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Combinatorics of Equivariant Cohomology: Flags and Regular Nilpotent Hessenberg Varieties

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COMBINATORICS OF EQUIVARIANT COHOMOLOGY:
FLAGS AND REGULAR NILPOTENT HESSENBERG VARIETIES

A Dissertation Presented

by

ELIZABETH DRELLICH

Submitted to the Graduate School of the
University of Massachusetts Amherst in partial fulfillment
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

February 2015

Department of Mathematics and Statistics

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ABSTRACT

COMBINATORICS OF EQUIVARIANT COHOMOLOGY:
FLAGS AND REGULAR NILPOTENT HESSENBERG VARIETIES

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The field of Schubert Calculus deals with computations in the cohomology rings of certain algebraic varieties, including flag varieties and Schubert varieties. In the equivariant setting, GKM theory turns multiplication in the cohomology ring of certain varieties into a combinatorial computation. This dissertation uses combinatorial tools, including Billey's formula, to do Schubert calculus computations in several varieties. First we address do computations in the equivariant cohomology of full and partial flag varieties, the classical spaces in Schubert calculus. We then do computations in the equivariant cohomology of a family of non-classical spaces: regular nilpotent Hessenberg varieties. The final chapter gives a complete presentation for the cohomology ring of the Peterson variety, a type of regular nilpotent Hessenberg variety.

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CHAPTER 1

INTRODUCTION

Schubert calculus, in the style of Goresky-Kottwitz-MacPherson (GKM), uses combinatorial tools to understand the equivariant cohomology of certain algebraic varieties. This thesis presents Schubert-calculus-type calculations with partial flag varieties, regular nilpotent Hessenberg varieties, and Peterson varieties. Partial flag varieties are GKM spaces and thus the traditional spaces with which to do such calculations. Regular nilpotent Hessenberg varieties, of which the Peterson varieties are a subfamily, are not.

In the late 1800's, Hermann Schubert asked questions about intersections of subspaces: given four lines in projective 3-space, how many lines intersect all four? If the four lines are in general position, by which Schubert meant that the four lines are in two intersecting pairs, there will be two lines which intersect all four. Schubert published his results in 1879 [32], but his calculations involved case-by-case dimension counts and his questions were ambiguously posed. When Hilbert proposed his 23 problems in 1900, the fifteenth was to make Schubert's enumerative geometry rigorous.

To make these enumerative geometry calculations rigorous, the field of Schubert calculus needed to pose clearer questions. Instead of asking how many lines intersect a certain number of given lines, we instead needed to ask about the intersections of algebraic varieties called Schubert varieties and Grassmannians. Then the answers to these questions no longer need tweaks and modifications to get the correct dimension count.

By the 1970's, questions of intersections were known to be entirely contained in the questions of the ring and module structure of the cohomology of the varieties [22]. The cohomology of an

algebraic variety encodes the intersections of its subspaces so an understanding of the cohomology is sufficient to answer questions about the number of points in the intersection of Schubert varieties. In the cohomology setting instead of intersecting subspaces one multiplies classes. In modern parlance, to “do Schubert calculus” with a space is to give the structure of its cohomology, complete with generators for the ring, a basis for it as a module, and rules for multiplying within the ring.

Since the 1970’s, mathematicians have done Schubert calculus with a range of spaces. Schubert’s original questions are answered by studying Grassmannians [9][23][29][34] and Schubert varieties [4][5][30]. The field has expanded to include flag varieties which can be treated as a specific case of Schubert variety, as well as partial flag varieties [10][33] and affine Grassmannians [25][26][27].

In 1997, Goresky-Kottwitz-MacPherson published a paper describing an approach to equivariant cohomology that is known as GKM theory [15]. Equivariant cohomology uses additional information from an appropriate group action on the space. For every result in Schubert calculus using ordinary cohomology there is an analogous statement for equivariant cohomology. Other cohomology theories are also used. There is work in quantum and quantum equivariant Schubert calculus as well as work using cobordism and k -theoretic techniques [12].

GKM theory gives a combinatorial structure that provides a module basis for the equivariant cohomology and ring generators for the ring. It says that all of the information about the equivariant cohomology happens at the fixed points of the space. Moreover, explicit combinatorial calculations tell us exactly what occurs at those fixed points. But in order to get these structures the space and group action need to meet certain criteria, which will be discussed Chapter 2. Such pairs are called GKM spaces. This thesis uses applicable parts of GKM theory when appropriate, and reconstructs GKM-style combinatorial structures when the theory does not apply.

This thesis presents results about partial flag varieties and regular nilpotent Hessenberg varieties. Chapter 3 gives a new basis for the cohomology of the flag variety G/B as a product of the cohomology of G/P and P/B where P is any parabolic subgroup of G . In Chapter 4 we

give several results about regular nilpotent Hessenberg varieties, including partial results about a conjectured basis for the equivariant cohomology of regular nilpotent Hessenberg varieties. The last chapter specializes to a particular regular nilpotent Hessenberg variety, the Peterson variety. We give, in all Lie types, a module basis and ring generators for the equivariant cohomology of the Peterson variety, as well as multiplication rules for the ring.

CHAPTER 2

TECHNICAL BACKGROUND

The notation conventions presented in this chapter will be used throughout the document. Certain definitions and theorems used in multiple chapters are also provided here.

2.1 Notation conventions

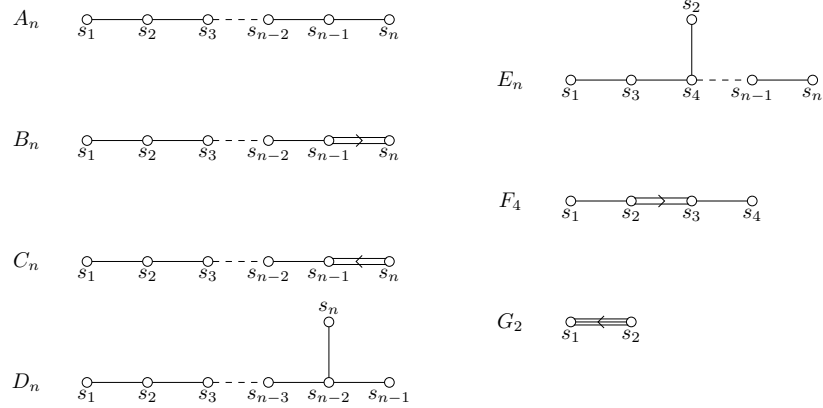
All root systems and Weyl groups are constructed by fixing a complex reductive linear algebraic group G , a Borel subgroup B , and a maximal torus $T \subseteq B \subseteq G$. This choice gives

- a root system Φ
- positive roots $\Phi^+ \subset \Phi$
- simple roots $\Delta \subset \Phi^+$
- an associated Weyl group W
- associated Lie algebras $\mathfrak{t} \subseteq \mathfrak{b} \subseteq \mathfrak{g}$
- root spaces $\mathfrak{g}_\alpha \subset \mathfrak{g}$ for each root $\alpha \in \Phi$.

We also choose a basis element $E_\alpha \in \mathfrak{g}_\alpha$ for each of the root spaces. Some of our constructions rely on a specific ordering of the roots $\alpha_1, \alpha_2, \dots, \alpha_{|\Delta|} \in \Delta$. This ordering is expressed in the Dynkin diagrams of Δ as shown in Figure 1. For Lie type A we will use the convention that $G = GL_n(\mathbb{C})$, B is the set of upper-triangular matrices in G , and T is the set of diagonal matrices in G .

Figure 1: Dynkin diagrams of root systems.

The Dynkin diagrams show the order on the simple reflections. The same order is imposed on the corresponding simple roots throughout this paper.



We recall standard notation about roots and weight lattices as found in Humphreys' Introduction to Lie Algebras and Representation Theory [17, pp 35-37]. The roots α are the (non-trivial) weights of the adjoint representation of \mathfrak{t} . As such they live in the dual \mathfrak{t}^* . Throughout this thesis and especially in Chapters 4 and 5, many computations will take place in the ring $\mathbb{C}[\alpha_i : \alpha_i \in \Delta]$. This ring is a subring of the Cartan subalgebra $\mathbb{C}[\mathfrak{t}^*]$ which we reference in Chapter 3.

2.2 GKM Theory

Named after Goresky-Kottwitz-MacPherson and their 1997 paper, GKM theory allows us to study the equivariant cohomology of certain geometric spaces by looking at polynomials associated to the fixed points of the space [15]. Many others including Braden, Brion, Carrell, Knutson, Rosu, and Tao, contributed to the development of this theory [15].

In order for GKM theory to apply to a space X and a torus group T acting on that space the pair must have three properties:

1. X is equivariantly formal with respect to T ,
2. X contains finitely many points, X^T , which are fixed by T , and
3. X has finitely many one dimensional orbits under the action of T .

If a space X satisfies these properties with respect to a group action of torus T , then it is called a GKM space. In this case GKM theory gives two main properties of the T -equivariant cohomology of X .

1. The T -equivariant cohomology of X injects into the T -equivariant cohomology of the T -fixed points of X .

$$H_T^*(X) \hookrightarrow \bigoplus_{X^T} H_T^*(pt)$$

Moreover a straightforward algebraic calculation describes this injection. Thus we can study the equivariant cohomology $H_T^*(X)$ by looking only at the equivariant cohomology of the T -fixed points of X .

2. The pair of X and T has an associated GKM graph constructed using the set X^T as vertices and the one-dimensional orbits as edges. Equivariant Schubert classes arising from this graph give a basis for $H_T^*(X)$ as a module over $H_T^*(pt)$. A subset of these classes generate the equivariant cohomology as a ring.

When paired with the maximal torus in the Borel subgroup, all of the spaces classically studied by Schubert calculus, including Grassmannians, flag varieties, and Schubert varieties, are GKM spaces. The construction of the GKM graph and the equivariant Schubert classes will be discussed in depth in Sections 2.2.1 and 2.2.2 respectively.

2.2.1 Construction of GKM Graphs

Let X be a GKM space with respect to the group T acting on X . We build an edge-labeled graph \mathcal{G} such that:

- the vertex set $V(\mathcal{G})$ is the set X^T of T -fixed points of X ,
- the edge set $E(\mathcal{G})$ is the set of one (complex) dimensional T orbits in X , and
- each edge is labeled by an element of $\mathbb{C}[\mathfrak{t}^*]$ which we will take more care to define.

Each one-dimensional orbit is parameterized by a character of T . As this parameter tends towards zero or infinity, the orbit approaches a T -fixed point of X . We label the edge between these fixed points by the weight $\alpha \in \mathbb{C}[\mathfrak{t}^*]$ corresponding to that character of T .

Example 2.1. We construct the GKM graph for the flag variety $GL_2(\mathbb{C})/B$ under the action of $T = \begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix}$. The set of flags has coset representatives

$$GL_2(\mathbb{C})/B = \left\{ \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix} : a \in \mathbb{C} \right\} \cup \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

The flags fixed by T are $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Consider a flag that is not fixed by T . For example

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} t_1 & t_1 \\ t_2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{t_1}{t_2} & 1 \\ 1 & 0 \end{bmatrix}.$$

We associate the weight α_1 with the character $\frac{t_1}{t_2}$ and the orbit parameterized by this character is

$$\lim_{\frac{t_1}{t_2} \rightarrow 0} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \lim_{\frac{t_1}{t_2} \rightarrow \infty} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus the GKM graph contains two fixed points and an edge connecting them. We label that edge α_1 to indicate the parameter along that orbit.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bullet \xrightarrow{\alpha_1} \bullet \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Given a choice of B and coset representatives, the edges of the GKM graph are directed corresponding to the limits as $\frac{t_i}{t_j}$ goes to zero and infinity. When the fixed points are permutations, this direction corresponds to the Bruhat order. The vertex at the zero end of each edge is above the vertex at the infinity end in the Bruhat order.

The identification of $\frac{t_1}{t_2}$ with the simple root α_1 uses a formal algebraic isomorphism between the characters of T which are written as a multiplicative group, and the weights of the adjoint representation of \mathfrak{t} which are an additive group. We often conflate characters with weights. We use the characters of the torus action of T to build GKM graphs but once the graphs are constructed all computation occur inside the Cartan subalgebra $\mathbb{C}[\mathfrak{t}^*]$.

Example 2.2. When there are multiple one-dimensional orbits, labeling the edges of the graph becomes more complicated. The GKM space $GL_3(\mathbb{C})/B$ is a larger example that demonstrates several points.

1. Not all points of $GL_3(\mathbb{C})/B$ are in one-dimensional T orbits. For example

$$\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} t_1 & t_1 & t_1 \\ t_2 & 0 & 0 \\ 0 & t_3 & 0 \end{bmatrix} = \begin{bmatrix} \frac{t_1}{t_2} & \frac{t_1}{t_3} & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

is a two-dimensional orbit.

2. Multiple edges can have the same parameter. As before, the weight α_1 is associated with the character $\frac{t_1}{t_2}$. There are three distinct one-dimensional orbits parameterized by the $\frac{t_1}{t_2}$.

These orbits are:

$$\begin{bmatrix} \frac{t_1}{t_2} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \frac{t_1}{t_2} & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \frac{t_1}{t_2} & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

3. Not all edges have the same parameter. The weight α_2 corresponds to the character $\frac{t_2}{t_3}$.

The three one-dimensional orbits parameterized by $\frac{t_2}{t_3}$ are:

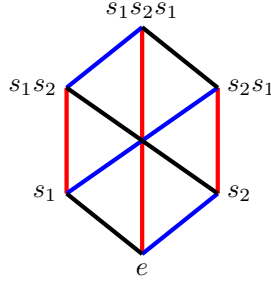
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{t_2}{t_3} & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ \frac{t_2}{t_3} & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ \frac{t_2}{t_3} & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

4. The weight associated to the parameter $\frac{t_1}{t_3}$ is $\alpha_1 + \alpha_2$ and its orbits are:

$$\begin{bmatrix} \frac{t_1}{t_3} & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \frac{t_1}{t_3} & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} \frac{t_1}{t_3} & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

5. The T -fixed points are indexed by a combinatorial object. In the case of $GL_3(\mathbb{C})/B$ the fixed points are flags that can be represented by matrices with one 1 in each row and each column. These are the permutation matrices, so the fixed points correspond to permutations on three elements.

We can write the permutations corresponding to the fixed points in terms of the simple reflections $s_1 = (1, 2)$ and $s_2 = (2, 3)$. The GKM graph for $GL_3(\mathbb{C})/B$ is



Here colors indicate the edge labels: black for α_1 , blue for α_2 , and red for $\alpha_1 + \alpha_2$.

We frequently draw the GKM graph without writing the fixed points which index the vertices. In our next construction we will label the vertices of the GKM graph and make a distinction between the fixed point and the label on the corresponding vertex.

2.2.2 Equivariant Schubert Classes as Generalized Splines

Definition 2.3. Let R be an arbitrary (commutative) ring and G a graph with edges labeled by ideals \mathcal{I} of R . A **generalized spline** on the edge-labeled graph is a set of labels on the vertices of G such that

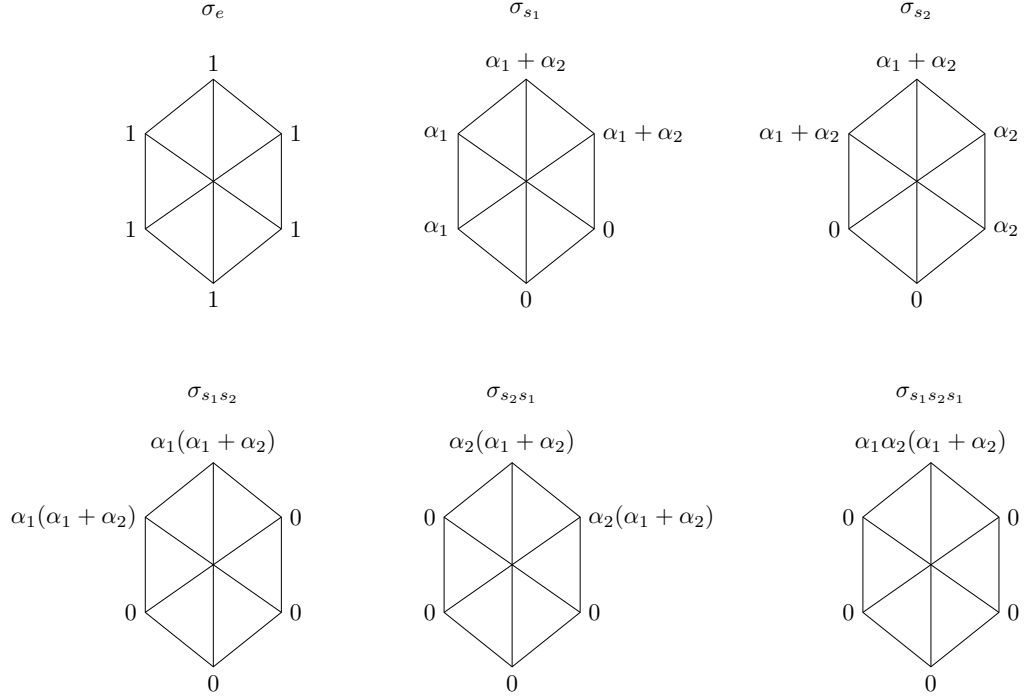
- each vertex v is labeled by an element of R
- if a and b are labels of a pair of vertices connected by an edge with label $\mathcal{I} \subseteq R$ then $a - b$ is in \mathcal{I} .

Let X be a GKM space with torus action T and GKM graph \mathcal{G} . Choose the ring $R = \mathbb{C}[\alpha_1, \dots, \alpha_n] \cong H_T^*(pt)$ and to each edge of \mathcal{G} associate the ideal $\mathcal{I} \subseteq R$ generated by its label. The set of generalized splines on the GKM graph \mathcal{G} is both a module and a ring over $\mathbb{C}[\alpha_1, \dots, \alpha_n]$. It is isomorphic as both a ring and a module to $H_T^*(X)$ which is a module over $H_T^*(pt)$. It is also a free module over $H_T^*(pt)$ and thus any basis will consist of $|X^T|$ elements, each of which is a generalized spline on the GKM graph \mathcal{G} .

Theorem 2.4 (Existence of the Schubert Basis [14]). Let $P \subseteq G$ be a parabolic subgroup. Then $X = G/P$ is a GKM space with respect to the action of T and a module basis of $H_T^*(X)$ can be given explicitly. There is a specific set of $|X^T|$ splines on the GKM graph \mathcal{G} called equivariant Schubert classes that form a basis of $H_T^*(X)$. These classes are indexed by the same combinatorial object as the fixed points.

A Schubert class is denoted σ_v where v is a vertex of \mathcal{G} .

Example 2.5. Continuing Example 2.2, the T -equivariant Schubert classes of $GL_3(\mathbb{C}/B)$ are indexed by the six elements of \mathfrak{S}_3 . The edge labels are understood to be those given in Example 2.2.



The index v of the equivariant Schubert class σ_v corresponds to the smallest (in Bruhat order) vertex which has a non-zero label.

The polynomial labeling the vertex w in the equivariant Schubert class σ_v is called $\sigma_v(w)$.

2.2.3 Billey's formula

We now restrict our focus to flag varieties G/B with the action of maximal torus T . In this case the T -fixed points correspond to the elements of the Weyl group W and the parameters on the one-dimensional orbits correspond to positive roots of Φ . If there are m simple roots in Φ then the GKM graph is an m -regular graph on $|W|$ vertices. Furthermore the Schubert classes are indexed by elements of W and, as demonstrated in Example 2.5, the index of the Schubert class is the smallest fixed point in that class with a non-zero label.

In the case of G/B it is possible to say exactly what each Schubert class is. Billey gave an explicit combinatorial formula for the polynomial $\sigma_v(w)$ at the fixed point w .

Definition 2.6 (Billey's Formula [4]). *Fix a reduced word for $w = s_{b_1}s_{b_2}\cdots s_{b_{\ell(w)}}$ and define $\mathbf{r}(\mathbf{i}, w) = s_{b_1}s_{b_2}\cdots s_{b_{i-1}}(\alpha_{b_i})$. Then*

$$\sigma_v(w) = \sum_{\substack{\text{reduced words} \\ v = s_{b_{j_1}}s_{b_{j_2}}\cdots s_{b_{j_{\ell(v)}}}} \binom{\ell(v)}{\prod_{i=1}^{\ell(v)} \mathbf{r}(\mathbf{j}_i, w)}. \quad (2.1)$$

Proposition 2.7 (Billey [4]). *Properties of the polynomial $\sigma_v(w)$:*

- *The polynomial $\sigma_v(w)$ is homogeneous of degree $\ell(v)$.*
- *If $v \not\leq w$ then $\sigma_v(w) = 0$.*
- *If $v \leq w$ then $\sigma_v(w) \neq 0$.*
- *The coefficients of $\sigma_v(w)$ are non-negative integers.*
- *The polynomial $\sigma_v(w)$ does not depend on the choice of reduced word for w .*

When v and w are words of relatively short length it is simple to calculate $\sigma_v(w)$ by hand.

Example 2.8. Let G/B have Weyl group $W = A_2$ and let $w = s_1s_2s_1$ and $v = s_1$. The word v is found as a subword of $s_1s_2s_1$ in the two places $\mathbf{s}_1s_2s_1$ and $s_1s_2\mathbf{s}_1$.

$$\sigma_v(w) = \mathbf{r}(\mathbf{1}, s_1s_2s_1) + \mathbf{r}(\mathbf{3}, s_1s_2s_1) = \alpha_1 + s_1s_2(\alpha_1) = \alpha_1 + \alpha_2$$

Using Billey's formula the entire basis of Schubert classes for the T -equivariant cohomology of any flag variety can be computed directly.

2.3 Hessenberg varieties

Hessenberg varieties are a large family of subvarieties of the flag variety G/B . They are defined by two parameters: a subspace of the Lie algebra and an element of the Lie algebra. For any Lie algebra \mathfrak{g} the relevant subspaces can be defined by root spaces.

Definition 2.9. *A Hessenberg space H is a subspace of the Lie algebra \mathfrak{g} which*

- *contains \mathfrak{b} and*

- is closed under the Lie bracket with \mathfrak{b} .

In type A_{n-1} a Hessenberg space H can be presented as either an $n \times n$ matrix or as a **Hessenberg function** $h : [n] \rightarrow [n]$. The matrix presentation of H in type A has the properties

- every entry on the diagonal is allowed to be non-zero and
- whenever an entry in H must be non-zero, all entries to the south and west must also be zero.

A Hessenberg function has the property $h(i) \geq \max\{i, h(i-1)\}$ for all $i = 1, 2, \dots, n$. The value $h(i)$ is the number of free variables in the i^{th} column of the matrix H .

Example 2.10. There are 5 Hessenberg spaces in $GL_3(\mathbb{C})$.

$$\begin{array}{ccccc}
 \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} & \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix} & \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} & \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & * & * \end{bmatrix} & \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \\
 H_1 & H_2 & H_3 & H_4 & H_5 \\
 h_1(i)=i & h_2(1)=h_2(2)=2 & h_3(1)=1 & h_4(1)=2 & h_5(i)=n \\
 & h_2(3)=3 & h_3(2)=h_3(3)=3 & h_4(2)=h_4(3)=3 &
 \end{array}$$

The Hessenberg space is one of two parameters which define a Hessenberg variety.

Definition 2.11. Let $X \in \mathfrak{g}$ be a linear operator and $H \subset \mathfrak{g}$ be a Hessenberg space. The corresponding **Hessenberg variety** is defined

$$\text{Hess}(X, H) = \{gB \in G/B : \text{Ad}(g^{-1})X \in H\}. \tag{2.2}$$

There are many families within the Hessenberg varieties. Specially named Hessenberg varieties include regular semisimple and regular nilpotent Hessenberg varieties, Springer varieties, and Peterson varieties.

Using the spaces from Example 2.10, the Hessenberg variety $\text{Hess}(X, H_1)$ is called a Springer variety. The variety $\text{Hess}(X, H_5)$ is the full flag variety.

In this dissertation we focus on regular nilpotent Hessenberg varieties, a family for which the operator X is a regular nilpotent operator. In the last chapter, we will focus on Peterson varieties, a subfamily of regular nilpotent Hessenberg varieties.

CHAPTER 3

A PRODUCT DECOMPOSITION OF $H_T^*(G/B)$

Classical Schubert calculus studies the cohomology of flag and partial flag varieties. Let $P \subset G$ be a parabolic subgroup. Then G/P is a partial flag variety. Like the full flag variety G/B when B is a Borel subgroup, partial flag varieties are GKM spaces under the action of the maximal torus T . A parabolic subgroup $P \subset G$ contains B and gives rise to two new varieties: P/B and G/P . The second one, G/P , is called a partial flag variety.

Since T acts on the variety P/B as well, we can study the T -equivariant cohomology of all three of these varieties. By the Künneth theorem the three cohomology rings are related:

$$H_T^*(G/B) \cong H_T^*(G/P) \otimes H_T^*(P/B).$$

This chapter gives a new proof of this fact via an explicit product of Schubert classes. We define $H_T^*(G/P)$ and $H_T^*(P/B)$ as submodules of $H_T^*(G/B)$ and give module bases for them in terms of Schubert classes. We then give an explicit module isomorphism.

Theorem 3.1. *Let P be a parabolic subgroup of G . Then the map*

$$\begin{array}{ccc} H_T^*(G/P) \otimes H_T^*(P/B) & \rightarrow & H_T^*(G/B) \\ p \otimes q & \mapsto & pq \end{array}$$

is a bilinear isomorphism.

This is an equivariant version of the Leray-Hirsch theorem. Guillemin-Sabatini-Zara showed that Leray-Hirsch holds in the equivariant setting for GKM spaces using a construction on the GKM graphs [16, Theorem 3.5], we will do it explicitly using a different technique which involves Schubert classes. These techniques which we introduce for the proof of Theorem 3.1, we will later apply to computations involving Hessenberg varieties. One of these varieties will be discussed in depth in Section 4.3.4.

3.1 $H_T^*(G/P)$ and $H_T^*(P/B)$ as submodules of $H_T^*(G/B)$

As we have already discussed, the set of Schubert classes $\{\sigma_v : v \in W\}$ is a module basis for $H_T^*(G/B)$ over $H_T^*(pt)$. Let W_P be the subgroup of the Weyl group W generated by the simple reflections $s_i \in P$ and let W^P be the set of minimal coset representatives of W/W_P . The group W_P is the Weyl group corresponding to P/B .

Classical GKM theory applies directly to P/B and says that the set $\{\sigma_v|_{W_P} : v \in W_P\}$ is a basis of the T -equivariant cohomology of P/B . The submodule of $H_T^*(G/B)$ generated by the set $\{\sigma_v : v \in W_P\}$ is isomorphic to the module $H_T^*(P/B)$. Therefore we can think of the equivariant cohomology of P/B as

$$H_T^*(P/B) \cong \text{span}\{\sigma_v : v \in W_P\} \subset H_T^*(G/B).$$

By applying GKM theory to the partial flag variety G/P , the set $\{\sigma_w|_{W^P} : w \in W^P\}$ is a module basis for $H_T^*(G/P)$ [36]. We can consider the corresponding Schubert classes in the larger ring $H_T^*(G/B)$ and think of the equivariant cohomology of G/P as

$$H_T^*(G/P) \cong \text{span}\{\sigma_w : w \in W^P\} \subset H_T^*(G/B).$$

Whereas the presentation for $H_T^*(P/B)$ is also a ring isomorphism, this presentation of $H_T^*(G/P)$ is strictly a module isomorphism. These inclusions into $H_T^*(G/B)$ are the presentations we use to prove Theorem 3.1.

3.2 Proof of Theorem 3.1

We prove the main theorem of this chapter by showing that the product of the basis of $H_T^*(G/P)$ and the basis of $H_T^*(P/B)$ is a basis for $H_T^*(G/B)$. We write this product as

$$\{\sigma_w\sigma_v : w \in W^P, v \in W_P\}.$$

The class $\sigma_w\sigma_v$ has polynomial degree $\ell(w) + \ell(v)$ and each word in W can be written uniquely as wv for some $w \in W^P$ and $v \in W_P$ [6]. Thus for any m the collection $\{\sigma_w\sigma_v : w \in W^P, v \in W_P\}$ has the same number of classes of polynomial degree m as the standard Schubert class basis $\{\sigma_v : v \in W\}$. Therefore to prove Theorem 3.1 it suffices to prove that the collection of classes $\{\sigma_w\sigma_v : w \in W^P, v \in W_P\}$ is linearly independent over $\mathbb{C}[t^*]$.

Theorem 3.2. *The set of Schubert class products $\{\sigma_w\sigma_v : w \in W^P, v \in W_P\}$ is a linearly independent set over $\mathbb{C}[t^*]$.*

In this section we will prove Theorem 3.2 by arranging these products in a matrix A with entries $\sigma_w(w'v')\sigma_v(w'v')$. The columns of this matrix correspond to pairs $(w', v') \in W^P \times W_P$ and the rows correspond to pairs (w, v) in the same set. In this way the columns of matrix A are indexed by T -fixed points and the rows correspond to products of Schubert classes $\sigma_w\sigma_v$.

We begin by establishing an order on $W^P \times W_P$. The elements of both W^P and W_P are partially ordered by length; fix a total order on W^P (respectively W_P) consistent with this partial order and extend this lexicographically to all of $W^P \times W_P$. For instance all rows and columns corresponding to pairs in (e, W_P) come before any pair in (s_i, W_P) .

For the remainder of this chapter we will consider the matrix A to have rows and columns ordered as above.

3.2.1 Key lemmas

The proof of Theorem 3.2 is at the end of this section. We begin with two lemmas. The first will prove that given the above ordering of its rows and columns, the matrix $A = (\sigma_w(w'v')\sigma_v(w'v'))$ is block upper-triangular. The second lemma will construct a matrix $M \cdot wN$ where M is an invertible matrix and wN is known to have linearly independent rows and columns. We can then prove Theorem 3.2 by showing that the diagonal blocks of A are scalar multiples of the matrix $M \cdot wN$.

Lemma 3.3. *The matrix $(\sigma_w(w'v')\sigma_v(w'v'))_{(w,v),(w',v') \in W^P \times W_P}$ is block upper-triangular.*

Proof. Choose $w, w' \in W^P$. Consider the blocks of A whose rows are indexed by pairs in (w, W_P) and whose columns are indexed by pairs in (w', W_P) . By construction this is a square $|W_P| \times |W_P|$ block. Its entries are $(\sigma_w(w'v')\sigma_v(w'v'))$ where v, v' range over all of W_P . The last letter in every reduced word for $w' \in W^P$ is a simple reflection $s_i \notin W_P$ (a fact shown by many, including Bjorner and Brenti [6]). Thus every reduced word for $w \in W^P$ inside $w'v$ is in fact in the prefix w' . The term $\sigma_w(w'v')$ thus equals $\sigma_w(w')$ which is zero unless $w \leq w'$. Therefore whenever $\ell(w) \geq \ell(w')$ and $w \neq w'$ the entire block is zero. \square

Example 3.4. Consider the varieties G/P and P/B in $GL_3(\mathbb{C})/B$ corresponding to $W_P = \langle s_2 \rangle$ and $W^P = \{e, s_1, s_2s_1\}$. Let σ_{W_P} denote $\{\sigma_v : v \in W_P\}$. Then the blocks of the matrix $(\sigma_w(w'v')\sigma_v(w'v'))$ are

$$\begin{array}{c} eW_P \quad s_1W_P \quad s_2s_1W_P \\ \sigma_e\sigma_{W_P} \\ \sigma_{s_1}\sigma_{W_P} \\ \sigma_{s_2s_1}\sigma_{W_P} \end{array} \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.$$

Example 3.5. This example treats pairs $w, w' \in W^P$ with the same length. Let W_P be the parabolic subgroup $\langle s_3 \rangle \subset \langle s_1, s_2, s_3 \rangle = A_3$. The elements of W^P with length two are s_1s_2, s_2s_1 , and s_3s_2 . The blocks of $(\sigma_w(w'v')\sigma_v(w'v'))$ where w, w' have length two have the form

$$\begin{array}{c} s_1s_2W_P \quad s_2s_1W_P \quad s_3s_2W_P \\ \sigma_{s_1s_2}\sigma_{W_P} \\ \sigma_{s_2s_1}\sigma_{W_P} \\ \sigma_{s_3s_2}\sigma_{W_P} \end{array} \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

In the next lemma we show that the rows of the diagonal blocks of the matrix $(\sigma_w(w'v')\sigma_v(w'v'))$ are linearly independent. It is not immediately obvious that the matrices in this lemma are in fact the diagonal blocks; that result is part of the content of the main theorem.

Lemma 3.6 (Linear independence of diagonal blocks). *Fix $w \in W^P$. Assume that the elements of W_P are ordered consistently with the partial order on length. Let M be the matrix defined by*

$$M_{vu} = \begin{cases} \sigma_{vu^{-1}}(w) & \text{if } u \text{ is a suffix of } v \\ 0 & \text{otherwise.} \end{cases}$$

Define the matrix N by $N = (\sigma_u(v'))_{u, v' \in W_P}$. Consider the algebra isomorphism $w : \mathbb{C}[\mathfrak{t}^] \rightarrow \mathbb{C}[\mathfrak{t}^*]$ induced from the action $t_\alpha \mapsto t_{w(\alpha)}$. Denote the image of N under this action of w by wN . Then the rows of the matrix $M \cdot wN$ are linearly independent over $\mathbb{C}[\mathfrak{t}^*]$.*

Note that w does not permute the rows or columns of N .

Proof. If $\ell(u) > \ell(v)$ then by construction $M_{vu} = 0$. If $\ell(u) = \ell(v)$ then $M_{vu} = 0$ unless $v = u$. Therefore M is an upper-triangular matrix. The entries on the diagonal have the form $M_{vv} = \sigma_e(w) = 1$. Since 1 is a unit in $\mathbb{C}[\mathfrak{t}^*]$ the matrix M is invertible.

Note that $N = (\sigma_u(v'))_{u,v' \in W_P}$ is the matrix of Schubert classes in $H_T^*(P/B)$. The rows of N are the Schubert class basis for $H_T^*(P/B)$ so the rows and columns of matrix N are linearly independent. The function w acts on the matrix N by sending each t_α to $t_{w(\alpha)}$. This operation is invertible and so preserves linear independence of the matrix rows. Thus the new matrix wN also has linearly independent rows.

Since M is invertible over $\mathbb{C}[\mathfrak{t}^*]$ and wN has linearly independent rows over $\mathbb{C}[\mathfrak{t}^*]$ the rows of the matrix product $M \cdot wN$ are also linearly independent over $\mathbb{C}[\mathfrak{t}^*]$. \square

3.2.2 Proof of Theorem 3.2

We now show that each of the diagonal blocks of A identified in Lemma 3.3 is a scalar multiple of the matrix $M \cdot wN$ defined by Lemma 3.6. This proves that the rows of matrix A are linearly independent and thus the collection of Schubert class products $\{\sigma_w \sigma_v : w \in W^P, v \in W_P\}$ is linearly independent over $\mathbb{C}[\mathfrak{t}^*]$.

Proof. Consider the matrix $A = (\sigma_w(w'v')\sigma_v(w'v'))_{(w,v),(w',v') \in W^P \times W_P}$ with rows and columns ordered lexicographically subordinate to the length partial order on W^P and W_P described above.

Partition the matrix A into blocks according to the pairs $w, w' \in W^P$. Lemma 3.3 proved that the matrix $(\sigma_w(w'v')\sigma_v(w'v'))$ is block-upper-triangular with this partition. Now consider the blocks along the diagonal, namely the blocks of the form

$$(\sigma_w(wv')\sigma_v(wv'))_{v,v' \in W_P} = \sigma_w(w) \cdot (\sigma_v(wv'))_{v,v' \in W_P}$$

for each $w \in W^P$. Billey's formula guarantees that $\sigma_w(w)$ is non-zero so it suffices to consider the matrix $(\sigma_v(wv'))_{v,v' \in W_P}$. We will show that

$$(\sigma_v(wv'))_{v,v' \in W_P} = M \cdot wN$$

where M and wN are the matrices of Lemma 3.6. Multiplying matrices gives $M \cdot wN =$

$$\begin{aligned}
& \left\{ \begin{array}{c} v \text{ ranges over } W_P \\ \left(\begin{array}{c} \overbrace{M_{vu}}^{u \text{ ranges over } W_P} \end{array} \right) \end{array} \right\} \cdot \left\{ \begin{array}{c} u \text{ ranges over } W_P \\ \left(\begin{array}{c} \overbrace{w(\sigma_u(v'))}^{v' \text{ ranges over } W_P} \end{array} \right) \end{array} \right\} \\
&= \left\{ \begin{array}{c} v \text{ ranges over } W_P \\ \left(\begin{array}{c} \overbrace{\sum_{u \in W_P} M_{vu} \cdot w(\sigma_u(v'))}^{v' \text{ ranges over } W_P} \end{array} \right) \end{array} \right\}
\end{aligned}$$

Next we show that for any $v, v' \in W_P$ the polynomial $\sigma_v(wv')$ can be decomposed as the sum $\sum_{u \in W_P} M_{vu} \cdot w(\sigma_u(v'))$. To do this, consider Billey's formula for $\sigma_v(wv')$ and group terms according to which part of v is a subword of w and which part is a subword of v' . More precisely:

$$\sigma_v(wv') = \sum_{\substack{u \text{ a suffix} \\ \text{of } v}} \overbrace{\sigma_{vu^{-1}}(w)}^{\text{part of } v \text{ found in } w} \cdot \underbrace{w\sigma_u(v')}_{\text{part of } v \text{ found in } v'}$$

By construction of M we have

$$\sum_{u \in W_P} M_{vu} \cdot w(\sigma_u(v')) = \sum_{\substack{u \text{ a suffix} \\ \text{of } v}} \sigma_{vu^{-1}}(w) \cdot w\sigma_u(v') = \sigma_v(wv').$$

Therefore the matrix $(\sigma_v(wv'))_{v, v' \in W_P}$ is equal to $M \cdot wN$ as desired.

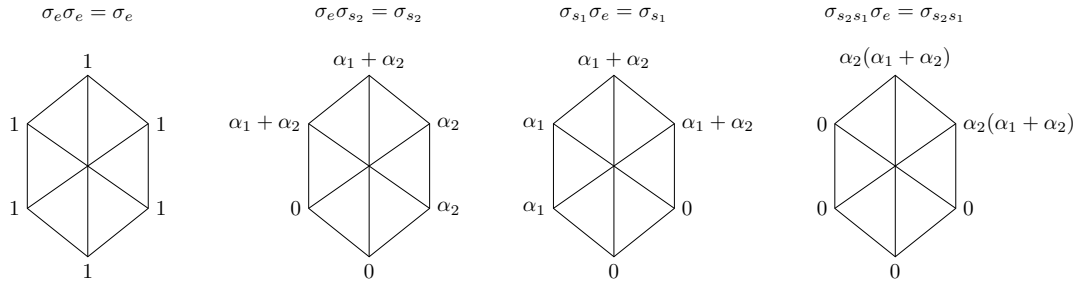
By Lemmas 3.3 and 3.6 the rows of matrix A are linearly independent over $\mathbb{C}[\mathfrak{t}^*]$. Thus the Schubert class products $\{\sigma_w \sigma_v : w \in W^P, v \in W_P\}$ are linearly independent over $\mathbb{C}[\mathfrak{t}^*]$. \square

As discussed at the beginning of Section 3.2 the map $W^P \times W_P \rightarrow W$ given by $(w, v) \mapsto wv$ is a bijection. The degree of the homogeneous class $\sigma_w \sigma_v$ is by definition $\ell(w) + \ell(v)$ so the homogeneous classes $\{\sigma_w \sigma_v : w \in W^P, v \in W_P\}$ form a linearly-independent set in $H_T^*(G/B)$ of the same degrees as the equivariant Schubert classes. Thus the collection of Schubert class products $\{\sigma_w \sigma_v : w \in W^P, v \in W_P\}$ form a basis for $H_T^*(G/B)$. This completes the proof of Theorem 3.1.

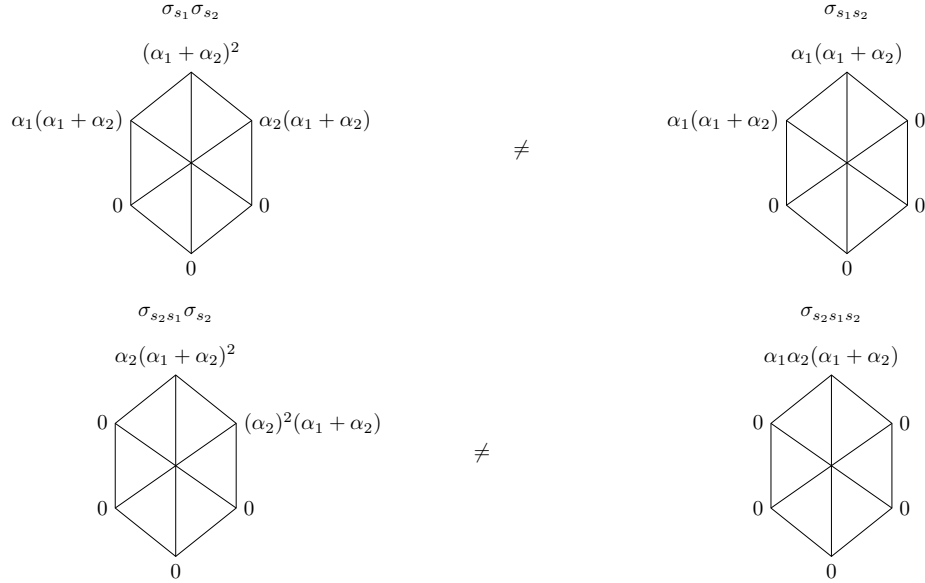
3.3 The Composite Basis for $H_T^*(G/B)$

The basis for $H_T^*(G/B)$ given by Theorem 3.1 is not the classical Schubert basis. For example when $G = GL_3(\mathbb{C})$ there are two classes that differ between this basis and the Schubert basis.

Example 3.7. This example uses $H_T^*(GL_3/B)$ to demonstrate that the basis $\{\sigma_w\sigma_v\}$ is not equal to the Schubert basis. We continue the example $W_P = \langle s_2 \rangle$ and $W^P = \{e, s_1, s_2s_1\}$. Four of the classes $\{\sigma_w\sigma_v : w \in W^P \text{ and } v \in W_P\}$ are also Schubert classes:



The remaining two classes are not Schubert classes.



The class $\sigma_{s_1}\sigma_{s_2}$ is equal to $\sigma_{s_1s_2} + \sigma_{s_2s_1}$ and the class $\sigma_{s_2s_1}\sigma_{s_1}$ is equal to $\sigma_{s_1s_2s_1} + \alpha_2\sigma_{s_2s_1}$.

If neither u nor v is the identity then $\sigma_u\sigma_v$ will generally not be a Schubert class.

CHAPTER 4

REGULAR NILPOTENT HESSENBERG VARIETIES

Regular nilpotent Hessenberg varieties are a family of Hessenberg varieties that have been studied in depth by Harada-Tymoczko [19], Brion-Carrell [7] and Peterson [unpublished]. These varieties inherit a significant amount of structure from the flag variety and have interesting internal symmetries.

Definition 4.1. *A regular nilpotent Hessenberg variety is a Hessenberg variety where the operator X is a regular nilpotent operator in \mathfrak{g} . Explicitly regular nilpotent Hessenberg varieties have the form $\text{Hess}(N, H)$ where*

$$N = \sum_{\alpha \in \Delta} E_{\alpha}. \tag{4.1}$$

In Lie type A_{n-1} the Jordan normal form of the operator N is the $n \times n$ matrix with ones on the upper diagonal and zeros in all other entries.

Since we are fixing one parameter of the Hessenberg variety, each Hessenberg space corresponds to exactly one regular nilpotent Hessenberg variety.

Example 4.2. There are 5 Hessenberg spaces in $GL_3(\mathbb{C})$ corresponding to the 5 type- A_2 regular nilpotent Hessenberg varieties.

$$\begin{array}{ccccc}
 \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} &
 \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix} &
 \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} &
 \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & * & * \end{bmatrix} &
 \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \\
 \mathcal{H}_1 & \mathcal{H}_2 & \mathcal{H}_3 & \mathcal{H}_4 & \mathcal{H}_5
 \end{array}$$

The variety $\text{Hess}(N, \mathcal{H}_1)$ is the regular nilpotent Springer variety, $\text{Hess}(N, \mathcal{H}_4)$ is the type- A_2 Peterson variety, and $\text{Hess}(N, \mathcal{H}_5) = GL_3(\mathbb{C})/B$ is the flag variety.

In this chapter we work exclusively in type A , although in the last chapter we will discuss the Peterson variety in all Lie types. In type A it is convenient to describe $Hess(N, H)$ in terms of flags. Recall that the regular nilpotent Hessenberg variety corresponding to a Hessenberg function h is

$$Hess(N, \mathcal{H}) = \{V_\bullet = V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n : NV_i \in V_{h(i)}\}. \quad (4.2)$$

We will also fix a basis for \mathbb{C}^n to simplify computations. Given a regular nilpotent operator N , there a basis with respect to which N is a single Jordan block corresponding to the 0 eigenvalue. We fix this to be our standard basis for the rest of this chapter. This will make some of the proofs in this chapter simpler by letting us use matrix notation to discuss vectors and flags explicitly.

4.1 Isomorphic Varieties

Given the set of regular nilpotent Hessenberg varieties, a natural first question is whether there are families of related varieties within the set. In Example 4.2 there is a clear relationship between the Hessenberg spaces \mathcal{H}_2 and \mathcal{H}_3 since flipping one of these spaces along the antidiagonal produces the other space. Not just the Hessenberg spaces are related; the associated Hessenberg varieties are isomorphic.

Theorem 4.3. *Let $H \subset GL_n(\mathbb{C})$ be a Hessenberg space and let ${}^T H$ be the Hessenberg space obtained by flipping along the antidiagonal. Then*

$$Hess(N, H) \cong Hess(N, {}^T H).$$

Proof. Let w_0 be the longest word in \mathfrak{S}_n . The permutation matrix corresponding to w_0 has ones on the antidiagonal and all other entries are zero. Any $n \times n$ matrix M can be flipped along its antidiagonal by taking its transpose and conjugating by the longest word:

$${}^T M = w_0 M^T w_0. \quad (4.3)$$

This operation can be done on subspaces of $n \times n$ matrices as well. The space ${}^T H = w_0 H^T w_0$ is a Hessenberg space whenever H is a Hessenberg space. To show that $Hess(N, H)$ is isomorphic to $Hess(N, {}^T H)$ we give the following map:

$$\begin{aligned} \phi: G/B &\rightarrow G/B \\ gB &\mapsto w_0(g^T)^{-1}w_0B \end{aligned} \quad (4.4)$$

To see this map is well-defined, suppose, $g_1 = g_2b$ for some $b \in B$. Then

$$\phi(g_1B) = w_0(g_1^T)^{-1}w_0B = w_0((g_2b)^T)^{-1}w_0B = w_0(g_2^T)^{-1}(b^T)^{-1}w_0B.$$

The term $(b^T)^{-1}$ is in B and $Bw_0 = w_0B$, (both are the set matrices in $GL_n(\mathbb{C})$ with zeros below the antidiagonal). So $\phi(g_1B) = w_0(g_2^T)^{-1}w_0B = \phi(g_2B)$ and the map ϕ is well-defined.

Since we are in type A , gB is in $\text{Hess}(N, H)$ if and only if $g^{-1}Ng \in H$. Observing that the matrix $w_0Nw_0 = N^T$ we take the transpose and conjugate by w_0 to get that

$$w_0(g^{-1}Ng)^T w_0 \in w_0H^T w_0 = {}^T H. \quad (4.5)$$

As a composition of continuous bijections, namely transpose, inverse, and conjugation by the longest word, the map ϕ is a continuous bijection from G/B to itself. In fact ϕ is its own inverse:

$$\phi^2(gB) = \phi(w_0(g^T)^{-1}w_0B) = w_0((w_0(g^T)^{-1}w_0)^T)^{-1}w_0 = (((g^T)^{-1})^T)^{-1}B = gB. \quad (4.6)$$

Thus ϕ is an isomorphism between $\text{Hess}(N, H)$ and $\text{Hess}(N, {}^T H)$. \square

While in some cases ${}^T H = H$, this map groups many regular nilpotent Hessenberg varieties into isomorphic pairs.

4.2 Decomposable Regular Nilpotent Hessenberg Varieties

To study the structure of regular nilpotent Hessenbergs more closely, we want to look at the most basic unit of these varieties. Recall that any type A Hessenberg space can be given by a Hessenberg function $h : [n] \rightarrow [n]$ where $h(i)$ is greater than or equal to both i and $h(i-1)$.

Definition 4.4. *A type- A regular nilpotent Hessenberg variety is called **decomposable** if for some $i < n$ the Hessenberg function $h(i) = i$. If $h(i) > i$ for all $i < n$ then the corresponding variety is called **indecomposable**.*

Theorem 4.5. *Every type- A regular nilpotent Hessenberg variety is the product of indecomposable regular nilpotent Hessenberg varieties.*

Proof. Let $h : [n] \rightarrow [n]$ be a Hessenberg function with $h(j) = j$ for some $j < n$ and let $\text{Hess}(N, H)$ be the corresponding regular nilpotent Hessenberg variety. We define two new type- A Lie algebras

and root systems by letting $G_1 = GL_j(\mathbb{C})$ and $G_2 = GL_{n-j}(\mathbb{C})$ and $\mathfrak{g}_1, \mathfrak{g}_2$ be their respective Lie algebras. For each, we define a Hessenberg function:

$$\begin{aligned} h_1 : [j] &\rightarrow [j] & \text{and} & & h_2 : [n-j] &\rightarrow [n-j] \\ i &\mapsto h(i) & & & i &\mapsto h(i+j) - j \end{aligned} \quad (4.7)$$

These determine two regular nilpotent Hessenberg varieties $Hess(N_1, H_1)$ and $Hess(N_2, H_2)$ where $N_1 = N|_{\mathfrak{g}_1}$ and $N_2 = N|_{\mathfrak{g}_2}$ are regular nilpotent operators in \mathfrak{g}_1 and \mathfrak{g}_2 respectively. We will show that

$$Hess(N, H) \cong Hess(N_1, H_1) \times Hess(N_2, H_2). \quad (4.8)$$

Define a map from

$$\begin{aligned} Hess(N_1, H_1) \times Hess(N_2, H_2) &\rightarrow Hess(N, H) \\ (V_\bullet^{(1)}, V_\bullet^{(2)}) &\mapsto V_\bullet^{(1)} \oplus V_\bullet^{(2)} \end{aligned} \quad (4.9)$$

If $V_\bullet^{(1)}$ and $V_\bullet^{(2)}$ are flags in the two smaller Hessenberg varieties, then V_\bullet is the flag in $Hess(N, H)$ where

$$V_i = \begin{cases} V_i^{(1)} & \text{if } i \leq j \\ V_j^{(1)} \oplus V_{i-j}^{(2)} & \text{if } i > j \end{cases} \quad (4.10)$$

In matrix notation $V_\bullet = \begin{bmatrix} V_\bullet^{(1)} & * \\ 0 & V_\bullet^{(2)} \end{bmatrix}$.

To see that $V_\bullet \in Hess(N, H)$ we observe that

$$\begin{aligned} NV_i &= N_1 V_i^{(1)} \subset V_{h_1(i)}^{(1)} = V_{h(i)} & \text{if } i \leq j \\ NV_i &\subseteq V_j^{(1)} \oplus N_2 V_{i-j}^{(2)} \subset V_j^{(2)} \oplus V_{h_2(i-j)}^{(2)} = V_{h(i)} & \text{if } i > j. \end{aligned} \quad (4.11)$$

As a direct sum of linear operators, this map is injective. It remains to be shown that every flag V_\bullet in $Hess(N, H)$ has this form.

Let $V_\bullet = V_1 \subsetneq V_2 \subsetneq \dots \subsetneq V_n = \mathbb{C}^n$ be a flag in $Hess(N, H)$. Let $v \in V_j$ be a vector. Without loss of generality say that $v = (v_1, v_2, \dots, v_p, 0, \dots, 0)$ where v_p is the last non-zero entry. By the definition of $Hess(N, H)$ the vector Nv is also in V_j as are all the vectors $N^k v$ for k a non-negative integer.

Since v_p is non-zero, the vectors $N^k v$ are non-zero when k is less than p . The collection of vectors $\{N^k v : k = 0, 1, \dots, p-1\}$ is a linearly independent set in the space V_j . We know V_j has dimension j by definition of the flag. Therefore p is less than or equal to j . This means any vector v in V_j must be in the span of the first j basis elements. We conclude that V_j is equal to the span of the first j basis elements and that the matrix form of V_\bullet looks like

$$V_\bullet = \begin{bmatrix} V_\bullet^{(a)} & * \\ 0 & V_\bullet^{(b)} \end{bmatrix}. \quad (4.12)$$

Here we define the two smaller flags to be $V_i^{(a)} = V_i$ and $V_i^{(b)} = V_{i+j}/V_j$. By the definition of h_1 the flag $V_\bullet^{(a)}$ is in $Hess(N_1, H_1)$. For $V_\bullet^{(b)}$ we have that $N_2 V_i^{(b)} = N_2(V_{i+j}/V_j)$ which is equal to $(NV_{i+j})/V_j$ as a subspace of \mathbb{C}^{n-j} . Similarly $V_{h_2(i)}^{(b)}$ is equal to $V_{h(i+j)}/V_j$ as a subspace of \mathbb{C}^{n-j} . Therefore for any $V_\bullet \in Hess(N, H)$

$$NV_{i+j} \subset V_{h(i+j)} \iff (NV_{i+j})/V_j \subset V_{h(i+j)}/V_j \iff N_2 V_i^{(b)} \subset V_{h_2(i)}^{(b)}. \quad (4.13)$$

Thus every flag V_\bullet in $Hess(N, H)$ is the product of a flag in $Hess(N_1, H_1)$ and a flag in $Hess(N_2, H_2)$. This process of decomposing the regular nilpotent Hessenberg variety into the product of smaller varieties can be repeated until each Hessenberg function preserves only the largest element of its domain. \square

While Theorem 4.5 is only stated in Lie type A , the corresponding constructions for other Lie types are similar given the appropriate definition for decomposable and indecomposable. Specifically, a regular nilpotent Hessenberg variety would be called decomposable if for some negative simple root α the root subspace \mathfrak{g}_α is not in the Hessenberg space H . Each omitted negative simple root α in a decomposable Hessenberg variety gives rise to two smaller Lie algebras, \mathfrak{g}_1 and \mathfrak{g}_2 , corresponding to the connected components of the Dynkin diagram with α removed.

4.3 Equivariant Cohomology of $Hess(N, H)$

Hessenberg varieties are generally not GKM spaces. However they are subvarieties of the flag variety. Under the action of both the maximal torus (which doesn't preserve Hessenberg varieties) and a smaller one-dimensional subtorus (which does) the flag variety is a GKM space. Given that they

are contained in one of the classical GKM spaces, it is natural to ask what parts of GKM theory can be applied to Hessenberg varieties. Is there enough residual structure to do Schubert calculus?

Harada-Reiner-Tymoczko conjectured that the flag variety structures that remain in the regular nilpotent Hessenberg varieties, including a restricted torus action, are powerful enough to determine the module structure of the equivariant cohomology.

4.3.1 A 1-dimensional Torus Action

The first question is whether the maximal torus T acts on a regular nilpotent Hessenberg variety $Hess(N, H) \subset Flags$. Unfortunately it does not. But Kostant gave a one-dimensional subtorus $S \subseteq T$ that does preserve regular nilpotent Hessenberg varieties [24].

In type A , the torus $S \subseteq T$ has form:

$$S = \begin{bmatrix} t & 0 & \cdots & 0 & 0 \\ 0 & t^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t^{n-1} & 0 \\ 0 & 0 & \cdots & 0 & t^n \end{bmatrix} \subseteq \begin{bmatrix} t_1 & 0 & \cdots & 0 & 0 \\ 0 & t_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & t_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & t_n \end{bmatrix} = T.$$

This circle action was given by Kostant and used by Harada-Tymoczko, among others, to study Hessenberg varieties. We can also define S in general Lie type.

Definition 4.6. [19, Lemma 5.1] *The characters $\alpha_1, \dots, \alpha_n \in \mathfrak{t}^*$ are a maximal \mathbb{Z} -linearly independent set. Let $\phi : T \rightarrow (\mathbb{C}^*)^n$ be the isomorphism of linear algebraic groups $t \mapsto (\alpha_1(t), \alpha_2(t), \dots, \alpha_n(t))$. Then define a one-dimensional torus S by*

$$S = \phi^{-1}(\{(c, c, \dots, c) : c \in \mathbb{C}^*\}).$$

The one-dimensional torus S is isomorphic to \mathbb{C}^* and $H_S^*(pt)$ is isomorphic to $\mathbb{C}[t]$.

Proposition 4.7. [19, Lemma 5.1] *The torus S acts on the regular nilpotent Hessenberg variety.*

Any point in $Hess(N, H)$ fixed by T will also be fixed by S . In fact these are the only points in

the regular nilpotent Hessenberg variety fixed by S :

$$(Hess(N, H))^S = Hess(N, H) \cap (G/B)^T.$$

These fixed points are the flags $V_\bullet \in Hess(N, H)$ which have matrix representations with exactly one 1 in each row and column - namely the permutation matrices in $GL_n(\mathbb{C})$. Accordingly the S -fixed points of $Hess(N, H)$ can be indexed by elements of \mathfrak{S}_n the same way that the T -fixed points of $Flags$ are. Unlike with the flag variety, the torus-fixed points of $Hess(N, H)$ do not necessarily form a group.

Commutative diagram

If $Hess(N, H)$ were a GKM space under the action of S then GKM theory would tell us that studying the S -equivariant cohomology at the fixed points is sufficient to understand the S -equivariant cohomology of the whole variety. We already know that regular nilpotent Hessenbergs are not GKM spaces, but nonetheless all of the information about the equivariant cohomology is contained in the equivariant cohomology of the fixed points.

Harada-Tymoczko gave this commutative diagram [19]:

$$\begin{array}{ccccc} H_T^*(G/B) & \longrightarrow & H_S^*(G/B) & \longrightarrow & H_S^*(Hess(N, H)) \\ \downarrow & & \downarrow & & \downarrow \\ \bigoplus_{(G/B)^T} H_T^*(pt) & \longrightarrow & \bigoplus_{(G/B)^S} H_S^*(pt) & \longrightarrow & \bigoplus_{(Hess(N, H))^S} H_S^*(pt) \end{array} \quad (4.14)$$

It is conjectured that the map from $H_S^*(G/B)$ to $H_S^*(Hess(N, H))$ is surjective. This would be a consequence of Conjecture 4.12 in the next section. If that map is surjective, then in the same way that $H_T^*(G/B)$ can be studied by looking at the cohomology of the T -fixed points, we need only look at the equivariant cohomology of the S -fixed points of $Hess(N, H)$ in order to understand its equivariant cohomology.

Billey's formula and Hessenberg Schubert classes

The best way to look at the equivariant cohomology at the fixed points is to look at the explicit polynomials associated with those fixed points. Billey's formula does this for T -equivariant

Schubert classes. We will take one of those classes and build a Hessenberg Schubert class out of it.

Let σ_v be an equivariant Schubert class in $\bigoplus_{(G/B)^T} H_T^*(pt)$. We represent σ_v as a GKM graph with vertices labeled by polynomials. Each vertex corresponds to a T -fixed point of G/B and the polynomial labeling the vertex w is denoted $\sigma_v(w)$, Billey's formula give $\sigma_v(w)$ explicitly. The first step is to find the image of the Schubert class in $\bigoplus_{(G/B)^S} H_S^*(pt)$. This is done by the ring homomorphism

$$\begin{aligned} \pi_1 : \mathbb{C}[\alpha_1, \alpha_2, \dots, \alpha_{n-1}] &\rightarrow \mathbb{C}[t] \\ \alpha_i &\mapsto t \quad \text{for all } i \end{aligned} \tag{4.15}$$

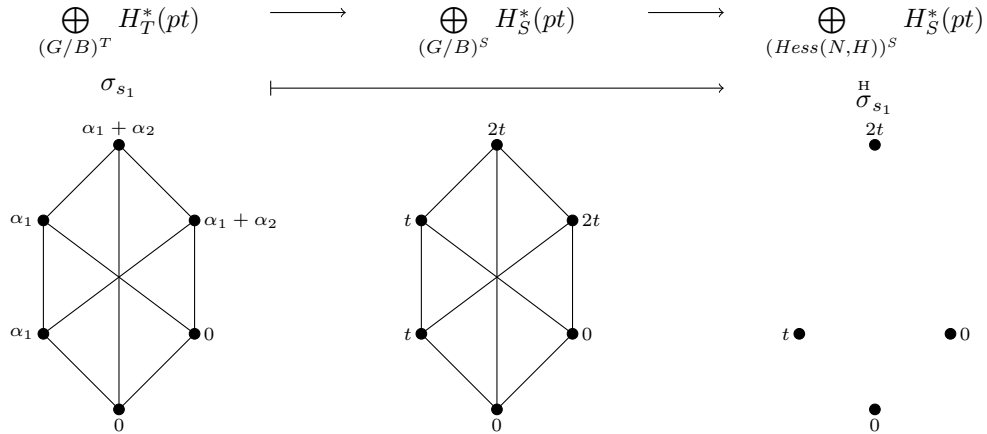
This homomorphism brings the T -equivariant Schubert classes to S -equivariant Schubert classes. The Hessenberg Schubert classes are obtained by forgetting the vertices corresponding to points that are not contained in the Hessenberg variety.

Definition 4.8. *The Hessenberg Schubert class $\overset{H}{\sigma}_v$ is a collection of polynomials $(\overset{H}{\sigma}_v(w))_{w \in (\text{Hess}(N,H))^S}$ where the polynomials are defined by*

$$\overset{H}{\sigma}_v(w) = \pi_1(\sigma_v(w)).$$

It is the image of σ_v in $\bigoplus_{(\text{Hess}(N,H))^S} H_S^*(pt)$.

Example 4.9. Let $h : [3] \rightarrow [3]$ be the Hessenberg Schubert function defined by $h(1) = 2$ and $h(2) = h(3) = 3$. The Schubert class σ_{s_1} is mapped to the Hessenberg Schubert class $\overset{H}{\sigma}_{s_1}$ as follows:



4.3.2 A conjectured basis for $H_S^*(\text{Hess}(N, H))$

While the Hessenberg Schubert classes are nice to have, the amazing property of the equivariant Schubert classes is that they form a module basis of the T -equivariant cohomology of the flag variety over $H_T^*(pt)$. Counting the number of Hessenberg Schubert classes and comparing to the number of polynomials associated with each, we see that we have far too many for the Hessenberg Schubert classes to be linearly independent, and thus do not have a basis.

If a basis of Hessenberg Schubert classes exists, then we could start to do explicit computations in the S -equivariant cohomology of regular nilpotent Hessenberg varieties. Harada-Reiner-Tymoczko conjectured a construction for such a basis in type A .

Definition 4.10. *We say a word $v \in W$ is in H if the permutation matrix of v is in H . For any Hessenberg space H we define $V_H = \{v \in W : v \in H\}$.*

Example 4.11. Consider the Hessenberg function $h(1) = 2, h(2) = h(3) = 3$. The word s_1s_2 in A_2 has matrix $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. This matrix is contained in H so we say $s_1s_2 \in H$.

Conjecture 4.12 (Harada-Reiner-Tymoczko). *Let $H \subset GL_n(\mathbb{C})$ be a Hessenberg space and $\text{Hess}(N, H)$ be the corresponding type- A regular nilpotent Hessenberg variety. The set of Hessenberg Schubert classes $\{\sigma_v^H : v \in V_H\}$ forms a basis of $H_S^*(\text{Hess}(N, H))$ as a module over $H_S^*(pt)$.*

There are several things to note about this conjecture that make it seem likely. First this construction will always give the correct number of basis elements. There are $\prod_{i=1}^n (h(i) - i + 1)$ points of $\text{Hess}(N, H)$ that are fixed by S . There are similarly $\prod_{i=1}^n (h(i) - i + 1)$ words v that fit into H .

Second, this construction gives a set of classes with degrees that correspond to the Betti numbers for $\text{Hess}(N, H)$ [19, Lemma 5.3]. Since the set $\{\sigma_v^H : v \in V_H\}$ has degrees corresponding to the Betti numbers, showing that the set is linearly independent over $\mathbb{C}[t] \cong H_S^*(pt)$ is sufficient to prove the conjecture.

4.3.3 Towards a proof of the conjecture

This conjecture has been proven in certain cases, though in general it remains open. The conjecture is trivially true for the regular nilpotent Springer variety, and has been successfully proven for the type A Peterson [19] and modified Peterson varieties [2]. It has also been confirmed by computers for small cases up to $n = 5$. It is also true for the full flag variety, which is itself a regular nilpotent Hessenberg variety.

Proposition 4.13. *For any regular nilpotent Hessenberg variety, Conjecture 4.12 is true if and only if it holds for each of the indecomposable components $\text{Hess}(N_i, H_i)$ of $\text{Hess}(N, H)$.*

Proof. Without loss of generality assume that $h(j) = j$ for some $j < n$. Then each permutation $v \in V_H$ has matrix form $\begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix}$ where v_1 is a permutation on the first j columns and v_2 is a permutation on the last $n - j$ columns. We know from the construction from the proof of Theorem 4.5 these words v_1 and v_2 correspond to flags in $\text{Hess}(N_1, H_1)$ and $\text{Hess}(N_2, H_2)$ respectively. The flag v_1 is contained in $H \cap \mathfrak{g}_1$ and thus $v_1 \in V_{H_1}$. Similarly the word v_2 is in the set $H \cap \mathfrak{g}_2$ and thus in V_{H_2} . For any pair v_1, v_2 in V_{H_1} and V_{H_2} respectively the word $v_1 v_2$ is in V_H and any $v \in V_H$ can be expressed as such a product. Therefore $V_H = V_{H_1} \times V_{H_2}$.

To determine the linear independence of the set $\{\overset{\text{H}}{\sigma}_v : v \in V_H\}$ it is convenient to look at them as the columns of a matrix. Let A_H be the matrix with columns indexed by classes $v \in V_H$ and rows indexed by fixed points $w \in (\text{Hess}(N, H))^S$. The entries of A_H are the polynomials $\overset{\text{H}}{\sigma}_v(w) \in \mathbb{C}[t]$. Proving the linear independence of the columns of matrix A_H is sufficient to prove Conjecture 4.12 for $\text{Hess}(N, H)$.

We consider the entry $\overset{\text{H}}{\sigma}_v(w)$ in the matrix A_H . The word v can be written as $v_1 v_2$ and the word w as $w_1 w_2$ where w_1 is an S -fixed point in $\text{Hess}(N_1, H_1)$ and w_2 is an S -fixed point in $\text{Hess}(N_2, H_2)$. Our entry can be rewritten as $\overset{\text{H}}{\sigma}_{v_1 v_2}(w_1 w_2)$. As words in the simple reflections, none of v_1, v_2, w_1, w_2 contain the reflection s_j . Moreover $v_1, w_1 \in \langle s_1, \dots, s_{j-1} \rangle$ and $v_2, w_2 \in \langle s_{j+1}, \dots, s_{n-1} \rangle$. Abusing notation to use $w_1(\overset{\text{H}}{\sigma}_{v_2}(w_2))$ for the image of the class $w_1(\sigma_{v_2}(w_2))$ under the map π_1 that sends $\sigma_v(w)$ to $\overset{\text{H}}{\sigma}_v(w)$ we have

$$\overset{\text{H}}{\sigma}_{v_1 v_2}(w_1 w_2) = \overset{\text{H}}{\sigma}_{v_1}(w_1) w_1(\overset{\text{H}}{\sigma}_{v_2}(w_2)) = \overset{\text{H}}{\sigma}_{v_1}(w_1) \overset{\text{H}}{\sigma}_{v_2}(w_2). \quad (4.16)$$

The matrix A_H now has columns indexed by $v_1v_2 \in V_{H_1} \times V_{H_2}$ and rows indexed by words $w_1w_2 \in (\text{Hess}(N_1, H_1))^S \times (\text{Hess}(N_2, H_2))^S$.

Looking at A_{H_1} as the matrix with columns indexed by $v_1 \in V_{H_1}$ and rows indexed by permutations $w_1 \in (\text{Hess}(N_1, H_1))^S$ and similarly for A_{H_2} we see that $A_H = A_{H_1} \otimes A_{H_2}$, Thus the matrix A_H is linearly independent over $\mathbb{C}[t]$ if and only if each of the matrices on the right-hand-side are linearly independent. \square

Since Conjecture 4.12 is true for a regular nilpotent Hessenberg variety if and only if it is true for each of its indecomposable components, we can construct Hessenberg varieties for which the conjecture holds, and also can restrict our investigation of the conjecture to indecomposable regular nilpotent Hessenbergs.

4.3.4 The case $h(1) = 3, h(i) = n$ for $i > 1$

Bayegan and Harada proved that Conjecture 4.12 holds when the Hessenberg function h has a specific form. Their *modified Peterson variety* has $h(1) = 3$ and $h(i) = i + 1$ for $1 < i < n$ and $h(n) = n$. Following their model of investigating a specific Hessenberg space rather than all spaces simultaneously, we discuss the Hessenberg function $h(1) = 3$ and $h(i) = n$ for all $i > 1$. We denote the corresponding Hessenberg space H_3 for convenience.

Example 4.14. *The type- A_4 space H_3 has form*

$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \end{bmatrix}$$

By examining the properties of the matrix A_{H_3} and exploiting Billey's formula we will reduce the question of whether A_{H_3} is a linearly independent matrix to the question of whether a much smaller matrix is linearly independent. This is done in several steps:

1. Explicitly determining the classes V_{H_3} and the fixed points $W_{H_3} = (\text{Hess}(N, H_3))^S$ and expressing them as the disjoint union of three sets.

2. Using row reduction and properties of Billey's formula to reduce the problem of linear independence to showing the linear independence of a smaller matrix β .

$$\begin{array}{c}
\begin{array}{ccc} Z & s_1Z & s_2s_1Z \end{array} \\
\begin{array}{c} Z \\ Ys_2 \end{array} \begin{pmatrix} F & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix} \longrightarrow \begin{array}{ccc} Z & s_1\tilde{Z} \cup s_2s_1\tilde{Z} & s_1\tilde{Z}s_2 \cup s_2s_1\tilde{Z}s_2 \\ \begin{array}{c} Z \\ Ys_2 \end{array} \begin{pmatrix} F & 0 & 0 \\ * & \beta & \gamma \\ * & \beta & Q\beta + \gamma \end{pmatrix}
\end{array}
\end{array}$$

$$\begin{array}{ccc} Z & s_1\tilde{Z} \cup s_2s_1\tilde{Z} & s_1\tilde{Z}s_2 \cup s_2s_1\tilde{Z}s_2 \\ \begin{array}{c} Z \\ Ys_2 \end{array} \begin{pmatrix} F & 0 & 0 \\ * & \beta & \gamma \\ * & 0 & Q\beta \end{pmatrix}
\end{array}$$

3. Showing that β is a block-upper-triangular matrix.

$$\beta = \begin{pmatrix} \beta_{n-1} & * & \cdots & * & * \\ 0 & \beta_{n-2} & \cdots & * & * \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & \beta_3 & * \\ 0 & 0 & \cdots & 0 & \beta_2 \end{pmatrix}$$

4. Showing that each block β_j on the diagonal of β is block-upper-triangular where b and f are linearly independent blocks.

$$\beta_j = \begin{pmatrix} \tilde{\beta}_j & * & * \\ 0 & b & * \\ 0 & 0 & f \end{pmatrix}$$

These steps will reduce the problem of whether the $3(n-1)! \times 3(n-1)!$ matrix A_{H_3} is linearly independent to a question of whether the much smaller matrix $\tilde{\beta}_j$ is linearly independent.

Step 1: Fixed Points and Classes

Lemma 4.15. *The Schubert classes V_{H_3} are:*

$$V_{H_3} = Z \cup s_1Z \cup s_2s_1Z$$

where $Z = \langle s_2, s_3, \dots, s_{n-1} \rangle$

Proof. The element $v \in A_{n-1}$ is in V_{H_3} if and only if $v(1) \in \{1, 2, 3\}$. We can partition V_{H_3} into three sets $V_i = \{v \in A_{n-1} : v(1) = i\}$ for $i = 1, 2, 3$. For each i the set $|V_i|$ has $(n-1)!$ elements.

The set V_1 is all of the permutations that fix 1, so $V_1 = \langle s_2, \dots, s_{n-1} \rangle = Z$. If $v \in V_1 = Z$ then $s_1 v \in V_2$ since $s_1 v(1) = s_1(1) = 2$. Therefore $s_1 Z \subset V_2$ and because $|V_2| = |s_1 Z| = |Z| = (n-1)!$ we have $s_1 Z = V_2$. Similarly if $v \in V_2$ then $s_2 v \in V_3$ since $s_2 v(1) = s_2(2) = 3$. Therefore $s_2 V_2 = s_2 s_1 Z \subset V_3$ and since $|s_2 s_1 Z| = |Z| = (n-1)!$ we have $s_2 s_1 Z = V_3$. \square

Lemma 4.16. *The S -fixed points of $Hess(N, H_3)$ are:*

$$W_{H_3} = Z \cup Y \cup Y s_2$$

where $Y = \{s_i s_{i-1} \cdots s_1 s_i s_{i-1} \cdots s_2 y : i \in \{1, 2, \dots, n-1\}, y \in \langle s_3, s_4, \dots, s_{n-1} \rangle\}$ and $Z = \langle s_2, \dots, s_{n-1} \rangle$ as above.

We will use the following notational conventions in the proof. Let $E_{(i,j)}$ be the $n \times n$ matrix with all entries equaling zero except for the entry in row i , column j . The regular nilpotent matrix can be written as $N = \sum_{i=1}^{n-1} E_{(i,i+1)}$.

Any permutation $\omega \in \mathfrak{S}_n$ can be written uniquely as, from left to right, a string of consecutive simple reflections descending to s_1 followed by a string of consecutive simple reflections descending to s_2 and so on [6]. Each word ω corresponds to a unique set of integers $\{k_1, k_2, \dots, k_n\}$ where $i-1 \leq k_i \leq n$. We allow $k_i = i-1$ in order to signify that the string descending to s_i is empty.

Example 4.17. These words are written in descending string form. They are marked with a vertical line where a descending string ends. Note that for ω_2 the string descending to s_2 is empty.

$$\omega_1 = s_4 s_3 s_2 s_1 | s_2 | s_4 s_3 | s_4 \quad \omega_2 = s_3 s_2 s_1 || s_3$$

Proof of Lemma 4.16. We partition W_{H_3} into three parts. Let W_1 be the set of $w \in W_{H_3}$ where $w^{-1} N w$ has all zeros in the first column. Let W_2 be the set of $w \in W_{H_3}$ where $w^{-1} N w$ has a 1 in the second row of the first column. Lastly let W_3 be the set of $w \in W_{H_3}$ where $w^{-1} N w$ has a 1 in the third row of the first column. By the definition of $Hess(N, H_3)$ if w is a permutation matrix in the variety then these are the only three options for the first column of $w^{-1} N w$.

First we observe that there are $(n-1)!$ elements in the set W_i for each i . If the first column has all zeros, then there are $n-1$ places for a 1 to occur in the second column, since it cannot be on the diagonal. Then there are $n-2$ places for a one to occur in the next column and so on. If there is a 1 in the third row of the first column, then there can be a 1 in $n-2$ places in the second column, as it cannot occur in rows 2 or 3. Thus there are $n-1$ possibilities for the second column (it could have no 1s), and similarly $n-2$ for the third, etc. Thus both W_1 and W_3 have $(n-1)!$ elements. Finally there are the same number of fixed points as there are classes corresponding to any type- A Hessenberg space, and Lemma 4.15 gives us $3(n-1)!$ classes. Hence W_2 must also contain $(n-1)!$ elements.

If $w \in Z$ then $w^{-1}Nw$ has no non-zero entries in the first column. So Z is contained in W_1 and since they are the same size, the two sets are equal.

With our notational conventions we can define W_2 explicitly. Let $x_i = s_i s_{i-1} \cdots s_2 s_1 s_i s_{i-1} \cdots s_3 s_2$. Then $x_i^{-1} E_{(i,i+1)} x_i = E_{(2,1)}$. If $y \in \langle s_3, s_4, \dots, s_{n-1} \rangle$ then $y^{-1} E_{(2,1)} y = E_{(2,1)}$. So

$$(x_i y)^{-1} E_{(i,i+1)} (x_i y) = y^{-1} x_i^{-1} E_{(i,i+1)} x_i y = y^{-1} E_{(2,1)} y = E_{(2,1)}$$

Thus $x_i y \in W_2$ and so $Y = \{x_i y\} \subset W_2$. By writing the permutation $x_i y$ as reflections descending to s_1 then reflections descending to s_2 and so on, we see that $x_i y = x_j y'$ if and only if $i = j$ and $y = y'$. So

$$|Y| = (\text{number of } x_i)(\text{number of } y) = (n-1)(n-2)! = (n-1)! = |W_2|$$

and hence $Y = W_2$. Lastly, since $s_2 E_{(2,1)} s_2 = E_{(3,1)}$, we can multiply Y on the right by s_2 to obtain a set of permutations contained in W_3 . Furthermore $|Y s_2| = |Y| = (n-1)! = |W_3|$ so we have that $Y s_2 = W_3$. \square

Step 2: Row Reduction using Billey's Formula

Using these explicit descriptions of classes and fixed points, we present the matrix A_{H_3} as follows:

$$A_{H_3} = Y \begin{array}{c} Z \\ Z s_1 \\ Z s_2 s_1 \end{array} \begin{pmatrix} F & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}$$

Here F is the full set of S -equivariant Schubert classes on the full flag variety $GL_{n-1}(\mathbb{C})/B$ with fixed points indexed by the elements of $Z = \langle s_2, \dots, s_{n-1} \rangle$. Its columns are a basis for the equivariant cohomology of that flag variety and therefore linearly independent. Thus showing the linear independence of $A_{\mathcal{H}}$ has been reduced to showing the linear independence of the bottom right 2×2 block above. We rearrange the columns of that block one more time to get

$$Y \begin{pmatrix} s_1 Z & s_2 s_1 Z \\ * & * \\ * & * \end{pmatrix} = Y_{s_2} \begin{pmatrix} s_1 \tilde{Z} \cup s_2 s_1 \tilde{Z} & s_1 \tilde{Z} s_2 \cup s_2 s_1 \tilde{Z} s_2 \\ * & * \\ * & * \end{pmatrix}$$

where $\tilde{Z} = \{z \in Z : s_2 \notin D_R(z)\}$. Since \tilde{Z} is one of the two cosets of $Z/\langle s_2 \rangle$, we have that $|\tilde{Z}| = \frac{1}{2}(n-1)!$. Therefore this rearrangement still leaves the matrix with four $(n-1)! \times (n-1)!$ blocks, each represented above by an asterisk.

A well known identity for Billey's formula is:

$$\sigma_v(w) = \begin{cases} \sigma_v(ws) & \text{if } s \notin D_R(v) \\ \sigma_v(ws) - \sigma_{vs}(w) \cdot v(\alpha_s) & \text{if } s \in D_R(v) \end{cases}$$

Noting that s_2 is not in the right descent set of v for all $v \in s_1 \tilde{Z} \cup s_2 s_1 \tilde{Z}$ and is in the right descent set of v all $v \in s_1 \tilde{Z} s_2 \cup s_2 s_1 \tilde{Z} s_2$ the blocks in this last matrix can be related to each other by

$$Y \begin{pmatrix} s_1 \tilde{Z} \cup s_2 s_1 \tilde{Z} & s_1 \tilde{Z} s_2 \cup s_2 s_1 \tilde{Z} s_2 \\ \beta & \gamma \\ \beta & \gamma + Q\beta \end{pmatrix}.$$

Here Q is a diagonal matrix with the root $-y(\alpha_2)$ as the entry in row $y \in Y$. This matrix row reduces to

$$Y_{s_2} \begin{pmatrix} s_1 \tilde{Z} \cup s_2 s_1 \tilde{Z} & s_1 \tilde{Z} s_2 \cup s_2 s_1 \tilde{Z} s_2 \\ \beta & \gamma \\ 0 & Q\beta \end{pmatrix}.$$

Since Q is a diagonal matrix with all eigenvalues non-zero, it has no effect on whether $Q\beta$ is linearly independent. Therefore proving the linear independence of A_{H_3} now reduces to proving the linear independence of the block β .

Step 3: β is a Block Diagonal

To discuss the matrix β in more detail, we need an explicit description for the sets indexing its rows and columns.

Lemma 4.18. *Let z be an element of $Z = \langle s_2, s_3, \dots, s_{n-1} \rangle$ with descending string form*

$$z = s_{k_2} s_{k_2-1} \cdots s_3 s_2 s_{k_3} s_{k_3-1} \cdots s_3 s_{k_4} s_{k_4-1} \cdots s_4 \cdots s_{n-1}$$

If $k_2 < k_3$ then s_2 is not in the right descent set of z and z is in \tilde{Z} . If k_2 is greater than or equal to k_3 then s_2 is in the right descent set of z so $z \notin \tilde{Z}$.

Proof. Let $z = s_{k_2} s_{k_2-1} \cdots s_3 s_2 s_{k_3} s_{k_3-1} \cdots s_3 s_{k_4} s_{k_4-1} \cdots s_4 \cdots s_{n-1}$. Because s_2 commutes with s_i for $i > 3$, we have that $s_2 \in D_R(z)$ if and only if $s_2 \in D_R(s_{k_2} s_{k_2-1} \cdots s_3 s_2 s_{k_3} s_{k_3-1} \cdots s_3)$. If $k_2 = k_3 = k$ then

$$\begin{aligned} & s_{k_2} s_{k_2-1} \cdots s_3 s_2 s_{k_3} s_{k_3-1} \cdots s_3 \\ &= s_k s_{k-1} s_k s_{k-2} s_{k-1} s_{k-3} s_{k-2} \cdots s_4 s_2 s_3 \\ &= s_{k-1} s_k s_{k-2} s_{k-1} s_{k-3} s_{k-2} \cdots s_3 s_4 s_2 s_3 s_2. \end{aligned}$$

If $k_2 > k_3 = k$ then

$$\begin{aligned} & s_{k_2} s_{k_2-1} \cdots s_{k_3+1} s_{k_3} s_{k_3-1} \cdots s_3 s_2 s_{k_3} s_{k_3-1} \cdots s_3 \\ &= s_{k_2} s_{k_2-1} \cdots s_{k+1} s_k s_{k-1} s_k s_{k-2} s_{k-1} s_{k-3} s_{k-2} \cdots s_4 s_2 s_3 \\ &= s_{k_2} s_{k_2-1} \cdots s_{k+1} s_{k-1} s_k s_{k-2} s_{k-1} s_{k-3} s_{k-2} \cdots s_3 s_4 s_2 s_3 s_2. \end{aligned}$$

Since $1 \leq k_2 \leq n-1$ and $2 \leq k_3 \leq n-1$ the number of pairs (k_2, k_3) such that $k_2 \geq k_3$ can be counted by noticing that if $k_2 = m$ then there are $m-1$ possible values of k_3 . So the number of such pairs is:

$$\sum_{m=1}^{n-1} (m-1) = \sum_{m=1}^{n-1} m - \sum_{m=1}^{n-1} 1 = \frac{(n-1)(n)}{2} - (n-1) = \frac{1}{2}(n^2 - n - 2n + 2) = \frac{1}{2}(n-1)(n-2)$$

which is exactly half of the total number of possible pairs (k_2, k_3) such that $1 \leq k_2 \leq n-1$ and $2 \leq k_3 \leq n-1$. As exactly half of the elements $z \in Z$ have $s_2 \notin D_R(z)$ the set of z with $k_2 \geq k_3$ are exactly these elements. The remaining elements $z \in Z$ have both $k_2 < k_3$ and $s_2 \in D_R(z)$. \square

Because every permutation in \tilde{Z} can be expressed in descending string form with $k_2 < k_3$ the permutations indexing the columns of the matrix β can be partitioned based on the first three

descending strings. For $a \in \{1, 2\}$ and $1 \leq b < c \leq n$ define the following words:

$$\lambda_a = \begin{cases} s_1 & \text{if } a = 1 \\ s_2 s_1 & \text{if } a = 2 \end{cases}$$

$$\mu_b = \begin{cases} e & \text{if } b = 1 \\ s_2 & \text{if } b = 2 \\ s_b \cdots s_2 & \text{if } b > 2 \end{cases}$$

$$\nu_c = \begin{cases} e & \text{if } c = 2 \\ s_3 & \text{if } c = 3 \\ s_c \cdots s_3 & \text{if } c > 3 \end{cases}$$

Using this notation we define these sets into which the columns are partitioned:

$$A(a, b, c) = \{\lambda_a \mu_b \nu_c \omega : \omega \in \langle s_4, \dots, s_{n-1} \rangle\}.$$

In addition to partitioning the columns of β we want to impose a partial order on the sets $A(a, b, c)$. We will say that $A(a_1, b_1, c_1) < A(a_2, b_2, c_2)$ if $c_1 < c_2$. The columns of β are ordered from largest to smallest.

A similar method will partition and order the rows of β so that the matrix can be written as square blocks. The permutations indexing the rows of β look like $x_i y$ where the term x_i is $s_i s_{i-1} \cdots s_1 s_i s_{i-1} \cdots s_2$ and $y \in \langle s_3, \dots, s_{n-1} \rangle$. By expanding out the first descending string of each y we get that the rows are indexed by

$$Y = \{s_i s_{i-1} \cdots s_1 s_i s_{i-1} \cdots s_2 s_l s_{l-1} \cdots s_3 s_l s_{l-1} \cdots s_4 \cdots s_{n-1}\}.$$

Recall from Example 4.17 that $x_i = s_i \cdots s_1 s_i \cdots s_2$. For any $1 \leq i \leq n-1$ and $2 \leq l \leq n-1$ the set $B(i, l)$ is defined by

$$B(i, l) = \{x_i s_l s_{l-1} \cdots s_3 \omega : \omega \in \langle s_4, \dots, s_{n-1} \rangle\}.$$

The set $B(i_1, l_1)$ is less than $B(i_2, l_2)$ if $\max(i_1, l_1) < \max(i_2, l_2)$ and the rows of matrix β are given in decreasing order with respect to this partial order.

Lemma 4.22. *If $j_1 < j_2$ then all entries in the block of β with rows in $B(j_1)$ and columns in $A(j_2)$ are zero. This means that*

$$\beta = \begin{pmatrix} \beta_{n-1} & * & \cdots & * & * \\ 0 & \beta_{n-2} & \cdots & * & * \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & \beta_3 & * \\ 0 & 0 & \cdots & 0 & \beta_2 \end{pmatrix}$$

and the linear independence of β depends only on the linear independence of the blocks β_j on the diagonal.

Proof. Let $j_1 < j_2$ and consider an arbitrary entry in the block $B(j_1) \times A(j_2)$. Recalling that the row corresponds to an S fixed point w of the Hessenberg variety and the column to a Hessenberg Schubert class σ_v that entry is $\sigma_v^H(w)$. The term $\sigma_v^H(w)$ is zero unless $v \leq w$. We will show that for any $v \in A(j_2)$ and $w \in (B(j_1))$, v is not a subword of w . Since $v \in A(a, b, j_2)$ it has descending string form

$$v = \overbrace{s_a s_1 s_b s_{b-1} \cdots s_2 s_{j_2} s_{j_2-1} \cdots s_3}^{\text{prefix}} \omega_v$$

for some ω_v which has no $s_1, s_2,$ or s_3 reflections in it. The fixed point w is in $B(i, l)$ for some pair i and l where $j_1 \geq i$ and l . So the descending string form of w looks like

$$w = \overbrace{s_i \cdots s_1 s_i \cdots s_2 s_l \cdots s_3}^{\text{prefix}} \omega_w$$

for an ω_w with no $s_1, s_2,$ or s_3 reflections. The s_{j_2} in the prefix of v cannot commute past the s_3 at the end of the prefix. So if v is a subword of w then that s_{j_2} must be a subword of the prefix of w . But both i and l are less than j_2 so there is no s_{j_2} in the prefix of w . Therefore $v \not\leq w$ and by Billey's formula $\sigma_v(w) = 0$ and thus $\sigma_v^H(w) = 0$. \square

Step 4: Block Diagonality of β_j

Each block β_j is a $(2(j-1)(n-3)!) \times (2(j-1)(n-3)!)^2$ square matrix, which while still quite large is a significant improvement over the matrix we started with. Our original matrix A_{H_3} was $3(n-1)!$ by $3(n-1)!$ so especially for small j , this is quite an improvement. While these are still partial results, at this point we would ideally induct on j . While that step remains open, we can go one step farther toward showing that each β_j is linearly independent.

First we will identify the last two blocks of rows and columns of the matrix. These are the rows and columns that have a specific prefix preceding a word ω from $\Omega = \langle s_4, \dots, s_{n-1} \rangle$.

$$\begin{array}{c}
 \text{remainder of } B(j) \\
 \left(\begin{array}{c|cc}
 \text{remainder of } A(j) & \{s_2s_1\omega\} & \{s_1\omega\} \\
 \hline
 & & \\
 & \tilde{\beta}_j & \\
 \hline
 \{s_2s_1s_2\omega\} & a & \begin{array}{c} b \\ c \end{array} \\
 \{s_1\omega\} & d & \begin{array}{c} e \\ f \end{array}
 \end{array} \right)
 \end{array}$$

All entries in the blocks labeled with a, d , and e are zero. Each entry can be computed using Billey's formula. Consider an entry in the block a . It has form $\sigma_v^H(s_2s_1s_2\omega)$ where v is in the remainder of $A(j)$ not in the last two blocks. If v did not contain a reflection s_3 it would have been in one of those right-most columns. But since the fixed point $s_2s_1s_2\omega$ does not contain s_3 the polynomial $\sigma_v(s_2s_1s_2\omega)$ is zero. Thus $\sigma_v^H(s_2s_1s_2\omega)$ is also zero.

Similarly for any v in the remainder of $A(j)$ the polynomial $\sigma_v^H(s_1\omega)$ is also zero. Thus both block a and block d are zero. As for block e , these entries have form $\sigma_v^H(w)$ where v has an s_2 and w does not.

The block f has entries that look like $\sigma_{s_1\omega_1}^H(s_1\omega_2)$ for some $\omega_1, \omega_2 \in \Omega$. Since there is exactly one way to find s_1 in $s_1\omega_2$ the polynomial $\sigma_{s_1\omega_1}(s_1\omega_2) = \sigma_{s_1}(s_1) \cdot s_1(\sigma_{\omega_1}(\omega_2))$. Thus the whole block f is $\sigma_{s_1}^H(s_1) = t$ multiplied by the image in one variable of the matrix $(s_1(\sigma_{\omega_1}(\omega_2)))$ where ω_1 and ω_2 range over Ω . But since $(\sigma_{\omega_1}(\omega_2))$ will be a polynomial in $\alpha_4, \dots, \alpha_{n-1}$ the permutation s_1 does not affect it. So the block f is $t(\sigma_{\omega_1}^H(\omega_2))$ which is t times the matrix of Hessenberg Schubert classes associated with the full flag variety in $n-3$ dimensions. Thus the columns of f are linearly independent.

Similarly the entries of b look like $\sigma_{s_2s_1\omega_1}^H(s_2s_1s_2\omega_2) = \sigma_{s_2s_1}^H(s_2s_1s_2) \cdot \sigma_{\omega_1}^H(\omega_2)$. Again we have a single polynomial, in this case $\sigma_{s_2s_1}^H(s_2s_1s_2) = 2t^2$, multiplied by the matrix A_H for a full flag

variety. Thus b is also a linearly independent matrix.

This work has reduced the problem of whether Conjecture 4.12 is true for the Hessenberg variety $Hess(N, H_3)$ to a much smaller case with a clear direction for an inductive step. While these results are still partial, there is some promise that the matrix multiplication techniques from Chapter 3 will be the right tool to solve this particular problem.

CHAPTER 5

PETERSON VARIETIES

Peterson varieties are the best understood subfamily of the regular nilpotent Hessenberg varieties. This chapter will do “Schubert calculus” in the equivariant cohomology rings of Peterson varieties. Peterson varieties were introduced by D. Peterson in the 1990s when he used them to construct the small quantum cohomology of partial flag varieties. Since then Kostant used Peterson varieties to describe the quantum cohomology of flag manifolds [24]; Rietsch described the totally non-negative part of type A Peterson varieties in 2006 using mirror symmetry constructions [31]; Insko-Yong explicitly described the singular locus of type A Peterson varieties and intersected them with Schubert varieties [21].

Using work by Harada-Tymoczko [19] and Precup [28], we construct a basis for the S -equivariant cohomology of Peterson varieties in *all* Lie types. This construction gives a basis of **Peterson Schubert classes**. Classical Schubert calculus asks how to multiply Schubert classes; we ask how to multiply Peterson Schubert classes. We give a Monk’s formula for multiplying a ring generator and a module generator, and a Giambelli’s formula for expressing any Peterson Schubert class in terms of the ring generators.

In type A the equivariant cohomology of the Peterson variety was understood by Harada-Tymoczko who gave a basis and a Monk’s rule for the equivariant cohomology ring [18]. A type A Giambelli’s formula was given by Bayegan-Harada [1]. This chapter extends those results to all Lie types.

For any Lie type the Peterson subspace in \mathfrak{g} is

$$H_{Pet} = \mathfrak{b} \oplus \bigoplus_{\alpha \in -\Delta} \mathfrak{g}_\alpha.$$

The regular nilpotent operator $N \in \mathfrak{g}$ is

$$N = \sum_{\alpha \in \Delta} E_{\alpha}.$$

Definition 5.1. *The Peterson variety Pet is a subvariety of the flag variety defined by*

$$Pet = \{gB \in G/B : Ad(g^{-1})(N) \in H_{Pet}\}.$$

The Peterson variety is the Hessenberg variety $Hess(N, H_{Pet})$.

Peterson varieties are irreducible and not smooth [21].

Harada and Tymoczko gave the S -fixed points of Pet explicitly. Let $K \subseteq \Delta$ be a subset of the simple roots. Define $W_K \subseteq W$ as the parabolic subgroup generated by K and let w_K be the longest element of W_K .

Proposition 5.2. *[19, Proposition 5.8] An element $w \in W$ is an S -fixed point of Pet if and only if $w = w_K$ for some set $K \subseteq \Delta$.*

Although Pet has a torus action and torus-fixed points indexed by Weyl group elements, it is not a GKM space. We now start building the GKM-like structures for the Peterson variety.

5.1 Peterson Schubert classes as a basis of $H_S^*(Pet)$

There is a well known projection from $H_T^*(G/B)$ to $H_S^*(Pet)$. It is not obvious what the image should be, but we show that this map is surjective in Theorem 5.3. Then we identify a specific subset of Schubert classes whose images under this map form a basis of $H_S^*(Pet)$. This builds on the work of Harada-Tymoczko [19]. The following commutative diagram is key to our argument.

$$\begin{array}{ccc}
H_T^*(G/B) & \hookrightarrow & \bigoplus_{(G/B)^T} H_T^*(pt) \\
\downarrow & & \downarrow \pi_1 \\
H_S^*(G/B) & \hookrightarrow & \bigoplus_{(G/B)^S} H_S^*(pt) \\
\downarrow & & \downarrow \pi_2 \\
H_S^*(Pet) & \hookrightarrow & \bigoplus_{(Pet)^S} H_S^*(pt)
\end{array} \tag{5.1}$$

A priori $H_T^*(G/B)$ is a module over $\mathbb{C}[\alpha_i : \alpha_i \in \Delta]$. We will use both π_1 and π_2 to indicate both the full maps and their restriction to a single class. Restricting to the cohomology of a single

point, the map $\pi_1 : H_T^*(pt) \rightarrow H_S^*(pt)$ is the ring homomorphism which takes simple roots $\alpha_i \in \Delta$ to the variable t . Again restricting to the cohomology of a single point, the map π_2 forgets the components of the class corresponding to T -fixed points of G/B that are not in the Peterson variety.

5.1.1 Peterson Schubert classes

The image of a Schubert class $\sigma_v \in H_T^*(G/B)$ in $H_S^*(Pet)$ is denoted p_v and called a **Peterson Schubert class**. The class p_v has one polynomial for each S -fixed point of Pet so a Peterson Schubert class can be thought of as a $2^{|\Delta|}$ -tuple of polynomials in $\mathbb{C}[t]$. Below is an example in type A_2 .

$$\begin{array}{c}
 \sigma_{s_1} \\
 \begin{array}{c} 1 \\ s_1 \\ s_2 \\ s_1 s_2 \\ s_2 s_1 \\ s_1 s_2 s_1 \end{array} \begin{pmatrix} 0 \\ \alpha_1 \\ 0 \\ \alpha_1 \\ \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_2 \end{pmatrix} \xrightarrow{\pi_1} \begin{pmatrix} 0 \\ t \\ 0 \\ t \\ 2t \\ 2t \end{pmatrix} \xrightarrow{\pi_2} \begin{pmatrix} 0 \\ t \\ 0 \\ \\ \\ 2t \end{pmatrix} \\
 p_{s_1}
 \end{array}$$

Theorem 5.3. *The “poset pinball” machinery given by Harada-Tymoczko [19, Theorem 5.4] holds for Peterson varieties of all Lie types. Specifically the map $H_T^*(G/B) \rightarrow H_S^*(Pet)$ is injective.*

The result of this theorem is that we can use the maps π_1 and π_2 to study $H_S^*(Pet)$.

Proof. Precup proved that Pet is paved by complex affines for any Lie type [28, Theorem 5.4]. In the same paper Precup showed that the compact cohomology of the Peterson variety is only supported in even dimensions [28, Lemma 2.7]. Because the Peterson variety is compact this implies its regular cohomology vanishes in odd degree. So Pet is equivariantly formal [35]. These results show that Harada-Tymoczko’s Lemma 5.3 extends to all Lie types and the remainder of their proof of Theorem 5.4 is type-independent [19, Theorem 5.4]. \square

5.1.2 A basis of Peterson Schubert classes

The S -fixed points of Pet are indexed by subsets $K \subseteq \Delta$ so we want to index the Peterson Schubert classes by $K \subseteq \Delta$.

Definition 5.4. A subset of simple roots $K \subseteq \Delta$ is called **connected** if the induced Dynkin diagram of K is a connected subgraph of the Dynkin diagram of Δ .

Any subset $K \subseteq \Delta$ can be written as $K = K_1 \times \cdots \times K_m$ where each K_i is a maximally connected subset. Each connected subset corresponds to its own Lie type.

Definition 5.5. Let $K \subseteq \Delta$ be a connected subset. We define $v_K \in W_K$ to be

$$v_K = \prod_{\text{Root}_K(i)=1}^{|K|} s_i$$

where $\text{Root}_K(i)$ is the index of the corresponding root in a root system of the same Lie type as K , ordered as in Figure 1. If $K = K_1 \times \cdots \times K_m$ and each K_i is maximally connected then $v_K = v_{K_1} v_{K_2} \cdots v_{K_m}$.

When Δ is not of type D or E this definition gives $v_K = s_{a_1} s_{a_2} \cdots s_{a_m}$ where $K = \{\alpha_{a_1}, \alpha_{a_2}, \dots, \alpha_{a_m}\}$ and $a_1 < a_2 < \cdots < a_m$. This is the definition given in type A by Harada-Tymoczko [18]. Example 5.6 illustrates how Definition 5.5 differs from the type A definition.

Example 5.6. Let $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ be a the set of simple roots of a type E_6 root system and let $K = \Delta \setminus \{\alpha_6\}$. The subset $K \subseteq \Delta$ represented by a marked set of vertices in the Dynkin diagram and compared to the Dynkin diagram for D_5 . The word v_K is $s_1 s_3 s_4 s_5 s_2$.



Note that v_K is a Coxeter element of W_K . Because of the labeling imposed on the simple roots in Figure 1, each subset K of simple roots corresponds to exactly one word v_K .

Lemma 5.7. For any set of simple roots Δ and any subsets $J, K \subseteq \Delta$ the Peterson Schubert class satisfies $p_{v_J}(w_K) = 0$ unless $J \subseteq K$.

Proof. Suppose $J \not\subseteq K$ and that $\alpha_j \in J \setminus K$. Then $s_j \leq v_J$ and $s_j \not\leq w_K$ in the Bruhat order. For $\sigma_{v_J}(w_K)$ to be non-zero there must be some subword of w_K that is equal to v_J and therefore

$v_J \leq w_K$. But $s_j \leq v_J$ implies that $s_j \leq w_K$ which is a contradiction. Thus $\sigma_{v_J}(w_K) = 0$ by Proposition 2.7 and by construction $p_{v_J}(w_K) = 0$. \square

Lemma 5.8. *For any set of simple roots Δ and any subset $K \subseteq \Delta$*

$$p_{v_K}(w_K) \neq 0.$$

Proof. Since $v_K \in W_K$ we must have $v_K \leq w_K$. By Proposition 2.7 the polynomial $\sigma_{v_K}(w_K) \in \mathbb{C}[\Delta]$ is not equal to zero. We have defined $p_{v_K}(w_K)$ to be $\pi_1(\sigma_{v_K}(w_K))$. Since $\sigma_{v_K}(w_K)$ has positive integer coefficients by the same proposition, its image in $\mathbb{C}[t]$ must also have positive integer coefficients. \square

Theorem 5.9. *The Peterson Schubert classes $\{p_{v_K} : K \subseteq \Delta\}$ are a basis of $H_S^*(Pet)$.*

Proof. This is a version of Harada-Tymoczko's Theorem 5.9 [19]. With Precup's work we now extend the proof to all Peterson varieties. Impose a partial order on the sets $\{K \subseteq \Delta\}$ by inclusion. Use that partial order to order the classes $\{p_{v_K}\}$ and the S -fixed points $w_K \in Pet$. Lemma 5.7 implies that the collection $\{p_{v_K}\}$ is lower-triangular and Lemma 5.8 implies that the collection has full rank. Thus $\{p_{v_K}\}$ is a linearly independent set.

By the properties of Billey's formula, the polynomial degree of p_{v_K} is $|K|$ and its cohomology degree is $2|K|$. As there are $\binom{n}{|K|}$ subsets of Δ with size $|K|$, there are exactly $\binom{n}{|K|}$ Peterson Schubert varieties with cohomology degree $2|K|$. Precup's paving by affines reveals that the dimensions of the corresponding pavings are also $\binom{n}{|K|}$ [28, Corollary 4.13]. As a linearly independent set with the right number of elements of each degree, the set $\{p_{v_K}\}$ is a module basis of $H_S^*(Pet)$ [18, Proposition A.1]. \square

Example 5.10. Below we give the Peterson Schubert classes which form a basis of the S -equivariant cohomology of Pet in Lie type C_3 . The classes and fixed points are indexed by the subsets $K \subseteq \Delta$.

K	p_{v_0}	$p_{v_{\{\alpha_1\}}}$	$p_{v_{\{\alpha_2\}}}$	$p_{v_{\{\alpha_3\}}}$	$p_{v_{\{\alpha_1, \alpha_2\}}}$	$p_{v_{\{\alpha_1, \alpha_3\}}}$	$p_{v_{\{\alpha_2, \alpha_3\}}}$	$p_{v_{\{\alpha_1, \alpha_2, \alpha_3\}}}$
\emptyset	$\begin{pmatrix} 1 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$
$\{\alpha_1\}$	$\begin{pmatrix} 1 \\ \end{pmatrix}$	$\begin{pmatrix} t \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$
$\{\alpha_2\}$	$\begin{pmatrix} 1 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} t \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$
$\{\alpha_3\}$	$\begin{pmatrix} 1 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} t \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$
$\{\alpha_1, \alpha_2\}$	$\begin{pmatrix} 1 \\ \end{pmatrix}$	$\begin{pmatrix} 2t \\ \end{pmatrix}$	$\begin{pmatrix} 2t \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 2t^2 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$
$\{\alpha_1, \alpha_3\}$	$\begin{pmatrix} 1 \\ \end{pmatrix}$	$\begin{pmatrix} t \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} t \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} t^2 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$
$\{\alpha_2, \alpha_3\}$	$\begin{pmatrix} 1 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 3t \\ \end{pmatrix}$	$\begin{pmatrix} 4t \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$	$\begin{pmatrix} 6t^2 \\ \end{pmatrix}$	$\begin{pmatrix} 0 \\ \end{pmatrix}$
$\{\alpha_1, \alpha_2, \alpha_3\}$	$\begin{pmatrix} 1 \\ \end{pmatrix}$	$\begin{pmatrix} 5t \\ \end{pmatrix}$	$\begin{pmatrix} 8t \\ \end{pmatrix}$	$\begin{pmatrix} 9t \\ \end{pmatrix}$	$\begin{pmatrix} 20t^2 \\ \end{pmatrix}$	$\begin{pmatrix} 45t^2 \\ \end{pmatrix}$	$\begin{pmatrix} 36t^2 \\ \end{pmatrix}$	$\begin{pmatrix} 60t^3 \\ \end{pmatrix}$

5.2 Monk's Formula

Now that we have a basis for $H_S^*(Pet)$ in terms of Peterson Schubert classes, we can examine the structure of $H_S^*(Pet)$ through its multiplication rules. First we determine a minimal set of Peterson Schubert classes that generate the ring $H_S^*(Pet)$.

Lemma 5.11. *The Peterson Schubert classes p_{s_i} generate the ring $H_S^*(Pet)$ as an algebra over $H_S^*(pt)$.*

Proof. It is well known that the Schubert classes σ_{s_i} generate $H_T^*(G/B)$. Fulton gives a complete proof for type A [11, Section 10.2]. A consequence of Theorem 5.9 is that the map

$$\phi : H_T^*(G/B) \rightarrow H_S^*(Pet)$$

is a surjective ring homomorphism. Thus the image $\{p_{s_i}\}$ of the generators $\{\sigma_{s_i}\}$ of $H_T^*(G/B)$ is a generating set for $H_S^*(Pet)$. \square

Monk's rule is an explicit formula for multiplying an arbitrary module generator class p_{v_K} by a ring generator class p_{s_i} . For the Peterson variety, a Monk's formula gives a set of constants $c_{i,K}^J \in H_S^*(pt)$ such that

$$p_{s_i} p_{v_K} = \sum_{J \subseteq \Delta} c_{i,K}^J \cdot p_{v_J}. \quad (5.2)$$

The Peterson Schubert classes $\{p_{v_K} : K \subseteq \Delta\}$ are a module basis for $H_S^*(Pet)$ and the product of p_{s_i} and p_{v_K} is also in that module. Thus a unique set of constants $\{c_{i,K}^J\}$ solve this equation. Because $H_S^*(pt) = \mathbb{C}[t]$ these structure constants are complex polynomials in t .

Monk's rule requires that we sum over all Peterson Schubert classes, but we simplify the formula significantly. First we eliminate many subsets $J \subseteq \Delta$ by showing that $c_{i,K}^J = 0$.

Lemma 5.12. *If $|J| > |K| + 1$ then $c_{i,K}^J = 0$.*

Proof. The polynomial degree of $p_v \in \mathbb{C}[t]$ is the length of a reduced word for v . Therefore the Peterson Schubert class p_{v_K} has degree $|K|$ and the polynomial degree of $p_{s_i p_{v_K}}$ is $|K| + 1$. The polynomial degrees on the right- and left-hand sides of Equation (5.2) must be equal. Take only the parts of each side of Equation (5.2) that have degree higher than $|K| + 1$. Hence it follows that

$$0 = \sum_{\substack{J \subseteq \Delta \\ |J| > |K| + 1}} c_{i,K}^J \cdot p_{v_J}.$$

The Peterson Schubert classes p_{v_J} are linearly independent by Theorem 5.9. Therefore whenever $|J| > |K| + 1$ the coefficient $c_{i,K}^J$ is zero. \square

We can further refine Equation (5.2) by removing another set of subsets $J \subseteq \Delta$ from the equation.

Lemma 5.13. *The constant $c_{i,K}^J = 0$ unless $K \subseteq J$.*

Proof. Suppose that L is the smallest counter example, i.e., $L \subseteq \Delta$ does not contain K and for all $H \subsetneq L$ the coefficient $c_{i,K}^H = 0$. Evaluate Monk's formula at the S -fixed point w_L to get

$$p_{s_i}(w_L) \cdot p_{v_K}(w_L) = \sum_{\substack{J \subseteq \Delta \\ |J| \leq |K| + 1}} c_{i,K}^J \cdot p_{v_J}(w_L).$$

The word $v_K \not\leq w_L$ by hypothesis so the left-hand side is 0. If $J \not\subseteq L$ then $p_{v_J}(w_L) = 0$ and thus

$$0 = \sum_{\substack{J \subseteq L \subseteq \Delta \\ |J| \leq |K| + 1}} c_{i,K}^J \cdot p_{v_J}(w_L).$$

By construction if $J \subsetneq L$ then $c_{i,K}^J = 0$ so we are left with

$$0 = c_{i,K}^L \cdot p_{v_L}(w_L).$$

By Lemma 5.8 the evaluation $p_{v_L}(w_L) \neq 0$. Since $H_S^*(pt) = \mathbb{C}[t]$ is an integral domain we conclude that $c_{i,K}^L = 0$. \square

Having determined coefficients that are always zero, we can give Monk's formula for Peterson varieties. Our coefficients are complex polynomials in t . We say such a polynomial is non-negative and rational if it is contained in $\mathbb{Q}_{\geq 0}[t]$.

Theorem 5.14 (Monk's formula for Peterson varieties). *The Peterson Schubert classes satisfy*

$$p_{s_i} \cdot p_{v_K} = p_{s_i}(w_K) \cdot p_{v_K} + \sum_{\substack{J \text{ such that} \\ K \subsetneq J \subseteq \Delta \\ |J|=|K|+1}} c_{i,K}^J \cdot p_{v_J}$$

where the coefficients $c_{i,K}^J$ are non-negative rational numbers given by

$$c_{i,K}^J = (p_{s_i}(w_J) - p_{s_i}(w_K)) \cdot \frac{p_{v_K}(w_J)}{p_{v_J}(w_J)}.$$

We need one more lemma in order to prove Monk's rule.

Lemma 5.15. *Consider the map $\pi_1 : H_T^*(G/B) \rightarrow H_S^*(G/B)$ from Equation (5.1). Let v, w be elements of the Weyl group. The image under π_1 of the evaluation $\sigma_v(w)$ of a Schubert class σ_v at the fixed point w is the monomial $c \cdot t^m$ where c is a non-negative integer and m is the length of v .*

Proof. By the properties of Billey's formula given in Proposition 2.7, the polynomial $\sigma_v(w)$ is homogeneous of degree $\ell(v)$ with non-negative integer coefficients. Its image $\pi_1(\sigma_v(w))$ is $ct^{\ell(v)}$ where c is the sum of the integer coefficients of $\sigma_v(w)$. \square

Now we prove Theorem 5.14.

Proof. By Lemma 5.12 the general Monk's formula in Equation (5.2) simplifies to

$$p_{s_i} \cdot p_{v_K} = \sum_{|J| \leq |K|+1} c_{i,K}^J \cdot p_{v_J}$$

and Lemma 5.13 further refines the equation to

$$p_{s_i} \cdot p_{v_K} = c_{i,K}^K \cdot p_{v_K} + \sum_{\substack{K \subsetneq J \subseteq \Delta \\ |J|=|K|+1}} c_{i,K}^J \cdot p_{v_J}. \quad (5.3)$$

If we evaluate both sides of Equation (5.3) at the S -fixed point w_K and use the fact that $p_{v_J}(w_K) = 0$ whenever J is not a subset of K , we obtain

$$p_{s_i}(w_K) \cdot p_{v_K}(w_K) = c_{i,K}^K \cdot p_{v_K}(w_K).$$

The polynomial $p_{v_K}(w_K)$ is non-zero by Lemma 5.8. Since $\mathbb{C}[t]$ is an integral domain we may divide both sides by $p_{v_K}(w_K)$. This leaves $c_{i,K}^K = p_{s_i}(w_K)$. By Lemma 5.15 the polynomial $p_{s_i}(w_K)$ is a degree-one monomial with an integer coefficient.

Next fix a subset $L \subseteq \Delta$ such that $K \subsetneq L$ and let $|L| = |K| + 1$. Evaluating at the S -fixed point w_L gives

$$p_{s_i}(w_L) \cdot p_{v_K}(w_L) = p_{s_i}(w_K) \cdot p_{v_K}(w_L) + \sum_{\substack{J \text{ such that} \\ K \subsetneq J \subseteq \Delta \\ |J|=|K|+1}} c_{i,K}^J \cdot p_{v_J}(w_L).$$

But $p_{v_J}(w_L) = 0$ unless $J \subseteq L$ by the properties of Billey's formula so in fact

$$p_{s_i}(w_L) \cdot p_{v_K}(w_L) = p_{s_i}(w_K) \cdot p_{v_K}(w_L) + c_{i,K}^L \cdot p_{v_L}(w_L).$$

Solving for $c_{i,K}^L$ gives

$$c_{i,K}^L = (p_{s_i}(w_L) - p_{s_i}(w_K)) \cdot \frac{p_{v_K}(w_L)}{p_{v_L}(w_L)}. \quad (5.4)$$

If the term $(p_{s_i}(w_L) - p_{s_i}(w_K)) = 0$ then the constant $c_{i,K}^L$ is clearly non-negative and rational. Suppose that $(p_{s_i}(w_L) - p_{s_i}(w_K)) \neq 0$. By Lemma 5.15 the term $(p_{s_i}(w_L) - p_{s_i}(w_K))$ has degree one. By the same lemma $\frac{p_{v_K}(w_L)}{p_{v_L}(w_L)}$ has degree $|K| - |L| = |K| - (|K| + 1) = -1$. It remains to show that $c_{i,K}^L$ is non-negative.

It suffices to show that $(p_{s_i}(w_L) - p_{s_i}(w_K))$ is non-negative because $\frac{p_{v_K}(w_L)}{p_{v_L}(w_L)}$ will always be non-negative. The word w_L can be written as $w_K \cdot \tilde{w}$ for some reduced word $\tilde{w} \in W_L$ [6]. Let $s_{b_1} s_{b_2} \cdots s_{b_m}$ be a reduced word for w_K and $s_{b_{m+1}} s_{b_{m+2}} \cdots s_{b_n}$ be a reduced word for \tilde{w} . The length $\ell(s_i) = 1$ for each i so Billey's formula says

$$\begin{aligned} \sigma_{s_i}(w_K \cdot \tilde{w}) &= \sum_{s_{b_j}=s_i} \mathbf{r}(\mathbf{j}, w_L) \\ &= \sum_{\substack{s_{b_j}=s_i \\ j \leq m}} \mathbf{r}(\mathbf{j}, w_L) + \sum_{\substack{s_{b_j}=s_i \\ j > m}} \mathbf{r}(\mathbf{j}, w_L) \\ &= \sigma_{s_i}(w_K) + \sum_{\substack{s_{b_j}=s_i \\ j > m}} \mathbf{r}(\mathbf{j}, w_L). \end{aligned}$$

Since π_1 is a ring homomorphism from $\mathbb{C}[\Delta]$ to $\mathbb{C}[t]$, we obtain

$$p_{s_i}(w_J) - p_{s_i}(w_K) = \pi_1(\sigma_{s_i}(w_J) - \sigma_{s_i}(w_K)) = \pi_1 \left(\sum_{\substack{s_{b_j}=s_i \\ j > m}} \mathbf{r}(\mathbf{j}, w_J) \right).$$

By the properties of Billey's formula each term $\mathbf{r}(\mathbf{j}, w_L)$ is a positive root in Φ . Therefore its image $\pi_1(\mathbf{r}(\mathbf{j}, w_J))$ is ct for some positive integer c . The t will be canceled by $\frac{p_{v_K}(w_L)}{p_{v_L}(w_L)}$ which has degree -1 . Thus $(p_{s_i}(w_L) - p_{s_i}(w_K))$ is non-negative and so is the coefficient $c_{i,K}^J$. \square

The coefficient $c_{i,K}^L$ given in the previous theorem is a priori a rational number. Frequently, but not always, $c_{i,K}^L$ will be an integer.

In classical Schubert calculus the structure constants are generally non-negative integers. Frequently they are in bijection with dimensions of irreducible representations. However, structure constants for the Peterson variety are *not* necessarily integers. For example in type D_5 let $K = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $J = \Delta$. Then

$$c_{5,K}^J = \frac{5}{2}.$$

Conjecture 5.16. *We conjecture that in this basis, non-integral structure constants only occur in Lie types D and E.*

5.3 Giambelli's Formula

Giambelli's formula tells us how to express an arbitrary module-basis element in terms of the ring generators. For the basis of $H_T^*(Flags)$ consisting of Schubert classes it looks like

$$\sigma_\lambda = \det(\sigma_{\lambda_i + j - i})_{1 \leq i, j \leq r}$$

where σ_λ is the Schubert class corresponding to the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ [11]. While easy to write down, this formula is hard to compute for a given Schubert class. Giambelli's formula for Peterson varieties simplifies to a single product.

Lemma 5.17. *For a Peterson Schubert class p_{v_K} there is a constant C satisfying*

$$C \cdot p_{v_K} = \prod_{\alpha_i \in K} p_{s_i}. \tag{5.5}$$

Proof. If $|K| = m$ let $K = \{\alpha_{a_1}, \alpha_{a_2}, \dots, \alpha_{a_m}\}$. Define a sequence of nested subsets $\emptyset = K_0 \subsetneq K_1 \subsetneq K_2 \subsetneq \dots \subsetneq K_m = K$ by

$$K_i = \{\alpha_{a_1}, \alpha_{a_2}, \dots, \alpha_{a_i}\}.$$

From Equation (5.3)

$$p_{s_{a_{i+1}}} \cdot p_{v_{K_i}} = c_{a_{i+1}, K_i}^{K_{i+1}} \cdot p_{v_{K_i}} + \sum_{\substack{K_i \subsetneq J \subseteq \Delta \\ |J|=|K_i|+1}} c_{a_{i+1}, K_i}^J \cdot p_{v_J}.$$

Theorem 5.14 says $c_{a_{i+1}, K_i}^{K_{i+1}} = p_{s_{a_{i+1}}}(w_{K_i})$. Because $\alpha_{a_{i+1}} \notin K_i$ the coefficient $p_{s_{a_{i+1}}}(w_{K_i}) = 0$. If $\alpha_{a_{i+1}} \notin J$ the term $p_{s_{a_{i+1}}}(w_J) = 0$. Thus if $J \neq K_{i+1}$ the coefficient $c_{a_{i+1}, K_i}^J = 0$. Now Equation (5.3) reduces to

$$p_{s_{a_{i+1}}} \cdot p_{v_{K_i}} = c_{a_{i+1}, K_i}^{K_{i+1}} \cdot p_{v_{K_{i+1}}}.$$

Solving for $p_{v_{K_{i+1}}}$ gives

$$\frac{p_{s_{a_{i+1}}} \cdot p_{v_{K_i}}}{c_{a_{i+1}, K_i}^{K_{i+1}}} = p_{v_{K_{i+1}}}.$$

By induction on i we see

$$p_{v_K} = \frac{\prod_{i=1}^{|K|} p_{s_{a_i}}}{\prod_{i=1}^{|K|} c_{a_i, K_{i-1}}^{K_i}}.$$

This gives that

$$C = \prod_{i=1}^{|K|} c_{a_{i+1}, K_i}^{K_{i+1}}.$$

□

Knowing that Giambelli's formula is a single product rather than a determinantal formula, we want to give the constant C explicitly. To find this C we consider the simplest non-trivial Peterson Schubert classes, those that are connected.

Definition 5.18. *In Definition 5.4 a subset of simple roots $K \subseteq \Delta$ was called connected if the induced Dynkin diagram of K is a connected subgraph of the Dynkin diagram of Δ . The class p_{v_K} is called connected whenever K is connected.*

Every Peterson Schubert class can be expressed in terms of connected classes.

Theorem 5.19. *If $J, K \subset \Delta$ are each connected subsets such that $J \cup K$ is disconnected then*

$$p_{v_{J \cup K}} = p_{v_J} \cdot p_{v_K}. \tag{5.6}$$

Proof. We show that equality holds when Equation (5.6) is evaluated at any S -fixed point w_L . If L does not contain $J \cup K$ we can suppose without loss of generality that $J \not\subseteq L$. Then both $p_{v_{J \cup K}}(w_L)$ and $p_{v_J}(w_L)$ are zero.

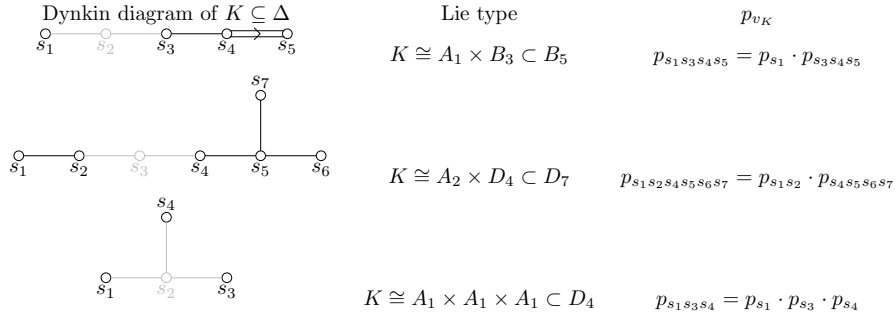


Figure 2: Connected root subsystems and connected Peterson Schubert classes. The subsystem $K \subseteq \Delta$ is drawn as a marked set of vertices in the Dynkin diagram. The associated Peterson Schubert class is given at right.

Now suppose $J \cup K \subseteq L$. Even though $J \cup K$ is disconnected, L may be connected or disconnected. Fix a reduced word for w_L

$$\tilde{w}_L = s_{a_1} s_{a_2} \cdots s_{a_{\ell(w_L)}}$$

and let $b \prec \tilde{w}_L$ mean that b is a subword of \tilde{w}_L . The indexing set of the subword b is the set $I(b) \subseteq \{1, 2, \dots, \ell(w_L)\}$ such that

$$b = s_{a_{j_1}} s_{a_{j_2}} \cdots s_{a_{j_{|I(b)|}}} \text{ for } j_1 < j_2 < \cdots < j_{|I(b)|} \text{ with each } j_k \in I(b).$$

The subwords of \tilde{w}_L are in bijection with the subsets of $\{1, 2, \dots, \ell(w_L)\}$. Given two subwords $b_1, b_2 \prec \tilde{w}_L$ we define their union $b_1 \cup b_2$ to be the subword

$$b_1 \cup b_2 = s_{a_{j_1}} s_{a_{j_2}} \cdots s_{a_{j_{|I(b_1) \cup I(b_2)|}}}$$

for $j_1 < j_2 < \cdots < j_{|I(b_1) \cup I(b_2)|}$ with each $j_k \in I(b_1) \cup I(b_2)$. Let $b_J, b_K \prec \tilde{w}_L$ be reduced words for v_J and v_K respectively. Since J and K are disconnected $I(b_J) \cap I(b_K) = \emptyset$ and v_J commutes entirely with v_K [6]. Thus $b_J \cup b_K$ is a reduced word for $v_J \cdot v_K = v_{J \cup K}$.

Conversely let $b \prec \tilde{w}_L$ be a reduced word for $v_{J \cup K}$. We can partition $I(b)$ into

$$I(b)_J = \{j_k \in I(b) : \alpha_{a_{j_k}} \in J\} \text{ and } I(b)_K = \{j_k \in I(b) : \alpha_{a_{j_k}} \in K\}.$$

Since $v_J \leq v_{J \cup K}$ and b is a reduced word for $v_{J \cup K}$, some reduced word for v_J must be a subword of b . Let $b_J \prec b$ be that subword. Since no reflections associated with K are in b_J , $I(b_J) \subseteq I(b)_J$.

A parallel argument shows that there is some subword $b_K \prec b$ equal to v_K and that $I(b_K) \subseteq I(b)_K$.

By our previous argument $b_J \cup b_K$ is a reduced word for $v_J \cdot v_K = v_{J \cup K}$. So $\ell(b_J \cup b_K) = \ell(v_{J \cup K})$ which equals $\ell(b)$. Thus $I(b_J) = I(b)_J$ and $I(b_K) = I(b)_K$ and $b = b_J \cup b_K$.

A subword $b \prec \tilde{w}_L$ is a reduced word for $v_{J \cup K}$ if and only if $b = b_J \cup b_K$ for subwords $b_J, b_K \prec \tilde{w}_L$ reduced words for v_J and v_K . Billey's formula in Equation (2.1) is a sum over such subwords. We use it to rewrite the left- and right-hand sides of Equation (5.6). The left-hand side becomes:

$$p_{v_{J \cup K}}(w_L) = \sum_{\substack{b \prec \tilde{w}_L \\ b = v_{J \cup K} \\ |I(b)| = |J \cup K|}} \left(\prod_{j \in I(b)} \mathbf{r}(\mathbf{j}, \tilde{w}_L) \right). \quad (5.7)$$

Similarly the right-hand side becomes $p_{v_J}(w_L) \cdot p_{v_K}(w_L) =$

$$\left[\sum_{\substack{b \prec \tilde{w}_L \\ b = v_J \\ |I(b)| = |J|}} \left(\prod_{j \in I(b)} \mathbf{r}(\mathbf{j}, \tilde{w}_L) \right) \right] \cdot \left[\sum_{\substack{b \prec \tilde{w}_L \\ b = v_K \\ |I(b)| = |K|}} \left(\prod_{j \in I(b)} \mathbf{r}(\mathbf{j}, \tilde{w}_L) \right) \right]. \quad (5.8)$$

Both Equations (5.7) and (5.8) expand out to the expression

$$\sum_{\substack{b_J \prec \tilde{w}_L \\ b_J = v_J \\ |I(b_J)| = |J|}} \sum_{\substack{b_K \prec \tilde{w}_L \\ b_K = v_K \\ |I(b_K)| = |K|}} \left[\left(\prod_{j \in I(b_J)} \mathbf{r}(\mathbf{j}, \tilde{w}_L) \right) \cdot \left(\prod_{j \in I(b_K)} \mathbf{r}(\mathbf{j}, \tilde{w}_L) \right) \right].$$

□

Any subset $K \subset \Delta$ gives rise to a Peterson Schubert class that is the product of connected Peterson Schubert classes. Understanding the connected Peterson Schubert classes thus gives full information on all Peterson Schubert classes. The next theorem gives Giambelli's formula explicitly for connected Peterson Schubert classes.

Theorem 5.20. *If $K \subseteq \Delta$ is a connected root subsystem of type A_n, B_n, C_n, F_4 , or G_2 then*

$$|K|! \cdot p_{v_K} = \prod_{\alpha_i \in K} p_{s_i}.$$

If K is a connected root subsystem of type D_n then

$$\frac{|K|!}{2} \cdot p_{v_K} = \prod_{\alpha_i \in K} p_{s_i}.$$

If K is a connected root subsystem of type E_n then

$$\frac{|K|!}{3} \cdot p_{v_K} = \prod_{\alpha_i \in K} p_{s_i}.$$

Our proof of this theorem is combinatorial and treats each Lie type as its own case. The uniformity across Lie types suggests that a uniform proof exists. Such a proof might shed light on the topology of these varieties. In fact, the theorem can be stated in an even more uniform manner.

Theorem 5.21. *If $K \subseteq \Delta$ is a connected root subsystem of any Lie type and $|\mathcal{R}(v_K)|$ is the number of reduced words for v_K then*

$$\frac{|K|!}{|\mathcal{R}(v_K)|} \cdot p_{v_K} = \prod_{\alpha_i \in K} p_{s_i}.$$

Proof. Given Theorem 5.20 it is sufficient to show that $|\mathcal{R}(v_K)| = 1$ if K is type A, B, C, F , or G , that $|\mathcal{R}(v_K)| = 2$ for type D and that $|\mathcal{R}(v_K)| = 3$ for type E . Given one reduced word any other reduced word can be obtained by a series of braid moves and commutations [6]. If K is type A, B, C, F , or G then s_i and s_{i+1} do not commute for any i . Therefore $s_1 s_2 \cdots s_{n-1} s_n$ is the only reduced word for v_K .

If K is of type D then s_i and s_{i+1} commute if and only if $i = n - 1$. Also s_{n-2} and s_n do not commute. The only two reduced words for v_K are $s_1 s_2 \cdots s_{n-2} s_{n-1} s_n$ and $s_1 s_2 \cdots s_{n-2} s_n s_{n-1}$ so $|\mathcal{R}(v_K)| = 2$.

If K is type E_n then we start with the word $v_K = s_1 s_2 s_3 s_4 \cdots s_n$ with the labels given as in Figure 1. The reflection s_2 commutes with s_1 and s_3 but not s_4 . The reflection s_3 does not commute with s_1 . When $i > 2$, s_i and s_{i+1} do not commute. Thus v_K has exactly 3 reduced words: $s_1 s_2 s_3 s_4 \cdots s_n$ and $s_1 s_3 s_2 s_4 \cdots s_n$ and $s_2 s_1 s_3 s_4 \cdots s_n$. \square

We can now give Giambelli's formula explicitly for all Peterson Schubert classes.

Corollary 5.22. *If $K \subseteq \Delta$ and $K = K_1 \times K_2 \times \cdots \times K_j$ where each K_ℓ is a connected root system then*

$$C_K \cdot p_{v_K} = \prod_{i \in K} p_{s_i} \tag{5.9}$$

where $C_K = \prod_{\ell=1}^j \frac{|K_\ell|!}{|\mathcal{R}(v_{K_\ell})|}$.

Proof. The corollary follows immediately from Theorems 5.19 and 5.21. \square

5.4 Modified Excited Young Diagrams

To compute the constant term of Giambelli's formula we must evaluate the Peterson Schubert class p_v at fixed points. These are related both to the work by Woo -Yong [37], and the work by Ikeda-Naruse [20]. For the remainder of this paper, K will be a connected root system identified by Lie type.

Since we only evaluate at fixed points w_K where $K \subseteq \Delta$ is a connected root system, the first step is to write w_K explicitly as a skew diagram λ_{w_K} . The i^{th} column from the left represents the simple reflection s_i . Reading left-to-right and top-to-bottom gives a reduced word for w_K . Figure 3 gives several examples.

The goal is to compute $p_v(w_K)$. To start we use Equations (2.1) and (5.1) to rewrite it as

$$\begin{aligned} \pi(\sigma_v(w_K)) &= \pi \left(\sum_{\substack{\text{reduced words} \\ v=s_{b_{j_1}} s_{b_{j_2}} \cdots s_{b_{j_{\ell(v)}}}} \left(\prod_{i=1}^{\ell(v)} \mathbf{r}(\mathbf{j}_i, w_K) \right) \right) \\ &= \sum_{\substack{\text{reduced words} \\ v=s_{b_{j_1}} s_{b_{j_2}} \cdots s_{b_{j_{\ell(v)}}}} \left(\prod_{i=1}^{\ell(v)} \pi(\mathbf{r}(\mathbf{j}_i, w_K)) \right). \end{aligned}$$

In order to do this we label the \mathbf{i}^{th} box of λ_{w_K} with $\frac{1}{t} \cdot \pi_1(\mathbf{r}(\mathbf{i}, w_K))$. The term $\pi_1(\mathbf{r}(\mathbf{i}, w_K))$ is a degree-one monomial in $\mathbb{C}[t]$ whose coefficient is the height of the root $\mathbf{r}(\mathbf{i}, w_K)$. Thus the labels are positive integers. Call this labeled shape $\lambda_{p(w_K)}$. We give an example in type B_3 :

$$\begin{array}{cc} \lambda_{w_K} & \lambda_{p(w_K)} \\ \begin{array}{ccc} & & s_3 \\ & s_2 & s_3 \\ s_1 & s_2 & s_3 \\ s_1 & s_2 & \\ s_1 & & \end{array} & \begin{array}{ccc} & & 1 \\ & 3 & 2 \\ 4 & 5 & 3 \\ 1 & 2 & \\ 1 & & \end{array} \end{array}.$$

A v -excitation μ of λ_{w_K} is any collection of $\ell(v)$ boxes in the labeled diagram such that the labels of the boxes of λ_{w_K} when read left-to-right and top-to-bottom give a reduced word for v . For

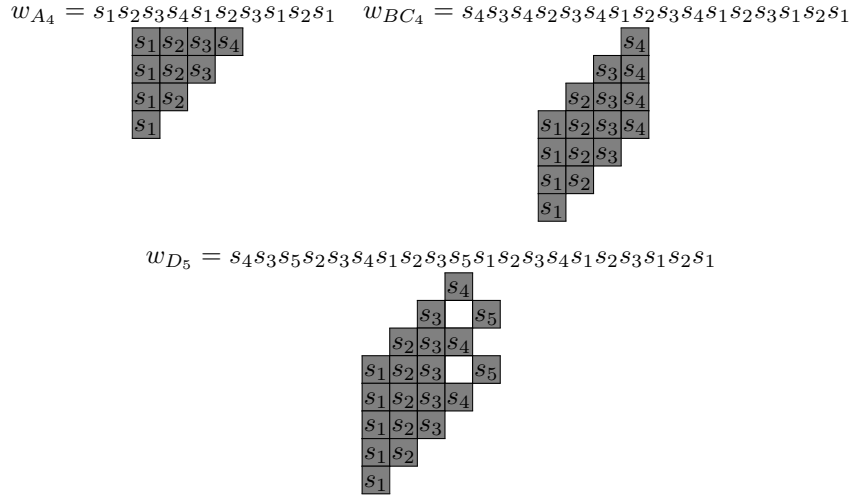
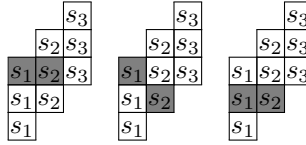
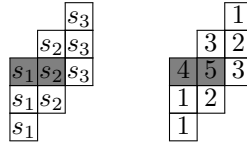


Figure 3: Skew diagrams of w_K . We use the reduced words for w_K given by Sage.

example if K is type B_3 there are three $s_1 s_2$ -excitations of λ_{w_K} :



If μ is a v -excitation of λ_{w_K} then $M_p(\mu)$ is the product of the entries in the boxes of $\lambda_{p(w_K)}$ filled by μ .



For this $s_1 s_2$ -excitation μ of w_{B_3} the coefficient is $M_p(\mu) = (4)(5) = 20$. Now $p_v(w_K)$ can be computed by this diagrammatic construction:

$$p_v(w_K) = \sum_{\substack{\mu \text{ a } v\text{-excitation} \\ \text{of } w_K}} M_p(\mu) \cdot t^{\ell(v)}. \quad (5.10)$$

5.5 Proof of the Giambelli's formula

Theorems 5.20 and 5.21 gave two versions of the main theorem of this paper. Having used Theorem 5.20 to prove Theorem 5.21, we now prove Theorem 5.20 case by case according to Lie type. This section first gives a proof for type A which will be used in subsequent sections for the

proofs of the other classical types. Last we prove the theorem for the exceptional Lie types. This involves computer-generated proofs for the E series and explicit calculations for types F_4 and G_2 .

5.5.1 Type A

$$A_n : \quad \circ \text{---} \circ \text{---} \circ \text{---} \cdots \text{---} \circ \text{---} \circ$$

$s_1 \quad s_2 \quad s_3 \quad s_{n-2} \quad s_{n-1} \quad s_n$

While Giambelli's formula for type A was fully addressed by Bayegan-Harada [1], we give a proof using our excited Young diagrams. The diagrams λ_{w_K} and $\lambda_{p(w_K)}$ are

$$\lambda_{w_K} = \begin{array}{cccc} s_1 & s_2 & s_3 & \cdots & s_{n-2} & s_{n-1} & s_n \\ s_1 & s_2 & s_3 & \cdots & s_{n-2} & s_{n-1} & \\ s_1 & s_2 & s_3 & \cdots & s_{n-2} & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ s_1 & s_2 & s_3 & & & & \\ s_1 & s_2 & & & & & \\ s_1 & & & & & & \end{array} \quad \lambda_{p(w_K)} = \begin{array}{cccc} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ 1 & 2 & 3 & \cdots & n-2 & n-1 & \\ 1 & 2 & 3 & \cdots & n-2 & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ 1 & 2 & 3 & & & & \\ 1 & 2 & & & & & \\ 1 & & & & & & \end{array}$$

Proof. Lemma 5.17 showed $C \cdot p_{v_K} = \prod_{1 \leq i \leq n} p_{s_i}$. Evaluating this equation at the fixed point w_K gives

$$C \cdot p_{v_K}(w_K) = \prod_{1 \leq i \leq n} p_{s_i}(w_K). \tag{5.11}$$

There is only one filling w_K with $v_K = s_1 s_2 \cdots s_n$, specifically

$$\mu = \begin{array}{cccc} s_1 & s_2 & s_3 & \cdots & s_{n-2} & s_{n-1} & s_n \\ s_1 & s_2 & s_3 & \cdots & s_{n-2} & s_{n-1} & \\ s_1 & s_2 & s_3 & \cdots & s_{n-2} & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ s_1 & s_2 & s_3 & & & & \\ s_1 & s_2 & & & & & \\ s_1 & & & & & & \end{array}$$

Thus $p_{v_K}(w_K) = M(\mu) \cdot t^n = n! \cdot t^n$. We must also evaluate $p_{s_i}(w_K)$ for each $s_i \in K$. From $\lambda_{p(w_K)}$ each s_i -excitation μ of w_K has $M(\mu) = i$. From the diagram, there are $n - i + 1$ such excitations. So

$$p_{s_i}(w_K) = \sum_{\substack{\mu \text{ a } s_i\text{-excitation} \\ \text{of } w_K}} M(\mu) \cdot t^{\ell(s_i)} = (n - i + 1) i \cdot t.$$

Solving Equation (5.11) for C we obtain

$$C \cdot n! \cdot t^n = \prod_{1 \leq i \leq n} [(n-i+1)i \cdot t] = \prod_{1 \leq i \leq n} (n-i+1) \cdot \prod_{1 \leq i \leq n} (i \cdot t) = (n!)^2 \cdot t^n$$

which gives $C = n! = |K|!$ as desired. \square

The type A result greatly simplifies the proof for the other classical Lie types.

Lemma 5.23. *Let K be a root system of type $B_n, C_n,$ or D_n . Then $J = K \setminus \{s_n\}$ is a root subsystem of type A_{n-1} and*

$$c_{n,J}^K \cdot (n-1)! \cdot p_{v_K} = \prod_{s_i \in K} p_{s_i} \quad (5.12)$$

where $c_{n,J}^K = p_{s_n}(w_K) \cdot \frac{p_{v_J}(w_K)}{p_{v_K}(w_K)}$.

Proof. By the proof of Giambelli's formula for type A and Theorem 5.14 respectively,

$$(n-1)! \cdot p_{v_J} = \prod_{s_i \in J} p_{s_i} \quad \text{and} \quad p_{s_n} \cdot p_{v_J} = c_{n,J}^K \cdot p_{v_K}. \quad (5.13)$$

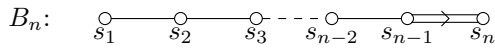
Combining these gives Equation (5.12). By Theorem 5.14

$$c_{n,J}^K = (p_{s_n}(w_K) - p_{s_n}(w_J)) \cdot \frac{p_{v_J}(w_K)}{p_{v_K}(w_K)}.$$

By construction the root α_n is not in J so $p_{s_n}(w_J) = 0$ giving the desired result. \square

Now we will prove Giambelli's formula for the other classical Lie types.

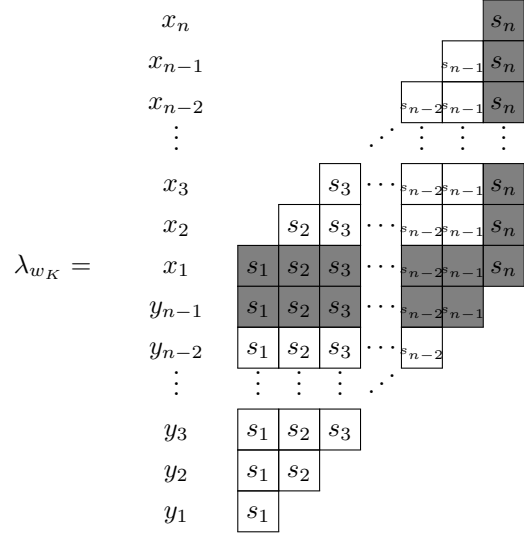
5.5.2 Type B



Proposition 5.24. *Theorem 5.20 holds when K is a type B root system.*

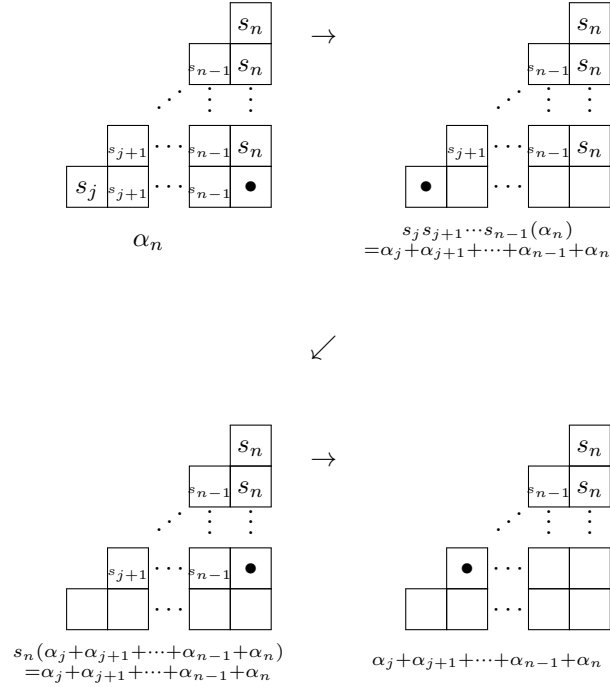
Proof. Let K be a type B_n root system and $J \subset K = K \setminus \{s_n\}$. By Lemma 5.23 showing that $c_{n,J}^K = n$ is sufficient to prove Giambelli's formula for type B .

If K is of type B_n the diagram of the reduced word for w_K is given below. Each row is labeled by the word of reflections in that row. For example $x_2 = s_2 s_3 \cdots s_{n-1} s_n$.



To compute $c_{n,J}^K$ we need to compute $p_v(w_K)$ where v is s_n , $s_1 s_2 \cdots s_{n-1}$, and $s_1 s_2 \cdots s_n$. All of the v -excitations of w_K for these words are contained in the shaded area of the λ_{w_K} above. So we only need the entries of $\lambda_{p(w_K)}$ in those shaded boxes. Start with the box labeled s_n in row x_j of the diagram. The reflections that come after do not act on the root, so we look at $x_n x_{n-1} \cdots x_j$ and calculate the root as it moves through the diagram. A bullet, \bullet , marks the location of the root in the diagram at each step. The initial root is below the first diagram and we follow how it

changes.



By the time the bullet gets to the position in the lower left, the root is $\alpha_j + \cdots + \alpha_n$ which is invariant under all simple reflections except s_j and s_{j-1} . Neither of those reflections act on the bullet as it continues through the diagram. Thus the label on the box s_n in row x_j of $\lambda_{p(w_K)}$ is $n - j + 1$.

We can start the bullet in any box of the diagram. Suppose that the h^{th} simple reflection of w_k is the i^{th} box in row y_{n-1} . Then $\mathbf{r}(\mathbf{h}, w_K)$ will be

$$x_n \cdots x_1 s_1 s_2 \cdots s_{i-1} s_{i-1}(\alpha_i) = x_n \cdots x_1 \left(\sum_{m=1}^i \alpha_m \right) = \sum_{m=1}^i \alpha_{n-m}.$$

Thus the entry in the corresponding box of $\lambda_{p(w_K)} = \frac{1}{\ell} \pi \left(\sum_{m=1}^i \alpha_{n-m} \right) = i$.

Another bullet-pushing argument shows that

$$x_n x_{n-1} \cdots x_2(\alpha_1) = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + 2\alpha_n.$$

If $1 < i < n$ then $x_n x_{n-1} \cdots x_2(\alpha_i) = \alpha_{n-i+1}$. Let the h^{th} simple reflection of w_K to be the i^{th} box of row x_1 for some $i \neq n$. The root $\mathbf{r}(\mathbf{h}, w_K)$ is

$$\begin{aligned} x_n x_{n-1} \cdots x_2 s_1 s_2 \cdots s_{i-1}(\alpha_i) &= x_n x_{n-1} \cdots x_2 \left(\sum_{m=1}^i \alpha_m \right) \\ &= \alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} + 2\alpha_n + \sum_{m=2}^i \alpha_{n-m+1}. \end{aligned}$$

Thus the entry in the corresponding box of $\lambda_{p(w_K)}$ is $n + i$. Now we can label the relevant boxes of $\lambda_{p(w_K)}$:

$$\lambda_{p(w_K)} = \begin{array}{ccccccc} & & & & & & 1 \\ & & & & & & 2 \\ & & & & & & 3 \\ & & & \dots & & & \vdots \\ & & & & & & \vdots \\ x_3 & & & & \dots & & n-2 \\ x_2 & & & & \dots & & n-1 \\ x_1 & n+1 & n+2 & n+3 & \dots & 2n-2 & 2n-1 & n \\ y_{n-1} & 1 & 2 & 3 & \dots & n-2 & n-1 \end{array}$$

With this labeling established, we can see that $p_{s_n}(w_K) = \frac{n(n+1)}{2}t$. We also observe that there is only one v_K -excitation of w_K since v_K contains, in order, the reflections s_1 through s_n which appear in that order in row x_1 and in no other subword of w_K . So

$$p_{v_K}(w_K) = (n+1)(n+2) \cdots (2n-2)(2n-1)n \cdot t^n = \frac{(2n-1)!}{(n-1)!} t^n.$$

The last piece is to calculate $p_{v_J}(w_K)$. All subwords of w_K that are reduced words for v_J are entirely contained within the $x_1 y_{n-1}$ subword of w_K . We look at the excited Young diagrams in just those two rows of $\lambda_{p(w_K)}$. A v_J -excitation μ of these two rows is determined by how many boxes it takes from row x_1 . The excitation μ that uses i boxes from row x_1 looks like:

$$\begin{array}{ccccccc} n+1 & n+2 & n+3 & \cdots & n+i & n+i+1 & \cdots & 2n-2 & 2n-1 & n \\ 1 & 2 & 3 & \cdots & i & i+1 & \cdots & n-2 & n-1 \end{array}$$

This v_J -excitation contributes coefficient $M(\mu) = \frac{(n+i)!}{n \cdot i!}$ to $p_{v_J}(w_K)$. Since $0 \leq i \leq n-1$,

$$p_{v_J}(w_K) = \sum_{i=0}^{n-1} \frac{(n+i)!}{n \cdot i!} t^{n-1}.$$

Putting all of the pieces together we have

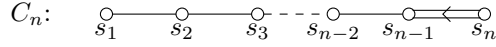
$$\begin{aligned} c_{n,J}^K &= \frac{p_{s_n}(w_K) \cdot p_{v_J}(w_K)}{p_{v_K}(w_K)} \\ &= \frac{\frac{n(n+1)}{2} t}{\frac{(2n-1)!}{(n-1)!} t^n} \cdot \sum_{i=0}^{n-1} \frac{(n+i)!}{n \cdot i!} t^{n-1} \\ &= \frac{(n+1)!}{2n \cdot (2n-1)!} \cdot \sum_{i=0}^{n-1} \frac{(n+i)!}{i!}. \end{aligned}$$

By a combinatorial identity for the sum, this can be rewritten as

$$c_{n,J}^K = \frac{(n+1)!}{2n \cdot (2n-1)!} \cdot \frac{n \cdot (2n)!}{(n+1)!} = n.$$

□

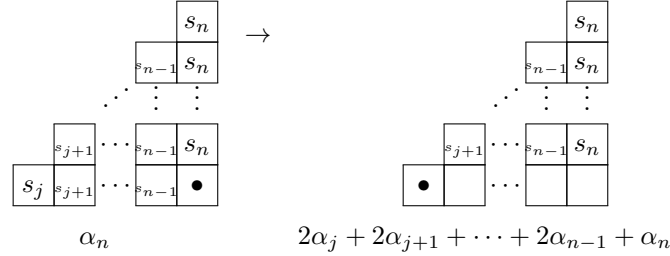
5.5.3 Type C



Proposition 5.25. *Theorem 5.20 holds when K is a type C root system.*

Proof. This proof mirrors the proof in type B . Let K be a type C_n root system and define $J \subset K$ to be $J = K \setminus \{s_n\}$. By Lemma 5.23 showing that $c_{n,J}^K = n$ is sufficient to prove Giambelli's formula for type C .

The longest word w_K is the same for type C_n as for type B_n , in fact the only changes from type B are the box labels of $\lambda_{p(w_K)}$. First we find the label for the box corresponding to reflection s_n in row x_j .



The root $2\alpha_j + 2\alpha_{j+1} + \cdots + 2\alpha_{n-1} + \alpha_n$ is invariant under all reflections except s_j and s_{j-1} which will not act on the root. Thus the label in $\lambda_{p(w_K)}$ is $2(n-j) + 1$. Adding up all the labels of these boxes gives that

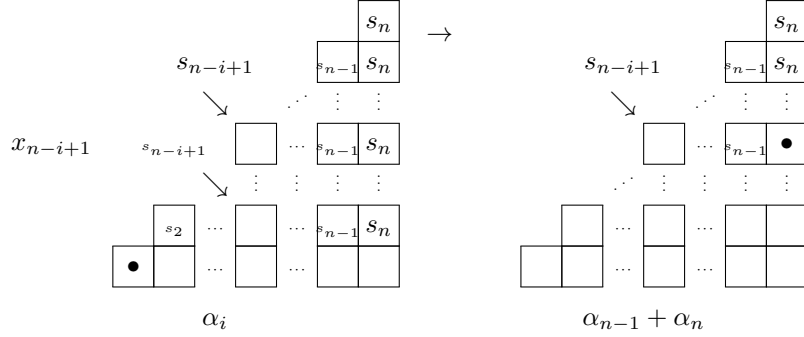
$$p_{s_n}(w_K) = \sum_{j=1}^n (2j-1) \cdot t = n^2 \cdot t.$$

To compute $p_{v_J}(w_K)$ and $p_{v_K}(w_K)$ we need to compute the $\lambda_{p(w_K)}$ diagram labels of rows y_{n-1} and x_1 . If the h^{th} box is box $i \neq n$ of row x_1 then

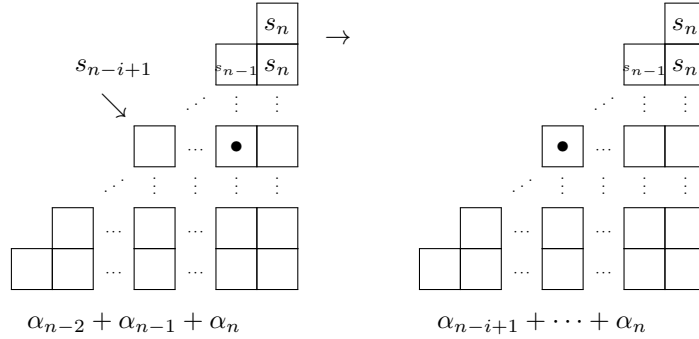
$$\mathbf{r}(\mathbf{h}, w_K) = x_n x_{n-1} \cdots x_2 s_1 s_2 \cdots s_{i-1}(\alpha_i) = x_n x_{n-1} \cdots x_2 \left(\sum_{m=1}^i \alpha_m \right).$$

By moving a root through the diagram, $x_n x_{n-1} \cdots x_2(\alpha_1) = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and if $1 < i < n$

then $x_n x_{n-1} \cdots x_2(\alpha_i)$ gets pushed through the diagram as follows:



Since we are working in type C the reflection s_{n-1} sends α_n to $2\alpha_{n-1} + \alpha_n$ so the next row of the diagram acts like this:



Row x_{n-i+2} eliminates everything except α_{n-i+1} which is preserved for the rest of the diagram.

So

$$\mathbf{r}(\mathbf{h}, w_K) = x_n x_{n-1} \cdots x_2 \left(\sum_{m=1}^i \alpha_m \right) = \sum_{m=1}^n \alpha_m + \sum_{m=2}^i \alpha_{n-m+1}$$

and the entry in $\lambda_{p(w_K)}$ is $n+i-1$. Like in type B , $x_n x_{n-1} \cdots x_2 x_1(\alpha_i) = \alpha_{n-i}$ for $i \neq n$. We can fill in the entries of rows x_1 and y_{n-1} of $\lambda_{p(w_K)}$ as follows:

$$\begin{array}{c} x_1 \\ y_{n-1} \end{array} \begin{array}{ccccccc} \text{light gray} & \text{light gray} & \text{light gray} & \cdots & \text{light gray} & \text{light gray} & \cdots \\ n & n+1 & n+2 & \cdots & n+i-1 & n+i & \cdots \\ 1 & 2 & 3 & \cdots & i & i+1 & \cdots \\ \text{light gray} & \text{light gray} & \text{light gray} & \cdots & \text{light gray} & \text{light gray} & \cdots \\ 2n-2 & 2n-1 & n & \cdots & n-2 & n-1 & \cdots \end{array}$$

A v_J -excitation of w_K is marked in light gray. Summing over the number of boxes of μ that are in row x_1 as in the previous section gives

$$p_{v_J}(w_K) = \sum_{i=0}^{n-1} \frac{(n+i-1)!}{i!} t^{n-1} = \frac{(2n-1)!}{n!} t^{n-1}.$$

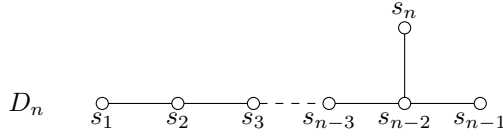
We also see that there is only one v_K -excitation of w_K so $p_{v_K}(w_K) = \frac{(2n-1)!}{(n-1)!} t^n$.

Putting all the pieces together we obtain

$$c_{n,J}^K = p_{s_n}(w_K) \frac{p_{v_J}(w_K)}{p_{v_K}(w_K)} = n^2 \frac{(2n-1)!}{(n-1)!} = \frac{n^2}{n} = n.$$

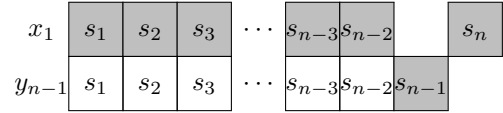
□

5.5.4 Type D

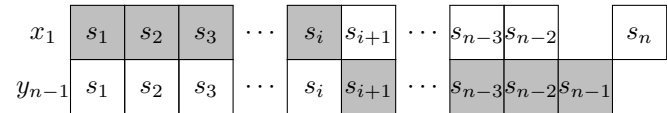


Proposition 5.26. *Theorem 5.20 holds when K is a connected type D root system.*

Proof. Let K be a connected type D_n root system and $J \subset K = K \setminus \{s_n\}$. By Lemma 5.23 it suffices to show that $c_{n,J}^K = \frac{n}{2}$. If K is a root system of type D_n then the shape of λ_{w_K} depends on whether n is even or odd. Figure 4 gives the two diagrams for type D_n . In each of these shapes, there is only one v_K -excitation of w_K . This subword occurs in the rows x_1 and y_{n-1} and looks like:



The v_J -excitations of w_K are in the same two rows and there are $n - 1$ such excitations. Each excitation μ looks like:

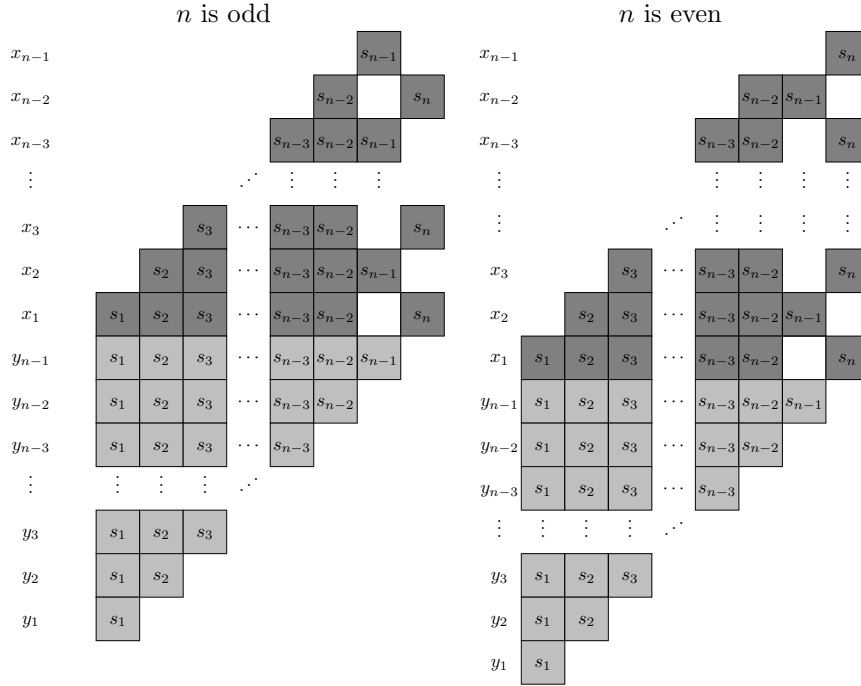


We need to find the labels of these boxes in $\lambda_{p(w_K)}$ in order to compute $p_{v_K}(w_K)$ and $p_{v_J}(w_K)$. Denote by x the word obtained from the first $n - 1$ rows of w_K , i.e. $x = x_{n-1}x_{n-2}\cdots x_2x_1$. We compute $x(\alpha_i)$ for $i < n$.

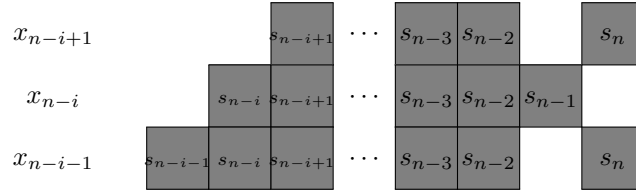
First we examine the action of x on α_i for $1 < i \leq n - 2$. Suppose that $i < n - 2$. Then we take the action of x row by row to get to the root α_{n-2} . The first reflection in x_1 to not preserve α_i is s_{i+1} which sends it to $\alpha_i + \alpha_{i+1}$. The next reflection, s_i then brings the root to α_{i+1} which the rest of the reflections in x_1 preserve. Similarly $x_2(\alpha_{i+1}) = \alpha_{i+2}$ if $i + 1 < n - 2$. This pattern continues until

$$x_{n-i-2}x_{n-i-3}\cdots x_1(\alpha_i) = \alpha_{n-2}$$

Figure 4: The diagrams of λ_{w_K} for K a type D_n root system.



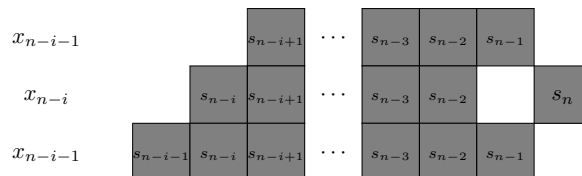
The action of the next three rows, x_{n-i-1} , x_{n-i} , and x_{n-i+1} depend on whether $n - i$ is even or odd. If $n - i$ is odd then the next three rows have form



The actions of these rows are:

$$\begin{aligned} x_{n-i-1}(\alpha_{n-2}) &= \alpha_{n-1} \\ x_{n-i}(\alpha_{n-1}) &= \alpha_{n-i} + \cdots + \alpha_{n-2} + \alpha_{n-1} \\ x_{n-i+1}(\alpha_{n-i} + \cdots + \alpha_{n-2} + \alpha_{n-1}) &= \alpha_{n-i} \end{aligned}$$

If instead $n - i$ is even the next three rows look like



and act by:

$$\begin{aligned} x_{n-i-1}(\alpha_{n-2}) &= \alpha_n \\ x_{n-i}(\alpha_{n-1}) &= \alpha_{n-i} + \cdots + \alpha_{n-2} + \alpha_n \\ x_{n-i+1}(\alpha_{n-i} + \cdots + \alpha_{n-2} + \alpha_n) &= \alpha_{n-i}. \end{aligned}$$

Whether $n - i$ is odd or even, the root α_{n-i} is invariant under the action of s_j for $j > n - i + 1$ so $x(\alpha_i) = \alpha_{n-i}$ for all i greater than 1 and less than $n - 2$.

If we start with the root α_1 then $x_{n-3}x_{n-4} \cdots x_1(\alpha_1) = \alpha_{n-2}$. The rest of the computation is

$$x_{n-1}x_{n-2}(\alpha_{n-2}) = \begin{cases} s_{n-1}s_{n-2}s_n(\alpha_{n-2}) = \alpha_n & \text{if } n \text{ is odd} \\ s_n s_{n-2} s_{n-1}(\alpha_{n-2}) = \alpha_{n-1} & \text{if } n \text{ is even.} \end{cases}$$

Next we address $x(\alpha_{n-1})$. Going row by row,

$$x_1(\alpha_{n-1}) = \alpha_1 + \alpha_2 + \cdots + \alpha_{n-2} + \alpha_{n-1} \text{ and}$$

$$x_2(\alpha_1 + \alpha_2 + \cdots + \alpha_{n-2} + \alpha_{n-1}) = \alpha_1$$

which is invariant under s_j for $j > 2$. Thus $x(\alpha_{n-1}) = \alpha_1$.

The result of these computations is that for $i < n$ the word x takes each root α_i to a root α_j and therefor $\pi_1(x(\alpha_i)) = t$ for all $i < n$. Since the label in the i^{th} box of row y_{n-1} is the height of the root $x s_1 s_2 \cdots s_{i-1}(\alpha_i) = x(\alpha_1 + \cdots + \alpha_i)$ that box in $\lambda_{p(w_K)}$ is labeled i .

We also want to find the root corresponding to the i^{th} box of row x_1 . A non-reduced way to write the word preceding that box in w_K is $x s_n s_{n-2} s_{n-3} \cdots s_{i+1} s_i$ and the corresponding root is

$$\begin{aligned} x s_n s_{n-2} s_{n-3} \cdots s_{i+1} s_i(\alpha_i) &= x(-\alpha_i - \alpha_{i+1} - \cdots - \alpha_{n-2} - \alpha_n) \\ &= -x(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-2}) - x(\alpha_n). \end{aligned}$$

We compute the action of x on the root α_n :

$$\begin{aligned} x_1(\alpha_n) &= -(\alpha_1 + \alpha_2 + \cdots + \alpha_{n-2} + \alpha_n) \\ s_{n-1}(-(\alpha_1 + \alpha_2 + \cdots + \alpha_{n-2} + \alpha_n)) &= -(\alpha_1 + \alpha_2 + \cdots + \alpha_{n-2} + \alpha_{n-1} + \alpha_n) \end{aligned}$$

$$s_{n-2}(-(\alpha_1 + \alpha_2 + \cdots + \alpha_{n-2} + \alpha_{n-1} + \alpha_n)) = -(\alpha_1 + \alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n)$$

Each subsequent reflection places the coefficient 2 in front of another simple root until

$$x_2 x_1(\alpha_n) = -(\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n).$$

This root is invariant under the action of s_i for $i > 2$ and thus

$$x(\alpha_n) = -(\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n)$$

Thus the root associated with the i^{th} box of row x_1 is $x s_n s_{n-2} s_{n-3} \cdots s_{i+1}(\alpha_i) =$

$$-x(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{n-2}) + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n.$$

which has height in the root poset $2n - 3 - (n - i - 1) = n + i - 2$. We can now label the rows

x_1 and y_{n-1} in the diagram $\lambda_{p(w_K)}$.

$$\begin{array}{c} x_1 \\ y_{n-1} \end{array} \begin{array}{|c|c|c|} \hline n-1 & n & n+1 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \cdots \begin{array}{|c|c|c|} \hline 2n-5 & 2n-4 & \\ \hline n-3 & n-2 & n-1 \\ \hline \end{array} \begin{array}{|c|} \hline 2n-3 \\ \hline \end{array}.$$

This means that $p_{v_K}(w_K) = \frac{(2n-3)!}{(n-2)!} (n-1)t^n$. Furthermore when μ is a v_J -excitation with i boxes in the top row we see a contribution to $p_{v_J}(w_K)$ of $M(\mu) = (n-1) \frac{(n+i-2)!}{i!}$. Therefore

$$p_{v_J}(w_K) = \left[\sum_{i=0}^{n-2} (n-1) \frac{(n+i-2)!}{i!} \right] t^{n-1} = \frac{(2n-3)!}{(n-2)!} t^{n-1}.$$

The constant $c_{n,J}^K$ is

$$p_{s_n}(w_K) \frac{p_{v_J}(w_K)}{w_{v_K}(w_K)} = p_{s_n}(w_K) \frac{\frac{(2n-3)!}{(n-2)!} t^{n-1}}{\frac{(2n-3)!}{(n-2)!} (n-1)t^n} = p_{s_n}(w_K) \frac{1}{(n-1)t}. \quad (5.14)$$

The polynomial $p_{s_n}(w_K)$ is computed in the even and odd cases. In both cases, if m is less than $n-1$ and there is a box corresponding to s_n in row x_m , then

$$\begin{aligned} & x_{n-1} x_{n-2} x_{n-3} \cdots x_{m+1} s_m s_{m+1} \cdots s_{n-3} s_{n-2}(\alpha_n) \\ &= x_{n-1} x_{n-2} x_{n-3} \cdots x_{m+1} (\alpha_m + \alpha_{m+1} + \cdots + \alpha_{n-3} + \alpha_{n-2} + \alpha_n) \\ &= x_{n-1} x_{n-2} x_{n-3} \cdots x_{m+2} (\alpha_m + 2\alpha_{m+1} + \cdots + 2\alpha_{n-3} + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n). \end{aligned}$$

The root $\alpha_m + 2\alpha_{m+1} + \cdots + 2\alpha_{n-3} + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ is invariant under the action of s_i for $i > m+1$. Thus the root corresponding to that box is

$$\alpha_m + 2\alpha_{m+1} + \cdots + 2\alpha_{n-3} + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$$

and the entry in $\lambda_{p(w_K)}$ is $2(n - m) - 1$.

If n is odd then row x_{n-1} does not contain the reflection s_n and

$$\begin{aligned} p_{s_n}(w_K) &= \left[\sum_{\substack{1 \leq m \leq n-1 \\ m \text{ odd}}} 2(n - m) - 1 \right] \cdot t \\ &= \left[\sum_{l=1}^{\frac{n}{2}-1} 2n - 4l - 1 \right] \cdot t \\ &= \frac{n^2 - n}{2} \cdot t. \end{aligned}$$

If n is even then the row x_{n-1} contains only the reflection s_n and that box corresponds to the root α_n . Since $\pi(\alpha_n) = t$ we have

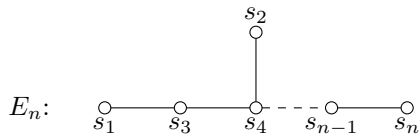
$$\begin{aligned} p_{s_n}(w_K) &= \left[1 + \sum_{\substack{1 \leq m \leq n-3 \\ m \text{ odd}}} 2(n - m) - 1 \right] \cdot t \\ &= \left[1 + \sum_{l=1}^{\frac{n}{2}-1} 2n - 2(2l - 1) - 1 \right] \cdot t \\ &= \left[1 + \sum_{l=1}^{\frac{n}{2}-1} 2n - 4l + 1 \right] \cdot t \\ &= \left[1 + \left(\frac{n}{2} - 1\right)(2n + 1) - 4 \frac{\left(\frac{n}{2} - 1\right)\left(\frac{n}{2}\right)}{2} \right] \cdot t \\ &= \frac{n^2 - n}{2} \cdot t \end{aligned}$$

Thus for any $n > 3$ regardless of parity, Equation (5.14) becomes

$$c_{n,J}^K = \frac{n^2 - n}{2} t \cdot \frac{1}{(n - 1)t} = \frac{n}{2}.$$

□

5.5.5 Type E



Proposition 5.27. *Theorem 5.20 holds when K is a type E root system.*

Proof. The Giambelli formula for the Peterson varieties in the exceptional types was calculated using Sage code. While the calculations for types F_4 and G_2 can easily be reproduced by hand, the E series computations heavily relied on computers. As such the type E computations will be given with the accompanying code to reproduce the results. Unlike for the infinite series Lie types, if K is a root system of type E_n , the word w_K does not give rise to a nice diagram. Instead we present that information as a table in Figure 5.

The first list L_{w_K} gives the ordered simple reflections s_{j_i} such that $w_K = s_{j_1} s_{j_2} \cdots s_{j_\ell}$ is the reduced word for w_K given by the algebraic combinatorics platform Sage. The second list $L_{p(w_K)}$ is created using a Sage program. For each simple reflection s_{j_i} of w_K we record $\frac{1}{t} \cdot \pi(\mathbf{r}(\mathbf{i}, w_K))$. The code for this program is available at <http://arxiv.org/abs/1311.2678>.

The only reduced words of v_K are

$$v_K = s_1 s_2 s_3 s_4 \cdots s_n = s_2 s_1 s_3 s_4 \cdots s_n = s_1 s_3 s_2 s_4 \cdots s_n.$$

Python code can find all sublists of L_{w_K} that are equal to one of the three corresponding lists. These sublists are the v_K -excitations μ of w_K . For each excitation μ , $M(\mu)$ is the product of the entries in $L_{p(w_K)}$ in the same positions as those in the sublist μ . We then sum over all such μ to get

$$p_{v_K}(w_K) = \left[\sum_{\mu \text{ a } v_K\text{-excitation of } L_{w_K}} M(\mu) \right] \cdot t^n.$$

We also evaluate $p_{s_i}(w_K)$ by summing all of the entries in $L_{p(w_K)}$ corresponding to entries which

Figure 5: The lists L_{w_K} and $L_{p(w_K)}$ for $K = E_6, E_7$, and E_8 . The bold simple reflection in $L_{w_{E_7}}$ and $L_{w_{E_8}}$ is the last occurrence of the reflections s_7 and s_8 respectively.

$$\begin{array}{l}
 L_{w_{E_6}} = \quad s_1, \quad s_3, \quad s_4, \quad s_5, \quad s_6, \quad s_2, \quad s_4, \quad s_5, \quad s_3, \quad s_4, \quad s_1, \quad s_3, \quad s_2, \quad s_4, \quad s_5, \quad s_6, \quad s_2, \quad s_4, \\
 \quad s_5, \quad s_3, \quad s_4, \quad s_1, \quad s_3, \quad s_2, \quad s_4, \quad s_5, \quad s_3, \quad s_4, \quad s_1, \quad s_3, \quad s_2, \quad s_4, \quad s_1, \quad s_3, \quad s_2, \quad s_1 \\
 L_{p(w_{E_6})} = \quad 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 4, \quad 5, \quad 6, \quad 6, \quad 7, \quad 7, \quad 8, \quad 8, \quad 9, \quad 10, \quad 11, \quad 1, \quad 2, \\
 \quad 3, \quad 3, \quad 4, \quad 4, \quad 5, \quad 5, \quad 6, \quad 7, \quad 1, \quad 2, \quad 2, \quad 3, \quad 3, \quad 4, \quad 1, \quad 2, \quad 1, \quad 1
 \end{array}$$

$$\begin{array}{l}
 L_{w_{E_7}} = \quad s_7, \quad s_6, \quad s_5, \quad s_4, \quad s_3, \quad s_2, \quad s_4, \quad s_5, \quad s_6, \quad s_7, \quad s_1, \quad s_3, \quad s_4, \quad s_5, \quad s_6, \quad s_2, \quad s_4, \\
 \quad s_5, \quad s_3, \quad s_4, \quad s_1, \quad s_3, \quad s_2, \quad s_4, \quad s_5, \quad s_6, \quad \mathbf{s_7}, \quad s_1, \quad s_3, \quad s_4, \quad s_5, \quad s_6, \quad s_2, \quad s_4, \\
 \quad s_5, \quad s_3, \quad s_4, \quad s_1, \quad s_3, \quad s_2, \quad s_4, \quad s_5, \quad s_6, \quad s_2, \quad s_4, \quad s_5, \quad s_3, \quad s_4, \quad s_1, \quad s_3, \quad s_2, \\
 \quad s_4, \quad s_5, \quad s_3, \quad s_4, \quad s_1, \quad s_3, \quad s_2, \quad s_4, \quad s_1, \quad s_3, \quad s_2, \quad s_1 \\
 L_{p(w_{E_7})} = \quad 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 5, \quad 6, \quad 7, \quad 8, \quad 9, \quad 6, \quad 7, \quad 8, \quad 9, \quad 10, \quad 9, \quad 10, \\
 \quad 11, \quad 11, \quad 12, \quad 12, \quad 13, \quad 13, \quad 14, \quad 15, \quad 16, \quad 17, \quad 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 4, \quad 5, \\
 \quad 6, \quad 6, \quad 7, \quad 7, \quad 8, \quad 8, \quad 9, \quad 10, \quad 11, \quad 1, \quad 2, \quad 3, \quad 3, \quad 4, \quad 4, \quad 5, \quad 5, \\
 \quad 6, \quad 7, \quad 1, \quad 2, \quad 2, \quad 3, \quad 3, \quad 4, \quad 1, \quad 2, \quad 1, \quad 1
 \end{array}$$

$$\begin{array}{l}
 L_{w_{E_8}} = \quad s_8, \quad s_7, \quad s_6, \quad s_5, \quad s_4, \quad s_3, \quad s_2, \quad s_4, \quad s_5, \quad s_6, \quad s_7, \quad s_1, \quad s_3, \quad s_4, \quad s_5, \quad s_6, \quad s_2, \\
 \quad s_4, \quad s_5, \quad s_3, \quad s_4, \quad s_1, \quad s_3, \quad s_2, \quad s_4, \quad s_5, \quad s_6, \quad s_7, \quad s_8, \quad s_7, \quad s_6, \quad s_5, \quad s_4, \quad s_3, \\
 \quad s_2, \quad s_4, \quad s_5, \quad s_6, \quad s_7, \quad s_1, \quad s_3, \quad s_4, \quad s_5, \quad s_6, \quad s_2, \quad s_4, \quad s_5, \quad s_3, \quad s_4, \quad s_1, \quad s_3, \\
 \quad s_2, \quad s_4, \quad s_5, \quad s_6, \quad s_7, \quad \mathbf{s_8}, \quad s_7, \quad s_6, \quad s_5, \quad s_4, \quad s_3, \quad s_2, \quad s_4, \quad s_5, \quad s_6, \quad s_7, \quad s_1, \\
 \quad s_3, \quad s_4, \quad s_5, \quad s_6, \quad s_2, \quad s_4, \quad s_5, \quad s_3, \quad s_4, \quad s_1, \quad s_3, \quad s_2, \quad s_4, \quad s_5, \quad s_6, \quad s_7, \quad s_1, \\
 \quad s_3, \quad s_4, \quad s_5, \quad s_6, \quad s_2, \quad s_4, \quad s_5, \quad s_3, \quad s_4, \quad s_1, \quad s_3, \quad s_2, \quad s_4, \quad s_5, \quad s_6, \quad s_2, \quad s_4, \\
 \quad s_5, \quad s_3, \quad s_4, \quad s_1, \quad s_3, \quad s_2, \quad s_4, \quad s_5, \quad s_3, \quad s_4, \quad s_1, \quad s_3, \quad s_2, \quad s_4, \quad s_1, \quad s_3, \quad s_2, \\
 \quad s_1 \\
 L_{p(w_{E_8})} = \quad 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 6, \quad 7, \quad 8, \quad 9, \quad 10, \quad 7, \quad 8, \quad 9, \quad 10, \quad 11, \quad 10, \\
 \quad 11, \quad 12, \quad 12, \quad 13, \quad 13, \quad 14, \quad 14, \quad 15, \quad 16, \quad 17, \quad 18, \quad 29, \quad 11, \quad 12, \quad 13, \quad 14, \quad 15, \\
 \quad 15, \quad 16, \quad 17, \quad 18, \quad 19, \quad 16, \quad 17, \quad 18, \quad 19, \quad 20, \quad 19, \quad 20, \quad 21, \quad 21, \quad 22, \quad 22, \quad 23, \\
 \quad 23, \quad 24, \quad 25, \quad 26, \quad 27, \quad 28, \quad 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 5, \quad 6, \quad 7, \quad 8, \quad 9, \quad 6, \\
 \quad 7, \quad 8, \quad 9, \quad 10, \quad 9, \quad 10, \quad 11, \quad 11, \quad 12, \quad 12, \quad 13, \quad 13, \quad 14, \quad 15, \quad 16, \quad 17, \quad 1, \\
 \quad 2, \quad 3, \quad 4, \quad 5, \quad 4, \quad 5, \quad 6, \quad 6, \quad 7, \quad 7, \quad 8, \quad 8, \quad 9, \quad 10, \quad 11, \quad 1, \quad 2, \\
 \quad 3, \quad 3, \quad 4, \quad 4, \quad 5, \quad 5, \quad 6, \quad 7, \quad 1, \quad 2, \quad 2, \quad 3, \quad 3, \quad 4, \quad 1, \quad 2, \quad 1, \\
 \quad 1
 \end{array}$$

are s_i in L_{w_K} . This gives the following data for the E series:

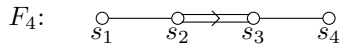
	Type E_6	Type E_7	Type E_8
$p_{v_K}(w_K)$	$887040t^6$	$661620960t^7$	$11179629901440t^8$
$p_{s_1}(w_K)$	$16t$	$34t$	$92t$
$p_{s_2}(w_K)$	$22t$	$49t$	$136t$
$p_{s_3}(w_K)$	$30t$	$66t$	$182t$
$p_{s_4}(w_K)$	$42t$	$96t$	$270t$
$p_{s_5}(w_K)$	$30t$	$75t$	$220t$
$p_{s_6}(w_K)$	$16t$	$52t$	$168t$
$p_{s_7}(w_K)$		$27t$	$114t$
$p_{s_8}(w_K)$			$172t$

This table along with Lemma 5.17 gives us that the E series Peterson varieties have the following Giambelli's formula.

$$\begin{aligned}
E_6 : C \cdot p_{v_K}(w_K) &= \prod_{1 \leq i \leq 6} p_{s_i}(w_K) \quad \text{where} \quad C = 240 = \frac{6!}{3} \\
E_7 : C \cdot p_{v_K}(w_K) &= \prod_{1 \leq i \leq 7} p_{s_i}(w_K) \quad \text{where} \quad C = 680 = \frac{7!}{3} \\
E_8 : C \cdot p_{v_K}(w_K) &= \prod_{1 \leq i \leq 8} p_{s_i}(w_K) \quad \text{where} \quad C = 13440 = \frac{8!}{3}
\end{aligned}$$

□

5.5.6 Type F_4



Proposition 5.28. *Theorem 5.20 holds when K is a type F_4 root system.*

Proof. If K is a root system of type F_4 then $J = K \setminus \{s_4\}$ is a root subsystem of type B_3 and therefore

$$c_{4,J}^K \cdot p_K = p_{s_4} p_{v_J} = \frac{1}{3!} \prod_{i=1}^4 p_i.$$

Evaluating at $w_K = s_4 s_3 s_2 s_3 s_1 s_2 s_3 s_4 s_3 s_2 s_3 s_1 s_2 s_3 s_4 s_3 s_2 s_3 s_1 s_2 s_3 s_1 s_2 s_1$ gives

$$\begin{aligned}
p_K(w_K) &= 18480t^4 = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot t^4 \\
p_1(w_K) &= 22t & p_2(w_K) &= 42t \\
p_3(w_K) &= 30t & p_4(w_K) &= 16t
\end{aligned}$$

Since

$$c_{4,J}^K \cdot p_K(w_K) = \frac{1}{3!} \prod_{i=1}^4 p_i(w_K),$$

we can solve for $c_{4,J}^K$ to see that $c_{4,J}^K = 4$. Thus $4! \cdot p_K = (|K|)! \cdot p_K = \prod_{i=1}^4 p_i$. □

5.5.7 Type G_2

$$G_2: \begin{array}{c} \circ \rightleftarrows \circ \\ s_1 \quad s_2 \end{array}$$

Proposition 5.29. *Theorem 5.20 holds when K is a type G_2 root system.*

Proof. Since in type G_2 the ring $H_S^*(Pet_{G_2})$ has only four Peterson Schubert classes, we give the basis explicitly evaluated at e, s_1, s_2 , and $s_1 s_2 s_1 s_2 s_1 s_2$:

$$\begin{array}{cccc} p_{\emptyset} & p_{\{1\}} & p_{\{2\}} & p_{\{1,2\}} \\ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ t \\ 0 \\ 6t \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ t \\ 10t \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 30t^2 \end{pmatrix} \end{array}$$

From this basis it is clear that $2 \cdot p_K = \prod_{i \in K} p_{s_i}$ and of course $|K|! = 2$. □

APPENDIX A

NOTATION

G	a complex reductive linear algebraic group
B	a Borel subgroup
T	a maximal torus
Φ	a root system
Φ^+	positive roots in Φ
Δ	positive simple roots in Φ
α_i	a simple root in Δ
W	a Weyl group
s_i	a simple reflection in W associated to α_i
w_0	the longest word in W
$\mathfrak{t}, \mathfrak{b}, \mathfrak{g}$	Lie algebras associated to T, B, G respectively
\mathfrak{g}	the root space in \mathfrak{g} associated to the root $\alpha \in \Phi$
E_α	a basis element in \mathfrak{g}_α
\mathfrak{t}^*	the dual of \mathfrak{t}
$\mathbb{C}[\mathfrak{t}^*]$	a Cartan subalgebra
X^T or $(X)^T$	the T -fixed points of X
$H_T^*(X)$	the T -equivariant cohomology of X
σ_v	the (equivariant) Schubert class associated to the word $v \in W$
$\sigma_v(w)$	the localization of σ_v at the T -fixed point $wB \in G/B$
$\mathbf{r}(\mathbf{i}, w)$	the root associated with i^{th} letter of w

H	a Hessenberg space in \mathfrak{g}
h	a Hessenberg function $h : [n] \rightarrow [n]$
$Hess(X, H)$	the Hessenberg variety associated to Hessenberg space H and operator $X \in \mathfrak{g}$
P	a parabolic subgroup of G
W_P	the parabolic subgroup of W , also the Weyl group of P/B
W^P	the minimal coset representatives of W/W_P
N	a regular nilpotent operator in \mathfrak{g}
${}^T M$	the matrix obtained by flipping matrix M along its antidiagonal
$Flags$	the flag variety G/B
S	a one dimensional subtorus of T
V_\bullet	a flag in G/B
π_1	a ring homomorphism from $\mathbb{C}[\alpha_1, \dots, \alpha_n]$ to $\mathbb{C}[t]$ induced by $\alpha_i \mapsto t$ for all i
$\overset{\mathbb{H}}{\sigma}_v$	the Hessenberg Schubert class associated with w
$\overset{\mathbb{H}}{\sigma}_v(w)$	the Hessenberg Schubert class $\overset{\mathbb{H}}{\sigma}$ localized at w
V_H	the set of $v \in W$ such that the permutation matrix of v is contained in H
W_H	the set of S -fixed points in $Hess(N, H)$
A_H	the matrix with entries $\overset{\mathbb{H}}{\sigma}_v(w)$
H_3	the type- A_{n-1} Hessenberg space defined by $h(1) = 3$ and $h(i) = n$ for $i > 1$
Z	the subgroup $\langle s_2, s_3, \dots, s_{n-1} \subset \mathfrak{S}_n$
Y	the subset $\{s_i s_{i-1} \cdots s_1 s_{i-1} s_{i-2} \cdots s_2 y : 1 \leq i \leq n-1 \text{ and } y \in \langle s_3, \dots, s_{n-1} \rangle\}$
$D_R(z)$	the right descent set of z
\tilde{Z}	the coset $\{z \in Z : s_2 \notin D_R(z)\}$
β	the block of A_{H_3} with rows indexed by Y and columns indexed by $s_1 \tilde{Z} \cup s_2 s_1 \tilde{Z}$
β_j	the j^{th} diagonal block of β

H_{Pet}	the Peterson Hessenberg space
Pet	the Peterson variety
w_K	the longest word of a parabolic subgroup $W_K \subset W$
p_v	the Peterson Schubert class associated to the word $v \in W$
\tilde{w}	a fixed reduced word for $w \in W$
v_K	a specific Coxeter element of W_K defined in Definition 5.5
$ \mathcal{R}(v) $	the number of reduced words for $v \in W$
λ_{w_K}	the skew diagram representing the word $w_K \in W$
$\lambda_{p(w_K)}$	the skew digram with each box containing the height of the corresponding positive root
μ	a v -excitation in λ_{w_K} for a fixed word $v \in W$
$M_p(\mu)$	the product of the entries in $\lambda_{p(w_K)}$ corresponding the the excitation μ

A P P E N D I X B

CODE FOR COMPUTING GIAMBELLI'S FORMULA FOR TYPE E PETERSON VARIETIES

This work was done in Sage 5.2.

First we define a function `root_finder` that will return the root $\mathbf{r}(i, w)$ for a fixed reduced word w .

```
def T(i,z):
    return alpha[int(z.reduced_word()[int(i-1)])]
def p(i,z):
    list=[]
    for j in Sequence([0..int(i-2)]):
        list.append(z.reduced_word()[int(j)])
    return W.from_reduced_word(list)
def root_finder(m,n):
    a=m.reduced_word();
    q=n
    for i in Sequence[0..int(m.length()-1)]:
        q=q.simple_reflection(a[int(m.length()-i-1)]);
    return q;
def r(i,w):
    if i<=w.length():
        return root_finder(p(i,w),T(i,w))
    else:
```

```
return 1
```

These functions can be used in any Lie type. To calculate the lists L_{w_K} and $L_{p(w_K)}$ we must specify which Lie type we want to work in. The code for calculating in type E_8 is given first with annotations, followed by the similar code for types E_7 and E_6 .

B.1 Type E_8

First we need to define the objects we will need for the computation. `n=8`

```
W=WeylGroup(['E',n],prefix="s")
R=RootSystem(['E',n]);
S=R.root_space();
B=S.basis();
space=R.root_lattice();
alpha=space.simple_roots();
Q=R.root_poset()
[s1,s2,s3,s4,s5,s6,s7,s8]=W.simple_reflections()
w=W.long_element()
```

Now that we have defined the root poset on the positive roots Φ^+ a simple function will give the value of $\frac{1}{7}\pi(\alpha)$ for any positive root α .

```
def height(m):
    return Q.rank(m)+1;
```

The lists L_{w_K} and $L_{p(w_K)}$ are obtained using this code:

```
worde8=w.reduced_word()
listpwk=[]
for i in Sequence[1..w.length()]:
    listpwk.append(height(r(i,w)))
print worde8, listpwk
```

The list `worde8` is $L_{w_{E_8}}$ and the list `listpwk` is $L_{p(w_{E_8})}$. For each simple reflection s_1, s_2, \dots, s_8 the evaluation of $p_{s_i}(w_{E_8})$ is calculated using this code. Here we have used $i = 3$, but this must be run eight times, setting i equal to one through eight.

```
i=3
q=0
for j in Sequence(0..(len(worde8)-1)):
    if worde8[j]==i:
        q=q+listpwk[j];
q
```

The final value is $q = \frac{1}{t} p_{s_i}(w_{E_8})$. The last component of Giambelli's formula is evaluating $p_{v_K}(w_K)$. This code finds all sublists of `worde8` that are reduced words for v_K . Since v_K must end in s_8 and it last occurs in the 57th spot in the list, we only need sublists of the first 57 terms. The CPU time was slightly over 10000 minutes for this step of the calculation. The same calculation can be done with fewer lines of code, but a longer run time.

```
X=Set([0..56])
Y=X.subsets(8)
Z=[]
for q in Y:
    if worde8[sorted(q)[7]]==8:
        if worde8[sorted(q)[6]]==7:
            if worde8[sorted(q)[5]]==6:
                if worde8[sorted(q)[4]]==5:
                    if worde8[sorted(q)[3]]==4:
                        if worde8[sorted(q)[2]]==3:
                            if worde8[sorted(q)[1]]==2:
                                if worde8[sorted(q)[0]]==1:
                                    Z.append(q)
```

```

for q in Y:
    if worde8[sorted(q)[7]]==8:
        if worde8[sorted(q)[6]]==7:
            if worde8[sorted(q)[5]]==6:
                if worde8[sorted(q)[4]]==5:
                    if worde8[sorted(q)[3]]==4:
                        if worde8[sorted(q)[2]]==2:
                            if worde8[sorted(q)[1]]==3:
                                if worde8[sorted(q)[0]]==1:
                                    Z.append(q)

```

```

for q in Y:
    if worde8[sorted(q)[7]]==8:
        if worde8[sorted(q)[6]]==7:
            if worde8[sorted(q)[5]]==6:
                if worde8[sorted(q)[4]]==5:
                    if worde8[sorted(q)[3]]==4:
                        if worde8[sorted(q)[2]]==3:
                            if worde8[sorted(q)[1]]==1:
                                if worde8[sorted(q)[0]]==2:
                                    Z.append(q)

```

Now the list Z contains all subwords of w_K that are reduced words for v_K and we can sum over them as follows:

```

q=0
for j in Sequence(0..(len(Z)-1)):
    a=1
    for k in Sequence(0..7):
        a=a*k

```



```
q=q+a
```

```
q
```

This value q is equal to $\frac{1}{i^8} p_{v_K}(w_K)$.

B.2 Type E_7

The basic setup:

```
n=7
W=WeylGroup(['E',n],prefix="s")
R=RootSystem(['E',n]);
S=R.root_space();
B=S.basis();
space=R.root_lattice();
alpha=space.simple_roots();
Q=R.root_poset()
[s1,s2,s3,s4,s5,s6,s7]=W.simple_reflections()
w=W.long_element()
```

```
def height(m):
    return Q.rank(m)+1;
```

The lists L_{w_K} and $L_{p(w_K)}$ are obtained using this code:

```
worde7=w.reduced_word()
listpwk=[]
for i in Sequence[1..w.length()]:
    listpwk.append(height(r(i,w)))
print worde7, listpwk
```

The list `worde7` is $L_{w_{E_7}}$ and the list `listpwk` is $L_{p(w_{E_7})}$. For each simple reflection s_1, s_2, \dots, s_7

the evaluation of $p_{s_i}(w_{E_7})$ is calculated using this code. Here we have used $i = 3$, but this must be run seven times, setting i equal to one through seven.

```
i=3
q=0
for j in Sequence(0..(len(worde7)-1)):
    if worde7[j]==i:
        q=q+listpwk[j];
q
```

The final value is $q = \frac{1}{t} p_{s_i}(w_{E_7})$. The last component of Giambelli's formula is evaluating $p_{v_K}(w_K)$. This code finds all sublists of `worde8` that are reduced words for v_K . Since v_K must end in s_7 which last appears in the 27^{th} spot in the list, we only need sublists of the first 27 terms.

```
X=Set([0..26])
Y=X.subsets(7)
Z=[]
for q in Y:
    if worde7[sorted(q)[6]]==7:
        if worde7[sorted(q)[5]]==6:
            if worde7[sorted(q)[4]]==5:
                if worde7[sorted(q)[3]]==4:
                    if worde7[sorted(q)[2]]==3:
                        if worde7[sorted(q)[1]]==2:
                            if worde7[sorted(q)[0]]==1:
                                Z.append(q)
```

```
for q in Y:
    if worde7[sorted(q)[6]]==7:
        if worde7[sorted(q)[5]]==6:
            if worde7[sorted(q)[4]]==5:
                if worde7[sorted(q)[3]]==4:
```

```

        if worde7[sorted(q)[2]]==2:
            if worde7[sorted(q)[1]]==3:
                if worde7[sorted(q)[0]]==1:
                    Z.append(q)

for q in Y:
    if worde7[sorted(q)[6]]==7:
        if worde7[sorted(q)[5]]==6:
            if worde7[sorted(q)[4]]==5:
                if worde7[sorted(q)[3]]==4:
                    if worde7[sorted(q)[2]]==3:
                        if worde7[sorted(q)[1]]==1:
                            if worde7[sorted(q)[0]]==2:
                                Z.append(q)

```

Now the list Z contains all subwords of w_K that are reduced words for v_K and we can sum over them as follows:

```

q=0
for j in Sequence(0..(len(Z)-1)):
    a=1
    for k in Sequence(0..6):
        a=a*k
    q=q+a
q

```

This value q is equal to $\frac{1}{t^7} p_{v_K}(w_K)$.

B.3 Type E_6

Basic setup:

n=6

```

W=WeylGroup(['E',n],prefix="s")
R=RootSystem(['E',n]);
S=R.root_space();
B=S.basis();
space=R.root_lattice();
alpha=space.simple_roots();
Q=R.root_poset()
[s1,s2,s3,s4,s5,s6]=W.simple_reflections()
w=W.long_element()

```

```

def height(m):
    return Q.rank(m)+1;

```

The lists L_{w_K} and $L_{p(w_K)}$ are obtained using this code:

```

worde6=w.reduced_word()
listpwk=[]
for i in Sequence[1..w.length()]:
    listpwk.append(height(r(i,w)))
print worde6, listpwk

```

The list `worde6` is $L_{w_{E_6}}$ and the list `listpwk` is $L_{p(w_{E_6})}$. For each simple reflection s_1, s_2, \dots, s_6 the evaluation of $p_{s_i}(w_{E_6})$ is calculated using this code. Here we have used $i = 3$, but this must be run six times, setting i equal to one through six.

```

i=3
q=0
for j in Sequence(0..(len(worde6)-1)):
    if worde6[j]==i:
        q=q+listpwk[j];

```

q

The final value is $q = \frac{1}{t} p_{s_i}(w_{E_6})$. The last component of Giambelli's formula is evaluating $p_{v_K}(w_K)$. This code finds all sublists of `worde8` that are reduced words for v_K . Since v_K must end in s_7 and the last occurrence is in the 16th spot in the list, we only need sublists of the first 16 terms.

```
X=Set([0..15])
Y=X.subsets(6)
Z=[]
for q in Y:
    if worde6[sorted(q)[5]]==6:
        if worde6[sorted(q)[4]]==5:
            if worde6[sorted(q)[3]]==4:
                if worde6[sorted(q)[2]]==3:
                    if worde6[sorted(q)[1]]==2:
                        if worde6[sorted(q)[0]]==1:
                            Z.append(q)
```

```
for q in Y:
    if worde6[sorted(q)[5]]==6:
        if worde6[sorted(q)[4]]==5:
            if worde6[sorted(q)[3]]==4:
                if worde6[sorted(q)[2]]==2:
                    if worde6[sorted(q)[1]]==3:
                        if worde6[sorted(q)[0]]==1:
                            Z.append(q)
```

```
for q in Y:
    if worde6[sorted(q)[5]]==6:
        if worde6[sorted(q)[4]]==5:
            if worde6[sorted(q)[3]]==4:
```

```

if worde6[sorted(q)[2]]==3:
    if worde6[sorted(q)[1]]==1:
        if worde6[sorted(q)[0]]==2:
            Z.append(q)

```

Now the list Z contains all subwords of w_K that are reduced words for v_K and we can sum over them as follows:

```

q=0
for j in Sequence(0..(len(Z)-1)):
    a=1
    for k in Sequence(0..5):
        a=a*k
    q=q+a
q

```

This value q is equal to $\frac{1}{t^6} p_{v_K}(w_K)$.

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