Indeed, as shown in Appendix B, we first establish the existence of IRS without any strong restrictions (Theorem B1), and then construct a non-interior RS based on the IRS (Theorem B2).

4 Persistent Exploitation

Theorem 1 shows the condition of the emergence of exploitation — pure workers are exploited if there is possibility of positive real profit. If this condition is met and exploitation emerges, will it disappear at some point or persist? To answer this question, we turn to the long-term behavior of RS to explore the exploitative status at any finite period and the labor allocation in the infinite limit with the help of the following definitions.

Definition 4 (persistence)**.** A time-related property P_t is said to be *persistent* if for any $t > 0$, P_t holds.

Definition 5 (persistent exploitation)**.** We say exploitation at an RS is persistent if WP_t exploitation exists for any t, i.e., $\forall t, \exists \nu \in \mathcal{N}, \Lambda_t^\nu > vb$.

Definition 6 (asymptotic equivalence)**.** Two sequence u_t and v_t are *asymptotically equivalent*, denoted by $u_t \sim v_t$, if

$$\lim_{t\to\infty} \frac{u_t}{v_t} = 1$$

Definition 7. At an RS, we say the labor allocation (Λ_t^ν) is *asymptotically egalitarian* if the labor time for all agent are asymptotically equivalent for each other, i.e.,

$$\forall \nu, \mu \in \mathcal{N}, \Lambda_t^\nu \sim \Lambda_t^\mu$$

An RS is *asymptotically egalitarian* if labor allocation is asymptotically egalitarian.

Note that persistence characterizes the situation in any finite periods, while asymptotics is used to describe the limiting behavior. For example, for the sequence $u_t = 1+1/t$, the property $u_t > 1$ is persistent since it holds for any $t > 0$, however, we have u_t converges to 1 and thus is asymptotically equivalent to the sequence $v_t = 1$. Similarly, it is possible that an RS is asymptotically egalitarian but with persistent exploitation by definition. Indeed, we will show that any RS is asymptotically egalitarian, while exploitation, if exists, is persistent. Specifically, if there is PPRP then pure workers are exploited WP_t for any t, even though labor time tends to be equalized in the limit. To establish this result, we will first provide some asymptotic properties of prices system at RS.

The Asymptotic Behavior of Prices.

Lemma 4.1. *Let $\{(p, w), \Xi\}$ be an RS for $\mathcal{E}(\overline{\Omega})$ with WM. Then the sequence $(\frac{p_t}{w_t})$ decreasingly converge to the labor value v. That is*

$$\frac{p_t}{w_t} \downarrow v, \text{ as } t \to \infty$$

Proof. By Lemma 3.2 and 3.3, we have

$$\frac{p_t}{w_t} - \frac{p_{t+1}}{w_{t+1}} = \frac{(1+\pi_{t+1})p_t - p_{t+1}}{w_t(1+\pi_{t+1})} \geqq 0$$

Therefore, the sequence $\{p_t/w_t\}$ is decreasing. Moreover, by Lemma 3.1 and 3.3, we have

$$w_t = \Pi_{\tau=2}^t (1+\pi_\tau) w_1 = \Pi_{\tau=1}^t (1+\pi_\tau) w_0$$
$$p_t = \Pi_{\tau=1}^t (1+\pi_\tau) p_0 A^t + \Pi_{\tau=1}^t (1+\pi_\tau) w_0 L \sum_{\tau=0}^{t-1} A^\tau$$

where $w_0 = w_1/(1+\pi_1)$. Then

$$\frac{p_t}{w_t} = \frac{p_0}{w_0} A^t + L \sum_{\tau=0}^{t-1} A^\tau \to 0 + L(I-A)^{-1} = v, \quad t \to \infty$$

since the Frobenius root $\rho(A) < 1$ by the productiveness (Assumption 1). $\square$

Lemma 4.2. *Let $\{(p, w), \Xi\}$ be an RS for $\mathcal{E}(\overline{\Omega})$ with WM, and π_t be associated profit rate. Define $R \equiv 1/\rho(A) - 1$ where $\rho(A)$ is the Frobenius root of A.*

(i) *If $\limsup_{t\to\infty} \pi_t < R$, then $\lim_{t\to\infty}[(1+\pi_t)p_{t-1} - p_t] = 0$.*

(ii) *If $\liminf_{t\to\infty} \pi_t > R$, then $\liminf_{t\to\infty}[(1+\pi_t)p_{t-1} - p_t]_i > 0$ for all i.*

Proof. See Kaneko and Yoshihara (2019, Proposition 5). □

Labor Allocation in the Limit. First, we examine labor allocation among agents in the limit in the special case with interior RS.

Proposition 3. *Let $\{(p, w), \Xi\}$ be a IRS for $\mathcal{E}(\overline{\Omega})$ with WM. Then*

$$\lim_{t\to\infty} \Lambda_t^\nu = vb, \forall \nu$$

Proof. By Definition 2 and Lemma 3.4, we have

$$\Lambda_t^\nu = \frac{p_t b - [(1+\pi_t)p_{t-1} - p_t]A(x_t^\nu + y_t^\nu)}{w_t} \leq \frac{p_t b}{w_t} \to vb, \text{ as } t \to \infty$$

Then $\forall \varepsilon > 0$, let $\varepsilon' = \varepsilon/(N-1)$,

$$\exists T, \forall t > T, \frac{p_t b}{w_t} < vb + \varepsilon'$$

and therefore, $\Lambda_t^\nu < vb + \varepsilon' < vb + \varepsilon$ for all ν. Since $\sum_\mu \Lambda_t^\mu = vNb$ by Proposition 1, we have

$$\begin{aligned}
\Lambda_t^\nu &= Nvb - \sum_{\mu\neq\nu} \Lambda_t^\mu \\
&> Nvb - \sum_{\mu\neq\nu} (vb + \varepsilon') \\
&= Nvb - (N-1)(vb + \varepsilon') \\
&= vb - \varepsilon
\end{aligned}$$

Thus we have $vb - \varepsilon < \Lambda_t^\nu < vb + \varepsilon$ for all $\varepsilon > 0$. Then $\lim_{t\to\infty} \Lambda_t^\nu = vb, \forall \nu$. □

Proposition 3 is a replication of Veneziani (2007)'s main conclusion with different time structure, and a generalization of Kaneko and Yoshihara (2019)'s results to the case with labor market. Actually, the assumption of no saving turns out to be not necessary, as shown in the following theorem.

Theorem 3. *Let $\{(p, w), \Xi\}$ be an RS for $\mathcal{E}(\overline{\Omega})$ with WM. Suppose that there is PPRP, then*

(i) *WL exploitation exists in the limit, i.e.,*

$$\lim_{T\to\infty} \sum_{t=1}^{T} (\Lambda_t^\nu - \Lambda_t^\mu) \neq 0$$

unless the the initial distribution of endowment is egalitarian in the sense that $(v - \frac{p_0}{w_0})\omega_1^\nu = constant, \forall \nu$.

(ii) *Any RS is asymptotically egalitarian. Indeed,*

$$\lim_{t\to\infty} \Lambda_t^\nu = vb, \forall \nu$$

Proof. For any ν and μ in $\mathcal{N}$, by Lemma 3.4 we have

$$\begin{aligned}\Lambda^\nu - \Lambda^\mu &\equiv \sum_{t=1}^{T}(\Lambda_t^\nu - \Lambda_t^\mu) = \left(\frac{p_T}{w_T}\omega_{T+1}^\nu - \frac{p_0}{w_0}\omega_1^\nu\right) - \left(\frac{p_T}{w_T}\omega_{T+1}^\mu - \frac{p_0}{w_0}\omega_1^\mu\right) \\ &= \left(\frac{p_T}{w_T} - \frac{p_0}{w_0}\right)(\omega_1^\nu - \omega_1^\mu)\end{aligned}$$

by $\omega_{T+1}^\nu = \omega_1^\nu$ and $\omega_{T+1}^\mu = \omega_1^\mu$ due to Lemma 2.1. Since

$$\frac{p_T}{w_T} \to v, T \to \infty$$

by Lemma 4.1, we have

$$\lim_{T\to\infty} \sum_{t=1}^{T}(\Lambda_t^\nu - \Lambda_t^\mu) = \lim_{T\to\infty}\left(\frac{p_T}{w_T} - \frac{p_0}{w_0}\right)(\omega_1^\nu - \omega_1^\mu) = \left(v - \frac{p_0}{w_0}\right)(\omega_1^\nu - \omega_1^\mu) \equiv S^* \quad (11)$$

and $S^* \neq 0$ unless $\left(v - \frac{p_0}{w_0}\right)\omega_1^\nu = \left(v - \frac{p_0}{w_0}\right)\omega_1^\mu$, which establishes (i).

For (ii), we first show that $\lim_{t\to\infty}(\Lambda_t^\nu - \Lambda_t^\mu) = 0$ by the fact that the infinite series $\sum_{t=1}^{\infty}(\Lambda_t^\nu - \Lambda_t^\mu)$ is convergent for any $\nu, \mu \in \mathcal{N}$. Indeed, denote the partial sum of the series by

$$S_t = \sum_{t'=1}^{t}(\Lambda_{t'}^\nu - \Lambda_{t'}^\mu)$$

Then, as shown above in (11), we have $\lim_{t\to\infty} S_t = S^*$, and therefore

$$\Lambda_t^\nu - \Lambda_t^\mu = S_t - S_{t-1} \to S^* - S^* = 0, \text{ as } t \to \infty$$

In addition, $\sum_{\nu\in\mathcal{N}} \Lambda_t^\nu = vNb$ holds for any t, then $\lim_{t\to\infty} \Lambda_t^\nu = 0, \forall \nu \in \mathcal{N}$. Indeed, we have

$$\sum_{\mu\in\mathcal{N}} \Lambda_t^\mu + \sum_{\substack{\mu\neq\nu \\ \mu\in\mathcal{N}}}(\Lambda_t^\nu - \Lambda_t^\mu) \to v(Nb), t \to \infty$$

and

$$\sum_{\mu\in\mathcal{N}} \Lambda_t^\mu + \sum_{\substack{\mu\neq\nu \\ \mu\in\mathcal{N}}}(\Lambda_t^\nu - \Lambda_t^\mu) = N\Lambda_t^\nu$$

Thus

$$\lim_{t\to\infty} \Lambda_t^\nu = vb, \forall \nu$$

□

Part (i) shows that WL exploitation will never disappear even in the limit if there is PPRP and inegalitarian initial distribution of wealth. In other words, the system is exploitative as a whole overall regardless of individual saving behaviors.

The second part of Theorem 3 shows that any RS is asymptotically egalitarian — the sequence of labor allocations converges to equal labor distribution in the limit, irrespective of the asymptotic behavior of the real profit.

As shown in Lemma 4.2, it is possible to have an RS with real profit rate not convergent to zero. In other words, there could exists an RS with PPRP in the limit but no WP_t exploitation there, which is not observed in the setting with stationary prices since profit rate always converges to zero (Veneziani, 2007, Theorem 2, p. 201). The existence of such an RS is another evidence to support our claim in previous section — it is more appropriate to use WL exploitation rather than WP_t exploitation when discussing the relationship between exploitation and positive profit. With Example 1 we have an RS with WP_t exploitation even when there is no PPRP. Theorem 3 here together with Lemma 4.2 show that we could have an RS with PPRP in the limit but no WP_t exploitation.

Persistence of Exploitation. Furthermore, if one adopts the definition of strong persistence which requires the existence of WP_t exploitation in the limit, then Theorem 3 can be interpreted as an impossibility theorem (e.g., Kaneko & Yoshihara, 2019; Veneziani, 2007). However, if we care about not just the limit that will never be reached but the exploitation status within finite periods, then we should also examine the weak persistence, which would be done below.

Proposition 4. *Let $\{(p, w), \Xi\}$ be an RS for $\mathcal{E}(\overline{\Omega})$ with WM. Suppose that there is PPRP and that $\mathcal{W}_t \neq \emptyset$ for all t. Then for any finite t,*

$$\Lambda_t^\nu > vb, \forall \nu \in \mathcal{W}_t$$

Proof. By Lemma 3.2 with PPRP we have $p_t > w_t v$, then

$$\Lambda_t^\nu = \frac{p_t b + p_t \omega_{t+1}^\nu}{w_t} \geqq \frac{p_t b}{w_t} > vb, \forall t, \forall \nu \in \mathcal{W}_t$$

□

Proposition 4 shows that pure workers are exploited persistently as long as PPRP. Therefore, exploitation is persistent in any RS with pure workers all the time. For example, the pure workers at the beginning in any IRS would remain pure workers, so exploitation is persistent in IRS if there is pure workers exploited. In conclusion, if there is PPRP then pure workers are exploited persistently, even though the RS is asymptotically egalitarian.

Both results, the persistence of exploitation and the asymptotically egalitarian labor allocation, provide useful information about the long-term behavior of exploitation. Persistent exploitation captures the Marxian perception of capitalism as an enduring exploitative system — exploitation will never disappear at any time as long as capitalism survives, while asymptotically egalitarian labor allocation tells us more about the asymptotic trends of exploitation.

It means that labor time tends to be equalized, which implies the disappearance of WP_t exploitation in the limit. Moreover, it implies that after some point the degree of exploitation can be arbitrary small, measured by the difference between labor supplied and labor received. In other words, exploitation exists all the time but tends to be smaller and smaller in the long-run, if we agree on this particular measure of the size of exploitation.

These asymptotic trends of exploitation implied by asymptotically egalitarian labor allocation at RS is not contradictory with the Marx's view of capitalism as one stage in human history that is neither sustainable nor permanent. Therefore, it is sufficient to show the persistence of exploitation in the sense that it persists in any finite period to model ensuring exploitation in capitalism. We may ask too much in the search of mechanism to generate RS that is not asymptotically egalitarian as done in current literature. That being said, to study the limiting behavior is also meaningful and informative.

5 Conclusion

In this paper we present an intertemporal model of pr-industrial economies with linear technology to study the persistence of exploitation, and show that pure workers are exploited all the time except in the infinite limit if there is possibility of positive profits. In our point of view, this is sufficient to conclude that exploitation is persistent since it does exists in any period, even though labor allocation tends to be equalized. Therefore, in the subsistence economy with positive real profit and inegalitarian distribution of endowment, exploitation can not just emerge as shown in Roemer (1982b) but also persist without any imperfectness of the market.

We has also explored the relationship among exploitation, positive profits and class status and generalized all of the **FMT**, **PECP** and **CECP** with appropriate definition of UEL exploitation respectively. Contrary to the existing literature focusing on interior RS, our discussion suggests that the distinction between exploitation within period and that in the whole life is useful in general.

There are several limitations of this model. First, we show the long-run behavior of exploitation by that of the prices system, while the latter is derived as a characteristics of the RS — the prices system must have such a property by the definition of RS. It says little about what drives the prices to behavior in this way, and therefore provides little insight on what makes exploitation disappear in the limit.

Second, we only focus on the pre-industrial economy with leisure preference according to which each producer try to minimize labor time after obtaining the subsistence good. The most direct extension of this model is to study the case with monotonic preference (see Appendix C and Galanis et al., 2019) and furthermore the accumulating economies where producer's object function is to maximize revenue (e.g., Roemer, 1982a, Ch. 4). For the accumulating economies, we could expect that labor would become scarce relative to capital accumulated after some point if the size of population is assumed to be fixed, which would drive the profit rate to zero in finite periods. Then we would need some other mechanism to ensure the persistence of exploitation, which is left for further exploration.

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Appendices

A Proofs

Proof of Lemma 2.1. By (ii) and (iii) of Defintion 1, we have

$$A(x_t + y_t) \leqq \omega_t$$
$$(I - A)(x_t + y_t) \geqq Nb + \omega_{t+1} - \omega_t$$

and then

$$\omega_t \geqq (I - A)\overline{\omega} + A\omega_{t+1} \tag{12}$$

Therefore, we have $\omega_{t+1} \geqq \overline{\omega} \Rightarrow \omega_t \geqq \overline{\omega}$. Since $\omega_{T+1} \geqq \omega_1 = \overline{\omega}$, we have $\omega_t \geqq \overline{\omega}$ for all t by induction.

Again, by (12), we have $\omega_t = \overline{\omega} \Rightarrow \omega_{t+1} = \overline{\omega}$. Indeed, $\omega_t = \overline{\omega} \Rightarrow A(\omega_{t+1} - \overline{\omega}) \leqq 0$. Suppose that $\omega_{t+1} \geq \overline{\omega}$, then $A(\omega_{t+1} - \overline{\omega}) \geq 0$ contradicted. Thus, we have $\omega_{t+1} \ngeq \overline{\omega}$. Therefore, $\omega_{t+1} = \overline{\omega}$ since $\omega_t \geqq \overline{\omega}, \forall t$ as shown above. Since $\omega_1 = \overline{\omega}$, we have $\omega_t = \overline{\omega}$ for all t by induction.

Together with $\omega^\nu_{T+1} \geqq \omega^\nu_1$ for all ν, the equality must hold. Indeed, suppose not, then $\omega_{T+1} > \overline{\omega}$, contradicted. □

Proof of Corollary 2.1. Consider the RS such that $x^\nu_t = 0, \forall \nu, \forall t$ without loss of generality, then $y_t = Nb + \overline{\omega} > 0, \forall t$ by (2) of Proposition 1. First, we have $w_t > 0$ from the equilibrium of the labor market. Suppose $w_t = 0$, then $z_t = 0$, contradicted with $z_t = Ly_t > 0$.

Next, we show that $p_t \geqq w_t L$. Suppose, on the contrary, that $p_{jt} < w_t L_j$ for some j, then we have $y^\nu_{jt} = 0$ for all ν. Otherwise, by setting $y^\nu_{jt} = 0$ the agent ν is able to reduce z^ν_t if $\Lambda^\nu \neq 0$, or the increase $p_T \omega^\nu_{T+1}$ when $\Lambda^\nu = 0$ by WM. Therefore, $y_{jt} = 0$, contradicted with $y_t > 0$.

Finally, since $L > 0$ by Assumption 1, $p_t \geqq w_t L > 0$. □

Proof of Lemma 3.1. By Proposition 1, we have $x_t + y_t = Nb + \overline{\omega} > 0, \forall t$. Then for any t, there exist some ν such that $x^\nu_t + y^\nu_t > 0$. Suppose, on the contrary, that $\pi_{jt} > \pi_{it}$, then if $\Lambda^\nu > 0$, we have $x^\nu_{it} + y^\nu_{it} = 0$ contradicted. If $\Lambda^\nu = 0$, by WM, we have $x^\nu_{it} + y^\nu_{it} = 0$ contradicted. □

Proof of Lemma 3.2. From $x_t + y_t > 0$ we have $x^\nu_t + y^\nu_t > 0$ for some ν. Suppose that $(1+\pi_t)p_{j,t-1} < p_{jt}$ for some j. Then it is feasible to replace $x^\nu_t + y^\nu_t$ by some δ^ν_t with $\delta^\nu_{jt} > 0$ to satisfy the capital constraint but making the reproducibility constraint slack. Therefore, we have $(1+\pi_t)p_{t-1} \geqq p_t$, which implies

$$(1+\pi_t)p_{t-1}A \geqq p_t A \Rightarrow p_t - w_t L \geqq p_t A \Rightarrow p_t(I - A) \geqq w_t L$$

Therefore, we have either

$$(1+\pi_t)p_{t-1} = p_t \Rightarrow p_t = w_t v$$

or

$$(1+\pi_t)p_{t-1} \geq p_t \Rightarrow (1+\pi_t)p_{t-1}A \geq p_t A$$

since A is indecomposable and thus $p_t > w_t v$. Finally, $\pi_t > -1$ follows from $(1+\pi_t)p_{t-1} \geqq p_t \geq 0$. □

Proof of Lemma 3.3. Consider a one-period perturbation at period $t+1$, let $\omega'_{t+1} = \omega_{t+1} + \Delta$. Then in the perturbation path,

$$\Lambda'_t + \Lambda'_{t+1} = \Lambda^\nu_t + \Lambda^\nu_{t+1} + \left(\frac{p_t \Delta}{w_t} - \frac{(1+\pi_{t+1})p_t \Delta}{w_{t+1}}\right)$$

By the optimality, we have $\frac{p_t \Delta}{w_t} - \frac{(1+\pi_{t+1})p_t\Delta}{w_{t+1}} \geq 0, \forall \Delta$. Therefore,

$$\frac{p_t}{w_t} = \frac{(1+\pi_{t+1})p_t}{w_{t+1}} \Rightarrow w_{t+1} = (1+\pi_{t+1})w_t$$

□

Proof of Proposition 2. By Lemma 3.1 and 3.3 we have

$$\begin{aligned}(1+\pi_t)p_{t-1} - p_t &= (1+\pi_t)[(1+\pi_{t-1})p_{t-2}A + w_{t-1}L] - (1+\pi_t)p_{t-1}A - w_t L \\ &= (1+\pi_t)[(1+\pi_{t-1})p_{t-2} - p_{t-1}]A\end{aligned}$$

Then

$$(1+\pi_t)p_{t-1} - p_t = \prod_{\tau=2}^{t}(1+\pi_\tau)[(1+\pi_1)p_0 - p_1]A^{t-1} \tag{13}$$

Together with $1 + \pi_t > 0, \forall t$ from Lemma 3.2, we have (6). □

Proof of Lemma 3.4. By Corollary 2.2 and Lemma 3.1, we have

$$\begin{aligned}(1+\pi_t)p_{t-1}\omega_t^\nu &= (1+\pi_t)p_{t-1}A(x_t^\nu + y_t^\nu)\\ &= (p_t - w_t L)(x_t^\nu + y_t^\nu)\\ &= p_t x_t^\nu + (p_t - w_t L)y_t^\nu - w_t L x_t^\nu\\ &= p_t b + p_t\omega_{t+1}^\nu - w_t z_t^\nu - w_t L x_t^\nu\\ &= p_t b + p_t\omega_{t+1}^\nu - w_t \Lambda_t^\nu\end{aligned}$$

Therefore, $\Lambda_t^\nu = \frac{p_t b + p_t\omega_{t+1}^\nu - (1+\pi_t)p_{t-1}\omega_t^\nu}{w_t}$ by $w_t > 0$.

By Lemma 3.3, we have

$$\begin{aligned}&\Lambda_t^\nu + \Lambda_{t+1}^\nu\\ =&\frac{p_t b}{w_t} + \frac{p_t\omega_{t+1}^\nu}{w_t} - \frac{(1+\pi_t)p_{t-1}\omega_t^\nu}{w_t} + \frac{p_{t+1}b}{w_{t+1}} + \frac{p_{t+1}\omega_{t+2}^\nu}{w_{t+1}} - \frac{(1+\pi_{t+1})p_t\omega_{t+1}^\nu}{w_{t+1}}\\ =&\frac{p_t b}{w_t} + \frac{p_t\omega_{t+1}^\nu}{w_t} - \frac{p_{t-1}\omega_t^\nu}{w_{t-1}} + \frac{p_{t+1}b}{w_{t+1}} + \frac{p_{t+1}\omega_{t+2}^\nu}{w_{t+1}} - \frac{p_t\omega_{t+1}^\nu}{w_t}\\ =&\frac{p_t b}{w_t} + \frac{p_{t+1}b}{w_{t+1}} - \frac{p_{t-1}\omega_t^\nu}{w_{t-1}} + \frac{p_{t+1}\omega_{t+2}^\nu}{w_{t+1}}\end{aligned}$$

Therefore,

$$\Lambda^\nu = \sum_{t=1}^T \Lambda_t^\nu = \sum_{t=1}^T \frac{p_t b}{w_t} - \frac{(1+\pi_1)p_{kT-1}\omega_1^\nu}{w_1} + \frac{p_T\omega_{T+1}^\nu}{w_T}$$

□

Proof of Theorem 1. By Lemma 3.2, we have

$$(1+\pi_1)p_0 \geq p_1 \Leftrightarrow (1+\pi_t)p_{t-1} \geq p_t, \forall t \Rightarrow p_t > w_t v, \forall t$$

(i) By Lemma 3.4 (2) and $\omega_{T+1}^\nu = \omega_1^\nu = 0$, we have

$$\Lambda^\nu = \sum_{t=1}^T \frac{p_t b}{w_t} > vTb$$

(ii) By Lemma 3.4 (1) and $\omega_t^\nu = 0$, we have

$$\Lambda_t^\nu = \frac{p_t b + p_t\omega_{t+1}^\nu}{w_t} \geqq \frac{p_t b}{w_t} > \frac{w_t v b}{w_t} = vb$$

(iii) By Lemma 3.4, we have

$$\Lambda_t^\nu = \frac{p_t}{w_t}(b + \omega_{t+1}^\nu - \omega_t^\nu) = v(b + \omega_{t+1}^\nu - \omega_t^\nu)$$

Therefore,

$$\Lambda^\nu = \sum_t \Lambda_t^\nu = vTb + v(\omega_{T+1}^\nu - \omega_1^\nu) = vTb$$

as $\omega_{T+1}^\nu = \omega_1^\nu$ by Lemma 2.1.

(iv) By Definition 2, we have

$$\Lambda_t^\nu = \frac{p_t b - [(1+\pi_t)p_{t-1} - p_t]A(x_t^\nu + y_t^\nu)}{w_t}$$
$$= \frac{p_t b}{w_t} = vb, \quad \forall t, \forall \nu.$$

□

Proof of Lemma 3.5. Let

$$\psi_t \equiv \frac{p_t(Nb) + p_t\overline{\omega} - (1+\pi_t)p_{t-1}\overline{\omega}}{w_t} \tag{14}$$

(1) From the proof of Lemma 3.4, we have

$$\sum_{t=1}^T \psi_t = \sum_{t=1}^T \frac{p_t(Nb)}{w_t} - \frac{(1+\pi_1)p_0\overline{\omega}}{w_1} + \frac{p_t\overline{\omega}}{w_T} = \sum_\nu \Psi(\omega_1^\nu)$$

Therefore, it is sufficient to show that $\psi_t = v(Nb)$. Indeed, we have

$$\begin{aligned}\psi_t &= \frac{p_t(Nb) + p_t\overline{\omega} - (1+\pi_t)p_{t-1}A(I-A)^{-1}(Nb)}{w_t} \\ &= \frac{p_t(Nb) + p_t\overline{\omega} - (p_t - w_tL)(I-A)^{-1}(Nb)}{w_t} \\ &= \frac{p_t}{w_t}[I + A(I-A)^{-1} - (I-A)^{-1}](Nb) + v(Nb) \\ &= v(Nb)\end{aligned}$$

(2) By equation (9) and condition (2) in Theorem 2, we have $z_t = \psi_t$ defined by (14) while $\psi_t = v(Nb)$ as shown above.

□

B Existence of RS

Theorem B1 (Existence of IRS)**.** *Given any sequence of profit rate $\pi = \{\pi_t\}_{t=1}^T$ such that $\pi_t + 1 > 0$ for every t, and any sequence of prices $\mathbf{p}(p_0, w_1, \pi)$, let $(1+\pi_1)\frac{p_0 b}{w_1} \leqq 1$ hold. Let $\Omega \in \mathbf{\Delta}(p_0, w_1, \pi)$ be such that*

$$\max_{\nu\in\mathcal{N}} \max_{1\leqq t\leqq T} \left(\frac{p_t}{w_t} - \frac{p_{t-1}}{w_{t-1}}\right)\omega_1^\nu \leqq \frac{p_T b}{w_T}$$

Then, there exists an IRS for $\mathcal{E}(\Omega)$ associated with the price sequence $\mathbf{p}(p_0, w_1, \pi)$.

Proof. Given the price sequence $\mathbf{p}(p_0, w_1, \pi)$, every agent ν has the following profile of optimal labor supplies:

$$\begin{aligned}
\Lambda_t^{*\nu} &= \frac{p_t b + p_t \omega_{t+1}^{\nu} - (1+\pi_t) p_{t-1} \omega_t^{\nu}}{w_t} \text{ for each } t = 1, \ldots, T; \text{ and} \\
\Lambda^{*\nu} &= \sum_{t=1}^{T} \frac{p_t b}{w_t} - \frac{(1+\pi_1) p_0 \omega_1^{\nu}}{w_1} + \frac{p_T \omega_{T+1}^{\nu}}{w_T} \\
&= \sum_{t=1}^{T} \frac{p_t b}{w_t} - \frac{(1+\pi_1) p_0 \omega_1^{\nu}}{w_1} + \frac{p_T \omega_1^{\nu}}{w_T}.
\end{aligned}$$

By the property of the optimal lifetime labor supply Λ^{ν}, non-saving action $\omega_t^{\nu} = \omega_1^{\nu}$ for each $t = 2, \ldots, T$ is consistent with ν's optimal action. Indeed, this is a necessary condition for IRS. Then, each agent's one-period optimal labor supply at period t is given by

$$\begin{aligned}
\Lambda_t^{*\nu} &= \frac{p_t b}{w_t} + \frac{p_t \omega_1^{\nu}}{w_t} - \frac{(1+\pi_t) p_{t-1} \omega_1^{\nu}}{w_t} \\
&= \frac{p_t b}{w_t} + \left(\frac{p_t}{w_t} - \frac{p_{t-1}}{w_{t-1}} \right) \omega_1^{\nu} \text{ by Lemma 3.3.} \\
&\leqq \frac{p_t b}{w_t} \text{ by } \left(\frac{p_t}{w_t} - \frac{p_{t-1}}{w_{t-1}} \right) \leqq 0 \text{ from Lemma 4.1.} \\
&\leqq \frac{p_0 b}{w_0} \leqq 1 \text{ by } (1+\pi_1) \frac{p_0 b}{w_1} \leqq 1.
\end{aligned}$$

Moreover, as $\max_{\nu \in \mathcal{N}} \max_{1 \leqq t \leqq T} \left(\frac{p_t}{w_t} - \frac{p_{t-1}}{w_{t-1}} \right) \omega_1^{\nu} \leqq \frac{p_T b}{w_T}$ holds, we have $\Lambda_t^{*\nu} \geqq 0$ for each $t = 2, \ldots, T$ and every $\nu \in \mathcal{N}$. Thus, $\left\{ (\Lambda_t^{*\nu})_{\nu \in \mathcal{N}} \right\}_{t=1,\ldots,T} \in Z(p_0, w_1, \pi, \Omega)$ holds.

Then, according to (9), each agent's wealth at each period $t = 1, \ldots, T$ is given by $W_t^{\nu} \equiv p_{t-1} \omega_1^{\nu} \geqq 0$. Moreover, by the definition of Υ, $\sum_{\nu \in \mathcal{N}} W_t^{\nu} = p_{t-1} \overline{\omega}$ holds at each period t. Therefore, by Theorem 2, there exists an IRS for $\mathcal{E}(\Omega)$ associated with the price sequence $\mathbf{p}(p_0, w_1, \pi)$. ☐

Theorem B2 (The existence of RS)**.** *Given any sequence of profit rate $\pi = \{\pi_t\}_{t=1}^{T}$ such that $\pi_t + 1 > 0$ for every t, and any sequence of prices $\mathbf{p}(p_0, w_1, \pi)$, let $(1+\pi_1) \frac{p_0 b}{w_1} \leqq 1$ hold. Let $\Omega \in \mathbf{\Delta}(p_0, w_1, \pi)$ be such that*

$$\max_{\nu \in \mathcal{N}} \max_{1 \leqq t \leqq T} \left(\frac{p_t}{w_t} - \frac{p_{t-1}}{w_{t-1}} \right) \omega_1^{\nu} \leqq \frac{p_T b}{w_T}$$

Let an IRS for $\mathcal{E}(\Omega)$ associated with the price sequence $\mathbf{p}(p_0, w_1, \pi)$ and the path of labor time $\left\{ (\Lambda_t^{*\nu})_{\nu \in \mathcal{N}} \right\}_{t=1,\ldots,T} \in Z(p_0, w_1, \pi, \Omega)$. Let $\left\{ (l_t^{*\nu})_{\nu \in \mathcal{N}} \right\}_{t=1,\ldots,T-1}$ be a sequence of profiles of N real numbers, which satisfies the following condition:

(1) $\sum_{\nu \in \mathcal{N}} l_t^{*\nu} = 0$ *for every* $t = 1, \ldots, T-1$;

(2) $\Lambda_t^{*\nu} + l_t^{*\nu} \in [0, 1]$ *for any* $\nu \in \mathcal{N}$ *and every* $t = 1, \ldots, T-1$;

(3) $\left|\sum_{t=1}^{t'} l_t^{*\nu}\right| \leqq \frac{p_{t'}\omega_1^\nu}{w_{t'}}$ *whenever* $\sum_{t=1}^{t'} l_t^{*\nu} < 0, \forall \nu \in \mathcal{N}$ *and every* $t' = 1, \ldots, T-1$;

(4) $\sum_{t=1}^{T-1} l_t^{*\nu} \leqq 1 - \frac{p_T b}{w_T} - \left(\frac{p_T}{w_T} - \frac{p_{T-1}}{w_{T-1}}\right)\omega_1^\nu$ *and* $\sum_{t=1}^{T-1} l_t^{*\nu} \leqq \frac{p_T b}{w_T} + \left(\frac{p_T}{w_T} - \frac{p_{T-1}}{w_{T-1}}\right)\omega_1^\nu$ *for any* $\nu \in \mathcal{N}$.

Then, there exists a RS associated with the price sequence $\mathbf{p}(p_0, w_1, \pi)$ *and a sequence of labor allocations* $\left\{(\Lambda_t^{*\nu} + l_t^{*\nu})_{\nu\in\mathcal{N},\ t=1,\ldots,T}, \left(\Lambda_T^{*\nu} - \sum_{t=1}^{T-1} l_t^{*\nu}\right)_{\nu\in\mathcal{N}}\right\} \in Z(p_0, w_1, \pi, \Omega)$.

Proof. Given the specified $\left\{(l_t^{*\nu})_{\nu\in\mathcal{N}}\right\}_{t=1,\ldots,T-1}$, there exists a suitable profile $(\triangle\omega_1^\nu)_{\nu\in\mathcal{N}} \in \mathbb{R}^{nN}$ such that $\sum_{\nu\in\mathcal{N}} \triangle\omega_1^\nu = \mathbf{0}$, and $l_1^{*\nu} = \frac{p_1}{w_1}\triangle\omega_1^\nu$ and $p_1\omega_2^\nu \equiv p_1(\omega_1^\nu + \triangle\omega_1^\nu)$ for any $\nu \in \mathcal{N}$. Likewise, there exists a suitable profile $(\triangle\omega_2^\nu)_{\nu\in\mathcal{N}} \in \mathbb{R}_+^{nN}$ such that $\sum_{\nu\in\mathcal{N}} \triangle\omega_2^\nu = \mathbf{0}$, and $l_2^{*\nu} = \frac{p_2}{w_2}(\triangle\omega_1^\nu + \triangle\omega_2^\nu) - \frac{p_1}{w_1}\triangle\omega_1^\nu$ and $p_2\omega_3^\nu \equiv p_1(\omega_1^\nu + \triangle\omega_1^\nu + \triangle\omega_2^\nu)$ for any $\nu \in \mathcal{N}$. Likewise, for $t = 2, \ldots, T-1$, there exists a suitable profile $(\triangle\omega_t^\nu)_{\nu\in\mathcal{N}} \in \mathbb{R}_+^{nN}$ such that $\sum_{\nu\in\mathcal{N}} \triangle\omega_t^\nu = \mathbf{0}$, and $l_t^{*\nu} = \frac{p_t}{w_t}\left(\triangle\omega_1^\nu + \cdots + \triangle\omega_{t-1}^\nu + \triangle\omega_t^\nu\right) - \frac{p_{t-1}}{w_{t-1}}\left(\triangle\omega_1^\nu + \cdots + \triangle\omega_{t-1}^\nu\right)$ and $p_t\omega_{t+1}^\nu \equiv p_t\left(\omega_1^\nu + \triangle\omega_1^\nu + \cdots + \triangle\omega_{t-1}^\nu + \triangle\omega_t^\nu\right)$ for any $\nu \in \mathcal{N}$. Finally, $l_T^{*\nu} = -\frac{p_{T-1}}{w_{T-1}}\left(\triangle\omega_1^\nu + \cdots + \triangle\omega_{T-1}^\nu\right)$ and $p_T\omega_{T+1}^\nu \equiv p_T\left(\omega_1^\nu + \triangle\omega_1^\nu + \cdots + \triangle\omega_{T-1}^\nu + \triangle\omega_T^\nu\right)$ with $\triangle\omega_T^\nu \equiv -\left(\triangle\omega_1^\nu + \cdots + \triangle\omega_{T-1}^\nu\right)$.

Then, for each agent ν, the following new profile of labor supplies $\{\Lambda_t^\nu\}_{t=1,\ldots,T}$ can be defined:

$$\begin{aligned}
\Lambda_1^\nu &= \Lambda_1^{*\nu} + l_1^{*\nu} = \frac{p_1 b}{w_1} + \frac{p_1}{w_1}\omega_2^\nu - \frac{p_0}{w_0}\omega_1^\nu = \frac{p_1 b - (1+\pi_1)\,p_0\omega_1^\nu + p_1\omega_2^\nu}{w_1}; \\
\Lambda_t^\nu &= \Lambda_t^{*\nu} + l_t^{*\nu} = \frac{p_t b}{w_t} + \frac{p_t\omega_{t+1}^\nu}{w_t} - \frac{p_{t-1}\omega_t^\nu}{w_{t-1}} = \frac{p_t b - (1+\pi_t)\,p_{t-1}\omega_t^\nu + p_t\omega_{t+1}^\nu}{w_t} \text{ for } t = 2, \ldots, T-1; \\
\text{and } \Lambda_T^\nu &= \Lambda_T^{*\nu} + l_t^{*\nu} = \frac{p_T b}{w_T} + \frac{p_T\omega_{T+1}^\nu}{w_T} - \frac{p_{T-1}\omega_T^\nu}{w_{T-1}} = \frac{p_T b - (1+\pi_T)\,p_{T-1}\omega_T^\nu + p_T\omega_1^\nu}{w_T}.
\end{aligned}$$

By the definition of $\left\{(l_t^{*\nu})_{\nu\in\mathcal{N}}\right\}_{t=1,\ldots,T-1}$, $0 \leqq \Lambda_t^\nu \leqq 1$ holds for every $t = 1, \ldots, T$ and any $\nu \in \mathcal{N}$, and $\sum_{\nu\in\mathcal{N}} \Lambda_t^\nu = vNb$ holds for every $t = 1, \ldots, T$. Thus, $\left\{(\Lambda_t^\nu)_{\nu\in\mathcal{N}}\right\}_{t=1,\ldots,T} \in Z(p_0, w_1, \pi, \Omega)$ holds.

Then, according to (9), each agent's wealth at each period $t = 1, \ldots, T$ is given by $W_t^\nu \equiv p_{t-1}\omega_t^\nu$. Note that $p_{t-1}\omega_t^\nu = p_{t-1}\left(\omega_1^\nu + \triangle\omega_1^\nu + \cdots + \triangle\omega_{t-1}^\nu\right)$ and $w_{t-1}\left|\sum_{k=1}^{t-1} l_k^{*\nu}\right| = p_{t-1}\left(\triangle\omega_1^\nu + \cdots + \triangle\omega_{t-1}^\nu\right) \leqq p_{t-1}\omega_1^\nu$ holds if $\sum_{k=1}^{t-1} l_k^{*\nu} < 0$. Thus, $p_{t-1}\omega_t^\nu \geqq 0$ holds for any $\nu \in \mathcal{N}$ and every $t = 1, \ldots, T-1$. Moreover, by the definition of Υ, $\sum_{\nu\in\mathcal{N}} W_t^\nu = p_{t-1}\overline{\omega}$ holds at each period t. Therefore, by Theorem 2, there exists a RS for $\mathcal{E}(\Omega)$ associated with the price sequence $\mathbf{p}(p_0, w_1, \pi)$. □

C The Economies with Monotonic Preferences

Suppose that the agent's preference is monotonic instead of leisure preference as we assumed in the main text. Formally, assume the preference is represented by $u(c_t^\nu)$ which is a continuous, strictly increasing, strictly quasi-concave, and homogeneous of degree one. Then the maximization problem becomes

$$\begin{aligned}
\max \quad & \sum_{t=1}^{T} u(c_t^\nu) \\
\text{s.t.} \quad & p_t(x_t^\nu + \delta_t^\nu) + (p_t - w_t L)y_t^\nu + w_t z_t^\nu \geqq p_t c_t^\nu + p_t \omega_{t+1}^\nu \\
& p_{t-1}A(x_t^\nu + y_t^\nu) + p_{t-1}\delta_t^\nu \leq p_{t-1}\omega_t^\nu \\
& Lx_t^\nu + z_t^\nu \leq 1 \\
& \omega_{T+1}^\nu \geqq \omega_1^\nu \\
& x_t^\nu, y_t^\nu, \omega_t^\nu \geqq \mathbf{0}, z_t^\nu \geq 0
\end{aligned}$$

Lemma C.1. *Suppose that* $\{(p, w), \Xi)\}$ *where* $\Xi = (x^\nu, y^\nu, z^\nu, \delta^\nu, c^\nu, \omega^\nu)_{\nu \in \mathcal{N}}$ *is a RS for the economy* $\mathcal{E}(\Omega, u)$*. Then*

$$\frac{u(c_t^\nu)}{p_t c_t^\nu} = (1 + \pi_{t+1}) \frac{u(c_{t+1}^\nu)}{p_{t+1} c_{t+1}^\nu}$$

Proof. Since u is homogeneous of degree one, we have the agent's utility as a linear function of the expenditure at utility maximizers, i.e., $v_t^\nu = k_t e_t^\nu$ where e_t^ν is expenditure and $k_t = \frac{u(c_t^\nu)}{p_t c_t^\nu}$. Now consider a one-period perturbation, if the agent save one dollar in period t, she would loss k_t at time t, but thanks to the return of that one dollar in next period, $(1 + \pi_{t+1})$, she would gain $(1 + \pi_{t+1})k_{t+1}$. Therefore, at RS, we should have

$$k_t = (1 + \pi_{t+1})k_{t+1} \Rightarrow \frac{u(c_t^\nu)}{p_t c_t^\nu} = (1 + \pi_{t+1}) \frac{u(c_{t+1}^\nu)}{p_{t+1} c_{t+1}^\nu}$$

□

Lemma C.1 shows the Euler equation in the economy with monotonic preference. In contrast to the one in the main text, equation (4), it provides no information about the wages sequence. Moreover, if we consider the stationary RS as in Galanis et al. (2019) that $c_t^\nu = c^\nu$ for all t, then there exists a constant α_t such that $p_{t+1} = \alpha_t p_t$ since u is homogeneous degree one. Therefore, the Euler equation becomes

$$\frac{u(c^\nu)}{p_t c^\nu} = (1 + \pi_{t+1}) \frac{u(c^\nu)}{\alpha_t p_t c^\nu} \Rightarrow \alpha_t = (1 + \pi_{t+1})$$

Then $p_{t+1} = (1 + \pi_{t+1})p_t$. In other words, at stationary RS, there is no possibility of real profit rate when there is no positive rate of time preference, the same result observed in Galanis et al. (2019).