August 2015

Obstruction criteria for modular deformation problems

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OBSTRUCTION CRITERIA FOR MODULAR DEFORMATION PROBLEMS

A Dissertation Presented

by

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Submitted to the Graduate School of the

University of Massachusetts Amherst in partial fulfillment

of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2015

Department of Mathematics and Statistics

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DEDICATION

In loving memory of my Aunt Jean.

ACKNOWLEDGEMENTS

I cannot accurately express the depth of my gratitude towards my advisor, Tom Weston, for all of the guidance he has provided me over the past six years. I have learned an enormous amount from him, not only about math, but also about being a mathematician, the history of baseball, how to properly typeset χ and $\mathbf{Z}$, and how to beat the Oakland Raiders in Tecmo Super Bowl. His patience and his expertise provided an indispensable resource as I struggled to learn about Galois representations. I will miss our weekly meetings more than any other aspect of grad school, and being his student will always be a tremendous source of pride for me.

At UMass I was fortunate to take classes with many great professors, especially Eduardo Cattani, Richard Ellis, and Jenia Tevelev. Special thanks are reserved for Farshid Hajir and Siman Wong, who were invaluable sources of advice on all matters, mathematical and professional.

I never would have attended grad school were it not for the excellent tutelage and encouragement of the math faculty of The College of New Jersey. In particular, Tom Hagedorn was my first mathematical role model, and his influence will forever be present in every aspect of my mathematical career.

While I was an undergrad at TCNJ, I was lucky to enjoy the friendship of Sam Taylor, Chris Westenberger, Chris Geddis, and Brendan Kelly. They each had just as large an impact on the way I learned math as my professors did.

I am extremely grateful to Keenan Kidwell and Steven Pogman, both for the math they have patiently explained to me and for the many fun times we have shared. Their friendship is as valuable a take-away from grad school as the degree. My fellow UMass students were the source of much camaraderie and many recreational diversions, especially Nico, Tobias, Holley, Alden, Luke, Dan, and Adam.

Western Massachusetts is a great place to spend six years. The climbing community and The El Dojo provided excellent non-mathematical outlets for the stresses of grad school, and I made many wonderful friends this way, especially Mark Lodato and Saba Karimeddiny. The other great thing about living in the Pioneer Valley is the abundance of good coffee, food, and beer, and the best place to find all three is The Foundry in Northampton.

In all of my endeavors, I have enjoyed the endless support of my family. My parents have given me every opportunity to follow my dreams and to achieve my goals. They have supported me in every way possible. They have never pushed me in any direction other than the directions I have chosen for myself, and they have given their full support to every ambition I have ever entertained, from my kindergarten dreams of being a paleontologist, to my short-lived high school plan of becoming an airline pilot, and everything in between. This thesis is as much a culmination of their hard work as it is mine. My brothers, by directing seemingly-unlimited amounts of love and admiration my way, have provided me with an infinite source of inspiration and motivation, helping me to keep my chin up in even the most trying times.

Finally, at the end of each working day, I have been welcomed home by Nicole, Binx, and Jasper. Nicole has always believed in me, even on days when I have not, and without her love and encouragement, it is hard to imagine this thesis being written. I am excited for all of our future adventures, wherever and whatever they may be. Binx and Jasper, in addition to teaching me patience and stoicism at times, were sources of great comfort and joy. Sometimes, no matter who you are, you just need to pet a furry friend.

ABSTRACT

OBSTRUCTION CRITERIA FOR MODULAR DEFORMATION PROBLEMS

MAY 2015

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For a cuspidal newform $f = \sum a_n q^n$ of weight $k \geq 3$ and a prime $\mathfrak{p}$ of $\mathbf{Q}(a_n)$, the deformation problem for its associated mod $\mathfrak{p}$ Galois representation is unobstructed for all primes outside some finite set. Previous results give an explicit bound on this finite set for f of squarefree level; we modify this bound and remove the squarefree hypothesis. We also show that if the $\mathfrak{p}$-adic deformation problem for f is unobstructed, then f is not congruent mod $\mathfrak{p}$ to a newform of lower level.

TABLE OF CONTENTS

LIST OF NOTATION

$\bar{\mathbf{Q}}$	a fixed algebraic closure of $\mathbf{Q}$
$\bar{\mathbf{Q}}_\ell$	a fixed algebraic closure of $\mathbf{Q}_\ell$ with a fixed embedding $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_\ell$
$\mathbf{Q}_S$	maximal extension of $\mathbf{Q}$ that is unramified outside the finite set of primes S
$G_{\mathbf{Q}}$	$\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$
$G_{\mathbf{Q},S}$	$\mathrm{Gal}(\mathbf{Q}_S/\mathbf{Q})$
G_ℓ	$\mathrm{Gal}(\bar{\mathbf{Q}}_\ell/\mathbf{Q}_\ell)$
I_ℓ	inertia group in G_ℓ
ϵ_p	the p-adic cyclotomic character
$\bar{\psi}$	the reduction mod $\mathfrak{p}$ of a character ψ
ad ρ	the adjoint representation of ρ
$\mathrm{ad}^0\rho$	the trace-zero adjoint representation of ρ
$\mathbf{D}(f,S)$	the deformation problem associated to the modular form f and the finite set of primes S
$\mathbf{D}(f)$	the deformation problem associated to the modular form f and the minimal set of primes

CHAPTER 1

INTRODUCTION

With only a little exaggeration, modern number theory can broadly and succinctly be described as the study of the absolute Galois group $G_{\mathbf{Q}} = \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ of the rational numbers. This object, which encodes information about all solutions to all polynomial equations with rational coefficients, is extremely complicated. To study it, number theorists generally consider its representations, which are continuous[1] homomorphisms

$$\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(K)$$

from $G_{\mathbf{Q}}$ to some matrix group.

Modular forms provide an abundant source of interesting 2-dimensional Galois representations, thanks to work of Deligne and Serre (see Theorem 2.1). Given a newform f and a choice of a prime p, one can construct a representation

$$\rho_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O})$$

where $\mathcal{O}$ is a finitely generated $\mathbf{Z}_p$-module. Reducing modulo the maximal ideal of $\mathcal{O}$ yields a residual representation

$$\bar{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$$

where $k = \mathcal{O}/p$ is a finite field of characteristic p. In principle, the representations ρ_f and especially

[1] As $G_{\mathbf{Q}} = \varprojlim \mathrm{Gal}(L/\mathbf{Q})$ is the projective limit of the finite Galois groups $\mathrm{Gal}(L/\mathbf{Q})$, where $L/\mathbf{Q}$ ranges over all finite extensions of $\mathbf{Q}$, it is endowed with the profinite topology, hence the requirement that ρ be continuous.

$\bar{\rho}_f$ are easier to understand than the full Galois group $G_{\mathbf{Q}}$.

Given a residual representation $\bar{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$, there are many different representations $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\bar{\mathbf{Z}}_p)$ which may lift $\bar{\rho}$, in the sense of making the following diagram commute:

$$\begin{tikzcd} G_{\mathbf{Q}} \arrow[r, "\rho"] \arrow[dr, "\bar{\rho}"'] & \mathrm{GL}_2(\bar{\mathbf{Z}}_p) \arrow[d] \\ & \mathrm{GL}_2(k) \end{tikzcd}$$

In [13], Mazur took up the problem of describing all such lifts ρ of a fixed residual representation $\bar{\rho}$. As described in greater detail in Chapter 3, he showed that they were parametrized by a finitely generated $\mathbf{Z}_p$-module $\mathcal{R}$ called a *universal deformation ring.*

The deformation theory of Galois representations, and especially of modular Galois representations, has been an active area of research ever since Mazur's original paper. Most notably, deformation theory has played a central role in the proofs of both the Taniyama-Shimura Conjecture [1, 22] and Serre's Conjecture [10, 11]. In both proofs, it is necessary to show that a certain residual Galois representation

$$\bar{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\bar{\mathbf{F}}_p)$$

is given by the reduction of a modular Galois representation. As modular forms can be parametrized by a geometric algebra called a Hecke algebra, denoted $\mathbf{T}$, the strategy was to produce an isomorphism between the relevant deformation ring $\mathcal{R}$ and the appropriate Hecke algebra $\mathbf{T}$; theorems of this type are now commonly known as "$\mathcal{R} = \mathbf{T}$ theorems". Understanding the structure of a deformation ring $\mathcal{R}$ is thus an inherently interesting and important problem.

This thesis presents two results regarding the structure of deformation rings associated with modular Galois representations. In the simplest case, a deformation ring $\mathcal{R}$ is isomorphic to a power series ring in three variables, and in this case the deformation problem is called *unobstructed*; this terminology was coined by Mazur in [13]. The first original theorem in this thesis, Theorem 4.15, describes conditions (which depend only on the modular form f associated with $\bar{\rho}_f$) guaranteeing that the deformation problem is unobstructed. The second theorem, Theorem 5.5, shows that the deformation ring $\mathcal{R}$ is heavily influenced by the ramification behavior which is permitted to appear in deformations of $\bar{\rho}_f$.

For the reader who is familiar with algebraic number theory, this thesis is reasonably self-contained. Some definitions, important theorems, and technical lemmas in the theory of Galois cohomology have been relegated to an appendix. Chapters 2 and 3 lay the theoretical foundation for the rest of the thesis, recalling the important properties of modular forms, their associated Galois representations, and of deformation rings. Chapters 4 and 5 are devoted to proving Theorems 4.15 and 5.5, respectively. Chapter 6 gives some examples of applications of the main theorems.

CHAPTER 2

MODULAR GALOIS REPRESENTATIONS

In this chapter we recall the basic facts about Galois representations associated with modular forms. We also record the aspects of the Langlands Correspondence and the classification of automorphic representations which will be used in the sequel.

2.1 Modular Forms and Galois Representations

We begin by establishing some notation and language for modular forms. Proofs are omitted, and we recall only the facts which are essential to the rest of this thesis.

2.1.1 Modular Forms

Let $\mathcal{H} = \{\ z \in \mathbf{C} \mid \mathrm{Im}\ z > 0\}$ denote the complex upper half-plane. If $f : \mathcal{H} \to \mathbf{C}$ is a complex function, then $\mathrm{SL}_2(\mathbf{Z})$ acts on f by

$$\tau \cdot f(z) = f\left(\frac{az+b}{cz+d}\right), \quad \tau = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z}).$$

For any integer $N \geq 1$, we have the following *congruence subgroups*:

$$\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z}) \mid c \equiv 0 \pmod{N}\}$$

$$\Gamma_1(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z}) \mid a \equiv d \equiv 1 \pmod{N},\ c \equiv 0 \pmod{N}\}$$

Let Γ be one of Γ_0 or Γ_1. A *modular form of weight k and level* $\Gamma(N)$ is a holomorphic function $f : \mathcal{H} \to \mathbf{C}$ such that, for every $\tau \in \Gamma(N)$, the identity

$$\tau \cdot f(z) = (cz + d)^k f(z) \tag{2.1}$$

is satisfied. It is also required that f is holomorphic at infinity and at the cusps.

Since $\Gamma(N)$ will contain a matrix of the form $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ for some integer a, f is a-periodic, which implies that f has a q-expansion

$$f(q) = \sum_{n \geq 0} a_n(f) q^n$$

with $a_n(f) \in \mathbf{C}$ for each n. If $a_0(f) = 0$ then f is said to be a *cuspform*. The space of cuspforms of weight k and level $\Gamma(N)$ is denoted $S_k(\Gamma(N))$. If the particular congruence subgroup $\Gamma_0(N)$ or $\Gamma_1(N)$ is clear from context, or if it is irrelevant, we may just say that f is of weight k and level N, and write $f \in S_k(N)$.

A cuspidal modular form f is called a *newform* if

- it is in the new subspace $S_k(N)^{\text{new}}$,
- it is normalized, i.e. $a_1(f) = 1$, and
- it is a simultaneous eigenform for all of the Hecke operators T_n and diamond operators $\langle n \rangle$.

In this case, for every prime p we have $T_p(f) = a_p(f)f$, and for every $d|N$ we have $\langle d \rangle f = \omega(d)f$ for some Dirichlet character ω. We call χ the *nebentypus character* of f. The $a_i(f)$ are algebraic integers, the associated field extension $K_f = \mathbf{Q}(\{a_i\}_{i \geq 1})$ is a number field, and this field contains the values $\omega(d)$. For a prime $\mathfrak{p}$ lying over a rational prime p, we say f is *ordinary at* $\mathfrak{p}$ if $a_p(f) \neq 0 \pmod{\mathfrak{p}}$; otherwise it is called *nonordinary* at $\mathfrak{p}$.

2.1.2 Modular Galois Representations

Fix, now and forever, algebraic closures and embeddings $\bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_\ell$ for every prime ℓ. Denote the corresponding Galois groups by $G_\ell \hookrightarrow G_{\mathbf{Q}}$.

Let $f \in S_k(N)$ be a newform of weight $k \geq 2$ and nebentypus ω. For the entirety of this subsection, fix a rational prime p as well as a prime $\mathfrak{p}$ of $\bar{\mathbf{Q}}$ lying over it. Let $K_{\mathfrak{p}} = K_{f,\mathfrak{p}}$ denote the completion at $\mathfrak{p}$ of the number field associated to f, so there is an embedding $K_{\mathfrak{p}} \hookrightarrow \bar{\mathbf{Q}}_p$. Let $\epsilon_p : G_{\mathbf{Q}} \to \mathbf{Z}_p^{\times}$ denote the p-adic cyclotomic character.

For each such choice of $\mathfrak{p}$, it is possible to associate to f a Galois representation $\rho_{f,\mathfrak{p}}$. The following theorem is due to Deligne.

Theorem 2.1 (Deligne) Let $f = \sum a_n q^n$ be a newform of weight $k \geq 2$, level N, and nebentypus character ω. There is a continuous homomorphism

$$\rho_{f,\mathfrak{p}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(K_{\mathfrak{p}})$$

which is unramified outside the primes dividing Np. The determinant of $\rho_{f,\mathfrak{p}}$ is $\epsilon_p^{k-1}\omega$, and for every prime $\ell \nmid Np$, we have

$$\mathrm{trace}(\rho_{f,\mathfrak{p}}(\mathrm{Frob}_\ell)) = a_\ell.$$

In fact, $\rho_{f,\mathfrak{p}}$ can be conjugated so that its image is in $\mathrm{GL}_2(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of $K_{\mathfrak{p}}$. This depends on a choice of a $G_{\mathbf{Q}}$-stable lattice. Once this choice is made, it makes sense to compose $\rho_{f,\mathfrak{p}}$ with the reduction mod $\mathfrak{p}$ map, which yields the *residual* (mod $\mathfrak{p}$) Galois representation

$$\bar{\rho}_{f,\mathfrak{p}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(k_{\mathfrak{p}}),$$

where $k_{\mathfrak{p}}$ is the residue field of $\mathcal{O}$. For almost all primes $\mathfrak{p}$, the residual representation $\bar{\rho}_{f,\mathfrak{p}}$ is absolutely irreducible, and in this case, the residual representation is actually independent (up to isomorphism) of the choice of lattice. In most of our applications, we will only consider primes $\mathfrak{p}$ for which $\bar{\rho}_{f,\mathfrak{p}}$ is absolutely irreducible, in which case we can harmlessly assume $\rho_{f,\mathfrak{p}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O})$ to be integral.

2.2 Automorphic Forms and Local Langlands

Given a modular Galois representation $\rho_{f,\mathfrak{p}}$, it is useful to consider the restriction of $\rho_{f,\mathfrak{p}}$ or $\bar{\rho}_{f,\mathfrak{p}}$ to the decomposition group $G_\ell \subset G_{\mathbf{Q}}$ for each prime ℓ. By associating the modular form f to an automorphic form ϕ_f, the Local Langlands Correspondence (for $n = 2$) allows us to determine these restrictions explicitly. In this section, we (very briefly) recall the important facts about automorphic forms and the Local Langlands Correspondence which will be needed in later chapters. The primary source for this section is [4].

Every modular form f, being a holomorphic function on $\mathcal{H}$, corresponds to an automorphic form $\phi_f : \mathrm{GL}_2(\mathbf{A}) \to \mathbf{C}$; here $\mathbf{A}$ denotes the adeles of $\mathbf{Q}$. This automorphic form gives rise to an infinite-dimensional, irreducible, admissable automorphic representation π of $\mathrm{GL}_2(\mathbf{A})$. This representation can be decomposed as a restricted tensor product $\pi = \otimes \pi_\ell$, indexed by places ℓ, where each local component π_ℓ is an irreducible admissable representation of $\mathrm{GL}_2(\mathbf{Q}_\ell)$.

The Local Langlands correspondence for $n = 2$ tells us that such representations π_ℓ are in bijection with (isomorphism classes of) 2-dimensional complex Frobenius-semi-simple Weil-Deligne representations of $W_{\mathbf{Q}_\ell}$, the Weil-Deligne subgroup of G_ℓ. Grothendieck's Monodromy Theorem [8, Proposition 2.17] allows us to associate π_ℓ with a representation $\rho_\ell : G_\ell \to \mathrm{GL}_2(\bar{K}_{\mathfrak{p}})$, and in our setting, we in fact have $\rho_\ell = \rho_{f,\mathfrak{p}}|_{G_\ell}$. See [8, Section 2] for more details.

Since each local component π_ℓ is irreducible, π_ℓ (or, equivalently, $\rho_{f,\mathfrak{p}}|_{G_\ell}$) is one of the following types:

(i) $\pi_\ell = \pi(\chi_1, \chi_2)$ is principal series associated to characters $\chi_i : G_\ell \to \bar{K}_{\mathfrak{p}}^\times$, where $\chi_1\chi_2^{-1} \neq |\cdot|^{\pm}$ ($|\cdot|$ denotes the absolute value character);

(ii) $\pi_\ell \simeq \mathrm{St} \otimes \chi$ is special (twist of Steinberg) associated to a character $\chi_i : G_\ell \to \bar{K}_{\mathfrak{p}}^\times$; or

(iii) π_ℓ is supercuspidal.

While the definitions of types (i)–(iii) can be found in [4, Section 11.2], it will be sufficient for our purposes to understand what the corresponding local Galois representations look like. In fact, we may consider the base change of such a representation to the algebraic closure $\bar{K}_{\mathfrak{p}}$. The following can be found in [21].

When π_ℓ is principal series associated to characters χ_1, χ_2, the local Galois representation has the form

$$\rho_{f,\mathfrak{p}}|_{G_\ell} \otimes \bar{K}_{\mathfrak{p}} \simeq \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}. \tag{2.2}$$

When π_ℓ is special with associated character χ, the local Galois representation has the form

$$\rho_{f,\mathfrak{p}}|_{G_\ell} \otimes \bar{K}_{\mathfrak{p}} \simeq \begin{pmatrix} \epsilon_p \chi & * \\ 0 & \chi \end{pmatrix}. \tag{2.3}$$

We postpone until Section 4.3.3 discussion of the supercuspidal representations, which are more complicated to describe.

CHAPTER 3

DEFORMATION THEORY

3.1 Basic Definitions

Consider an odd, continuous Galois representation $\bar{\rho} : G_{\mathbf{Q},S} \to \mathrm{GL}_2(k)$, where k is some finite field and S is a finite set of primes containing the characteristic of k and the infinite place. Here $G_{\mathbf{Q},S}$ denotes the Galois group of the maximal extension of $\mathbf{Q}$ unramified outside S. Let $\mathcal{C}$ be the category whose objects are local rings which are inverse limits of artinian local rings with residue field k, and whose morphisms $A \to B$ are continuous local homomorphisms inducing the identity map on residue fields. If $A \in \mathcal{C}$, then we say $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(A)$ is a *lift* of $\bar{\rho}$ if the following diagram commutes:

$$\begin{tikzcd} G_{\mathbf{Q}} \arrow[r, "\rho"] \arrow[dr, "\bar{\rho}_f"'] & \mathrm{GL}_2(A) \arrow[d] \\ & \mathrm{GL}_2(k) \end{tikzcd}$$

The vertical arrow is induced by the reduction map $A \to k$; we consider two lifts equivalent if they are conjugate to one another by a matrix in the kernel of this induced map. An equivalence class of lifts is called a *deformation* of $\bar{\rho}$.

There is an associated *deformation functor*

$$D_{\bar{\rho}}^{S} : \mathcal{C} \to \mathrm{Sets}$$

which sends a ring A to the set of deformations of $\bar{\rho}$ to A. When $\bar{\rho}$ is absolutely irreducible, this functor is representable by a ring $\mathcal{R}_{\bar{\rho}} \in \mathcal{C}$ [3, Lemma 9.5]. Thus, there exists a *universal deformation* $\rho^{univ} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{R}_{\bar{\rho}})$ such that, for any deformation $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(A)$ of $\bar{\rho}$, there is a unique map $\mathrm{GL}_2(\mathcal{R}_{\bar{\rho}}) \to \mathrm{GL}_2(A)$ making the following diagram commute:

$$\begin{tikzcd} G_{\mathbf{Q}} \arrow[r, "\rho^{univ}"] \arrow[dr, "\rho"'] & \mathrm{GL}_2(\mathcal{R}_{\bar{\rho}}) \arrow[d] \\ & \mathrm{GL}_2(A) \end{tikzcd}$$

In the normal way, these universal objects are unique up to unique isomorphism.

3.2 Deformations of Modular Galois Representations

Let f be a newform of level N. Let $\mathfrak{p}$ be a prime, lying above a rational prime p, such that $\bar{\rho}_{f,\mathfrak{p}}$ is absolutely irreducible. Write $\mathcal{O}$ for the ring of integers of $K_{\mathfrak{p}}$, and write $k_{\mathfrak{p}}$ for its residue field, which has characteristic p. Let S denote the set of the primes dividing Np. As described in 2.1.2, Deligne's theorem (Theorem 2.1) produces a Galois representation

$$\rho_{f,\mathfrak{p}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathcal{O}).$$

Since $\rho_{f,\mathfrak{p}}$ is unramified outside of $Np\infty$, it actually factors through $G_{\mathbf{Q},S} = \mathrm{Gal}(\mathbf{Q}_S/\mathbf{Q})$, where $\mathbf{Q}_S$ is the maximal extension of $\mathbf{Q}$ unramified outside S. So we obtain a diagram

$$\begin{tikzcd} G_{\mathbf{Q},S} \arrow[r, "\rho_{f,\mathfrak{p}}"] \arrow[dr, "\bar{\rho}_{f,\mathfrak{p}}"'] & \mathrm{GL}_2(\mathcal{O}) \arrow[d] \\ & \mathrm{GL}_2(k_{\mathfrak{p}}) \end{tikzcd}$$

Thus, any deformation of the residual representation $\bar{\rho}_{f,\mathfrak{p}}$ must also be unramified outside S.

Later in this thesis, particularly in Chapter 5, we will enlarge S to include more finite primes than simply those dividing Np. This will have the effect of imposing stronger ramification conditions on the deformations of $\bar{\rho}_{f,\mathfrak{p}}$. See also Section 3.4 below for relevant notation.

3.3 Structure of Deformation Rings

We return to the notation from Section 3.1. For $i = 1, 2$, let d_i be the k-dimension of the Galois cohomology group $H^i(G_{\mathbf{Q},S}, \mathrm{ad}\ \bar{\rho})$. Mazur showed [13, Section 1.10] that $d_1 - d_2 = 3$ and the universal deformation ring is of the form

$$R_{\bar{\rho}} \simeq W(k)[\![T_1, \ldots, T_{d_1}]\!]/(r_1, \ldots, r_{d_2}),$$

where $W(k)$ is the ring of Witt vectors of k. Thus, if $d_2 = 0$, then $R_{\bar{\rho}}$ is simply a power series ring in three variables. In this case, the deformation problem for $\bar{\rho}$ is said to be *unobstructed.*

Let $p > 2$ be the characteristic of k, and let ϵ_p be the p-adic cyclotomic character. For any $G_{\mathbf{Q},S}$-module M, define

$$\text{Ш}^1(G_{\mathbf{Q},S}, M) := \ker \left[H^1(G_{\mathbf{Q},S}, M) \to \bigoplus_{\ell \in S} H^1(G_\ell, M) \right].$$

Recall that for any representation $\rho : G \to \mathrm{GL}_2(K)$ of a group G over a field K, the *adjoint representation* $\mathrm{ad}\ \rho : G \to \mathrm{GL}_4(K)$ is defined by letting $g \in G$ act on $\mathrm{End}(\rho) \simeq \mathrm{GL}_2(K)$ via conjugation by $\rho(g)$. The trace-zero component of $\mathrm{ad}\ \rho$ is denoted by $\mathrm{ad}^0\rho : G \to \mathrm{GL}_3(K)$.

The following proposition [21, Lemma 6] is extremely useful for determining whether certain deformation problems are unobstructed.

Proposition 3.1 We have

$$\dim_k H^2(G_{\mathbf{Q},S}, \mathrm{ad}\ \bar{\rho}) \leq \dim_k \text{Ш}^1(G_{\mathbf{Q},S}, \bar{\epsilon}_p \otimes \mathrm{ad}^0 \bar{\rho}) + \sum_{p \in S} \dim_k H^0(G_p, \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}) \tag{3.1}$$

with equality if $p > 3$.

Proof. The natural trace pairing

$$\mathrm{ad}\ \bar{\rho} \otimes \mathrm{ad}\ \bar{\rho} \to k$$

identifies $\bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}$ with the Cartier dual $(\mathrm{ad}\ \bar{\rho})^*$ of $\mathrm{ad}\ \bar{\rho}$. Thus, by Poitou-Tate (A.5) we have an exact sequence

$$\begin{aligned}0 \longrightarrow H^0(G_{\mathbf{Q},S}, \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}) \longrightarrow & \bigoplus_{\ell \in S} H^0(G_\ell, \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}) \\ & \longrightarrow \mathrm{Hom}(H^2(G_{\mathbf{Q},S}, \mathrm{ad}\ \bar{\rho}), k) \longrightarrow \text{Ш}^1(G_{\mathbf{Q},S}, \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}) \longrightarrow 0.\end{aligned}$$

Since $\bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho} = \bar{\epsilon}_p \oplus (\bar{\epsilon}_p \otimes \mathrm{ad}^0 \bar{\rho})$ we have

$$\text{Ш}^1(G_{\mathbf{Q},S}, \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}) = \text{Ш}^1(G_{\mathbf{Q},S}, \bar{\epsilon}_p) \oplus \text{Ш}^1(G_{\mathbf{Q},S}, \bar{\epsilon}_p \otimes \mathrm{ad}^0 \bar{\rho}).$$

The first term vanishes by [19, Lemma 10.6], so our exact sequence becomes

$$\begin{aligned}0 \longrightarrow H^0(G_{\mathbf{Q},S}, \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}) \longrightarrow & \bigoplus_{\ell \in S} H^0(G_\ell, \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}) \\ & \longrightarrow \mathrm{Hom}(H^2(G_{\mathbf{Q},S}, \mathrm{ad}\ \bar{\rho}), k) \longrightarrow \text{Ш}^1(G_{\mathbf{Q},S}, \bar{\epsilon}_p \otimes \mathrm{ad}^0 \bar{\rho}) \longrightarrow 0.\end{aligned}$$

Thus we have the inequality

$$\begin{aligned}&\dim_k H^2(G_{\mathbf{Q},S}, \mathrm{ad}\ \bar{\rho}) \leq \\ &\qquad \dim_k \text{Ш}^1(G_{\mathbf{Q},S}, \bar{\epsilon}_p \otimes \mathrm{ad}^0 \bar{\rho}) + \sum_{\ell \in S} \dim_k H^0(G_\ell, \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}) - H^0(G_{\mathbf{Q},S}, \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}).\end{aligned}$$

By [21, Lemma 3], the group $H^0(G_{\mathbf{Q},S}, \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho})$ vanishes if $p \neq 3$, and the proposition follows.

□

When $\bar{\rho} = \bar{\rho}_{f,\mathfrak{p}}$ for some newform $f \in S_k(N)$, the term $\text{Ш}^1(G_{\mathbf{Q},S}, \bar{\epsilon}_p \otimes \mathrm{ad}^0 \bar{\rho})$ can be controlled by a certain set $\mathrm{Cong}(f)$ of *congruence primes* for f, as described in [21, Section 4.1]. Our focus will instead be on the local invariants $H^0(G_\ell, \bar{\epsilon}_p \otimes \mathrm{ad}\, \bar{\rho})$ for $\ell \in S$, which we refer to as *obstructions at* ℓ. Thus, we simply state the following result of Weston which handles the $\text{Ш}^1(G_{\mathbf{Q},S}, \bar{\epsilon}_p \otimes \mathrm{ad}^0 \bar{\rho})$ term.

Write M for the conductor of the nebentypus character ω of f, so $M \mid N$. Denote by ω_0 the primitive character associated to ω. We say $\mathfrak{p}$ is a *congruence prime* for f if there is a newform g of weight k and level $d \mid N$ such that:

- g has nebentypus character lifting ω_0;
- g is not a Galois conjugate of f;
- $\bar{\rho}_{f,\bar{\mathfrak{p}}} \simeq \bar{\rho}_{g,\bar{\mathfrak{p}}}$ for some prime $\bar{\mathfrak{p}}$ of $\bar{\mathbf{Q}}_p$ above $\mathfrak{p}$.

We denote by $\mathrm{Cong}(f)$ the set of congruence primes for f. Then the main result of [21, Section 4.1] is the following.

Proposition 3.2 Let $f \in S_k(N)$ be a newform, and let $\mathfrak{p}$ be a prime of K_f lying over the rational prime p. Assume that:

(1) $\bar{\rho}_{f,\mathfrak{p}}$ is absolutely irreducible;

(2) $p > k$;

(3) Either $N > 1$ or $p \nmid (2k-3)(2k-1)$;

(4) $p \nmid N$;

(5) $p \nmid \phi(N)$ (that is, $\ell \not\equiv 1 \pmod{p}$ for all $\ell \mid N$);

(6) $\bar{\rho}_{f,\mathfrak{p}}$ is ramified at p for all $p \mid \frac{N}{M}$;

Then $\text{Ш}^1(G_{\mathbf{Q},S}, \bar{\epsilon}_p \otimes \mathrm{ad}^0 \bar{\rho}) \neq 0$ only if $\mathfrak{p} \in \mathrm{Cong}(f)$.

Proof. See Lemma 7 and Section 4.1 of [21]. □

3.4 Some Useful Notation

We introduce some non-standard notation which will be useful throughout this thesis. Given a modular form f, a prime $\mathfrak{p} \mid p$ of $\bar{K}_f$, and a finite set of places S, let us write $\mathbf{D}(f, S)$ for the corresponding deformation problem

$$\begin{tikzcd} G_{\mathbf{Q},S} \arrow[r, "\rho^{univ}"] \arrow[dr, "\bar{\rho}_{f,\mathfrak{p}}"'] & \mathrm{GL}_2(\mathcal{R}_{\bar{\rho}_{f,\mathfrak{p}}}) \arrow[d] \\ & \mathrm{GL}_2(k_{\mathfrak{p}}) \end{tikzcd} .$$

We suppress $\mathfrak{p}$ from the notation, as it will always be clear from context. If S contains only the primes dividing the level of f and the infinite place, then we may simply write $\mathbf{D}(f)$, and we call this the *minimal deformation problem* for f.

CHAPTER 4

EXPLICIT OBSTRUCTION CRITERIA

Let $f = \sum a_n q^n$ be a newform in $S_k(N)$ with nebentypus character ω of conductor M. Let S be a finite set of places of $\mathbf{Q}$ containing all places dividing $N\infty$, and let N_S denote the product of the primes in S. Fix a prime $\mathfrak{p}$ of K_f dividing an odd rational prime p, and let $\hat{S} = S \cup \{p\}$. The main result of [21] is [21, Theorem 18], which is the following.

Theorem 4.1 Suppose that N is squarefree. Assume also that $\bar{\rho}_{f,\mathfrak{p}}$ is absolutely irreducible and $p > 3$. If

$$H^2(G_{\mathbf{Q},\hat{S}}, \mathrm{ad}\ \bar{\rho}_{f,\mathfrak{p}}) \neq 0$$

then one of the following holds:

(1) $p \leq k$;

(2) $p \mid N$;

(3) $p \mid \phi(N_S)$;

(4) $p \mid \ell + 1$ for some $\ell \mid \frac{N}{M}$;

(5) $a_\ell^2 \equiv (\ell+1)^2 \ell^{k-2} \omega(\ell)\ (mod\ \mathfrak{p})$ for some $\ell \mid \frac{N_S}{N}$, $\ell \neq p$;

(6) $p = k + 1$ and f is ordinary at $\mathfrak{p}$;

(7) $k = 2$ and $a_p^2 \equiv \omega(p)\ (mod\ \mathfrak{p})$;

(8) $N = 1$ and $p \mid (2k-3)(2k-1)$;

(9) $\mathfrak{p} \in Cong(f)$.

It follows that for f of squarefree level, if none of the above conditions are satisfied, then the deformation problem $\mathbf{D}(f, S)$ is unobstructed. This beautiful theorem gave the first general criteria for determining when a modular deformation problem is unobstructed. Note that these conditions describe only a finite set of primes, leading to the following corollary:

Corollary 4.2 For a newform f of squarefree level, the deformation problem $\mathbf{D}(f, S)$ associated to a prime $\mathfrak{p}$ is unobstructed for all but finitely many primes $\mathfrak{p}$.

Thus, this theorem says that away from an exceptional finite set of primes, modular deformation rings are generally power series in three variables. Moreover, one can explicitly determine a set which contains this exceptional set of obstructed primes from information about the level, weight, and q-expansion of the modular form being considered.

In this chapter, we improve Weston's result by removing the hypothesis that the level of f be squarefree. Thus, for a newform f of any weight $k \geq 2$ and any level $N \geq 1$, one is able to bound the set of obstructed primes.

Using the results from the Langlands correspondence described in Chapter 1, we will determine explicit obstruction criteria for modular deformation problems. That is, given a residual modular Galois representation $\bar{\rho}_{f,\mathfrak{p}}$, we will find conditions on the residue prime $\mathfrak{p}$ which guarantee that $H^0(G_\ell, \bar{\epsilon}_p \otimes \text{ad } \bar{\rho}_{f,\mathfrak{p}}) = 0$ for every prime ℓ in some finite set S. As explained in Section 3.3, the nonvanishing of these cohomology groups is precisely what prevents the deformation problem from being unobstructed. If $H^0(G_\ell, \bar{\epsilon}_p \otimes \text{ad } \bar{\rho}) \neq 0$, then then corresponding deformation problem is said to have obstructions at ℓ. Of course, this cohomology group is simply the G_ℓ-invariants of the representation $\bar{\epsilon}_p \otimes \text{ad } \bar{\rho}$. Our strategy is to use the Local Langlands correspondence to determine $(\bar{\epsilon}_p \otimes \text{ad } \bar{\rho})|_{G_\ell}$ explicitly.

While modular Galois representations can be hard to describe, it is often easier to understand their semisimplifications. Thus, when studying the existence of obstructions at a prime ℓ, we will frequently use the fact that for any mod $\mathfrak{p}$ representation $\bar{\rho} : G_\ell \to \mathrm{GL}_2(k_\mathfrak{p})$, we have

$$\dim_{k_\mathfrak{p}} H^0(G_\ell, \bar{\epsilon}_p \otimes \mathrm{ad}^0 \bar{\rho}) \leq \dim_{k_\mathfrak{p}} H^0(G_\ell, \bar{\epsilon}_p \otimes \mathrm{ad}^0(\bar{\rho})^{ss}), \tag{4.1}$$

where $\mathrm{ad}^0(\bar{\rho})^{ss}$ denotes the semisimplification of $\mathrm{ad}^0 \bar{\rho}$.

4.1 Notation and Strategy

We fix some notation to be used throughout this chapter. Let $f = \sum a_n q^n$ be a newform of level N and weight $k \geq 2$. Let ω be its nebentypus character, and let M be the conductor of ω. Let K be the number field associated to f, and fix a prime $\mathfrak{p}$ in K with residue field $k_\mathfrak{p}$ of characteristic p such that $(N, p) = 1$ and $\bar{\rho}_{f,\mathfrak{p}}$ is absolutely irreducible. Let S be a finite set of places containing the primes dividing $N\infty$, and let $\hat{S} = S \cup \{p\}$. We wish to study the conditions under which the deformation problem $\mathbf{D}(f, \hat{S})$ associated to

$$\bar{\rho}_{f,\mathfrak{p}} : G_{\mathbf{Q},\hat{S}} \to \mathrm{GL}_2(k_\mathfrak{p})$$

is unobstructed. As described in Section 3.3, as long as $\mathfrak{p} \notin \mathrm{Cong}(f)$, then this amounts to determining when $H^0(G_\ell, \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}) \neq 0$ for $\ell \in S$. We also remind the reader that the set $\hat{S}$ controls the ramification behavior which is permitted in deformations of $\bar{\rho}_{f,\mathfrak{p}}$

Let π be the automorphic representation associated to f, and write $\pi = \otimes \pi_\ell$ for its decomposition into admissible complex representations π_ℓ of $\mathrm{GL}_2(\mathbf{Q}_\ell)$. By the Local Langlands correspondence, the classification of each π_ℓ allows us to study $\bar{\rho}_{f,\mathfrak{p}}|_{G_\ell}$ in an explicit fashion. In [21], the requirement that N be squarefree aided in the determination of π_ℓ for each $\ell \in S$; in particular, π_ℓ had to be either an unramified principal series, a principal series with one ramified character and one unramified character, or a special representation associated to an unramified character, and these were the only possibilities. When $\ell^2 \mid N$, it is not so easy to determine the structure of π_ℓ. However, this turns out to be unnecessary, as described in Section 4.3.

4.2 Local representation at ℓ, where $\ell \in S$, $\ell \nmid N$

Before considering the primes $\ell \in S$ which divide the level of the modular form, let us consider the simpler cases where the prime ℓ does not divide N. The restriction of a modular Galois representation to G_ℓ when $\ell \nmid N$ depends entirely on whether $\ell = p$ or $\ell \neq p$, and we consider these cases separately. In particular, when $\ell \nmid N$ we do not encounter any novelties in the associated deformation problem, and this case was handled entirely by Weston in [21, Sections 3.1 and 3.4]. We include proofs for completeness, following [21] closely.

4.2.1 Local representation at ℓ where $\ell \neq p$, $\ell \nmid N$

For a prime ℓ not dividing Np, the local representation is well-understood. Fix $\alpha, \beta \in \bar{K}$ satisfying

$$\alpha + \beta = a_\ell \quad \text{and} \quad \alpha\beta = \ell^{k-1}\omega(\ell).$$

The semisimplification of the local representation $\rho_{f,\mathfrak{p}}|_{G_\ell} \otimes \bar{K}_\mathfrak{p}$ is the direct sum of two characters

$$\rho_{f,\mathfrak{p}}|_{G_\ell}^{ss} \otimes \bar{K}_\mathfrak{p} \simeq \chi_1 \oplus \chi_2,$$

where the $\chi_i : G_\ell \to \bar{K}_\mathfrak{p}^\times$ are unramified and satisfy

$$\chi_1(\mathrm{Frob}_\ell) = \alpha \quad \text{and} \quad \chi_2(\mathrm{Frob}_\ell) = \beta.$$

The reductions mod $\mathfrak{p}$ of these characters are denoted $\bar{\chi}_i : G_\ell \to \bar{k}_\mathfrak{p}$. As explained in [2, Section 1], if $\alpha \neq \beta$ then $\rho_{f,\mathfrak{p}} \otimes \bar{K}_\mathfrak{p}$ is split, i.e. $\rho_{f,\mathfrak{p}} \otimes \bar{K}_\mathfrak{p} \simeq \chi_1 \oplus \chi_2$. In this case, it is still possible that the residual representation is not split, but if $\alpha \not\equiv \beta \pmod{\mathfrak{p}}$ then we indeed have

$$\bar{\rho}_{f,\mathfrak{p}}|_{G_\ell} \otimes \bar{k}_\mathfrak{p} \simeq \bar{\chi}_1 \oplus \bar{\chi}_2.$$

The utility of this description comes from the fact that the existence of eigenvectors for $\bar{\epsilon}_p \otimes \mathrm{ad}^0\ \bar{\rho}_{f,\mathfrak{p}}$ with $k_\mathfrak{p}$-rational eigenvalues is invariant under base change, so to show that $H^0(G_\ell, \bar{\epsilon}_p \otimes \mathrm{ad}^0\ \bar{\rho}_{f,\mathfrak{p}}) = 0$ it suffices to study the invariants of $\bar{\rho}_{f,\mathfrak{p}}|_{G_\ell} \otimes \bar{k}_\mathfrak{p}$. The following proposition of Weston [21, Lemma 9] settles this case.

Proposition 4.3 Assume $\ell \nmid Np$ and $\ell \not\equiv 1 \pmod{p}$. Then $H^0(G_\ell, \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}_{f,\mathfrak{p}}) \neq 0$ if and only if $a_\ell^2 \equiv (\ell+1)^2 \ell^{k-2} \omega(\ell) \pmod{\mathfrak{p}}$.

Proof. By the preceding discussion and observation (4.1), it suffices to show that $(\bar{\epsilon}_p \otimes \mathrm{ad}^0\ \bar{\rho}_{f,\mathfrak{p}}|_{G_\ell}^{ss}) \otimes \bar{k}_\mathfrak{p}$ has no G_ℓ-invariants. A simple computation shows that

$$(\bar{\epsilon}_p \otimes \mathrm{ad}^0\ \bar{\rho}_{f,\mathfrak{p}}|_{G_\ell}^{ss}) \otimes \bar{k}_\mathfrak{p} \simeq \bar{\epsilon}_p \oplus \bar{\epsilon}_p \bar{\chi}_1 \bar{\chi}_2^{-1} \oplus \bar{\epsilon}_p \bar{\chi}_1^{-1} \bar{\chi}_2.$$

Since G_ℓ is topologically generated by Frob_ℓ and $\bar{\epsilon}_p(\mathrm{Frob}_\ell) \equiv \ell \pmod{\mathfrak{p}}$, the assumption that $\ell \not\equiv 1 \pmod{p}$ shows that the first summand has no invariants. Thus, there can be obstructions at ℓ if and only if one of $\bar{\epsilon}_p \bar{\chi}_1 \bar{\chi}_2^{-1}$ or $\bar{\epsilon}_p \bar{\chi}_1^{-1} \bar{\chi}_2$ is trivial. Evaluating these characters at Frob_ℓ, we see that this is equivalent to the condition

$$\frac{\alpha}{\beta} \equiv \ell^{\pm 1} \pmod{\mathfrak{p}}. \tag{4.2}$$

This congruence can be rewritten as

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} \equiv \ell + \frac{1}{\ell} \pmod{\mathfrak{p}},$$

or

$$\frac{(\alpha+\beta)^2}{\alpha\beta} \equiv \frac{(\ell+1)^2}{\ell} \pmod{\mathfrak{p}}.$$

Since $\alpha + \beta = a_\ell$ and $\alpha\beta = \ell^{k-1}\omega(\ell)$, this is equivalent to

$$a_\ell^2 \equiv (\ell+1)^2 \ell^{k-2} \omega(\ell) \pmod{\mathfrak{p}} \tag{4.3}$$

as desired.

Conversely, if (4.3) holds, then so does (4.2), and then the assumption that $\ell \not\equiv 1 \pmod{p}$ forces $\alpha \not\equiv \beta \pmod{\mathfrak{p}}$ and the converse follows. $\square$

4.2.2 Local representation at ℓ where $\ell = p$, $\ell \nmid N$

Recall from Section 2.1.1 the following definitions. For a modular form $f = \sum a_n q^n$ and any prime $\mathfrak{p} \mid p$ of K, recall that f is called *ordinary* at $\mathfrak{p}$ if $a_p \neq 0 \pmod{\mathfrak{p}}$; otherwise, f is called *nonordinary* at $\mathfrak{p}$. When $p \nmid N$, the structure of the local representation $\rho_{f,\mathfrak{p}}|_{G_p}$ depends on whether f is ordinary or nonordinary at $\mathfrak{p}$. See [6, pp. 214–215] for the details of the following classifications.

- If f is ordinary at $\mathfrak{p}$, the semisimplification of $\rho_{f,\mathfrak{p}} \otimes \bar{K}_{\mathfrak{p}}$ when restricted to inertia is

$$\rho_{f,\mathfrak{p}}|_{I_p} \otimes \bar{K}_{\mathfrak{p}} \simeq \epsilon_p^{k-1} \oplus 1.$$

- If f is nonordinary at $\mathfrak{p}$, then $\bar{\rho}_{f,\mathfrak{p}}|_{G_p}$ is absolutely irreducible.

The first lemma is [21, Lemma 14].

Lemma 4.4 Assume $p \nmid N$. If f is ordinary at $\mathfrak{p}$ and $H^0(G_p, \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}_{f,\mathfrak{p}}) \neq 0$, then $k \equiv 0, 2 \pmod{p-1}$.

Proof. It suffices to study the semisimplfication of $\bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}_{f,\mathfrak{p}}|_{I_p} \otimes \bar{k}_{\mathfrak{p}}$, and by the preceding discussion this is isomorphic to

$$\bar{\epsilon}_p \oplus \bar{\epsilon}_p \oplus \bar{\epsilon}_p^k \oplus \bar{\epsilon}_p^{2-k}.$$

But $\bar{\epsilon}_p$ is ramified at p, hence it is nontrivial. In particular, it has order $p-1$, so none of $\bar{\epsilon}_p, \bar{\epsilon}_p^k, \bar{\epsilon}_p^{2-k}$ are trivial if $k \not\equiv 0, 2 \pmod{p-1}$. $\square$

In the special case where $k = 2$, the above lemma is vacuous, and we need another lemma.

Lemma 4.5 Assume $p \nmid N$ and $p > 2k$. Then $H^0(G_p, \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}_{f,\mathfrak{p}}) = 0$ unless $k = 2$ and $a_p^2 \equiv \omega(p) \pmod{\mathfrak{p}}$.

Proof. The proof is not hard, but it involves the theory of Fontaine-Laffaille and filtered Dieudonné modules, taking us too far afield. Instead, see [20, Proposition 4.4]. $\square$

The final lemma of this section is [21, Lemma 15].

Lemma 4.6 Assume $p \nmid N$. If f is nonordinary at $\mathfrak{p}$ and $p > 3$, then $H^0(G_p, \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}_{f,\mathfrak{p}}) = 0$.

Proof. As mentioned at the beginning of this section, under the given hypotheses $\rho_{f,\mathfrak{p}}|_{G_p}$ is absolutely irreducible. If the projective image of $\bar{\rho}_{f,\mathfrak{p}}$ is dihedral, then the G_p-representation ad $\bar{\rho}_{f,\mathfrak{p}}$ is the sum of the trivial character, a quadratic character, and an irreducible two-dimensional G_p-representation. If the projective image of $\bar{\rho}_{f,\mathfrak{p}}$ is not dihedral, then ad $\bar{\rho}_{f,\mathfrak{p}}$ is the sum of the trivial character and an irreducible three-dimensional G_p-representation. In either case, since we assumed $p > 3$, the character $\bar{\epsilon}_p$ has order at least 3, hence it is neither trivial nor quadratic, and the result follows from [21, Lemma 3]. $\square$

When $\ell = p$ and $p \mid N$, the local Galois representation is not nearly as well-understood; indeed, this is the subject of p-adic Hodge theory. We hypothesize this case out of existence when we prove the main theorem.

4.3 Twists and ℓ-primitive newforms

Recall that for any primitive Dirichlet character χ of conductor M, we may twist the newform f to obtain another newform $f \otimes \chi = \sum b_n q^n$, where $b_n = \chi(n)a_n$ for almost all n. The level of $f \otimes \chi$ is at most NM^2, but it may be smaller. For any newform f and any prime ℓ, one says that f is *ℓ-primitive* if the ℓ-part of its level is minimal among all its twists by Dirichlet characters. We have the following simple but important lemma.

Lemma 4.7 Let f be a newform and let f_ℓ be an ℓ-primitive twist. Then

$$H^0(G_\ell, \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}_{f,\mathfrak{p}}) = H^0(G_\ell, \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}_{f_\ell,\mathfrak{p}}).$$

In particular, f has local obstructions at ℓ if and only if f_ℓ has local obstructions at ℓ.

Proof. For some Dirichlet character χ we have $f_\ell = f \otimes \chi$. Then it is well-known that $\rho_{f_\ell,\mathfrak{p}} \simeq \chi \otimes \rho_{f,\mathfrak{p}}$, and a straightforward matrix calculation shows that $\mathrm{ad}(\bar{\rho}_{f_\ell,\mathfrak{p}}) \simeq \mathrm{ad}(\bar{\rho}_{f,\mathfrak{p}})$. The lemma follows. $\square$

By Lemma 4.7, when studying local obstructions at ℓ for a newform f, we may assume that f is ℓ-primitive. The utility of considering ℓ-primitive newforms is given by the following result, which comes from [12, Proposition 2.8].

Proposition 4.8 Let π_ℓ be the local component of an ℓ-primitive newform $f \in S_k(\Gamma_1(N\ell^r))$ with $\ell \nmid N$ and $r \geq 1$. Then one of the following holds.

(1) $\pi_\ell \simeq \pi(\chi_1, \chi_2)$ is principal series, where χ_1 is unramified and χ_2 is ramified;

(2) $\pi_\ell \simeq \mathrm{St} \otimes \chi$, is special with χ unramified;

(3) π_ℓ is supercuspidal.

Proof. See [12, Proposition 2.8] for the proof. $\square$

If the level of a newform f is divisible by ℓ^2, it may be difficult to explicitly determine an ℓ-minimal twist. Loeffler and Weinstein have made this computationally feasible in many cases; see [12]. We will avoid this extra difficulty and simply determine where obstructions might occur in all three cases of the above proposition. The arguments used by Weston in [21] are robust enough to be adapted to the non-squarefree setting when we are in cases (1) and (2) of Proposition 4.8. Thus, we handle these cases first, following Weston's arguments closely.

4.3.1 Principal Series Obstruction Conditions

In this section we follow [21, Section 3.2] closely. Suppose we are in case (1) of Proposition 4.8, so $\ell \mid N$ and $\pi_\ell \simeq \pi(\chi_1, \chi_2)$ is principal series associated to two continuous characters $\chi_i : G_\ell \to \bar{K}_\mathfrak{p}$, where χ_1 is ramified and χ_2 is unramified. Thus, by (2.2), the semisimplification of the associated Galois representation satisfies

$$\rho_{f,\mathfrak{p}}|_{G_\ell}^{ss} \otimes \bar{K}_{\mathfrak{p}} \simeq \chi_1 \oplus \chi_2. \tag{4.4}$$

We obtain the following obstruction criteria; see also [21, Lemma 10].

Lemma 4.9 Suppose π_ℓ is principal series, associated to one ramified character and one unramified character. Assume also that $\ell \neq p$ and $\ell \not\equiv 1 \pmod{p}$. Then $H^0(G_\ell, \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}_{f,\mathfrak{p}}) = 0$.

Proof. Just as in the proof of Proposition 4.3, it suffices to show that neither $\bar{\epsilon}_p\bar{\chi}_1\bar{\chi}_2^{-1}$ or $\bar{\epsilon}_p\bar{\chi}_1^{-1}\bar{\chi}_2$ is trivial.

By Theorem 2.1, the determinant of $\rho_{f,\mathfrak{p}}|_{G_\ell}$ is $\epsilon_p^{k-1}\omega|_{G_\ell}$. Then (4.4) implies that

$$\chi_1\chi_2|_{G_\ell} = \epsilon_p^{k-1}\omega|_{G_\ell}.$$

Since both ϵ_p and χ_2 are unramified on G_ℓ, upon restricting to inertia we see that $\chi_1|_{I_\ell} = \omega|_{I_\ell}$ is a non–trivial character taking values in $\mu_{\ell-1}$. By hypothesis $\ell \not\equiv 1 \pmod{p}$, so $\mu_{\ell-1}$ injects into $k_{\mathfrak{p}}^\times$, and the reduction $\bar{\chi}_1 : G_\ell \to \bar{k}_{\mathfrak{p}}^\times$ is still a ramified character.

It follows that, upon restricting to I_ℓ, neither $\bar{\epsilon}_p\bar{\chi}_1\bar{\chi}_2^{-1}$ or $\bar{\epsilon}_p\bar{\chi}_1^{-1}\bar{\chi}_2$ is trivial, hence neither character is trivial on the full Galois group G_ℓ. This proves the result. $\square$

4.3.2 Special Obstruction Conditions

In this section we follow [21, Section 3.3] closely. Suppose we are in case (2) of Proposition 4.8, so π_ℓ is the special representation associated to an unramified character $\chi : G_\ell \to \bar{K}_{\mathfrak{p}}^\times$. Then by (2.3) the associated Galois representation has the form

$$\rho_{f,\mathfrak{p}}|_{G_\ell} \otimes \bar{K}_{\mathfrak{p}} \simeq \begin{pmatrix} \epsilon_p\chi & * \\ 0 & \chi \end{pmatrix} \tag{4.5}$$

with the upper right corner ramified.

We begin by collecting some technical lemmas. The following is [20, Lemma 5.1].

Lemma 4.10 If $\ell^2 \not\equiv 1 \pmod p$, then

$$\bar\rho_{f,\mathfrak{p}}|_{G_\ell} \otimes \bar k_{\mathfrak{p}} \simeq \begin{pmatrix} \bar\epsilon_p \bar\chi & * \\ 0 & \bar\chi \end{pmatrix}.$$

Proof. By (4.5) the semisimplification of $\bar\rho_{f,\mathfrak{p}} \otimes \bar k_{\mathfrak{p}}$ is $\bar\epsilon_p \bar\chi \oplus \bar\chi$, so the only way the lemma can fail is if

$$\bar\rho_{f,\mathfrak{p}}|_{G_\ell} \otimes \bar k_{\mathfrak{p}} \simeq \begin{pmatrix} \bar\chi & \nu \\ 0 & \bar\epsilon_p \bar\chi \end{pmatrix}.$$

for some nontrivial $\nu : G_\ell \to \bar k_{\mathfrak{p}}$. Suppose $g, h \in G_\ell$. Then

$$\begin{pmatrix} \bar\chi(g) & \nu(g) \\ 0 & \bar\epsilon_p \bar\chi(g) \end{pmatrix} \begin{pmatrix} \bar\chi(h) & \nu(h) \\ 0 & \bar\epsilon_p \bar\chi(h) \end{pmatrix} = \begin{pmatrix} \bar\chi(gh) & \bar\chi(g)\nu(h) + \bar\epsilon_p \bar\chi(h)\nu(g) \\ 0 & \bar\epsilon_p \bar\chi(gh) \end{pmatrix}.$$

Since $\bar\rho_{f,\mathfrak{p}}$ is a homomorphism, we must have

$$\nu(gh) = \bar\chi(g)\nu(h) + \bar\epsilon_p \bar\chi(h)\nu(g),$$

or equivalently

$$\bar\epsilon_p^{-1} \bar\chi^{-1} \nu(gh) = \bar\epsilon_p^{-1}(g) \bar\epsilon_p^{-1} \bar\chi^{-1} \nu(h) + \bar\epsilon_p^{-1} \bar\chi^{-1} \nu(h)$$

Thus $\bar\epsilon_p^{-1} \bar\chi^{-1} \nu$ is a 1-cocycle in $H^1(G_\ell, \bar k_{\mathfrak{p}}(-1))$, but this group vanishes by Lemma A.2. □

Following Weston, our next step is to translate the existence of obstructions at ℓ into a statement about the ramification of $\bar\rho_{f,\mathfrak{p}}$ at ℓ. The following is adapted from [21, Lemma 11].

Lemma 4.11 Assume π_ℓ is special, associated to an unramified character χ, with $\ell \neq p$ and $\ell^2 \not\equiv 1 \pmod p$. Then $H^0(G_\ell, \bar\epsilon_p \otimes \mathrm{ad}\, \bar\rho_{f,\mathfrak{p}}) \neq 0$ if and only if $\bar\rho_{f,\mathfrak{p}}$ is unramified at ℓ.

Proof. By Lemma 4.10, the assumption that $\ell^2 \not\equiv 1 \pmod p$ implies

$$\bar{\rho}_{f,\mathfrak{p}}|_{G_\ell} \otimes \bar{k}_{\mathfrak{p}} \simeq \begin{pmatrix} \bar{\epsilon}_p \bar{\chi} & \nu \\ 0 & \bar{\chi} \end{pmatrix}$$

for some $\nu : G_\ell \to \bar{k}_{\mathfrak{p}}$. Just as in the proof of Lemma 4.10, one can verify that $\bar{\chi}^{-1}\nu \in H^1(G_\ell, \bar{k}_{\mathfrak{p}}(1))$. As $\bar{\epsilon}_p$ and $\bar{\chi}$ are unramified at ℓ, the representation $\bar{\rho}_{f,\mathfrak{p}}|_{G_\ell}$ is unramified if and only if $\bar{\chi}^{-1}\nu$ is unramified. The fact that $\bar{\epsilon}_p$ is unramified at ℓ also implies that I_ℓ acts trivially on $\bar{k}_{\mathfrak{p}}$, so if $\phi \in H^1(G_\ell, \bar{k}_{\mathfrak{p}}(1))$ is nonzero, then its restriction to I_ℓ is actually a homomorphism

$$\phi|_{I_\ell} \in H^1(I_\ell, \bar{k}_{\mathfrak{p}}(1)) = \mathrm{Hom}(I_\ell, \bar{k}_{\mathfrak{p}}(1)).$$

Since char $\bar{k}_{\mathfrak{p}} = p$, any such homomorphism must factor through the maximal pro-p quotient of I_ℓ, which is isomorphic to $\mathbf{Z}_p(1)$. Thus $\phi|_{I_\ell}$ is nonzero, and this shows that any nonzero element of $H^1(G_\ell, \bar{k}_{\mathfrak{p}}(1))$ is ramified.

Thus, $\bar{\rho}_{f,\mathfrak{p}}$ is unramified if and only if it is semisimple, and a simple matrix computation shows that $\bar{\rho}_{f,\mathfrak{p}}$ is semisimple if and only if $H^0(G_\ell, \bar{\epsilon}_p \otimes \mathrm{ad}^0\ \bar{\rho}_{f,\mathfrak{p}}) \neq 0$. □

With this lemma in hand, we can give obstruction criteria for the present case. Recall from Section 3.3 the definition of the congruence primes $\mathrm{Cong}(f)$ for a modular form f.

Proposition 4.12 Assume π_ℓ is special, associated to an unramified character χ, with $\ell \neq p$, $\ell^2 \not\equiv 1 \pmod p$, and $\bar{\rho}_{f,\mathfrak{p}}$ absolutely irreducible. Then $H^0(G_\ell, \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}_{f,\mathfrak{p}}) \neq 0$ if and only if there is a prime $\bar{\mathfrak{p}}$ of $\bar{K}_{\mathfrak{p}}$ over $\mathfrak{p}$ such that $\bar{\mathfrak{p}} \in \mathrm{Cong}(f)$.

Proof. By Lemma 4.11, the existence of obstructions at ℓ is equivalent to $\bar{\rho}_{f,\mathfrak{p}}$ being unramified, and this is equivalent to $\bar{\mathfrak{p}} \in \mathrm{Cong}(f)$ by [6, (B) of p.221]. □

4.3.3 Supercuspidal Obstruction Conditions

We now consider case (3) of Proposition 4.8, where π_ℓ is *supercuspidal*. When $p > 2$, a supercuspidal π_ℓ is always induced from a quadratic extension of $\mathbf{Q}_p$, and these will be the focus of Proposition 4.14 below. When $\ell = 2$, there are additional supercuspidal representations, called extraordinary representations, and we consider these first. The case where π_ℓ is extraordinary was actually already dealt with in [20, Proposition 3.2] and causes no problems if $p \geq 5$. We reproduce Weston's proof here.

Proposition 4.13 Suppose π_ℓ is extraordinary, so $\ell = 2$, Then $H^0(G_\ell, \bar{\epsilon}_p \otimes \text{ad } \bar{\rho}_{f,\mathfrak{p}}) = 0$ if $k_\mathfrak{p}$ has residue characteristic at least 5.

Proof. Let $\rho : G_2 \to \mathrm{GL}_2(\bar{\mathbf{Q}}_p)$ be the representation of G_2 which is in Langlands correspondence with π_2. In this case, the projective image of inertia, proj $\rho(I_2)$, in $\mathrm{PGL}_2(\bar{\mathbf{Q}}_p)$ is isomorphic to either A_4 or S_4, and the composition

$$\text{proj } \rho(I_2) \hookrightarrow \mathrm{PGL}_2(\bar{\mathbf{Q}}_p) \xrightarrow{\text{ad}^0} \mathrm{GL}_3(\bar{\mathbf{Q}}_p)$$

is an irreducible representation of proj ρ. Since proj $\rho(I_2)$ has order 12 or 24, it follows that $\text{ad}^0\bar{\rho}_{f,\mathfrak{p}}$ is an irreducible $\bar{\mathbf{F}}_p$-representation of I_2 since $\text{char}(k_\mathfrak{p}) \geq 5$, thus $H^0(I_2, \bar{\epsilon}_p \otimes \text{ad}^0\ \bar{\rho}_{f,\mathfrak{p}}) = 0$ and the proposition follows. ☐

We will henceforth assume $p \geq 5$, so by this proposition there are no obstructions in the extraordinary case.

Now we deal with the final remaining possibility for π_ℓ, which is the supercuspidal, non-extraordinary case. This is the truly novel case that arises when removing the squarefree hypothesis and thus is the crux of this chapter. Recall that for any character ψ, we write $\bar{\psi}$ for its reduction mod $\mathfrak{p}$.

Proposition 4.14 Suppose f is a newform of weight $k \geq 2$ such that π_ℓ is supercuspidal but not extraordinary. Suppose also that $\ell > 5$. If $\ell^4 \not\equiv 1 \pmod{p}$, then $H^0(G_\ell, \bar{\epsilon}_p \otimes \text{ad } \bar{\rho}_{f,\mathfrak{p}}) = 0$.

Proof. The Langlands correspondence (cf. [20, Proposition 3.2] or [12, Remark 3.11]) implies that there is a quadratic extension $E/\mathbf{Q}_\ell$ such that

$$\rho_{f,\mathfrak{p}}|_{G_\ell} \simeq \mathrm{Ind}_{G_E}^{G_\ell}\chi$$

where G_E =Gal$(\bar{E}/E)$ is the absolute Galois group of E and $\chi : G_E \to \bar{\mathbf{Q}}_p$ is a continuous character. Let $\chi_E : \mathrm{Gal}(E/\mathbf{Q}_\ell) \to \{\pm 1\}$ be the nontrivial character for $E/\mathbf{Q}_\ell$. Let χ^c be the Galois conjugate character of χ, and let $\psi = \chi \cdot (\chi^c)^{-1}$. We have

$$\bar{\epsilon}_p \otimes \mathrm{ad}^0 \bar{\rho}_{f,\mathfrak{p}}|_{G_\ell}^{\mathrm{ss}} \simeq \bar{\epsilon}_p \chi_E \ \oplus \ \left(\bar{\epsilon}_p \otimes \mathrm{Ind}_{G_E}^{G_\ell}\ \bar{\psi}\right).$$

Since $p > 3$, the first summand has no G_ℓ-invariants, so we may focus on the second summand. By Mackey's criterion, the induced representation $\mathrm{Ind}_{G_E}^{G_\ell}\ \bar{\psi}$ is irreducible if and only if $\bar{\psi} \neq \bar{\psi}^c$. If it is irreducible, then so is its twist and we are done.

So suppose that $\bar{\psi} = \bar{\psi}^c$. We first note that, since $\bar{\psi} = \bar{\chi}(\bar{\chi}^c)^{-1}$, we have $\bar{\psi}^c = \bar{\chi}^c\bar{\chi}^{-1}$, hence

$$\bar{\psi}^2 = \bar{\psi}\bar{\psi}^c = [\bar{\chi}(\bar{\chi}^c)^{-1}] \cdot [\bar{\chi}^c(\bar{\chi})^{-1}] = 1.$$

Thus, $\bar{\psi}$ is a quadratic character on G_E.

Restricting the induced representation to G_E we have

$$(\mathrm{Ind}_{G_E}^{G_\ell}\bar{\psi})|_{G_E} \simeq \bar{\psi} \oplus \bar{\psi}^c = \bar{\psi} \oplus \bar{\psi}$$

where the first equality is a generality about induced representations and the second comes from our assumption that $\bar{\psi} = \bar{\psi}^c$. So we already have $H^0(G_E, \bar{\epsilon}_p \otimes \mathrm{Ind}_{G_E}^{G_\ell}\bar{\psi}) = 0$ unless $\bar{\psi} = \bar{\epsilon}_p^{-1}|_{G_E}$, in which case $\bar{\epsilon}_p|_{G_E}$ is quadratic. Since G_E has index 2 in G_ℓ, this would imply that on G_ℓ the cyclotomic character $\bar{\epsilon}_p$ has order at most 4. Evaluating at Frob_ℓ, this implies that $\ell^4 \equiv 1 \pmod{p}$. So if $\ell^4 \not\equiv 1 \pmod{p}$, then the representation has no G_E-invariants and hence it has no G_ℓ-invariants, completing the proof. $\square$

4.4 Main Theorem

We are now ready to prove the first main theorem of this thesis, which removes the squarefree hypothesis from Theorem 4.1. For a newform f of level N, recall that $\mathrm{Cong}(f)$ is the set of congruence primes for f as defined in Section 3.3. We use ϕ to denote Euler's totient function.

Theorem 4.15 Assume that $\bar{\rho}_{f,\mathfrak{p}}$ is absolutely irreducible and $p \geq 5$. If $H^2(G_{\mathbf{Q},\hat{S}}, \bar{\epsilon}_p \otimes \mathrm{ad}\, \bar{\rho}_{f,\mathfrak{p}}) \neq 0$ then one of the following holds:

1. $p \leq k$;
2. $p \mid N$;
3. $p \mid \phi(N_S)$, where N_S is the product of the primes in S;
4. $p \mid (\ell + 1)$ for some $\ell \mid N$;
5. $a_\ell^2 \equiv (\ell+1)^2 p^{k-2} \omega(\ell) \pmod{\mathfrak{p}}$ for some $\ell \in S$, $\ell \nmid N$, $p \neq \ell$;
6. $p = k + 1$ and f is ordinary at ℓ;
7. $k = 2$ and $a_p^2 \equiv \omega(p) \pmod{\mathfrak{p}}$;
8. $N = 1$ and $p \mid (2k-3)(2k-1)$;
9. $\mathfrak{p} \in \mathrm{Cong}(f)$;
10. $\ell^4 \equiv 1 \pmod{p}$ for some ℓ such that $\ell^2 \mid N$.

Remark. We note that conditions (1)–(9) are essentially the same conditions from [21, Theorem 18]; these conditions deal with the non-supercuspidal primes in S, while condition (10) deals with the (potentially) supercuspidal primes.

Proof. Since the behavior of the local representation $\bar{\rho}_{f,\mathfrak{p}}|_{G_p}$ is extremely difficult to control when $p \mid N$, this finite set of primes is hypothesized away by condition (2).

By equation (3.1), if $H^2(G_{\mathbf{Q},\hat{S}}, \bar{\epsilon}_p \otimes \text{ad } \bar{\rho}_{f,\mathfrak{p}}) \neq 0$ then either $\dim_k Ш^1(G_{\mathbf{Q},\hat{S}}, \bar{\epsilon}_p \otimes \text{ad}^0 \bar{\rho}) \neq 0$ or $H^0(G_\ell, \bar{\epsilon}_p \otimes \text{ad } \bar{\rho}_{f,\mathfrak{p}}) \neq 0$ for some $\ell \in \hat{S}$. By [21, Lemma 17], the former is only possible if $\mathfrak{p} \in \text{Cong}(f)$. This is accounted for in condition (9).

Now let $\ell \in \hat{S}$. If $N > 1$ and $\ell \nmid Np$, then by Proposition 4.3, there can exist obstructions at ℓ only if conditions (3) and (5) are satisfied. If $N = 1$, then by [21, Proposition 17(3)], we need only exclude the finitely primes in condition (8).

Suppose $\ell = p$. If f is ordinary at $\mathfrak{p}$ and $k > 2$, then Lemma 4.4 shows that condition (1) or (6) must hold. If f is ordinary at $\mathfrak{p}$ at $k = 2$, then Lemma 4.5 shows that condition (7) must hold. Finally, if f is nonordinary at $\mathfrak{p}$, then since $p > 3$, Lemma 4.6 shows that there can be no obstructions at p.

Now suppose $\ell \in S$ (so $\ell \neq p$) and $\ell \mid N$. By Lemma 4.7 we may assume f is ℓ-minimal. Let π_ℓ denote the corresponding local automorphic representation.

If π_ℓ is principal series, then Lemma 4.9 shows that there are no obstructions at ℓ as long as $\ell \not\equiv 1 \pmod{p}$, and condition (3) ensures that this is the case.

If π_ℓ is special, then by Proposition 4.12 there can exist obstructions at ℓ only if condition (3) or (4), along with condition (9), is satisfied.

If π_ℓ is supercuspidal, then by Proposition 4.14, there can exist obstructions at ℓ only if condition (10) is satisfied.

This exhausts the possibilities for $\ell \in \hat{S}$ and the structure of π_ℓ, and the proof is complete. $\square$

We note that this theorem is extremely amenable to computer-aided computation. The only primes which are not explicitly described in conditions (1)–(10) are the primes $\mathfrak{p}$ for which $\bar{\rho}_{f,\mathfrak{p}}$ is reducible. The following well-known result (cf. [21, Lemma 21]) describes these final primes to be excluded and is easy to implement by computer.

Lemma 4.16 Let $f = \sum a_n q^n$ be a newform of weight k and level N with associated number field

K. Let $\mathfrak{p}$ be a prime of K dividing the rational prime p. Suppose $\mathfrak{p}$ is a reducible prime for f, so that $\bar{\rho}_{f,\mathfrak{p}} \otimes \bar{k}_{\mathfrak{p}} \simeq \chi_1 \oplus \chi_2$ for some characters $\chi_1, \chi_2 : G_{\mathbf{Q}} \to \bar{k}_{\mathfrak{p}}^{\times}$. If $p \nmid N$, then each χ_i has conductor dividing Np. If also $p > k$, then one of the χ_i has conductor dividing N, so that $a_\ell \equiv \ell^{k-1} + 1 \pmod{\mathfrak{p}}$ for all $\ell \equiv 1 \pmod{N}$.

CHAPTER 5

LEVEL RAISING AND OBSTRUCTIONS

5.1 Minimal Deformation Problems and Optimal Levels

For any odd, continuous, absolutely irreducible representation $\bar{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(k_{\mathfrak{p}})$, with $k_{\mathfrak{p}}$ a finite field of characteristic p, let $\mathcal{H}(\bar{\rho})$ be the set of newforms of level prime to p giving rise to this representation, so if $f \in \mathcal{H}(\bar{\rho})$ then $\bar{\rho}_{f,\mathfrak{p}} \simeq \bar{\rho}$. Among all such newforms, there is a least level appearing, which we call the *optimal* level for $\mathcal{H}(\bar{\rho})$. In fact, this optimal level is the prime-to-p Artin conductor of $\bar{\rho}$ (see the introduction of [2]).

Let f and g be newforms in $\mathcal{H}(\bar{\rho})$ with associated minimal sets of primes S and S', respectively. We have an isomorphism of residual Galois representations $\bar{\rho}_{f,\mathfrak{p}} \simeq \bar{\rho}_{g,\mathfrak{p}}$, and if $S \subset S'$ then we have an equality of deformation problems $\mathbf{D}(f, S') = \mathbf{D}(g)$. Furthermore, since $S \subset S'$, if $\mathbf{D}(f)$ is obstructed then so is $\mathbf{D}(g)$. In fact, we prove the following theorem:

Theorem 5.1 If $\mathbf{D}(f)$ is unobstructed, then f is of optimal level for $\mathcal{H}(\bar{\rho})$.

In Section 5.3 we present the proof of this theorem; our strategy is to prove the contrapositive. By Proposition 5.2 below, we know the factorization of any nonoptimal level. If g is a newform of nonoptimal level, we compare it to an optimal level newform f. Since, as discussed above, $\mathbf{D}(g)$ inherits any obstructions that $\mathbf{D}(f)$ might have, we may assume that $\mathbf{D}(f)$ is unobstructed, and we show that even in this case, $\mathbf{D}(g)$ is necessarily obstructed.

This theorem is motivated by the following heuristic: If $\bar{\rho}$ is ℓ-ordinary and ℓ-distinguished, and if $\mathcal{H}(\bar{\rho})$ is the set of of all ℓ-ordinary, ℓ-stabilized newforms with mod ℓ Galois representation isomorphic to $\bar{\rho}$, then $\mathcal{H}(\bar{\rho})$ is a dense set of classical points in a *Hida family* $\mathbf{H}$. This object has a geometric interpretation in which its irreducible components have associated integers, called levels, which correspond to levels of modular forms. The components of non-minimal level have associated (full) Hecke algebras of higher $\Lambda = \mathbf{Z}_\ell[\![T]\!]$-rank than the minimal-level component (cf. [7, Section 2.4] for more details). Thus, if a general enough $\mathcal{R} = \mathbf{T}$ theorem is known (or believed), then this forces the deformation ring to grow as well. Our theorem shows that this sort of behavior is not a special property of Hida families, and that it actually occurs independent of any geometric structure.

It is worth pointing out that this theorem does *not* follow immediately from Theorem 4.15, because condition (9) of that theorem is not a sharp obstruction criterion, i.e. it does not *guarantee* the existence of obstructions.

5.2 Preliminaries

In this section we record the results which we will use to prove this chapter's main theorem. First, let us fix some notation.

Let $f = \sum a_n q^n$ be a newform of weight $k \geq 2$, level N (prime to p), and nebentypus ω, and let M be the conductor of ω. Let S be a finite set of places containing the primes which divide $Np\infty$. Let $K = \mathbf{Q}(a_n)$ be the number field associated to f, and fix a prime $\mathfrak{p}$ of $\bar{K}$ which lies over the rational prime p. We have $f \in \mathcal{H}(\bar{\rho})$, where $\bar{\rho}_{f,\mathfrak{p}} \simeq \bar{\rho}$.

Suppose f is of optimal level for $\mathcal{H}(\bar{\rho})$. If $g \in \mathcal{H}(\bar{\rho})$ is of nonoptimal level, we will want to know what form its level can have. The following is a result of Carayol (see the introduction of [2]).

Proposition 5.2 Suppose $\bar{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\bar{\mathbf{F}}_p)$ is modular of weight $k \geq 2$ and level N' prime to p. Then

$$N' = N \cdot \prod \ell^{\alpha(\ell)}$$

where N is the conductor of $\bar{\rho}$, and for each ℓ with $\alpha(\ell) > 0$, one of the following holds:

1. $\ell \nmid Np$, $\ell(\mathrm{tr}\ \bar{\rho}(\mathrm{Frob}_\ell)^2) = (1+\ell)^2 \det\ \bar{\rho}(\mathrm{Frob}_\ell)$ in $\bar{\mathbf{F}}_p$ and $\alpha(\ell) = 1$;

2. $\ell \equiv -1 \bmod p$ and one of the following holds:

   (a) $\ell \nmid N$, tr $(\bar{\rho}(\mathrm{Frob}_\ell) = 0$ in $\bar{\mathbf{F}}_p$ and $\alpha(\ell) = 2$, or

   (b) $\ell \,||\, N$, det $\bar{\rho}$ is unramified at ℓ, and $\alpha(\ell) = 1$;

3. $\ell \equiv 1 \bmod p$ and one of the following holds:

   (a) $\ell \nmid N$ and $\alpha(\ell) = 2$, or

   (b) $\ell^2 \nmid N$, or the power of ℓ dividing N is the same as the power dividing the conductor of det $\bar{\rho}$, and $\alpha(\ell) = 1$.

Our goal, then, is to show that each of the possible supplementary primes appearing in Proposition 5.2 gives rise to an obstruction. We collect some lemmas in this direction.

The first lemma is proved in [21, Section 3].

Lemma 5.3 If $p \mid (\ell - 1)$ for some $\ell \in S$, then $\mathbf{D}(f, S)$ is obstructed.

Proof. Since $p \mid \ell - 1$, we have $H^0(G_\ell, \bar{\epsilon}_p) \neq 0$. Since $\bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}_{f,\mathfrak{p}} \simeq \bar{\epsilon}_p \oplus (\bar{\epsilon}_p \otimes \mathrm{ad}^0\ \bar{\rho}_{f,\mathfrak{p}})$ this shows that $H^0(G_\ell, \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}_{f,\mathfrak{p}}) \neq 0$ and so $\mathbf{D}(f, S)$ is obstructed. □

We record one final lemma before proving our theorem.

Lemma 5.4 If $\ell \,||\, N$, $\ell \nmid M$, and $\ell^2 \equiv 1 \pmod p$, then $\mathbf{D}(f, S)$ is obstructed.

Proof. As explained in [20, Section 5.2], in this case π_ℓ is special, associated to an unramified character. By (2.3) this translates on the Galois side to the existence of an unramified character $\chi : G_\ell \to \bar{K}_{\mathfrak{p}}^\times$ such that

$$\rho_{f,\mathfrak{p}}|_{G_\ell} \otimes \bar{K}_\lambda \simeq \begin{pmatrix} \epsilon_p \chi & * \\ 0 & \chi \end{pmatrix},$$

with the upper right corner ramified. Upon reduction this matrix becomes either

$$A = \begin{pmatrix} \bar{\epsilon}_p \bar{\chi} & \nu \\ 0 & \bar{\chi} \end{pmatrix} \quad \text{or} \quad B = \begin{pmatrix} \bar{\chi} & \nu \\ 0 & \bar{\epsilon}_p \bar{\chi} \end{pmatrix}$$

for some $\nu : G_\ell \to \bar{k}_{\mathfrak{p}}$. We note that by Lemma 4.10, possibility B can only occur if $\ell^2 \equiv 1 \pmod{p}$.

Let $C = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. One computes

$$ACA^{-1} = \begin{pmatrix} 0 & \bar{\epsilon}_p \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad BCB^{-1} = \begin{pmatrix} 0 & \bar{\epsilon}_p^{-1} \\ 0 & 0 \end{pmatrix},$$

and so

$$(\bar{\epsilon}_p \otimes \text{ad } \bar{\rho}_{f,\mathfrak{p}}) \cdot C = \begin{pmatrix} 0 & \bar{\epsilon}_p^j \\ 0 & 0 \end{pmatrix}$$

for $j = 0$ or 2, so $C \in H^0(G_\ell, \bar{\epsilon}_p \otimes \text{ad } \bar{\rho}_{f,\mathfrak{p}})$. If $j = 0$, this is obvious; if $j = 2$, this follows from ϵ_p is ramified only at p and hence factors through a group which is topologically generated by Frob_ℓ, but $\epsilon_p(\text{Frob}_\ell) = l$, and $\ell^2 \equiv 1 \pmod{p}$. So in either case $H^0(G_\ell, \bar{\epsilon}_p \otimes \text{ad } \bar{\rho}_{f,\mathfrak{p}}) \neq 0$, hence $\mathbf{D}(f, S)$ is obstructed. $\square$

5.3 Optimal Level Deformation Problems

Let $f \in S_k(\Gamma_1(N))$ and $g \in S_k(\Gamma_1(N'))$ be newforms in $\mathcal{H}(\bar{\rho})$ with f of optimal level and $N' > N$; by Proposition 5.2, $N \mid N'$. Let S (resp. S') be the set of places of $\mathbf{Q}$ dividing $Np\infty$ (resp. $N'p\infty$), so $S \subset S'$.

Write $f = \sum a_n q^n$. Let K be a field containing the Fourier coefficients of both f and g, and let $\mathfrak{p}$ be a prime of K over p such that $f \equiv g \pmod{\mathfrak{p}}$ and hence $\bar{\rho} \simeq \bar{\rho}_{f,\mathfrak{p}} \simeq \bar{\rho}_{g,\mathfrak{p}}$. Write $k_{\mathfrak{p}}$ for the residue field of $\mathfrak{p}$.

Using this notation, we are now ready to prove Theorem 5.1. For the reader's convenience, we restate the theorem.

Theorem 5.5 If $\mathbf{D}(f)$ is unobstructed, then f is of optimal level for $\mathcal{H}(\bar{\rho})$.

Proof. We will prove the contrapositive. Keeping the notation from the beginning of Section 5.3, let f and g be newforms in $\mathcal{H}(\bar{\rho})$, with f of optimal level and g of non-optimal level. We will show that $\mathbf{D}(g)$ is obstructed.

If $\mathbf{D}(f)$ is obstructed, then as noted earlier, this implies $\mathbf{D}(g)$ is also obstructed. So in proving the theorem, we may assume that $\mathbf{D}(f)$ is unobstructed.

We consider separately the primes $\ell \in S'$ which appear in cases (1), (2), and (3) of Proposition 5.2. Note that we have an equivalence of deformation problems $\mathbf{D}(g, S') = \mathbf{D}(f, S')$. We write $\mathbf{D}$ for these equivalent deformation problems.

First, suppose $\ell \mid N'$ is as in case (3), so in particular $\ell \equiv 1 \pmod{p}$. Then by Lemma 5.3 we see that $\mathbf{D}$ is obstructed.

Next, suppose $\ell \mid N'$ is as in case (1) of the proposition, so ℓ is a prime such that $\ell \nmid Np$, $\alpha(\ell) = 1$, and $\ell a_\ell^2 \equiv (1+\ell)^2 \omega(\ell) \ell^{k-1} \pmod{p}$, or equivalently (since ℓ is invertible in $\bar{k}_{\mathfrak{p}}$),

$$a_\ell^2 \equiv (\ell+1)^2 \ell^{k-2} \omega(\ell) \pmod{\mathfrak{p}}.$$

Then by Lemma 4.3 we see that $\mathbf{D}$ is obstructed.

Finally, suppose $\ell \mid N'$ is as in case (2), so $\ell \equiv -1 \pmod{p}$ and one of the following holds:

(a) $\ell \nmid N$, $a_\ell \equiv 0 \pmod{\mathfrak{p}}$, and $\alpha(\ell) = 2$; or

(b) $\ell \mid\mid N$, det ρ is unramified at ℓ, and $\alpha(\ell) = 1$.

If ℓ were as in case (b), then actually $\ell \in S$ and $\ell \nmid M$, hence Lemma 5.4 shows that in fact $\mathbf{D}(f, S)$ is obstructed. This contradicts our hypothesis on $\mathbf{D}(f)$, so we can ignore this case.

Finally, we must consider case (a), so that $\ell \equiv -1 \pmod{p}$, $\ell \nmid N$, $a_\ell \equiv 0 \pmod{\mathfrak{p}}$, and $\alpha(\ell) = 2$. Recalling that $\mathbf{D} = \mathbf{D}(f, S')$, Lemma 4.3 gives the obstruction since $a_\ell \equiv (\ell+1) \equiv 0 \pmod{\mathfrak{p}}$. $\square$

Remark. It is not the case that every minimal, optimal level deformation problem is unobstructed. Indeed, for any prime ℓ we have

$$\bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}_{f,\mathfrak{p}} \simeq \bar{\epsilon}_p \oplus (\bar{\epsilon}_p \otimes \mathrm{ad}^0\ \bar{\rho}_{f,\mathfrak{p}})$$

and

$$H^0(G_\ell, \bar{\epsilon}_p) \neq 0 \Leftrightarrow \ell \equiv 1 \pmod{p},$$

so condition (3) of Theorem 4.15 is sharp.

Example. Let $p = 5$, $\ell = 11$, and $k = 3$. The space $S_3(\Gamma_1(11), 3)$ contains one newform defined over $\mathbf{Q}$ and four newforms which are Galois conjugates defined over $\mathbf{Q}(\alpha)$, where α is a root of $x^4 + 5x^3 + 15x^2 + 15x + 5$. The minimal set S for any of these newforms is $S = \{11, \infty\}$. Since $S_3(\Gamma_1(1))$ is empty, all of these newforms are of optimal level for their respective mod p representations, but since $\ell \equiv 1$ mod p their minimal deformation problems are obstructed.

Remark. The techniques in this paper cannot rule out the possibility that two (or more) congruent modular forms of optimal level can exist for an unobstructed modular deformation problem.

Combining this result with the fact that Theorem 4.15 excludes only a finite set of primes, we have the following corollary.

Corollary 5.6 Let f be a newform of level N and weight $k \geq 2$. For infinitely many primes p, f represents an optimal modular realization of a mod p representation $\bar{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\bar{\mathbf{F}}_p)$.

Proof. For infinitely many such p, $\mathbf{D}(f)$ is unobstructed by Theorem 4.15, and by Theorem 5.5, this implies that f is of optimal level among modular forms realizing $\bar{\rho}$. $\square$

Remark. Actually, there is a much simpler proof of this fact: If f is of nonoptimal level for its mod p representation, then there is a modular form g of lower level such that f is congruent to g (mod $\mathfrak{p}$). But such a congruence can occur for only finitely many primes ℓ, which follows from the q-expansion principle and the fact that these spaces of modular forms are finite dimensional.

We also get another corollary. For any integer N, let $d(N)$ be the number of prime divisors of N, i.e. $d(N) = \sum_{p|N} 1$.

Corollary 5.7 Fix a prime $\mathfrak{p}$ with residue field k of characteristic $p > 3$, suppose $\bar{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$ is a modular mod $\mathfrak{p}$ representation of prime-to-p conductor N, and let f be a newform of level N such that $\bar{\rho} \simeq \bar{\rho}_{f,\mathfrak{p}}$. If g is a newform of level N' such that $f \equiv g \pmod{\mathfrak{p}}$, then

$$\dim_k H^2(G_{\mathbf{Q},S}, \mathrm{ad}\ \bar{\rho}) \geq d\left(N'/N\right)$$

where S is any finite set of places containing the primes which divide $N'p\infty$.

Proof. The proof of Theorem 5.5 shows that if g is of nonoptimal level N', then for every prime ℓ dividing N'/N, we have $H^0(G_\ell, \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}_{f,\mathfrak{p}}) \neq 0$. By equation (3.1) we have

$$\dim_k H^2(G_{\mathbf{Q},S}, \mathrm{ad}\ \bar{\rho}) \geq \sum_{\ell \in S} \dim_k H^0(G_\ell), \bar{\epsilon}_p \otimes \mathrm{ad}\ \bar{\rho}_{f,\mathfrak{p}})$$

and the corollary follows.

$\square$

CHAPTER 6

EXAMPLES

Example.

Let $k = 2$, $p = 11$, and $\ell = 7$. Consider the CM elliptic curve E with Cremona label $49a1$; it is given by

$$E : y^2 + xy = x^3 - x^2 - 2x - 1,$$

and its associated modular form $f \in S_2(\Gamma_0(49))$ has q-expansion

$$f = q + q^2 - q^4 - 3q^8 - 3q^9 + \cdots .$$

The mod p Galois representation $\bar{\rho}_{f,p}$ is irreducible, and one checks that none of conditions (1)–(8) of Theorem 4.15 are satisfied. Using Sage [16], one also verifies that $p \notin \mathrm{Cong}(f)$ and so f is of optimal level for this representation.

Loeffler and Weinstein have incorporated their results from [12] into the Local Components package of [16]. Using this, one discovers that π_ℓ is supercuspidal. However, since $\ell^4 \equiv 3 \pmod{p}$, condition (10) is also not satisfied. Thus $\mathbf{D}(f)$ is unobstructed.

Example.

This example shows that condition (10) of Theorem 4.15 is necessary but not sufficient for producing local obstructions at supercuspidal primes. Let us fix $k = 3$, $p = 5$, and $\ell = 7$. Note that $\ell^2 \equiv -1 \bmod p$ and $\ell^4 \equiv 1 \bmod p$.

Using [16], one finds a newform f in $S_3(\Gamma_1(49))$ with a q-expansion that begins

$$f = q + \left(-\frac{1}{92}\alpha^3 + \frac{5}{92}\alpha^2 - \frac{41}{91}\alpha + \frac{229}{92}\right) q^2 + \left(-\frac{1}{184}\alpha^3 + \frac{5}{184}\alpha^2 - \frac{133}{184}\alpha + \frac{229}{184}\right) q^3 + \cdots$$

where α is a root of $x^4 - 4x^3 + 82x^2 - 188x + 1841$; let $K = \mathbf{Q}(\alpha)$.

Using the Local Components package of [16], one discovers that π_ℓ is supercuspidal. Let $E = \mathbf{Q}_\ell(s)$ be the unramified quadratic extension of $\mathbf{Q}_\ell$. Let $L = K(\beta)$ where β satisfies the polynomial $x^2 + \left(\frac{3}{1288}\alpha^3 + \frac{11}{184}\alpha^2 - \frac{153}{1288}\alpha + \frac{467}{184}\right) x - 1$. Then the character χ associated to π_ℓ is characterized by

$$\chi : E^\times \to L$$

$$s \mapsto \beta, \quad 7 \mapsto 7.$$

(Here we are viewing χ as a character of $E^\times$ instead of G_E via local class field theory.) Let $\mathfrak{p}$ be either of the two primes of L which lies over $\mathfrak{p}$. Then using [16], one verifies that χ and its conjugate χ^c are equivalent mod $\mathfrak{p}$ by checking that $\beta - \beta^c$ has positive $\mathfrak{p}$-valuation. In the notation of Proposition 4.14, this shows that $\bar{\psi} = 1$; the induction of this character is a symmetric representation, and so $\bar{\epsilon}_p \otimes \mathrm{Ind}_{G_E}^{G_p} \bar{\psi}$ is an invariant G_ℓ-representation, hence $H^0(G_\ell, \bar{\epsilon}_p \otimes \mathrm{ad}\, \bar{\rho}_{f,\mathfrak{p}}) = 0$.

APPENDIX

COHOMOLOGY AND DUALITY THEOREMS

Galois Cohomology

For the reader's convenience, we recall here some definitions and well-known facts from Galois cohomology which are used extensively throughout the rest of this thesis. The main source for the first two sections is [17, Chapter VII]

Let $G = \mathrm{Gal}(K/L)$ be a Galois group and let M be a G-module. We let $H^0(G, M) = M^G$ denote the G-invariant elements of M. H^0 is functorial in the sense that, if M and N are both G-modules with a compatible morphism $f : M \to N$, then there is an induced morphism $H^0(G, M) \to H^0(G, N)$. This functor is left-exact, so given an exact sequence of G-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

the sequence

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G$$

is also exact, and the right-derived functors of $H^0(G, \cdot)$ define the higher Galois cohomology groups $H^i(G, \cdot)$, so that there is a long exact sequence of Galois cohomology groups:

$$\begin{aligned} 0 &\longrightarrow H^0(G, A) \longrightarrow H^0(G, B) \longrightarrow H^0(G, C) \\ &\longrightarrow H^1(G, A) \longrightarrow H^1(G, B) \longrightarrow H^1(G, C) \\ &\longrightarrow H^2(G, A) \longrightarrow H^2(G, B) \longrightarrow H^2(G, C) \longrightarrow \cdots \end{aligned}$$

In this appendix we collect several important results in Galois cohomology which are used at crucial points in this thesis. In the following sections, μ denotes the group scheme of roots of unity.

Inflation-Restriction

Let G be a group and $H \triangleleft G$ a normal subgroup. Let A be a G-module. For any i, there are natural maps

$$\mathrm{Res} : H^i(G, A) \to H^i(H, A)$$
$$\mathrm{Inf} : H^i(G/H, A) \to H^i(G, A),$$

called *restriction* and *inflation*, respectively. The following result is a useful tool for computing Galois cohomology groups.

Proposition A.1 (Inflation-Restriction Exact Sequence) The following sequence is exact:

$$0 \to H^1(G/H, A^H) \xrightarrow{\mathrm{Inf}} H^1(G, A) \xrightarrow{\mathrm{Res}} H^1(H, A).$$

Let G_ℓ be a decomposition group above a rational prime ℓ, let I_ℓ be its inertia group, and let $G_{\mathbf{F}} = G_\ell/I_\ell$. Let $k_{\mathfrak{p}}$ denote a finite field of characteristic $p \neq \ell$, and let $\bar{k}_{\mathfrak{p}}$ be its algebraic closure. The following technical lemma is known to experts, but a published proof is hard to find. The proof below was communicated to me by Adam Gamzon. Recall that the Tate twist $\bar{k}_{\mathfrak{p}}(n)$ denotes the G_ℓ-module which is identical to $\bar{k}$ as a set, but whose Galois action is given by $\bar{\epsilon}_p^n$.

Lemma A.2 If $\ell^2 \not\equiv 1 \pmod{p}$, then $H^1(G_\ell, \bar{k}_{\mathfrak{p}}(-1)) = 0$.

Proof. The Inflation-Restriction Exact Sequence in Galois cohomology yields

$$H^1(G_{\mathbf{F}}, \bar{k}_{\mathfrak{p}}(-1)^{I_\ell}) \to H^1(G_\ell, \bar{k}_{\mathfrak{p}}(-1)) \to H^1(I_\ell, \bar{k}_{\mathfrak{p}}(-1))^{G_{\mathbf{F}}},$$

where $G_{\mathbf{F}} \simeq G_\ell/I_\ell$. Since $\bar{\epsilon}_p$ is unramified at ℓ, I_ℓ acts trivially on $\bar{k}(-1)$, hence this exact sequence becomes

$$H^1(G_{\mathbf{F}}, \bar{k}_{\mathfrak{p}}(-1)) \to H^1(G_\ell, \bar{k}_{\mathfrak{p}}(-1)) \to \mathrm{Hom}(I_\ell, \bar{k}_{\mathfrak{p}}(-1))^{G_{\mathbf{F}}}.$$

We will show that both end terms are zero. First we compute $\mathrm{Hom}(I_\ell, \bar{k}_{\mathfrak{p}}(-1))^{G_{\mathbf{F}}}$. Since the characteristic of $\bar{k}$ is p, any homomorphism in this group must factor through the maximal pro-p quotient of I_ℓ, which is isomorphic as a G_ℓ-module to $\mathbf{Z}_p(1) \simeq \epsilon_p$. Take any $\phi \in \mathrm{Hom}(\mathbf{Z}_p(1), \bar{k}_{\mathfrak{p}}(-1))^{G_{\mathbf{F}}}$ and let $g \in G_{\mathbf{F}}$ denote the Frobenius automorphism. Then since ϕ is invariant under the action of g, for any $\alpha \in I_\ell$ we have

$$\phi(\alpha) = (g \cdot \phi)(\alpha) = g \cdot \phi(g^{-1} \cdot \alpha) = \bar{\epsilon}_p^{-2}(g)\phi(\alpha) = \ell^{-2}\phi(\alpha).$$

Thus, if $\ell^2 \not\equiv 1 \pmod{p}$ then $\mathrm{Hom}(I_\ell, \bar{k}_{\mathfrak{p}}(-1))^{G_{\mathbf{F}}} = 0$.

Now we compute the first term. Suppose $\phi \in H^1(G_{\mathbf{F}}, \bar{k}(-1))$. The cocycle condition implies

$$\phi(g^k) = \phi(g)(1 + p^{-1} + \cdots + p^{-(k-1)})$$

for all k. Let n be the order of p^{-1} in $\bar{k}^\times$, and set $m = \left(p^{-(n-1)} + \cdots + p^{-1} + \frac{p^{-1}}{p^{-1}-1}\phi(g)\right)$. We will now show that for any r, $\phi(g^r)$ is equal to the coboundary $\bar{\epsilon}_p^{-1}m - m = p^{-r}m - m$. Note that in $\bar{k}_{\mathfrak{p}}$ we have the identity

$$1 + p^{-1} + \cdots + p^{-(n-1)} = 0$$

so it suffices to check the claim for $\sigma = g^r$ with $1 \le r \le n-1$. We have

$$p^{-r}m - m = \left(p^{-(r-1)} + \cdots + 1 + p^{-(n-1)} + \cdots + p^{-(r+1)} + \frac{p^{-(r+1)}}{p^{-1}-1}\right)\phi(g) - m,$$

and since $p^{-1} + \cdots + p^{-(n-1)} = -1$ this is equal to

$$\left(-p^{-r} + 1 + p\left(\frac{p^{-r}-1}{p^{-1}-1}\right)\right)\phi(g) = (1 + \cdots + p^{-(r-1)})\phi(g) = \phi(g^r)$$

as claimed.

So $H^1(G_{\mathbf{F}}, \bar{k}(-1)) = 0$, and this completes the proof of the lemma. $\square$

Local Duality

Fix a finite extension K of $\mathbf{Q}_p$ and let $G_K = \mathrm{Gal}(\bar{K}/K)$ be its absolute Galois group. Let M be a finite G_K-module. The Pontryagin dual of M is defined as

$$M^* = \mathrm{Hom}(M, \mu(\bar{K})).$$

Then there is a canonical pairing

$$M \times M^* \to \mu(\bar{K})$$

which induces a map (via cup-product)

$$H^i(G_K, M) \times H^{2-i}(G_K, M^*) \to H^2(G_K, \mu(\bar{K}) \simeq \mathbf{Q}/\mathbf{Z}$$

for $i = 0, 1, 2$.

Proposition A.3 (Tate Local Duality)

1. The map $H^i(G_K, M) \times H^{2-i}(G_K, M^*) \to \mathbf{Q}/\mathbf{Z}$ is a perfect pairing of finite groups.

2. $H^i(G_K, M) = 0$ for $i \geq 3$.

Proof. See [14] Chapter 1, Corollary 2.3. $\square$

Thus, there is an identification $H^i(G_K, M)^\vee \simeq H^{2-i}(G_K, M^*)$ where $H^i(G_K, M)^\vee$ is the Pontryagin dual of $H^i(G_K, M)$. Let $d = [K : \mathbf{Q}_p]$ be the degree of K over $\mathbf{Q}_p$. For any finite set X, let $[X]$ denote its order.

Proposition A.4 (Euler-Poincare Characteristic) The Euler-Poincare characteristic of M (with respect to K) is

$$\chi(K, M) = \frac{[H^0(G_K, M)] \cdot [H^2(G_K, M)]}{[H^1(G_K, M)]} = p^{-\nu_p([M]) \cdot d}.$$

Proof. See [15] Theorem 7.3.1. □

Poitou-Tate Duality

We now relate the Galois cohomology of a global field to the cohomology of its local completions. Fix a finite extension K of $\mathbf{Q}$, and for any set S of places of K containing the infinite place, let K_S denote the maximal extension of K unramified outside S. Set $G_S = \mathrm{Gal}(K_S/K)$. Write S_f for the finite places in S and S_∞ for $S \setminus S_f$. Let μ denote the group scheme of roots of unity.

Let M be a finite G_S-module whose order is an S-unit in K. Let $\mathcal{O}_S^\times$ denote the group of S-units of K_S, and again define the Pontryagin dual of M by $M^* = \mathrm{Hom}(M, \mathcal{O}_S^\times) = \mathrm{Hom}(M, \mu(K_S))$.

In addition to the Galois cohomology groups $H^i(G_S, M)$, we define some additional topological groups. For a place $v \in S$, let K_v denote the corresponding completion, and write $H^i(K_v, M)$ in place of $H^i(G_{K_v}, M)$. Then we make the following definitions:

$$\begin{aligned}
P^0(G_S, M) &= \prod_{v \in S_f} H^0(K_v, M) \times \prod_{v \in S_\infty} \hat{H}^0(K_v, M),\\
P^1(G_S, M) &= \tilde{\prod} H^1(K_v, M),\\
P^2(G_S, M) &= \bigoplus_{v \in S} H^2(K_v, M).
\end{aligned}$$

In the above definitions, $\hat{H}^i$ denotes the modified (Tate) Galois cohomology groups, and $\tilde{\prod}$ denotes the topological restricted product.

We thus have homomorphisms $H^i(G_S, M) \to P^i(G_S, M)$, whose kernels we denote by $Ш^i(G_S, M)$. This is the notation used in Section 3.3.

We come now to the main theorem of this appendix.

Theorem A.5 (Poitou-Tate Exact Sequence) Let M be a finite G_S-module such that the order of M is an S-unit in K. Then there is a canonical exact sequence of topological groups

$$\begin{aligned}
0 \longrightarrow H^0(G_S, M) \longrightarrow P^0(G_S, M) \longrightarrow H^2(G_S, M^*)^\vee \\
\longrightarrow H^1(G_S, M) \longrightarrow P^1(G_S, M) \longrightarrow H^1(G_S, M^*)^\vee \\
\longrightarrow H^2(G_S, M) \longrightarrow P^2(G_S, M) \longrightarrow H^0(G_S, M^*)^\vee \longrightarrow 0
\end{aligned}$$

Proof. See [15] Chapter VIII Section 6. $\square$

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