Quality Competition in Supply Chain Networks with Applications to Information Asymmetry, Product Differentiation, Outsourcing, and Supplier Selection

Dong Li

Follow this and additional works at: https://scholarworks.umass.edu/dissertations_2

Part of the Business Administration, Management, and Operations Commons, Management Sciences and Quantitative Methods Commons, and the Operations Research, Systems Engineering and Industrial Engineering Commons

Recommended Citation
Li, Dong, "Quality Competition in Supply Chain Networks with Applications to Information Asymmetry, Product Differentiation, Outsourcing, and Supplier Selection" (2015). Doctoral Dissertations. 424.
https://scholarworks.umass.edu/dissertations_2/424

This Open Access Dissertation is brought to you for free and open access by the Dissertations and Theses at ScholarWorks@UMass Amherst. It has been accepted for inclusion in Doctoral Dissertations by an authorized administrator of ScholarWorks@UMass Amherst. For more information, please contact scholarworks@library.umass.edu.
QUALITY COMPETITION IN SUPPLY CHAIN NETWORKS WITH APPLICATIONS TO INFORMATION ASYMMETRY, PRODUCT DIFFERENTIATION, OUTSOURCING, AND SUPPLIER SELECTION

A Dissertation Presented

by

DONG “MICHELLE” LI

Submitted to the Graduate School of the University of Massachusetts Amherst in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY

September 2015

Isenberg School of Management
© Copyright by Dong “Michelle” Li 2015
All Rights Reserved
QUALITY COMPETITION
IN SUPPLY CHAIN NETWORKS
WITH APPLICATIONS TO
INFORMATION ASYMMETRY, PRODUCT
DIFFERENTIATION, OUTSOURCING, AND SUPPLIER
SELECTION

A Dissertation Presented
by
DONG “MICHELLE” LI

Approved as to style and content by:

Anna Nagurney, Chair

Hari Jagannathan Balasubramanian, Member

Christian Rojas, Member

Adams Steven, Member

George R. Milne, Program Director
Isenberg School of Management
To my family.
ACKNOWLEDGMENTS

I would like to express my sincere gratitude to many people who have helped and supported me in my doctoral studies at the University of Massachusetts Amherst.

First and foremost, I am very grateful to my esteemed advisor, John F. Smith Memorial Professor Anna Nagurney, for her many years of encouragement, support, and inspiration. It is the extraordinary leadership, mentorship, and guidance from her that made this dissertation possible. A lot have I learned from Professor Nagurney’s dedication to research and education as well as her incomparable work ethics, which will for sure continue to guide my future career.

I shall also thank Professors Hari Balasubramanian, Christian Rojas, and Adams Steven as my outstanding dissertation committee members. They not only made helpful and insightful comments and suggestions on my dissertation, but also gave tremendous support during my job search. In addition, I am thankful to Professors Iqbal Agha, D. Anthony Butterfield, Ahmed Ghoniem, George Milne, and Senay Solak, for their support and encouragement during my doctoral studies.

Moreover, my thanks go to my colleagues and friends at the Virtual Center for Supernetworks: Professors Ladimer Nagurney, Min Yu, and Amir Masoumi, and to Sara Saberi and Shivani Shukla, for their support and contributions to my professional and personal development. I would also like to thank Susan Boyer, Cerrianne Fisher, Rebecca Jerome, Daniel Kasal, Dianne Kelly, Audrey Kieras, Matthew LaClaire, Sarah Malek, Priscilla Mayoussier, Ellen Pekar, and Lynda Vassallo, as well, for their administrative assistance.
Special thanks are given to my parents and my dearest friends for always being there for me during my happiness and sorrow, believing in me, and persistently supporting my dreams.

Last, but not least, this research was supported, in part, by the National Science Foundation (NSF) grant CISE #1111276, for the NeTS: Large: Collaborative Research: Network Innovation Through Choice project awarded to the University of Massachusetts Amherst, and, by the John F. Smith Memorial Fund at the Isenberg School of Management. This support is gratefully acknowledged.
ABSTRACT

QUALITY COMPETITION IN SUPPLY CHAIN NETWORKS WITH APPLICATIONS TO INFORMATION ASYMMETRY, PRODUCT DIFFERENTIATION, OUTSOURCING, AND SUPPLIER SELECTION

SEPTEMBER 2015

DONG “MICHELLE” LI

Bachelor of Management, NANKAI UNIVERSITY

Ph.D., UNIVERSITY OF MASSACHUSETTS AMHERST

Directed by: Professor Anna Nagurney

The quality of the products produced and delivered in supply chain networks is essential for consumers’ safety, well-being, and benefits, and for firms’ profitability and reputation. However, because of the complexity of today’s large-scale highly globalized supply chain networks, along with issues such as the growth in outsourcing and in global procurement, as well as the information asymmetry associated with quality, supply chain networks are more exposed to both domestic and international quality failures.

In this dissertation, I contribute to the equilibrium and dynamic modeling and analysis of quality competition in supply chain networks under scenarios of information asymmetry, product differentiation, outsourcing, and under supplier selection.
The first part of the dissertation consists of a review of the relevant literature, the research motivation, and an overview of methodologies.

The second part of the dissertation formulates quality competition with minimum quality standards under the scenario of information asymmetry, specifically, when there is no product differentiation by brands or labels. In the third part, in contrast, quality competition is modeled under product differentiation, when firms engage in distinguishing their products from their competitors’.

The fourth part concentrates on quality competition in supply chain networks with outsourcing. The models yield the optimal make-or-buy and contractor selection decisions for the firm(s) and the optimal pricing and quality decisions for the contractors. The impacts of firms’ attitudes towards disrepute are also studied numerically.

In the fifth part, a multitiered supply chain network model of quality competition with suppliers is developed. It consists of competing suppliers and competing firms who purchase components for the assembly of their products and, if capacity permits, produce their own components. The optimal supplier-selection decisions, optimal component production and quality, and the optimal quality preservation levels of the assembly processes are provided. Such issues as the values of the suppliers to the firms, the impacts of capacity disruptions, and the potential investments in capacity enhancements are explored numerically.

The models and analysis in this dissertation can be applied to numerous industries, ranging from the food industry to the pharmaceutical industry, automobile industry, and to the high technology industry.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>v</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>vi</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>x</td>
</tr>
<tr>
<td><strong>CHAPTER</strong></td>
<td></td>
</tr>
<tr>
<td>1. INTRODUCTION AND RESEARCH MOTIVATION</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Definitions and Quantification of Quality and Cost of Quality</td>
<td>3</td>
</tr>
<tr>
<td>1.2 Literature Review</td>
<td>8</td>
</tr>
<tr>
<td>1.2.1 Models with Quality Information Asymmetry Between Firms and Consumers</td>
<td>8</td>
</tr>
<tr>
<td>1.2.2 Models of Quality Competition</td>
<td>10</td>
</tr>
<tr>
<td>1.2.3 Models of Quality in Manufacturing Outsourcing</td>
<td>12</td>
</tr>
<tr>
<td>1.2.4 Models with Suppliers’ Quality</td>
<td>14</td>
</tr>
<tr>
<td>1.3 Dissertation Overview</td>
<td>15</td>
</tr>
<tr>
<td>1.3.1 Additional Motivation and Contributions in Chapter 3</td>
<td>16</td>
</tr>
<tr>
<td>1.3.2 Additional Motivation and Contributions in Chapter 4</td>
<td>17</td>
</tr>
<tr>
<td>1.3.3 Additional Motivation and Contributions in Chapter 5</td>
<td>19</td>
</tr>
<tr>
<td>1.3.4 Additional Motivation and Contributions in Chapter 6</td>
<td>21</td>
</tr>
<tr>
<td>1.3.5 Additional Motivation and Contributions in Chapter 7</td>
<td>23</td>
</tr>
<tr>
<td>1.3.6 Concluding Comments</td>
<td>26</td>
</tr>
<tr>
<td>2. METHODOLOGIES</td>
<td>28</td>
</tr>
<tr>
<td>2.1 Variational Inequality Theory</td>
<td>29</td>
</tr>
<tr>
<td>2.2 The Relationships between Variational Inequalities and Game Theory</td>
<td>34</td>
</tr>
<tr>
<td>2.3 Projected Dynamical Systems</td>
<td>36</td>
</tr>
<tr>
<td>2.4 Multicriteria Decision-Making</td>
<td>41</td>
</tr>
</tbody>
</table>
5.1.2 The Behavior of the Contractors and Their Optimality Conditions .............................................. 121
5.1.3 The Equilibrium Conditions for the Supply Chain Network with Outsourcing and Quality and Price Competition ...... 125

5.2 The Underlying Dynamics and Stability Analysis ........................................... 126
5.3 Explicit Formulae for the Euler Method Applied to the Supply Chain Network with Outsourcing and Quality and Price Competition .............................................. 130
5.4 Additional Numerical Examples and Sensitivity Analysis .............................................. 136
5.5 Summary and Conclusions .............................................................................. 145

6. A SUPPLY CHAIN NETWORK MODEL WITH PRODUCT DIFFERENTIATION, OUTSOURCING, AND QUALITY AND PRICE COMPETITION .............................................. 147

6.1 The Supply Chain Network Model with Product Differentiation, Outsourcing, and Quality and Price Competition ............. 148

6.1.1 The Behavior of the Firms and Their Optimality Conditions .............................................. 152
6.1.2 The Behavior of the Contractors and Their Optimality Conditions .............................................. 156
6.1.3 The Equilibrium Conditions for the Supply Chain Network with Product Differentiation, Outsourcing, and Quality and Price Competition .............................................. 158

6.2 Explicit Formulae for the Euler Method Applied to the Supply Chain Network with Product Differentiation, Outsourcing, and Quality and Price Competition .............................................. 160
6.3 Numerical Examples and Sensitivity Analysis .............................................. 161
6.4 Summary and Conclusions .............................................................................. 173

7. A SUPPLY CHAIN NETWORK MODEL WITH SUPPLIER SELECTION AND QUALITY AND PRICE COMPETITION .............................................. 176

7.1 The Supply Chain Network Model with Supplier Selection and Quality and Price Competition .............................................. 177

7.1.1 The Behavior of the Firms and Their Optimality Conditions .............................................. 184
7.1.2 The Behavior of the Suppliers and Their Optimality Conditions .............................................. 190
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Categories of Quality-Related Costs</td>
<td>8</td>
</tr>
<tr>
<td>5.1</td>
<td>Notation for the Supply Chain Network Model with Outsourcing and Quality and Price Competition</td>
<td>118</td>
</tr>
<tr>
<td>6.1</td>
<td>Notation for the Supply Chain Network Model with Product Differentiation, Outsourcing, and Quality and Price Competition</td>
<td>151</td>
</tr>
<tr>
<td>6.2</td>
<td>Total Costs of Firm 1 with Different Sets of $\omega_1$ and $\omega_2$</td>
<td>173</td>
</tr>
<tr>
<td>6.3</td>
<td>Total Costs of Firm 2 with Different Sets of $\omega_1$ and $\omega_2$</td>
<td>173</td>
</tr>
<tr>
<td>7.1</td>
<td>Notation for the Supply Chain Network Model with Supplier Selection and Quality and Price Competition</td>
<td>181</td>
</tr>
<tr>
<td>7.2</td>
<td>Functions for the Supply Chain Network Model with Supplier Selection and Quality and Price Competition</td>
<td>182</td>
</tr>
<tr>
<td>7.3</td>
<td>Maximum Acceptable Investments ($\times 10^3$) for Capacity Changing when the Capacity of the Firm Maintains 80 but that of the Supplier Varies</td>
<td>212</td>
</tr>
<tr>
<td>7.4</td>
<td>Maximum Acceptable Investments ($\times 10^3$) for Capacity Changing when the Capacity of the Supplier Maintains 120 but that of the Firm Varies</td>
<td>212</td>
</tr>
</tbody>
</table>
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>The Supply Chain Network Topology with Multiple Manufacturing Plants</td>
</tr>
<tr>
<td>3.2</td>
<td>The Supply Chain Network Topology for Example 3.1</td>
</tr>
<tr>
<td>3.3</td>
<td>Equilibrium Product Shipments, Equilibrium Quality Levels, Average Quality at the Demand Market, and Price at the Demand Market as $q_{11}$ and $q_{21}$ Vary in Example 3.1</td>
</tr>
<tr>
<td>3.4</td>
<td>Demand at $R_1$ and the Profits of the Firms as $q_{11}$ and $q_{21}$ Vary in Example 3.1</td>
</tr>
<tr>
<td>3.5</td>
<td>The Supply Chain Network Topology for Examples 3.2 and 3.3</td>
</tr>
<tr>
<td>3.6</td>
<td>The Supply Chain Network Topology for Example 3.4</td>
</tr>
<tr>
<td>3.7</td>
<td>The Equilibrium Demands, Average Quality Levels, Prices at the Demand Markets, and the Profits of the Firms as $\beta$ Varies in Example 3.4</td>
</tr>
<tr>
<td>4.1</td>
<td>The Supply Chain Network Topology</td>
</tr>
<tr>
<td>4.2</td>
<td>The Supply Chain Network Topology for Example 4.1</td>
</tr>
<tr>
<td>4.3</td>
<td>The Supply Chain Network Topology for Example 4.2</td>
</tr>
<tr>
<td>4.4</td>
<td>Product Shipments for Example 4.1</td>
</tr>
<tr>
<td>4.5</td>
<td>Quality Levels for Example 4.1</td>
</tr>
<tr>
<td>4.6</td>
<td>Product Shipments for Example 4.2</td>
</tr>
<tr>
<td>4.7</td>
<td>Quality Levels for Example 4.2</td>
</tr>
<tr>
<td>4.8</td>
<td>The Supply Chain Network Topology for Example 4.3</td>
</tr>
</tbody>
</table>
6.3 Equilibrium Product Flows and Quality Levels as \( \omega \) Increases for Example 6.1 .......................................................... 167

6.4 Equilibrium Prices, Disrepute Costs, and Total Costs of the Firms as \( \omega \) Increases for Example 6.1 .......................................................... 168

6.5 Equilibrium Product Flows and Quality Levels as \( \omega \) Increases for Example 6.2 .......................................................... 170

6.6 Equilibrium Prices, Disrepute Costs, and Total Costs of the Firms as \( \omega \) Increases for Example 6.2 .......................................................... 171

7.1 The Multitiered Supply Chain Network Topology with Suppliers ...... 179

7.2 Supply Chain Network Topology for Example 7.1 ......................... 203

7.3 Equilibrium Component Quantities, Equilibrium Component Quality Levels, Equilibrium Product Quantity (Demand), and Product Quality as the Capacity of the Supplier Varies ............................... 206

7.4 Equilibrium Quality Preservation Level, Equilibrium Lagrange Multiplier, Demand Price, Equilibrium Contracted Price, the Supplier’s Profit, and the Firm’s Profit as the Capacity of the Supplier Varies .......................................................... 207

7.5 Equilibrium Component Quantities, Equilibrium Component Quality Levels, Equilibrium Product Quantity (Demand), and Product Quality as the Capacity of the Firm Varies ............................... 210

7.6 Equilibrium Quality Preservation Level, Equilibrium Lagrange Multiplier, Demand Price, Equilibrium Contracted Price, the Supplier’s Profit, and the Firm’s Profit as the Capacity of the Firm Varies .......................................................... 211

7.7 Supply Chain Network Topology for Example 7.2 ......................... 214

7.8 Supply Chain Network Topology With Disruption to Supplier 1 ...... 219

7.9 Supply Chain Network Topology With Disruption to Supplier 2 ...... 221

7.10 Supply Chain Network Topology With Disruption to Suppliers 1 and 2 ......................................................................................... 222
CHAPTER 1
INTRODUCTION AND RESEARCH MOTIVATION

Supply chains are networks consisting of multiple decision-makers, such as manufacturers, transporters/distributors, and retailers, that participate in the processes of the production, delivery, and sales of goods as well as services so as to satisfy consumers at the demand markets (cf. Nagurney (2006)). Nowadays, as an increasing number of firms from around the globe interact and compete with one another to provide products to geographically distributed locations, supply chain networks are far more complex than ever before. As a consequence, they are also more exposed to both domestic and international failures running the gamut from poor product quality to unfilled demand (see, e.g., Nagurney, Yu, and Qiang (2011), Nagurney, Masoumi, and Yu (2012), Liu and Nagurney (2013), and Yu and Nagurney (2013)).

Examples of recent vivid product quality failures have included adulterated infant formula (Barboza (2008)), inferior pharmaceuticals (see Masoumi, Yu, and Nagurney (2012)), bacteria-laden food (see, e.g., Marsden (2004)), and even low-performing high tech products (see Goettler and Gordon (2011)) as well as inferior durable goods referred to a “lemons” in the case of automobiles as noted in Akerlof’s (1970) fundamental study. At the same time, quality has been recognized as “the single most important force leading to the economic growth of companies in international markets” (Feigenbaum (1982)), and, in the long run, as the most important factor affecting a business unit’s performance and competitiveness, relative to the quality levels of its competitors (Buzzell and Gale (1987)). High quality products make an important contribution to a firm’s long-term profitability, due to the fact that consumers expect
good products and services, and are willing to pay higher prices for them. Products of high quality can also ensure the reputation of the brand, since firms can obtain certifications/labels and declarations. For instance, the ISO (International Organization for Standardization) 9000 series guarantees the safety and reliability of the quality management processes of firms and, hence, the quality of their products. In addition, firms who fail to produce and deliver products of good quality may have to pay for the accompanying consequences, such as the costs of returns, replacements, the loss of customer satisfaction and loyalty, and the loss of their reputation, which can be priceless. Most importantly, poor quality products, whether inferior durable goods, such as automobiles, or consumables such as pharmaceuticals and food, may negatively affect the safety and the well-being of consumers, with, possibly, associated fatal consequences.

It is, hence, puzzling and paradoxical that, since firms should have sufficient incentive to produce high quality products, why do low quality products still exist?

The reality of today’s supply chain networks, given their global reach from sourcing locations to points of demand, is further challenged by such issues as the growth in outsourcing and in global procurement as well as the information asymmetry associated with what producers know about the quality of their products and what consumers know. Although much of the related literature has focused on the micro aspects of supply chain networks, considering two or three decision-makers, it is essential to capture the scale of supply chain networks that occurs in practice and to evaluate and analyze the competition in a quantifiable manner. My focus, hence, is to provide computable supply chain network models, the associated analysis, and computational procedures, that enable decision-makers to evaluate the full complexity of supply chain networks with an emphasis on product quality by capturing the objective functions that decision-makers are faced with, whether that of cost minimization, profit-maximization, etc., along with the constraints.
In this dissertation, I contribute to the equilibrium and dynamic modeling and analysis of quality competition in supply chain networks in an environment of increasing competitiveness in order to explore such critical issues as: the role of information asymmetry and of product differentiation, as well as the impacts of outsourcing and of supplier selection. Specifically, the dissertation addresses the following fundamental questions:

1. What are the equilibrium product quality levels of competing firms and how to compute their values?
2. How do these quality levels evolve over time until the equilibrium is achieved?
3. How stable are the equilibria?
4. What are the impacts on product quality, costs, and profits, of minimum quality regulations?

In this dissertation, the equilibria and the associated dynamics of production, quality, and prices, are determined, under scenarios of, respectively, information asymmetry, product differentiation, outsourcing, and under supplier selection.

Since quality competition is the main theme of the dissertation, I first present definitions of quality and the quantification of quality and associated cost. I then provide the literature review and give the outline of the chapters in this dissertation.

1.1. Definitions and Quantification of Quality and Cost of Quality

Different definitions of quality have been presented at various times by researchers in different fields. The definitions can be classified into four main categories: 1). quality is excellence, 2). quality is value, 3). quality is meeting and/or exceeding customers’ expectations, and 4). quality is the conformance to a design or specification.

According to the view that quality is excellence (e.g., Tuchman (1980), Garvin (1984), and Pirsig (1992)), this perspective requires the investment of the best effort
possible to produce the most admirable and uncompromising achievements possible. Although striving for excellence may bring significant marketing benefits for firms, one has to admit that excellence is a very abstract and subjective term, and it is very difficult to articulate precisely what excellence is, let alone explain clearly what are the standards for excellence, and how excellence can be measured, modeled, achieved, and compared in practice.

Feigenbaum (1951), Abbott (1955), and Cronin and Taylor (1992) criticized the quality-as-excellence definition and argued that the definition of quality should be value. According to them, quality is the value of a product under certain conditions, which include the actual use and the price of the product. Many attributes of quality can be included in value, such as price and durability, but quality is actually not synonymous with value (Stahl and Bounds (1991)). When consumers purchase products, they consider not only their quality but also their prices, which are two separate concepts. The term *value*, hence, has the disadvantage of blending these two distinct concepts together.

The extent to which a product or service meets and/or exceeds a customer’s expectations is another definition of quality (e.g., Feigenbaum (1983), Parasuraman, Zeithaml, and Berry (1985), Buzzell and Gale (1987), and Grönroos (1990)). It is argued that customers are the only ones who judge quality, and the quality of a product should be just the perception of quality by consumers. This definition allows firms to focus on factors that consumers care about. However, it is also very subjective and, hence, very difficult to quantify and to measure. Different customers may have different preferences as to the attributes of a product and it is often the case that even consumers themselves may not know what their expectations are (Cameron and Whetten (1983)).

Shewhart (1931), Juran (1951), Levitt (1972), Gilmore (1974), Crosby (1979), Deming (1986), and Chase and Aquilano (1992), most of whom are operations man-
agement scholars, are the major advocates of the conformance-to-specification definition of quality. They define quality as “the degree to which a specific product conforms to a design or specification,” which is how well the product is conforming to an established specification. A major advantage of this definition is that it makes quality relatively straightforward to quantify, which is essential for firms and researchers who are eager to measure it, manage it, model it, compare it across time, and to also make associated decisions (Shewhart (1931)).

Also, at first glance, it may seem that the conformance-to-specification definition of quality focuses too much on internal quality measurement rather than on consumers’ desires at the demand markets. However, with notice that consumers’ needs and desires for a product are actually governed by specific requirements or standards and these can be correctly translated to a specification by, for example, engineers (Oliver (1981)), this feature makes the conformance-to-specification definition quite general. In addition to consumers’ needs, the specification of a product can also include both international and domestic standards (Yip (1989)), and, in order to gain marketing advantages in the competition with other firms, the competitors’ product specifications.

All of the above four definitions of quality are still in use today (Wankhade and Dabade (2010)). As one may notice from the above, each definition has both strengths and weaknesses in criteria such as measurement, generalizability, and consumer relevance.

In this dissertation, since the conformation-to-specification definition not only makes quality quantifiable, but also is sufficiently general to include many dimensions of quality, I define quality as “the degree to which a specific product conforms to a design or specification.” Quality, hence, may vary from a 0% conformance level to a 100% conformance level (see, e.g., Juran and Gryna (1988), Campanella (1990), Feigenbaum (1983), Porter and Rayner (1992), and Shank and Govindarajan (1994)).
When the quality of a particular product is at a 0% conformance level, the product has no quality; when the quality achieves a 100% conformance level, the product is of perfect quality. Therefore, quality levels are quantified as values between 0 and the perfect quality level in Chapters 5, 6, and 7.

Quality levels with lower and upper bounds can also be found in Akerlof (1970) \((q \in [0, 2])\), Leland (1979) \((q \in [0, 1])\), Chan and Leland (1982) \((q \in [q_0, q_H])\), Lederer and Rhee (1995) \((q \in [0, 1])\), Acharyya (2005) \((q \in [q_0, \bar{q}])\), and Chambers, Kouvelis, and Semple (2006) \((q \in [0, q_{max}])\). Reyniers and Tapiero (1995), Tagaras and Lee (1996), Baiman, Fischer, and Rajan (2000), Hwang, Radhakrishnan, and Su (2006), Hsieh and Liu (2010), and Lu, Ng, and Tao (2012) modelled quality as probabilities, which are between 0 and 1. These models believe that, due to the laws of physics, the state of technology, and the ability of improving quality, there should be a quality ceiling.

However, in the majority of economics and management science papers on quality competition (see, e.g., Mussa and Rosen (1978), Gal-or (1983), Cooper and Ross (1984), Riordan and Sappington (1987), Rogerson (1988), Ronnen (1991), Banker, Khosla, and Sinha (1998), Johnson and Myatt (2003), Xu (2009), Xie et al. (2011), and Kaya (2011)), quality levels are only assumed to be nonnegative, and there are no upper bounds on quality. These models believe that, there is no “best” quality, because there can always be a quality level that is even better than the best. Since Chapters 3 and 4 are inspired by these papers, no upper bounds are assumed therein. Moreover, the inclusion of upper bounds presents no technical difficulties.

Based on the conformance-to-specification definition of quality, the cost of quality, consumers’ sensitivity to quality, and the cost of quality disrepute (that is, the loss of reputation due to low quality), all of which are crucial elements in modeling quality competition in supply chain networks, can be quantified and measured, and the equilibrium quality level of each firm can also be determined. Following the
conformance-to-specification definition of quality, quality cost is defined as the “cost incurred in ensuring and assuring quality as well as the loss incurred when quality is not achieved” (ASQC (1971) and BS (1990)). Although, according to traditional cost accounting, quality cost may not be practically quantified in cost terms (Chiatmrong (2003)), there are a variety of schemes by which quality costing can be implemented by firms, some of which have been described in Juran and Gryna (1988) and Feigenbaum (1991).

Based on the quality management literature, four categories of quality-related costs occur in the process of quality management. These are: the prevention cost, the appraisal cost, the internal failure cost, and the external failure cost. They have been developed and are widely applied in organizations (see, e.g., Crosby (1979), Harrington (1987), Juran and Gryna (1993), and Rapley, Prickett, and Elliot (1999)). Quality cost is usually understood as the sum of the four categories of quality-related costs, and, it is widely believed that, the functions of the four quality-related costs are convex functions of quality conformance level. Therefore, the cost of quality is also convex in quality (see, e.g., Feigenbaum (1983), Juran and Gryna (1988), Campanella (1990), Porter and Rayner (1992), Shank and Govindarajan (1994), and Alzaman, Ramudhin, and Bulgak (2010)). Please see Table 1 for more details of the four quality-related costs.

Among the four quality-related costs, the external failure cost, which is the compensation cost incurred when customers are dissatisfied with the quality of the products, such as warranty charges and the complaint adjustment cost, is strongly related to consumers’ satisfaction in terms of the firm’s product, and, hence, can be utilized to measure the disrepute cost of the firm in addition to the cost of quality.

It is notable that, in addition to the cost of quality, the expenditures on R&D have also widely been recognized as a cost depending on the quality level of the firm,
Table 1.1. Categories of Quality-Related Costs

<table>
<thead>
<tr>
<th>Category</th>
<th>Definition</th>
<th>Examples</th>
<th>Shape of the Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prevention costs</td>
<td>Investments to ensure the required quality level in the process of production</td>
<td>Costs in quality engineering, receiving inspection, equipment repair/maintenance, and quality training</td>
<td>Continuous, convex, monotonically increasing. When the quality of conformance is 0%, this cost is zero.</td>
</tr>
<tr>
<td>Appraisal costs</td>
<td>Costs incurred in identifying poor quality before shipment</td>
<td>Incoming inspection and testing cost, in-process inspection and testing cost, final inspection and testing cost, and evaluation of stock cost</td>
<td>Continuous, convex, monotonically increasing. When the quality of conformance is 0%, this cost is zero.</td>
</tr>
<tr>
<td>Internal failure</td>
<td>Failure costs incurred when defects are discovered before shipment</td>
<td>Scrap cost, rework cost, failure analysis cost, re-inspection and retesting cost</td>
<td>Continuous, convex, monotonically decreasing. When the quality of conformance is 100%, this cost is zero.</td>
</tr>
<tr>
<td>External failure</td>
<td>Failure costs associated with defects that are found after delivery of defective goods or services</td>
<td>Warranty charges cost, complaint adjustment cost, returned material cost, and allowances cost</td>
<td>Continuous, convex, monotonically decreasing. When the quality of conformance is 100%, this cost is zero.</td>
</tr>
</tbody>
</table>

which is independent of production and sales (cf. Klette and Griliches (2000), Hoppe and Lehmann-Grube (2001), and Symeonidis (2003)).

1.2. Literature Review

The literature review below discusses, respectively, models with information asymmetry in quality, models of quality competition, models of quality in manufacturing outsourcing, and models with suppliers’ quality.

1.2.1 Models with Quality Information Asymmetry Between Firms and Consumers

To-date, markets with asymmetric information have been studied by many notable economists, including Akerlof (1970), Spence (1975), and Stiglitz (1987, 2002), all of whom shared the Nobel Prize in Economic Sciences. The seminal contribution in the area of quality information asymmetry between firms and consumers is the classic
work of Akerlof (1970)’s, which has stimulated the research in this domain. Following Akerlof (1970), Leland (1979) modeled perfect competition in a market with quality information asymmetry, and argued that such markets may benefit from minimum quality standards. Smallwood and Conlisk (1979) investigated market share equilibria in a multiperiod model considering quality positively related to the probability of repeated purchases. Shapiro (1982) analyzed a monopolist’s behavior in a market with imperfect information in quality, and noted that it was a reason for quality deterioration. Chan and Leland (1982) developed a model with price and quality competition among firms in which they could acquire price/quality information at a cost and the average cost functions were identical for all firms. Schwartz and Wilde (1985) considered markets where consumers were imperfectly informed about prices and quality, and provided equilibria under cases where all consumers preferred higher quality and lower quality.

Bester (1998) studied price and quality competition between two firms and noted that imperfect information quality reduced the sellers’ incentives for differentiation. Besancenot and Vranceanu (2004) studied quality information asymmetry among firms who decided on prices and quality, and consumers who searched for the best offer in a sequential way. Armstrong and Chen (2009) presented a model in which some consumers shopped without attention to quality, and firms might cheat to exploit these consumers. Baltzer (2012) considered two firms involved in price and quality competition with specific underlying functional forms to study the impact of minimum quality standards and labels.

Moreover, price and advertising have long been viewed as indicators of quality for consumers. Examples are as follows. Wolinsky (1983) was concerned with markets with price and quality competition in which consumers had imperfect information, and concluded that price indicated quality. Cooper and Ross (1984) modeled perfect quality competition among firms to examine the degree that prices conveyed informa-

1.2.2 Models of Quality Competition

The noncooperative competition problem among firms, each of which acts in its own self-interest, is a classical problem in economics, and it is also an example of a game theory problem, with the governing equilibrium conditions constituting a Nash equilibrium (cf. Nash (1950, 1951)). Well-known formalisms for oligopolistic competition include, in addition, to the Cournot (1838)-Nash framework in which firms select their optimal production quantities, the Bertrand (1883) framework, in which firms choose their product prices, as well as the von Stackelberg (1934) framework, in which decisions are made sequentially in a leader-follower type of game.

However, as argued by Abbott (1955) and Dubovik and Janssen (2012), if one focuses solely on the price or quantity competition among firms, one ignores a critical component of consumers’ decision processes and the very nature of competition – that of quality. Both price/quantity and quality have to be considered as strategic variables for firms in a competitive market. In particular, as noted by Banker, Khosla, and Sinha (1998), Hotelling’s (1929) paper, which considered price and quality competition between two firms and modeled quality as a location decision, has inspired the study of quality competition in economics as well as in marketing and operations research / management science.
Pioneers in the study of quality competition assumed that firms as well as their decisions were identical. Examples are as follows. Abbott (1953) analyzed quality equilibrium in a single-fixed-price market with entry, where firms only competed in quality. Mussa and Rosen (1978) modeled a firm’s decisions on the price and quality of its quality differentiated product line, and compared the associated monopoly and competitive solutions. Dixit (1979) studied quantity and quality competition by considering several cases of oligopolistic equilibria and comparing them with the social optimum. De Vany and Saving (1983) modeled quantity and capacity competition for monopolists, where quality was related to capacity captured by the waiting cost.


gated the impact of minimum quality standards on the price and quality competition between two firms. However, most models in this area, as mentioned above, are developed under duopoly settings.

Oligopoly models with quality competition that considered more than two firms have been proposed in both economics and management science. In addition to Dixit (1979), Leland (1977) considered the quality choices of a finite number of firms competing for consumers, and used the “characteristics” approach to model consumers’ choices. Gal-or (1983) developed an oligopoly model with quality heterogenous consumers, in which both prices and quality levels of the firms were determined at the equilibrium. Lederer and Rhee (1995) modeled quality competition among firms where the prices and the quality levels of the products were not related. Karmarkar and Pitblado (1997) considered the competition among several identical firms where the consumer’s utility which was a function related to quality. Scarpa (1998) developed a price and quality competition model with three firms to study the effects of a minimum quality standard in a vertically differentiated market. In addition, Banker, Khosla, and Sinha (1998) modeled quality competition among firms with quadratic cost functions in one demand market, and investigated the impact of number of competitors on quality. Brekke, Siciliani, and Straume (2010) investigated the relationship between competition and quality via a spatial price-quality competition model.

1.2.3 Models of Quality in Manufacturing Outsourcing

Outsourcing, a strategy capable of bringing potentially large benefits to firms, has been attributed to product quality issues in global supply chain networks. As argued by Marucheck et al. (2011), although such problems have long been viewed as a technical problem in the domain of regulators, epidemiologists, and design engineers, there has been a growing consciousness that operations management can provide fresh
and effective approaches to managing product quality and safety. In this section, I provide a literature review of contributions to the study of quality in manufacturing outsourcing from the domain of operations management, and the related fields of operations research and management science. Most of the studies, as noted earlier, focus exclusively on supply chains with a limited number of firms and contractors and without product differentiation.

The impact of outsourcing on quality, suggestions as to how to mitigate associated quality issues, and the associated decision-making problems have been studied by various scholars. Riordan and Sappington (1987) modeled the quantity and quality choices of a firm with one contractor under information asymmetry, and analyzed the firm’s choice of organizational mode. Sridhar and Balachandran (1997) developed a model with one firm and two sequential contractors with information asymmetry to select one of them as the inside contractor. Zhu, Zhang, and Tsung (2007) investigated the roles of different parties in quality improvement by focusing on a model between two entities. The cost of goodwill loss caused by bad quality was also considered. Kaya and Özer (2009) modeled quality in outsourcing with one firm, one contractor, and information asymmetry to determine how the firm’s pricing strategy affects quality risk. Xie, Yue, and Wang (2011) utilized a quality standard to regulate quality in a global supply chain with one firm and one contractor under cases of vertical integration and decentralized settings.

In addition, Kaya (2011) considered a model in which the supplier makes the quality decision and another model in which the manufacturer decides on the quality with quadratic quality cost functions. Gray, Roth, and Leiblein (2011) studied the effects of location decisions on quality risk based on real data from the drug industry. Lu, Ng, and Tao (2012) developed a model with one firm and one contractor, and argued that contract enforcement would help to mitigate the low quality led by outsourcing. Handley and Gray (2013) studied 95 contracting relationships and found
that external failures had a positive effect on the contractors’ perception of quality importance. Steven, Dong, and Corsi (2013) investigated empirically how outsourcing was related to product recalls, and concluded that the relationship was positive. Moreover, the paper by Xiao, Xia, and Zhang (2014) examined outsourcing decisions for two competing manufacturers who have quality improvement opportunities and product differentiation.

1.2.4 Models with Suppliers’ Quality

The quality of a finished/final product depends not only on the quality of the firm that produces and delivers it, but also on the quality of the components provided by the firm’s suppliers (Robinson and Malhotra (2005) and Foster (2008)). It is actually the suppliers that determine the quality of the materials that they purchase as well as the standards of their manufacturing activities.

Therefore, there has been increasing attention on supply chain networks with suppliers’ quality in both management science and economics. However, in the literature, most models are based on a single firm - single supplier - single component supply chain network without the preservation/decay of quality in the assembly processes of the products, and the possible in-house component production by the firms is not considered. Given the reality of many finished product supply chains, these models may be limiting in terms of both scope and practice. Specifically, although focused, simpler models, may yield closed form analytical solutions, more general frameworks, that are computationally tractable, are also needed, given the size and complexity of real-world global supply chains.

A literature review of models focusing on suppliers’ quality in multitiered supply chain networks is given. Specifically, in the literature, the relationships and contracts between firms and suppliers in terms of quality and the associated decision-making problems are analyzed. Reyniers and Tapiero (1995) modeled the effect of contract

In addition, Lin et al. (2005) conducted empirical research to study the correlation between quality management and supplier selection, based on data from practicing managers. Hwang, Radhakrishnan, and Su (2006) examined a quality management problem in a supply chain network with one supplier, and provided evidence of the increasing use of certification. Chao, Iravani, and Savaskan (2009) considered two contracts with recall cost sharing between a manufacturing and a supplier to induce quality improvement. Hsieh and Liu (2010) studied the supplier’s and the manufacturer’s quality investment with different degrees of information revealed. Moreover, Xie et al. (2011) investigated quality and price decisions in a risk-averse supply chain with two entities under uncertain demand.

1.3. Dissertation Overview

This dissertation consists of eight chapters. Chapter 2 provides a review of the methodologies that are utilized in this dissertation, mainly variational inequality theory (Nagurney (1999)) and projected dynamical systems theory (Dupuis and Nagurney (1993) and Nagurney and Zhang (1996)). The summary and conclusions of the dissertation with future research plans are given in Chapter 8. Below, I detail the
contributions in Chapters 3 through 7 and provide additional background for these chapters.

1.3.1 Additional Motivation and Contributions in Chapter 3

Inspired by Akerlof (1970), the main topic in Chapter 3 is that of quality competition with quality information asymmetry between firms and consumers. The motivation for this chapter is as follows. Supply chain networks have transformed the manner in which goods are produced, transported, and consumed around the globe and have created more choices and options for consumers during different seasons. At the same time, given the distances that may be involved as well as the types of products that are consumed, there may be information asymmetry associated with knowledge about the quality of the products. Stiglitz (2002) defined information asymmetry as the “fact that different people know different things.” Specifically, when there is no differentiation by brands or labels, products from different firms are viewed as being homogeneous for consumers. Therefore, producers in certain industries are aware of their product quality whereas consumers may not be aware of the product quality of specific firms a priori.

Information asymmetry becomes increasingly complex when manufacturers (producers) have, at their disposal, multiple manufacturing plants, which may be on-shore or off-shore, with the ability to monitor the quality in the latter sometimes challenging. Indeed, major issues and quality problems associated with distinct manufacturing plants, for example, in the pharmaceutical industry, have been the focus of increasing attention. Since 2009, quality failures in several manufacturing plants of Hospira, a leading manufacturer of injectable drugs, led to several major recalls of products produced at manufacturing plants in, for example, North Carolina, California, and Costa Rica (see Thomas (2013)). As another example, in 2011, Ben Venue, a division of the German pharmaceutical company Boehringer Ingelheim, was forced to shut
one of its plants, in Bedford, Ohio, due to quality issues investigated by the Food and Drug Administration (FDA) (Lopatto (2013)).

In Chapter 3, static and dynamic models of quality competition among firms, each of which may have multiple plants at its disposal, are presented under information asymmetry in quality in a supply chain network context using variational inequality theory and projected dynamical systems theory, respectively. These models capture quality levels both on the supply side as well as on the demand side, with linkages through the transportation costs, yielding an integrated economic network framework. I model the competition among firms, which are spatially separated, in a Cournot-Nash manner. The firms compete in product shipment and product quality in multiple demand markets with each one seeking to maximize its profit, where quality is associated with both the manufacturing plants and the transportation processes. Elastic demand in both prices and quality levels is assumed at the demand markets. In addition, since minimum quality standards are imposed in many industries, from pharmaceuticals to food to automobiles, by national authorities in order to guarantee consumers’ welfare and safety (cf. Boom (1995)), in Chapter 3, I also investigate quantitatively the effects of the imposition of minimum quality standards. The effectiveness of the imposition of minimum quality standards on quality has been studied in economics, with or without information asymmetry, by Leland (1979), Shapiro (1983), Besanko, Donnenfeld, and White (1988), Ronnen (1991), and Balianoune-Lutz and Lutz (2010). However, this has not been done previously in a general supply chain network framework. Chapter 3 is based on Nagurney and Li (2014a).

1.3.2 Additional Motivation and Contributions in Chapter 4

In Chapter 4, unlike in Chapter 3, product differentiation is considered in quality competition among firms, but without any quality information asymmetry between
firms and consumers. Please note that, in this dissertation, the product differentiation is specifically horizontal product differentiation. In addition, I investigate the costs associated with research and development (R&D). R&D plays a significant role in the improvement of technology, and, hence, quality (Bernstein and Nadiri (1991), Motta (1993), Cohen and Klepper (1996), and Acharyya (2005)). In the process of R&D, for example, firms may gain competitive advantages from increased specialization of scientific and technological knowledge, skills and resources, and the state of knowledge of a firm may, typically, be reflected in the quality of its product (Lilien and Yoon (1990), Aoki (1991), Berndt et al. (1995), and Shankar, Carpenter, and Krishnamurthi (1998)).

Because of the influence of R&D on quality improvement, value adding, and profit enhancement, firms may invest in R&D activities, which is referred to as the cost of quality improvement. Eli Lilly, one of the world’s largest pharmaceutical companies, invested billions of dollars in profits back into its R&D (Steiner et al. (2007)). The 18th biggest R&D spender, Apple, invested 2.6 billion US dollars in 2011 in R&D and maintained a net profit of 13 billion dollars every three months (Krantz (2012)).

The framework for the competitive supply chain network model in this chapter is, again, that of Cournot-Nash competition in which the firms compete by determining their optimal product shipments as well as the quality levels of their products. I present both the static model, in an equilibrium context, using variational inequality theory, and its dynamic counterpart, using, as was done in Chapter 3, projected dynamical systems theory. Stability analysis is conducted and numerical examples given, along with the dynamic trajectories of the product shipments and quality levels as they evolve over time. Chapter 4 is based on Nagurney and Li (2014b).
1.3.3 Additional Motivation and Contributions in Chapter 5

As mentioned above, the supply chain network models established in Chapters 3 and 4 focus on two parties, firms and demand markets. In Chapters 5 and 6, in contrast, quality competition in supply chain networks with outsourcing is analyzed and formulated with another party added to the supply chain networks, that of the contractors of the firms. In all the modeling chapters I also identify the underlying network structure, which enhances the understanding of and the applicability of the models.

Outsourcing of manufacturing/production has long been noted in supply chain management and it has become prevalent in numerous manufacturing industries. One of the main arguments for the outsourcing of production, as well as distribution, is cost reduction (Insinga and Werle (2000), Cecere (2005), and Jiang, Belohlav, and Young (2007)). Outsourcing, as a supply chain strategy, may also increase operational efficiency and agility (Klopack (2000) and John (2006)), enhance a firm’s competitiveness (cf. Narasimhan and Das (1999)), and even yield benefits from supportive government policies (Zhou (2007)).

In the pharmaceutical industry, for example, in 2010, up to 40% of the drugs that Americans consumed were imported, and more than 80% of the active ingredients for drugs sold in the United States were outsourced (Ensinger (2010)), with the market for outsourced pharmaceutical manufacturing expanding at the rate of 10% to 12% annually in the US (Olson and Wu (2011)). In the fashion industry, according to the ApparelStats Report released by the American Apparel and Footwear Association, 97.7% of the apparel sold in the United States in 2011 was produced outside the US (AAFA (2012)). In addition, in the electronics industry, in the fourth quarter alone of 2012, 100% of the 26.9 million iPhones sold by Apple were designed in California, but assembled in China (Apple (2012) and Rawson (2012)).
However, parallel to the dynamism of and growth in outsourcing, the nation’s growing reliance on sometimes uninspected contractors has raised public and governmental awareness and concern, with outsourcing firms being faced with quality-related risks (cf. Doig et al. (2001), Helm (2006), and Steven, Dong, and Corsi (2013)). In 2003, the suspension of the license of Pan Pharmaceuticals, the world’s fifth largest contract manufacturer of health supplements, due to quality failure, caused costly consequences in terms of product recalls and credibility losses (Allen (2003)). In 2008, fake heparin made by a Chinese manufacturer not only led to recalls of drugs in over ten European countries (Payne (2008)), but also resulted in the deaths of 81 Americans (Harris (2011)). Furthermore, in 2009, more than 400 peanut butter products were recalled after 8 people died and more than 500 people in 43 states, half of them children, were sickened by salmonella poisoning, the source of which was a peanut butter plant in Georgia (Harris (2009)).

Therefore, with the increasing volume of outsourcing, it is imperative for firms to be prepared to adopt best practices aimed at safeguarding the quality of their supply chain networks and their reputations. Outsourcing makes supply chain networks more complex and, hence, more vulnerable to quality risks (cf. Bozarth et al. (2009)). In outsourcing, since contract manufacturers are not of the same brand names as the original firms, they may have fewer incentives to be concerned with quality (Amaral, Billington, and Tsay (2006)), which may lead them to expend less effort to ensure high quality. Therefore, quality should be incorporated into the make-or-buy as well as the contractor-selection decisions of firms.

In Chapter 5, a supply chain network model with outsourcing and quality competition among contractors, which takes into account the quality concerns in the context of global outsourcing, is developed by utilizing a game theory approach. The firm is engaged in determining the optimal product flows associated with its supply chain network activities in the form of manufacturing and distribution. It seeks to mini-
mize its total cost, with the associated function also capturing the firm’s weighted disrepute cost caused by possible quality issues associated with the contractors, in addition to multimarket demand satisfaction. Simultaneously, the contractors, who pay opportunity costs for their pricing strategies and compete with one another in prices and quality, seek to maximize their profits. In-house quality levels are assumed to be perfect. Unlike in Chapters 3 and 4, the demands at the demand markets are assumed to be fixed. This is relevant to, for example, the pharmaceutical industry. The variational inequality formulation of the equilibrium conditions for the model is provided with the dynamics modeled as projected dynamical system. The equilibrium model yields the product flows associated with the supply chain in-house and outsourcing network activities and provided the optimal make-or-buy and contractor-selection decisions for the firm. An average quality level is used to facilitate the quantification of the disrepute cost of the firm. In addition, the contractor equilibrium prices that they charge the firm and their equilibrium quality levels are also given. Chapter 5 is based on Nagurney, Li, and Nagurney (2013).

1.3.4 Additional Motivation and Contributions in Chapter 6

One may notice that, in addition to the increasing volume of outsourcing mentioned above, interestingly, the supply chain networks weaving the original manufacturers and the contractors are becoming increasingly complex. Firms may no longer outsource exclusively to specific contractors, and there may be contractors engaging with multiple firms, who are actually competitors. The US head office of Volvo was outsourcing the production of components to companies such as Minda HUF, Visteon, and Arvin Meritor, who actually obtained almost 100% of their components, ranging from the engine parts to the electric parts, from Indian contractors (Klum (2007)). Furthermore, in the IT industry, Apple, Compaq, Dell, Gateway, Lenovo, and Hewlett-Packard are consumers of Quanta Computer Incorporated, a Taiwan-
based Chinese manufacturer of notebook computers (Landler (2002)). Also, Foxconn, another Taiwan-based manufacturer, is currently producing tablet computers for Apple, Google, Android, and Amazon (Nystedt (2010) and Topolsky (2010)).

Indeed, if quality issues in outsourcing are to be considered in complex supply chain networks with multiple firms and contractors, product differentiation cannot be ignored. When consumers observe a brand of a product, they consider the quality, function, and reputation of that particular brand name. With outsourcing, chances are that the product was manufactured by a completely different company than the brand indicates, but the level of quality and the reputation associated with the outsourced product still remain with the “branded” original firm. If a product is recalled for a faulty part and that part was outsourced, the original firm is the one that carries the burden of correcting its damaged reputation.

Therefore, Chapter 6, which extends the model in Chapter 5, investigates further the supply chain network problem with outsourcing and quality competition. A supply chain network game theory model with multiple original firms competing with one another is developed, and the products of the distinct original firms are differentiated by their brands. Moreover, the in-house quality levels are no longer assumed to be perfect, but, rather, are strategic variables of the firms, since in-house quality failures may also occur (cf. Beamish and Bapuji (2008)).

In the model in Chapter 6, the original firms compete in terms of in-house quality levels and product flows while satisfying the fixed demands at multiple demand markets. The contractors, however, aiming at maximizing their own profits, are engaged in the competition for the outsourced production and distribution in terms of prices that they charge and their quality levels. The equilibrium conditions are formulated as a variational inequality problem. The solution of the model provides each original firm with its equilibrium in-house quality level as well as its equilibrium in-house and
outsourced production and shipment quantities that minimize its total cost and its weighted cost of disrepute. This chapter is based on Nagurney and Li (2015).

1.3.5 Additional Motivation and Contributions in Chapter 7

In recent years, there have been numerous examples of finished product failures due to the poor quality of a suppliers’ components. For example, the toy manufacturer, Mattel, in 2007, recalled 19 million toy cars because of suppliers’ lead paint and poorly designed magnets, which could harm children if ingested (Story and Barboza (2007)). In 2013, four Japanese car-makers, along with BMW, recalled 3.6 million vehicles because the airbags supplied by Takata Corp., the world’s second-largest supplier of airbags, were at risk of rupturing and injuring passengers (Kubota and Klayman (2013)). The recalls are still ongoing and have expanded to other companies as well (Tabuchi and Jensen (2014)). Most recently, the defective ignition switches in General Motors (GM) vehicles, which were produced by Delphi Automotive in Mexico, have been linked to 13 deaths, due to the fact that the switches could suddenly shut off engines with no warning (Stout, Ivory, and Wald (2014) and Bomey (2014)).

In addition, serious quality shortcomings and failures associated with suppliers have also occurred in finished products such as aircraft (Drew (2014)), pharmaceuticals (Rao (2014)), and also food (Strom (2013) and McDonald (2014)). In 2009, over 400 peanut butter products were recalled after 8 people died and more than 500 people, half of them children, were sickened by salmonella poisoning, the source of which was a peanut butter plant in Georgia (Harris (2009)).

Product quality is an important feature that enables firms to maintain and even to improve their competitive advantage and reputation. However, numerous finished products are made of raw materials as well as components and it is usually the case that the components and materials are produced and supplied not by the firms that process them into products but by suppliers in globalized supply chain networks. For
example, Sara Lee bread, an everyday item, is made with flour from the US, vitamin supplements from China, gluten from Australia, honey from Vietnam and India, and other ingredients from Switzerland, South America, and Russia (Bailey (2007)), let alone automobiles and aircrafts which are made up of thousands of different components.

Suppliers may have less reason to be concerned with quality. In Mattel’s case, some of the suppliers were careless, others flouted rules, and others simply avoided obeying the rules (Tang (2008)). With non-conforming components, it may be challenging and very difficult for firms to produce high quality finished products even if they utilize the most superior production and transportation delivery techniques.

Furthermore, since suppliers may be located both on-shore and off-shore, supply chain networks of firms may be more vulnerable to disruptions around the globe than ever before. Photos of Honda automobiles under 15 feet of water were some of the most appalling images of the impacts of the Thailand floods of 2011. Asian manufacturing plants affected by the catastrophe were unable to supply components for cars, electronics, and many other products (Kageyama (2011)). In the same year, the triple disaster in Fukushima affected far more than the manufacturing industry in Japan. A General Motors plant in Louisiana had to shut down due to a shortage of Japanese-made components after the disaster took place (Lohr (2011)). Under such disruptions, suppliers may not even be capable of performing their production tasks, let alone guaranteeing the quality of the components.

Moreover, the number of suppliers that a firm may be dealing with can be vast. For example, according to Seetharaman (2013), Ford, the second largest US car manufacturer, had 1,260 suppliers at the end of 2012 with Ford purchasing approximately 80 percent of its parts from its largest 100 suppliers. Due to increased demand, many of the suppliers, according to the article were running “flat out” with the consequence that there were quality issues. In the case of Boeing, according to Denning (2013),
complex products such as aircraft involve a necessary degree of outsourcing, since the firm lacks the necessary expertise in some areas, such as, for example, engines and avionics. Nevertheless, as noted therein, Boeing significantly increased the amount of outsourcing for the 787 Dreamliner airplane over earlier planes to about 70 percent, whereas for the 737 and 747 airplanes it had been at around 35-50 percent. Problems with lithium-ion batteries produced in Japan grounded several flights and resulted in widespread media coverage and concern for safety of the planes because of that specific component (see Parker (2014)).

In Chapter 7, I formulate the supply chain network problem with multiple competing firms and their potential suppliers. The proposed game theory model captures the relationships between firms and suppliers in supply chain networks with quality competition. Along with the general multitiered supply chain network model, I also provide a computational procedure so as not to limit the number of suppliers and competing firms.

Specifically, the potential suppliers may either provide distinct components to the firms, or provide the same component, in which case, they compete noncooperatively with one another in terms of quality and prices. The firms, in turn, are responsible for assembling the products under their brand names using the components needed and transporting the products to multiple demand markets. They also have the option of producing their own components, if necessary. The firms compete in product quantities, the quality preservation levels of their assembly processes, the contracted component quantities produced by the suppliers, and in in-house component quantities and quality levels. Each of the firms aims to maximize profits. The quality of an end product is determined by the qualities and quality levels of its components, produced both by the firms and the suppliers, the importance of the quality of each component to that of the end product, and the quality preservation level of its assembly process. Chapter 7 is based on Li and Nagurney (2015).
1.3.6 Concluding Comments

The main contributions of the results in this dissertation to the existing literature are summarized below.

1. Shumpeter (1943) stated that “The essential point to grasp is that in dealing with capitalism we are dealing with an evolutionary process.” In addition, Cabral (2012) articulated the need for new dynamic oligopoly models, combined with network features, as well as quality. As one can see from the literature review (cf. Section 1.2), almost no dynamic model has been constructed with quality competition. In this dissertation, in Chapters 3, 4, and 5, the underlying dynamics, along with the stability results, are presented, in addition to the equilibrium models.

2. Most models in the existing literature (cf. Section 1.2) consider a limited number of entities, such as manufacturing plants, suppliers, components, and/or demand markets. However, supply chain network problems, in reality, are actually large-scale network problems. The models described and developed in this dissertation, hence, contribute to the existing literature in terms of generality and scope. They are all network-based and the entities are not restricted to a specific number.

3. The underlying functions in the models in this dissertation are of general form. First, given the impact of quality on production and transportation, quality is considered in both the production and transportation cost functions in Chapters 3, 5, 6, and 7. In addition, the demand price functions and the cost functions are not limited to specific functional forms, i.e., linear or quadratic. Furthermore, the functions need not be separable and can depend on vectors of quantity and quality. Such features capture more realistically the nature of competition in resources and technologies.

4. In the supply chain network models with outsourcing, in Chapters 5 and 6, the reputation loss caused by nonconforming quality is considered in order to model the firms’ concerns as to their reputation associated with possible low quality due to
outsourcing. Moreover, the opportunity cost of each contractor in terms of its pricing
strategies is also modeled, which has not been considered previously in the supply
chain literature in quality competition and outsourcing (cf. Section 1.2).

5. In the multitiered supply chain network model with suppliers’ quality in Chap-
ster 7, the in-house production by the firms in terms of the components needed for
assembling their branded products is allowed, limited by capacities. A multiplier is
used to model the preservation/decay of quality in the assembly process of the prod-
uct of each firm. In addition, the network topology and the model in Chapter 7 are
also able to capture the case of outsourcing as a special case.

6. In this dissertation, both qualitative results, including stability analysis results
for the dynamics, as well as an effective, and easy to implement, computational
procedure are provided, along with numerical examples with possible applications
and managerial insights.
In this chapter, the fundamental theories and methodologies that are utilized in this dissertation are provided. First, variational inequality theory, which is utilized throughout this dissertation as the essential methodology to analyze the equilibria of quality competition in supply chain networks with information asymmetry in quality, product differentiation, outsourcing, and supplier selection, is presented. In addition, some of the relationships between variational inequality and game theory, which are used in this dissertation to model the competition among firms in quantity and quality and the competition among contractors/suppliers in price and quality, are further discussed. I also recall the theory of projected dynamical systems, which is utilized in Chapters 3, 4 and 5 in this dissertation to analyze the dynamics of quantities and quality levels and/or the dynamics of prices. The relationship between variational inequality problems and projected dynamical systems is also provided, followed by qualitative properties and stability analysis of projected dynamical systems.

For completeness, concepts of multicriteria decision-making and the weighted sum method are recalled briefly; these are here utilized in Chapters 5 and 6 to model supply chain network problems with quality competition and outsourcing.

Finally, I review the algorithms: the Euler method and the modified projection method. The Euler method is employed to solve the variational inequality problems in Chapters 3 through 6 in this dissertation. It is also utilized to provide discrete-time realizations of the continuous-time adjustment processes associated with the projected
dynamical systems in Chapters 3, 4, and 5. The modified projection method is used to solve the variational inequality problem in Chapter 7.

Additional theorems and proofs associated with finite-dimensional variational inequality theory can be found in Nagurney (1999). Further details and proofs concerning projected dynamical systems theory can be found in Dupuis and Nagurney (1993) and Nagurney and Zhang (1996).

2.1. Variational Inequality Theory

In this section, I briefly recall the theory of variational inequalities. All definitions and theorems are taken from Nagurney (1999).

Definition 2.1

The finite-dimensional variational inequality problem, $\text{VI}(F, K)$, is to determine a vector $X^* \in K \subset \mathbb{R}^n$, such that

$$\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in K,$$

(2.1a)

where $F$ is a given continuous function from $K$ to $\mathbb{R}^n$, $K$ is a given closed convex set, and $\langle \cdot, \cdot \rangle$ denotes the inner product in $n$-dimensional Euclidean space.

In (2.1a), $F(X) \equiv (F_1(X), F_2(X), \ldots, F_n(X))^T$, and $X \equiv (X_1, X_2, \ldots, X_n)^T$. $F(X)$ and $X$ are both column vectors. Recall that for two vectors $u, v \in \mathbb{R}^n$, the inner product $\langle u, v \rangle = \|u\|\|v\|\cos\theta$, where $\theta$ is the angle between the vectors $u$ and $v$, and (2.1a) is equivalent to

$$\sum_{i=1}^{n} F_i(X) \cdot (X_i - X_i^*) \geq 0, \quad \forall X \in K.$$  

(2.1b)

The variational inequality problem is a general problem that encompasses a wide spectrum of mathematical problems, including, optimization problems, complementarity problems, and fixed point problems (see Nagurney (1999)). It has been shown
that optimization problems, both constrained and unconstrained, can be reformulated as variational inequality problems. The relationship between variational inequalities and optimization problems, which is explored in this dissertation, is now briefly reviewed.

**Proposition 2.1**

Let $X^*$ be a solution to the optimization problem:

\[
\text{Minimize } f(X) \tag{2.2}
\]

subject to:

\[X \in \mathcal{K},\]

where $f$ is continuously differentiable and $\mathcal{K}$ is closed and convex. Then $X^*$ is a solution of the variational inequality problem:

\[
\langle \nabla f(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \tag{2.3}
\]

where $\nabla f(X)$ is the gradient vector of $f$ with respect to $X$, where $\nabla f(X) \equiv \left(\frac{\partial f(X)}{\partial X_1}, \ldots, \frac{\partial f(X)}{\partial X_n}\right)^T$.

**Proposition 2.2**

If $f(X)$ is a convex function and $X^*$ is a solution to $\text{VI}(\nabla f, \mathcal{K})$, then $X^*$ is a solution to the optimization problem (2.2). In the case that the feasible set $\mathcal{K} = \mathbb{R}^n$, then the unconstrained optimization problem is also a variational inequality problem.

The variational inequality problem can be reformulated as an optimization problem under certain symmetry conditions. The definitions of positive-semidefiniteness, positive-definiteness, and strong positive-definiteness are recalled next, followed by a theorem presenting the above relationship.
Definition 2.2

An \( n \times n \) matrix \( M(X) \), whose elements \( m_{ij}(X); i, j = 1, \ldots, n \), are functions defined on the set \( S \subset \mathbb{R}^n \), is said to be positive-semidefinite on \( S \) if

\[
v^T M(X)v \geq 0, \quad \forall v \in \mathbb{R}^n, \ X \in S. \tag{2.4}\]

It is said to be positive-definite on \( S \) if

\[
v^T M(X)v > 0, \quad \forall v \neq 0, \ v \in \mathbb{R}^n, \ X \in S. \tag{2.5}\]

It is said to be strongly positive-definite on \( S \) if

\[
v^T M(X)v \geq \alpha \|v\|^2, \text{ for some } \alpha > 0, \quad \forall v \in \mathbb{R}^n, \ X \in S. \tag{2.6}\]

Theorem 2.1

Assume that \( F(X) \) is continuously differentiable on \( K \) and that the Jacobian matrix

\[
\nabla F(X) = \begin{bmatrix}
\frac{\partial F_1}{\partial X_1} & \cdots & \frac{\partial F_1}{\partial X_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial F_n}{\partial X_1} & \cdots & \frac{\partial F_n}{\partial X_n}
\end{bmatrix} \tag{2.7}
\]

is symmetric and positive-semidefinite. Then there is a real-valued convex function \( f : K \rightarrow \mathbb{R}^1 \) satisfying

\[
\nabla f(X) = F(X) \tag{2.8}
\]

with \( X^* \) the solution of \( \text{VI}(F, K) \) also being the solution of the mathematical programming problem:
Minimize \( f(X) \)

subject to:

\[ X \in K, \]

where \( f(X) = \int F(X)^Tdx \), and \( \int \) is a line integral.

Thus, variational inequality is a more general problem formulation than optimization problem formulation, since it can also handle a function \( F(X) \) with an asymmetric Jacobian (see Nagurney (1999)). Next, the qualitative properties of variational inequality problems, especially, the conditions for existence and uniqueness of a solution, are recalled.

**Theorem 2.2**

If \( K \) is a compact convex set and \( F(X) \) is continuous on \( K \), then the variational inequality problem admits at least one solution \( X^* \).

**Theorem 2.3**

If the feasible set \( K \) is unbounded, then \( \text{VI}(F,K) \) admits a solution if and only if there exists an \( R > 0 \) and a solution of \( \text{VI}(F,S), X^*_R \), such that \( \|X^*_R\| < R \), where \( S = \{X : \|X\| \leq R\} \).

**Theorem 2.4**

Suppose that \( F(X) \) satisfies the coercivity condition

\[
\frac{(F(X) - F(X_0), X - X_0)}{\|X - X_0\|} \to \infty
\]

as \( \|X\| \to \infty \) for \( X \in K \) and for some \( X_0 \in K \). Then \( \text{VI}(F,K) \) always has a solution.

According to Theorem 2.4, the existence condition of a solution to a variational inequality problem can be guaranteed by the coercivity condition. Next, certain
monotonicity conditions are utilized to discuss the qualitative properties of existence and uniqueness. Some basic definitions of monotonicity are reviewed first.

**Definition 2.3 (Monotonicity)**

$F(X)$ is monotone on $\mathcal{K}$ if

$$\langle F(X^1) - F(X^2), X^1 - X^2 \rangle \geq 0, \quad \forall X^1, X^2 \in \mathcal{K}. \quad (2.10)$$

**Definition 2.4 (Strict Monotonicity)**

$F(X)$ is strictly monotone on $\mathcal{K}$ if

$$\langle F(X^1) - F(X^2), X^1 - X^2 \rangle > 0, \quad \forall X^1, X^2 \in \mathcal{K}, X^1 \neq X^2. \quad (2.11)$$

**Definition 2.5 (Strong Monotonicity)**

$F(X)$ is strongly monotone on $\mathcal{K}$ if

$$\langle F(X^1) - F(X^2), X^1 - X^2 \rangle \geq \alpha \|X^1 - X^2\|^2, \quad \forall X^1, X^2 \in \mathcal{K}, \quad (2.12)$$

where

$$\alpha > 0.$$

**Definition 2.6 (Lipschitz Continuity)**

$F(X)$ is Lipschitz continuous on $\mathcal{K}$ if there exists an $L > 0$, such that

$$\langle F(X^1) - F(X^2), X^1 - X^2 \rangle \leq L \|X^1 - X^2\|^2, \quad \forall X^1, X^2 \in \mathcal{K}. \quad (2.13)$$

$L$ is called the Lipschitz constant.
Theorem 2.5
Suppose that \( F(X) \) is strictly monotone on \( K \). Then the solution to the \( \text{VI}(F,K) \) problem is unique, if one exists.

Theorem 2.6
Suppose that \( F(X) \) is strongly monotone on \( K \). Then there exists precisely one solution \( X^* \) to \( \text{VI}(F,K) \).

In summary of Theorems 2.2, 2.5, and 2.6, strong monotonicity of the function \( F \) guarantees both existence and uniqueness, in the case of an unbounded feasible set \( K \). If the feasible set \( K \) is compact, that is, closed and bounded, the continuity of \( F \) guarantees the existence of a solution. The strict monotonicity of \( F \) is then sufficient to guarantee its uniqueness provided its existence.

2.2. The Relationships between Variational Inequalities and Game Theory

In this section, some of the relationships between variational inequalities and game theory are briefly discussed.

Nash (1950, 1951) developed noncooperative game theory, involving multiple players, each of whom acts in his/her own interest. In particular, consider a game with \( m \) players, each player \( i \) having a strategy vector \( X_i = \{X_{i1}, \ldots, X_{in}\} \) selected from a closed, convex set \( K^i \subset \mathbb{R}^n \). Each player \( i \) seeks to maximize his/her own utility function, \( U_i : K \to \mathbb{R} \), where \( K = K^1 \times K^2 \times \ldots \times K^m \subset \mathbb{R}^{mn} \). The utility of player \( i \), \( U_i \), depends not only on his/her own strategy vector, \( X_i \), but also on the strategy vectors of all the other players, \( (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_m) \). An equilibrium is achieved if no one can increase his/her utility by unilaterally altering the value of its strategy vector. The formal definition of the Nash equilibrium is recalled as following.
Definition 2.7 (Nash Equilibrium)

A Nash equilibrium is a strategy vector

\[ X^* = (X_1^*, \ldots, X_m^*) \in \mathcal{K}, \]  

such that

\[ U_i(X_i^*, \hat{X}_i^*) \geq U_i(X_i, \hat{X}_i^*), \quad \forall X_i \in \mathcal{K}_i, \forall i, \]  

(2.15)

where \( \hat{X}_i^* = (X_1^*, \ldots, X_{i-1}^*, X_{i+1}^*, \ldots, X_m^*) \).

It has been shown by Hartman and Stampacchia (1966) and Gabay and Moulin (1980) that given continuously differentiable and concave utility functions, \( U_i, \forall i \), the Nash equilibrium problem can be formulated as a variational inequality problem defined on \( \mathcal{K} \).

Theorem 2.7 (Variational Inequality Formulation of Nash Equilibrium)

Under the assumption that each utility function \( U_i \) is continuously differentiable and concave, \( X^* \) is a Nash equilibrium if and only if \( X^* \in \mathcal{K} \) is a solution of the variational inequality

\[ \langle F(X^*), X - X^* \rangle \geq 0, \quad X \in \mathcal{K}, \]  

(2.16)

where \( F(X) \equiv (-\nabla_{X_1} U_1(X), \ldots, -\nabla_{X_m} U_m(X))^T \), and \( \nabla_{X_i} U_i(X) = (\frac{\partial U_i(X)}{\partial X_{i1}}, \ldots, \frac{\partial U_i(X)}{\partial X_{im}}) \).

The conditions for existence and uniqueness of a Nash equilibrium are now introduced. As stated in the following theorem, Rosen (1965) presented existence under the assumptions that \( \mathcal{K} \) is compact and each \( U_i \) is continuously differentiable.

Theorem 2.8 (Existence Under Compactness and Continuous Differentiability)

Suppose that the feasible set \( \mathcal{K} \) is compact and each \( U_i \) is continuously differentiable.

Then existence of a Nash equilibrium is guaranteed.
Gabay and Moulin (1980), on the other hand, relaxed the assumption of the compactness of \( K \), and proved existence of a Nash equilibrium after imposing a coercivity condition on \( F(X) \).

**Theorem 2.9 (Existence Under Coercivity)**

Suppose that \( F(X) \), as given in Theorem 2.7, satisfies the coercivity condition (2.9). Then there always exists a Nash equilibrium.

Furthermore, Karamardian (1969) demonstrated existence and uniqueness of a Nash equilibrium under the strong monotonicity assumption.

**Theorem 2.10 (Existence and Uniqueness Under Strong Monotonicity)**

Assume that \( F(X) \), as given in Theorem 2.7, is strongly monotone on \( K \). Then there exists precisely one Nash equilibrium \( X^* \).

Additionally, based on Theorem 2.5, uniqueness of a Nash equilibrium can be guaranteed under the assumptions that \( F(X) \) is strictly monotone and an equilibrium exists.

**Theorem 2.11 (Uniqueness Under Strict Monotonicity)**

Suppose that \( F(X) \), as given in Theorem 2.7, is strictly monotone on \( K \). Then the Nash equilibrium, \( X^* \), is unique, if it exists.

### 2.3. Projected Dynamical Systems

In this section, the theory of projected dynamical systems (cf. Dupuis and Nagurney (1993) and Nagurney and Zhang (1996)) is recalled, followed by the relationship between projected dynamical systems and variational inequality problems. Finally, some properties of the dynamic trajectories and the stability analysis of projected dynamical systems are provided. All the definitions and theorems can be found in Dupuis and Nagurney (1993) and Nagurney and Zhang (1996).
Definition 2.8

Given \( X \in \mathcal{K} \) and \( v \in \mathbb{R}^n \), define the projection of the vector \( v \) at \( X \) (with respect to \( \mathcal{K} \)) by

\[
\Pi_{\mathcal{K}}(X, v) = \lim_{\delta \to 0} \frac{(P_{\mathcal{K}}(X + \delta v) - X)}{\delta}
\]

with \( P_{\mathcal{K}} \) denoting the projection map:

\[
P_{\mathcal{K}}(X) = \arg\min_{X' \in \mathcal{K}} \|X' - X\|,
\]

where \( \| \cdot \| = \langle x, x \rangle \).

The class of ordinary differential equations that this dissertation focuses on takes on the following form:

\[
\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \quad X(0) = X_0 \in \mathcal{K},
\]

where \( \dot{X} \) denotes the rate of change of vector \( X \), \( \mathcal{K} \) is closed convex set, corresponding to the constraint set in a particular application, and \( F(X) \) is a vector field defined on \( \mathcal{K} \). I refer to the ordinary differential equation in (2.19) as ODE(\( F, \mathcal{K} \)).

The classical dynamical system, in contrast to (2.19), takes the form:

\[
\dot{X} = -F(X), \quad X(0) = X_0 \in \mathcal{K}.
\]

Definition 2.9 (The Projected Dynamical Systems)

Define the projected dynamical system (referred to as PDS(\( F, \mathcal{K} \))) \( X_0(t) : \mathcal{K} \times \mathbb{R} \mapsto \mathcal{K} \) as the family of solutions to the Initial Value Problem (IVP) (2.19) for all \( X_0 \in \mathcal{K} \).
Definition 2.10 (An Equilibrium Point)

The vector $X^* \in \mathcal{K}$ is a stationary point or an equilibrium point of the projected dynamical system $PDS(F, \mathcal{K})$ if

$$
\dot{X} = 0 = \Pi_{\mathcal{K}}(X^*, -F(X^*)).
$$

(2.21)

In other words, $X^*$ is a stationary point or an equilibrium point if, once the projected dynamical system is at $X^*$, it will remain at $X^*$ for all future times. Definition 2.10 demonstrates that $X^*$ is an equilibrium point of the projected dynamical system $PDS(F, \mathcal{K})$ if the vector field $F$ vanishes at $X^*$. However, it is only true when $X^*$ is an interior point of the constraint set $\mathcal{K}$. When $X^*$ lies on the boundary of $\mathcal{K}$, we may have $F(X^*) \neq 0$. Note that for classical dynamical systems, the necessary and sufficient condition for an equilibrium point is that the vector field vanish at that point, that is, $-F(X^*) = 0$.

Now, I recall the equivalence between the set of equilibria of a projected dynamical system and the set of solutions of the corresponding variational inequality problem by presenting the following theorem (see Dupuis and Nagurney (1993)).

Theorem 2.12

Assume that $\mathcal{K}$ is a convex polyhedron. Then the equilibrium points of the $PDS(F, \mathcal{K})$ coincide with the solutions of $VI(F, \mathcal{K})$. Hence, for $X^* \in \mathcal{K}$ and satisfying

$$
\dot{X} = 0 = \Pi_{\mathcal{K}}(X^*, -F(X^*))
$$

(2.22)

also satisfies

$$
\langle F(X^*), X - X^* \rangle \geq 0, \forall X \in \mathcal{K}.
$$

(2.23)
Before addressing the conditions for existence and uniqueness of the trajectory of a projected dynamical system, I recall the following fundamental assumption, which is implied by Lipschitz continuity (Definition 2.6).

**Assumption 2.1**

There exists a $B < \infty$ such that the vector field $-F : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the linear growth condition $\| F(X) \| \leq B(1 + \| X \|), \forall X \in \mathcal{K}$, and also

$$\langle -F(X) + F(y), X - y \rangle \leq B \| X - y \|^2, \forall X, y \in \mathcal{K}. \quad (2.24)$$

**Theorem 2.13 (Existence, Uniqueness, and Continuous Dependence)**

Assume the above assumption. Then

(i) For any $X_0 \in \mathcal{K}$, there exists a unique solution $X_0(t)$ to the initial value problem;

$$\dot{X} = \Pi_{\mathcal{K}}(X, -F(X)), \quad X(0) = X_0; \quad (2.25)$$

(ii) If $X_n \to X_0$ as $n \to \infty$, then $X_n(t)$ converges to $X_0(t)$ uniformly on every compact set of $[0, \infty)$.

The second statement of Theorem 2.13 is sometimes called the continuous dependence of the solution path to ODE$(F, \mathcal{K})$ on the initial value. Therefore, the PDS$(F, \mathcal{K})$ is well-defined and inhabits $\mathcal{K}$ whenever Assumption 2.1 holds.

The stability of a system is defined as the ability of the system to maintain or restore its equilibrium when acted upon by forces tending to displace it. Since the ordinary differential equation of PDS$(F, \mathcal{K})$ (2.19) has a discontinuous right-hand side, the question of the stability of the system arises. I now recall stability concepts for projected dynamical systems at their equilibrium points.
Definition 2.11 (Stability)

An equilibrium point $X^*$ is stable, if for any $\epsilon > 0$, there exists a $\delta > 0$, such that for all $X \in B(X^*, \delta)$ and all $t \geq 0$

\[ X \cdot t \in B(X^*, \epsilon). \] (2.26)

The equilibrium point $X^*$ is unstable, if it is not stable.

$B(X, r)$ is used to denote the open ball with radius $r$ and center $X$.

Definition 2.12 (Exponential Stability)

An equilibrium point $X^*$ is exponentially stable, if there exists a $\delta > 0$ and constants $B > 0$ and $\mu > 0$ such that

\[ \|X \cdot t - X^*\| \leq B\|X - X^*\|e^{-\mu t}, \quad \forall t \geq 0, \forall X \in B(X^*, \delta); \] (2.27)

$X^*$ is globally exponentially stable, if the above holds true for all $X^0 \in \mathcal{K}$.

Definition 2.13 (Monotone Attractor)

An equilibrium point $X^*$ is a monotone attractor, if there exists a $\delta > 0$ such that for all $X \in B(X^*, \delta)$,

\[ d(X, t) = \|X \cdot t - X^*\| \] (2.28)

is a nonincreasing function of $t$; $X^*$ is a global monotone attractor, if $d(X, t)$ is nonincreasing in $t$ for all $X \in \mathcal{K}$.

An equilibrium $X^*$ is a strictly monotone attractor, if there exists a $\delta > 0$ such that for all $X \in B(X^*, \delta)$, $d(X, t)$ is monotonically decreasing to zero in $t$; $X^*$ is a strictly global monotone attractor, if $d(X, t)$ is monotonically decreasing to zero in $t$ for all $X \in \mathcal{K}$.

The stability of a projected dynamical system is actually determined by the monotonicity of the $F(X)$ in the associated variational inequality problem. Next, I recall the local and global stability directly under various monotonicity conditions.
Theorem 2.14 (Monotone Attractor)

Suppose that $X^*$ solves $VI(F, K)$. If $F(X)$ is locally monotone at $X^*$, then $X^*$ is a monotone attractor for the $PDS(F, K)$; if $F(X)$ is monotone, then $X^*$ is a global monotone attractor.

Theorem 2.15 (Strict Monotone Attractor)

Suppose that $X^*$ solves $VI(F, K)$. If $F(X)$ is locally strictly monotone at $X^*$, then $X^*$ is a strictly monotone attractor for the $PDS(F, K)$; if $F(X)$ is strictly monotone at $X^*$, then $X^*$ is a strictly global monotone attractor.

Theorem 2.16 (Exponential Stability)

Suppose that $X^*$ solves $VI(F, K)$. If $F(X)$ is locally strongly monotone at $X^*$, then $X^*$ is exponentially stable for the $PDS(F, K)$; if $F(X)$ is strongly monotone at $X^*$, then $X^*$ is a globally exponentially stable.

2.4. Multicriteria Decision-Making

The goal of multicriteria decision-making (MCDM) is to evaluate a set of alternatives in terms of a number of conflicting criteria (Keeney and Raiffa (1976), Cohon (1978), and Triantaphyllou (2000)), according to the preferences of the decision-maker (Gal, Stewart, and Hanne (1999), and Jones, Mirrazavi, and Tamiz (2002)). In this section, the multiobjective optimization problem and the weighted sum method are briefly reviewed.

The multiobjective optimization problem with $n$ decision variables can be generalized as (see Marler and Arora (2004)):

\[
\text{Minimize} \quad G(X) = [G_1(X), G_2(X), \ldots, G_k(X)]^T
\]  \hspace{1cm} (2.29)

subject to:

\[
g_j(X) \leq 0, \quad j = 1, 2, \ldots, m,
\]  \hspace{1cm} (2.30)
\[ h_l(X) = 0, \quad l = 1, 2, \ldots, e, \]

where \( k \) is the number of objective functions, \( m \) is the number of inequality constraints, \( e \) is the number of equality constraints, and \( X \) is the \( n \)-dimensional vector of decision variables. The feasible set \( \mathcal{K}^1 \) is defined as:

\[ \mathcal{K}^1 \equiv \{ X | (2.30) \text{ and } (2.31) \text{ are satisfied} \}. \]  

The Pareto optimality of a solution to a multiobjective problem is defined by Pareto (1971), as follows.

**Definition 2.14 (Pareto Optimal)**

A point, \( X^* \in \mathcal{K}^1 \), is Pareto optimal iff there does not exist another point, \( X^* \in \mathcal{K}^1 \), such that \( \mathbf{G}(X) \leq \mathbf{G}(X^*) \), and \( G_i(X) < G_i(X^*) \) for at least one function.

The weighted sum method, which is the most common approach to multiobjective optimization problems (see Marler and Arora (2004)), is as following. Associated with a vector of weights, denoted by \( w \), representing the decision-maker’s preferences, the multiobjective objective function (2.29) can be expressed as:

\[ \Gamma = \sum_{i=1}^{k} w_i G_i(X). \]  

As noted by Zadeh (1963), the optimal solution to (2.33) is Pareto optimal if all of the weights are positive.

### 2.5. Algorithms

In this section, I review the algorithms. The Euler method, which is based on the general iterative scheme of Dupuis and Nagurney (1993), and the modified projection method of Korpelevich (1977) are presented.
2.5.1 The Euler Method

The Euler method can be utilized to compute the solution to a variational problem (cf. (2.1a)), and can also be used for the computation of the solution to the related projected dynamic system (cf. (2.19)) (see Dupuis and Nagurney (1993) and Nagurney and Zhang (1996)). This algorithm not only provides a discretization of the continuous time trajectory defined by (2.19) but also yields a stationary, that is, an equilibrium point that satisfies variational inequality (2.1a). It has been utilized to solve a plethora of dynamic network problems (see, e.g., Nagurney and Zhang (1996)).

Specifically, recall that, at an iteration $\tau$ of the Euler method (see also Nagurney and Zhang (1996)), where $\tau$ denotes an iteration counter, one computes:

$$X^{\tau+1} = P_K(X^{\tau} - \alpha_{\tau}F(X^{\tau})),$$

where $F$ is the function in (2.1a), and $P_K$ is the projection on the feasible set $K$, defined by

$$P_K(X) = \arg\min_{X' \in K} \|X' - X\|. \quad (2.35)$$

I now state the complete statement of the Euler method.

**Step 0: Initialization**

Set $X^0 \in K$.

Let $\tau = 1$ and set the sequence $\{\alpha_{\tau}\}$ so that $\sum_{\tau=1}^{\infty} \alpha_{\tau} = \infty$, $\alpha_{\tau} > 0$ for all $\tau$, and $\alpha_{\tau} \to 0$ as $\tau \to \infty$.

**Step 1: Computation**

Compute $X^\tau \in K$ by solving the variational inequality subproblem:

$$\langle X^\tau + \alpha_{\tau}F(X^{\tau-1}) - X^{\tau-1}, X - X^\tau \rangle \geq 0, \quad \forall X \in K. \quad (2.36)$$
Step 2: Convergence Verification

If \( \max |X_\tau^l - X_{\tau-1}^l| \leq \epsilon \), for all \( l \), with \( \epsilon > 0 \), a pre-specified tolerance, then stop; otherwise, set \( \tau := \tau + 1 \), and go to Step 1.

The VI subproblem (2.36) is actually a quadratic programming problem. It can be solved by the exact equilibration algorithm of Dafermos and Sparrow (1969) when it has a special network structure. In this dissertation, where appropriate, I also explore the network structure. This algorithm has been applied to many different applications of networks with special structure (cf. Nagurney and Zhang (1996) and Nagurney (1999)). See also Nagurney and Zhang (1997) for an application to fixed-demand traffic network equilibrium problems. The exact equilibration algorithm is used in Chapters 5 and 6 to solve some subproblems induced in the Euler method for fixed-demand supply chain network problems with outsourcing.

An assumption is recalled, followed by the convergence conditions of the Euler method in Theorem 2.17 and Corollary 2.1.

Assumption 2.2

Suppose we fix an initial condition \( X_0 \in \mathcal{K} \) and define the sequence \( \{X_\tau, \tau \in \mathbb{N}\} \) by (2.34). We assume the following conditions:

1. \( \sum_{i=0}^{\infty} a_i = \infty \), \( a_i > 0 \), \( a_i \to 0 \) as \( i \to \infty \).

2. \( d(F_\tau(x), \overline{F}(x)) \to 0 \) uniformly on compact subsets of \( \mathcal{K} \) as \( \tau \to \infty \).

3. Define \( \phi_y \) to be the unique solution to \( \dot{X} = \Pi_{\mathcal{K}}(X, -F(X)) \) that satisfies \( \phi_y(0) = y \in \mathcal{K} \). The \( \omega \)-limit set of \( \mathcal{K} \)

\[
\omega(\mathcal{K}) = \bigcup_{y \in \mathcal{K}} \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} \{\phi_y(s)\}}
\]

is contained in the set of stationary points of \( \dot{X} = \Pi_{\mathcal{K}}(X, -F(X)) \).

4. The sequence \( \{X_\tau, \tau \in \mathbb{N}\} \) is bounded.
5. The solution to \( \dot{X} = \Pi_K(X, -F(X)) \) are stable in the sense that given any compact set \( K_1 \) there exists a compact set \( K_2 \) such that \( \bigcup_{y \in K \cap K_1} \bigcup_{t \geq 0} \{ \phi_y(t) \} \subset K_2 \).

**Theorem 2.17**

Let \( S \) denote the set of stationary point of the projected dynamical system (2.19), equivalently, the set of solutions to the variational inequality problem (2.1a). Assume Assumption 2.1 and Assumption 2.2. Suppose \( \{X_\tau, \tau \in N\} \) is the scheme generated by (2.34). Then \( d(X_\tau, S) \to 0 \) as \( \tau \to \infty \), where \( d(X_\tau, S) \to 0 = \inf_{X \in S} \|X_\tau - X\| \).

**Corollary 2.1**

Assume the conditions of Theorem 2.17, and also that \( S \) consists of a finite set of points. Then \( \lim_{\tau \to \infty} X_\tau \) exists and equals to a solution to the variational inequality.

Theorem 2.17 indicates that Assumption 2.2 is the elementary condition under which the Euler method (2.34) converges. Proposition 2.3 and Proposition 2.4 below suggest some alternative conditions that are better known in variational inequality theory as sufficient conditions for Part 3 and Part 5 of Assumption 2.2.

**Proposition 2.3**

If the vector field \( F(X) \) is strictly monotone at some solution \( X^* \) to the variational inequality problem (2.1a), then Part 3 of Assumption 2.2 holds true.

**Proposition 2.4**

If the vector field \( F(X) \) is monotone at some solution \( X^* \) to the variational inequality problem (2.1a), then Part 5 of Assumption 2.2 holds true.

In the subsequent chapters, as appropriate, I adapt the convergence proofs to specific supply chain network applications.

**2.5.2 The Modified Projection Method**

The modified projection method of Korpelevich (1977) can be utilized to solve a variational inequality problem in standard form (cf. (2.1a)). This method is guaranteed to converge if the monotonicity (cf. (2.10)) and Lipschitz continuity (cf. (2.13))
of the function $F$ that enters the variational inequality (cf. (2.1a)) can be satisfied, and a solution to the variational inequality exists.

I now recall the modified projection method, and let $\tau$ denote an iteration counter.

**Step 0: Initialization**

Set $X^0 \in \mathcal{K}$. Let $\tau = 1$ and let $\alpha$ be a scalar such that $0 < \alpha \leq \frac{1}{L}$, where $L$ is the Lipschitz continuity constant (cf. (2.13)).

**Step 1: Computation**

Compute $\bar{X}^\tau$ by solving the variational inequality subproblem:

$$\langle \bar{X}^\tau + \alpha F(X^\tau - 1) - X^\tau, X - \bar{X}^\tau \rangle \geq 0, \quad \forall X \in \mathcal{K}. \quad (2.37)$$

**Step 2: Adaptation**

Compute $X^\tau$ by solving the variational inequality subproblem:

$$\langle X^\tau + \alpha F(\bar{X}^\tau) - X^\tau - 1, X - X^\tau \rangle \geq 0, \quad \forall X \in \mathcal{K}. \quad (2.38)$$

**Step 3: Convergence Verification**

If $\max |X_l^\tau - X_l^{\tau - 1}| \leq \epsilon$, for all $l$, with $\epsilon > 0$, a prespecified tolerance, then stop; else, set $\tau := \tau + 1$, and go to Step 1.

**Theorem 2.18 (Convergence of the Modified Projection Method)**

If $F(X)$ is monotone and Lipschitz continuous (and a solution exists), the modified projection algorithm converges to a solution of variational inequality (2.1a).

In the following chapters, I derive the variational inequality formulations and the projected dynamical systems of the supply chain network models with quality competition with application to information asymmetry, production differentiation, outsourcing, and supplier selection. The computational algorithms introduced in this
chapter, which are the Euler method and the modified projection method, are also adapted accordingly.
CHAPTER 3
A SUPPLY CHAIN NETWORK MODEL WITH
INFORMATION ASYMMETRY IN QUALITY, MINIMUM
QUALITY STANDARDS, AND QUALITY
COMPETITION

In this chapter, I construct a supply chain network model with information asymmetry between firms and consumers in product quality. The competing, profit-maximizing firms with, possibly, multiple manufacturing plants which may be located on-shore or off-shore, are aware of their quality levels but do not have differentiated brands or labels to distinguish their products from their competitors’. Therefore, consumers at the demand markets view the products as well as their quality levels as homogeneous. Such a framework is relevant to products ranging from certain foods to pharmaceuticals.

In this model, firms compete with one another in both the shipment amounts produced as well as the quality levels at their manufacturing plants with the objective of profit maximization. Quality is associated not only with the manufacturing plants but also tracked through the transportation process, which is assumed to preserve the product quality. I propose both the equilibrium model, with the equivalent variational inequality formulation, and its dynamic counterpart, using projected dynamical systems theory, and demonstrate how minimum quality standards can be incorporated into the framework. Both theoretical results, in the form of existence and uniqueness results as well as stability analysis, are presented. The Euler method (cf. Section 2.5.1) is used to solve the model, and the convergence results are also given. The numerical examples, accompanied by sensitivity analysis, reveal interesting results
and insights for firms, consumers, as well as policy-makers, who impose the minimum quality standards.

This chapter is based on Nagurney and Li (2014a), and is organized as follows. In Section 3.1, I present both the static models (without and with minimum quality standards), along with their variational inequality formulations, as well as the dynamic version using projected dynamical systems theory. In Section 3.2, I then provide qualitative properties of the equilibrium solutions and establish that the set of stationary points of the projected dynamical systems formulation coincides with the set of solutions to the corresponding variational inequality problem. In Section 3.3, I describe the closed form expressions for the Euler method applied in computing the equilibrium product shipments and quality levels, and establish convergence. In Section 3.4, I provide numerical examples and conduct sensitivity analyses, which yield valuable insights for firms, consumers, and policy-makers. In Section 3.5, I summarize the results and present the conclusions.

### 3.1. The Supply Chain Network Model with Information Asymmetry in Quality and Quality Competition

In this section, I construct the supply chain network equilibrium model in which the firms compete in product quantities and quality levels and there is information asymmetry in quality. I first consider the case without minimum quality standards and then demonstrate how standards can be incorporated. Next, the dynamic counterpart of the latter is developed, which contains the former as a special case. The static equilibrium model(s) are given in Section 3.2.1 and the dynamic version in Section 3.2.2.
3.1.1 The Equilibrium Model Without and With Minimum Quality Standards

I first present the model without minimum quality standards and then show how it can be extended to include minimum quality standards, which are useful policy instruments in practice.

I consider $I$ firms, with a typical firm denoted by $i$, which compete with one another in a noncooperative Cournot-Nash manner in the production and distribution of the product. Each firm $i$ has, at its disposal, $n_i$ manufacturing plants. The firms determine the quantities to produce at each of their manufacturing plants and the quantities to ship to the $n_R$ demand markets. They also control the quality level of the product at each of their manufacturing plants. Information asymmetry occurs in that the firms are aware of the quality levels of the product produced at each of their manufacturing plants but the consumers are only aware of the average quality levels of the product at the demand markets.

I consider the supply chain network topology depicted in Figure 3.1. The top nodes correspond to the firms, the middle nodes to the manufacturing plants, and the bottom nodes to the common demand markets. I assume that the demand at each demand market is positive; otherwise, the demand market (node) will be removed from the supply chain network.

In Figure 3.1, the first set of links connecting the top two tiers of nodes corresponds to the process of manufacturing at each of the manufacturing plants of firm $i$; $i = 1, \ldots, I$. Such plants are denoted by $M_i^1, \ldots, M_i^{n_i}$, respectively, for firm $i$, with a typical one denoted by $M_i^j$; $j = 1, \ldots, n_i$. The manufacturing plants may be located not only in different regions of a country but also in different countries. The next set of links connecting the two bottom tiers of the supply chain network corresponds to transportation links joining the manufacturing plants with the demand markets, with a typical demand market denoted by $R_k$; $k = 1, \ldots, n_R$. 
Figure 3.1. The Supply Chain Network Topology with Multiple Manufacturing Plants

The nonnegative product amount produced at firm \(i\)'s manufacturing plant \(M_i^j\) and shipped to demand market \(R_k\) is denoted by \(Q_{ijk}\); \(i = 1, \ldots, I; j = 1, \ldots, n_i; k = 1, \ldots, n_R\). For each firm \(i\), I group its \(Q_{ijk}\)'s into the vector \(Q_i \in R_+^{n_i n_R}\), and then group all such vectors for all firms into the vector \(Q \in R_+^{\sum_i^I n_i n_R}\).

The nonnegative production output of firm \(i\)'s manufacturing plant \(M_i^j\) is denoted by \(s_{ij}\), the demand for the product at demand market \(R_k\) is denoted by \(d_k\); \(k = 1, \ldots, n_R\), and the quality level or, simply, quality, of the product produced by firm \(i\)'s manufacturing plant \(M_i^j\) is denoted by \(q_{ij}\). Note that different manufacturing plants owned by a firm may have different quality levels. This is highly reasonable since, for example, different plants may have different resources available in terms of skilled labor and facilities as well as labor expertise and even infrastructure.

The output at firm \(i\)'s manufacturing plant \(M_i^j\) and the demand for the product at each demand market \(R_k\) must satisfy, respectively, the conservation of flow equations (3.1) and (3.2):

\[
s_{ij} = \sum_{k=1}^{n_R} Q_{ijk}, \quad i = 1, \ldots, I; j = 1, \ldots, n_i,
\]

(3.1)
\[ d_k = \sum_{i=1}^{I} \sum_{j=1}^{n_i} Q_{ijk}, \quad k = 1, \ldots, n_R. \] 

Hence, the output produced at firm \( i \)'s manufacturing plant \( M_i \) is equal to the sum of the amounts shipped to the demand markets, and the quantity consumed at a demand market is equal to the sum of the amounts shipped by the firms to that demand market. I group all \( s_{ij} \)s into the vector \( s \in R^{I \sum_{i=1}^{I} n_i} \) and all \( d_k \)s into the vector \( d \in R^{n_R \times I} \). For each firm \( i \), its own quality levels are grouped into the vector \( q_i \in R^{n_i} \), and all such vectors for all firms are grouped into the vector \( q \in R^{I \sum_{i=1}^{I} n_i} \).

The product shipments must be nonnegative, that is:

\[ Q_{ijk} \geq 0, \quad i = 1, \ldots, I; j = 1, \ldots, n_i; k = 1, \ldots, n_R, \] 

(3.3)

In addition, the quality levels cannot be lower than 0\% conformance level (cf. Section 1.1), thus,

\[ q_{ij} \geq 0, \quad i = 1, \ldots, I; j = 1, \ldots, n_i. \] 

(3.4)

The production cost at firm \( i \)'s manufacturing plant \( M_i \) is denoted by \( f_{ij} \). Given the impact of quality on production, I express the production cost function of each firm as a function that depends on both production quantities and quality levels (see, e.g., Spence (1975), Dixit (1979), Rogerson (1988), Shapiro (1982), Lederer and Rhee (1995), Teng and Thompson (1996), and Salameh and Jaber (2000)). The cost of quality (Section 1.1) is also included in the production cost. Hence, in this model, I allow for the general situation where \( f_{ij} \) may depend upon the production pattern and the vector of quality levels, that is,

\[ f_{ij} = f_{ij}(s, q), \quad i = 1, \ldots, I; j = 1, \ldots, n_i. \] 

(3.5a)

In view of (3.1), I define the plant manufacturing cost functions \( \hat{f}_{ij}; i = 1, \ldots, I; j = 1, \ldots, n_i; \) in shipment quantities and quality levels, that is,
\[ \hat{f}_{ij} = \hat{f}_{ij}(Q, q) \equiv f_{ij}(s, q). \]  

Moreover, given the impact of quality on transportation (cf. Floden, Barthel, and Sorkina (2010) and Saxin, Lammgard, and Floden (2005) and the references therein), I also associate quality with respect to the distribution/transport activities of the products to the demand markets. I accomplish this by having the transportation/distribution functions depend explicitly on both flows and quality levels, with the assumption of quality preservation, that is, the product will be delivered at the same quality level that it was produced at. Let \( \hat{c}_{ijk} \) denote the total transportation cost associated with shipping the product produced at firm \( i \)'s manufacturing plant \( M_i \) to demand market \( R_k \), that is,

\[ \hat{c}_{ijk} = \hat{c}_{ijk}(Q, q), \quad i = 1, \ldots, I; j = 1, \ldots, n_i; k = 1, \ldots, n_R. \]  

Note that, according to (3.6), the transportation cost is such that the quality of the product is not degraded as it undergoes the shipment process. Transportation cost functions in both quantities and quality levels were utilized by Nagurney and Wolf (2014), Saberi, Nagurney, and Wolf (2014), and Nagurney et al. (2014) but in the context of Internet applications and not supply chains.

Since the products of the \( I \) firms are, in effect, homogeneous, common resources and technologies may be utilized in the processes of manufacturing and transportation. In this model, in order to capture the competition for resources and technologies on the supply side, I allow for general production cost functions, which may depend on the entire production pattern \( s \) and the entire vector of quality levels \( q \) (cf. (3.5a)), and general transportation cost functions, which may depend on the vectors \( Q \) and \( q \) (cf. (3.6)). These cost functions measure not only the monetary costs in the corresponding processes, but also other important factors, such as the time spent in conforming the processes. Such general production and transportation cost functions
in both quantities and quality levels are utilized, for the first time, in a supply chain context. The production cost functions (3.5a) and the transportation functions (3.6) are assumed to be convex and twice continuously differentiable.

Since firms do not differentiate the products as well as their quality levels, consumers’ perception of the quality of all such products, which may come from different firms, is the average quality level of all products, $\hat{q}_k; k = 1, \ldots, n_R$, where

$$\hat{q}_k = \frac{\sum_{i=1}^{I} \sum_{j=1}^{n_i} Q_{ijk} q_{ij}}{d_k}, \quad k = 1, \ldots, n_R \quad (3.7)$$

with the average (perceived) quality levels grouped into the vector $\hat{q} \in R^{n_R}$. Here, I utilize the average quality as proposed by the Nobel laureate George Akerlof (see Akerlof (1970)) and further developed by the Nobel laureates Stiglitz (1987) and Spence (1975). In this model, the quality of a product is not directly observable by consumers, but it can be estimated by the average quality of all such products in the same demand market (cf. Nagurney, Li, and Nagurney (2014)). This average quality can be conveyed among consumers through word of mouth, their own consumption experiences, advertising, etc.

The demand price at demand market $R_k$ is denoted by $\rho_k$. Consumers located at the demand markets, which are spatially separated, respond not only to the quantities available of the products but also to their quality levels. Recall that Kaya and Özer (2009) and Kaya (2011) considered demand functions in price and quality, which are of the following form: $q = a - bp + e + \varepsilon$, where $e$ is the quality effort. Xie et al. (2011) utilized a demand function in the investigation of quality investment, of the form $D = a + \alpha x - \beta p$, where $x$ is quality. Anderson and de Palma (2001) captured the utility $u$ of each consumer as $u = q - p + \varepsilon$ in their research on asymmetric oligopolies, with $q$ being quality. In addition, Banker, Khosla, and Sinha (1998) considered a duopoly case and utilized a demand function given by $q_i = k_i \alpha - \beta p_i + \gamma p_j + \lambda x_i - \mu x_j$, where $x$ denotes quality. In the model in this chapter, because the products are
homogeneous, the demand price function of all such products at a demand market depends, in general, upon the entire demand pattern, as well as on the average quality levels at all the demand markets, that is,

\[ \rho_k = \rho_k(d, \hat{q}), \quad k = 1, \ldots, n_R. \]  

(3.8a)

The generality of this expression enables modeling and application flexibility. Each demand price function is, typically, assumed to be monotonically decreasing in product quantity but increasing in terms of the average product quality. Demand functions that are functions of the prices and the average quality levels were also used by Akerlof (1970). Therein the producers, in the form of a supply market, are aware of their product quality levels (cf. (3.5a)), while consumers at the demand markets are aware only of the average quality levels. However, multiple manufacturing plants, transportation, and multiple demand markets are not considered in Akerlof (1970), and, the profit-maximizing behavior of individual, competing firms, as in this chapter, is not modeled.

In light of (3.2) and (3.7), one can define the demand price function \( \hat{\rho}_k; k = 1, \ldots, n_R \), in quantities and quality levels of the firms, so that

\[ \hat{\rho}_k = \hat{\rho}_k(Q, q) \equiv \rho_k(d, \hat{q}), \quad k = 1, \ldots, n_R. \]  

(3.8b)

Functions (3.8a and b) are not limited to linear demand price functions, and they are assumed to be continuous and twice continuously differentiable.

The strategic variables of firm \( i \) are its product shipments \( \{Q_i\} \) and its quality levels \( q_i \). The profit/utility \( U_i \) of firm \( i; i = 1, \ldots, I \), is given by:

\[ U_i = \sum_{k=1}^{n_R} \rho_k(d, \hat{q}) \sum_{j=1}^{n_i} Q_{ijk} - \sum_{j=1}^{n_i} f_{ij}(s, q) - \sum_{k=1}^{n_R} \sum_{j=1}^{n_i} \hat{c}_{ijk}(Q, q). \]  

(3.9a)

55
which is the difference between its total revenue and its total costs (production and transportation). By making use of (3.5b) and (3.8b), (3.9a) is equivalent to

\[
U_i = \sum_{k=1}^{n_R} \hat{\rho}_k(Q, q) \sum_{j=1}^{n_i} Q_{ijk} - \sum_{j=1}^{n_i} \hat{f}_{ij}(Q, q) - \sum_{k=1}^{n_R} \sum_{j=1}^{n_i} \hat{c}_{ijk}(Q, q). \tag{3.9b}
\]

In view of (3.1) - (3.9b), one may write the profit as a function solely of the product shipment pattern and quality levels, that is,

\[
U = U(Q, q), \tag{3.10}
\]

where \(U\) is the \(I\)-dimensional vector with components: \(\{U_1, \ldots, U_I\}\).

Let \(K^i\) denote the feasible set corresponding to firm \(i\), where \(K^i \equiv \{(Q_i, q_i)\mid Q_i \geq 0, \text{ and } q_i \geq 0\}\) and define \(K \equiv \prod_{i=1}^{I} K^i\).

I consider Cournot-Nash competition, in which the \(I\) firms produce and deliver their product in a noncooperative fashion, each one trying to maximize its own profit. They seek to determine a nonnegative product shipment and quality level pattern \((Q^*, q^*)\) \(\in K\) for which the \(I\) firms will be in a state of equilibrium as defined below (cf. Definition 2.7).

**Definition 3.1: A Supply Chain Network Cournot-Nash Equilibrium with Information Asymmetry in Quality**

A product shipment and quality level pattern \((Q^*, q^*)\) \(\in K\) is said to constitute a supply chain network Cournot-Nash equilibrium with information asymmetry in quality if for each firm \(i; i = 1, \ldots, I\),

\[
U_i(Q^*_i, q^*_i, \hat{Q}^*_i, \hat{q}^*_i) \geq U_i(Q_i, q_i, \hat{Q}^*_i, \hat{q}^*_i), \quad \forall (Q_i, q_i) \in K^i, \tag{3.11}
\]

where \(\hat{Q}^*_i \equiv (Q^*_1, \ldots, Q^*_{i-1}, Q^*_{i+1}, \ldots, Q^*_I)\) and \(\hat{q}^*_i \equiv (q^*_1, \ldots, q^*_{i-1}, q^*_{i+1}, \ldots, q^*_I)\).
According to (3.11), an equilibrium is established if no firm can unilaterally improve upon its profits by selecting an alternative vector of product shipments and quality level of its product.

I now present alternative variational inequality formulations of the above supply chain network Cournot-Nash equilibrium in the following theorem.

**Theorem 3.1: Variational Inequality Formulations**

Assume that for each firm $i$ the profit function $U_i(Q, q)$ is concave with respect to the variables in $Q_i$ and $q_i$, and is continuous and continuously differentiable. Then the product shipment and quality pattern $(Q^*, q^*) \in K$ is a supply chain network Cournot-Nash equilibrium with quality information asymmetry according to Definition 3.1 if and only if it satisfies the variational inequality

$$
- \sum_{i=1}^{I} \sum_{j=1}^{n_i} \sum_{k=1}^{n_R} \frac{\partial U_i(Q^*, q^*)}{\partial Q_{ijk}} \times (Q_{ijk} - Q^*_{ijk}) - \sum_{i=1}^{I} \sum_{j=1}^{n_i} \frac{\partial U_i(Q^*, q^*)}{\partial q_{ij}} \times (q_{ij} - q^*_{ij}) \geq 0,
$$

\[\forall (Q, q) \in K; \quad (3.12)\]

that is,

$$
\sum_{i=1}^{I} \sum_{j=1}^{n_i} \sum_{k=1}^{n_R} \left[ -\hat{\rho}_k(Q^*, q^*) - \sum_{l=1}^{n_R} \frac{\partial \hat{\rho}_l(Q^*, q^*)}{\partial Q_{ijk}} \sum_{h=1}^{n_i} Q^*_{ihh} + \sum_{h=1}^{n_i} \frac{\partial \hat{f}_{ih}(Q^*, q^*)}{\partial Q_{ijk}} \right] \times (Q_{ijk} - Q^*_{ijk})
$$

$$
+ \sum_{h=1}^{n_i} \sum_{l=1}^{n_R} \frac{\partial \hat{c}_{ihh}(Q^*, q^*)}{\partial Q_{ijk}} \left[ -n_i \sum_{k=1}^{n_R} \frac{\partial \hat{\rho}_k(Q^*, q^*)}{\partial q_{ij}} \sum_{h=1}^{n_i} Q^*_{ihh} + \sum_{h=1}^{n_i} \frac{\partial \hat{f}_{ih}(Q^*, q^*)}{\partial q_{ij}} \right] \right] \times (q_{ij} - q^*_{ij}) \geq 0, \quad \forall (Q, q) \in K; \quad (3.13)
$$

equivalently, $(d^*, s^*, Q^*, q^*) \in K^1$ is an equilibrium production, shipment, and quality level pattern if and only if it satisfies the variational inequality

$$
\sum_{k=1}^{n_R} [-\rho_k(d^*, \hat{q}^*)] \times (d_k - d^*_k) + \sum_{i=1}^{I} \sum_{j=1}^{n_i} \left[ \sum_{h=1}^{n_i} \frac{\partial f_{ih}(s^*, q^*)}{\partial s_{ij}} \right] \times (s_{ij} - s^*_{ij})
$$
\[
+ \sum_{i=1}^{l} \sum_{j=1}^{n_{i}} \sum_{k=1}^{n_{R}} \left[ -\sum_{t=1}^{n_{R}} \frac{\partial \rho_{t}(d^{*}, q^{*})}{\partial Q_{ijk}} \sum_{h=1}^{n_{i}} Q_{ihl}^{*} + \sum_{h=1}^{n_{i}} \sum_{l=1}^{n_{R}} \frac{\partial c_{ihl}(Q^{*}, q^{*})}{\partial Q_{ijk}} \right] \times (Q_{ijk} - Q_{ijk}^{*}) \\
+ \sum_{i=1}^{l} \sum_{j=1}^{n_{i}} \left[ -\sum_{k=1}^{n_{R}} \frac{\partial p_{k}(Q^{*}, q^{*})}{\partial q_{ij}} \sum_{h=1}^{n_{i}} Q_{ihk}^{*} + \sum_{h=1}^{n_{i}} \sum_{k=1}^{n_{R}} \frac{\partial f_{ih}(s^{*}, q^{*})}{\partial q_{ij}} + \sum_{h=1}^{n_{i}} \sum_{k=1}^{n_{R}} \frac{\partial c_{ihk}(Q^{*}, q^{*})}{\partial q_{ij}} \right] \times (q_{ij} - q_{ij}^{*}) \geq 0, \quad \forall (d, s, Q, q) \in K^{1},
\]

where \( K^{1} \equiv \{(d, s, Q, q) | Q \geq 0, q \geq 0, \text{and } (3.1), (3.2), \text{and } (3.7) \text{ hold}\} \).

**Proof:** (3.12) follows directly from Gabay and Moulin (1980) and Dafermos and Nagurney (1987), as given in Theorem 2.7. For firm \( i \)'s manufacturing plant \( M_{i}^{j} \); \( i = 1, \ldots, I; j = 1, \ldots, n_{i} \) and demand market \( R_{k} \); \( k = 1, \ldots, n_{R} \):

\[
- \frac{\partial U_{i}(Q, q)}{\partial Q_{ijk}} = -\frac{\partial [\sum_{t=1}^{n_{R}} \rho_{t}(Q, q) \sum_{h=1}^{n_{i}} Q_{ihl} - \sum_{h=1}^{n_{i}} \sum_{l=1}^{n_{R}} c_{ihl}(Q, q)]}{\partial Q_{ijk}} = -\sum_{t=1}^{n_{R}} \frac{\partial \rho_{t}(Q, q)}{\partial Q_{ijk}} \sum_{h=1}^{n_{i}} Q_{ihl} + \sum_{h=1}^{n_{i}} \frac{\partial \hat{f}_{ih}(Q, q)}{\partial Q_{ijk}} + \sum_{h=1}^{n_{i}} \sum_{l=1}^{n_{R}} \frac{\partial c_{ihl}(Q, q)}{\partial Q_{ijk}}.
\]

Also, for firm \( i \)'s manufacturing plant \( M_{i}^{j} \); \( i = 1, \ldots, I; j = 1, \ldots, n_{i} \):

\[
- \frac{\partial U_{i}(Q, q)}{\partial q_{ij}} = -\frac{\partial [\sum_{k=1}^{n_{R}} \hat{p}_{k}(Q, q) \sum_{h=1}^{n_{i}} Q_{ihk} - \sum_{h=1}^{n_{i}} \sum_{k=1}^{n_{R}} \hat{c}_{ihk}(Q, q)]}{\partial q_{ij}} = -\sum_{k=1}^{n_{R}} \frac{\partial \hat{p}_{k}(Q, q)}{\partial q_{ij}} \sum_{h=1}^{n_{i}} Q_{ihk} + \sum_{h=1}^{n_{i}} \frac{\partial \hat{f}_{ih}(Q, q)}{\partial q_{ij}} + \sum_{h=1}^{n_{i}} \sum_{k=1}^{n_{R}} \frac{\partial \hat{c}_{ihk}(Q, q)}{\partial q_{ij}}.
\]

Thus, variational inequality (3.13) is immediate. In addition, by re-expressing the production cost functions and the demand price functions in (3.15) and (3.16) as in (3.5b) and (3.8b) and using the conservation of flow equations (3.1) and (3.2) and \( \frac{\partial f_{as}(s, q)}{\partial Q_{ijk}} = \frac{\partial f_{as}(s, q)}{\partial s_{ij}} \frac{\partial s_{ij}}{\partial Q_{ijk}} \), the equivalence of variational inequalities (3.13) and (3.14) holds true. \( \square \)
I now describe an extension of the above framework that incorporates minimum quality standards. I integrate the framework with minimum quality standards and the framework without, and present the equilibrium conditions of both through a unified variational inequality formulation.

I retain the above notation, firm behavior, and constraints, but now nonnegative lower bounds on the quality levels at the manufacturing plants are imposed, denoted by \( q_{ij} \); \( i = 1, \ldots, I; j = 1, \ldots, n_i \) so that (3.4) is replaced by:

\[
q_{ij} \geq q_{ij} \quad i = 1, \ldots, I; j = 1, \ldots, n_i
\]

(3.17)

with the understanding that, if the lower bounds are all identically equal to zero, then (3.17) collapses to (3.4) and, if the lower bounds are positive, then they represent minimum quality standards.

I define a new feasible set \( K^2 \equiv \{(Q, q)|Q \geq 0 \text{ and } (3.17) \text{ holds}\} \). Then the following Corollary is immediate.

**Corollary 3.1: Variational Inequality Formulations with Minimum Quality Standards**

*Assume that for each firm \( i \) the profit function \( U_i(Q, q) \) is concave with respect to the variables in \( Q \) and \( q_i \), and is continuous and continuously differentiable. Then the product shipment and quality pattern \( (Q^*, q^*) \in K^2 \) is a supply chain network Cournot-Nash equilibrium with quality information asymmetry in the presence of minimum quality standards if and only if it satisfies the variational inequality

\[
- \sum_{i=1}^{I} \sum_{j=1}^{n_i} \sum_{k=1}^{n_K} \frac{\partial U_i(Q^*, q^*)}{\partial Q_{ijk}} \times (Q_{ijk} - Q_{ijk}^*) - \sum_{i=1}^{I} \sum_{j=1}^{n_i} \frac{\partial U_i(Q^*, q^*)}{\partial q_{ij}} \times (q_{ij} - q_{ij}^*) \geq 0,
\]

\[ \forall (Q, q) \in K^2, \]

(3.18)
that is,

$$
\sum_{i=1}^{I} \sum_{j=1}^{n_i} \sum_{k=1}^{n_R} \left[ -\hat{\rho}_k(Q^*, q^*) - \sum_{l=1}^{n_R} \frac{\partial \hat{\rho}_l(Q^*, q^*)}{\partial Q_{ijk}} \sum_{h=1}^{n_i} Q_{ihl} + \sum_{h=1}^{n_i} \frac{\partial \hat{f}_{ih}(Q^*, q^*)}{\partial q_{ij}} \sum_{k=1}^{n_R} Q_{ihk} \right] \\
+ \sum_{l=1}^{n_R} \frac{\partial \hat{c}_{ihl}(Q^*, q^*)}{\partial Q_{ijk}} \times (Q_{ijk} - Q^*_{ijk}) \times (Q_{ijk} - Q^*_{ijk}) \geq 0, \quad \forall (Q, q) \in K^2.
$$

Variational inequality (3.19) contains variational inequality (3.13) as a special case when the minimum quality standards are all zero, and it will play a crucial role in the next section when the underlying dynamics associated with the firms’ adjustment processes in product shipments and quality levels is described.

I now put variational inequality (3.19) into standard form (cf. (2.1a)): determine $X^* \in K$ where $X$ is a vector in $\mathbb{R}^N$, $F(X)$ is a continuous function such that $F(X): X \mapsto K \subset \mathbb{R}^N$, and

$$
\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in K,
$$

(3.20)

where $\langle \cdot, \cdot \rangle$ is the inner product in the $N$-dimensional Euclidean space, and $K$ is closed and convex. I define the vector $X \equiv (Q, q)$ and the vector $F(X) \equiv (F^1(X), F^2(X))$. Also, here $N = \sum_{i=1}^{I} n_i n_R + \sum_{i=1}^{I} n_i$. $F^1(X)$ consists of components $F^1_{ijk} = \frac{\partial U_i(Q, q)}{\partial Q_{ijk}}$; $i = 1, \ldots, I; j = 1, \ldots, n_i; k = 1, \ldots, n_R$, and $F^2(X)$ consist of components $F^2_{ij} = \frac{\partial U_i(Q, q)}{\partial q_{ij}}$; $i = 1, \ldots, I; j = 1, \ldots, n_i$. In addition, I define the feasible set $K \equiv K^2$. Hence, (3.19) can be put into standard form (2.1a).
3.1.2 The Dynamic Model

I now describe the underlying dynamics for the evolution of product shipments and quality levels under information asymmetry in quality until the equilibrium satisfying variational inequality (3.19) is achieved.

Observe that, for a current vector of product shipments and quality levels at time $t$, $X(t) = (Q(t), q(t))$, $-F^1_{ijk}(X(t)) = \frac{\partial U_i(Q(t), q(t))}{\partial Q_{ijk}}$ is the marginal utility (profit) of firm $i$ with respect to the volume produced at its manufacturing plant $j$ and distributed to demand market $R_k$. $-F^2_{ij}(X(t)) = \frac{\partial U_i(Q(t), q(t))}{\partial q_{ij}}$ is firm $i$’s marginal utility with respect to the quality level of its manufacturing plant $j$. In this framework, the rate of change of the product shipment between firm $i$’s manufacturing plant $j$ and demand market $R_k$ is in proportion to $-F^1_{ij}(X)$, as long as the product shipment $Q_{ijk}$ is positive.

Namely, when $Q_{ijk} > 0$,

$$
\dot{Q}_{ijk} = \frac{\partial U_i(Q, q)}{\partial Q_{ijk}}, \tag{3.21}
$$

where $\dot{Q}_{ijk}$ denotes the rate of change of $Q_{ijk}$. However, when $Q_{ijk} = 0$, the non-negativity condition (3.3) forces the product shipment $Q_{ijk}$ to remain zero when $\frac{\partial U_i(Q, q)}{\partial Q_{ijk}} \leq 0$. Hence, one is only guaranteed of having possible increases of the shipment, that is, when $Q_{ijk} = 0$,

$$
\dot{Q}_{ijk} = \max\{0, \frac{\partial U_i(Q, q)}{\partial Q_{ijk}}\}. \tag{3.22}
$$

One may write (3.21) and (3.22) concisely as:

$$
\dot{Q}_{ijk} = \begin{cases} 
\frac{\partial U_i(Q, q)}{\partial Q_{ijk}}, & \text{if } Q_{ijk} > 0 \\
\max\{0, \frac{\partial U_i(Q, q)}{\partial Q_{ijk}}\}, & \text{if } Q_{ijk} = 0.
\end{cases} \tag{3.23}
$$

As for the quality levels, when $q_{ij} > q_{ij}$, then

$$
\dot{q}_{ij} = \frac{\partial U_i(Q, q)}{\partial q_{ij}}, \tag{3.24}
$$

61
where \( \dot{q}_{ij} \) denotes the rate of change of \( q_{ij} \); when \( q_{ij} = q_{ij} \),
\[
\dot{q}_{ij} = \max\{q_{ij}, \frac{\partial U_i(Q, q)}{\partial q_{ij}}\},
\]
(3.25)
since \( q_i \) cannot be lower than \( q_{ij} \) according to the feasible set \( \mathcal{K} = K^2 \).

Combining (3.24) and (3.25), one may write:
\[
\dot{q}_{ij} = \begin{cases} 
\frac{\partial U_i(Q, q)}{\partial q_{ij}}, & \text{if } q_{ij} > q_{ij} \\
\max\{q_{ij}, \frac{\partial U_i(Q, q)}{\partial q_{ij}}\}, & \text{if } q_{ij} = q_{ij}.
\end{cases}
\]
(3.26)

Applying (3.23) to all firm and manufacturing plant pairs \((i, j); i = 1, \ldots, I; j = 1, \ldots, n_i \) and all demand markets \( R_k; k = 1, \ldots, n_R \), and then applying (3.26) to all firm and manufacturing plant pairs \((i, j); i = 1, \ldots, I; j = 1, \ldots, n_i \), and combining the resultants, yields the following pertinent ordinary differential equation (ODE) (cf. (2.19)) for the adjustment processes of the product shipments and quality levels, in vector form:
\[
\dot{X} = \Pi\mathcal{K}(X, -F(X)),
\]
(3.27)
where \(-F(X) = \nabla U(Q, q)\), and \(\nabla U(Q, q)\) is the vector of marginal utilities as described above.

I now further interpret ODE (3.27) in the context of the supply chain network competition model with information asymmetry in quality. First, observe that ODE (3.27) guarantees that the product shipments are always nonnegative and the quality levels never go below the minimum quality standards. In addition, ODE (3.27) states that the rate of change of the product shipments and the quality levels is greatest when the firm’s marginal utilities are greatest. If the marginal utility of a firm with respect to its quality level is positive, then the firm will increase its quality level; if it is negative, then it will decrease the quality level, and the quality levels will also never be outside their lower bounds. A similar adjustment behavior holds for the
firms in terms of their product shipments. This type of behavior is rational from an economic standpoint. Therefore, ODE (3.27) corresponds to reasonable continuous adjustment processes for the supply chain network competition model with information asymmetry in quality.

3.2. Qualitative Properties

Since ODE (3.27) is nonstandard due to its discontinuous right-hand side, I further discuss the existence and uniqueness of (3.27). The fundamental theory with regards to existence and uniqueness of projected dynamical systems is provided in Theorem 2.13.

As given in Theorem 2.12, the necessary and sufficient condition for a product shipment and quality level pattern \( X^* = (Q^*, q^*) \) to be a supply chain network equilibrium with information asymmetry in quality, according to Definition 3.1, is that \( X^* = (Q^*, q^*) \) is a stationary point of the adjustment processes defined by ODE (3.27), that is, \( X^* \) is the point at which \( \dot{X} = 0 \).

I now investigate whether, and, under what conditions, the dynamic adjustment processes defined by (3.27) approach a Cournot-Nash equilibrium. Recall that Lipschitz continuity of \( F(X) \) (cf. Definition 2.6) guarantees the existence of a unique solution to (2.19) (cf. Theorem 2.13), where \( X^0(t) \) satisfies ODE (2.19) with initial shipment and quality level pattern \( (Q^0, q^0) \).

In addition, if the utility functions are twice differentiable and the Jacobian matrix of \( F(X) \), denoted by \( \nabla F(X) \), is positive-definite, then the corresponding \( F(X) \) is strictly monotone, and the solution to variational inequality (3.20) is unique, if it exists, according to Theorem 2.5.

Assumption 3.1

Suppose that in the supply chain network model with information asymmetry in quality there exists a sufficiently large \( M \), such that for any \( (i, j, k) \),
\[
\frac{\partial U_i(Q,q)}{\partial Q_{ijk}} < 0,
\]  
(3.28)

for all shipment patterns \(Q\) with \(Q_{ijk} \geq M\) and that there exists a sufficiently large \(\bar{M}\), such that for any \((i,j)\),

\[
\frac{\partial U_i(Q,q)}{\partial q_{ij}} < 0,
\]  
(3.29)

for all quality level patterns \(q\) with \(q_{ij} \geq \bar{M} \geq q_{ij}^\prime\).

I now give an existence result.

**Proposition 3.1**

Any supply chain network problem with information asymmetry in quality that satisfies Assumption 3.1 possesses at least one equilibrium shipment and quality level pattern satisfying variational inequality (3.19) (or (3.20)).

**Proof:** The proof follows from Theorem 2.3 and Proposition 1 in Zhang and Nagurney (1995). □

For completeness, a uniqueness result is now presented, see also Theorem 2.5.

**Proposition 3.2**

Suppose that \(F\) is strictly monotone at any equilibrium point of the variational inequality problem defined in (3.20). Then it has at most one equilibrium point.

In addition, an existence and uniqueness result is recalled (cf. Theorem 2.6).

**Theorem 3.2**

Suppose that \(F\) is strongly monotone. Then there exists a unique solution to variational inequality (3.20); equivalently, to variational inequality (3.19).

Moreover, the stability properties of the utility gradient processes are summarized (cf. Theorems 2.14, 2.15, and 2.16), under various monotonicity conditions on the marginal utilities (cf. Definitions 2.3, 2.4, and 2.5).
Theorem 3.3

(i). If $F(X)$ is monotone, then every supply chain network equilibrium with information asymmetry, $X^*$, provided its existence, is a global monotone attractor for the projected dynamical system. If $F(X)$ is locally monotone at $X^*$, then it is a monotone attractor for the projected dynamical system.

(ii). If $F(X)$ is strictly monotone, then there exists at most one supply chain network equilibrium with information asymmetry in quality, $X^*$. Furthermore, given existence, the unique equilibrium is a strictly global monotone attractor for the projected dynamical system. If $F(X)$ is locally strictly monotone at $X^*$, then it is a strictly monotone attractor for the projected dynamical system.

(iii). If $F(X)$ is strongly monotone, then the unique supply chain network equilibrium with information asymmetry in quality, which is guaranteed to exist, is also globally exponentially stable for the projected dynamical system. If $F(X)$ is locally strongly monotone at $X^*$, then it is exponentially stable.

3.3. Explicit Formulae for the Euler Method Applied to the Supply Chain Network Model with Information Asymmetry in Quality and Quality Competition

Here, I describe the realization of the Euler method, which is fully discussed in Section 2.5.1, for the computation of the solution to variational inequality (3.19).

The Euler method yields, at each iteration, explicit formulae for the computation of the product shipments and quality levels. In particular, for the product shipments, for $i = 1, \ldots, I$; $j = 1, \ldots, n_i$; $k = 1, \ldots, n_R$:
\[ Q_{ijk}^{\tau+1} = \max\{0, Q_{ijk}^{\tau} + a_\tau (\hat{\rho}_k(Q^\tau, q^\tau) + \sum_{l=1}^{n_R} \frac{\partial \hat{\rho}_l(Q^\tau, q^\tau)}{\partial Q_{ijk}} \sum_{h=1}^{n_i} Q_{ihl}^\tau - \sum_{h=1}^{n_i} \frac{\partial \hat{f}_{ih}(Q^\tau, q^\tau)}{\partial Q_{ijk}} \} \]

and the following closed form expressions for the quality levels for \( i = 1, \ldots, I; j = 1, \ldots, n_i; \)

\[ q_{ij}^{\tau+1} = \max\{q_{ij}^\tau, q_{ij}^\tau + a_\tau (\sum_{k=1}^{n_R} \frac{\partial \hat{\rho}_k(Q^\tau, q^\tau)}{\partial q_{ij}} \sum_{h=1}^{n_i} Q_{ihk}^\tau - \sum_{h=1}^{n_i} \frac{\partial \hat{f}_{ih}(Q^\tau, q^\tau)}{\partial q_{ij}} \} \]

I now provide the convergence result. The proof follows using similar arguments as those in Nagurney and Zhang (1996) and Theorem 2.17.

**Theorem 3.4**

In the supply chain network model with information asymmetry in quality, let \( F(X) = -\nabla U(Q, q) \), where all \( U_i; i = 1, \ldots, I, \) are grouped into the vector \( U(Q, q) \), be strictly monotone at any equilibrium shipment pattern and quality levels and assume that Assumption 3.1 is satisfied. Furthermore, assume that \( F \) is uniformly Lipschitz continuous. Then there exists a unique equilibrium product shipment and quality level pattern \( (Q^*, q^*) \in \mathcal{K}^2 \), and any sequence generated by the Euler method as given by (2.34), with explicit formulae at each iteration given by (3.30) and (3.31), where \( \{a_\tau\} \) satisfies \( \sum_{\tau=0}^{\infty} a_\tau = \infty \), \( a_\tau > 0, a_\tau \to 0 \), as \( \tau \to \infty \) converges to \( (Q^*, q^*) \).

**3.4. Numerical Examples and Sensitivity Analysis**

In this section, I present numerical supply chain network examples with information asymmetry in quality, which were solved via the Euler method (cf. (3.30) and (3.31)). I provide a spectrum of examples, accompanied by sensitivity analysis. The
Euler method was implemented using Matlab on a Lenovo E46A. The convergence tolerance is $10^{-6}$, so that the algorithm is deemed to have converged when the absolute value of the difference between each successive product shipment and quality level is less than or equal to $10^{-6}$. The sequence $\{a_\tau\}$ is set to: $\{0.3, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \ldots\}$. I initialized the algorithm by setting the product shipments equal to 20 and the quality levels equal to 0.

**Example 3.1**

The supply chain network topology of Example 3.1 is given in Figure 3.2. There are two firms, both of which have a single manufacturing plant and serve the same demand market $R_1$. The data are as follows.

![Figure 3.2. The Supply Chain Network Topology for Example 3.1](image)

The production cost functions at the manufacturing plants, $M^1_1$ and $M^1_2$, are:

\[
\hat{f}_{11}(Q_{111}, q_{11}) = 0.8Q^2_{111} + 0.5Q_{111} + 0.25Q_{111}q_{11} + 0.5q^2_{11}, \quad (3.32)
\]

\[
\hat{f}_{21}(Q_{211}, q_{21}) = Q^2_{211} + 0.8Q_{211} + 0.3Q_{211}q_{21} + 0.65q^2_{21}. \quad (3.33)
\]

The total transportation cost functions from the plants to the demand market $R_1$ are:

\[
\hat{c}_{111}(Q_{111}, q_{11}) = 1.2Q^2_{111} + Q_{111} + 0.25Q_{211} + 0.25q^2_{11}, \quad (3.34)
\]
\[ \hat{c}_{211}(Q_{211}, q_{21}) = Q_{211}^2 + Q_{211} + 0.35Q_{111} + 0.3q_{21}^2. \]  \hspace{1cm} (3.35)

The demand price function at the demand market \( R_1 \) is:

\[ \hat{\rho}_1(Q, q) = 2250 - (Q_{111} + Q_{211}) + 0.8\hat{q}_1. \]  \hspace{1cm} (3.36)

with the average quality expression given by:

\[ \hat{q}_1 = \frac{Q_{111}q_{11} + Q_{211}q_{21}}{Q_{111} + Q_{211}}. \]  \hspace{1cm} (3.37)

Also, I assume there are no positive imposed minimum quality standards, so that:

\[ q_{11} = q_{21} = 0. \]

The Euler method converges in 437 iterations and yields the following equilibrium solution. The equilibrium product shipments are:

\[ Q_{111}^* = 323.42, \quad Q_{211}^* = 322.72, \]

with the equilibrium demand at the demand market being, hence, \( d_1^* = 646.14 \).

The equilibrium quality levels are:

\[ q_{11}^* = 32.43, \quad q_{21}^* = 16.91, \]

with the average quality level at \( R_1, \hat{q}_1 \), being 24.68.

The incurred demand market price at the equilibrium is:

\[ \hat{\rho}_1 = 1623.60. \]

The profits of the firms are, respectively, 311,926.68 and 313,070.55.
In terms of qualitative analysis, the Jacobian matrix of $F(X) = -\nabla U(Q,q)$, denoted by $J(Q_{111,111}, q_{111,111}, q_{211,211})$, for this problem and evaluated at the equilibrium point $X^* = (Q_{111,111}^*, Q_{211,211}^*, q_{111,111}^*, q_{211,211}^*)$ is:

$$J(Q_{111,111}, Q_{211,211}, q_{111,111}, q_{211,211}) = \begin{pmatrix}
5.99 & 1.01 & -0.35 & -0.20 \\
0.99 & 6.01 & -0.20 & -0.30 \\
-0.35 & 2.00 & 1.50 & 0 \\
0.20 & -0.30 & 0 & 1.90
\end{pmatrix}.$$  

The eigenvalues of $\frac{1}{2}(J + J^T)$ are: 1.47, 1.88, 5.03, and 7.02, and are all positive. Thus, the equilibrium solution is unique, and the conditions for convergence of the algorithm are also satisfied (cf. Theorem 3.4). Moreover, according to Theorem 3.3, the equilibrium solution $X^*$ is exponentially stable.

Then I conducted sensitivity analysis by varying $q_{111}$ and $q_{211}$ beginning with their values set at 0 and increasing them to reflect the imposition of minimum quality standards set to 200, 400, 600, 800, and 1000. I display the results of this sensitivity analysis in Figures 3.3 and 3.4.

As the imposed minimum quality standard of a firm increases, its equilibrium quality level increases (cf. Figures 3.3.c and 3.3.d), which results in increasing production and transportation costs for the firm. Thus, in order to alleviate increasing costs, its equilibrium shipment quantity decreases as does its profit (cf. Figures 3.4.b and 3.4.c). However, due to competition, its competitor’s product shipment increases or at least remains the same (cf. Figures 3.3.a and 3.3.b).

Moreover, since consumers at the demand market do not differentiate between the products from different firms, the average quality level at the demand market as well as the price, which are determined by the quality levels of both firms (cf. (3.36) and (3.37)), are for both firms’ products. Firms prefer a higher average quality, since, at the same demand level, a higher average quality results in a higher price of the
Figure 3.3. Equilibrium Product Shipments, Equilibrium Quality Levels, Average Quality at the Demand Market, and Price at the Demand Market as $q_{11}$ and $q_{21}$ Vary in Example 3.1
Figure 3.4. Demand at $R_1$ and the Profits of the Firms as $q_{11}$ and $q_{21}$ Vary in Example 3.1

product. However, once a firm increases its own quality level, of course, the average quality level and, hence, the price increases, but its total cost will also increase due to the higher quality. Furthermore, the price increase is not only for the firm’s own product, but also for its competitor’s product. If a firm increases its own quality, both the firm and its competitor would get the benefits of the price increase, but only the firm itself would pay for the quality improvement. Thus, a firm prefers a free
ride, that is, it prefers that the other firm improve its product quality and, hence, the price, rather than have it increase its own quality.

Consequently, a firm may not be willing to increase its quality levels, while the other firm is, unless it is beneficial both cost-wise and profit-wise. This explains why, as the minimum quality standard of one firm increases, its competitor’s quality level increases slightly or remains the same (cf. Figures 3.3.c and 3.3.d).

When there is an enforced higher minimum quality standard imposed on a firm’s plant(s), the firm is forced to achieve a higher quality level, which may bring its own profit down but raise the competitor’s profit (cf. Figures 3.4.b and 3.4.c), even though the latter firm may actually face a lower minimum quality standard. When the minimum quality standard of a firm increases to a very high value, but that of its competitor is low, the former firm will not be able to afford the high associated cost with decreasing profit, and, hence, it will produce no product for the demand market and will be forced to leave the market.

The above results and discussion indicate the same result, but in a much more general supply chain network context, as found in Ronnen (1991), who, in speaking about minimum quality standards, on page 492, noted that: “low-quality sellers can be better off ... and high-quality sellers are worse off.” Also the computational results support the statement on page 490 in Akerlof (1970) that “good cars may be driven out of the market by lemons.” Moreover, the results also show that the lower the competitor’s quality level, the more harmful the competitor is to the firm with the high minimum quality standard, as shown in Figures 3.4.b and 3.4.c. The implications of the sensitivity analysis for policy-makers are clear – the imposition of a one-sided quality standard can have a negative impact on the firm in one’s region (or country). Moreover, policy-makers, who are concerned about the products at particular demand markets, should prevent firms located in regions with very low minimum quality standards from entering the market; otherwise, they may not only
bring the average quality level at the demand market(s) down and hurt the consumers, but such products may also harm the profits of the other firms with much higher quality levels and even drive them out of the market.

Therefore, it would be beneficial and fair for both firms and consumers if the policy-makers at the same or different regions or even countries could impose the same or at least similar minimum quality standards on plants serving the same demand market(s). In addition, the minimum quality standards should be such that they will not negatively impact either the high quality firms’ survival or the consumers at the demand market(s).

**Example 3.2**

Example 3.2 is built from Example 3.1. In Example 3.2, there is an additional manufacturing plant available for each of the two firms, and I assume that the new plant for each firm has the same associated data as its original one. This would represent a scenario in which each firm builds an identical plant in proximity to its original one. Thus, the forms of the production cost functions associated with the new plants, $M_1^2$ and $M_2^2$, and the total transportation cost functions associated with the new links to $R_1$ are the same as those for their counterparts in Example 3.1 (but depend on the corresponding variables). This example has the topology given in Figure 3.5.

The data associated with the new plants are as below.

The production cost functions at the new manufacturing plants, $M_1^2$ and $M_2^2$, are:

$$\hat{f}_{12}(Q_{121}, q_{12}) = 0.8Q_{121}^2 + 0.5Q_{121} + 0.25Q_{121}q_{12} + 0.5q_{12}^2,$$

$$\hat{f}_{22}(Q_{221}, q_{22}) = Q_{221}^2 + 0.8Q_{221} + 0.3Q_{221}q_{22} + 0.65q_{22}^2.$$

The total transportation cost functions on the new links are:

$$\hat{c}_{121}(Q_{121}, q_{12}) = 1.2Q_{121}^2 + Q_{121} + 0.25Q_{221} + 0.25q_{12}^2,$$
The demand price function retains its functional form, but with the new potential shipments added so that:

\[ \hat{\rho}_1 = 2250 - (Q_{111} + Q_{211} + Q_{121} + Q_{221}) + 0.8\hat{q}_1, \]

with the average quality at \( R_1 \) expressed as:

\[ \hat{q}_1 = \frac{Q_{111}q_{11} + Q_{211}q_{21} + Q_{121}q_{12} + Q_{221}q_{22}}{Q_{111} + Q_{211} + Q_{121} + Q_{221}}. \]

Also, at the new manufacturing plants I have that, as in the original ones:

\[ q_{12} = q_{22} = 0. \]

The Euler method converges in 408 iterations to the following equilibrium solution. The equilibrium product shipments are:

\[ Q_{111}^* = 225.96, \quad Q_{121}^* = 225.96, \quad Q_{211}^* = 225.54, \quad Q_{221}^* = 225.54. \]

The equilibrium demand at \( R_1 \) is, hence, \( d_1^* = 903. \)
The equilibrium quality levels are:

\[ q_{11}^* = 22.65, \quad q_{12}^* = 22.65, \quad q_{21}^* = 11.83, \quad q_{22}^* = 11.83, \]

with the average quality level, \( \hat{q}_1 \), now equal to 17.24. Note that the average quality level has dropped precipitously from its value of 24.68 in Example 3.1.

The incurred demand market price at \( R_1 \) is:

\[ \hat{\rho}_1 = 1,360.78. \]

The profits of the firms are, respectively, 406,615.47 and 407,514.97.

I now discuss the results. Since, for each firm, its new manufacturing plant and the original one are assumed to be identical, the equilibrium product shipments and the quality levels associated with the two plants are identical for each firm.

The availability of an additional manufacturing plant for each firm leads to the following results. First, the total cost of manufacturing and transporting the same amount of products is now less than in Example 3.1 for each firm, which can be verified by substituting \( Q_{111} + Q_{121} \) for \( Q_{111} \) and \( Q_{211} + Q_{221} \) for \( Q_{211} \) in (3.32) - (3.35) and comparing the total cost of each firm in Example 3.1 with that in Example 3.2. Hence, although the product shipments produced by the same manufacturing plant decrease in comparison to the associated values in Example 3.1, the total amount supplied by each firm increases, as does the total demand. The strategy of building an identical plant at the same location as the original one appears to be cost-wise and profitable for the firms; however, at the expense of a decrease in the average quality level at the demand market, as reflected in the results for Example 3.2. Policy-makers may wish to take note of this.
The Jacobian matrix of \( F(X) = -\nabla U(Q, q) \), \( J(Q_{111}, Q_{121}, Q_{211}, Q_{221}, q_{11}, q_{12}, q_{21}, q_{22}) \), and evaluated at \( X^* \) for Example 3.2, is

\[
J(Q_{111}, Q_{121}, Q_{211}, Q_{221}, q_{11}, q_{12}, q_{21}, q_{22}) = \begin{bmatrix}
5.99 & 1.99 & 1.00 & 1.00 & -0.25 & -0.10 & -0.10 & -0.10 \\
1.00 & 6.00 & 1.00 & 1.00 & -0.10 & -0.25 & -0.10 & -0.10 \\
1.00 & 1.00 & 6.00 & 2.01 & -0.10 & -0.10 & -0.20 & -0.10 \\
1.00 & 1.00 & 2.00 & 6.00 & -0.10 & -0.10 & -0.10 & -0.20 \\
-0.25 & -0.10 & 0.10 & 0.10 & 1.50 & 0 & 0 & 0 \\
-0.10 & -0.25 & 0.10 & 0.10 & 0 & 1.50 & 0 & 0 \\
0.10 & 0.10 & -0.20 & -0.10 & 0 & 0 & 1.90 & 0 \\
0.10 & 0.10 & -0.10 & -0.20 & 0 & 0 & 0 & 1.90 \\
\end{bmatrix}
\]

The Jacobian matrix for this example is strictly diagonally dominant, which guarantees its positive-definiteness. Thus, the equilibrium solution \( X^* \) is unique, the conditions for convergence of the algorithm are satisfied, and the equilibrium solution is exponentially stable.

**Example 3.3**

Example 3.3 is constructed from Example 3.2, but now the new plant for firm 1, \( M^2_1 \), is located in a country where the production cost is much lower but the total transportation cost to the demand market \( R_1 \) is higher, in comparison to the data in Example 3.2. In addition, the location of the second plant of firm 2, \( M^2_2 \), also changes, resulting in both a higher production cost and a higher transportation cost to \( R_1 \). Thus, the new manufacturing plants for each firm now have different associated cost functions as given below.

The production cost functions of the new plants, \( M^2_1 \) and \( M^2_2 \), are:

\[
\hat{f}_{12}(Q_{121}, q_{12}) = 0.3Q_{121}^2 + 0.1Q_{121} + 0.3Q_{121}q_{12} + 0.4q_{12}^2
\]
\( \hat{f}_{22}(Q_{221}, q_{22}) = 1.2Q_{221}^2 + 0.5Q_{221} + 0.3Q_{221}q_{22} + 0.5q_{22}^2. \)

The total transportation cost functions on the new links are now:

\( \hat{c}_{121}(Q_{121}, q_{12}) = 1.8Q_{121}^2 + Q_{121} + 0.25Q_{221} + 0.25q_{12}^2, \)

\( \hat{c}_{221}(Q_{221}, q_{22}) = 1.5Q_{221}^2 + 0.8Q_{221} + 0.3Q_{121} + 0.3q_{22}^2. \)

The Euler method converges in 498 iterations, yielding the equilibrium product shipments:

\[
Q_{111}^* = 232.86, \quad Q_{121}^* = 221.39, \quad Q_{211}^* = 240.82, \quad Q_{221}^* = 178.45,
\]

with an equilibrium demand \( d_1^* = 873.52. \) The equilibrium quality levels are:

\[
q_{111}^* = 25.77, \quad q_{121}^* = 19.76, \quad q_{211}^* = 10.64, \quad q_{221}^* = 9.37,
\]

with the average quality level at \( R_1, \hat{q}_1, \) equal to 16.73. The incurred demand market price is

\( \hat{\rho}_1 = 1,389.86. \)

The profits of the firms are, respectively, 415,706.05 and 378,496.95,

Although the production cost of firm 1’s foreign plant, \( M_1^2, \) is lower than that of the original plant, \( M_1^1, \) because of the high transportation cost to the demand market, the quantity produced at and shipped from \( M_1^2 \) decreases, in comparison to the value in Example 3.2. In addition, because of the higher manufacturing cost at firm 2’s foreign plant, \( M_2^2, \) the total supply of the product from firm 2 now decreases. The other results are: the demand at demand market \( R_1 \) decreases and the average quality there decreases slightly.
The Jacobian matrix of $F(X) = -\nabla U(Q, q)$ at equilibrium, denoted by $J(Q_{111}, Q_{121}, Q_{211}, Q_{221}, q_{11}, q_{12}, q_{21}, q_{22})$, for this example, is

$$J(Q_{111}, Q_{121}, Q_{211}, Q_{221}, q_{11}, q_{12}, q_{21}, q_{22}) = \begin{pmatrix}
5.99 & 1.99 & 1.01 & 1.01 & -0.27 & -0.10 & -0.11 & -0.08 \\
1.99 & 6.20 & 1.00 & 1.00 & -0.10 & -0.21 & -0.11 & -0.08 \\
0.99 & 1.00 & 6.01 & 2.01 & -0.11 & -0.11 & -0.20 & -0.08 \\
0.99 & 1.00 & 2.01 & 7.41 & -0.11 & -0.11 & -0.11 & -0.17 \\
-0.27 & -0.10 & 0.11 & 0.11 & 1.50 & 0 & 0 & 0 \\
-0.10 & -0.21 & 0.11 & 0.11 & 0 & 1.30 & 0 & 0 \\
0.11 & 0.11 & -0.20 & -0.11 & 0 & 0 & 1.90 & 0 \\
0.08 & 0.08 & -0.08 & -0.17 & 0 & 0 & 0 & 1.60
\end{pmatrix}.$$  

This Jacobian matrix is strictly diagonally dominant. Thus, it is positive-definite, and the uniqueness of the computed equilibrium is guaranteed. Also, the conditions for convergence of the algorithm are satisfied. The equilibrium solution for Example 3.3 has the same qualitative properties as the solution to Example 3.2.

**Example 3.4**

Example 3.4 considers the following scenario. Please refer to Figure 3.6 for the supply chain network topology for this example. There is a new demand market, $R_2$, added to Example 3.3, which is located closer to both firms’ manufacturing plants than the original demand market $R_1$. The total transportation cost functions for transporting the product to $R_2$ for both firms, respectively, are:

$$\hat{c}_{112}(Q_{112}, q_{11}) = 0.8Q_{112}^2 + Q_{112} + 0.2Q_{212} + 0.05q_{11}^2, \quad (3.38)$$

$$\hat{c}_{122}(Q_{122}, q_{12}) = 0.75Q_{122}^2 + Q_{122} + 0.25Q_{222} + 0.03q_{12}^2, \quad (3.39)$$
The production cost functions at the manufacturing plants have the same functional forms as in Example 3.3, but now they include the additional shipments to the new demand market, $R_2$, that is:

\[
\hat{c}_{212}(Q_{212}, q_{21}) = 0.6Q_{212}^2 + Q_{212} + 0.3Q_{112} + 0.02q_{21}^2, \quad (3.40)
\]

\[
\hat{c}_{222}(Q_{222}, q_{22}) = 0.5Q_{222}^2 + 0.8Q_{222} + 0.25Q_{122} + 0.05q_{22}^2. \quad (3.41)
\]

The production cost functions at the manufacturing plants have the same functional forms as in Example 3.3, but now they include the additional shipments to the new demand market, $R_2$, that is:

\[
\hat{f}_{12}(Q_{121}, Q_{122}, q_{12}) = 0.3(Q_{121} + Q_{122})^2 + 0.1(Q_{121} + Q_{122}) + 0.3(Q_{121} + Q_{122})q_{12} + 0.4q_{12}^2,
\]

\[
\hat{f}_{22}(Q_{221}, Q_{222}, q_{22}) = 1.2(Q_{221} + Q_{222})^2 + 0.5(Q_{221} + Q_{222}) + 0.3(Q_{221} + Q_{222})q_{22} + 0.5q_{22}^2.
\]

\[
\hat{f}_{11}(Q_{111}, Q_{112}, q_{11}) = 0.8(Q_{111} + Q_{112})^2 + 0.5(Q_{111} + Q_{112}) + 0.25(Q_{111} + Q_{112})q_{11} + 0.5q_{11}^2,
\]

\[
\hat{f}_{21}(Q_{211}, Q_{212}, q_{21}) = (Q_{211} + Q_{212})^2 + 0.8(Q_{211} + Q_{212}) + 0.3(Q_{211} + Q_{212})q_{21} + 0.65q_{21}^2.
\]

In addition, consumers at the new demand market $R_2$ are more sensitive to the quality of the product than consumers at the original demand market $R_1$. The demand price functions for both the demand markets are, respectively:

\[
\hat{\rho}_1 = 2250 - (Q_{111} + Q_{211} + Q_{121} + Q_{221}) + 0.8q_{11},
\]
\[ \hat{\rho}_2 = 2250 - (Q_{112} + Q_{122} + Q_{212} + Q_{222}) + 0.9 \hat{q}_2, \]

where
\[ \hat{q}_1 = \frac{Q_{111}q_{11} + Q_{211}q_{21} + Q_{121}q_{12} + Q_{221}q_{22}}{Q_{111} + Q_{211} + Q_{121} + Q_{221}}, \]

and
\[ \hat{q}_2 = \frac{Q_{112}q_{11} + Q_{212}q_{21} + Q_{122}q_{12} + Q_{222}q_{22}}{Q_{112} + Q_{212} + Q_{122} + Q_{222}}. \]

The Euler method converges in 597 iterations, and the equilibrium solution is as below. The equilibrium product shipments are:

\[ Q_{111}^* = 208.70, \quad Q_{121}^* = 211.82, \quad Q_{211}^* = 203.90, \quad Q_{221}^* = 129.79, \]
\[ Q_{112}^* = 165.39, \quad Q_{122}^* = 352.11, \quad Q_{212}^* = 182.30, \quad Q_{222}^* = 200.05. \]

The equilibrium demands at the two demand markets are now \( d_1^* = 754.21 \) and \( d_2^* = 899.85. \)

The equilibrium quality levels are:

\[ q_{11}^* = 53.23, \quad q_{12}^* = 79.08, \quad q_{21}^* = 13.41, \quad q_{22}^* = 13.82. \]

The value of \( \hat{q}_1 \) is 42.94 and that of \( \hat{q}_2 \) is 46.52.

The incurred demand market prices are:

\[ \hat{\rho}_1 = 1,530.15, \quad \hat{\rho}_2 = 1,392.03. \]

The profits of the firms are, respectively, 882,342.15 and 651,715.83.

Due to the addition of \( R_2 \), which has associated lower transportation costs, each firm ships more product to demand market \( R_2 \) than to \( R_1 \), and, at the same time, some of the previous demand at \( R_1 \) is shifted to \( R_2 \). Hence, the total demand \( d_1 + d_2 \) is now 88.76\% larger than the total demand \( d_1 \) in Example 3.2.
In addition, firm 1 is the one with larger market shares, and is able to achieve higher profit by attaining higher quality levels. Thus, as the total demand increases, the quality levels of firm 1 increase significantly. However, since it is not cost-wise for firm 2 to do so, due to its higher costs and lower market shares, firm 2 prefers a “free ride” from firm 1 with its quality levels basically remaining the same. The average quality levels, nevertheless, increase substantially anyway, which leads to the increase in the prices and both firms’ profits.

The Jacobian matrix of \(-\nabla U(Q, q)\), for Example 3.4, evaluated at the equilibrium, and denoted by \(J(Q_{111}, Q_{121}, Q_{211}, Q_{221}, Q_{112}, Q_{122}, Q_{212}, Q_{222}, q_{11}, q_{12}, q_{21}, q_{22})\), is

\[
J(Q_{111}, Q_{121}, Q_{211}, Q_{221}, Q_{112}, Q_{122}, Q_{212}, Q_{222}, q_{11}, q_{12}, q_{21}, q_{22}) =
\begin{pmatrix}
5.99 & 1.98 & 1.02 & 1.02 & 1.60 & 0 & 0 & 0 & -0.29 & -0.10 & -0.10 & -0.06 \\
1.98 & 6.17 & 1.04 & 1.04 & 0 & 0.60 & 0 & 0 & -0.10 & -0.25 & -0.10 & -0.06 \\
0.98 & 0.96 & 6.03 & 2.03 & 0 & 0 & 2.00 & 0 & -0.12 & -0.13 & -0.17 & -0.08 \\
0.98 & 0.96 & 2.03 & 7.43 & 0 & 0 & 0 & 2.40 & -0.12 & -0.13 & -0.12 & -0.13 \\
1.60 & 0 & 0 & 0 & 5.19 & 1.98 & 1.02 & 1.02 & -0.34 & -0.15 & -0.08 & -0.09 \\
0 & 0.60 & 0 & 0 & 1.98 & 4.07 & 1.03 & 1.03 & -0.07 & -0.37 & -0.08 & -0.09 \\
0 & 0 & 2.00 & 0 & 0.98 & 0.97 & 5.24 & 2.04 & -0.10 & -0.20 & -0.19 & -0.12 \\
0 & 0 & 0 & 2.40 & 0.98 & 0.97 & 2.04 & 5.44 & -0.10 & -0.20 & -0.10 & -0.20 \\
-0.29 & -0.10 & 0.12 & 0.12 & -0.34 & -0.07 & 0.10 & 0.10 & 1.60 & 0 & 0 & 0 \\
-0.10 & -0.25 & 0.13 & 0.13 & -0.15 & -0.37 & 0.20 & 0.20 & 1.36 & 0 & 0 & 0 \\
0.10 & 0.10 & -0.17 & -0.12 & 0.08 & 0.08 & -0.19 & -0.10 & 0 & 0 & 1.94 & 0 \\
0.06 & 0.06 & -0.08 & -0.13 & 0.09 & 0.09 & -0.12 & -0.20 & 0 & 0 & 0 & 1.70 \\
\end{pmatrix}
\]

The eigenvalues of \(\frac{1}{2}(J + J^T)\) are all positive and are: 1.29, 1.55, 1.66, 1.71, 1.93, 2.04, 3.76, 4.73, 6.14, 7.55, 8.01, and 11.78. Therefore, both the uniqueness of the equilibrium solution and the conditions for convergence of the algorithm are guaranteed. The equilibrium solution to Example 3.4 is exponentially stable.

I now explore the impact of the firms’ proximity to the second demand market \(R_2\). I multiply the coefficient of the second \(Q_{ijk}\) term, that is, the linear one, in each of the transportation cost functions \(\hat{c}_{ijk} (3.38) – (3.41)\) by a positive factor \(\beta\), but retain the other transportation cost functions as in Example 3.4. I vary \(\beta\) from 0 to 50, 100, 150, 200, 250, 300, and 350. The results are reported in Figure 3.7.

As \(\beta\) increases, that is, as \(R_2\) is located farther, the transportation costs to \(R_2\) increase. In order to decrease their total costs and increase their profits, firms ship
Figure 3.7. The Equilibrium Demands, Average Quality Levels, Prices at the Demand Markets, and the Profits of the Firms as $\beta$ Varies in Example 3.4

less of the product to $R_2$ while their shipments to $R_1$ increase, as shown in Figure 3.7.a. In addition, at the same time, firms cannot afford higher quality as the total costs of both firms increase, so the average quality levels at both demand markets decrease, as indicated in Figure 3.7.b. Due to the changes in the demands and the average quality levels, the price at $R_1$ decreases, but that at $R_2$ increases, and the profits of both firms decrease, as in Figures 3.7.c and 3.7.d. When $\beta = 350$, demand market $R_2$ will be removed from the supply chain network, due to the demand there
dropping to zero. Thus, when $\beta = 350$, the results of Example 3.4 are the same as those for Example 3.3.

The numerical examples in this section, along with the sensitivity analysis results, reveal the type of questions that can be explored and addressed through computations. Moreover, the analyses demonstrate the impacts of minimum quality standards even “across borders” as well as the importance of the location of manufacturing plants vis-a-vis the demand markets. The insights gained from the numerical examples are useful to firms, to consumers at demand markets, as well as to policy-makers.

3.5. Summary and Conclusions

In this chapter, I developed a framework for the modeling, analysis, and computation of solutions to competitive supply chain network problems in static and dynamic settings in which there is information asymmetry in quality. I also demonstrated how this framework can capture the inclusion of policy interventions in the form of minimum quality standards.

This chapter adds to the literature on information asymmetry with imperfect competition, and which has focused on analytical results for stylized problems. It also contributes to the literature on supply chains with quality competition and reveals the spectrum of insights that can be obtained through computations, supported by theoretical analysis. Finally, it contributes to the integration of economics with operations research and the management sciences.

The model discussed in this chapter focuses on information asymmetry in quality without product differentiation. As revealed by the model and the numerical examples, without product differentiation, no matter how high a firm’s quality level(s) is, the quality level perceived by consumers at the demand market is only the average quality level of all firms. Thus, it is important for firms to differentiate their products from the other products, in order to prevent the “free ride” from the other
firms. In the next chapter, Chapter 4, a dynamic Cournot-Nash model with quality competition and product differentiation is established. Numerical examples are also provided.
CHAPTER 4

A SUPPLY CHAIN NETWORK MODEL WITH TRANSPORTATION COSTS, PRODUCT DIFFERENTIATION, AND QUALITY COMPETITION

In this chapter, I develop both static and dynamic models of Cournot-Nash competition that include production and transportation costs, product differentiation, and quality levels in a supply chain network framework. The production costs capture the cost of quality, which, in turn, can also represent the R&D cost. I first present the equilibrium version of the model and derive alternative variational inequality formulations. The projected dynamical systems model, which provides a continuous-time evolution of the firms’ product shipments and product quality levels, is also constructed. In addition, I establish stability analysis results using theorems given in Section 2.3, and construct a discrete-time version of the continuous-time adjustment process using the Euler method discussed in Section 2.5.1. This algorithm is then utilized to compute solutions to several numerical examples.

The framework established in this chapter can serve as the foundation for the modeling and analysis of competition among firms in industries ranging from food to pharmaceuticals to durable goods and high tech products, as well as Internet services, where quality and product differentiation are seminal.

In this chapter, the supply chain network model of Cournot-Nash competition with transportation costs, product differentiation, and quality levels is established without minimum quality standards (cf. Chapter 3). However, using the methodologies provided in Chapter 2, minimum quality standards can be easily embedded in
both the static and the dynamic models as well as the computational procedure of the models.

This chapter is based on Nagurney and Li (2014b), and is organized as follows. In Section 4.1, I present the static version of the supply chain network model, and establish alternative variational inequality formulations of the governing Cournot-Nash equilibrium conditions. I then present its dynamic counterpart and show that the set of equilibria coincides with the set of stationary points of the projected dynamical systems supply chain network model. In Section 4.2, I present stability analysis results, and illustrate the concepts with several numerical examples. In Section 4.3, I propose the discrete-time adjustment process, which provides an approximation to the continuous-time trajectories of the firm’s product shipments and quality levels over time. In Section 4.4, the algorithm is applied to track the trajectories over time and to compute the stationary points; equivalently, the equilibria. I summarize the results and present the conclusions in Section 4.5.

4.1. The Supply Chain Network Model with Transportation Costs, Product Differentiation, and Quality Competition

In this section, I develop a supply chain network model with product differentiation and quality competition. It is assumed that the firms compete under the Cournot-Nash equilibrium concept of non-cooperative behavior and select both their product shipments as well as the quality levels of their products. The consumers, in turn, signal their preferences for the products through the demand price functions associated with the demand markets, which are spatially separated. The demand price functions are, in general, functions of the demands for the products at all the demand markets as well as the quality levels of the products.
Figure 4.1. The Supply Chain Network Topology

I first develop the equilibrium model and derive the variational inequality formulation. Then, the underlying dynamics associated with the firms’ production outputs as well as quality levels and the projected dynamical systems model whose set of stationary points corresponds to the set of solutions of the variational inequality problem is presented.

Please refer to Figure 4.1 for the underlying network structure of the supply chain network with product differentiation.

There are $I$ firms and $n_R$ demand markets that are generally spatially separated. There is a distinct (but substitutable) product produced by each of the $I$ firms and is consumed at the $n_R$ demand markets. Let $s_i$ denote the nonnegative product output produced by firm $i$ and let $d_{ik}$ denote the demand for the product of firm $i$ at demand market $R_k$. Let $Q_{ik}$ denote the nonnegative shipment of firm $i$‘s product to demand market $R_k$. I group the production outputs into the vector $s \in R^I_+$, the demands into the vector $d \in R^{I \times n_R}_+$, and the product shipments into the vector $Q \in R^{I \times n_R}_+$. Here $q_i$ denotes the quality level, or, simply, the quality, of product $i$, which is produced by firm $i$. The quality levels of all firms are grouped into the vector $q \in R^I_+$. All vectors here are assumed to be column vectors, except where noted.

The following conservation of flow equations must hold:
\[ s_i = \sum_{k=1}^{n_R} Q_{ik}, \quad i = 1, \ldots, I; \quad (4.1) \]
\[ d_{ik} = Q_{ik}, \quad i = 1, \ldots, I; \quad k = 1, \ldots, n_R, \quad (4.2) \]
\[ Q_{ik} \geq 0, \quad i = 1, \ldots, I; \quad k = 1, \ldots, n_R, \quad (4.3) \]

and since the quality levels must also be nonnegative, as in Chapter 3 (cf. (3.4)),

\[ q_i \geq 0, \quad i = 1, \ldots, I. \quad (4.4) \]

Hence, the quantity of the product produced by each firm is equal to the sum of the amounts shipped to all the demand markets, and the quantity of a firm’s product consumed at a demand market is equal to the amount shipped from the firm to that demand market. Both the shipment volumes and the quality levels must be nonnegative.

A production cost \( \hat{f}_i \) is associated with each firm \( i \). Although the products produced by the firms are differentiated by brands, they are still substitutes. Common factors maybe utilized in the process of manufacturing. Hence, I allow for the general situation where the production cost of a firm \( i \) may depend upon the entire production pattern and on its own quality level, that is,

\[ \hat{f}_i = \hat{f}_i(s, q_i), \quad i = 1, \ldots, I. \quad (4.5) \]

It is assumed here that the functions in (4.5) also capture the quality cost (cf. Section 1.1), since, as a special case, the above functions can take on the form

\[ \hat{f}_i(s, q_i) = f_i(s, q_i) + g_i(q_i), \quad i = 1, \ldots, I, \quad (4.6) \]

where the first term depends on both quality and production outputs and the second term only depends on the quality. Interestingly, the second term in (4.6) can also be
interpreted as the R&D cost of the firm, which is assumed to depend on the quality level of its products (cf. Section 1.1).

The production cost functions (4.5) (and (4.6)) are assumed to be convex and twice continuously differentiable.

Similar as in Chapter 3 (cf. 3.8a and b), the demand price function for a product at a demand market is presented by function (4.7):

\[ \rho_{ik} = \rho_{ik}(d, q), \quad i = 1, \ldots, I; k = 1, \ldots, n_R. \]  

Unlike functions (3.8a) and (3.8b) in Chapter 3, (4.7) is the demand price function under product differentiation. It also depends, in general, not only on the entire consumption pattern, but also on all the levels of quality of all the products.

The demand price functions are, typically, assumed to be monotonically decreasing in product quantity but increasing in terms of product quality.

Let \( \hat{c}_{ik} \) denote the total transportation cost associated with shipping firm \( i \)'s product to demand market \( R_k \), where the total transportation cost is given by the function:

\[ \hat{c}_{ik} = \hat{c}_{ik}(Q_{ik}), \quad i = 1, \ldots, I; k = 1, \ldots, n_R. \]  

In contrast to the transportation cost functions in Chapter 3 (cf. (3.6)), in (4.8), I assume that transportation activities will not affect product quality.

The demand price functions (4.7) and the total transportation cost functions (4.8) are assumed to be continuous and twice continuously differentiable.

The strategic variables of firm \( i \) are its product shipments \( \{Q_i\} \) where \( Q_i = (Q_{i1}, \ldots, Q_{in_R}) \) and its quality level \( q_i \). The profit or utility \( U_i \) of firm \( i; i = 1, \ldots, I \), is, hence, given by the expression

\[ U_i = \sum_{k=1}^{n_R} \rho_{ik}d_{ik} - \hat{f}_i - \sum_{k=1}^{n_R} \hat{c}_{ik}, \]  

89
which is the difference between its total revenue and its total cost.

In view of (4.1) - (4.9), one may write the profit as a function solely of the shipment pattern and quality levels, that is,

$$ U = U(Q, q), \quad (4.10) $$

where $U$ is the $I$-dimensional vector with components: $\{U_1, \ldots, U_I\}$.

Let $K^i$ denote the feasible set corresponding to firm $i$, where $K^i \equiv \{(Q_i, q_i)|Q_i \geq 0, \text{ and } q_i \geq 0\}$ and define $K \equiv \prod_{i=1}^{I} K^i$.

I consider the market mechanism in which the $I$ firms supply their products in a non-cooperative fashion, each one trying to maximize its own profit. They seek to determine a nonnegative product shipment and quality level pattern $(Q^*, q^*)$ for which the $I$ firms will be in a state of equilibrium as defined below (cf. Definition 2.7).

**Definition 4.1: A Supply Chain Network Cournot-Nash Equilibrium with Product Differentiation and Quality Levels**

A product shipment and quality level pattern $(Q^*, q^*) \in K$ is said to constitute a supply chain network Cournot-Nash equilibrium if for each firm $i; i = 1, \ldots, I$,

$$ U_i(Q^*_i, q^*_i, \hat{Q}^*_i, \hat{q}^*_i) \geq U_i(Q_i, q_i, \hat{Q}^*_i, \hat{q}^*_i), \quad \forall (Q_i, q_i) \in K^i, \quad (4.11) $$

where

\[ \hat{Q}^*_i \equiv (Q^*_1, \ldots, Q^*_{i-1}, Q^*_{i+1}, \ldots, Q^*_I); \quad \text{and} \quad \hat{q}^*_i \equiv (q^*_1, \ldots, q^*_{i-1}, q^*_{i+1}, \ldots, q^*_I). \quad (4.12) \]

According to (4.11), an equilibrium is established if no firm can unilaterally improve upon its profits by selecting an alternative vector of product shipments and quality level of its product.
I now present alternative variational inequality formulations of the above supply chain network Cournot-Nash equilibrium with product differentiation in the following theorem.

**Theorem 4.1**

Assume that, for each firm $i$, the profit function $U_i(Q, q)$ is concave with respect to the variables $\{Q_{i1}, \ldots, Q_{im_R}\}$, and $q_i$, and is continuous and continuously differentiable. Then $(Q^*, q^*) \in K$ is a supply chain network Cournot-Nash equilibrium according to Definition 4.1 if and only if it satisfies the variational inequality

$$-\sum_{i=1}^{I} \sum_{k=1}^{n_R} \frac{\partial U_i(Q^*, q^*)}{\partial Q_{ik}} \times (Q_{ik} - Q^*_{ik}) - \sum_{i=1}^{I} \frac{\partial U_i(Q^*, q^*)}{\partial q_i} \times (q_i - q^*_i) \geq 0, \forall (Q, q) \in K,$$

(4.13)

or, equivalently, $(s^*, Q^*, d^*, q^*) \in K^1$ is an equilibrium production, shipment, consumption, and quality level pattern if and only if it satisfies the variational inequality

$$\sum_{i=1}^{I} \frac{\partial \hat{f}_i(s^*, q^*_i)}{\partial s_i} \times (s_i - s^*_i) + \sum_{i=1}^{I} \sum_{k=1}^{n_R} \left[ \frac{\partial \hat{c}_{ik}(Q^*_{ik})}{\partial Q_{ik}} - \sum_{l=1}^{n_R} \frac{\partial \rho_{il}(d^*, q^*)}{\partial d_{ik}} \times d^*_il \right] \times (Q_{ik} - Q^*_{ik})$$

$$-\sum_{i=1}^{I} \sum_{k=1}^{n_R} \rho_{ik}(d^*, q^*) \times (d_{ik} - d^*_{ik})$$

$$+ \sum_{i=1}^{I} \left[ \frac{\partial \hat{f}_i(s^*, q^*_i)}{\partial q_i} - \sum_{l=1}^{n_R} \frac{\partial \rho_{il}(d^*, q^*)}{\partial q_i} \times d^*_il \right] \times (q_i - q^*_i) \geq 0, \forall (s, Q, d, q) \in K^1,$$

(4.14)

where $K^1 \equiv \{(s, Q, d, q) | Q \geq 0, q \geq 0, \text{ and } (4.1) \text{ and } (4.2) \text{ hold}\}$.

**Proof:** (4.13) follows directly from Gabay and Moulin (1980) and Dafermos and Nagurney (1987), as given in Theorem 2.7.

In order to obtain variational inequality (4.14) from variational inequality (4.13), it is noted that:
\[-\frac{\partial U_i}{\partial Q_{ik}} = \left[ \frac{\partial \hat{f}_i}{\partial s_i} + \frac{\partial \hat{c}_{ik}}{\partial Q_{ik}} - \rho_{ik} - \sum_{l=1}^{n_R} \frac{\partial \rho_{il}}{\partial d_{ik}} \times d_{il} \right]; \quad i = 1, \ldots, I; \ k = 1, \ldots, n_R; \]

(4.15)

and

\[-\frac{\partial U_i}{\partial q_i} = \left[ \frac{\partial \hat{f}_i}{\partial q_i} - \sum_{k=1}^{n_R} \frac{\partial \rho_{il}}{\partial q_i} \times d_{il} \right]; \quad i = 1, \ldots, I. \]

(4.16)

Multiplying the right-most expression in (4.15) by \((Q_{ik} - Q_{ik}^*)\) and summing the resultant over all \(i\) and all \(k\); similarly, multiplying the right-most expression in (4.16) by \((q_i - q_i^*)\) and summing the resultant over all \(i\) yields, respectively:

\[\sum_{i=1}^{I} \sum_{k=1}^{n_R} \left[ \frac{\partial \hat{f}_i}{\partial s_i} + \frac{\partial \hat{c}_{ik}}{\partial Q_{ik}} - \rho_{ik} - \sum_{l=1}^{n_R} \frac{\partial \rho_{il}}{\partial d_{ik}} \times d_{il} \right] \times (Q_{ik} - Q_{ik}^*) \]

(4.17)

and

\[\sum_{i=1}^{I} \left[ \frac{\partial \hat{f}_i}{\partial q_i} - \sum_{l=1}^{n_R} \frac{\partial \rho_{il}}{\partial q_i} \times d_{il} \right] \times (q_i - q_i^*). \]

(4.18)

Finally, summing (4.17) and (4.18) and then using constraints (4.1) and (4.2), yields variational inequality (4.14). \(\Box\)

I now put the above supply chain network equilibrium problem with product differentiation and quality levels into standard variational inequality form, as in (2.1a), that is,

Determine \(X^* \in \mathcal{K} \subset \mathbb{R}^N\), such that

\[\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K}, \]

(4.19)

where \(F\) is a given continuous function from \(\mathcal{K}\) to \(\mathbb{R}^N\) and \(\mathcal{K}\) is a closed and convex set.
I define the \((In_R+I)\)-dimensional vector \(X \equiv (Q, q)\) and the \((In_R+I)\)-dimensional row vector \(F(X) = (F^1(X), F^2(X))\) with the \((i, k)\)-th component, \(F^1_{ik}\), of \(F^1(X)\) given by

\[
F^1_{ik}(X) \equiv -\frac{\partial U_i(Q,q)}{\partial Q_{ik}}, \tag{4.20}
\]

the \(i\)-th component, \(F^2_i\), of \(F^2(X)\) given by

\[
F^2_i(X) \equiv -\frac{\partial U_i(Q,q)}{\partial q_i}, \tag{4.21}
\]

and with the feasible set \(\mathcal{K} \equiv K\). Then, clearly, variational inequality (4.13) can be put into standard form (2.1a).

In a similar manner, one can establish that variational inequality (4.14) can also be put into standard variational inequality form (2.1a).

I now propose a dynamic adjustment process for the evolution of the firms’ product shipments and product quality levels. Observe that, for a current product shipment and quality level pattern at time \(t\), \(X(t) = (Q(t), q(t))\), \(-F^1_{ik}(X(t)) = \frac{\partial U_i(Q(t),q(t))}{\partial Q_{ik}}\), given by (4.20), is the marginal utility (profit) of firm \(i\) with respect to its product shipment to demand market \(R_k\). Similarly, \(-F^2_i(X(t)) = \frac{\partial U_i(Q(t),q(t))}{\partial q_i}\), given by (4.21), is the firm’s marginal utility (profit) with respect to its quality level. In this framework, the rate of change of the product shipment between a firm and demand market pair \((i, k)\) is in proportion to \(-F^1_{ik}(X)\), as long as the product shipment \(Q_{ik}\) is positive. Namely, when \(Q_{ik} > 0\),

\[
\dot{Q}_{ik} = \frac{\partial U_i(Q,q)}{\partial Q_{ik}}, \tag{4.22}
\]

where \(\dot{Q}_{ik}\) denotes the rate of change of \(Q_{ik}\). However, when \(Q_{ik} = 0\), the nonnegativity condition (4.3) forces the product shipment \(Q_{ik}\) to remain zero when \(\frac{\partial U_i(Q,q)}{\partial Q_{ik}} \leq 0\).
Hence, in this case, One is only guaranteed of having possible increases of the shipment. Namely, when $Q_{ik} = 0$,

$$
\dot{Q}_{ik} = \max\{0, \frac{\partial U_i(Q, q)}{\partial Q_{ik}}\}.
$$

(4.23)

One may write (4.22) and (4.23) concisely as:

$$
\dot{Q}_{ik} = \begin{cases} 
\frac{\partial U_i(Q, q)}{\partial Q_{ik}}, & \text{if } Q_{ik} > 0 \\
\max\{0, \frac{\partial U_i(Q, q)}{\partial Q_{ik}}\}, & \text{if } Q_{ik} = 0.
\end{cases}
$$

(4.24)

As for the quality levels, when $q_i > 0$, then

$$
\dot{q}_i = \frac{\partial U_i(Q, q)}{\partial q_i},
$$

(4.25)

where $\dot{q}_i$ denotes the rate of change of $q_i$; otherwise:

$$
\dot{q}_i = \max\{0, \frac{\partial U_i(Q, q)}{\partial q_i}\},
$$

(4.26)

since $q_i$ must be nonnegative.

Combining (4.25) and (4.26), one may write:

$$
\dot{q}_i = \begin{cases} 
\frac{\partial U_i(Q, q)}{\partial q_i}, & \text{if } q_i > 0 \\
\max\{0, \frac{\partial U_i(Q, q)}{\partial q_i}\}, & \text{if } q_i = 0.
\end{cases}
$$

(4.27)

Applying (4.24) to all firm and demand market pairs $(i, k)$; $i = 1, \ldots, I$; $k = 1, \ldots, n_R$, and applying (4.27) to all firms $i$; $i = 1, \ldots, I$, and combining the resultants,
yields the following pertinent ordinary differential equation (ODE) (cf. (2.19)) for the adjustment processes of the product shipments and quality levels, in vector form, as:

\[
\dot{X} = \Pi_K(X, -F(X)),
\]

where, \( F(X) = -\nabla U(Q, q) \), where \( \nabla U(Q, q) \) is the vector of marginal utilities with components given by (4.20) and (4.21).

I now interpret the ODE (4.28) in the context of the supply chain network model with product differentiation and quality competition. First, note that ODE (4.28) ensures that the production shipments and quality levels are always nonnegative. Indeed, if one were to consider, instead, the ordinary differential equation: \( \dot{X} = -F(X) \), or, equivalently, \( \dot{X} = \nabla U(X) \), such an ODE would not ensure that \( X(t) \geq 0 \), for all \( t \geq 0 \), unless additional restrictive assumptions were to be imposed. ODE (4.28), however, retains the interpretation that if \( X \) at time \( t \) lies in the interior of \( \mathcal{K} \), then the rate at which \( X \) changes is greatest when the vector field \( -F(X) \) is greatest. Moreover, when the vector field \( -F(X) \) pushes \( X \) to the boundary of the feasible set \( \mathcal{K} \), then the projection \( \Pi_K \) ensures that \( X \) stays within \( \mathcal{K} \). Hence, the product shipments and quality levels are always nonnegative.

Although the use of the projection on the right-hand side of ODE (4.28) guarantees that the product shipments and the quality levels are always nonnegative, it also raises the question of existence of a solution to ODE (4.28), since this ODE is nonstandard due to its discontinuous right-hand side. The fundamental theory of existence and uniqueness of projected dynamical systems is given in Theorem 2.13.

In addition, as in Theorem 2.12, the necessary and sufficient condition for a product shipment and quality level pattern \( X^* = (Q^*, q^*) \) to be a supply chain network Cournot-Nash equilibrium, according to Definition 4.1, is that \( X^* = (Q^*, q^*) \) is a stationary point of the adjustment process defined by ODE (4.28), that is, \( X^* \) is the point at which \( \dot{X} = 0 \).
Consider now the competitive system consisting of firms who, in order to maximize their utilities, adjust their product shipment and quality level patterns by instantly responding to the marginal utilities, according to (4.28). The following questions naturally arise and are of interest. Does the utility gradient process defined by (4.28), approach a Cournot-Nash equilibrium, and how does it approach an equilibrium in term of the convergence rate? Also, for a given Cournot-Nash equilibrium, do all the disequilibrium shipment and quality level patterns that are close to this equilibrium always stay near by? Motivated by these questions, I now present some stability analysis results for the Cournot-Nash equilibrium, under the above utility gradient process.

4.2. Stability Under Monotonicity

As provided in Theorem 2.13, Lipschitz continuity of $F(X)$ (cf. Definition 2.6) guarantees the existence of a unique solution to (4.28), where $X^0(t)$ satisfies ODE (4.28) with initial shipment and quality level pattern $(Q^0, q^0)$. In other words, $X^0(t)$ solves the initial value problem (4.28).

In the context of the supply chain network problem with product differentiation and quality competition, where $F(X)$ is the vector of negative marginal utilities as in (4.20) – (4.21), if the utility functions are twice differentiable and the Jacobian of the negative marginal utility functions (or, equivalently, the negative of the Hessian matrix of the utility functions) is positive-definite, then the corresponding $F(X)$ is strictly monotone.

In a practical supply chain network model, it is reasonable to expect that the utility of any firm $i$, $U_i(Q, q)$, would decrease whenever its output has become sufficiently large, that is, when $U_i$ is differentiable, $\frac{\partial U_i(Q, q)}{\partial Q_{ik}}$ is negative for sufficiently large $Q_{ik}$; the same holds for sufficiently large $q_i$. Hence, the following assumption is not unreasonable:
Assumption 4.1

Suppose that in the supply chain network model there exists a sufficiently large $M$, such that for any $(i,k)$,

$$
\frac{\partial U_i(Q,q)}{\partial Q_{ik}} < 0,
$$

(4.29)

for all shipment patterns $Q$ with $Q_{ik} \geq M$ and that there exists a sufficiently large $\bar{M}$, such that for any $i$,

$$
\frac{\partial U_i(Q,q)}{\partial q_i} < 0,
$$

(4.30)

for all quality level patterns $q$ with $q_i \geq \bar{M}$.

I now give an existence result.

Proposition 4.1

Any supply chain network problem, as described above, that satisfies Assumption 4.1 possesses at least one equilibrium shipment and quality level pattern.

Proof: The proof follows from Theorem 2.3 and Zhang and Nagurney (1995). □

For completeness, a uniqueness result is now presented (cf. Theorem 2.5).

Proposition 4.2

Suppose that $F$ is strictly monotone at any equilibrium point of the variational inequality problem defined in (4.19). Then it has at most one equilibrium point.

In addition, an existence and uniqueness result is recalled (cf. Theorem 2.6).

Theorem 4.2

Suppose that $F$ is strongly monotone. Then there exists a unique solution to variational inequality (4.19); equivalently, to variational inequality (4.14).

Additionally, the following theorem presents the stability results of the projected dynamical system described in (4.28) (cf. Theorems 2.14, 2.15, and 2.16).
Theorem 4.3

(i). If \( -\nabla U(Q, q) \) is monotone, then every supply chain network Cournot-Nash equilibrium, provided its existence, is a global monotone attractor for the utility gradient process.

(ii). If \( -\nabla U(Q, q) \) is strictly monotone, then there exists at most one supply chain network Cournot-Nash equilibrium. Furthermore, provided existence, the unique spatial Cournot-Nash equilibrium is a strictly global monotone attractor for the utility gradient process.

(iii). If \( -\nabla U(Q, q) \) is strongly monotone, then there exists a unique supply chain network Cournot-Nash equilibrium, which is globally exponentially stable for the utility gradient process.

I now present two examples in order to illustrate some of the above concepts and results.

Example 4.1

Consider a supply chain network problem consisting of two firms and one demand market, as depicted in Figure 4.2.

\[ f_1(s, q_1) = s_1^2 + 2s_1s_2 + 2q_1^2 + 39, \quad f_2(s, q_2) = 2s_2^2 + 2s_1s_2 + q_2^2 + 37, \]

Figure 4.2. The Supply Chain Network Topology for Example 4.1
the total transportation cost functions are:

\[ \hat{c}_{11}(Q_{11}) = Q_{11}^2 + 10, \quad \hat{c}_{21}(Q_{21}) = 7Q_{21}^2 + 10, \]

and the demand price functions are:

\[ \rho_{11}(d, q) = 100 - d_{11} - 0.4d_{21} + 0.3q_1 + 0.05q_2, \]
\[ \rho_{21}(d, q) = 100 - 0.6d_{11} - 1.5d_{21} + 0.1q_1 + 0.5q_2. \]

The utility function of firm 1 is, hence:

\[ U_1(Q, q) = \rho_{11}d_{11} - \hat{f}_1 - \hat{c}_{11}, \]

whereas the utility function of firm 2 is:

\[ U_2(Q, q) = \rho_{21}d_{21} - \hat{f}_2 - \hat{c}_{21}. \]

The Jacobian matrix of \(-\nabla U(Q, q)\), denoted by \(J(Q_{11}, Q_{21}, q_1, q_2)\), is

\[
J(Q_{11}, Q_{21}, q_1, q_2) = \begin{pmatrix}
6 & 1.4 & -0.3 & -0.5 \\
2.6 & 21 & -0.1 & -0.5 \\
-0.3 & 0 & 4 & 0 \\
0 & -0.5 & 0 & 2
\end{pmatrix}.
\]

This Jacobian matrix is positive-definite, since it is strictly diagonally dominant, and, hence, minus the gradient of the utility functions, that is, \(-\nabla U(Q, q)\) is strongly monotone (see also Nagurney (1999)). Thus, both the existence and uniqueness of the solution to variational inequality (4.13) with respect to this example are guaranteed.
(Theorem 4.2). Moreover, the equilibrium solution, which is: $Q_{11}^* = 16.08$, $Q_{21}^* = 2.79$, $q_1^* = 1.21$, and $q_2^* = 0.70$, is globally exponentially stable, according to Theorem 4.3.

**Example 4.2**

Another example with two firms and two demand markets, as depicted in Figure 4.3, is presented.

![Figure 4.3](image)

**Figure 4.3.** The Supply Chain Network Topology for Example 4.2

The production cost functions of the two firms are:

$$\hat{f}_1(s,q_1) = s_1^2 + s_1 s_2 + 2q_1^2 + 39, \quad \hat{f}_2(s,q_2) = 2s_2^2 + 2s_1 s_2 + q_2^2 + 37,$$

and the total transportation cost functions are:

$$\hat{c}_{11}(Q_{11}) = Q_{11}^2 + 10, \quad \hat{c}_{12}(Q_{12}) = 5Q_{12}^2 + 7,$$

$$\hat{c}_{21}(Q_{21}) = 7Q_{21}^2 + 10, \quad \hat{c}_{22}(Q_{22}) = 2Q_{22}^2 + 5,$$

and the demand price functions are:

$$\rho_{11}(d,q) = 100 - d_{11} - 0.4d_{21} + 0.3q_1 + 0.05q_2,$$

$$\rho_{12}(d,q) = 100 - 2d_{12} - d_{22} + 0.4q_1 + 0.2q_2,$$
\[ \rho_{21}(d,q) = 100 - 0.6d_{11} - 1.5d_{21} + 0.1q_1 + 0.5q_2, \]
\[ \rho_{22}(d,q) = 100 - 0.7d_{12} - 1.7d_{22} + 0.01q_1 + 0.6q_2. \]

The utility function of firm 1 is:

\[ U_1(Q,q) = \rho_{11}d_{11} + \rho_{12}d_{12} - \hat{f}_1 - (\hat{c}_{11} + \hat{c}_{12}) \]

with the utility function of firm 2 being:

\[ U_2(Q,q) = \rho_{21}d_{21} + \rho_{22}d_{22} - \hat{f}_2 - (\hat{c}_{21} + \hat{c}_{22}). \]

The Jacobian of \(-\nabla U(Q,q)\), denoted by \( J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, q_1, q_2) \), is

\[
J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, q_1, q_2) = 
\begin{pmatrix}
6 & 2 & 1.4 & 1 & -0.3 & -0.05 \\
2 & 16 & 1 & 2 & -0.4 & -0.2 \\
2.6 & 2 & 21 & 4 & -0.1 & -0.5 \\
2 & 2.7 & 4 & 7.4 & -0.01 & -0.6 \\
-0.3 & -0.4 & 0 & 0 & 4 & 0 \\
0 & 0 & -0.5 & -0.6 & 0 & 2
\end{pmatrix}.
\]

This Jacobian matrix is positive-definite. Hence, \(-\nabla U(Q,q)\) is strongly monotone, and both the existence and the uniqueness of the solution to variational inequality (4.13) with respect to this example are guaranteed. Moreover, the equilibrium solution (stationary point) is: \(Q_{11}^* = 14.27, Q_{12}^* = 3.81, Q_{21}^* = 1.76, Q_{22}^* = 4.85, q_1^* = 1.45, q_2^* = 1.89\), and it is globally exponentially stable.

The stationary points of both Examples 4.1 and 4.2 are computed using the Euler method. In the next section, I present the induced closed form expressions at each iteration, along with the convergence result.
4.3. Explicit Formulae for the Euler Method Applied to the Supply Chain Network Model with Transportation Costs, Product Differentiation and Quality Competition

The projected dynamical system yields continuous-time adjustment processes. However, as provided in Section 2.5.1, for computational purposes, a discrete-time algorithm, the Euler method, which serves as an approximation to the continuous-time trajectories, is needed.

The explicit formulae yielded by the Euler method are as following. In particular, the following closed form expression for all the product shipments, \( i = 1, \ldots, I; k = 1, \ldots, n_R \), is:

\[
Q_{ik}^{\tau+1} = \max\{0, Q_{ik}^{\tau} + a_\tau (\rho_{ik}(d^{\tau}, q^{\tau}) + \sum_{l=1}^{n_R} \frac{\partial \rho_{il}(d^{\tau}, q^{\tau})}{\partial d_{ik}} d_{il}^{\tau} - \frac{\partial \hat{f}_i(s^{\tau}, q^{\tau})}{\partial s_i} - \frac{\partial \hat{c}_{ik}(Q_{ik}^{\tau})}{\partial Q_{ik}^{\tau}})\},
\]

(4.31)

and the following closed form expression for all the quality levels, \( i = 1, \ldots, I \), is:

\[
q_i^{\tau+1} = \max\{0, q_i^{\tau} + a_\tau \left( \sum_{l=1}^{n_R} \frac{\partial \rho_{il}(d^{\tau}, q^{\tau})}{\partial q_i} d_{il}^{\tau} - \frac{\partial \hat{f}_i(s^{\tau}, q^{\tau})}{\partial q_i} \right)\}
\]

(4.32)

with the demands being updated according to:

\[
d_{ik}^{\tau+1} = Q_{ik}^{\tau+1}, \quad i = 1, \ldots, I; k = 1, \ldots, n_R,
\]

(4.33)

and the supplies being updated according to:

\[
s_i^{\tau+1} = \sum_{k=1}^{n_R} Q_{ik}^{\tau+1}, \quad i = 1, \ldots, I.
\]

(4.34)

I now provide the convergence result. The proof is direct from Theorem 2.17 and Nagurney and Zhang (1996).
Theorem 4.4

In the supply chain network problem with product differentiation and quality levels let 
\[ F(X) = -\nabla U(Q,q) \] 
be strictly monotone at any equilibrium pattern and assume that Assumption 4.1 is satisfied. Also, assume that \( F \) is uniformly Lipschitz continuous. Then there exists a unique equilibrium product shipment and quality level pattern \((Q^*, q^*) \in K\) and any sequence generated by the Euler method as given by (2.34), where \( \{a_\tau\} \) satisfies \( \sum_{\tau=0}^\infty a_\tau = \infty\), \( a_\tau > 0\), \( a_\tau \to 0\), as \( \tau \to \infty \) converges to \((Q^*, q^*)\).

In the next section, I apply the Euler method to compute solutions to numerical supply chain network problems.

4.4. Numerical Examples

I implemented the Euler method (cf. (4.31) - (4.34)), using Matlab on a Lenovo E46A. The convergence criterion is \( \epsilon = 10^{-6}\); that is, the Euler method is considered to have converged if, at a given iteration, the absolute value of the difference between each successive product shipment and quality level is no more than \( \epsilon \).

The sequence \( \{a_\tau\} \) is: \(.1(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \ldots)\). I initialized the algorithm by setting each product shipment \( Q_{ik} = 2.5\), \( \forall i, k \), and by setting the quality level of each firm \( q_i = 0.00\), \( \forall i \).

In Section 4.3, I discussed stability analysis and present results for two numerical examples. I now provide additional results for these two examples.

Example 4.1 Revisited

The Euler method requires 39 iterations for convergence to the equilibrium pattern for Example 4.1. A graphical depiction of the iterates, consisting of the product shipments and the quality levels, is given, respectively, in Figures 4.4 and 4.5. The utility/profit of firm 1 is 723.89 and that of firm 2 is 34.44.

One can see from these figures, that, as predicted by the stability analysis results, the convergence is exponentially fast.
As shown in Figure 4.5, the equilibrium quality level of firm 1 is 42.15% higher than that of firm 2, which happens because customers are more quality-sensitive to firm 1’s product, as described in the demand price functions.

Example 4.2 Revisited

For Example 4.2, the Euler method requires 45 iterations for convergence. A graphical depiction of the product shipment and quality level iterates is given in Figures 4.6 and 4.7. One can see from the figures that the convergence to the equilibrium solution is exponentially fast. The profit of firm 1 is 775.19, whereas that of firm 2 is 145.20.
In the next example, there is another firm, firm 3, entering the market, and its quality cost is much higher than those of firms 1 and 2.

Figure 4.6. Product Shipments for Example 4.2

Figure 4.7. Quality Levels for Example 4.2

Example 4.3

The third numerical supply chain network example consists of three firms and two demand markets, as depicted in Figure 4.8.

This example is built from Example 4.2 with the production cost functions of the original two firms expanded and the original demand price functions as well. I also add new data for the new firm. The complete data for this example are given below.
Figure 4.8. The Supply Chain Network Topology for Example 4.3

The production cost functions are:

\[ \hat{f}_1(s, q_1) = s_1^2 + s_1s_2 + s_1s_3 + 2q_1^2 + 39, \]
\[ \hat{f}_2(s, q_2) = 2s_2^2 + 2s_1s_2 + 2s_3s_2 + q_2^2 + 37, \]
\[ \hat{f}_3(s, q_3) = s_3^2 + s_1s_3 + s_3s_2 + 8q_3^2 + 60. \]

The total transportation cost functions are:

\[ \hat{c}_{11}(Q_{11}) = Q_{11}^2 + 10, \quad \hat{c}_{12}(Q_{12}) = 5Q_{12}^2 + 7, \]
\[ \hat{c}_{21}(Q_{21}) = 7Q_{21}^2 + 10, \quad \hat{c}_{22}(Q_{22}) = 2Q_{22}^2 + 5, \]
\[ \hat{c}_{31}(Q_{31}) = 2Q_{31}^2 + 9, \quad \hat{c}_{32}(Q_{32}) = 3Q_{32}^2 + 8, \]

and the demand price functions are:

\[ \rho_{11}(d, q) = 100 - d_{11} - 0.4d_{21} - 0.1d_{31} + 0.3q_1 + 0.05q_2 + 0.05q_3, \]
\[ \rho_{12}(d, q) = 100 - 2d_{12} - d_{22} - 0.1d_{32} + 0.4q_1 + 0.2q_2 + 0.2q_3, \]
\[ \rho_{21}(d, q) = 100 - 0.6d_{11} - 1.5d_{21} - 0.1d_{31} + 0.1q_1 + 0.5q_2 + 0.1q_3, \]
\[ \rho_{22}(d, q) = 100 - 0.7d_{12} - 1.7d_{22} - 0.1d_{32} + 0.01q_1 + 0.6q_2 + 0.01q_3, \]
\[ \rho_{31}(d, q) = 100 - 0.2d_{11} - 0.4d_{21} - 1.8d_{31} + 0.2q_1 + 0.2q_2 + 0.7q_3, \]

\[ \rho_{32}(d, q) = 100 - 0.1d_{12} - 0.3d_{22} - 2d_{32} + 0.2q_1 + 0.1q_2 + 0.4q_3. \]

The utility function expressions of firm 1, firm 2, and firm 3 are, respectively:

\[ U_1(Q, q) = \rho_{11}d_{11} + \rho_{12}d_{12} - \hat{f}_1 - (\hat{c}_{11} + \hat{c}_{12}), \]

\[ U_2(Q, q) = \rho_{21}d_{21} + \rho_{22}d_{22} - \hat{f}_2 - (\hat{c}_{21} + \hat{c}_{22}), \]

\[ U_3(Q, q) = \rho_{31}d_{31} + \rho_{32}d_{32} - \hat{f}_3 - (\hat{c}_{31} + \hat{c}_{32}). \]

The Jacobian of \(-\nabla U(Q, q)\), denoted by \(J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, q_1, q_2, q_3)\), is

\[
J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, q_1, q_2, q_3) = \\
\begin{pmatrix}
6 & 2 & 1.4 & 1 & 1.1 & 1 & -0.3 & -0.05 & -0.05 \\
2 & 16 & 1 & 2 & 1 & 1.1 & -0.4 & -0.2 & -0.2 \\
2.6 & 2 & 21 & 4 & 2.1 & 2 & -0.1 & -0.5 & -0.5 \\
2 & 2.7 & 4 & 7.4 & 2 & 2.1 & -0.01 & -0.6 & -0.01 \\
1.2 & 1 & 1.4 & 1 & 9.6 & 2 & -0.2 & -0.2 & -0.7 \\
1 & 1.1 & 1 & 1.3 & 2 & 12 & -0.2 & -0.1 & -0.4 \\
-0.3 & -0.4 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & -0.5 & -0.6 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & -0.7 & -0.4 & 0 & 0 & 16
\end{pmatrix}.
\]

This Jacobian matrix is positive-definite. Thus, \(-\nabla U(Q, q)\) is strongly monotone, and both the existence and uniqueness of the solution to variational inequality (4.13) with respect to this example are guaranteed.

The Euler method converges to the equilibrium solution: \(Q_{11}^* = 12.63, Q_{12}^* = 3.45, Q_{21}^* = 1.09, Q_{22}^* = 3.21, Q_{31}^* = 6.94, Q_{32}^* = 5.42, q_1^* = 1.29, q_2^* = 1.23, q_3^* = 0.44, \) in 42
iterations. The profits of the firms are: $U_1 = 601.67$, $U_2 = 31.48$, and $U_3 = 403.97$.

Graphical depictions of the product shipment and the quality level iterates are given, respectively, in Figures 4.9 and 4.10.

![Figure 4.9. Product Shipments for Example 4.3](image)

![Figure 4.10. Quality Levels for Example 4.3](image)

The properties of the Jacobian matrix are verified above in order to also evaluate the stability of the utility gradient process as well as to check whether conditions for convergence of the algorithm are satisfied. One should realize, however, that the algorithm does not require strong monotonicity of minus the gradient of the utility functions for convergence (cf. Theorem 4.4). Moreover, if the algorithm converges, it converges to a stationary point of the projected dynamical systems; equivalently, to a
solution of the variational inequality problem governing the Cournot-Nash equilibrium conditions for the supply chain network model.

In addition, with these examples, I illustrate the types of problems with not unrealistic features and underlying functions that can be theoretically effectively analyzed as to their qualitative properties and also their solutions computed.

Example 4.4

Example 4.4 is constructed from Example 4.3 to consider the following scenario. The consumers at demand market $R_2$ have become more sensitive as to the quality of the products. To reflect this, the new demand price functions associated with demand market $R_2$ are now:

\[ \rho_{12}(d, q) = 100 - 2d_{12} - d_{22} - 0.1d_{32} + 0.49q_1 + 0.2q_2 + 0.2q_3, \]

\[ \rho_{22}(d, q) = 100 - 0.7d_{12} - 1.7d_{22} - 0.1d_{32} + 0.01q_1 + 0.87q_2 + 0.01q_3, \]

and

\[ \rho_{32}(d, q) = 100 - 0.1d_{12} - 0.3d_{22} - 2d_{32} + 0.2q_1 + 0.1q_2 + 1.2q_3. \]

The Jacobian of $-\nabla U(Q, q)$ is now:

\[
J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, q_1, q_2, q_3) = \begin{pmatrix}
6 & 2 & 1.4 & 1 & 1.1 & 1 & -0.3 & -0.05 & -0.05 \\
2 & 16 & 1 & 2 & 1 & 1.1 & -0.49 & -0.2 & -0.2 \\
2.6 & 2 & 21 & 4 & 2.1 & 2 & -0.1 & -0.5 & -0.5 \\
2 & 2.7 & 4 & 7.4 & 2 & 2.1 & -0.01 & -0.87 & -0.01 \\
1.2 & 1 & 1.4 & 1 & 9.6 & 2 & -0.2 & -0.2 & -0.7 \\
1 & 1.1 & 1 & 1.3 & 2 & 12 & -0.2 & -0.1 & -1.2 \\
-0.3 & -0.49 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & -0.5 & -0.87 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & -0.7 & -1.2 & 0 & 0 & 16 \\
\end{pmatrix}
\]

This Jacobian matrix is also positive-definite. Therefore, for this example, the existence and the uniqueness of the equilibrium product shipment and quality level pattern are guaranteed. Moreover, the utility gradient processes for both Examples 4.3 and 4.4 are globally exponentially stable.
The computed equilibrium solution is now: \( Q_{11}^* = 13.41, Q_{12}^* = 3.63, Q_{21}^* = 1.41, Q_{22}^* = 4.08, Q_{31}^* = 3.55, Q_{32}^* = 2.86, q_1^* = 1.45, q_2^* = 2.12, q_3^* = 0.37. \) The profits of the firms are now: \( U_1 = 682.44, U_2 = 82.10, \) and \( U_3 = 93.19. \)

**Figure 4.11.** Product Shipments for Example 4.4

![Product Shipments](image)

**Figure 4.12.** Quality Levels for Example 4.4

![Quality Levels](image)

The Euler method requires 47 iterations for convergence. Please refer to Figures 4.11 and 4.12 to view the iterates of the product shipments and the quality levels generated by the Euler method. Due to the fact that, in each iteration, the values of \( Q_{12} \) and \( Q_{22} \) are very close, the trajectories of these two almost overlap in Figure 4.11.
The results are now discussed. As consumers become more sensitive to the quality of the substitutable product, the equilibrium quality levels of the three firms change, with those of firm 1 and firm 2 increasing, relative to their values in Example 4.3. Since it costs much more for firm 3 to achieve higher quality levels than firm 1 and firm 2, the profit of firm 3 decreases by 76.9%, while the profits of firms 1 and 2 increase by 13.4% and 160.8%, respectively. Hence, the pressure on the consumers’ side through the demand price functions in quality can result not only in higher quality but also in higher profits for those firms that have lower quality costs.

**Example 4.5**

Example 4.5 is constructed, for completeness. The data are as in Example 4.4 except for the production cost functions, which are now:

\[
\hat{f}_1(s,q_1) = 2s_1^2 + 0.005s_1q_1 + 2q_1^2 + 30, \quad \hat{f}_2(s,q_2) = 4s_2^2 + 0.005s_2q_2 + q_2^2 + 30, \\
\hat{f}_3(s,q_3) = 4s_3^2 + 0.005s_3q_3 + 8q_3^2 + 50.
\]

The Jacobian of \(-\nabla U(Q,q)\), denoted by \(J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, q_1, q_2, q_3)\), is

\[
J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, q_1, q_2, q_3) = \begin{pmatrix}
8 & 4 & 0.4 & 0 & 0.1 & 0 & -0.295 & -0.05 & -0.05 \\
4 & 18 & 0 & 1 & 0 & 0.1 & -0.395 & -0.2 & -0.2 \\
0.6 & 0 & 25 & 8 & 0.1 & 0 & -0.1 & -0.495 & -0.1 \\
0 & 0.7 & 8 & 15.4 & 0 & 0.1 & -0.01 & -0.595 & -0.01 \\
0.2 & 0 & 0.4 & 0 & 9.6 & 2 & -0.2 & -0.2 & -0.695 \\
0 & 0.1 & 0 & 0.3 & 2 & 12 & -0.2 & -0.1 & -0.395 \\
-0.295 & -0.395 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & -0.495 & -0.595 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & -0.695 & -0.395 & 0 & 0 & 16
\end{pmatrix}.
\]

This Jacobian matrix is positive-definite, and both the existence and the uniqueness of the equilibrium solution to this example are guaranteed.
The Euler method converges to the equilibrium solution: $Q_{11}^* = 10.95$, $Q_{12}^* = 2.84$, $Q_{21}^* = 2.04$, $Q_{22}^* = 5.34$, $Q_{31}^* = 4.47$, $Q_{32}^* = 3.49$, $q_1^* = 1.09$, $q_2^* = 2.10$, $q_3^* = 0.28$ in 46 iterations. The profits of the firms are: $U_1 = 1222.89$, $U_2 = 668.03$, and $U_3 = 722.03$. Graphical depictions of the product shipment and the quality level iterates are given in Figures 4.13 and 4.14.

![Graphical depiction of product shipment](image1)

**Figure 4.13.** Product Shipments for Example 4.5

![Graphical depiction of quality levels](image2)

**Figure 4.14.** Quality Levels for Example 4.5

### 4.5. Summary and Conclusions

In this chapter, I developed a new supply chain network model and presented both its static and dynamic realizations. The model handles product differentiation.
and includes transportation costs as well as quality cost functions and demand price functions that capture both demand for the substitutable products as well as their quality levels. The model is a Cournot-Nash model in which the strategic variables associated with each firm are its product shipments as well as the quality level of each firm’s product.

I derived the governing equilibrium conditions and provided alternative variational inequality formulations. A continuous-time adjustment process was also proposed. I provided qualitative properties of existence and uniqueness of the dynamic trajectories and also gave results, using a monotonicity approach, for stability analysis.

I presented the closed form expressions for the product shipment and quality levels at each iteration, which provide discrete-time realizations of the continuous-time product shipment and quality level trajectories. A convergence result was also given. Through several numerical examples, I illustrated the model and the theoretical results.

Given that supply chain networks with differentiated products as well as quality issues are relevant to many industries, ranging from food to high tech, and even the Internet, I believe that the results in this chapter are relevant to many application domains.

Unlike Chapters 3 and 4, in the next chapter, Chapter 5, I consider a supply chain network problem with outsourcing and price and quality competition among the contractors. The optimal make-or-buy and contractor-selection decisions are provided. In addition, in Chapters 3 and 4, demands are assumed to be elastic. In contrast, the analysis in Chapter 5 assumes non-elastic demand, in order to capture the original firm’s projected/forecasted demand in multiple demand markets.
CHAPTER 5
A SUPPLY CHAIN NETWORK MODEL WITH OUTSOURCING AND QUALITY AND PRICE COMPETITION

In this chapter, I develop both the equilibrium and the dynamic versions of the supply chain network game theory model which takes into account the quality concerns in the context of global outsourcing. Unlike Chapters 3 and 4, this chapter captures the behaviors of a firm and its potential contractors with consideration of the transactions between them and the quality of the outsourced product. The demand of the product of the firm is assumed to be fixed at multiple demand markets, since the firm can be expected to have good demand forecasting abilities. Specifically, the demand of some products, such as pharmaceuticals, vaccines, and food, is actually inelastic because they are necessities for consumers.

In addition, the impact of outsourced quality on a firm’s reputation is also considered in this chapter through the incorporation of the disrepute cost, which is a cost determined by the quality of the product produced by its contractors and the amount of product that is outsourced.

In this model, the firm that is engaged in determining its in-house and outsourced optimal product flows seeks to minimize its total cost and its weighted disrepute cost, and to satisfy the demand of its product in multiple demand markets. The potential contractors of the firm, however, compete with one another by determining the prices that they charge the firm for manufacturing and delivering the product to the demand markets and the quality levels in order to maximize their profits. The opportunity cost of each contractor in terms of the price it charges the firm is considered.
This game theory model allows for the determination of the optimal product flows associated with the supply chain in-house and outsourcing network activities and provides the firm with its optimal make-or-buy decisions and the optimal contractor-selection. In this chapter, I state the governing equilibrium conditions and derive the equivalent variational inequality formulation of the model, and propose the dynamic adjustment processes for the evolution of the product flows, the quality levels, and the prices, along with stability analysis results. The algorithm, which is the Euler method (cf. Section 2.5.1), yields a discretization of the continuous-time adjustment processes. I also present convergence results and compute solutions to numerical examples to illustrate the generality and applicability of the framework.

This chapter is based on Nagurney, Li, and Nagurney (2013). It is organized as follows. In Section 5.1, I describe the decision-making behavior of the firm and the competing contractors. I then develop the game theory model, state the equilibrium conditions, and derive the equivalent variational inequality formulation. It is assumed that the projected demand for the product is known at the various demand markets since the firm can be expected to have good in-house forecasting abilities. Hence, I focus on cost minimization associated with the firm but profit maximization for the contractors who compete in prices and quality.

In Section 5.2, I provide a dynamic version of the model through a description of the underlying adjustment processes associated with the product flows, the quality levels, and the contractor prices. The projected dynamical system has stationary points that coincide with the solutions for relevant corresponding the variational inequality problem. Stability analysis results are also provided.

In Section 5.3, I describe the closed form expressions yielded by the Euler method at each iteration, for the contractor prices and the quality levels, with the product flows being solved exactly using an equilibration algorithm. The Euler method provides a discrete-time version of the continuous-time adjustment processes given in
Section 5.2. I illustrate the concepts through small examples with sensitivity analysis results. I then apply the Euler method in Section 5.4 to demonstrate the modeling and computational framework on larger examples. The case of a disruption in the supply chain network and two cases focused on opportunity costs are also discussed. I summarize the results and give the conclusions in Section 5.5.

5.1. The Supply Chain Network Model with Outsourcing and Quality and Price Competition

In this section, I develop the supply chain network model with outsourcing and with price and quality competition among the contractors. It is assumed that a firm is involved in the processes of in-house manufacturing and distribution of a product, and may also contract its manufacturing and distribution activities to contractors, who may be located overseas. I seek to determine the optimal product flows of the firm to its demand markets, along with the prices the contractors charge the firm for production and distribution, and the quality levels of their products.

For clarity and definiteness, I consider the network topology of the firm depicted in Figure 5.1. In the supply chain network, there are \( n_M \) manufacturing facilities or plants that the firm owns and \( n_R \) demand markets. Some of the links from the top-tiered node 0, representing the firm, are connected to its manufacturing facility/plant nodes, which are denoted, respectively, by: \( M_1, \ldots, M_{n_M} \) and these, in turn, are connected to the demand nodes: \( R_1, \ldots, R_{n_R} \).

As also depicted in Figure 5.1, the outsourcing of the product in terms of its production and delivery is captured. There are \( n_O \) contractors available to the firm. The firm may potentially contract to any of these contractors who then also distribute the outsourced product that they manufacture to the \( n_R \) demand markets. The first set of outsourcing links directly link the top-most node 0 to the \( n_O \) contractor nodes, \( O_1, \ldots, O_{n_O} \), which correspond to their respective manufacturing activities, and the
The external failure cost mentioned in Table 5.1 is the compensation cost incurred when customers are unsatisfied with the quality of the products. I assume that the disrepute cost of the firm, \( dc(q') \), is a monotonically decreasing function of the average quality level.

As in Table 5.1, the transaction costs between the original firms and the contractors are also considered. It is the “cost of making each contract” (cf. Coase (1937) and also Aubert, Rivard, and Patry (1996)), which includes the costs of evaluating suppliers, negotiation costs, the monitoring and the enforcement of the contract in
Table 5.1. Notation for the Supply Chain Network Model with Outsourcing and Quality and Price Competition

<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{jk}$</td>
<td>the nonnegative amount of product produced at manufacturing plant $j$ and delivered to demand market $R_k$. The ${Q_{jk}}$ elements are grouped into the vector $Q \in R_+^{n_Mn_R}$.</td>
</tr>
<tr>
<td>$d_k$</td>
<td>the demand for the product at demand market $R_k$, assumed known and fixed.</td>
</tr>
<tr>
<td>$q_j$</td>
<td>the nonnegative quality level of the product produced by contractor $j$. The ${q_j}$ elements are grouped into the vector $q \in R_+^{n_O}$.</td>
</tr>
<tr>
<td>$\pi_{jk}$</td>
<td>the price charged by contractor $j$ for producing and delivering a unit of the product to $k$. I group the ${\pi_{jk}}$ elements for contractor $j$ into the vector $\pi_j \in R_+^{n_R}$ and then group all such vectors for all the contractors into the vector $\pi \in R_+^{n_On_R}$.</td>
</tr>
<tr>
<td>$f_j(\sum_{k=1}^{n_R} Q_{jk})$</td>
<td>the total production cost at manufacturing plant $j$; $j = 1, \ldots, n_M$ owned by the firm.</td>
</tr>
<tr>
<td>$q'$</td>
<td>the average quality level.</td>
</tr>
<tr>
<td>$tc_j(\sum_{k=1}^{n_R} Q_{n_M+j,k})$</td>
<td>the total transaction cost associated with the firm transacting with contractor $j$; $j = 1, \ldots, n_O$.</td>
</tr>
<tr>
<td>$\hat{c}<em>{jk}(Q</em>{jk})$</td>
<td>the total transportation cost associated with delivering the product manufactured at $j$ to $k$; $j = 1, \ldots, n_M$; $k = 1, \ldots, n_R$.</td>
</tr>
<tr>
<td>$sc_{jk}(Q, q)$</td>
<td>the total cost of contractor $j$; $j = 1, \ldots, n_O$, to produce and distribute the product to demand market $R_k$; $k = 1, \ldots, n_R$.</td>
</tr>
<tr>
<td>$\hat{c}_j(q)$</td>
<td>quality cost faced by contractor $j$; $j = 1, \ldots, n_O$.</td>
</tr>
<tr>
<td>$oc_{jk}(\pi)$</td>
<td>the opportunity cost associated with pricing the product by contractor $j$; $j = 1, \ldots, n_O$ and delivering it to $R_k$; $k = 1, \ldots, n_R$.</td>
</tr>
<tr>
<td>$dc(qt)$</td>
<td>the cost of disrepute, which corresponds to the external failure quality cost (cf. Section 1.1).</td>
</tr>
</tbody>
</table>
order to ensure the quality (Franceschini et al. (2003), Heshmati (2003), and Liu and Nagurney (2013)).

In addition, according to regulations (FDA (2002), U.S. Department of Health and Human Services, CDER, FDA, and CBER (2009), and the European Commission Health and Consumers Directorate (2010)), before signing the contract, a firm should have reviewed and evaluated the contractors’ ability to perform the outsourcing tasks. Therefore, the production/distribution costs and the quality cost information of the contractors are assumed to be known by the firm.

5.1.1 The Behavior of the Firm and Its Optimality Conditions

In this model, I assume that in-house activities can ensure a 100% perfect quality conformance level. The quality conformance level of contractor \( j \) is denoted by \( q_j \), which varies from a 0% conformance level to a 100% conformance level (cf. Section 1.1), such that

\[
0 \leq q_j \leq q^U, \quad j = 1, \ldots, n_O,
\]

(5.1)

where \( q^U \) is the value representing perfect quality achieved by the firm in its in-house manufacturing.

The quality level associated with the product of the firm is, hence, an average quality level that is determined by the quality levels decided upon by the contractors and the outsourced product amounts. Thus, the average quality level for the firm’s product, both in-house and outsourced, can be expressed as

\[
q^t = \frac{\sum_{j=n_M+1}^{n} \sum_{k=1}^{n_R} Q_{jk}q_{j-n_M} + (\sum_{j=1}^{n_M} \sum_{k=1}^{n_R} Q_{jk})q^U}{\sum_{k=1}^{n_R} d_k}.
\]

(5.2)

(5.2) is a variant of the average quality measure developed in Chapter 3 to assess the average quality level of homogeneous products from multiple firms with information asymmetry in quality, but without outsourcing.
The decision-making problem for the firm is to decide how much to do in-house and how much to outsource to each potential contractor, that is, to select the product flows $Q$. The firm can choose to outsource all of its products, or outsource some of its products to any number of contracts, or choose not to outsource. In contrast, the contractors, who compete with one another in quality and the prices they charge the firm, select their respective quality level $q_j$ and price vector $\pi_j$.

The objective of the firm is to maximize its utility (cf. (5.3) below), represented by minus its total costs that include the production costs, the transportation costs, the payments to the contractors, the total transaction costs, along with the weighted cost of disrepute, with the nonnegative term $\omega$ denoting the weight that the firm imposes on the disrepute cost function (cf. Section 2.4). The firm’s utility function is denoted by $U_0$ and, hence, the firm seeks to

$$\text{Maximize}_Q \quad U_0(Q, q^*, \pi^*) = -\sum_{j=1}^{n_M} f_j \left( \sum_{k=1}^{n_R} Q_{jk} \right) - \sum_{j=1}^{n_M} \sum_{k=1}^{n_R} \hat{c}_{jk}(Q_{jk}) - \sum_{j=1}^{n_O} \sum_{k=1}^{n_R} \pi^*_{jk} Q_{n_M+j,k}$$

$$- \sum_{j=1}^{n_O} t_{c_j} \left( \sum_{k=1}^{n_R} Q_{n_M+j,k} \right) - \omega dc(q^*).$$  \hspace{1cm} (5.3)

subject to:

$$\sum_{j=1}^{n} Q_{jk} = d_k, \quad k = 1, \ldots, n_R,$$  \hspace{1cm} (5.4)

$$Q_{jk} \geq 0, \quad j = 1, \ldots, n; k = 1, \ldots, n_R,$$  \hspace{1cm} (5.5)

with $q^*$ in (5.3) as in (5.2).

Note that (5.3) is equivalent to minimizing the total costs. Also, according to (5.4) the demand at each demand market must be satisfied. All the cost functions in (5.3) are continuous, twice continuously differentiable, and convex. I define the feasible set $K^0$ as follows: $K^0 \equiv \{Q \mid Q \in R^{nnR}_{+} \text{ with (5.4) satisfied}\}$. $K^0$ is closed and convex. The following theorem is immediate.
Theorem 5.1

The optimality conditions for the firm, faced with (5.3) and subject to (5.4) and (5.5), with $q^t$ as in (5.2) embedded into $dc(q^t)$, and under the above imposed assumptions, coincide with the solution of the following variational inequality (cf. Propositions 2.1 and 2.2 and Bazaraa, Sherali, and Shetty (1993)): determine $Q^* \in K^0$

$$- \sum_{h=1}^{n} \sum_{l=1}^{n_R} \frac{\partial U_0(Q^*, q^*, \pi^*)}{\partial Q_{hl}} \times (Q_{hl} - Q_{hl}^*) \geq 0, \forall Q \in K^0, \quad (5.6)$$

with notice that: for $h = 1, \ldots, n_M; l = 1, \ldots, n_R$:

$$- \frac{\partial U_0}{\partial Q_{hl}} = \left[ \frac{\partial f_h(\sum_{k=1}^{n_R} Q_{hk})}{\partial Q_{hl}} + \frac{\partial \hat{c}_h(Q_{hl})}{\partial Q_{hl}} + \omega \frac{\partial dc(q^t)}{\partial q^t} \right]$$

and for $h = n_M + 1, \ldots, n; l = 1, \ldots, n_R$:

$$- \frac{\partial U_0}{\partial Q_{hl}} = \left[ \frac{\partial f_h(\sum_{k=1}^{n_R} Q_{hk})}{\partial Q_{hl}} + \frac{\partial \hat{c}_h(Q_{hl})}{\partial Q_{hl}} + \omega \frac{\partial dc(q^t)}{\partial q^t} \frac{q^U}{\sum_{k=1}^{n_R} d_k} \right],$$

5.1.2 The Behavior of the Contractors and Their Optimality Conditions

The objective of the contractors is profit maximization. Their revenues are obtained from the purchasing activities of the firm, while their costs are the costs of production and distribution, the quality cost, and the opportunity cost.

Opportunity cost is defined as “the loss of potential gain from other alternatives when one alternative is chosen” (New Oxford American Dictionary (2010)). I capture the contractors’ opportunity costs as functions of the prices charged, since, if the
values are too low, they may not recover all of their costs, whereas if they are too high, the firm may select another contractor. In this model, opportunity costs are the only costs that depend on the prices that the contractors charge the firm and, hence, there is no double counting.

This is the first time that opportunity cost is considered in this dissertation. The concept of opportunity cost (cf. Mankiw (2011)) is very relevant to both economics and operations research. It has been emphasized in pharmaceutical firm competition by Grabowski and Vernon (1990), Palmer and Raftery (1999), and Cockburn (2004). Gan and Litvinov (2003) also constructed opportunity cost functions that are functions of prices as I consider here (see Table 5.1) in an energy application.

Interestingly, Leland (1979), inspired by the work of the Nobel laureate Akerlof (1970) on quality, introduced opportunity costs that are functions of quality levels.

It is emphasized that general opportunity cost functions include both explicit and implicit costs (Mankiw (2011)) with the explicit opportunity costs requiring monetary payment, and including possible anticipated regulatory costs, wage expenses, and the opportunity cost of capital (see Porteus (1986)), etc. Implicit opportunity costs are those that do not require payment, but to the decision-maker, still need to be monetized, for the purposes of decision-making, and can include the time and effort put in (see Payne, Bettman, and Luce (1996)), and the profit that the decision-maker could have earned, if he had made other choices (Sandoval-Chavez and Beruvides (1998)).

Please note that, as presented in Table 5.1, \( sc_{jk}(Q,q) \) is contractor \( j \)'s cost function associated with producing and delivering the firm’s product to demand market \( R_k \). It only captures the cost of production and delivery, and, similar as the production cost functions (3.5a and 3.5b) and the transportation cost functions (3.6) in Chapter 3, it depends on both the quantities and the quality levels. However, \( \hat{c}_j(q) \), which is a convex function in quality levels, is the quality cost associated with quality
management (cf. Section 1.1), and is over and above the cost of production and delivery activities. Thus, \(sc_{jk}(Q, q)\) and \(\hat{c}_j(q)\) are two entirely different costs, and they do not overlap.

Each contractor has, as its strategic variables, its quality level, and the prices that it charges the firm for production and distribution to the demand markets. I denote the utility/profit of each contractor \(j\) by \(U_j\), with \(j = 1, \ldots, n_O\). Hence, each contractor \(j; j = 1, \ldots, n_O\) seeks to:

\[
\text{Maximize}_{q_j, \pi_j} U_j(Q^*, q, \pi) = \sum_{k=1}^{n_R} \pi_{jk} Q^*_{n_M + j, k} - \sum_{k=1}^{n_R} sc_{jk}(Q^*, q) - \hat{c}_j(q) - \sum_{k=1}^{n_R} oc_{jk}(\pi) \tag{5.7}
\]

subject to:

\[
\pi_{jk} \geq 0, \quad k = 1, \ldots, n_R, \tag{5.8}
\]

and (5.1) for each \(j\).

I assume that the cost functions in each contractor’s utility function are continuous, twice continuously differentiable, and convex. Moreover, I assume that the contractors compete in a noncooperative in the sense of Nash, with each one trying to maximize its own profits.

I define the feasible sets \(K^j \equiv \{(q_j, \pi_j) | \pi_j \text{satisfies (5.8) and } q_j \text{satisfies (5.1) for } j\}\). I also define the feasible sets \(K^1 \equiv \prod_{j=1}^{n_O} K^j\) and \(K \equiv K^0 \times K^1\). All the above-defined feasible sets are closed and convex.

**Definition 5.1: A Bertrand-Nash Equilibrium with Price and Quality Competition**

A quality level and price pattern \((q^*, \pi^*) \in K^1\) is said to constitute a Bertrand-Nash equilibrium if for each contractor \(j; j = 1, \ldots, n_O\)

\[
U_j(Q^*, q^*_j, \hat{q}^*_j, \pi^*_j, \hat{\pi}^*_j) \geq U_j(Q^*, q_j, \hat{q}_j, \pi_j, \hat{\pi}_j), \quad \forall (q_j, \pi_j) \in K^j, \tag{5.9}
\]
where

\[ \hat{q}_j^* \equiv (q_1^*, \ldots, q_{j-1}^*, q_{j+1}^*, \ldots, q_{n_O}) , \quad (5.10) \]

\[ \hat{\pi}_j^* \equiv (\pi_1^*, \ldots, \pi_{j-1}^*, \pi_{j+1}^*, \ldots, \pi_{n_O}^*) . \quad (5.11) \]

According to (5.9) (cf. Definition 2.7), a Bertrand-Nash equilibrium is established if no contractor can unilaterally improve upon its profits by selecting an alternative vector of quality levels and prices charged to the firm.

Next, I present the variational inequality formulation of the Bertrand-Nash equilibrium according to Definition 5.1 (see Bertrand (1883), Nash (1950, 1951), and Nagurney (2006)), which follows directly from Theorem 2.7.

**Theorem 5.2**

Assume that, for each contractor \( j; j = 1, \ldots, n_O \), the profit function \( U_j(Q,q,\pi) \) is concave with respect to the variables \( \{\pi_{j1}, \ldots, \pi_{jn_R}\} \) and \( q_j \), and is continuous and twice continuously differentiable. Then \((q^*, \pi^*) \in K^1 \) is a Bertrand-Nash equilibrium according to Definition 5.1 if and only if it satisfies the variational inequality:

\[
- \sum_{j=1}^{n_O} \frac{\partial U_j(Q^*, q^*, \pi^*)}{\partial q_j} \times (q_j - q_j^*) - \sum_{j=1}^{n_O} \sum_{k=1}^{n_R} \frac{\partial U_j(Q^*, q^*, \pi^*)}{\partial \pi_{jk}} \times (\pi_{jk} - \pi_{jk}^*) \geq 0,
\]

\[ \forall (q, \pi) \in K^1 . \quad (5.12) \]

with notice that: for \( j = 1, \ldots, n_O \):

\[
- \frac{\partial U_j}{\partial q_j} = \sum_{k=1}^{n_R} \frac{\partial sc_{jk}(Q^*, q)}{\partial q_j} + \frac{\partial \hat{c}_j(q)}{\partial q_j}, \quad (5.13)
\]

and for \( j = 1, \ldots, n_O; k = 1, \ldots, n_R \):

\[
- \frac{\partial U_j}{\partial \pi_{jk}} = \sum_{r=1}^{n_R} \frac{\partial oc_{jr}(\pi)}{\partial \pi_{jk}} - Q_{n_M+j,k}^* . \quad (5.14)
\]
5.1.3 The Equilibrium Conditions for the Supply Chain Network with Outsourcing and Quality and Price Competition

In equilibrium, the optimality conditions for all contractors and the optimality conditions for the firm must hold simultaneously, according to the definition below.

**Definition 5.2: Supply Chain Network Equilibrium with Outsourcing and with Price and Quality Competition**

The equilibrium state of the supply chain network with outsourcing is one where both variational inequalities (5.6) and (5.12) hold simultaneously.

The following theorem is then immediate.

**Theorem 5.3**

The equilibrium conditions governing the supply chain network model with outsourcing are equivalent to the solution of the variational inequality problem: determine \((Q^*, q^*, \pi^*) \in \mathcal{K}\), such that:

\[
- \sum_{h=1}^{n} \sum_{l=1}^{n_R} \frac{\partial U_0(Q^*, q^*, \pi^*)}{\partial Q_{hl}} \times (Q_{hl} - Q_{hl}^*) - \sum_{j=1}^{n_O} \frac{\partial U_j(Q^*, q^*, \pi^*)}{\partial q_j} \times (q_j - q_j^*) \\
- \sum_{j=1}^{n_O} \sum_{k=1}^{n_R} \frac{\partial U_j(Q^*, q^*, \pi^*)}{\partial \pi_{jk}} \times (\pi_{jk} - \pi_{jk}^*) \geq 0, \quad \forall (Q, q, \pi) \in \mathcal{K}.
\]

**(5.15)**

**Proof:** Summation of variational inequalities (5.6) and (5.12) yields variational inequality (5.15). A solution to variational inequality (5.15) satisfies the sum of (5.6) and (5.12) and, hence, is an equilibrium according to Definition 5.2. □

I now put variational inequality (5.15) into standard form (cf. (2.1a)): determine \(X^* \in \mathcal{K}\) where \(X\) is a vector in \(\mathbb{R}^N\), \(F(X)\) is a continuous function such that \(F(X) : X \mapsto \mathcal{K} \subset \mathbb{R}^N\), and

\[
\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K},
\]

\[(5.16)\]
where $\langle \cdot, \cdot \rangle$ is the inner product in the $N$-dimensional Euclidean space, and $\mathcal{K}$ is closed and convex. I define the vector $X \equiv (Q, q, \pi)$. Also, here $N = nn_R + n_O + n_O n_R$.

Note that (5.16) may be rewritten as:

$$\sum_{i=1}^{N} F_i(X^*) \times (X_i - X_i^*) \geq 0, \ \forall X \in \mathcal{K}. \quad (5.17)$$

The components of $F$ are then as follows. The first $nn_R$ components of $F$ are given by: $-\frac{\partial U_0(Q,q,\pi)}{\partial Q_{hl}}$ for $h = 1, \ldots, n; \ l = 1, \ldots, n_R$; the next $n_O$ components of $F$ are given by: $-\frac{\partial U_j(Q,q,\pi)}{\partial q_j}$ for $j = 1, \ldots, n_O$, and the subsequent $n_O n_R$ components of $F$ are given by: $-\frac{\partial U_j(Q,q,\pi)}{\partial \pi_{jk}}$ with $j = 1, \ldots, n_O; \ k = 1, \ldots, n_R$. Hence, (5.15) can be put into standard form (2.1a).

5.2. The Underlying Dynamics and Stability Analysis

I now describe the underlying dynamics until the equilibrium satisfying variational inequality (5.15) is achieved. I identify dynamic adjustment processes for the evolution of the firm’s product flows, both in-house and outsourced, and the contractors’ quality levels and prices charged to the firm. In Section 5.3, I provide a discrete-time version of the continuous time adjustment processes in the form of an algorithm.

Observe that, for a current vector of product flows, quality levels, and prices at time $t$, $X(t) = (Q(t), q(t), \pi(t))$, $-F_i(X(t)) = \frac{\partial U_0(Q(t),q(t),\pi(t))}{\partial Q_{hl}}$, for $i = 1, \ldots, nn_R$ and $h = 1, \ldots, n; \ l = 1, \ldots, n_R$, and is given by minus the value of the expressions following (5.6), is the marginal utility of the firm with respect to the product flow produced at $h$ and distributed to demand market $R_l$. Similarly, $-F_i(X(t)) = \frac{\partial U_j(Q(t),q(t),\pi(t))}{\partial q_j}$, given by minus the value in (5.13), is contractor $j$’s marginal utility (profit) with respect to its quality level associated with the product that it produced and distributed to the demand markets, with $j = 1, \ldots, n_O$ and $i = nn_R + 1, \ldots, nn_R + n_O$. Finally, $-F_i(X(t)) = \frac{\partial U_j(Q(t),q(t),\pi(t))}{\partial \pi_{jk}}$, given by minus the value in (5.14), is contractor $j$’s
marginal utility (profit) with respect to its price charged for delivering the product that it produced to \( k \), with \( j = 1, \ldots, n_O; k = 1, \ldots, n_R \), and \( i = nn_R + n_O + 1, \ldots, n \).

It is imperative that the constraints be satisfied, consisting of the demand constraints for the firm, and the nonnegativity assumption on the product flows, as well as the nonnegativity assumption on the contractors’ quality levels and the prices that they charge.

Specifically, I propose the following pertinent ordinary differential equation (ODE) (cf. (2.19)) for the adjustment processes of the product flows, quality levels, and contractor prices, in vector form, as:

\[
\dot{X} = \Pi_K(X, -F(X)),
\]

(5.18)

where, \(-F(X) = \nabla U(Q, q, \pi)\), where \(\nabla U(Q, q, \pi)\) is the vector of marginal utilities as described above.

I now further interpret ODE (5.18) in the context of the supply chain network game theory model with price and quality competition among the contractors. Observe that ODE (5.18) guarantees that the product flows, quality levels, and the contractor prices are always nonnegative and that the demand constraints (5.4) are also satisfied, in contrast to the classic dynamical systems given in Section 2.3.

Recall now the definition of \( F(X) \) for the model, which captures the behavior of all the decision-makers in the supply chain network with outsourcing and competition in an integrated manner. ODE (5.18) states that the rate of change of the product flows, quality levels, and contractor prices is greatest when the firm’s and contractors’ marginal utilities are greatest. If the marginal utility of a contractor with respect to its quality level is positive, then the contractor will increase its quality level; if it is negative, then it will decrease the quality. Note that the quality levels will also never be outside their upper bound. A similar adjustment behavior holds for the contractors in terms of the prices that they charge, but without an upper bound on
the prices. This type of behavior is rational from an economic standpoint. Therefore, ODE (5.18) corresponds to reasonable continuous adjustment processes for the supply chain network game theory model with outsourcing and quality and price competition.

The question of existence of a solution to ODE (5.18) raises, since this ODE is nonstandard due to its discontinuous right-hand side. The fundamental theory with regards to existence and uniqueness of projected dynamical systems as defined by (5.18) is provided in Theorem 2.13.

In addition, the necessary and sufficient condition for a pattern \( X^\ast = (Q^\ast, q^\ast, \pi^\ast) \) to be an equilibrium, according to Definition 5.2, is that \( X^\ast = (Q^\ast, q^\ast, \pi^\ast) \) is a stationary point of the adjustment processes defined by ODE (5.18), that is, \( X^\ast \) is the point at which \( \dot{X} = 0 \), as given in Theorem 2.12.

I now turn to the questions as to whether and under what conditions does the adjustment processes defined by ODE (5.18) approach an equilibrium. I first note that Lipschitz continuity of \( F(X) \) (cf. Definition 2.6) guarantees the existence of a unique solution to (2.19), where \( X^0(t) \) satisfies ODE (5.18) with product flow, quality level, and price pattern \( (Q^0, q^0, \pi^0) \) (cf. Theorem 2.13).

Definitions of stability are recalled (see Definitions 2.11, 2.12, and 2.13). The monotonicity of a function \( F \) is closely related to the positive-definiteness of its Jacobian \( \nabla F \) (cf. Nagurney (1999)). Specifically, if \( \nabla F \) is positive-semidefinite, then \( F \) is monotone; if \( \nabla F \) is positive-definite, then \( F \) is strictly monotone; and, if \( \nabla F \) is strongly positive-definite, then \( F \) is strongly monotone. Definitions of positive-semidefiniteness, positive-definiteness, and strong positive-definiteness are given in Definition 2.2, with those of monotonicity in Definitions 2.3, 2.4, and 2.5.

In the context of this supply chain network game theory model with outsourcing, where \( F(X) \) is the vector of negative marginal utilities, I note that if the utility functions are twice differentiable and the Jacobian of the negative marginal utility functions (or, equivalently, the negative of the Hessian matrix of the utility functions)
for the integrated model is positive-definite, then the corresponding \( F(X) \) is strictly monotone.

I now recall an existence and uniqueness result (cf. Theorem 2.6).

**Theorem 5.4**

*Suppose that \( F \) is strongly monotone. Then there exists a unique solution to variational inequality (5.16); equivalently, to variational inequality (5.15).*

The following theorem summarizes the stability properties of the utility gradient processes, under various monotonicity conditions on the marginal utilities, as given in Theorems 2.14, 2.15, and 2.16.

**Theorem 5.5**

(i). If \( F(X) \) is monotone, then every supply chain network equilibrium, as defined in Definition 5.2, provided its existence, is a global monotone attractor for the utility gradient processes.

(ii). If \( F(X) \) is strictly monotone, then there exists at most one supply chain network equilibrium. Furthermore, given existence, the unique equilibrium is a strictly global monotone attractor for the utility gradient processes.

(iii). If \( F(X) \) is strongly monotone, then the unique supply chain network equilibrium, which is guaranteed to exist, is also globally exponentially stable for the utility gradient processes.
5.3. Explicit Formulae for the Euler Method Applied to the Supply Chain Network with Outsourcing and Quality and Price Competition

For computational purposes, the Euler method (cf. Section 2.5.1) is utilized. It is the first time that this algorithm is being adapted and applied for the solution of supply chain network game theory problems under Bertrand-Nash competition and with outsourcing.

Note that, as given in Section 2.5.1, at each iteration \( \tau \) of the Euler method, \( X^{\tau+1} \) in (2.34) is actually the solution to the strictly convex quadratic programming problem given by:

\[
X^{\tau+1} = \text{Minimize}_{X \in \mathcal{K}} \quad \frac{1}{2} \langle X, X \rangle - \langle X^\tau - a_\tau F(X^\tau), X \rangle.
\] (5.19)

Because the strictly convex quadratic programming problem (5.19) with product flows has a special network structure, one can apply the exact equilibration algorithm for solving the product flows at each iteration \( \tau \) (cf. Section 2.5.1).

Furthermore, in light of the structure of the underlying feasible set \( \mathcal{K} \), one can obtain the values for the quality variables explicitly according to the following closed form expressions for contractor \( j; j = 1, \ldots, n_O \):

\[
q^{\tau+1}_j = \min \{ q^U, \max \{ 0, q^\tau_j + a_\tau \left( -\sum_{k=1}^{n_R} \frac{\partial s_{jk}(Q^\tau, q^\tau)}{\partial q_j} - \frac{\partial \hat{c}_j(q^\tau)}{\partial q_j} \) \} \}. \] (5.20)

Also, one can obtain the following explicit formulae for the contractor prices: for \( j = 1, \ldots, n_O; k = 1, \ldots, n_R \):

\[
\pi^{\tau+1}_{jk} = \max \{ 0, \pi^\tau_{jk} + a_\tau \left( -\sum_{r=1}^{n_R} \frac{\partial o_{jr}(\pi^\tau)}{\partial \pi_{jk}} + Q^\tau_{n_M+j,k} \) \}. \] (5.21)

I now provide the convergence result. The proof is direct from Theorem 2.17 and Nagurney and Zhang (1996).
Theorem 5.6

In the supply chain network model with outsourcing, let \( F(X) = -\nabla U(Q,q,\pi) \) be strongly monotone. Also, assume that \( F \) is uniformly Lipschitz continuous. Then there exists a unique equilibrium product flow, quality level, and price pattern \((Q^*,q^*,\pi^*)\) \( \in \mathcal{K} \) and any sequence generated by the Euler method (2.34), where \( \{a_\tau\} \) satisfies \( \sum_{\tau=0}^\infty a_\tau = \infty \), \( a_\tau > 0 \), \( a_\tau \to 0 \), as \( \tau \to \infty \) converges to \((Q^*,q^*,\pi^*)\).

Note that convergence also holds if \( F(X) \) is monotone (cf. Theorem 8.6 in Nagurney and Zhang (1996)) provided that the price iterates are bounded. The product flow iterates as well as the quality level iterates will be bounded due to the constraints. Clearly, in practice, contractors cannot charge unbounded prices for production and delivery. Hence, one can also expect the existence of a solution, given the continuity of the functions that make up \( F(X) \), under less restrictive conditions than strong monotonicity.

I now provide a small example to clarify ideas, along with a variant, and also conduct a sensitivity analysis exercise. The supply chain network consists of the firm, a single contractor, and a single demand market, as depicted in Figure 5.2.

Figure 5.2. The Supply Chain Network Topology for an Illustrative Numerical Example

The data are as follows. The firm’s production cost function is:
\[ f_1(Q_{11}) = Q_{11}^2 + Q_{11} \]

and its total transportation cost function is:

\[ c_{11}(Q_{11}) = .5Q_{11}^2 + Q_{11}. \]

The firm’s transaction cost function associated with the contractor is given by:

\[ tc_1(Q_{21}) = .05Q_{21}^2 + Q_{21}. \]

The demand for the product at demand market \( R_1 \) is 1,000 and \( q^U \) is 100 and the weight \( \omega \) is 1.

The contractor’s total cost of production and distribution function is:

\[ sc_{11}(Q_{21}, q_1) = Q_{21}q_1. \]

Its total quality cost function is given by:

\[ c_1(q_1) = 10(q_1 - 100)^2. \]

The contractor’s opportunity cost function is:

\[ oc_{11}(\pi_{11}) = .5(\pi_{11} - 10)^2. \]

The firm’s cost of disrepute function is:

\[ dc(q^t) = 100 - q^t \]

where \( q^t \) (cf. (5.2)) is given by: \( \frac{Q_{21}q_1 + Q_{11}100}{1000} \).
The convergence tolerance is set to $10^{-3}$ so that the algorithm is deemed to have converged when the absolute value of the difference between each product flow, each quality level, and each price is less than or equal to $10^{-3}$. So, in effect, a stationary point has been achieved. The Euler method was initialized with $Q_{11}^0 = Q_{21}^0 = 500$, $q_1^0 = 1$, and $\pi_{11}^0 = 0$. The sequence $\{a_r\}$ is set to: $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \ldots\}$.

The Euler method converges in 87 iterations and yields the following product flow, quality level, and price pattern:

$$Q_{11}^* = 270.50, \quad Q_{21}^* = 729.50, \quad q_1^* = 63.52, \quad \pi_{11}^* = 739.50.$$  

The total cost incurred by the firm is 677,128.65 with the contractor earning a profit of 213,786.67. The value of $q^t$ is 73.39.

The Jacobian matrix of $F(X) = -\nabla U(Q, q, \pi)$, for this example, denoted by $J(Q_{11}, Q_{21}, q_1, \pi_{11})$, is

$$J(Q_{11}, Q_{21}, q_1, \pi_{11}) = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 0.1 & 0.001 & 1 \\ 0 & 1 & 20 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$  

This Jacobian matrix is positive-definite, and, hence, $-\nabla U(Q, q, \pi)$ is strongly monotone (see also Nagurney (1999)). Thus, both the existence and the uniqueness of the solution to variational inequality (5.15) with respect to this example are guaranteed (Theorem 5.4). Moreover, the equilibrium solution, reported above, is globally exponentially stable (Theorem 5.5).

I then construct a variant of this example. The transportation cost function is reduced by a factor of 10 so that it is now:

$$\hat{c}_{11}(Q_{11}) = .05Q_{11}^2 + .1Q_{11}$$
with the remainder of the data as in the original example above. This could capture the situation of the firm moving its production facility closer to the demand market.

The Euler method again requires 87 iterations for convergence and yields the following equilibrium solution:

\[ Q_{11}^* = 346.86, \quad Q_{21}^* = 653.14, \quad q_1^* = 67.34, \quad \pi_{11}^* = 663.15. \]

The firm’s total cost is now 581,840.07 and the contractor’s profit is 165,230.62. The value of \( q' \) is now 78.67. The average quality increases, with the quantity of the product produced by the firm having increased. Also, the price charged by the contractor decreases but the quality level of its product increases.

For both the examples, the underlying constraints are satisfied, consisting of the demand constraint, the nonnegativity constraints, as well as the upper bound on the contractor’s quality level. In addition, the variational inequality for this problem is satisfied.

It is easy to verify that the Jacobian of \( F \) for the variant is positive-definite with the only change in the Jacobian matrix above being that the 3 is replaced by 2.1.

I proceed to conduct a sensitivity analysis exercise. I return to the original example and increase the demand for the product at \( R_1 \) in increments of 1,000. The results of the computations are reported in Figures 5.3, 5.4, and 5.5 for the equilibrium product flows, the quality levels and the average quality \( q' \), and, finally, the equilibrium prices.

It is interesting to observe that, when the demand increases past a certain point, the contractor’s equilibrium quality level decreases to zero and stays at that level. Such unexpected insights may be obtained through a modeling and computational framework developed in this chapter. Of course, the results in this subsection are based on constructed examples. One may, of course, conduct other sensitivity analysis exercises and also utilize different underlying cost functions in order to tailor the general framework to specific firms’ needs and situations.
Figure 5.3. Equilibrium Product Flows as the Demand Increases for the Illustrative Example

Figure 5.4. Equilibrium Contractor Prices as the Demand Increases for the Illustrative Example
5.4. Additional Numerical Examples and Sensitivity Analysis

In this section, I applied the Euler method (cf. (5.19)-(5.21)) to compute solutions to examples that are larger than those in the preceding section. I report all of the input data as well as the output. The Euler method was initialized as in the illustrative example, except that the initial product flows are equally distributed among the available options for each demand market. I use the same convergence tolerance as previously.

Example 5.1

Example 5.1 consists of the topology given in Figure 5.6. There are two manufacturing plants owned by the firm and two possible contractors. The firm must satisfy the demands for its product at the two demand markets. The demand for the product at demand market $R_1$ is 1,000 and it is 500 at demand market $R_2$. $q^U$ is 100 and the weight $\omega$ is 1.
The production cost functions at the plants are:

\[ f_1 \left( \sum_{k=1}^{2} Q_{1k} \right) = (Q_{11} + Q_{12})^2 + 2(Q_{11} + Q_{12}), \]

\[ f_2 \left( \sum_{k=1}^{2} Q_{2k} \right) = 1.5(Q_{21} + Q_{22})^2 + 2(Q_{21} + Q_{22}). \]

The total transportation cost functions are:

\[ \hat{c}_{11}(Q_{11}) = 1.5Q_{11}^2 + 10Q_{11}, \quad \hat{c}_{12}(Q_{12}) = Q_{12}^2 + 25Q_{12}, \]

\[ \hat{c}_{21}(Q_{21}) = 1Q_{21}^2 + 5Q_{21}, \quad \hat{c}_{22}(Q_{22}) = 2.5Q_{22}^2 + 40Q_{22}. \]

The transaction cost functions are:

\[ tc_1(Q_{31} + Q_{32}) = .5(Q_{31} + Q_{32})^2 + .1(Q_{31} + Q_{32}), \]

\[ tc_2(Q_{41} + Q_{42}) = .25(Q_{41} + Q_{42})^2 + .2(Q_{41} + Q_{42}). \]

The contractors’ total cost of production and distribution functions are:

\[ sc_{11}(Q_{31}, q_1) = Q_{31}q_1, \quad sc_{12}(Q_{32}, q_1) = Q_{32}q_1, \]
\[ sc_{21}(Q_{41}, q_2) = 2Q_{41}q_2, \quad sc_{22}(Q_{42}, q_2) = 2Q_{42}q_2. \]

Their total quality cost functions are given by:

\[ \hat{c}_1(q_1) = 5(q_1 - 100)^2, \quad \hat{c}_2(q_2) = 10(q_1 - 100)^2. \]

The contractors’ opportunity cost functions are:

\[ oc_{11}(\pi_{11}) = .5(\pi_{11} - 10)^2, \quad oc_{12}(\pi_{12}) = (\pi_{12} - 10)^2, \]
\[ oc_{21}(\pi_{21}) = (\pi_{21} - 5)^2, \quad oc_{22}(\pi_{22}) = .5(\pi_{22} - 20)^2. \]

The firm’s cost of disrepute function is:

\[ dc(q^t) = 100 - q^t \]

where \( q^t \) (cf. (5.2)) is given by: \( \frac{Q_{31}q_1 + Q_{32}q_1 + Q_{41}q_2 + Q_{42}q_2 + Q_{11}100 + Q_{12}100 + Q_{21}100 + Q_{22}100}{1500} \).

The Euler method converges in 153 iterations and yields the following equilibrium solution. The computed product flows are:

\[ Q_{11}^* = 95.77, \quad Q_{12}^* = 85.51, \quad Q_{21}^* = 118.82, \quad Q_{22}^* = 20.27, \]
\[ Q_{31}^* = 213.59, \quad Q_{32}^* = 224.59, \quad Q_{41}^* = 571.83, \quad Q_{42}^* = 169.63. \]

The computed quality levels of the contractors are:

\[ q_1^* = 56.18, \quad q_2^* = 25.85, \]

and the computed prices are:

\[ \pi_{11}^* = 223.57, \quad \pi_{12}^* = 122.30, \quad \pi_{21}^* = 290.92, \quad \pi_{22}^* = 189.61. \]
The total cost of the firm is 610,643.26 and the profits of the contractors are: 5,733.83 and 9,294.44. The value of $q^t$ is 50.55.

In order to investigate the stability of the computed equilibrium for Example 5.1 (and a similar analysis holds for the subsequent examples), I construct the Jacobian matrix as follows. The Jacobian matrix of $F(X) = -\nabla U(Q,q,\pi)$, for this example, denoted by $J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, Q_{41}, Q_{42}, q_1, q_2, \pi_{11}, \pi_{12}, \pi_{21}, \pi_{22})$, is

$$J(Q_{11}, Q_{12}, Q_{21}, Q_{22}, Q_{31}, Q_{32}, Q_{41}, Q_{42}, q_1, q_2, \pi_{11}, \pi_{12}, \pi_{21}, \pi_{22}) = \begin{pmatrix} 5 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -6.67 \times 10^{-4} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -6.67 \times 10^{-4} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & -6.67 \times 10^{-4} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & -6.67 \times 10^{-4} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 20 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$  

This Jacobian matrix is positive-semidefinite. Hence, according to Theorem 5.5, since $F(X)$ is monotone, every supply chain network equilibrium, as defined in Definition 5.2, is a global monotone attractor for the utility gradient process.

Interestingly, in this example, the firm pays relatively higher prices for the products with a lower quality level from contractor $O_2$. This happens because the firm’s demand is fixed in each demand market, and, therefore, there is no pressure for quality improvement from the demand side (as would be the case if the demands were elastic (cf. Chapter 4). As reflected in the transaction costs with contractors $O_1$ and $O_2$, the firm is willing to pay higher prices to $O_2$, despite a lower quality level.

I then conduct sensitivity analysis. In particular, I investigate the effect of increases in the demands at both demand markets $R_1$ and $R_2$. The results for the new equilibrium product flows are depicted in Figure 5.7, with the results for the
Figure 5.7. Equilibrium Product Flows as the Demand Increases for Example 5.1

Figure 5.8. Equilibrium Contractor Prices as the Demand Increases for Example 5.1
equilibrium prices and the equilibrium quality levels in Figure 5.8 and Figure 5.9, respectively. The contractors consistently provide the majority of the product to the demand markets.

Observe from Figure 5.8 that as the demand increases the quality levels for both contractors drops to zero.

**Example 5.2**

A disruption to the original supply chain in Example 5.1 is considered. No business is immune from supply chain disruptions and, particularly, as noted by Purcell (2010), pharmaceuticals are especially vulnerable since they are high-value, highly regulated products. Moreover, pharmaceuticals disruptions may not only increase costs but may also create hazards and expose the pharmaceutical companies to damage to their brands and reputations (see also Nagurney, Yu, and Qiang (2011), Masoumi, Yu, and Nagurney (2012), and Qiang and Nagurney (2012)).
Specifically, I consider the following disruption. The data are as in Example 5.1 but contractor $O_2$ is not able to provide any production and distribution services. This could arise due to a natural disaster, adulteration in its production process, and/or an inability to procure an ingredient. Hence, the topology of the disrupted supply chain network is as in Figure 5.10.

The Euler method converges in 73 iterations and yields the following new equilibrium solution. The computed product flows are:

\[
Q_{11}^* = 218.06, \quad Q_{12}^* = 141.79, \quad Q_{21}^* = 260.20, \quad Q_{22}^* = 25.96, \\
Q_{31}^* = 521.74, \quad Q_{32}^* = 332.25.
\]

The computed quality level of the remaining contractor is:

\[
q_1^* = 14.60,
\]

and the computed prices are:

\[
\pi_{11}^* = 531.74, \quad \pi_{12}^* = 176.12.
\]
With the decrease in competition among the contractors, since there is now only one, rather than two, as in Example 5.1, the quality level of contractor \( O_1 \) drops, but the prices that it charges increases. The new average quality level is \( q^t=51.38 \). The total cost of the firm is now 1,123,226.62 whereas the profit of the first contractor is now 123,460.67, a sizable increase relative to that with competition as in Example 5.1.

**Example 5.3**

The data for Example 5.3 are the same as for Example 5.1, except that the opportunity cost functions \( oc_{11}, oc_{12}, oc_{21}, oc_{22} \) are all equal to 0.00.

The Euler method converges in 2915 iterations and yields the following equilibrium solution. The computed product flows are:

\[
Q^*_1 = 451.17, \quad Q^*_2 = 396.50, \quad Q^*_3 = 548.73, \quad Q^*_4 = 103.39,
\]

\[
Q^*_{11} = 0.00, \quad Q^*_{12} = 0.00, \quad Q^*_{21} = 0.00, \quad Q^*_{22} = 0.00.
\]

The computed quality levels of the contractors are:

\[
q^*_1 = 100.00, \quad q^*_2 = 100.00,
\]

and the computed prices are:

\[
\pi^*_1 = 3,060.70, \quad \pi^*_2 = 2,515.08, \quad \pi^*_3 = 3,060.56, \quad \pi^*_4 = 2,515.16.
\]

The total cost of the firm is 2,171,693.16 and the profits of the contractors are 0.00 and 0.00. The value of \( q^t \) is 100.00.

Because the opportunity costs are all zero, in order to improve the total profit, the contractors will charge the firm very large prices. Thus, the original firm would rather produce by itself than outsource to the contractors.
One can see, from this example, that, in addition to the total revenue term, each contractor must consider an outsourcing price related term, such as the opportunity cost, in its objective function. Without considering such a function, the outsourcing quantities will all be zero (cf. (5.14)), and, hence, a contractor would not secure any contracts from the firm.

**Example 5.4**

The data for Example 5.4 are the same as for Example 5.1, except the opportunity cost functions and the demand. The demand in $R_1$ is now 700, and the demand in $R_2$ is 100.

The contractors' opportunity cost functions now become:

\[oc_{11}(\pi_{11}) = 0.5(\pi_{11} - 2)^2 - 15265.29, \quad oc_{12}(\pi_{12}) = (\pi_{12} - 1)^2 - 513.93,\]

\[oc_{21}(\pi_{21}) = (\pi_{21} - 1)^2 - 35751.25, \quad oc_{22}(\pi_{22}) = 0.5(\pi_{22} - 2)^2 - 613.20.\]

The Euler method converges in 93 iterations and yields the following equilibrium solution. The computed product flows are:

\[Q_{11}^* = 69.12, \quad Q_{12}^* = 19.65, \quad Q_{21}^* = 77.99, \quad Q_{22}^* = 0.00,\]

\[Q_{31}^* = 174.72, \quad Q_{32}^* = 45.34, \quad Q_{41}^* = 378.17, \quad Q_{42}^* = 35.00.\]

The computed quality levels of the contractors are:

\[q_1^* = 77.99, \quad q_2^* = 58.68,\]

and the computed prices are:

\[\pi_{11}^* = 176.73, \quad \pi_{12}^* = 23.67, \quad \pi_{21}^* = 190.08, \quad \pi_{22}^* = 37.02.\]
The total cost of the firm is 204,701.28 and the profits of the contractors are: 12,366.75 and 7,614.84. The incurred opportunity costs at the equilibrium prices are all zero. Thus, in this example, the equilibrium prices that the contractors charge the firm are such that they are able to adequately recover their costs, and secure contracts.

The value of $q_t$ is 72.61.

5.5. Summary and Conclusions

In this chapter, I developed a supply chain network game theory model, in both equilibrium and dynamic versions, to capture contractor selection, based on the competition among the contractors in the prices that they charge as well as the quality levels of the products that they produce. I introduced a disrepute cost associated with the average quality at the demand markets. It is assumed that the firm is cost-minimizing whereas the contractors are profit-maximizing.

I utilized variational inequality theory for the formulation of the governing Bertrand-Nash equilibrium conditions and then revealed interesting dynamics associated with the evolution of the firm’s product flows, and the contractors’ prices and quality levels. I provided stability analysis results as well as an algorithm that can be interpreted as a discretization of the continuous-time adjustment processes. The methodological framework was illustrated through a series of numerical examples for which I reported the complete input and output data for transparency purposes. The numerical studies included sensitivity analysis results as well as a disruption to the supply chain network in that a contractor is no longer available for production and distribution. I also discussed the scenario in which the opportunity costs on the contractors’ side are identically equal to zero and the scenario in which the opportunity costs at the equilibrium are all zero.
This chapter is a contribution to the literature on outsourcing with a focus on quality. It also is an interesting application of game theory and associated methodologies.

In this chapter, only a single original firm is considered, and the in-house quality level of the single original firm is assumed to be perfect. In contrast, in the next chapter, Chapter 6, I consider quantity and quality competition among multiple firms, whose products are differentiated by their brands. Moreover, instead of assuming perfect in-house quality levels, quality levels are strategic variables of the firms in Chapter 6. Numerical examples and sensitivity analysis are also provided.
In this chapter, I develop a supply chain network game theory model with product differentiation, possible outsourcing of production and distribution, quantity and quality competition among the firms, and quality and price competition among the contractors. The model in this chapter extends the one in Chapter 5.

This chapter, in contrast to Chapter 5, focuses on the supply chain network consisting of multiple competing firms with product differentiation and their potential contractors. For each contractor, the number of contracted original firms is not predetermined. In addition, unlike Chapter 5, the in-house quality levels are no longer assumed to be perfect, and are considered as the strategic variables for the firms.

I assume that the original firms compete in the sense of Cournot-Nash. Each firm aims at determining its equilibrium in-house quality level, in-house production quantities and shipments, and outsourcing quantities, which satisfy demand requirements, so as to minimize its total cost and the weighted cost of disrepute. The contractors, in turn, are competing a la Bertrand-Nash in determining their optimal quality and price levels in order to maximize their individual total profits. This model provides each original firm with the equilibrium in-house quality level and the equilibrium make-or-buy and contractor-selection policy, with the demand for its product being satisfied in multiple demand markets. The demand of the product is still assumed to be fixed in this model, as in Chapter 5.
In this chapter, the governing equilibrium conditions of the model are formulated as a variational inequality problem. The Euler method (cf. Section 2.5.1), which provides a discrete-time adjustment process and tracks the evolution of the product flows, quality levels, and prices over time, is, again, utilized and convergence results given. Numerical examples are provided to illustrate how such a supply chain network game theory model can be applied in practice.

This chapter is based on Nagurney and Li (2015). It is organized as follows. In Section 6.1, I describe the decision-making behavior of the original firms, who compete in quantity and quality, and that of the contractors, who compete in price and quality. Then, I construct the supply chain network game theory model with product differentiation, possible outsourcing of production and distribution, and quality and price competition. I obtain the governing equilibrium conditions, and derive the equivalent variational inequality formulation. In Section 6.2, I describe the closed form expressions yielded by the Euler method for the prices and the quality levels, with the product flows being solved exactly by an equilibration algorithm. Convergence results are also provided. It is applied in Section 6.3 to compute solutions to numerical examples, along with sensitivity analysis, in order to demonstrate the generality and the applicability of the proposed framework. In Section 6.4, I summarize the results and present the conclusions.

6.1. The Supply Chain Network Model with Product Differentiation, Outsourcing, and Quality and Price Competition

In this section, I develop the supply chain network game theory model with product differentiation, outsourcing, price and quality competition among the contractors, and quantity and quality competition among the original firms. I consider a finite number of \( I \) original firms, with a typical firm denoted by \( i \), who compete noncooper-
tively. The products of the $I$ firms are not homogeneous but, rather, are differentiated by brands. Firm $i$; $i = 1, \ldots, I$, is involved in the processes of in-house manufacturing and distribution of its brand name product, and may subcontract its manufacturing and distribution activities to contractors who may be located overseas. I seek to determine the optimal product flows from each firm to its demand markets, along with the prices the contractors charge the firms, and the quality levels of the in-house manufactured products and the outsourced products.

For clarity and definiteness, I consider the supply chain network topology of the $I$ firms depicted in Figure 6.1. Each firm $i$; $i = 1, \ldots, I$, is considering in-house and outsourcing manufacturing facilities and serves the same $n_R$ demand markets. A link from each top-tiered node $i$, representing original firm $i$, is connected to its in-house manufacturing facility node $M^i$. The in-house distribution activities of firm $i$, in turn, are represented by links connecting $M^i$ to the demand nodes: $R_1, \ldots, R_{n_R}$.

![Figure 6.1. The Supply Chain Network Topology with Outsourcing and Multiple Competing Firms](image-url)
In this model, I capture the possible outsourcing of the products from the $I$ firms in terms of their production and delivery. As depicted in Figure 6.1, there are $n_O$ contractors available to each of the $I$ firms. Each firm may potentially contract to any of these contractors who then produce and distribute the product to the same $n_R$ demand markets. In Figure 6.1, hence, there are additional links from each topmost node $i; i = 1, \ldots, I$, to the $n_O$ contractor nodes, $O_1, \ldots, O_{n_O}$, each of which corresponds to the transaction activity of firm $i$ with contractor $j$. The next set of links, which emanates from the contractor nodes to the demand markets, reflect the production and delivery of the outsourced products to the $n_R$ demand markets.

As shown in Figure 6.1, the outsourced flows of different firms are represented by links with different colors, for convenience and clarity of depiction, which indicates that, in the processes of transaction and outsourcing of manufacturing and distribution, the outsourced products are still differentiated by brands. In Figure 6.1, red links are used to denote the outsourcing flows of firm 1 with blue links referring to those of firm $I$. The mathematical notation given in Table 6.1 explicitly handles such options.

In Figure 6.1, the top set of links consists of the manufacturing links, whether in-house or outsourcing, whereas the next set of links consists of the associated distribution links. For simplicity, let $n = 1 + n_O$, where $n_O$ is the number of potential contractors, denote the number of manufacturing plants, whether in-house or belonging to the contractors’.

The notation for the model is given in Table 6.1. The vectors are assumed to be column vectors. The optimal/equilibrium solution is denoted by “*”.

For consistency, I define and quantify quality levels, quality costs, production and delivery/transportation costs, transaction cost, and the disrepute cost in a manner similar to that in Chapter 5. However, unlike Chapter 5, the in-house quality levels are now strategic variables for the firms, therefore, in this model, in addition to the
Table 6.1. Notation for the Supply Chain Network Model with Product Differentiation, Outsourcing, and Quality and Price Competition

<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{ijk}$</td>
<td>the nonnegative amount of firm $i$’s product produced at manufacturing plant $j$, whether in-house or contracted, and delivered to demand market $R_k$, where $j = 1, \ldots, n$. For firm $i$, its own ${Q_{ijk}}$ elements are grouped into the vector $Q_i$, and all such vectors for all original firms are grouped into the vector $Q$, where $Q \in R_{+}^{n \times n \times R}$. All in-house quantities are grouped into the vector $Q^1 \in R_{+}^{n \times R}$, with all outsourcing quantities into the vector $Q^2 \in R_{+}^{n \times R_{O}}$.</td>
</tr>
<tr>
<td>$d_{ik}$</td>
<td>the demand for firm $i$’s product at demand market $R_k$; $k = 1, \ldots, n_R$.</td>
</tr>
<tr>
<td>$q_i$</td>
<td>the nonnegative quality level of firm $i$’s product produced in-house. I group the ${q_i}$ elements into the vector $q^1_i \in R_{+}^{n}$.</td>
</tr>
<tr>
<td>$q_{ij}$</td>
<td>the nonnegative quality level of firm $i$’s product produced by contractor $j$; $j = 1, \ldots, n_O$. I group all the ${q_{ij}}$ elements for firm $i$’s product into the vector $q^2_i \in R_{+}^{n_O}$. For each contractor $j$, I group its own ${q_{ij}}$ elements into the vector $q_j$, and then group all such vectors for all contractors into the vector $q^2 \in R_{+}^{n_O}$.</td>
</tr>
<tr>
<td>$q$</td>
<td>All the $q_i$ and $q_{ij}$ are grouped into the vector $q \in R_{+}^{n}$.</td>
</tr>
<tr>
<td>$\pi_{ijk}$</td>
<td>the price charged by contractor $j$; $j = 1, \ldots, n_O$, for producing and delivering a unit of firm $i$’s product to demand market $R_k$. The ${\pi_{ijk}}$ elements for contractor $j$ are grouped into the vector $\pi_j \in R_{+}^{1 \times n_{R}}$, and group all such vectors for all the contractors into the vector $\pi \in R_{+}^{n_{O} \times n_{R}}$.</td>
</tr>
<tr>
<td>$f_i(Q^1, q^1)$</td>
<td>the total in-house production cost of firm $i$.</td>
</tr>
<tr>
<td>$c_i(q^1)$</td>
<td>the total quality cost of firm $i$.</td>
</tr>
<tr>
<td>$tc_{ij}(\sum_{k=1}^{n_R} Q_{i,1+j,k})$</td>
<td>the total transaction cost associated with firm $i$ transacting with contractor $j$; $j = 1, \ldots, n_O$.</td>
</tr>
<tr>
<td>$\hat{c}_{ik}(Q^1, q^1)$</td>
<td>the total transportation cost associated with delivering firm $i$’s product manufactured in-house to demand market $R_k$; $k = 1, \ldots, n_R$.</td>
</tr>
<tr>
<td>$sc_{ij}(Q^2, q^2)$</td>
<td>the total cost of contractor $j$; $j = 1, \ldots, n_O$, to produce and distribute the product of firm $i$ to demand market $R_k$.</td>
</tr>
<tr>
<td>$\hat{c}_j(q^2)$</td>
<td>the total quality cost faced by contractor $j$; $j = 1, \ldots, n_O$.</td>
</tr>
<tr>
<td>$oc_{ijk}(\pi)$</td>
<td>the opportunity cost associated with pricing the product of firm $i$ at $\pi_{ijk}$, and delivering to demand market $R_k$, by contractor $j$; $j = 1, \ldots, n_O$.</td>
</tr>
<tr>
<td>$q'_i(Q_i, q_i, q^2_i)$</td>
<td>the average quality level of firm $i$’s product (cf. (6.3)).</td>
</tr>
<tr>
<td>$dc_i(q'_i(Q_i, q_i, q^2_i))$</td>
<td>the cost of disrepute of firm $i$.</td>
</tr>
</tbody>
</table>
outsourcing production and distribution costs, I also express the in-house production and distribution costs as functions that depend on both production quantities and quality levels. These functions are assumed to be convex in quality and quantity.

6.1.1 The Behavior of the Firms and Their Optimality Conditions

Unlike Chapter 5, the in-house quality levels are not assumed to be perfect anymore. As mentioned in Table 6.1, the quality level of firm \( i \)'s product produced in-house is denoted by \( q_i \), where \( i = 1, \ldots, I \), and the quality level of firm \( i \)'s product produced by contractor \( j \) is denoted by \( q_{ij} \), where \( j = 1, \ldots, n_O \). Both vary from a 0\% conformance level to a 100\% conformance level (cf. Section 1.1), so that, respectively,

\[
0 \leq q_{ij} \leq q^U, \quad i = 1, \ldots, I; \quad j = 1, \ldots, n_O, \tag{6.1}
\]

\[
0 \leq q_i \leq q^U, \quad i = 1, \ldots, I, \tag{6.2}
\]

where \( q^U \) is the value representing perfect quality level associated with the 100\% conformance level.

The average quality level of firm \( i \)'s product is, hence, an average quality level determined by the in-house quality level, the in-house product flows, the quality levels of the contractors, and the outsourced product flows. Thus, the average quality level for firm \( i \)'s product, both in-house and outsourced, can be expressed as

\[
q'_i(Q_i, q_i, q^2_i) = \frac{\sum_{k=1}^{n_R} \sum_{j=2}^{n} Q_{ijk} q_{i,j-1} + \sum_{k=1}^{n_R} Q_{i1k} q_i}{\sum_{k=1}^{n_R} d_{ik}}, \quad i = 1, \ldots, I. \tag{6.3}
\]

Compared to the average quality levels in Chapters 3 (cf. (3.7)) and 5 (cf. (5.2)), (6.3) captures the average quality levels at the demand markets in a competitive supply chain network with product differentiation and outsourcing.

Following Chapter 5, I assume that the disrepute cost of firm \( i \), \( dc_i(q_i'(Q_i, q_i, q^2_i)) \), is a monotonically decreasing function of the average quality level. Here, however, I
no longer assume, as was done in Chapter 5, which considered only a single firm, that
the firms produce their branded products with perfect quality. Hence, the average
quality level in (6.3) is a generalization of (5.2).

Each original firm $i$ selects the product flows $Q_i$ and the in-house quality level $q_i$,
whereas each contractor $j$, who competes for contracts in quality and price, selects
its outsourcing quality level vector $q_j$ and outsourcing price vector $\pi_j$.

Firm $i$’s utility function is denoted by $U^1_i$, where $i = 1, \ldots, I$. The objective of
original firm $i$ is to maximize its utility (cf. (6.4) below) represented by minus its
total costs that include the production cost, the quality cost, the transportation costs,
the payments to the contractors, the transaction costs, along with the weighted cost
of disrepute, with the nonnegative term $\omega_i$ denoting the weight that firm $i$ imposes
on the disrepute cost function (cf. Section 2.4).

As discussed in Chapter 5, the production and transportation cost functions only
capture the costs of production and delivery, and depend on both the quantities and
the quality levels. The production and the transportation costs and the quality cost
are entirely different costs, and they do not overlap.

Hence, firm $i$ seeks to

$$
\text{Maximize}_{Q_i, q_i} \quad U^1_i = -f_i(Q^1_i, q^1_i) - c_i(q^1_i) - \sum_{k=1}^{n_R} \hat{c}_{ik}(Q^1_i, q^1_i) - \sum_{j=1}^{n_O} \sum_{k=1}^{n_R} \pi_{ijk}^* Q_i,1+j,k \\
- \sum_{j=1}^{n_O} tc_{ij}(\sum_{k=1}^{n_R} Q_i,1+j,k) - \omega_i dc_i(q^1_i(q_i, q^1_i, q^2_i))
$$

(6.4)

subject to:

$$
\sum_{j=1}^{n} Q_{ijk} = d_{ik}, \quad i = 1, \ldots, I; k = 1, \ldots, n_R,
$$

(6.5)

$$
Q_{ijk} \geq 0, \quad i = 1, \ldots, I; j = 1, \ldots, n; k = 1, \ldots, n_R,
$$

(6.6)

and (6.2).
As revealed by constraint (6.5), a fixed demand is assumed for each firm, in order to capture the forecasted/projected demands at multiple demand markets. Note that, according to (6.4), the prices and the contractors’ quality levels are evaluated at their equilibrium values.

I assume that all the cost functions in (6.4) are continuous, twice continuously differentiable, and convex. The original firms compete in a noncooperative in the sense of Nash with each one trying to maximize its own utility. The strategic variables for each original firm \( i \) are all the in-house and the outsourcing flows produced and shipped by firm \( i \) and its in-house quality level.

I define the feasible set \( K_i \) as

\[
K_i = \{(Q_i, q_i)|Q_i \in R_{+}^{nn} \text{ with (6.5) satisfied and } q_i \text{ satisfying (6.2)}\}.
\]

All \( K_i; i = 1, \ldots, I \), are closed and convex. I also define the feasible set \( K^1 = \prod_{i=1}^{I} K_i \).

**Definition 6.1: Supply Chain Network Cournot-Nash Equilibrium with Product Differentiation, Outsourcing, and Quality Competition**

An in-house and outsourced product flow pattern and in-house quality level \((Q^*, q^1*)\) \( \in K^1 \) is said to constitute a Cournot-Nash equilibrium if for each firm \( i; i = 1, \ldots, I \),

\[
U^1_i(Q_i^*, \hat{Q}_i^*, q_i^*, \hat{q}_i^*, Q^{2*}, \pi_i^*) \geq U^1_i(Q_i, \hat{Q}_i^*, q_i, \hat{q}_i^*, Q^{2*}, \pi_i^*), \quad \forall (Q_i, q_i) \in K_i,
\]  

(6.7)

where

\[
\hat{Q}_i^* \equiv (Q_1^*, \ldots, Q_{i-1}^*, Q_{i+1}^*, \ldots, Q_I^*),
\]

\[
\hat{q}_i^* \equiv (q_1^*, \ldots, q_{i-1}^*, q_{i+1}^*, \ldots, q_I^*).
\]

According to (6.7) (cf. Definition 2.7), a Cournot-Nash equilibrium is established if no firm can unilaterally improve upon its utility by selecting an alternative vector of in-house or outsourced product flows and quality level.
Next, I derive the variational inequality formulation of the Cournot-Nash equilibrium with product differentiation and outsourcing according to Definition 6.1 (cf. Cournot (1838), Nash (1950, 1951), Gabay and Moulin (1980), and Theorem 2.7) in the following theorem.

**Theorem 6.1**

Assume that, for each firm $i; i = 1, \ldots, I$, the utility function $U^1_i(Q, q^1, q^2, \pi^*_i)$ is concave with respect to its variables $Q_i$ and $q_i$, and is continuous and twice continuously differentiable. Then $(Q^*, q^1*) \in \mathcal{K}^1$ is a Cournot-Nash equilibrium according to Definition 6.1 if and only if it satisfies the variational inequality:

$$
\begin{align*}
&\sum_{i=1}^{I} \sum_{h=1}^{n} \sum_{m=1}^{n_R} \frac{\partial U^1_i(Q^*, q^{1*}, q^{2*}, \pi^*_i)}{\partial Q_{ihm}} \times (Q_{ihm} - Q^*_{ihm}) \\
&- \sum_{i=1}^{I} \frac{\partial U^1_i(Q^*, q^{1*}, q^{2*}, \pi^*_i)}{\partial q_i} \times (q_i - q^*_i) \geq 0,
\end{align*}
$$

\forall (Q, q^1) \in \mathcal{K}^1,

(6.8)

with notice that: for $h = 1; i = 1, \ldots, I; m = 1, \ldots, n_R$:

$$
\frac{\partial U^1_i}{\partial Q_{ihm}} = \left[ \frac{\partial f_i}{\partial Q_{ihm}} + \sum_{k=1}^{n_R} \frac{\partial c_{ik}}{\partial Q_{ihm}} + \omega_i \frac{\partial d_i}{\partial q_i} \frac{\partial q^*_i}{\partial q_{ihm}} \right],
$$

for $h = 2, \ldots, n; i = 1, \ldots, I; m = 1, \ldots, n_R$:

$$
\frac{\partial U^1_i}{\partial Q_{ihm}} = \left[ \pi^*_{ih-1,m} + \frac{\partial t_{c_i,h-1}}{\partial Q_{ihm}} + \omega_i \frac{\partial d_{c_i}}{\partial q_i} \frac{\partial q^*_i}{\partial Q_{ihm}} \right],
$$

for $i = 1, \ldots, I$:

$$
\frac{\partial U^1_i}{\partial q_i} = \left[ \frac{\partial f_i}{\partial q_i} + \frac{\partial c_{i}}{\partial q_i} + \sum_{k=1}^{n_R} \frac{\partial c_{ik}}{\partial q_i} + \omega_i \frac{\partial d_{c_i}}{\partial q_i} \frac{\partial q^*_i}{\partial q_i} \right].
$$
6.1.2 The Behavior of the Contractors and Their Optimality Conditions

The objective of contractor $j; j = 1, \ldots, n_O$, is profit maximization. Their revenues are obtained from the purchasing activities of the original firms, while their costs are the costs of production and distribution, the total quality costs, and the opportunity costs (cf. Section 5.1.2).

Similar as in Chapter 5, in this model, each contractor has, as its strategic variables, its quality levels for producing and distributing the original firms’ products, and the prices that charges the firms. I denote the utility of each contractor $j$ by $U^2_j$, with $j = 1, \ldots, n_O$, and note that it represents the profit. Hence, each contractor $j; j = 1, \ldots, n_O$, seeks to:

$$\text{Maximize}_{q_j, \pi_j} \quad U^2_j = \sum_{k=1}^{n_R} \sum_{i=1}^I \pi_{ijk}^* Q_{i,1+k,j}^* - \sum_{k=1}^{n_R} \sum_{i=1}^I sc_{ijk}(Q^2, q^2) - \hat{c}_j(q^2)$$

$$- \sum_{k=1}^{n_R} \sum_{i=1}^I oc_{ijk}(\pi) \quad (6.9)$$

subject to:

$$\pi_{ijk} \geq 0, \quad j = 1, \ldots, n_O; k = 1, \ldots, n_R, \quad (6.10)$$

and (6.1) for each $j$. are at their equilibrium values.

According to (6.9), the original firms’ outputs are evaluated at the equilibrium, since the contractors do not control these variables, and, hence, must respond to these outputs.

The cost functions in each contractor’s utility function are continuous, twice continuously differentiable, and convex. The contractors compete in a noncooperative in the sense of Nash, with each one trying to maximize its own profits.

The feasible sets are defined as $K_j \equiv \{(q_j, \pi_j)|q_j \text{ satisfies } (6.1) \text{ and } \pi_j \text{ satisfies } (6.10) \text{ for } j\}$, $K^2 \equiv \Pi_{j=1}^{n_O} K_j$, and $\mathcal{K} \equiv \mathcal{K}^1 \times \mathcal{K}^2$. All the above-defined feasible sets are convex.
Definition 6.2: A Bertrand-Nash Equilibrium with Price and Quality Competition

A quality level and price pattern \((q^*, \pi^*) \in \mathcal{K}^2\) is said to constitute a Bertrand-Nash equilibrium if for each contractor \(j; j = 1, \ldots, n_O\),

\[
U_j^2(Q^*, q_j^*, \pi_j^*, \hat{q}_j^*) \geq U_j^2(Q^*, q_j, \pi_j, \hat{q}_j^*), \quad \forall (q_j, \pi_j) \in K_j, \tag{6.11}
\]

where

\[
\hat{q}_j^* \equiv (q_1^*, \ldots, q_{j-1}^*, q_j^*+1, q_{j+1}^*, \ldots, q_{n_O}^*),
\]

\[
\hat{\pi}_j^* \equiv (\pi_1^*, \ldots, \pi_{j-1}^*, \pi_j^*+1, \pi_{j+1}^*, \ldots, \pi_{n_O}^*).
\]

According to (6.11) (cf. Definition 2.7), a Bertrand-Nash equilibrium is established if no contractor can unilaterally improve upon its profits by selecting an alternative vector of quality levels or prices charged to the original firms.

I present the variational inequality formulation of the Bertrand-Nash equilibrium according to Definition 6.2 (see, Bertrand (1883), Nash (1950, 1951), Gabay and Moulin (1980), and Theorem 2.7) in the following theorem.

**Theorem 6.2**

Assume that, for each contractor \(j; j = 1, \ldots, n_O\), the profit function \(U_j^2(Q^*, q^2, \pi)\) is concave with respect to the variables \(\pi_j\) and \(q_j\), and is continuous and twice continuously differentiable. Then \((q^*, \pi^*) \in \mathcal{K}^2\) is a Bertrand-Nash equilibrium according to Definition 6.2 if and only if it satisfies the variational inequality:

\[
- \sum_{l=1}^I \sum_{j=1}^{n_O} \frac{\partial U_j^2(Q^*, q^2, \pi^*)}{\partial q_{lj}} \times (q_{lj} - q_{lj}^*)
- \sum_{l=1}^I \sum_{j=1}^{n_O} \sum_{k=1}^{n_R} \frac{\partial U_j^2(Q^*, q^2, \pi^*)}{\partial \pi_{ljk}} \times (\pi_{ljk} - \pi_{ljk}^*) \geq 0,
\]
∀(q^2, π) ∈ K^2. \hspace{1cm} (6.12)

with notice that: for \( j = 1, \ldots, n_O; \ l = 1, \ldots, I: \)

\[
-\frac{\partial U_j^2}{\partial q_{lj}} = \sum_{i=1}^{I} \sum_{k=1}^{n_R} \frac{\partial s_{ijk}}{\partial q_{lj}} + \frac{\partial \hat{c}_j}{\partial q_{lj}},
\]

and for \( j = 1, \ldots, n_O; l = 1, \ldots, I; k = 1, \ldots, n_R: \)

\[
-\frac{\partial U_j^2}{\partial \pi_{ljk}} = \sum_{i=1}^{I} \sum_{r=1}^{n_R} \frac{\partial o_{ijr}}{\partial \pi_{ljk}} - Q_{l,1+j,k}^*.
\]

6.1.3 The Equilibrium Conditions for the Supply Chain Network with Product Differentiation, Outsourcing, and Quality and Price Competition

In equilibrium, the optimality conditions for all contractors and the optimality conditions for all the original firms must hold simultaneously, according to the definition below.

**Definition 6.3: Supply Chain Network Equilibrium with Product Differentiation, Outsourcing, and Quality and Price Competition**

The equilibrium state of the supply chain network with product differentiation, outsourcing, and quality and price competition is one where both variational inequalities (6.8) and (6.12) hold simultaneously.

**Theorem 6.3**

The equilibrium conditions governing the supply chain network model with product
differentiation, outsourcing, and quality competition are equivalent to the solution of the variational inequality problem: determine \((Q^*, q^1, q^2, \pi^*) \in \mathcal{K}\), such that:

\[
- \sum_{i=1}^{I} \sum_{h=1}^{n} \sum_{m=1}^{n_R} \frac{\partial U_1^1(Q^*, q^1, q^2, \pi^*)}{\partial Q_{ihm}} (Q_{ihm} - Q_{ihm}^*) - \sum_{i=1}^{I} \frac{\partial U_1^1(Q^*, q^1, q^2, \pi^*)}{\partial q_i} (q_i - q_i^*) \\
- \sum_{l=1}^{I} \sum_{j=1}^{n_O} \frac{\partial U_2^2(Q^2, q^2, \pi^*)}{\partial q_{lj}} (q_{lj} - q_{lj}^*) \\
- \sum_{l=1}^{I} \sum_{j=1}^{n_O} \sum_{k=1}^{n_R} \frac{\partial U_2^2(Q^2, q^2, \pi^*)}{\partial \pi_{ljk}} (\pi_{ljk} - \pi_{ljk}^*) \geq 0, \\
\forall (Q, q^1, q^2, \pi) \in \mathcal{K}.
\] (6.13)

**Proof:** The proof follows a manner similar to that for Theorem 5.3. □

Variational inequality (6.13) can be put into standard form (cf. (2.1a)): determine \(X^* \in \mathcal{K}\) such that:

\[
\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K},
\] (6.14)

where \(\langle \cdot, \cdot \rangle\) denotes the inner product in \(N\)-dimensional Euclidean space, and \(N = \text{Inn}_R + I + \text{In}_O + \text{In}_O \text{In}_R\). If the column vectors \(X \equiv (Q, q^1, q^2, \pi)\) and \(F(X) \equiv (F_1(X), F_2(X), F_3(X), F_4(X))\), such that:

\[
F_1(X) = \left[ \frac{\partial U_1^1(Q, q^1, q^2, \pi)}{\partial Q_{ihm}} ; h = 1, \ldots, n; i = 1, \ldots, I; m = 1, \ldots, n_R \right],
\]

\[
F_2(X) = \left[ \frac{\partial U_1^1(Q, q^1, q^2, \pi)}{\partial q_i} ; i = 1, \ldots, I \right],
\]

\[
F_3(X) = \left[ \frac{\partial U_2^2(Q^2, q^2, \pi)}{\partial q_{lj}} ; l = 1, \ldots, I; j = 1, \ldots, n_O \right],
\]

\[
F_4(X) = \left[ \frac{\partial U_2^2(Q^2, q^2, \pi)}{\partial \pi_{ljk}} ; l = 1, \ldots, I; j = 1, \ldots, n_O; k = 1, \ldots, n_R \right],
\] (6.15)

then (6.13) can be re-expressed as (2.1a).
6.2. Explicit Formulae for the Euler Method Applied to the Supply Chain Network with Product Differentiation, Outsourcing, and Quality and Price Competition

The algorithm employed for the computation of the solution for the supply chain network game theory model is the Euler method (cf. Section 2.5.1).

Similar to the discussion in Chapter 5 (cf. (5.3)), at each iteration $\tau$, the values of the product flows can be determined by the exact equilibration algorithm of Dafermos and Sparrow (1969) (cf. Sections 2.5.1 and 5.3). Furthermore, one can determine the values for the in-house and the outsourced quality variables explicitly according to the following closed form expressions: for each original firm $i; i = 1, \ldots, I$:

$$
q_{i}^{\tau+1} = \min\{q^U, \max\{0, q_{i}^{\tau} - a_T \left( \frac{\partial f_i(Q_1^{\tau}, q_1^{\tau})}{\partial q_i} + \frac{\partial c_i(q_1^{\tau})}{\partial q_i} + \sum_{k=1}^{n_R} \frac{\partial \hat{c}_{ik}(Q_1^{\tau}, q_1^{\tau})}{\partial q_i} \right) + \omega_i \frac{\partial d c_i(q_i^{\tau})}{\partial q_i} \frac{\partial q_i^{\tau}}{\partial q_i} \}}
$$

(6.16)

and for the contractor and firm pairs: $l = 1, \ldots, I; j = 1, \ldots, n_O$:

$$
q_{lj}^{\tau+1} = \min\{q^U, \max\{0, q_{lj}^{\tau} - a_T \left( \sum_{i=1}^{I} \sum_{k=1}^{n_R} \frac{\partial s c_{ijkl}(Q_2^{\tau}, q_2^{\tau})}{\partial q_{lj}} + \frac{\partial \hat{c}_j(q_2^{\tau})}{\partial q_{lj}} \right) \}}
$$

(6.17)

Also, the following are the explicit formulae for the outsourced product prices: for $l = 1, \ldots, I; j = 1, \ldots, n_O; k = 1, \ldots, n_R$:

$$
\pi_{ljk}^{\tau+1} = \max\{0, \pi_{ljk}^{\tau} - a_T \left( \sum_{i=1}^{I} \sum_{r=1}^{n_R} \frac{\partial o c_{ijr}(\pi_{\tau})}{\partial \pi_{ljk}} - Q_{l,1+j,k}^{\tau} \right) \}.
$$

(6.18)

I now provide the convergence result. The proof follows using similar arguments as those in Theorem 2.17 and Nagurney and Zhang (1996).
Theorem 6.4

In the supply chain network game theory model with product differentiation, outsourcing, and quality competition, let $F(X) = -\nabla U(Q, q^1, q^2, \pi)$, where $I$ group all $U^1_i; i = 1, \ldots, I$, and $U^2_j; j = 1, \ldots, n_O$, into the vector $U(Q, q^1, q^2, \pi)$, be strongly monotone. Also, assume that $F$ is uniformly Lipschitz continuous (cf. Definition 2.6). Then there exists a unique equilibrium product flow, quality level, and price pattern $(Q^*, q^{1*}, q^{2*}, \pi^*) \in K$, and any sequence generated by the Euler method as given by (2.34), where $\{a_\tau\}$ satisfies $\sum_{\tau=0}^{\infty} a_\tau = \infty$, $a_\tau > 0, a_\tau \to 0$, as $\tau \to \infty$ converges to $(Q^*, q^{1*}, q^{2*}, \pi^*)$.

As discussed in Section 5.3, one can also expect the existence of a solution, given the continuity of the functions that make up $F(X)$, under less restrictive conditions that that of strong monotonicity.

The Euler method (cf. (5.19) and (6.16)-(6.18)), can be interpreted as a discrete-time adjustment process in which each iteration reflects a time step (cf. Section 2.5.1). The original firms determine, at each time step, their optimal production (and shipment) outputs and quality levels, whereas the contractors, at each time step (iteration), compute their optimal quality levels and the prices that they charge. The process evolves over time until the equilibrium product flows, quality levels, and contractor prices are achieved, at which point no one has any incentive to switch their strategies.

6.3. Numerical Examples and Sensitivity Analysis

In this section, I present numerical supply chain network examples for which I applied the Euler method (cf. (5.19) and (6.16)-(6.18)) to compute the equilibrium solutions. I present a spectrum of examples, accompanied by sensitivity analysis.

The supply chain network topology of the numerical examples is given in Figure 6.2. There are two original firms, both of which are located in North America. Their
products are substitutes but are differentiated by brands in the two demand markets, \( R_1 \) and \( R_2 \). Demand market \( R_1 \) is in North America, whereas demand market \( R_2 \) is in Asia. I use different colors to denote the outsourcing links of different original firms, with red links denoting the outsourcing links of firm 1 and blue links denoting those of firm 2.

![Figure 6.2. The Supply Chain Network Topology for the Numerical Examples](image)

Each original firm has one in-house manufacturing plant and two potential contractors. Contractor 1 and contractor 2 are located in North America and Asia, respectively. Each firm must satisfy the demands for its product at the two demand markets. The demands for firm 1’s product at \( R_1 \) and at \( R_2 \) are 50 and 100, respectively. The demands for firm 2’s product at \( R_1 \) and at \( R_2 \) are 75 and 150.

For the computation of solutions to the numerical examples, I implemented the Euler method using Matlab on a Lenovo Z580. The convergence tolerance is \( 10^{-6} \), so that the algorithm is deemed to have converged when the absolute value of the difference between each successive product flow, quality level, and price is less than or equal to \( 10^{-6} \). The sequence \( \{a_\tau\} \) is set to: \( \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\} \). I initialized the algorithm by equally distributing the product flows among the paths joining the firm
top-node to the demand market, by setting the quality levels equal to 1 and the prices equal to 0.

Example 6.1

The data are as follows.

The production cost functions at the in-house manufacturing plants are:

\[ f_1(Q^1, q^1) = (Q_{111} + Q_{112})^2 + 1.5(Q_{111} + Q_{112}) + 2(Q_{211} + Q_{212}) + 0.2q_1(Q_{111} + Q_{112}), \]

\[ f_2(Q^1, q^1) = 2(Q_{211} + Q_{212})^2 + 0.5(Q_{211} + Q_{212}) + (Q_{111} + Q_{112}) + 0.1q_2(Q_{211} + Q_{212}). \]

The total transportation cost functions for the in-house manufactured products are:

\[ \hat{c}_{11}(Q_{111}) = Q_{111}^2 + 5Q_{111}, \quad \hat{c}_{12}(Q_{112}) = 2.5Q_{112}^2 + 10Q_{112}, \]

\[ \hat{c}_{21}(Q_{211}) = 0.5Q_{211}^2 + 3Q_{211}, \quad \hat{c}_{22}(Q_{212}) = 2Q_{212}^2 + 5Q_{212}. \]

The in-house total quality cost functions for the two original firms are given by:

\[ c_1(q_1) = (q_1 - 80)^2 + 10, \quad c_2(q_2) = (q_2 - 85)^2 + 20. \]

The transaction cost functions are:

\[ tc_{11}(Q_{121} + Q_{122}) = 0.5(Q_{121} + Q_{122})^2 + 2(Q_{121} + Q_{122}) + 100, \]

\[ tc_{12}(Q_{131} + Q_{132}) = 0.7(Q_{131} + Q_{132})^2 + 0.5(Q_{131} + Q_{132}) + 150, \]

\[ tc_{21}(Q_{221} + Q_{222}) = 0.5(Q_{221} + Q_{222})^2 + 3(Q_{221} + Q_{222}) + 75, \]

\[ tc_{22}(Q_{221} + Q_{222}) = 0.75(Q_{231} + Q_{232})^2 + 0.5(Q_{231} + Q_{232}) + 100. \]

The contractors' total cost functions of production and distribution are:

\[ sc_{111}(Q_{121}, q_{11}) = 0.5Q_{121}q_{11}, \quad sc_{112}(Q_{122}, q_{11}) = 0.5Q_{122}q_{11}, \]
\[
sc_{121}(Q_{131}, q_{12}) = 0.5Q_{131}q_{12}, \quad sc_{122}(Q_{132}, q_{12}) = 0.5Q_{132}q_{12}.
\]
\[
sc_{211}(Q_{221}, q_{21}) = 0.3Q_{221}q_{21}, \quad sc_{212}(Q_{222}, q_{21}) = 0.3Q_{222}q_{21}.
\]
\[
sc_{221}(Q_{231}, q_{22}) = 0.25Q_{231}q_{22}, \quad sc_{222}(Q_{232}, q_{22}) = 0.25Q_{232}q_{22}.
\]

The total quality cost functions of the contractors are:

\[
\hat{c}_1(q_{11}, q_{21}) = (q_{11} - 75)^2 + (q_{21} - 75)^2 + 15,
\]
\[
\hat{c}_2(q_{12}, q_{22}) = 1.5(q_{12} - 75)^2 + 1.5(q_{22} - 75)^2 + 20.
\]

The contractors’ opportunity cost functions are:

\[
oc_{111}(\pi_{111}) = (\pi_{111} - 10)^2, \quad oc_{121}(\pi_{121}) = 0.5(\pi_{121} - 5)^2,
\]
\[
oc_{112}(\pi_{112}) = 0.5(\pi_{112} - 5)^2, \quad oc_{122}(\pi_{122}) = (\pi_{122} - 15)^2,
\]
\[
oc_{211}(\pi_{211}) = 2(\pi_{211} - 20)^2, \quad oc_{221}(\pi_{221}) = 0.5(\pi_{221} - 5)^2,
\]
\[
oc_{212}(\pi_{212}) = 0.5(\pi_{212} - 5)^2, \quad oc_{222}(\pi_{222}) = (\pi_{222} - 15)^2.
\]

The original firms’ disrepute cost functions are:

\[
dc_1(q'_1) = 100 - q'_1, \quad dc_2(q'_2) = 100 - q'_2,
\]

where

\[
q'_1 = \frac{Q_{121}q_{11} + Q_{131}q_{12} + Q_{111}q_1 + Q_{122}q_{11} + Q_{132}q_{12} + Q_{112}q_1}{d_{11} + d_{12}},
\]

and

\[
q'_2 = \frac{Q_{221}q_{21} + Q_{231}q_{22} + Q_{211}q_2 + Q_{222}q_{21} + Q_{232}q_{22} + Q_{212}q_2}{d_{21} + d_{22}}.
\]

\(\omega_1\) and \(\omega_2\) are 1. \(q^U\) is 100.

The Euler method converges in 255 iterations and yields the following equilibrium solution.
The computed product flows are:

\[ Q_{111}^* = 13.64, \quad Q_{121}^* = 26.87, \quad Q_{131}^* = 9.49, \quad Q_{112}^* = 9.34, \quad Q_{122}^* = 42.85, \]

\[ Q_{132}^* = 47.81, \quad Q_{211}^* = 16.54, \quad Q_{221}^* = 47.31, \quad Q_{231}^* = 11.16, \quad Q_{212}^* = 12.65, \]

\[ Q_{222}^* = 62.90, \quad Q_{232}^* = 74.45. \]

The computed quality levels of the original firms and the contractors are:

\[ q_{1}^* = 77.78, \quad q_{2}^* = 83.61, \quad q_{11}^* = 57.57, \quad q_{12}^* = 65.45, \quad q_{21}^* = 58.47, \quad q_{22}^* = 67.87. \]

The equilibrium prices are:

\[ \pi_{111}^* = 23.44, \quad \pi_{112}^* = 47.85, \quad \pi_{121}^* = 14.49, \quad \pi_{122}^* = 38.91, \]

\[ \pi_{211}^* = 31.83, \quad \pi_{212}^* = 67.90, \quad \pi_{221}^* = 16.16, \quad \pi_{222}^* = 52.23. \]

Notice that, although the North American contractor produces at a lower quality and at a higher price at equilibrium, it produces and distributes more than the off-shore contractor to \( R_1 \), who is located in North America. This happens for two reasons. First, because of the fixed demands, no pressure for quality improvement is imposed from the demand side. Secondly, as reflected in the transaction costs with the North America contractor, firms are willing to outsource more to this contractor. The Asian contractor, who produces at higher quality levels and at lower prices at equilibrium, produces and distributes more to \( R_2 \). This happens because the contractor who charges lower prices and produces at higher quality levels is highly preferable in the demand market, \( R_2 \), with the larger demand for which the original firms need to outsource more.
The total costs of the original firms’ are, respectively, 11,419.90 and 24,573.94, with their incurred disrepute costs being 36.32 and 34.69. The profits of the contractors are 567.84 and 440.92. The values of \( q'_1 \) and \( q'_2 \) are, respectively, 63.68 and 65.31.

I now conduct sensitivity analysis by varying the weights that the firms impose on their disrepute costs, \( \omega \), which is the vector of \( \omega_i; \ i = 1, 2 \), with \( \omega = (0, 0), (1000, 1000), (2000, 2000), (3000, 3000), (4000, 4000), (5000, 5000) \).

I display the equilibrium product flows and the equilibrium quality levels, both the in-house and the outsourced ones, and the average quality levels, in Figure 6.3, with the equilibrium prices changed by each contractor, the disrepute cost, and the total cost of each original firm displayed in Figure 6.4.

As the weights of the disrepute costs increase, there is more pressure for firms to improve quality. Thus, all the quality levels increase. In addition, because the in-house activities are more capable of guaranteeing higher quality, the outputs of both firms are shifted in-house as the weights increase. As a result, the outsourcing prices decrease (see (6.12)). Moreover, as shown in Figure 6.4, as expected, the values of the incurred disrepute costs decrease as \( \omega \) increases, but the total costs of the original firms increase.

**Example 6.2**

In Example 6.2, both firms consider quality levels as variables affecting their in-house transportation costs. The transportation cost functions of the original firms, hence, now depend on in-house quality levels as follows:

\[
\hat{c}_{11}(Q_{111}, q_1) = Q_{111}^2 + 1.5Q_{111}q_1, \quad \hat{c}_{12}(Q_{112}, q_1) = 2.5Q_{112}^2 + 2Q_{112}q_1, \\
\hat{c}_{21}(Q_{211}, q_2) = .5Q_{211}^2 + 3Q_{211}q_2, \quad \hat{c}_{22}(Q_{212}, q_2) = 2Q_{212}^2 + 2Q_{212}q_2.
\]

The remaining data are identical to those in Example 6.1.
Figure 6.3. Equilibrium Product Flows and Quality Levels as $\omega$ Increases for Example 6.1
The Euler method converges in 298 iterations and yields the following equilibrium solution.

The computed product flows are:

\[ Q^*_{111} = 0.00, \quad Q^*_{121} = 36.42, \quad Q^*_{131} = 13.58, \quad Q^*_{112} = 0.00, \quad Q^*_{122} = 46.42, \]

\[ Q^*_{132} = 53.58, \quad Q^*_{211} = 0.00, \quad Q^*_{221} = 60.13, \quad Q^*_{231} = 14.87, \quad Q^*_{212} = 3.83, \]

\[ Q^*_{222} = 65.50, \quad Q^*_{232} = 80.68. \]
The computed quality levels of the original firms and the contractors are:

\[ q_1^* = 80, \quad q_2^* = 80.90, \quad q_{11}^* = 54.29, \quad q_{12}^* = 63.81, \quad q_{21}^* = 56.16, \quad q_{22}^* = 67.04. \]

The equilibrium prices are:

\[ \pi_{111}^* = 28.21, \quad \pi_{112}^* = 51.42, \quad \pi_{121}^* = 18.58, \quad \pi_{122}^* = 41.79, \]
\[ \pi_{211}^* = 35.03, \quad \pi_{212}^* = 70.50, \quad \pi_{221}^* = 19.87, \quad \pi_{222}^* = 55.34. \]

The total costs of the original firms are, respectively, 13,002.64 and 27,607.44, with incurred disrepute costs of 41.45 and 38.80. The profits of the contractors are, respectively, 967.96 and 656.78. The average quality levels of the original firms, \( q_1' \) and \( q_2' \), are 58.55 and 61.20.

I also conduct sensitivity analysis by varying the weights associated with the disrepute costs, \( \omega \), for \( \omega = (0, 0), (1000, 1000), (2000, 2000), (3000, 3000), (4000, 4000), (5000, 5000). \) The results of this sensitivity analysis are displayed in Figures 6.5 and 6.6.

I now discuss the results of the sensitivity analysis for Example 6.2. As shown in Figures 6.5 and 6.6, as the weights of the disrepute costs increase, the changing trends of all the variables and costs in Figures 6.5 and 6.6 are the same as those in Example 6.1, except the trends for \( Q_{231}^* \), \( q_{22}^* \), and \( \pi_{221}^* \). As \( \omega \) increases from 0 to 2000, \( Q_{231}^* \) increases. However, as \( \omega \) increases further, \( Q_{231}^* \) decreases. The trends of \( q_{22}^* \) and \( \pi_{221}^* \) then change accordingly.

The reason is as follows. Because now in-house transportation costs more than before, in order to satisfy the fixed demand \( d_{21} \), firm 2 tends to shift more production and distribution to the contractor with a good quality level, when \( \omega \) is small. This is why \( Q_{231}^* \) increases as \( \omega \) increases from 0 to 2000. Nevertheless, as \( \omega \) increases further, firm 2 is under greater pressure to improve quality. Therefore, firm 2 then shifts more production in-house, and, as a result, \( Q_{231}^* \) decreases.
Figure 6.5. Equilibrium Product Flows and Quality Levels as $\omega$ Increases for Example 6.2
For firm 1, whose cost functions are completely distinct from those of firm 2, it is always more cost-wise for it to improve quality by shifting product flows in-house. Thus, the changing trends of all the variables and costs of firm 1 in Figures 6.5 and 6.6 are either monotonically increasing or decreasing.

I now compare Examples 6.1 and 6.2 with \( \omega \) in both examples equal to 0. Please refer to the preceding figures. This case is interesting and informative since it represents the scenario that neither firm 1 nor firm 2 cares about its possible reputation.
loss due to its product/brand having a lower quality. After the incorporation of quality levels into both firms’ in-house transportation costs, it now costs more for both firms to transport the same amounts of their products manufactured in-house and to maintain the same in-house quality levels as in Example 6.1, as reflected by the in-house transportation functions in Examples 6.1 and 6.2. Thus, in Example 6.2, the equilibrium in-house product flows are lower as compared to those in Example 6.1, and, in order to satisfy the demands, the equilibrium outsourced flows of both firms are higher. The corresponding results are: the contractors charge more to the firms; the outsourcing quality levels of the contractors are lower; the contractors’ total profits increase, and the firms’ total costs are higher than those in Example 6.1.

**Example 6.3**

In Example 6.3, I consider the scenario that the in-house transportation from the two firms to each demand market gets much more congested than before, and each firm’s in-house quantities also affect the other firm’s in-house transportation costs. The total in-house transportation cost functions of the two firms now become:

\[
\hat{c}_{11}(Q_{111}, Q_{211}, q_1) = Q_{111}^2 + 1.5Q_{111}q_1 + 7Q_{211},
\]

\[
\hat{c}_{12}(Q_{112}, Q_{212}, q_1) = 2.5Q_{112}^2 + 2Q_{112}q_1 + 10Q_{212},
\]

\[
\hat{c}_{21}(Q_{211}, Q_{111}, q_2) = .5Q_{211}^2 + 3Q_{211}q_2 + 8Q_{111},
\]

\[
\hat{c}_{22}(Q_{212}, Q_{112}, q_2) = 2Q_{212}^2 + 2Q_{212}q_2 + 10Q_{112}.
\]

The remaining data are identical to those in Example 6.1.

The total costs of firm 1 and firm 2 associated with different \( \omega \) values are displayed in Tables 6.2 and 6.3, respectively. Note that, in Table 6.2, the total cost of firm 1 increases monotonically, whether \( \omega_1 \) or \( \omega_2 \) increases. The same result is inferred from Table 6.3 for firm 2.
Table 6.2. Total Costs of Firm 1 with Different Sets of $\omega_1$ and $\omega_2$

<table>
<thead>
<tr>
<th>$\omega_2$</th>
<th>$\omega_1 = 0$</th>
<th>$\omega_1 = 1000$</th>
<th>$\omega_1 = 2000$</th>
<th>$\omega_1 = 3000$</th>
<th>$\omega_1 = 4000$</th>
<th>$\omega_1 = 5000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>12,999.09</td>
<td>45,135.09</td>
<td>61,322.22</td>
<td>71,463.36</td>
<td>77,437.89</td>
<td>80,462.63</td>
</tr>
<tr>
<td>1000</td>
<td>13,218.71</td>
<td>45,348.05</td>
<td>61,535.18</td>
<td>71,676.32</td>
<td>77,650.85</td>
<td>80,675.60</td>
</tr>
<tr>
<td>2000</td>
<td>13,425.67</td>
<td>45,571.40</td>
<td>61,758.53</td>
<td>71,899.67</td>
<td>77,874.20</td>
<td>80,898.94</td>
</tr>
<tr>
<td>3000</td>
<td>13,666.29</td>
<td>45,812.52</td>
<td>61,999.65</td>
<td>72,140.79</td>
<td>78,115.32</td>
<td>81,114.01</td>
</tr>
<tr>
<td>4000</td>
<td>14,091.85</td>
<td>46,034.08</td>
<td>62,221.20</td>
<td>72,362.34</td>
<td>78,336.88</td>
<td>81,361.62</td>
</tr>
<tr>
<td>5000</td>
<td>14,091.85</td>
<td>46,239.00</td>
<td>62,426.12</td>
<td>72,567.26</td>
<td>78,541.80</td>
<td>81,566.54</td>
</tr>
</tbody>
</table>

Table 6.3. Total Costs of Firm 2 with Different Sets of $\omega_1$ and $\omega_2$

<table>
<thead>
<tr>
<th>$\omega_2$</th>
<th>$\omega_1 = 0$</th>
<th>$\omega_1 = 1000$</th>
<th>$\omega_1 = 2000$</th>
<th>$\omega_1 = 3000$</th>
<th>$\omega_1 = 4000$</th>
<th>$\omega_1 = 5000$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>27,585.65</td>
<td>28,203.96</td>
<td>28,561.15</td>
<td>28,798.10</td>
<td>29,005.24</td>
<td>29,187.92</td>
</tr>
<tr>
<td>1000</td>
<td>62,896.33</td>
<td>63,626.00</td>
<td>63,983.19</td>
<td>64,220.14</td>
<td>64,427.28</td>
<td>64,609.96</td>
</tr>
<tr>
<td>2000</td>
<td>92,753.88</td>
<td>93,312.11</td>
<td>93,669.30</td>
<td>93,906.25</td>
<td>94,113.39</td>
<td>94,296.07</td>
</tr>
<tr>
<td>3000</td>
<td>116,378.40</td>
<td>116,981.94</td>
<td>117,339.13</td>
<td>117,576.08</td>
<td>117,783.22</td>
<td>117,965.90</td>
</tr>
<tr>
<td>4000</td>
<td>135,237.43</td>
<td>135,872.91</td>
<td>136,230.10</td>
<td>136,467.05</td>
<td>136,674.19</td>
<td>136,856.87</td>
</tr>
<tr>
<td>5000</td>
<td>150,231.01</td>
<td>150,886.51</td>
<td>151,243.69</td>
<td>151,480.65</td>
<td>151,687.79</td>
<td>151,870.47</td>
</tr>
</tbody>
</table>

The reason is the following. As discussed for Examples 6.1 and 6.2, when $\omega_i; i = 1, 2$, increases, the in-house production quantities of firm $i$ increase. Now, in Example 6.3, because of the in-house production costs (as in Example 6.1) and the new in-house transportation costs, the increase of firm $i$’s in-house quantities would also increase the other firm’s total cost.

According to the results in Tables 6.2 and 6.3, strategically, if a firm has to increase the weight of its own disrepute cost, it is more cost-wise to increase it before the other firm does. If the firm increases its weight at the same time as, or after the other firm does, it would incur more cost under the same disrepute cost weight.

6.4. Summary and Conclusions

In this chapter, I developed a supply chain network game theory model with product differentiation, outsourcing, and price and quality competition. The original firms compete with one another in in-house quality levels and in-house and outsourced pro-

173
duction (and shipment) flows in order to minimize their total costs and the weighted disrepute costs. The contractors, in turn, compete in their quality levels and the prices that they charge the original firms for manufacturing and distributing the products to the demand markets. This model provides the optimal make-or-buy as well as contractor selection decisions for each original firm.

I modeled the impact of quality on in-house and outsourced production and transportation and on the reputation of each firm through the quantification of the quality levels, quality cost, and the disrepute cost, with the production and the transportation cost functions depending on both quantities and quality levels.

Variational inequality theory was employed in the formulations of the equilibrium conditions of the original firms, the contractors, and the supply chain network game theory model with product differentiation, possible outsourcing of production and distribution, and quality and price competition. The algorithm adopted was the Euler method, which provides a discrete-time adjustment process and tracks the evolution of the in-house and outsourced production (and shipment) flows, the in-house and the outsourced quality levels, and the prices over time. It also yields closed form explicit formulae at each iteration with nice features for computation for all variables except for the production/shipment ones, which are computed via an exact equilibration algorithm.

In order to demonstrate the generality of the model and the computational scheme, I provided solutions to a series of numerical examples, accompanied by sensitivity analysis.

Unlike this chapter, in the next chapter, Chapter 6, a multitiered supply chain network model with suppliers’ quality is developed. Different from contractors, which produce and deliver products for firms, suppliers provide firms with the components needed to assemble their products. In Chapter 6, I include the quality of suppliers and firms in producing the components in the model. The preservation/decay of quality
in the assembly process of each firm is also considered and modeled as a decision variable. Numerical examples along with sensitivity analysis are provided.
This chapter, unlike Chapters 5 and 6, focuses on the supply chain network problem with multiple competing firms and their potential suppliers. The competitive behavior of each tier of decision-makers in the supply chain network is described along with their strategic variables, which include quality of the components and, in the case of the firms, the quality of the assembly process itself.

Specifically, the potential suppliers may either provide distinct components to the firms, or provide the same component, in which case, they compete noncooperatively with one another in terms of quality and prices. The firms, in turn, are responsible for assembling the products under their brand names using the components needed and transporting the products to multiple demand markets. They also have the option of producing their own components, if necessary. The firms compete in product quantities, the quality preservation levels of their assembly processes, the contracted component quantities produced by the suppliers, and in in-house component quantities and quality levels. Each of the firms aims to maximize profits. The quality of an end product is determined by the qualities and quality levels of its components, produced both by the firms and the suppliers, the importance of the quality of each component to that of the end product, and the quality preservation level of its assembly process. Consumers at the demand markets respond to both the prices and the quality of the end products. In addition, in this chapter, I provide a formula to
quantify the quality of the finished product based on the quality of the individual components.

The governing equilibrium conditions of the supply chain network are formulated as a variational inequality problem and qualitative properties are presented. The algorithm, which is the modified projection method (cf. Section 2.5.2), accompanied with convergence results, is then applied to numerical supply chain network examples, along with sensitivity analysis in which the impacts of capacity disruptions and complete supplier elimination are investigated.

This chapter is based on Li and Nagurney (2015). It is organized as follows. In Section 7.1, I develop the multitiered supply chain network model with competing suppliers and competing firms. The variational inequality formulations for each tier followed by a unified variational inequality are derived. In Section 7.2, the qualitative properties of the equilibrium pattern, in particular, existence and uniqueness results, are presented. In Section 7.3, the explicit formulae for the computation of this supply chain network problem, which is solved by the modified projection method (cf. Section 2.5.2), are presented, along with conditions for convergence, which is then applied in Section 7.4 to compute solutions to numerical supply chain network examples accompanied by sensitivity analysis. I summarize the results and present the conclusions in Section 7.5.

7.1. The Supply Chain Network Model with Supplier Selection and Quality and Price Competition

In this section, I develop a multitiered supply chain network game theory model with suppliers and firms that procure components from the suppliers for their products, which are differentiated by brand. I consider a supply chain network consisting of $I$ firms, with a typical firm denoted by $i$, $n_S$ suppliers, with a typical supplier de-
noted by \( j \), and a total of \( n_R \) demand markets, with a typical demand market denoted by \( R_k \).

The firms compete noncooperatively, and each firm corresponds to an individual brand representing the product that it produces. Product \( i \), which is the product produced by firm \( i \), requires \( n_i \) different components, and the total number of different components required by the \( I \) products is \( n_I \). Each supplier may be able to produce a variety of components for each firm.

The \( I \) firms are involved in the processes of assembling the products using the components needed, transporting the products to the demand markets, and, possibly, producing the components of the products. The suppliers, in turn, are involved in the processes of producing and delivering the components of the products to the firms. Both in-house and contracted component production activities are captured in the model. Firms’ and suppliers’ production capacities/abilities are also considered.

The network topology of the problem is depicted in Figure 7.1. The first two sets of links from the top are links corresponding to distinct supplier components. The links from the top-tiered nodes \( j; j = 1, \ldots, n_S \), representing the suppliers, are connected to the associated manufacturing nodes, denoted by nodes \( 1, \ldots, n_I \). These links represent the manufacturing activities of the suppliers. The next set of links that emanates from \( 1, \ldots, n_I \) to the firms, denoted by nodes \( 1, \ldots, I \), reflects the transportation of the components to the associated firms. In addition, the links that connect nodes \( 1^i, \ldots, n_{I_i}^i \), which are firm \( i \)'s component manufacturing nodes, and firm \( i \) are the manufacturing links of firm \( i \) for producing its components.

The rest of the links in Figure 7.1 are links corresponding to the finished products. The link connecting firm \( i \) and node \( i' \), which is the assembly node of firm \( i \), represents the activity of assembling firm \( i \)'s product using the components needed, which may be produced by firm \( i \), the suppliers, or both. Finally, the links joining nodes \( 1', \ldots, I' \)
with demand market nodes \(1, \ldots, n_R\) correspond to the transportation of the products to the demand markets.

In this chapter, the optimal component production quantities and quality levels, both by the firms and by the suppliers, the optimal product shipments from the firms to the demand markets, the optimal quality preservation levels of the assembly processes of the firms, and the prices that the suppliers charge the firms for producing and delivering the components are determined. The firms compete noncooperatively under the Cournot-Nash equilibrium concept in product shipments, in-house and contracted component production quantities, in-house component quality levels, and the quality preservation levels of the assembly processes. The suppliers, in turn,
compete in Bertrand fashion in the prices that they charge the firms and the quality levels of the components produced by them. It is assumed that there is no information asymmetry between the firms and the suppliers.

The notation for the variables and parameters in the model is given in Table 7.1. The functions in the model are given in Table 7.2. The transaction cost is defined in a manner similar to that in Chapters 5 and 6. The vectors are assumed to be column vectors. The optimal/equilibrium solution is denoted by “∗”.

The following conservation of flow equation must hold:

\[ Q_{ik} = d_{ik}, \quad i = 1, \ldots, I; k = 1, \ldots, n_R. \]  \hspace{1cm} (7.1)

Hence, the quantity of a firm’s brand-name product consumed at a demand market is equal to the amount shipped from the firm to that demand market. In addition, the shipment volumes must be nonnegative, that is:

\[ Q_{ik} \geq 0, \quad i = 1, \ldots, I; k = 1, \ldots, n_R. \]  \hspace{1cm} (7.2)

The quality levels of the components are quantified as values between 0 and the perfect quality (cf. Section 1.1), that is:

\[ q^U_{il} \geq q^S_{jil} \geq 0, \quad j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{\mu}, \]  \hspace{1cm} (7.3)

\[ q^U_{il} \geq q^F_{il} \geq 0, \quad i = 1, \ldots, I; l = 1, \ldots, n_{\mu}, \]  \hspace{1cm} (7.4)

where \( q^U_{il} \) is the value representing the prefect quality level associated with firm \( i \)'s component \( l; i = 1, \ldots, I; l = 1, \ldots, n_{\mu}. \)
Table 7.1. Notation for the Supply Chain Network Model with Supplier Selection and Quality and Price Competition

<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_{jil}^S$</td>
<td>the nonnegative amount of firm $i$’s component $l$ produced by supplier $j$; $j = 1, \ldots, n_S$; $i = 1, \ldots, I$; $l = 1, \ldots, n_l$. For firm $i$, I group its ${Q_{jil}^S}$ elements into the vector $Q_i^S \in R_+^{n_S \sum_i n_l}$. All the ${Q_{jil}^S}$ elements are grouped into the vector $Q^S \in R_+^{n_S \sum_i n_l}$.</td>
</tr>
<tr>
<td>$q_{jil}^S$</td>
<td>the quality of firm $i$’s component $l$ produced by supplier $j$. For supplier $j$, I group its ${q_{jil}^S}$ elements into the vector $q_j^S \in R_+^{n_l \sum_i n_l}$, and group all such vectors into the vector $q^S \in R_+^{n_S \sum_i n_l}$.</td>
</tr>
<tr>
<td>$CAP_{jil}^S$</td>
<td>the capacity of supplier $j$ for producing firm $i$’s component $l$.</td>
</tr>
<tr>
<td>$\pi_{jil}$</td>
<td>the price charged by supplier $j$ for producing one unit of firm $i$’s component $l$. For supplier $j$, its ${\pi_{jil}}$ elements are grouped into the vector $\pi_j \in R_+^{n_l \sum_i n_l}$, and group all such vectors into the vector $\pi^S \in R_+^{n_S \sum_i n_l}$.</td>
</tr>
<tr>
<td>$Q_{il}^F$</td>
<td>the nonnegative amount of firm $i$’s component $l$ produced by firm $i$ itself. For firm $i$, I group its ${Q_{il}^F}$ elements into the vector $Q_i^F \in R_+^{n_l}$, and group all such vectors into the vector $Q^F \in R_+^{\sum_i n_l}$.</td>
</tr>
<tr>
<td>$q_{il}^F$</td>
<td>the quality of firm $i$’s component $l$ produced by firm $i$ itself. For firm $i$, its ${q_{il}^F}$ elements are grouped into the vector $q_i^F \in R_+^{n_l}$, and group all such vectors into the vector $q^F \in R_+^{\sum_i n_l}$.</td>
</tr>
<tr>
<td>$CAP_{il}^F$</td>
<td>the capacity of firm $i$ for producing its component $l$.</td>
</tr>
<tr>
<td>$q_i$</td>
<td>the average quality of firm $i$’s component $l$, produced both by the firm and by the suppliers.</td>
</tr>
<tr>
<td>$Q_{ik}$</td>
<td>the nonnegative shipment of firm $i$’s product from firm $i$ to demand market $R_k$; $k = 1, \ldots, n_R$. For firm $i$, I group its ${Q_{ik}}$ elements into the vector $Q_i \in R_+^{n_R}$, and group all such vectors into the vector $Q \in R_+^{n_R}$.</td>
</tr>
<tr>
<td>$\alpha_i^F$</td>
<td>the quality preservation level of the assembly process of firm $i$. All ${\alpha_i^F}$ elements are grouped into the vector $\alpha^F_i \in R_+^{n_R}$.</td>
</tr>
<tr>
<td>$q_i$</td>
<td>the quality associated with firm $i$’s product. I group all ${q_i}$ elements into the vector $q \in R_+^{n_R}$.</td>
</tr>
<tr>
<td>$d_{ik}$</td>
<td>the demand for firm $i$’s product at demand market $R_k$. I group all ${d_{ik}}$ elements into the vector $d \in R_+^{n_R}$.</td>
</tr>
<tr>
<td>$\theta_{il}$</td>
<td>the amount of component $l$ needed by firm $i$ to produce one unit product $i$.</td>
</tr>
<tr>
<td>$\omega_{il}$</td>
<td>the ratio of the importance of the quality of firm $i$’s component $l$ in one unit product $i$ to the quality associated with one unit product $i$ (i.e., $q_i$).</td>
</tr>
</tbody>
</table>
Table 7.2. Functions for the Supply Chain Network Model with Supplier Selection and Quality and Price Competition

<table>
<thead>
<tr>
<th>Function</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_{il}^{F}(Q^F, q^F) )</td>
<td>firm ( i )'s production cost for producing its component ( l; i = 1, \ldots, I; l = 1, \ldots, n_l ).</td>
</tr>
<tr>
<td>( f_i(Q, \alpha^F) )</td>
<td>firm ( i )'s cost for assembling its product using the components needed.</td>
</tr>
<tr>
<td>( c_{ik}^{F}(Q, q) )</td>
<td>firm ( i )'s transportation cost for shipping its product to demand market ( R_k; k = 1, \ldots, n_R ).</td>
</tr>
<tr>
<td>( tc_{ijl}(Q^S) )</td>
<td>the transaction cost paid by firm ( i ) for transacting with supplier ( j; j = 1, \ldots, n_S ), for its component ( l ).</td>
</tr>
<tr>
<td>( f_{jl}^S(Q^S, q^S) )</td>
<td>supplier ( j )'s production cost for producing component ( l ).</td>
</tr>
<tr>
<td>( \hat{c}_{jil}(Q^S, q^S) )</td>
<td>supplier ( j )'s transportation cost for shipping firm ( i )'s component ( l ).</td>
</tr>
<tr>
<td>( oc_{jil}(\pi) )</td>
<td>the opportunity cost of supplier ( j ) associated with pricing firm ( i )'s component ( l ) at ( \pi_{jil} ) for producing and transporting it.</td>
</tr>
<tr>
<td>( \rho_{ik}(d, q) )</td>
<td>the demand price for firm ( i )'s product at demand market ( R_k ).</td>
</tr>
</tbody>
</table>

The average quality level of product \( i \)'s component \( l \) is determined by all the quantities and quality levels of that component, produced both by firm \( i \) and by the suppliers, that is:

\[
q_{il} = \frac{q_{il}^{F} Q_{il}^{F} + \sum_{j=1}^{n_S} Q_{jil}^{S} q_{jil}^{S}}{Q_{il}^{F} + \sum_{j=1}^{n_S} Q_{jil}^{S}}, \quad i = 1, \ldots, I; l = 1, \ldots, n_l. \tag{7.5}
\]

Unlike the average quality levels developed in Chapters 3, 6, and 7, (7.5) measures the average quality levels of components needed for assembling the final products, which can be produced by the firm, the suppliers, or both.

The quality failure rate of a finished product was modeled as a weighted summation of those of its components by Chao, Iravani, and Savaskan (2009). In Economides (1999), the quality of the composite good is the minimum quality of the quality levels of its components. Reyniers and Tapiero (1995), Tagaras and Lee (1996), Hwang, Radhakrishnan, and Su (2006), Baiman, Fischer, and Rajan (2000), Lim (2001), Hsieh and Liu (2010) considered the quality of a product as the product of the quality effort of the supplier and that of the firm. In this chapter, the quality level associated with
product $i$ is determined by the average quality levels of its components, the importance of the quality of the components to the quality of the product, and the quality preservation level of the assembly process of firm $i$. It is expressed as:

$$q_i = \alpha_i^F \left( \sum_{l=1}^{n_{il}} \omega_{il} q_{il} \right), \quad i = 1, \ldots, I; l = 1. \quad (7.6)$$

Note that $\alpha_i^F$ captures the percentage of the quality preservation of product $i$ in the assembly process of the firm and lies between 0 and 1, that is:

$$0 \leq \alpha_i^F \leq 1, \quad i = 1, \ldots, I. \quad (7.7)$$

The decay of quality can hence be captured by $1 - \alpha_i^F$. In Nagurney and Masoumi (2012), Masoumi, Yu, and Nagurney (2012), Nagurney and Nagurney (2012), Yu and Nagurney (2013), and in Nagurney et al. (2013), arc multipliers that are similar to $\alpha_i^F$ are used to model the perishability of particular products, such as pharmaceuticals, human blood, medical nuclear products, and fresh produce, in terms of the percentages of flows that reach the successor nodes in supply chain networks.

Assume that the importance of the quality levels of all components of product $i$ sums up to 1, that is:

$$\sum_{l=1}^{n_{il}} \omega_{il} = 1, \quad i = 1, \ldots, I. \quad (7.8)$$

In view of (7.1), (7.5), and (7.6), the transportation cost functions of the firms $c_{ik}(Q, q)$ and the demand price functions $\rho_{ik}(d, q); i = 1, \ldots, I; k = 1, \ldots, n_R$, are redefined in quantities and quality levels of the components, both by the firms and by the suppliers, the quantities of the products, and the quality preservation levels of the assembly processes, that is:

$$\hat{c}_{ik}^F = \hat{c}_{ik}^F (Q, Q^F, Q^S, q^F, q^S, \alpha^F) = c_{ik}(Q, q), \quad i = 1, \ldots, I; k = 1, \ldots, n_R, \quad (7.9)$$
\[
\hat{\rho}_{ik} = \hat{\rho}_{ik}(Q, Q^F, Q^S, q^F, q^S, \alpha^F) = \rho_{ik}(d, q), \quad i = 1, \ldots, I; k = 1, \ldots, n_R. \tag{7.10}
\]

The demand price functions (7.10) are, typically, assumed to be monotonically decreasing in product quantities but increasing in terms of product quality levels.

As noted in Table 7.2, the assembly cost functions, the production cost functions, the transportation cost functions, the transaction cost functions, and the demand price functions are general functions in vectors of quantities and/or quality levels, due to the competition among firms and the competition among suppliers for resources and technologies. Furthermore, as in Chapters 3, 5, and 6, it is assumed that transportation activities affect quality in terms of quality preservation, and, thus, quality does not deteriorate during transportation. However, in this chapter, quality may deteriorate in the assembly processes. As in Chapter 3 (cf. (3.5a and b)), the costs of quality (Section 1.1) of the suppliers and the firms are included in the associated production costs.

This model is also capable of handling the case of outsourcing (cf. Chapters 5 and 6) by setting each \( n_{i}^{\mu}; i = 1, \ldots, I, \) to 1. In such a case, the contractors do the outsourced jobs of producing products and transporting them to the firms, and the firms do the packaging and labeling for their products and may also produce in-house.

### 7.1.1 The Behavior of the Firms and Their Optimality Conditions

Given the prices \( \pi_i^* \) of the components that the suppliers charge firm \( i, \) and the quality \( q_{i}^S^* \) of the components produced by the suppliers, the objective of firm \( i; i = 1, \ldots, I, \) is to maximize its utility/profit \( U_i^F. \) It is the difference between its total revenue and its total cost. The total cost includes the assembly cost, the production costs, the transportation costs, the transaction costs, and the payments to the suppliers.
Hence, firm $i$ seeks to

$$
\text{Maximize}_{Q_i, Q_i^F, Q_i^S, q_i^F, q_i^S, \alpha_i^F} U_i^F = \sum_{k=1}^{n_R} \hat{\rho}_{ik}(Q, Q^F, Q^S, q^F, q^S, \alpha^F) d_{ik} - f_i(Q, \alpha^F) \\
- \sum_{l=1}^{n_{li}} f_i^F(Q^F, q^F) - \sum_{k=1}^{n_R} c_{ik}^F(Q, Q^F, Q^S, q^F, q^S, \alpha^F) - \sum_{j=1}^{n_S} \sum_{l=1}^{n_{ji}} t_{cil}(Q^S) - \sum_{j=1}^{n_S} \sum_{l=1}^{n_{ji}} \pi_{jil}^{*} Q_{jil}^S
$$

subject to:

$$
\sum_{k=1}^{n_R} Q_{ik} \theta_{il} \leq \sum_{j=1}^{n_S} Q_{jil}^S + Q_{jil}^F, \quad i = 1, \ldots, I; l = 1, \ldots, n_{li}, \quad (7.12)
$$

$$
CAP_{jil}^S \geq Q_{jil}^S \geq 0, \quad j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{li}, \quad (7.13)
$$

$$
CAP_{jil}^F \geq Q_{jil}^F \geq 0, \quad i = 1, \ldots, I; l = 1, \ldots, n_{li}, \quad (7.14)
$$

and (7.1), (7.2), (7.4), and (7.7).

All the cost functions and demand price functions in (7.11) are continuous and twice continuously differentiable. The cost functions are convex in quantities and/or quality levels and have bounded second-order partial derivatives. The demand price functions have bounded first-order and second-order partial derivatives. Constraint (7.12) captures the material requirements in the assembly process. Constraints (7.13) and (7.14) indicate that the component production quantities should be nonnegative and limited by the associated capacities, which can capture the abilities of producing. If a supplier or a firm is not capable of producing a certain component, the associated capacity should be 0.

The firms compete in the sense of Nash. The strategic variables for each firm $i$ are the product shipments to the demand markets, the in-house component production quantities, the contracted component production quantities, which are produced by the suppliers, the quality levels of the in-house produced components, and the quality preservation level of its assembly process.
The feasible set is defined as $\mathcal{K}_i^F$ as $\mathcal{K}_i^F \equiv \{(Q_i, Q_i^F, Q_i^S, q_i^F, \alpha_i^F) | (7.1), (7.2), (7.4), (7.7), \text{ and } (7.12)-(7.14) \text{ are satisfied}\}$. All $\mathcal{K}_i^F$; $i = 1, \ldots, I$, are closed and convex. I define the feasible set $\mathcal{K}_i^F \equiv \Pi_{i=1}^I \mathcal{K}_i^F$.

**Definition 7.1: A Cournot-Nash Equilibrium**

A product shipment, in-house component production, contracted component production, in-house component quality, and assembly quality preservation pattern $(Q^*, Q_i^F^*, Q_i^S^*, q_i^F^*, \alpha_i^F^*) \in \mathcal{K}_i^F$ is said to constitute a Cournot-Nash equilibrium if for each firm $i; i = 1, \ldots, I$,

$$U_i^F(Q_i, \hat{Q}_i^*, Q_i^F, \hat{Q}_i^F, Q_i^S, \hat{Q}_i^S, q_i^F, \hat{q}_i^F, \alpha_i^F, \hat{\alpha}_i^F, \pi_i^*, q_i^S) \geq$$

$$U_i^F(Q_i, \hat{Q}_i, Q_i^F, \hat{Q}_i^F, Q_i^S, \hat{Q}_i^S, q_i^F, \hat{q}_i^F, \alpha_i^F, \hat{\alpha}_i^F, \pi_i^*, q_i^S),$$

$$\forall (Q_i, Q_i^F, Q_i^S, q_i^F, \alpha_i^F) \in \mathcal{K}_i^F,$$  \hspace{1cm} (7.15)

where

$$\hat{Q}_i^* \equiv (Q_1^*, \ldots, Q_{i-1}^*, Q_{i+1}^*, \ldots, Q_I^*),$$

$$\hat{Q}_i^F^* \equiv (Q_1^{F*}, \ldots, Q_{i-1}^{F*}, Q_{i+1}^{F*}, \ldots, Q_I^{F*}),$$

$$\hat{Q}_i^S^* \equiv (Q_1^{S*}, \ldots, Q_{i-1}^{S*}, Q_{i+1}^{S*}, \ldots, Q_I^{S*}),$$

$$\hat{q}_i^F^* \equiv (q_1^{F*}, \ldots, q_{i-1}^{F*}, q_{i+1}^{F*}, \ldots, q_I^{F*}),$$

and

$$\hat{\alpha}_i^F^* \equiv (\alpha_1^{F*}, \ldots, \alpha_{i-1}^{F*}, \alpha_{i+1}^{F*}, \ldots, \alpha_I^{F*}).$$

According to (7.15) (cf. Definition 2.7), a Cournot-Nash equilibrium is established if no firm can unilaterally improve upon its profit by selecting an alternative vector of product shipments, in-house component production quantities, contracted component production quantities, in-house component quality levels, and the quality preservation level of its assembly process.
I now derive the variational inequality formulation of the Cournot-Nash equilibrium (see Cournot (1838), Nash (1950, 1951), and Gabay and Moulin (1980)) in the following theorem.

**Theorem 7.1**

Assume that, for each firm $i; i = 1, \ldots, I$, the utility function $U_i^F(Q, Q^F, Q^S, q^F, \alpha^F, \pi_i, q^S)$ is concave with respect to its variables in $Q_i$, $Q_i^F$, $Q_i^S$, $q_i^F$, and $\alpha_i^F$, and is continuous and twice continuously differentiable. Then $(Q^*, Q_i^F, Q_i^S, q_i^F, \alpha_i^F) \in \mathcal{K}_i^F$ is a Cournot-Nash equilibrium according to Definition 7.1 if and only if it satisfies the variational inequality:

$$
- \sum_{i=1}^{I} \sum_{k=1}^{n_R} \frac{\partial U_i^F(Q_i^*, Q_i^F^*, Q_i^S^*, q_i^F^*, \alpha_i^F^*, \pi_i^*, q_i^S^*)}{\partial Q_{ik}} \times (Q_{ik} - Q_i^{*}) \\
- \sum_{i=1}^{I} \sum_{l=1}^{n_i} \frac{\partial U_i^F(Q_i^*, Q_i^F^*, Q_i^S^*, q_i^F^*, \alpha_i^F^*, \pi_i^*, q_i^S^*)}{\partial Q_{il}^F} \times (Q_{il} - Q_i^{*}) \\
- \sum_{j=1}^{n_S} \sum_{i=1}^{I} \sum_{l=1}^{n_i} \frac{\partial U_i^F(Q_i^*, Q_i^F^*, Q_i^S^*, q_i^F^*, \alpha_i^F^*, \pi_i^*, q_i^S^*)}{\partial Q_{jil}^S} \times (Q_{jil}^S - Q_{jil}^{*}) \\
- \sum_{i=1}^{I} \sum_{l=1}^{n_i} \frac{\partial U_i^F(Q_i^*, Q_i^F^*, Q_i^S^*, q_i^F^*, \alpha_i^F^*, \pi_i^*, q_i^S^*)}{\partial q_{il}^F} \times (q_{il} - q_i^{*}) \\
- \sum_{i=1}^{I} \frac{\partial U_i^F(Q_i^*, Q_i^F^*, Q_i^S^*, q_i^F^*, \alpha_i^F^*, \pi_i^*, q_i^S^*)}{\partial \alpha_i^F} \times (\alpha_i^* - \alpha_i^{*}) \geq 0,
$$

with notice that: for $i = 1, \ldots, I$; $k = 1, \ldots, n_R$:

$$
\frac{\partial U_i^F}{\partial Q_{ik}} = \frac{\partial f_i(Q_i, q_i, \alpha_i, \pi_i)}{\partial Q_{ik}} + \sum_{h=1}^{n_R} \frac{\partial g_i^F(Q_i, Q_i^F, q_i^F, q_i^S, \alpha_i^F, \pi_i^F)}{\partial Q_{ik}} \times d_{ih} \\
- \sum_{h=1}^{n_R} \frac{\partial h_i(Q_i, Q_i^F, q_i^F, q_i^S, \alpha_i^F)}{\partial Q_{ik}} \times d_{ih},
$$

for $i = 1, \ldots, I$; $l = 1, \ldots, n_i$:

$$
\frac{\partial U_i^F}{\partial Q_{il}^F} = \sum_{n=1}^{n_i} \frac{\partial f_i^F(Q_i^F, q_i^F)}{\partial Q_{il}^F} + \sum_{h=1}^{n_R} \frac{\partial g_i^F(Q_i, Q_i^F, q_i^F, q_i^S, \alpha_i^F, \pi_i^F)}{\partial Q_{il}^F} \times d_{ih} - \sum_{h=1}^{n_R} \frac{\partial h_i(Q_i, Q_i^F, q_i^F, q_i^S, \alpha_i^F)}{\partial Q_{il}^F} \times d_{ih}.
$$
for \( j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_p; \):

\[
- \frac{\partial U_F}{\partial Q_{jil}} = \left[ \sum_{h=1}^{n_i} \frac{\partial c_{iwh}(Q^S)}{\partial Q_{jil}} + \sum_{h=1}^{n_R} \frac{\partial \tilde{c}_{ih}(Q, Q^F, Q^S, Q^F, q, q^S, \alpha^F)}{\partial Q^S_{jil}} \right]
- \sum_{h=1}^{n_R} \frac{\partial \tilde{\rho}_{ih}(Q, Q^F, Q^S, q^F, q^S, \alpha^F)}{\partial Q^S_{jil}} \times d_{ih},
\]

for \( i = 1, \ldots, I; l = 1, \ldots, n_p; \):

\[
- \frac{\partial U_F}{\partial Q^F_{il}} = \left[ \sum_{m=1}^{n_i} \frac{\partial f_{im}(Q^F, q^F)}{\partial Q^F_{il}} + \sum_{h=1}^{n_R} \frac{\partial \tilde{c}_{ih}(Q, Q^F, Q^S, q^F, q^S, \alpha^F)}{\partial Q^F_{il}} \right]
- \sum_{h=1}^{n_R} \frac{\partial \tilde{\rho}_{ih}(Q, Q^F, Q^S, q^F, q^S, \alpha^F)}{\partial Q^F_{il}} \times d_{ih},
\]

for \( i = 1, \ldots, I; \):

\[
- \frac{\partial U_F}{\partial \alpha^F_i} = \left[ \sum_{n=1}^{n_i} \frac{\partial f_{in}(Q^F, \alpha^F)}{\partial \alpha^F_i} \right] + \sum_{h=1}^{n_R} \frac{\partial \tilde{c}_{ih}(Q, Q^F, Q^S, q^F, q^S, \alpha^F)}{\partial \alpha^F_i} \times d_{ih},
\]

or, equivalently, in view of (7.1) and (7.12), \((Q^*, Q^F^*, Q^S^*, q^F^*, q^S^*, \alpha^F^*, \lambda^* \in K^F \) is a vector of the equilibrium product shipment, in-house component production, contracted component production, in-house component quality, and assembly quality preservation pattern and Lagrange multipliers if and only if it satisfies the variational inequality
if (see Bertsekas and Tsitsiklis (1989) page 287) the following holds:

\[
- \sum_{h=1}^{n_R} \frac{\partial \phi_h(Q^*, Q^{F*}, Q^{S*}, q^{F*}, q^{S*}, \alpha^{F*})}{\partial q_{ih}^*} \times Q_{ih}^* \times (q_{ih}^* - q_{ih}^{F*})
\]

\[
+ \sum_{i=1}^{l} \left[ \frac{\partial f_i(Q^{F*}, \alpha^{F*})}{\partial \alpha_i^*} + \sum_{h=1}^{n_R} \frac{\partial \phi_h(Q^*, Q^{F*}, Q^{S*}, q^{F*}, q^{S*}, \alpha^{F*})}{\partial \alpha_i^*} \right] + \sum_{h=1}^{n_R} \frac{\partial \phi_h(Q^*, Q^{F*}, Q^{S*}, q^{F*}, q^{S*}, \alpha^{F*})}{\partial \alpha_i^i} \times Q_{ih}^* \times (\alpha_i^* - \alpha_i^{F*})
\]

\[
- \sum_{h=1}^{n_R} \frac{\partial \phi_h(Q^*, Q^{F*}, Q^{S*}, q^{F*}, q^{S*}, \alpha^{F*})}{\partial q_{ih}^*} \times Q_{ih}^* \times (Q_{ih} - \lambda_i^*) \geq 0, \quad \forall (Q_i, Q_i^{F}, Q_i^{S}, q_i^{F}, q_i^{S}, \alpha_i^{F}, \lambda_i) \in K_i^{F},
\]

(7.17)

where \( K_i^{F} \equiv \Pi_{h=1}^{l} R_i^{F} \) and \( R_i^{F} \equiv \{(Q_i, Q_i^{F}, Q_i^{S}, q_i^{F}, q_i^{S}, \alpha_i^{F}, \lambda_i)|\lambda_i \geq 0 \) with (7.2), (7.4), (7.7), (7.13), and (7.14) satisfied}. \( \lambda_i \) is the \( n_i \)-dimensional vector with component \( l \) being the element \( \lambda^*_{il} \) corresponding to the Lagrange multiplier associated with the \((i, l)\)-th constraint (7.12). Both the above-defined feasible sets are convex.

**Proof:** For a given firm \( i \), under the imposed assumptions, (7.16) holds if and only if (see Bertsekas and Tsitsiklis (1989) page 287) the following holds:

\[
- \sum_{h=1}^{n_R} \frac{\partial \phi_h(Q^*, Q^{F*}, Q^{S*}, q^{F*}, q^{S*}, \alpha^{F*})}{\partial q_{ih}^*} \times Q_{ih}^* \times (q_{ih}^* - q_{ih}^{F*})
\]

\[
+ \sum_{i=1}^{l} \left[ \frac{\partial f_i(Q^{F*}, \alpha^{F*})}{\partial \alpha_i^*} + \sum_{h=1}^{n_R} \frac{\partial \phi_h(Q^*, Q^{F*}, Q^{S*}, q^{F*}, q^{S*}, \alpha^{F*})}{\partial \alpha_i^i} \right] + \sum_{h=1}^{n_R} \frac{\partial \phi_h(Q^*, Q^{F*}, Q^{S*}, q^{F*}, q^{S*}, \alpha^{F*})}{\partial \alpha_i^i} \times Q_{ih}^* \times (\alpha_i^* - \alpha_i^{F*})
\]

\[
- \sum_{h=1}^{n_R} \frac{\partial \phi_h(Q^*, Q^{F*}, Q^{S*}, q^{F*}, q^{S*}, \alpha^{F*})}{\partial q_{ih}^*} \times Q_{ih}^* \times (Q_{ih} - \lambda_i^*) \geq 0, \quad \forall (Q_i, Q_i^{F}, Q_i^{S}, q_i^{F}, q_i^{S}, \alpha_i^{F}, \lambda_i) \in K_i^{F},
\]

(7.18)
Variational inequality (7.18) holds for each firm $i; i = 1, \ldots, I$, and, hence, the summation of (7.18) yields variational inequality (7.17). □

### 7.1.2 The Behavior of the Suppliers and Their Optimality Conditions

Given the $Q^S*$ determined by the firms, the objective of supplier $j; j = 1, \ldots, n_S$, is to maximize its total profit, denoted by $U_j^S$. Its revenue is obtained from the payments of the firms, while its costs are the costs of production and delivery and the opportunity costs (cf. Section 5.1.2). The strategic variables of a supplier are the prices that it charges the firms and the quality levels of the components that it produces.

The decision-making problem for supplier $j$ is as the following:

$$
\text{Maximize}_{\pi_j, q_j^S} \quad U_j^S = \sum_{i=1}^{I} \sum_{l=1}^{n_{il}} \pi_j Q_{jil}^* - \sum_{l=1}^{n_l} f_j^S(Q^S, q^S) - \sum_{i=1}^{I} \sum_{l=1}^{n_{il}} c_{jil}^S(Q^S, q^S) - \sum_{i=1}^{I} \sum_{l=1}^{n_{il}} oc_{jil}(\pi) \quad (7.19)
$$

subject to:

$$
\pi_{jil} \geq 0, \quad j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{il}, \quad (7.20)
$$

and (7.3).

The cost functions of each supplier are continuous, twice continuously differentiable, and convex, and have bounded second-order partial derivatives.

The suppliers compete in a noncooperative in the sense of Nash, with each one trying to maximize its own profit. The feasible sets are defined as $K_j^S \equiv \{(\pi_j, q_j^S)|\pi_j \in R^{I \sum_{i=1}^{I} n_{il}}_+ \text{ and } q_j^S \text{ satisfies (7.3) for } j\}, \ K^S \equiv \Pi_{j=1}^{n_S} K_j^S$, and $\bar{K} \equiv \bar{K}^F \times K^S$. All the above-defined feasible sets are convex.
Definition 7.2: A Bertrand-Nash Equilibrium

A price and contracted component quality pattern \((\pi^*, q^S_*) \in K^S\) is said to constitute a Bertrand-Nash equilibrium if for each supplier \(j; j = 1, \ldots, n_S\),

\[
U^S_j(Q^S_*, \pi^*_j, \hat{\pi}^*_j, q^S_*, \hat{q}^S_*) \geq U^S_j(Q^S_*, \pi^*_j, \pi^*_j, q^S_*, \hat{q}^S_*) , \quad \forall (\pi^*_j, q^S_j) \in K^S_j , \quad (7.21)
\]

where

\[
\hat{\pi}^*_j \equiv (\pi^*_1, \ldots, \pi^*_j-1, \pi^*_j+1, \ldots, \pi^*_n_S) \\
\hat{q}^S_\ast \equiv (q^S_1, \ldots, q^S_{j-1}, q^S_{j+1}, \ldots, q^S_{n_S}).
\]

According to (7.21) (cf. Definition 2.7), a Bertrand-Nash equilibrium is established if no supplier can unilaterally improve upon its profit by selecting an alternative vector of prices that it charges the firms and the quality levels of the components that it produces.

The variational inequality formulation of the Bertrand-Nash equilibrium according to Definition 7.2 (see Bertrand (1883), Nash (1950, 1951), Gabay and Moulin (1980), Nagurney (2006)) is given in the following theorem.

**Theorem 7.2**

Assume that, for each supplier \(j; j = 1, \ldots, n_S\), the profit function \(U^S_j(Q^S_*, \pi, q^S)\) is concave with respect to the variables in \(\pi_j\) and \(q^S_\ast\), and is continuous and twice continuously differentiable. Then \((\pi^*, q^S_*) \in K^S\) is a Bertrand-Nash equilibrium according to Definition 7.2 if and only if it satisfies the variational inequality:

\[
- \sum_{j=1}^{n_S} \sum_{i=1}^{n_S} \sum_{l=1}^{n_S} \frac{\partial U^S_j(Q^S_*, \pi^*, q^S_*)}{\partial \pi_{jil}} \times (\pi_{jil}^* - \pi^*_{jil})
\]
\[- \sum_{j=1}^{n_S} \sum_{i=1}^{I} \sum_{l=1}^{n_{il}} \frac{\partial U^S_j(Q^{S*}, \pi^*, q^{S*})}{\partial q^S_{jil}} \times (q^S_{jil} - q^S_{jil}^{*}) \geq 0, \forall (\pi, q^S) \in \mathcal{K}^S, \quad (7.22)\]

with notice that: for \( j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{il} :\)

\[- \frac{\partial U^S_j}{\partial \pi_{jil}} = \sum_{g=1}^{g} \sum_{m=1}^{n_{gim}} \frac{\partial c_{jgm}(\pi)}{\partial \pi_{jil}} - Q^S_{jil}, \]

for \( j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{il} :\)

\[- \frac{\partial U^S_j}{\partial q^S_{jil}} = \sum_{m=1}^{n_{il}} \frac{\partial f^S_{jm}(Q^{S*}, q^S)}{\partial q^S_{jil}} + \sum_{g=1}^{g} \sum_{m=1}^{n_{gim}} \frac{\partial \hat{c}_{jgm}(Q^{S*}, q^S)}{\partial q^S_{jil}}. \]

7.1.3 The Equilibrium Conditions for the Supply Chain Network with Supplier Selection and Quality and Price Competition

In equilibrium, the optimality conditions for all firms and the optimality conditions for all suppliers must hold simultaneously, according to the definition below.

**Definition 7.3: Supply Chain Network Equilibrium with Supplier Selection and Quality and Price Competition**

The equilibrium state of the multitiered supply chain network with suppliers is one where both variational inequalities (7.16), or, equivalently, (7.17), and (7.22) hold simultaneously.

**Theorem 7.3**

The equilibrium conditions governing the multitiered supply chain network model with suppliers and quality competition are equivalent to the solution of the variational inequality problem: determine \((Q^*, Q^{F*}, Q^{S*}, q^{F*}, \alpha^{F*}, \pi^*, q^{S*}) \in \mathcal{K}, \) such that:

\[- \sum_{i=1}^{I} \sum_{k=1}^{n_R} \frac{\partial U^F_i(Q^*, Q^{F*}, Q^{S*}, q^{F*}, \alpha^{F*}, \pi^*, q^{S*})}{\partial Q_{ik}} \times (Q_{ik} - Q^*_{ik}) \]

192
\[-\sum_{i=1}^{n_i} \sum_{l=1}^{n_i} \frac{\partial U_i^F(Q^*, Q'^*, Q'^*, q'^*, \alpha'^*, \pi'^*, q'^*)}{\partial Q_{il}^F} \times (Q_{il}^F - Q_{il}^{F'}) - \sum_{j=1}^{n_s} \sum_{i=1}^{n_i} \sum_{l=1}^{n_i} \frac{\partial U_i^F(Q^*, Q'^*, Q'^*, q'^*, \alpha'^*, \pi'^*, q'^*)}{\partial q_{il}^F} \times (q_{il}^F - q_{il}^{F'}) - \sum_{i=1}^{n_i} \frac{\partial U_i^F(Q^*, Q'^*, Q'^*, q'^*, \alpha'^*, \pi'^*, q'^*)}{\partial \alpha_i^F} \times (\alpha_i^F - \alpha_i'^* ) - \sum_{j=1}^{n_s} \sum_{i=1}^{n_i} \sum_{l=1}^{n_i} \frac{\partial U_j^S(Q'^*, \pi'^*, q'^*)}{\partial \pi_{jl}^F} \times (\pi_{jl}^F - \pi_{jl}^{F'}) \times (q_{jl}^F - q_{jl}^{F'}) \geq 0, \quad \forall (Q, Q^*, Q'^*, q^*, \alpha^*, \pi, q) \in \mathcal{K}, \quad (7.23)\]

or, equivalently: determine \((Q^*, Q'^*, Q'^*, q'^*, \alpha'^*, \lambda^*, \pi'^*, q'^*) \in \mathcal{K}, \) such that:

\[-\sum_{h=1}^{n_R} \frac{\partial h(Q^*, Q'^*, Q'^*, q'^*, \alpha'^*)}{\partial Q_{ih}} \times Q_{ih}^* - \hat{h}(Q^*, Q'^*, Q'^*, q'^*, \alpha'^*) + \sum_{l=1}^{n_i} \lambda_i^h \theta_{il} \times (Q_{ih}^* - Q_{ih}) \]

\[+ \sum_{j=1}^{n_s} \sum_{i=1}^{n_i} \sum_{m=1}^{n_m} \frac{\partial e_j^{Fm}(Q'^*, q'^*)}{\partial q_{il}^F} + \sum_{h=1}^{n_R} \frac{\partial e_{ih}^F(Q^*, Q'^*, Q'^*, q'^*, q'^*, \alpha'^*)}{\partial Q_{il}^F} \times Q_{il}^* - Q_{il}^{F'}) \times (Q_{il}^F - Q_{il}^{F'}) \]

\[-\sum_{j=1}^{n_s} \sum_{i=1}^{n_i} \sum_{m=1}^{n_m} \frac{\partial e_{ihm}(Q'^*)}{\partial Q_{jil}^F} + \sum_{h=1}^{n_R} \frac{\partial e_{ih}^S(Q^*, Q'^*, q'^*, \alpha'^*)}{\partial Q_{jil}^F} \times Q_{jil}^* - Q_{jil}^{S'}) \times (Q_{jil}^S - Q_{jil}^{S'}) \]

\[+ \sum_{j=1}^{n_s} \sum_{i=1}^{n_i} \sum_{m=1}^{n_m} \frac{\partial e_{jim}(Q'^*, q'^*)}{\partial q_{jil}^F} + \sum_{h=1}^{n_R} \frac{\partial e_{jih}(Q^*, Q'^*, Q'^*, q'^*, q'^*, \alpha'^*)}{\partial q_{jil}^F} \times Q_{jil}^* - Q_{jil}^{F'}) \times (Q_{jil}^F - Q_{jil}^{F'}) \]

\[-\sum_{h=1}^{n_R} \frac{\partial {\hat{h}}(Q^*, Q'^*, Q'^*, q'^*, \alpha'^*)}{\partial Q_{ih}} \times Q_{ih}^* - \hat{h}(Q^*, Q'^*, Q'^*, q'^*, \alpha'^*) + \sum_{l=1}^{n_i} \lambda_i^h \theta_{il} \times (Q_{ih}^* - Q_{ih}) \]

\[+ \sum_{j=1}^{n_s} \sum_{i=1}^{n_i} \sum_{m=1}^{n_m} \frac{\partial {\hat{e}}_{j^{Fm}}(Q'^*, q'^*)}{\partial q_{ih}} + \sum_{h=1}^{n_R} \frac{\partial {\hat{e}}_{ih}^F(Q^*, Q'^*, Q'^*, q'^*, q'^*, \alpha'^*)}{\partial q_{ih}} \times Q_{ih}^* - Q_{ih}^{F'}) \times (Q_{ih}^F - Q_{ih}^{F'}) \]

\[-\sum_{h=1}^{n_R} \frac{\partial {\hat{e}}_{ih}(Q^*, Q'^*, Q'^*, q'^*, \alpha'^*)}{\partial \alpha_i^F} \times \alpha_i^F - \alpha_i'^* \times Q_{ih}^* \]

193
where $\mathcal{K} \equiv \mathcal{K}^F \times \mathcal{K}^S$.

**Proof:** The proof follows a manner similar to that for Theorem 5.3. □

I now put variational inequality (7.24) into standard form (cf. (2.1a)): determine $X^* \in \mathcal{K}$ where $X$ is a vector in $\mathbb{R}^N$, $F(X)$ is a continuous function such that $F(X): X \mapsto \mathcal{K} \subset \mathbb{R}^N$, and

$$
\langle F(X^*), X - X^* \rangle \geq 0, \quad \forall X \in \mathcal{K},
$$

where $\langle \cdot, \cdot \rangle$ is the inner product in the $N$-dimensional Euclidean space, $N = I n_\mu + 3 \sum_{i=1}^I n_\mu + 3 n_S \sum_{i=1}^I n_\mu + I$, and $\mathcal{K}$ is closed and convex. Define the vector $X \equiv (Q, Q^F, Q^S, q^F, q^S, \alpha^F, \alpha^S, \lambda, \pi, q^S)$ and the vector $F(X) \equiv (F^1(X), F^2(X), F^3(X), F^4(X), F^5(X), F^6(X), F^7(X), F^8(X))$, such that:

$$
F^1(X) = \left[ \frac{\partial f_i(Q, \alpha^F)}{\partial Q_{ik}} + \sum_{h=1}^{n_R} \frac{\partial^2 f_i(Q, Q^F, Q^S, q^F, q^S, \alpha^F)}{\partial Q_{ik} \partial Q_{jh}} - \sum_{h=1}^{n_R} \frac{\partial f_{ih}(Q, Q^F, Q^S, q^F, q^S, \alpha^F)}{\partial Q_{ik}} \right] \times Q_{ik} - \lambda_{il};
$$

$$
i = 1, \ldots, I; l = 1, \ldots, n_\mu, \]

$$
F^3(X) = \left[ \pi_{jl} + \sum_{h=1}^{n_R} \sum_{m=1}^{n_S} \frac{\partial \pi_{ihm}(Q^S)}{\partial Q_{jil}} + \sum_{h=1}^{n_R} \frac{\partial^2 \pi_{ihm}(Q^S)}{\partial Q_{jil} \partial Q_{jmh}} - \sum_{h=1}^{n_R} \frac{\partial \pi_{ih}(Q, Q^F, Q^S, q^F, q^S, \alpha^F)}{\partial Q_{jil}} \times Q_{jl} \right]
$$

$$\left. - \lambda_{il}; j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_\mu, \right],
$$

$$
F^4(X) = \left[ \sum_{m=1}^{n_S} \frac{\partial f_{im}(Q^F, q^F)}{\partial q_{jil}} + \sum_{h=1}^{n_R} \frac{\partial^2 f_{im}(Q^F, Q^S, q^F, q^S, \alpha^F)}{\partial q_{jil} \partial q_{jih}} - \sum_{h=1}^{n_R} \frac{\partial f_{ih}(Q, Q^F, Q^S, q^F, q^S, \alpha^F)}{\partial q_{jil}} \times Q_{jl} ; i = 1, \ldots, I; l = 1, \ldots, n_\mu, \right],
$$

$$
F^5(X) = \left[ \frac{\partial f_i(Q^F, \alpha^F)}{\partial \alpha_{il}} + \sum_{h=1}^{n_R} \frac{\partial^2 f_i(Q, Q^F, Q^S, q^F, q^S, \alpha^F)}{\partial \alpha_{il} \partial \alpha_{h}} - \sum_{h=1}^{n_R} \frac{\partial f_{ih}(Q, Q^F, Q^S, q^F, q^S, \alpha^F)}{\partial \alpha_{il}} \times Q_{ih} ; i = 1, \ldots, I, \right],
$$

$$
F^6(X) = \left[ \sum_{j=1}^{n_S} Q_{jil} + Q_{jil} - \sum_{k=1}^{n_\mu} \pi_{ikl}; i = 1, \ldots, I; l = 1, \ldots, n_\mu, \right],
$$

194
Hence, (7.24) can be put into standard form (7.25).

Similarly, I also put variational inequality (7.23) into standard form: determine $Y^* \in \overline{K}$ where $Y$ is a vector in $R^M$, $G(Y)$ is a continuous function such that $G(Y) : Y \mapsto \overline{K} \subset R^M$, and

\[
\langle G(Y^*), Y - Y^* \rangle \geq 0, \quad \forall Y \in \overline{K}, \tag{7.27}
\]

where $M = I n_R + 2 \sum_{i=1}^I n_i + 3 n_S \sum_{i=1}^I n_i + I$, and $\overline{K}$ is closed and convex. Define $Y \equiv (Q, Q^F, Q^S, q^F, \alpha^F, \pi, q^S)$, $G(Y) \equiv (\frac{\partial U^F}{\partial Q_{ik}}, \frac{\partial U^F}{\partial Q^F_{il}}, \frac{\partial U^F}{\partial \alpha^F_i}, \frac{\partial U^F}{\partial \pi_{jil}}, -\frac{\partial U^S}{\partial q^F_{il}}); j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_i$. Hence, (7.23) can be put into standard form (7.27).

The equilibrium solution $(Q^*, Q^F^*, Q^S^*, q^F^*, \alpha^F^*, \pi^*, q^S^*)$ to (7.27) and the $(Q^*, Q^F^*, Q^S^*, q^F^*, \alpha^F^*, \pi^*, q^S^*)$ in the equilibrium solution to (7.25) are equivalent for this multitiered supply chain network problem. In addition to $(Q^*, Q^F^*, Q^S^*, q^F^*, \alpha^F^*, \pi^*, q^S^*)$, the equilibrium solution to (7.25) also contains the equilibrium Lagrange multipliers $(\lambda^*)$.

7.2. Qualitative Properties

In this section, I present some qualitative properties of the solution to variational inequality (7.25) and (7.27), equivalently, (7.24) and (7.23). In particular, the existence result and the uniqueness result are presented. I also investigate the properties of the function $F$ given by (7.26) that enters variational inequality (7.25) and the function $G$ that enters variational inequality (7.27).

In a multitiered supply chain network with suppliers, it is reasonable to expect that the price charged by each supplier $j$ for producing one unit of firm $i$’s component $l$, $\pi_{jil}$, is bounded by a sufficiently large value, since, in practice, each supplier cannot
charge unbounded prices to the firms. Therefore, the following assumption is not unreasonable:

**Assumption 7.1**

*Suppose that in our multtiered supply chain network model with suppliers and quality competition, there exist a sufficiently large \( \Pi \), such that,*

\[
\pi_{ji} \leq \Pi, \quad j = 1, \ldots, n_S; i = 1, \ldots, I; l = 1, \ldots, n_{Pl}.
\]  

(7.28)

With this assumption, I have the following existence result.

**Theorem 7.4**

*With Assumption 7.1 satisfied, there exists at least one solution to variational inequality (7.25) and (7.27), equivalently, (7.24) and (7.23).*

**Proof:** I first prove that there exists at least one solution to variational inequality (7.27) (cf. (7.23)). Note that, all the component quality levels, \( q_{ji}^S \) and \( q_{il}^F \), and quality preservation levels \( \alpha_i \) are bounded due to constraints (7.3), (7.4), and (7.7). Because of constraint (7.12), all the product quantities \( Q_{ik} \) are also bounded, since the components quantities are nonnegative and capacitated (cf. (7.13) and (7.14)). Therefore, with Assumption 7.1, the feasible set of variational inequality (7.27) is bounded, and, hence, the existence of a solution to (7.27) (cf. (7.23)) is guaranteed (cf. Theorem 2.2). Because (7.27) and (7.25) (cf. (7.24)) are equivalent (Nagurney and Dhanda (2000)), the existence of (7.25) is also guaranteed. \( \square \)

I now present a monotonicity result.
Theorem 7.5

Under the assumptions in Theorems 7.1 and 7.2, the $F(X)$ that enters variational inequality (7.25), is monotone (cf. Definition 2.3), that is,

$$
<F(X') - F(X''), X' - X'' > \geq 0, \quad \forall X', X'' \in K,
$$
(7.29)

and the $G(Y)$ that enters variational inequality (7.27) is also monotone,

$$
<G(Y') - G(Y''), Y' - Y'' > \geq 0, \quad \forall Y', Y'' \in \bar{K}.
$$
(7.30)

Proof: Let $Y' = (Q', Q^{S'}, Q^{F'}, q^{F'}, \alpha^{F'}, \pi', q^{S'})$, $Y'' = (Q'', Q^{S''}, Q^{F''}, q^{F''}, \alpha^{F''}, \pi'', q^{S''})$, $X' = (Q', Q^{S'}, Q^{F'}, \alpha^{F'}, \lambda', \pi', q^{S'})$, and $X'' = (Q'', Q^{S''}, Q^{F''}, \alpha^{F''}, \lambda'', \pi'', q^{S''})$.

Then the left-hand-side of (7.30) can be expanded to:

$$
\sum_{i=1}^{n} \sum_{k=1}^{n} \left[ - \frac{\partial U^{F}_i(Q', Q^{S'}, Q^{F'}, q^{F'}, \alpha^{F'}, \pi', q^{S'})}{\partial q_{ik}} - \frac{\partial U^{F}_i(Q'', Q^{S''}, Q^{F''}, q^{F''}, \alpha^{F''}, \pi'', q^{S''})}{\partial q_{ik}} \right] \times (Q'_{ik} - Q''_{ik})
$$

$$
+ \sum_{j=1}^{n} \sum_{l=1}^{n} \left[ - \frac{\partial U^{F}_j(Q', Q^{S'}, Q^{F'}, q^{F'}, \alpha^{F'}, \pi', q^{S'})}{\partial q_{jl}} - \frac{\partial U^{F}_j(Q'', Q^{S''}, Q^{F''}, q^{F''}, \alpha^{F''}, \pi'', q^{S''})}{\partial q_{jl}} \right] \times (Q'_{jl} - Q''_{jl})
$$

$$
+ \sum_{i=1}^{n} \sum_{l=1}^{n} \left[ - \frac{\partial U^{S}_i(Q', Q^{S'}, q^{F'}, \alpha^{F'}, \pi', q^{S'})}{\partial \alpha_{il}} - \frac{\partial U^{S}_i(Q'', Q^{S''}, q^{F''}, \alpha^{F''}, \pi'', q^{S''})}{\partial \alpha_{il}} \right] \times (Q'_{il} - Q''_{il})
$$

$$
+ \sum_{j=1}^{n} \sum_{l=1}^{n} \left[ - \frac{\partial U^{S}_j(Q', Q^{S'}, q^{F'}, \alpha^{F'}, \pi', q^{S'})}{\partial q_{jl}} - \frac{\partial U^{S}_j(Q'', Q^{S''}, q^{F''}, \alpha^{F''}, \pi'', q^{S''})}{\partial q_{jl}} \right] \times (Q'_{jl} - Q''_{jl})
$$

$$
+ \sum_{i=1}^{n} \sum_{k=1}^{n} \left[ \lambda_{i}^{l} \theta_{i}^{l} - \lambda_{i}^{k} \theta_{i}^{k} \right] \times (Q'_{ik} - Q''_{ik})
$$

$$
+ \sum_{i=1}^{n} \sum_{l=1}^{n} [\lambda_{i}^{l} - \lambda_{i}^{l}] \times (Q'_{il} - Q''_{il})
$$

$$
+ \sum_{i=1}^{n} \sum_{k=1}^{n} [\lambda_{i}^{k} - \lambda_{i}^{k}] \times (Q'_{ik} - Q''_{ik})
$$

197
\[
\begin{align*}
\text{The expression in (7.31), equivalently, (7.32), is greater than or equal to zero, since it is assumed in Theorems 7.1 and 7.2 that the profit functions are concave with respect to associated variables. (7.31) is derived from the left-hand-side of (7.30), so (7.30) holds true, and, hence, the } G(Y) \text{ that enters variational inequality (7.27) is monotone. Because (7.32) is also the left-hand-side of (7.29), the } F(X) \text{ that enters variational inequality (7.25) is also monotone.} \n\end{align*}
\]
Theorem 7.6

Assume that the function \( G(Y) \) in variational inequality (7.27) is strictly monotone (cf. Definition 2.4) on \( \bar{K} \). Then, if variational inequality (7.27) admits a solution, \((Q^*, Q_{F}^*, Q_{S}^*, q_{F}^*, \alpha_{F}^*, \pi^*, q_{S}^*)\), that is the only solution.

Theorem 7.7

The function that enters the variational inequality problem (7.25) is Lipschitz continuous (cf. Definition 2.6), that is,

\[
\| F(X') - F(X'') \| \leq L \| X' - X'' \|, \quad \forall X', X'' \in K, \text{ where } L > 0. \tag{7.33}
\]

Proof: Since it is assumed that all the cost functions have bounded second-order partial derivatives, and the demand price functions have bounded first-order and second-order partial derivatives, the result is direct by applying a mid-value theorem from calculus to the \( F(X) \) that enters variational inequality (7.25). □

7.3. Explicit formulae for the Modified Projection Method Applied to the Supply Chain Network Model with Supplier Selection and Quality and Price Competition

In this section, the realization of the modified projection method (cf. Section 2.5.2), for the computation of the solution to variational inequality (7.24) is described. In particular, the explicit formulae for the solution to variational inequality (2.37) with \( F(X) \) and \( X \) as defined in variational inequality (7.25) are as the following:

\[
\bar{Q}_{ik}^{T-1} = \max\{0, Q_{ik}^{T-1} + a(-\frac{\partial f_i(Q^{T-1}, \alpha_{F}^{T-1})}{\partial Q_{ik}} - \sum_{h=1}^{n_B} \frac{\partial \hat{c}_{ih}^{F}(Q^{T-1}, Q_{F}^{T-1}, Q_{S}^{T-1}, q_{F}^{T-1}, q_{S}^{T-1}, \alpha_{F}^{T-1})}{\partial Q_{ik}}) \}
\]

\[
+ \sum_{h=1}^{n_B} \frac{\partial \hat{\rho}_{ih}(Q^{T-1}, Q_{F}^{T-1}, Q_{S}^{T-1}, q_{F}^{T-1}, q_{S}^{T-1}, \alpha_{F}^{T-1})}{\partial Q_{ik}}) \times \bar{Q}_{ih}^{T-1}
\]
\[
\begin{aligned}
\dot{\psi}_{il}^{T-1} &= \min\left\{ 0, \psi_{il}^{T-1} + a(-\sum_{m=1}^{n_R} \partial f_{im}^{T-1}(Q_{il}^{F-1}, q_{il}^{F-1}) \partial Q_{il}^{F-1}) \right. \\
&\left. + \sum_{h=1}^{n_R} \partial \psi_h(Q_{il}^{T-1}, Q_{il}^{F-1}, q_{il}^{F-1}, q_{il}^{S-1}, \alpha_{il}^{F-1}) \times Q_{ih}^{T-1} \right\}; \quad j = 1, \ldots, n_S; i = 1, \ldots, n_I; l = 1, \ldots, n_{ij}, \\
\dot{\psi}_{ij}^{T-1} &= \min\left\{ 0, \psi_{ij}^{T-1} + a(-\sum_{m=1}^{n_R} \partial f_{im}^{T-1}(Q_{ij}^{F-1}, q_{ij}^{F-1}) \partial Q_{ij}^{F-1}) \right. \\
&\left. + \sum_{h=1}^{n_R} \partial \psi_h(Q_{ij}^{T-1}, Q_{ij}^{F-1}, q_{ij}^{F-1}, q_{ij}^{S-1}, \alpha_{ij}^{F-1}) \times Q_{ih}^{T-1} \right\}; \quad j = 1, \ldots, n_S; i = 1, \ldots, n_I; l = 1, \ldots, n_{ij}, \\
\dot{\psi}_{i}^{T-1} &= \min\left\{ 0, \psi_{i}^{T-1} + a(-\sum_{m=1}^{n_R} \partial f_{im}^{T-1}(Q_{i}^{F-1}, q_{i}^{F-1}) \partial Q_{i}^{F-1}) \right. \\
&\left. + \sum_{h=1}^{n_R} \partial \psi_h(Q_{i}^{T-1}, Q_{i}^{F-1}, q_{i}^{F-1}, q_{i}^{S-1}, \alpha_{i}^{F-1}) \times Q_{ih}^{T-1} \right\}; \quad j = 1, \ldots, n_S; i = 1, \ldots, n_I; l = 1, \ldots, n_{ij}, \\
\dot{\psi}_{l}^{T-1} &= \min\left\{ 0, \psi_{l}^{T-1} + a(-\sum_{m=1}^{n_R} \partial f_{im}^{T-1}(Q_{l}^{F-1}, q_{l}^{F-1}) \partial Q_{l}^{F-1}) \right. \\
&\left. + \sum_{h=1}^{n_R} \partial \psi_h(Q_{l}^{T-1}, Q_{l}^{F-1}, q_{l}^{F-1}, q_{l}^{S-1}, \alpha_{l}^{F-1}) \times Q_{lh}^{T-1} \right\}; \quad j = 1, \ldots, n_S; i = 1, \ldots, n_I; l = 1, \ldots, n_{ij}, \\
\dot{\psi}_{i}^{T-1} &= \min\left\{ 0, \psi_{i}^{T-1} + a(-\sum_{m=1}^{n_R} \partial f_{im}^{T-1}(Q_{i}^{F-1}, q_{i}^{F-1}) \partial Q_{i}^{F-1}) \right. \\
&\left. + \sum_{h=1}^{n_R} \partial \psi_h(Q_{i}^{T-1}, Q_{i}^{F-1}, q_{i}^{F-1}, q_{i}^{S-1}, \alpha_{i}^{F-1}) \times Q_{ih}^{T-1} \right\}; \quad j = 1, \ldots, n_S; i = 1, \ldots, n_I; l = 1, \ldots, n_{ij}, \\
\dot{\psi}_{l}^{T-1} &= \min\left\{ 0, \psi_{l}^{T-1} + a(-\sum_{m=1}^{n_R} \partial f_{im}^{T-1}(Q_{l}^{F-1}, q_{l}^{F-1}) \partial Q_{l}^{F-1}) \right. \\
&\left. + \sum_{h=1}^{n_R} \partial \psi_h(Q_{l}^{T-1}, Q_{l}^{F-1}, q_{l}^{F-1}, q_{l}^{S-1}, \alpha_{l}^{F-1}) \times Q_{lh}^{T-1} \right\}; \quad j = 1, \ldots, n_S; i = 1, \ldots, n_I; l = 1, \ldots, n_{ij}, \\
\end{aligned}
\]

In addition, the explicit formulae for the solution to variational inequality (2.38) are as the following:
If Assumption 7.1 is satisfied, the modified projection method described above converges to the solution of variational inequality (7.25).
Proof: Existence of a solution to variational inequality (7.25) follows from Theorem 7.4, monotonicity follows from Theorem 7.5, and Lipschitz continuity, in turn, follows from Theorem 7.7. According to Theorem 2.18, the proof is complete. □

7.4. Numerical Examples and Sensitivity Analysis

In this section, I applied the modified projection method (cf. (7.34a)-(7.35h)) to several numerical examples accompanied by extensive sensitivity analysis. The modified projected method was implemented in Matlab on a Lenovo Z580. I set $a = 0.003$ in the algorithm with the convergence tolerance $\epsilon = 10^{-4}$. The product and component quantities were initialized to 30 and the prices, quality levels, quality preservation levels, and the Lagrange multipliers to 0.

Example 7.1

Consider the supply chain network topology given in Figure 7.2 in which firm 1 serves demand market $R_1$ and procure the components of its product from supplier 1. The firm also has the option of producing the components needed by itself. The product of firm 1 requires only one component $1^1$. 2 units of $1^1$ are needed for producing one unit of firm 1’s product. Thus,

$$\theta_{11} = 2.$$ 

Component $1^1$ corresponds to node 1 in the second tier and node $1^1$ in the third tier in Figure 2 below.

The capacity of the supplier is:

$$CAP^{S}_{111} = 120.$$ 

The firm’s capacity for producing its component is:

$$CAP^{F}_{11} = 80.$$
The value that represents the perfect component quality is:

\[ q_{11}^U = 75. \]

The supplier’s production cost is:

\[ f_{11}(Q_{111}^S, q_{111}^S) = 5Q_{111}^S + 0.8(q_{111}^S - 62.5)^2. \]

The supplier’s transportation cost is:

\[ c_{111}^S(Q_{111}^S, q_{111}^S) = 0.5Q_{111}^S + 0.2(q_{111}^S - 125)^2 + 0.3Q_{111}^S q_{111}^S, \]

and its opportunity cost is:

\[ oc_{111}(\pi_{111}) = 0.7(\pi_{111} - 100)^2. \]
The firm’s assembly cost is:

\[ f_1(Q_{11}, \alpha^F_1) = 0.75Q^2_{11} + 200\alpha^F_1 + 200\alpha^F_1 + 25Q_{11}\alpha^F_1. \]

The firm’s production cost for producing its component is:

\[ f^{F_1}(Q^{F_1}, q^{F_1}) = 2.5Q^{F_2} + 0.5(q^{F_1} - 60)^2 + 0.1Q^{F_1}q^{F_1}, \]

and its transaction cost is:

\[ tc_{111}(Q^S_{111}) = 0.5Q^S_{111} + Q^S_{111} + 100. \]

The firm’s transportation cost for shipping its product to the demand market is:

\[ c^{F_1}(Q_{11}, q_{11}) = 0.5Q^2_{11} + 0.02q_{11}^2 + 0.1Q_{11}q_{11}, \]

and the demand price function at demand market \( R_1 \) is:

\[ \rho_{11}(d_{11}, q_{11}) = -d_{11} + 0.7q_{11} + 1000, \]

where \( q_{11} = \alpha^F_1\omega_{11} Q^{F_1}q^{F_1}_{11} + Q^S_{111}q^{S}_{111} \) and \( \omega_{11} = 1. \)

The equilibrium solution obtained using the modified projection method is:

\[ Q^{*}_{11} = 89.26, \quad Q^{F*}_{11} = 60.16, \quad Q^{S*}_{111} = 118.38, \quad q^{F*}_{11} = 71.17, \]

\[ q^{S*}_{111} = 57.25, \quad \pi^{*}_{11} = 184.53, \quad \alpha^F_1 = 1.00, \quad \lambda^{*}_{11} = 305.25. \]

with the induced demand, demand price, and product quality being

\[ d_{11} = 89.26, \quad \rho_{11} = 954.10, \quad q_{11} = 61.94. \]

The profit of the firm is 33,331.69, and the profit of the supplier is 13,218.67.
For this example, the eigenvalues of the symmetric part of the Jacobian matrix of $G(Y)$ (cf. (7.27)) are 0.0016, 0.0101, 0.0140, 0.0169, 0.0439, 0.0503, 5.5468, which are all positive. Therefore, $\nabla G(Y)$ is positive-definite, and $G(Y)$ is strictly monotone. The uniqueness of the solution $(Q^*, Q^{F*}, Q^{S*}, q^{F*}, \alpha^{F*}, \pi^*, q^{S*})$ (cf. Theorem 7.6) and the convergence of the modified projection method (cf. Theorem 7.8) are then guaranteed.

In Example 7.1, the capacities of the firm and the supplier do not constrain the production of the components, since, at the equilibrium, the component quantities are lower than the associated capacities. However, in some cases, due to disruptions to capacities, such as disasters and strikes, firms and suppliers may not always be able to operate under desired capacities. In this sensitivity analysis, I investigate the impacts of the capacities that constrain the production of the components on the quantities, prices, quality levels, and the profits of the firm and the supplier.

First, I maintain the capacity of the firm at 80, and vary the capacity of the supplier from 0 to 20, 40, 60, 80, 100, and 120. The results of equilibrium quantities, quality levels, prices, and profits are shown in Figures 7.3 and 7.4. As indicated in Figure 7.3.b, when the capacity of the supplier is 0, the firm has to produce the components for its product by itself, at full capacity, which is 80. This production pressure limits the firm’s ability to produce with high quality, which causes a low in-house component quality (cf. Figure 7.3.d). Based on the data in this example, purchasing components from the supplier is always cheaper than producing them in-house. Therefore, as the capacity of the supplier increases, the firm buys more components from the supplier and tends to be more dependent on the supplier in component production. Thus, the contracted component quantity increases (cf. Figure 7.3.a), and the in-house component quantity decreases (cf. Figure 7.3.b). In addition, with more components provided by the supplier, the firm is now able
Figure 7.3. Equilibrium Component Quantities, Equilibrium Component Quality Levels, Equilibrium Product Quantity (Demand), and Product Quality as the Capacity of the Supplier Varies
Figure 7.4. Equilibrium Quality Preservation Level, Equilibrium Lagrange Multiplier, Demand Price, Equilibrium Contracted Price, the Supplier’s Profit, and the Firm’s Profit as the Capacity of the Supplier Varies
to assemble more products for profit maximization, which leads to an increase in demand (cf. Figure 7.3.e) and in profit (cf. Figure 7.4.f).

Since there is no competition on the supplier’s side, as the firm becomes more dependent on the supplier, it charges more to the firm to maximize its profit (cf. Figure 7.4.d). For the same reason, the supplier’s incentive to improve quality decreases, which leads to a reduction in contracted quality (cf. Figure 7.3.c). After the capacity of the supplier achieves a certain value (e.g., 100), as the capacity of the supplier increases, the contracted quantity and price keep increasing. This results in an extremely high payment to the supplier and a large transaction cost, and hence a decline in the profit of the firm (cf. Figure 7.4.f). The profit of the supplier always increases as its capacity expands (cf. Figure 7.4.e).

Moreover, when the supplier’s capacity is 20, the in-house component quality achieves a higher value than before (cf. Figure 7.3.d), because the firm is able to pay for quality improvement for more profit at this point. However, it decreases ever after, since, given the high payment to the supplier and the high transaction cost, the firm is unable to produce a higher quality anymore. This also explains the trend of the product quality (cf. Figure 7.3.f) and that of the demand price (cf. Figure 7.4.c). The highest product quality and the highest demand price are achieved when the supplier’s capacity is 20, after which they decrease.

Therefore, in the case of this example, the supplier would want to prevent disruptions to its own capacity in order to maintain a good profit. However, such disruptions may be beneficial for the firm’s profit and the quality of the product at the demand market. Hence, it may be wise for the firm to contract with competing suppliers who have capacities that are not so high to harm the profit of the firm.

As already noted, when the capacity of the supplier is 0, the quantity of the in-house produced component is bounded by the capacity of the firm, which is 80. This happens because the firm can actually produce more to improve its profit with higher
capacity. When the capacity of the firm is 80.78 or higher, the in-house component production does not have to operate at full capacity.

I then maintain the capacity of the supplier at 120, and vary the capacity of the firm from 0 to 20, 40, 60, and 80. The results are reported in Figures 7.5 and 7.6.

Most of the trends in Figures 7.5 and 7.6 follow a similar logic as that for Figures 7.3 and 7.4. However, as revealed in Figures 7.6.e and f, as the capacity of the firm increases, the profit of the supplier decreases, but that of the firm increases. Now, with higher capacity, the firm is more capable of producing more to satisfy the greater demand by itself, which weakens its dependence on the supplier and leads to a decline in the supplier’s profit. Therefore, disruptions to the firm’s capacity would benefit the profit of the supplier, but jeopardize the profit of the firm and the quality of the product at the demand market. Thus, the supplier would want to produce for firms who have low capacities and are, hence, more dependent on suppliers in component production.

As shown in Figure 7.5.a, when the capacity of the firm is 0, 20, and 40, the quantity of contracted component production is bounded by the capacity of the supplier. Actually, when the capacity of the supplier is no less than 141.71, 133.99, and 126.20, respectively, the supplier does not need to operate at full capacity.

This sensitivity analysis further sheds light on the investments in capacity changing for the supplier and for the firm. If the investment is higher than the associated profit improvement, it is not wise for the supplier or the firm to invest in themselves’ or each other’s capacity changing. Tables 7.3 and 7.4 below show the maximum acceptable investments for capacity changing for this sensitivity analysis. The first number in each cell is the maximum acceptable investment for the supplier, and the second is that for the firm. In the italic cells, the two numbers are with different signs.
Figure 7.5. Equilibrium Component Quantities, Equilibrium Component Quality Levels, Equilibrium Product Quantity (Demand), and Product Quality as the Capacity of the Firm Varies
Figure 7.6. Equilibrium Quality Preservation Level, Equilibrium Lagrange Multiplier, Demand Price, Equilibrium Contracted Price, the Supplier’s Profit, and the Firm’s Profit as the Capacity of the Firm Varies
Table 7.3. Maximum Acceptable Investments ($\times 10^3$) for Capacity Changing when the Capacity of the Firm Maintains 80 but that of the Supplier Varies

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>$CAP_{S11}=0$</th>
<th>20</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>100</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>-0.97, -5.89</td>
<td>1.90, 4.28</td>
<td>4.69, 7.20</td>
<td>6.60, 8.73</td>
<td>9.40, 8.88</td>
<td>12.25, 7.80</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>-2.86, -10.17</td>
<td>-2.00, -4.28</td>
<td>2.20, 2.92</td>
<td>4.70, 4.45</td>
<td>7.31, 4.00</td>
<td>10.36, 3.92</td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>-5.06, -13.09</td>
<td>-2.20, -2.92</td>
<td>2.50, 1.93</td>
<td>5.31, 1.98</td>
<td>8.16, 0.69</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>-7.57, -14.62</td>
<td>-4.69, 8.73</td>
<td>-2.50, 1.93</td>
<td>2.81, 0.93</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>-10.37, -14.77</td>
<td>-9.40, 8.88</td>
<td>-7.51, 4.60</td>
<td>-5.31, 1.68</td>
<td>-2.81, 0.15</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>120</td>
<td>-13.22, -13.69</td>
<td>-12.25, 7.80</td>
<td>-10.36, 4.74</td>
<td>-8.16, 0.93</td>
<td>-2.81, 1.08</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 7.4. Maximum Acceptable Investments ($\times 10^3$) for Capacity Changing when the Capacity of the Supplier Maintains 120 but that of the Firm Varies

<table>
<thead>
<tr>
<th>From</th>
<th>To</th>
<th>$CAP_{F11}=0$</th>
<th>20</th>
<th>40</th>
<th>60</th>
<th>80</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.00, 5.94</td>
<td>0.00, 9.77</td>
<td>-0.25, 11.10</td>
<td>-0.26, 11.10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>0.00, -5.94</td>
<td>0.00, 3.83</td>
<td>-0.25, 5.16</td>
<td>-0.26, 5.16</td>
<td></td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>0.25, -11.10</td>
<td>0.25, -5.16</td>
<td>0.25, -1.33</td>
<td>-0.01, 0.004</td>
<td></td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>0.26, -11.10</td>
<td>0.26, -5.16</td>
<td>0.26, -1.33</td>
<td>0.01, -0.004</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In Tables 7.3 and 7.4, for the cells in which both numbers are negative, it is not wise for the firm or the supplier to change the capacities at all, because their profits would decrease with the associated capacity change. For the italic cells that are with two opposite-sign numbers, the one with the negative number should prevent the other from investing in the associated capacity change, or, it should ask the other for a compensation which will prevent its profit from being compromised. This situation may occur only in 4 cases when the supplier’s capacity varies (cf. Table 7.3). However, in Table 7.4, it happens very often when the firm’s capacity varies, which is consistent with the results in the above sensitivity analysis. For the numbers that are 0, the associated profits will not be affected by the corresponding capacity changes.

In addition, if there is a capacity changing offer that costs more than the summation of the two numbers in the associated cell, it is not worthwhile for the supplier or the firm to accept the offer, since more profit cannot be obtained by doing so. If the offer costs less, the two parties should consider investing in the associated ca-
pacity change, and, if possible, negotiate on the separation of the payment between themselves.

**Example 7.2**

In Example 7.2, there are 2 firms competing with each other with differentiated but substitutable products in demand market $R_1$. The firms can procure the components for producing their products from suppliers 1 and 2 who also compete noncooperatively, and they can also produce the components needed by themselves.

Two components are required by the product of firm 1, components $1_1$ and $2_1$. 1 unit of $1_1$ and 2 units of $2_1$ are required for producing 1 unit of firm 1’s product. In order to produce 1 unit firm 2’s product, 2 units of $1_2$ and 1 unit of $2_2$ are needed. Therefore,

$$
\theta_{11} = 1, \quad \theta_{12} = 2, \quad \theta_{21} = 2, \quad \theta_{22} = 1.
$$

The ratio of the importance of the quality of the components to the quality of one unit product is:

$$
\omega_{11} = 0.2, \quad \omega_{12} = 0.8, \quad \omega_{21} = 0.4, \quad \omega_{22} = 0.6.
$$

The network topology of Example 7.2 is as in Figure 7.7. Components $1_1$ and $2_1$ are the same component, which correspond to nodes 1’s in the second tier of the figure. Components $2_1$ and $2_2$ are the same component, and they correspond to nodes 2’s in the second tier.

The other data are as follows:

The capacities of the suppliers are:

$$
CAP^S_{111} = 80, \quad CAP^S_{112} = 100, \quad CAP^S_{121} = 100, \quad CAP^S_{122} = 60,
$$

$$
CAP^S_{211} = 60, \quad CAP^S_{212} = 100, \quad CAP^S_{221} = 100, \quad CAP^S_{222} = 50.
$$
The firms’ capacities for in-house component production are:

\[ \text{CAP}^F_{11} = 30, \quad \text{CAP}^F_{12} = 30, \quad \text{CAP}^F_{21} = 30, \quad \text{CAP}^F_{22} = 30. \]

The values representing the perfect component quality are:

\[ q^U_{11} = 60, \quad q^U_{12} = 75, \quad q^U_{21} = 60, \quad q^U_{22} = 75. \]

The suppliers’ production costs are:

\[
\begin{align*}
&f^S_{11}(Q^S_{111}, Q^S_{121}, q^S_{111}, q^S_{121}, q^S_{211}, q^S_{221}) = 0.4(Q^S_{111} + Q^S_{121}) + 1.5(q^S_{111} - 50)^2 + 1.5(q^S_{121} - 50)^2 + q^S_{211} + q^S_{221}, \\
&f^S_{12}(Q^S_{112}, Q^S_{122}, q^S_{112}, q^S_{122}, q^S_{212}, q^S_{222}) = 0.4(Q^S_{112} + Q^S_{122}) + 2(q^S_{112} - 45)^2 + 2(q^S_{122} - 45)^2 + q^S_{212} + q^S_{222}, \\
&f^S_{21}(Q^S_{211}, Q^S_{221}, q^S_{211}, q^S_{221}, q^S_{111}, q^S_{121}) = Q^S_{211} + Q^S_{221} + 2(q^S_{211} - 31.25)^2 + 2(q^S_{221} - 31.25)^2 + q^S_{111} + q^S_{121}, \\
&f^S_{22}(Q^S_{212}, Q^S_{222}, q^S_{212}, q^S_{222}, q^S_{112}, q^S_{122}) = Q^S_{212} + Q^S_{222} + (q^S_{212} - 85)^2 + (q^S_{222} - 85)^2 + q^S_{112} + q^S_{122}.
\end{align*}
\]
Their transportation costs are:

\[ \hat{c}_{111}^S(Q_{111}^S, q_{111}^S) = 0.2Q_{111}^S + 1.2(q_{111}^S - 41.67)^2, \]

\[ \hat{c}_{112}^S(Q_{112}^S, q_{112}^S) = 0.1Q_{112}^S + 1.2(q_{112}^S - 37.5)^2, \]

\[ \hat{c}_{121}^S(Q_{121}^S, q_{121}^S) = 0.2Q_{121}^S + 1.4(q_{121}^S - 39.29)^2, \]

\[ \hat{c}_{122}^S(Q_{122}^S, q_{122}^S) = 0.1Q_{122}^S + 1.1(q_{122}^S - 36.36)^2, \]

\[ \hat{c}_{211}^S(Q_{211}^S, q_{211}^S) = 0.3Q_{211}^S + 1.3(q_{211}^S - 30.77)^2, \]

\[ \hat{c}_{212}^S(Q_{212}^S, q_{212}^S) = 0.4Q_{212}^S + 1.7(q_{212}^S - 32.35)^2, \]

\[ \hat{c}_{221}^S(Q_{221}^S, q_{221}^S) = 0.2Q_{221}^S + 1.3(q_{221}^S - 30.77)^2, \]

\[ \hat{c}_{222}^S(Q_{222}^S, q_{222}^S) = 0.1Q_{222}^S + 1.5(q_{222}^S - 30)^2. \]

The opportunity costs of the suppliers are:

\[ oc_{111}(\pi_{111}, \pi_{211}) = 5(\pi_{111} - 80)^2 + 0.5\pi_{211}, \]

\[ oc_{112}(\pi_{112}, \pi_{212}) = 9(\pi_{112} - 80)^2 + \pi_{212}, \]

\[ oc_{121}(\pi_{121}, \pi_{221}) = 5(\pi_{121} - 100)^2 + \pi_{221}, \]

\[ oc_{122}(\pi_{122}, \pi_{222}) = 7.5(\pi_{122} - 50)^2 + 0.1\pi_{222}, \]

\[ oc_{211}(\pi_{211}, \pi_{111}) = 5(\pi_{211} - 50)^2 + 2\pi_{111}, \]

\[ oc_{212}(\pi_{212}, \pi_{112}) = 8(\pi_{212} - 70)^2 + 0.5\pi_{112}, \]

\[ oc_{221}(\pi_{221}, \pi_{121}) = 9(\pi_{221} - 60)^2 + \pi_{121}, \]

\[ oc_{222}(\pi_{222}, \pi_{122}) = 8(\pi_{222} - 60)^2 + 0.5\pi_{122}. \]

The firms’ assembly costs are:

\[ f_1(Q_{11}, \alpha_1^F) = 3Q_{11}^2 + 0.5Q_{11}\alpha_1^F + 100\alpha_1^F + 50\alpha_1^F, \]
\[ f_2(Q_{21}, \alpha_{2}^F) = 2.75Q_{21}^2 + 0.6Q_{21}\alpha_{2}^F + 100\alpha_{2}^F + 50\alpha_{2}^F. \]

Their production costs for producing components are:

\[ f_{11}^F(Q_{11}^F, q_{11}^F) = Q_{11}^F + 0.0001Q_{11}^F q_{11}^F + 1.1(q_{11}^F - 36.36)^2, \]
\[ f_{12}^F(Q_{12}^F, q_{12}^F) = 1.25Q_{12}^F + 0.0001Q_{12}^F q_{12}^F + 1.2(q_{12}^F - 41.67)^2, \]
\[ f_{21}^F(Q_{21}^F, q_{21}^F) = Q_{21}^F + 0.0001Q_{21}^F q_{21}^F + 1.5(q_{21}^F - 33.33)^2, \]
\[ f_{22}^F(Q_{22}^F, q_{22}^F) = 0.75Q_{22}^F + 0.0001Q_{22}^F q_{22}^F + 1.25(q_{22}^F - 36)^2. \]

The transaction costs are:

\[ tc_{111}(Q_{111}^S) = 0.5Q_{111}^S + Q_{111}^S + 100, \quad tc_{112}(Q_{112}^S) = 0.5Q_{112}^S + 0.5Q_{112}^S + 150, \]
\[ tc_{211}(Q_{211}^S) = 0.75Q_{211}^S + 0.75Q_{211}^S + 150, \quad tc_{212}(Q_{212}^S) = Q_{212}^S + Q_{212}^S + 100, \]
\[ tc_{221}(Q_{221}^S) = 0.75Q_{221}^S + Q_{221}^S + 150, \quad tc_{222}(Q_{222}^S) = 0.5Q_{222}^S + 0.75Q_{222}^S + 100, \]
\[ tc_{222}(Q_{222}^S) = 0.8Q_{222}^S + 0.25Q_{222}^S + 100, \quad tc_{222}(Q_{222}^S) = 0.5Q_{222}^S + Q_{222}^S + 175. \]

The firms’ transportation costs are:

\[ c_{11}^F(Q_{11}, q_1) = 3Q_{11}^2 + 0.3Q_{11}q_1 + 0.25q_1, \quad c_{21}^F(Q_{21}, q_2) = 3Q_{21}^2 + 0.3Q_{21}q_2 + 0.1q_2, \]

and the demand price functions are:

\[ \rho_{11}(d_{11}, d_{21}, q_1, q_2) = -3d_{11} - 1.3d_{21} + q_1 + 0.74q_2 + 2200, \]
\[ \rho_{21}(d_{21}, d_{11}, q_2, q_1) = -3.5d_{21} - 1.4d_{11} + 1.1q_2 + 0.9q_1 + 1800, \]

where \( q_1 = \alpha_{1}^F (\omega_{11} Q_{111}^F q_{11}^S + Q_{111}^S q_{11}^S + Q_{211}^S q_{21}^S + \omega_{12} Q_{121}^F q_{12}^S + Q_{121}^S q_{12}^S + Q_{212}^S q_{21}^S) \) and \( q_2 = \alpha_{2}^F (\omega_{21} Q_{212}^F q_{21}^S + Q_{212}^S q_{21}^S + Q_{222}^S q_{22}^S + \omega_{22} Q_{222}^F q_{22}^S + Q_{222}^S q_{22}^S + Q_{222}^S q_{22}^S). \)
The modified projection method converges to the following equilibrium solution:

\[ Q^{*}_{11} = 93.56, \quad Q^{*}_{21} = 71.34, \]

\[ Q^{F*}_{11} = 30.00, \quad Q^{F*}_{12} = 30.00, \quad Q^{F*}_{21} = 30.00, \quad Q^{F*}_{22} = 30.00, \]

\[ Q^{S*}_{11} = 27.37, \quad Q^{S*}_{12} = 100.00, \quad Q^{S*}_{121} = 45.44, \quad Q^{S*}_{122} = 23.35, \]

\[ Q^{S*}_{21} = 36.19, \quad Q^{S*}_{212} = 57.12, \quad Q^{S*}_{221} = 67.24, \quad Q^{S*}_{222} = 17.99, \]

\[ q^{F*}_{11} = 38.26, \quad q^{F*}_{12} = 45.15, \quad q^{F*}_{21} = 34.93, \quad q^{F*}_{22} = 41.71, \]

\[ q^{S*}_{11} = 46.30, \quad q^{S*}_{12} = 42.19, \quad q^{S*}_{121} = 44.83, \quad q^{S*}_{122} = 41.94, \]

\[ q^{S*}_{21} = 31.06, \quad q^{S*}_{22} = 51.85, \quad q^{S*}_{221} = 51.06, \quad q^{S*}_{222} = 52.00, \]

\[ \pi^{*}_{11} = 82.74, \quad \pi^{*}_{12} = 85.56, \quad \pi^{*}_{121} = 104.54, \quad \pi^{*}_{122} = 51.56, \]

\[ \pi^{*}_{21} = 53.62, \quad \pi^{*}_{22} = 73.57, \quad \pi^{*}_{221} = 63.74, \quad \pi^{*}_{222} = 61.12, \]

\[ \alpha^{F*}_{1} = 1.00, \quad \alpha^{F*}_{2} = 1.00, \]

\[ \lambda^{s*}_{11} = 109.83, \quad \lambda^{s*}_{12} = 187.06, \quad \lambda^{s*}_{21} = 172.34, \quad \lambda^{s*}_{22} = 76.58, \]

and the induced demands, demand prices, and product quality levels are:

\[ d_{11} = 93.56, \quad d_{21} = 71.34, \quad \rho_{11} = 1,901.07, \quad \rho_{21} = 1,504.22, \]

\[ q_{1} = 44.06, \quad q_{2} = 41.13. \]

The firms’ profits are 94,610.69 and 57,787.69, respectively, and those of the suppliers are 15,671.13 and 6923.20.

The eigenvalues of the symmetric part of the Jacobian matrix of \( G(Y) \) (cf. (7.27)) are 0.0089, 0.0098, 0.0100, 0.0102, 0.0107, 0.0135, 0.0147, 0.0151, 0.0158, 0.0164, 0.0198, 0.0198, 0.0201, 0.0224, 0.0254, 0.0298, 0.0409, 0.0492, 0.0540, 0.0564, 0.0578, 0.0605, 0.0650, 0.0660, 0.1000, 0.1000, 0.1000, 0.1063, 0.1500, 0.1600, 0.1600, 0.1600,
0.1800, 0.1800, 2.0280, 2.1399, which are all positive. Therefore, the uniqueness of the solution \((Q^*, Q^{F^*}, Q^{S^*}, q^{F^*}, \alpha^{F^*}, \pi^*, q^{S^*})\) and the convergence of the modified projection method are guaranteed.

As mentioned in the Introduction, the manufacturing plants of suppliers may be located in different geographical locations around the globe, which increases the vulnerability of the supply chain networks of the firms to the disruptions that happen to the suppliers, such as those caused by natural disasters. In this analysis, I model and analyze the impacts of the disruptions to suppliers 1 and 2 on the profits of the firms and the demands, prices, and quality levels of the products.

I also evaluate the values of the two suppliers and which one of them is more important to the firms. With the values of the suppliers and the importance level of them to the firms, the firms can make more specific and targeted efforts in their supplier management strategies and in the contingency plans in handling the disruptions to their suppliers.

First, I present the following disruption. The data are as in Example 7.2, except that supplier 1 is no longer available for the firms to contract with or to produce or transport the components needed. The network topology with this disruption is presented in Figure 7.8.

The equilibrium solution achieved by the modified projection method is:

\[
Q_{11}^* = 65.00, \quad Q_{21}^* = 65.00,
\]

\[
Q_{11}^{F^*} = 30.00, \quad Q_{12}^{F^*} = 30.00, \quad Q_{21}^{F^*} = 30.00, \quad Q_{22}^{F^*} = 30.00,
\]

\[
Q_{11}^{S^*} = 0.00, \quad Q_{12}^{S^*} = 0.00, \quad Q_{121}^{S^*} = 0.00, \quad Q_{122}^{S^*} = 0.00,
\]

\[
Q_{211}^{S^*} = 35.00, \quad Q_{212}^{S^*} = 100.00, \quad Q_{221}^{S^*} = 100.00, \quad Q_{222}^{S^*} = 35.00,
\]

\[
q_{11}^{F^*} = 38.26, \quad q_{12}^{F^*} = 45.16, \quad q_{21}^{F^*} = 34.93, \quad q_{22}^{F^*} = 41.75,
\]
Figure 7.8. Supply Chain Network Topology With Disruption to Supplier 1

\[ q_{211}^* = 31.06, \quad q_{212}^* = 51.85, \quad q_{221}^* = 31.06, \quad q_{222}^* = 52.00, \]

\[ \pi_{211}^* = 53.50, \quad \pi_{212}^* = 76.25, \quad \pi_{221}^* = 65.56, \quad \pi_{222}^* = 62.19, \]

\[ \alpha_{F1}^* = 1.00, \quad \alpha_{F2}^* = 1.00, \]

\[ \lambda_{11}^* = 107.53, \quad \lambda_{12}^* = 448.93, \quad \lambda_{21}^* = 242.02, \quad \lambda_{22}^* = 95.98, \]

and the induced demands, demand prices, and product quality levels are:

\[ d_{11} = 65.00, \quad d_{21} = 65.00 \quad \rho_{11} = 1,998.07, \quad \rho_{21} = 1,569.17, \]

\[ q_1 = 47.12, \quad q_2 = 41.14. \]

The firms’ profits are 80,574.83 and 57,406.47, respectively, and supplier 2’s profit is 13,635.49. The uniqueness of the solution \((Q^*, Q_{F*}, Q_{S*}, q_{F*}, \alpha_{F*}, \pi^*, q_{S*})\) is guaranteed.
Without supplier 1 and with the firms’ limited in-house production capacities, there is no competition on the suppliers’ side and the firms have to depend more on the supplier in component production. Therefore, as shown by the results, 3 out of the 4 contracted component quantities produced by supplier 2 increase. supplier 2 charges the firms more than before, and its profit improves. Without supplier 1, the firms are not able to provide as many products as before, and hence, the demands at the demand market decrease. The quality of the products of firms 1 and 2 increase, and the prices at the demand market increase.

Under this disruption, the profit of firm 1 decreases by 14.84%, and that of firm 2 decreases by 0.66%. Therefore, from this perspective, supplier 1 is more important to firm 1 than to firm 2. The value of supplier 1 to firm 1 is 14,035.86, and that to firm 2 is 381.22, which are measured by the associated profit declines.

I then present the disruption in which supplier 2 is no longer available to the firms. The other data are the same as in Example 7.2. The network topology is as in Figure 7.9.

The modified projection method converges to the following equilibrium solution:

\[ Q^{*}_{11} = 65.00, \quad Q^{*}_{21} = 63.79, \]

\[ Q^{F*}_{11} = 30.00, \quad Q^{F*}_{12} = 30.00, \quad Q^{F*}_{21} = 30.00, \quad Q^{F*}_{22} = 30.00, \]

\[ Q^{S*}_{111} = 35.00, \quad Q^{S*}_{112} = 100.00, \quad Q^{S*}_{121} = 97.58, \quad Q^{S*}_{122} = 33.79, \]

\[ Q^{S*}_{211} = 0.00, \quad Q^{S*}_{212} = 0.00, \quad Q^{S*}_{221} = 0.00, \quad Q^{S*}_{222} = 0.00, \]

\[ q^{F*}_{11} = 38.26, \quad q^{F*}_{12} = 45.16, \quad q^{F*}_{21} = 34.93, \quad q^{F*}_{22} = 41.75, \]

\[ q^{S*}_{111} = 46.30, \quad q^{S*}_{112} = 42.19, \quad q^{S*}_{121} = 44.83, \quad q^{S*}_{122} = 41.94, \]

\[ \pi^{*}_{111} = 83.50, \quad \pi^{*}_{112} = 85.56, \quad \pi^{*}_{121} = 109.76, \quad \pi^{*}_{122} = 52.25, \]

\[ \alpha^{F*}_1 = 1.00, \quad \alpha^{F*}_2 = 1.00, \]
The induced demands, demand prices, and product quality levels are:

\[
\begin{align*}
    d_{11} &= 65.00, & d_{21} &= 63.79, & \rho_{11} &= 1,996.05, & \rho_{21} &= 1,570.59, \\
    q_1 &= 42.82, & q_2 &= 42.11.
\end{align*}
\]

The firms’ profits are 83,895.42 and 53,610.96, respectively, and supplier 1’s profit is 22,729.18. The uniqueness of the solution \((Q^*, Q_F^*, Q_S^*, q^F, q^S, \pi^*, \alpha^*)\) is guaranteed.

The impacts of the disruption to supplier 2 follow similar logic as those brought about by the disruption to supplier 1. The contracted component quantities by supplier 1 increase, and its profit increases. The demands at the demand market decrease. Firm 1’s product quality decreases, and firm 2’s increases. The prices at the demand market increase.
Without supplier 2, firm 1’s profit declines by 11.33%, and that of firm 2 reduces by 7.23%. Thus, supplier 2 is more important to firm 1 than to firm 2 under this disruption. The value of supplier 2 to firm 1 is 10,715.27, and that to firm 2 is 4,176.73.

In addition, according to the above results, supplier 1 is more important than supplier 2 to firm 1, and to firm 2, supplier 2 is more important.

For completeness, the disruption in which neither of the supplier is no longer available to the firms is also considered. The other data are the same as in Example 7.2. The network topology is depicted in Figure 7.10.

**Figure 7.10.** Supply Chain Network Topology With Disruption to Suppliers 1 and 2

The equilibrium solution obtained using the modified projection method is:

$$Q_{11}^* = 15.00, \quad Q_{21}^* = 15.00,$$

$$Q_{11}^{F*} = 15.00, \quad Q_{12}^{F*} = 30.00, \quad Q_{21}^{F*} = 30.00, \quad Q_{22}^{F*} = 30.00,$$

$$Q_{111}^{S*} = 0.00, \quad Q_{112}^{S*} = 0.00, \quad Q_{121}^{S*} = 0.00, \quad Q_{122}^{S*} = 0.00,$$

$$Q_{211}^{S*} = 0.00, \quad Q_{212}^{S*} = 0.00, \quad Q_{221}^{S*} = 0.00, \quad Q_{222}^{S*} = 0.00,$$
\[ q_{11}^{F^*} = 37.29, \quad q_{12}^{F^*} = 45.08, \quad q_{21}^{F^*} = 35.71, \quad q_{22}^{F^*} = 37.90, \]
\[ \alpha_1^{F^*} = 1.00, \quad \alpha_2^{F^*} = 1.00, \]
\[ \lambda_{11}^* = 30.46, \quad \lambda_{12}^* = 967.28, \quad \lambda_{21}^* = 772.88, \quad \lambda_{22}^* = 22.63. \]

The induced demands, demand prices, and the product quality levels are:

\[ d_{11} = 15.00, \quad d_{21} = 15.00, \quad \rho_{11} = 2,206.42, \quad \rho_{21} = 1,806.40, \]
\[ q_1 = 43.52, \quad q_2 = 37.02. \]

The firms’ profits are 30,016.91 and 24,391.32, respectively. The uniqueness of the solution \((Q^*, Q_{F}^{S^*}, Q_{F}^{S^*}, q_{F}^{S^*}, \alpha_{F}^{S^*}, \pi^*, q_{S}^{F^*})\) is guaranteed.

Compared to Example 7.2, without the suppliers, the demands at the demand market decrease, the firms’ product quality levels decrease, and the prices at the demand market increase. Firm 1’s profit deceases by 68.27%, firm 2’s reduces by 57.79%. The value of the suppliers to firm 1 is 64,593.78, and that to firm 2 is 33,396.37.

7.5. Summary and Conclusions

In this chapter, I develop a general multiteried supply chain network equilibrium model with a focus on quality in which suppliers compete to produce components that are utilized by competing firms as they assemble final products that are differentiated by brands. The firms can also produce components in-house, depending on their capacities. I model the competitive behavior of the two tiers of decision-makers as they identify their optimal strategies in terms of quantity and quality with the assembling firms also identifying their assembly quality preservation levels. The suppliers charge the firms prices for the components that they supply.

The novelty of our framework lies in its generality and its computability. Rather than focus, as some of the literature does, on one supplier-one manufacturer studies,
the number of components needed for the finished product, the number of suppliers, the number of firms, nor the number of demand markets are limited. Moreover, I provide a framework for tracking the quality of the product from the component level, through the assembly process into the final product, and ultimate distribution to the demand markets.

I derive the unified variational inequality formulation of the governing equilibrium conditions, provide qualitative properties of the equilibrium solution pattern, in terms of existence and uniqueness results, and propose an algorithm along with conditions for convergence. Our framework is illustrated with numerical examples, accompanied by sensitivity analysis that explores such critical issues as the impacts of capacity disruptions and the potential investments in capacity enhancements. I also conduct sensitivity analysis to reveal the impacts of specific supplier unavailability along with their values as reflected in the profits of the firms and in the quality of the finished products. With knowledge of the values of the suppliers to the firms, the firms can make more specific, targeted efforts in their supplier management strategies and in their contingency plans in the case of supplier disruptions.
CHAPTER 8
CONCLUSIONS AND FUTURE RESEARCH

8.1. Conclusions

The purpose of this dissertation was to contribute to the equilibrium and dynamics modeling and analysis of quality competition in supply chain networks under four different scenarios: information asymmetry, product differentiation, outsourcing, and supplier selection. In this dissertation, I demonstrated how to model, analyze, and solve quality competition problems, as appropriate to the specific scenarios, in a quantitative manner. The models developed are able to provide firms with the decisions in production quality, component quality, prices, and/or the selections of their contractors and suppliers, and the evolution of these decisions over time under the above mentioned four scenarios. The models can also be utilized by policy makers to study the impacts of quality standards, disrepute, and supply chain disruptions on the behavior and decisions of firms, contractors, and suppliers. The methodologies utilized in this dissertation included: variational inequality theory, projected dynamic systems, game theory, optimization theory, and multicriteria decision-making.

In Chapter 3, I focused on quality competition in supply chain networks under the information asymmetry associated with the knowledge of quality between firms and consumers, when there is no product differentiation. I introduced an average quality measure to capture consumers’ perception of quality at the demand markets under such information asymmetry. In this chapter, each firm might have multiple manufacturing plants, which might be located in different regions and therefore with different quality levels. The firms competed in product quantity and quality with
each firm aiming at maximizing its own profit in the competition. I also considered minimum quality standards in the model. A unified variation inequality formulation that provided the equilibrium conditions for the problem with minimum quality standards and the one without was presented. The impacts of different minimum quality standards were studied numerically. Suggestions to the policy makers and regulators in regards to the imposition of minimum quality standards were given accordingly.

In contrast to Chapter 3, Chapter 4 concentrated on quality competition under the scenario of product differentiation, as firms are engaging in distinguishing their product from the competitors’ in imperfect markets. I developed both equilibrium and dynamic models, along with qualitative properties of the solution. The costs of R&D activities and quality costs were included in the production cost functions. The impacts of consumers’ attitude towards quality, which can be captured by the quality coefficients in demand price functions, were studied numerically.

Chapters 5 and 6 were dedicated to quality competition in supply chain networks with outsourcing and fixed projected demand. The optimal make-or-buy and contractor-selection decisions of the firms and the pricing and quality decisions of the contractors were provided. An average quality level was constructed to measure the average product quality under outsourcing. The models allowed each firm to weight its disrepute cost, which measured the loss of reputation caused by poor quality products, in its decision-making problem with the weight capturing the firm’s attitude towards its reputation. In addition to the firms’ cost minimization behavior, the behavior of the contractors was also modeled. The contractors competed in the prices charged to the firms and their quality in producing and delivering each firm’s product, aiming at maximizing their profits. The equilibrium conditions of the entire system which includes both the firms and the contractors were presented. In the numerical examples, sensitivity analyses in terms of demand were conducted, and a contractor disruption case was modeled, solved, and discussed. Chapter 6 extended Chapter 5
in considering the competition among firms in product quantity and quality. The impacts of firms’ attitudes towards reputation were explored numerically. These two chapters can be applied to solve supply chain network problems with outsourcing in numerous industries, including the pharmaceutical industry, electronic industry, and the apparel industry, where outsourcing is prevalent.

Chapter 8 concentrated on quality competition in multitiered supply chain networks with supplier selection. Both the behavior of the firms and the behavior of the suppliers were modeled. Firms were responsible for assembling the products using the components needed, which can be produced by the firms in house and/or by the suppliers, and delivering the products to the demand markets. Firms competed in in-house and contracted component production, in-house component quality, product quantity and quality, and the quality preservation/decay level in the assembly process of the product, in order to maximize their profits. Suppliers, in turn, competed in the prices charged to the firms for producing and delivering the components and in component quality. The equilibrium decisions of the firms and the suppliers were presented. In this chapter, the quality of the final product was determined by the quality of its components, the importance of the quality of them to the final products, and the preservation/decay level of the assembly process. In the numerical examples, the value of each supplier to each firm, which was measured by the profit drop after associated supplier disruption, the impacts of capacity changing, and the investments in capacity enhancement were studied numerically. The network topology and the model in this model are able to capture the case of outsourcing as a special case.

In this dissertation, for each supply chain network model with quality competition, I stated the governing equilibrium conditions, derived the equivalent variational inequality formulation, proposed an algorithm, presented convergence results, and computed solutions to numerical examples in order to illustrate the generality and applicability of the framework. For models in Chapters 3, 4, and 5, I also proposed
the dynamic adjustment processes for the evolution of quality levels, and, as appropriate, product quantities and prices over time, with stability analysis results also presented. The framework extended the literature since the models herein are not limited to a fixed number of firms or demand markets nor to functions of specific form.

8.2. Future Research

Following the work in this dissertation, several potential directions can be suggested for future research. Some topics that I intend to pursue in the future are listed below.

1. For new products, their quality takes time to be revealed to consumers. The perception of the quality of new products would first depend only upon extrinsic attributes, such as warranty, brand name, and package, and, as time goes, the true quality which involves both extrinsic and intrinsic attributes, would be learned by consumers (Zeithaml (1988)). In the future, I plan to extend the models in Chapter 3 to capture the quality information asymmetry caused by the time delay of consumer’s learning process of quality in spatial price networks for new products. I hope to provide insights regarding the impacts of such information asymmetry on demand, price, quality, and consumer welfare over multiple periods in the planning horizon. By comparing the consumer welfare under perfect information and the one under information asymmetry, the value of perfect quality information for consumers can be measured.

2. I would also like to extend Chapters 5 and 6, which concentrated on quality competition in outsourcing, to incorporate the quality information asymmetry that lies between firms and their potential contractors and the transaction costs and risk caused by such information asymmetry. Specifically, I would like to further the understanding of firm-contractor interactions in outsourcing with a focus on contractible and
non-contractible information and the impacts of training and monitoring activities conducted by the firms on such information asymmetry.

3. The quality of perishable products, such as food, some pharmaceuticals, and human blood, decays, especially in the transportation and storage processes (Yu and Nagurney (2013), Masoumi, Yu, and Nagurney, (2012), and Nagurney, Masoumi, and Yu (2012)). In the future, I plan to formulate quality degradation in competitive transportation networks for perishable products. The study will focus on the influence of multimodal transportation, different transportation/storage technologies, and the costs of them on the quality preservation/decay of such products and on consumers’ demand to them at the demand markets.

4. Furthermore, the dynamics of quality competition in multitiered supply chain networks with suppliers, on which Chapter 8 focused, can be developed using projected dynamical systems theory (cf. Section 2.3) with stability results presented. Such a dynamic model, as those developed in Chapters 3, 4, and 5, can predict the evolution of the decisions of firms and those of the suppliers over time based on the discretization of the dynamics.

5. In addition, I would like to conduct empirical research to apply the models and results in this dissertation to real-life quality competition problems in supply chain networks under the four scenarios discussed and modeled. I also plan to develop models and provide solutions and suggestions for specific organizations with their own specific cost and demand information.


Li, D., Nagurney A., 2015. A general multitiered supply chain network model of quality competition with suppliers. Isenberg School of Management, University of Massachusetts Amherst.


