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Exact Solutions in Gravity: A journey through spacetime with the Kerr-Schild ansatz

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EXACT SOLUTIONS IN GRAVITY: A JOURNEY THROUGH SPACETIME WITH THE KERR-SCHILD ANSATZ

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by

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I have always wanted to understand the Universe. I still don't, but at least I was able to contribute a little. Time for something different...

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ABSTRACT

EXACT SOLUTIONS IN GRAVITY: A JOURNEY THROUGH SPACETIME WITH THE KERR-SCHILD ANSATZ

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The Kerr-Schild metric ansatz can be expressed in the form $g_{ab} = \bar{g}_{ab} + \lambda k_a k_b$, where $\bar{g}_{ab}$ is a background metric satisfying Einstein's equations, k_a is a null-vector, and λ is a free parameter. It was discovered in 1963 while searching for the elusive rotating black hole solutions to Einstein's equations, fifty years after the static solution was found and Einstein first formulated his theory of general relativity. While the ansatz has proved an excellent tool in the search for new exact solutions since then, its scope is limited, particularly with respect to higher dimensional theories. In this thesis, we present the analysis behind three possible modifications. In the first case a spacelike vector is added to the ansatz, and we show that many, although not all, of the simplifications that occur in the Kerr-Schild case continue to hold for the extended version of the ansatz. In the second case we look at the Kerr-Schild ansatz in the context of higher curvature theories of gravity; specifically Lovelock gravity

which organizes terms in the Lagrangian in such a way that the theory is ghost-free and the equations of motion remain second order. We find that the field equations reduce, in a similar manner as in the Kerr-Schild case, to a single equation of order λ^p for unique vacuum theories of order p in the curvature. Finally, we investigate the role of the Kerr-Schild ansatz in the context of Kaluza-Klein gravity theories.

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CHAPTER 1

INTRODUCTION TO THE KERR-SCHILD ANSATZ

1.1 Brief History of General Relativity

Black holes are the most majestic objects in the universe. They can rotate. They can be charged. They can be supermassive and lie at the heart of galaxies, or they can be microscopic and hurtle through space in complete isolation. Some are as old as the universe itself, while others are the fantastical endpoint of stellar collapse. Black holes can be used to confirm astronomical data, and they can be used as purely theoretical tools. If we allow ourselves to step outside the friendly confines of our familiar four dimensions, the black hole possibilities explode. In essence, the task of the black hole physicist becomes a complete categorization of the types of black holes given certain initial conditions. These can depend on the physical properties of the black hole itself, such as the aforementioned rotation and charge, or on the spacetime the black hole resides, such as the number of dimensions or the curvature. What was essentially a simple task in four dimensions, becomes a massive exercise in higher dimensions.

One of the most intriguing features of a black hole is the event horizon; a boundary where the geometry of spacetime is so warped that even light cannot escape. The event horizon encloses a region of spacetime surrounding the black hole that is cutoff from all forms of communication with the Universe surrounding it. We cannot know what happens beyond the horizon, and if any object should happen to cross it, that object will be lost to us forever[1]. This concept was first[2] theorized as early as 1783 by John Michell when he discussed escape velocities greater than the speed of light for classical bodies [1]. He writes

> "*If there should really exist in nature any bodies, whose density is not less than that of the sun, and whose diameters are more than 500 times the diameter of the sun, since their light could not arrive at us; or if there should exist any other bodies of a somewhat smaller size, which are not naturally luminous; of the existence of bodies under either of these circumstances, we could have no infomation from sight;*"

Michell named these objects "dark stars" since their gravitational pull was so strong that light could not escape from the surface; therefore, they could not be seen by the naked eye. He further hypothesized how we could detect[3] the presence of these dark stars, remarking;

> "*if any other luminous bodies should happen to revolve about them we might still perhaps from the motion of these revolving bodies infer the existence of the central ones with some degree of probability, as this might afford a clue to some of the apparent irregularities of the revolving bodies, which would not be easily explicable on any other hypothesis;*"

Michell was able to predict these objects almost 150 years before Einstein presented his gravitational equations. The idea was apparently so abstract that it made almost no impact with the surrounding scientific community.

Black holes are a physical consequence of Einstein's General Relativity, formulated in 1915, which relates the geometry of spacetime to the matter, or stress-energy, contained within it. This can be seen in Einstein's equation

$$R_{ab} - \frac{1}{2} R g_{ab} = T_{ab} \tag{1.1}$$

[1]Or maybe not, depending on what Quantum Mechanics has to say.

[2]Pierre Laplace came to a similar conclusion independently in 1796 [2].

[3]This is an amazingly accurate predicition considering that, currently, there are between one and two dozen stellar black hole candidates in the Milky Way galaxy, and all of them are in X-ray binary systems.

where the *left-hand side* is understood to contain the geometric information, *e.g.* the curvature of the spacetime, and the *right-hand side* is the Stress-Energy tensor containing the matter content of the spacetime. Einstein first demonstrated [2] the power of the new theory using the precession of the perihelion of Mercury. While the precession of orbits was known at the time, Newtonian theory was able to correctly account for this behavior in all of the planets, except Mercury. The disagreement between the astronomical observation of the precession, and what Newtonian theory predicted, was approximately 45 arcseconds per century. At one of four lectures to the Prussian Academy of Sciences in 1915, Einstein, using an approximate solution, was able to demonstrate[4] that general relativity indeed accounted for the descrepancy.

Within a month of the introduction of general relativity the first non-trivial (*i.e.* non-flat) exact solution, one describing a four dimensional static and spherically symmetric "black hole[5]" in a vacuum spacetime, was found by Karl Schwarzschild [3]. At the time, he was serving on the front lines for the German military to help with ballistic trajectories [2] during World War I. He sent his calculations directly to Einstein, who was impressed, remarking;

> "*I had not expected that one could formulate the exact solution of the problem in such a simple way. I liked very much your mathematical treatment of the subject.*"

Einstein would go on to present[6] the findings to the Prussian Academy of Sciences on behalf of Schwarzschild.

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[4]Einstein also showed that General Relativity would give twice the value of the Newtonian theory for the deflection of light around the sun. However, it would take three years before this prediction was confirmed using the 1919 eclipse by Sir Arthur Eddington and his team on the island of Principe, situated one degree North of the equator off the Eastern coast of Africa [2].

[5]The term "black hole" was first seen in print in a 1964 article by Ann Ewing for *Science News Letter* and later popularised by Wheeler during a talk at the Goddard Institute in 1967.

[6]A few weeks later, Einstein would present another paper by Schwarzschild [4], however it would be one of his last - within six months Schwarzschild would pass away after contracting an auto-immune disease on the front line.

A short time later, another exact solution was found independently by Hans Reissner in 1916 [5], Hermann Weyl in 1917 [6], and Gunnar Nordström in 1918 [7]. Reissner and Weyl solved Einstein's equation for the gravitational field surrounding a charged point source, whereas Nordström would expand the solution to a charged finite matter distribution. This result, which essentially added electric charge to the already known four dimensional static, spherically symmetric solution of Schwarzschild, would become known as the Reissner-Nordström black hole[7].

While it was well known at the time that stars could have angular momentum, there was no accompanying metric for one, *i.e.* a spherically symmetric rotating solution to Einstein's equations. Due to the highly non-linear nature of the Einstein field equations[8], the world would have to wait almost half a century to obtain such a solution for a rotating black hole. In 1963 a four-dimensional rotating solution was discovered by Roy Kerr [9]. This solution, now referred to as a Kerr black hole, was not obtained through a brute-force analysis but one of subtlety and nuance: the aforementioned spherical symmetry condition was relaxed to that of axial symmetry, and the key assumption that the metric was algebraically special[9] was added. The technique that was used would come to be known as the Kerr-Schild ansatz [12]. Besides being instrumental in the discovery of the Kerr solution, it would become extremely useful in the search for other exact black hole solutions. That it took so long to find a rotating exact solution to Einstein's equations is a testament not only to their complexity, but also to the ingenuity necessary in order to make progress in the field.

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[7]According to Thorne [2], the result was not fully understood as a charged black hole until the work of Brill and Graves [8] in 1960.

[8]Even when only considering solutions in a four dimensional vacuum, there are initially 16 coupled, second-order, non-linear, partial differential equations to solve. In practice, when taking certain spacetime symmetries into consideration (*i.e.* the spacetime is *torsion-free*) these 16 equations immediately reduce to, the much more manageable, 10 linearly-independent equations.

[9]This concept is discussed in detail in Section 1.4

1.2 Motivation

Black hole solutions to Einstein's equations were initially formulated in four dimensions along with uniqueness theorems, that in a sense, showed four dimensions to be quite special. The search has since been to try and formulate similar relationships for black holes existing in higher dimensions. Certain topologies or questions of stability that, in some cases, are unique to four dimensions become relaxed when considering their higher dimensional analogues; consequently, the spacetime can enjoy a much richer structure[10]. This essentially becomes a task of classification. How many parameters are necessary to categorize a black hole? How does this scheme change when extending into higher dimensions? In four dimensions, only three parameters are needed to classify a black hole - mass, charge, and rotation. In other words, in four dimensions we have a three parameter family of stationary black hole solutions leading to four different types of black holes depending on the values of these parameters. We could consider it as one black hole, the Kerr-Newman black hole[11] having non-zero values for all three parameters. The other three types of black holes would then be special cases of the Kerr-Newman black hole with some of those parameters set to zero: Schwarzschild would result from setting the charge and rotation parameters to zero, Reissner-Nordström would result from setting rotation equal to zero, and the aforementioned Kerr black hole would result from setting charge equal to zero.

The future of black holes is in extra dimensions. Given the high degree of non-linearity in solving Einstein's equations in four dimensions, the task in extra dimensions can quickly turn from monumental to impossible. This is where the Kerr-Schild formalism enters in such a powerful way. By using a special geometric property of the spacetime, the ansatz is able to drastically reduce the complexity of the equa-

[10]The topology of the event horizon in four dimensions is restricted to be spherical, however in five dimensions, *e.g.* a toroidal topology for the event horizon is allowed [28].

[11]Discovered in 1965, shortly after the Kerr solution

tions of motion, making the problems much more tractable. This feature cannot be overstated. After discovering the static Schwarzschild solution, physicists would spend almost half a century searching in vain for the rotating solution to Einstein's equations. The rotating solution, *i.e.* the Kerr black hole, would come with the initial construction of the Kerr-Schild ansatz [10][11] in 1963 in which Kerr and Schild [12] were investigating the properties of algebraically special spacetimes, or those spacetimes that enjoy special symmetries allowing them to be classified into specific families. These families in turn lead to uniqueness theorems for black holes and the larger goal of complete classification of black hole types. This scheme is well understood for four dimensions, but the situation becomes much different as soon as we generalize the number of dimensions. As previously stated, this can be difficult as exact solutions are hard to come by. The higher dimensional analogues to the Schwarzschild and Reissner-Nordström solutions were found by Tangherlini [13] in 1963. It would take an additional 23 years to find the higher dimensional generalization of the rotating Kerr solution, known as Myers-Perry black holes [17], where the Kerr-Schild ansatz would again play an integral role. To this point the ansatz has proved an invaluable tool for finding new black hole solutions in a multitude of settings. It is therefore natural to investigate its extension under different circumstances, *e.g.* in higher curvature theories of gravity as well as modifications to the ansatz itself.

1.3 The role of the Kerr-Schild ansatz

Following the prescription outlined by Xanthopolous [14], the spacetime metric is taken to have the following form

$$g_{ab} = \bar{g}_{ab} + \lambda h_{ab}, \qquad h_{ab} = k_a k_b \tag{1.2}$$

where $\bar{g}_{ab}$ is the metric of a background spacetime which is not necessarily taken as flat[12], k^a is a null vector with respect to the background metric ($\bar{g}_{ab}k^a k^b = 0$), and λ is a constant that will be used as an ordering parameter to aid in calculation. Initially this can be viewed as a small perturbation around the background spacetime. When calculating the inverse metric, which appears in the Christoffel symbols and by extension the Riemann curvature tensor and the Einstein equations, the perturbative expansion would generally include terms at all orders of λ. However due to the properties of the null vector k^a, we encounter a wonderful simplification when calculating the inverse metric such that it has the simple form

$$g^{ab} = \bar{g}^{ab} - \lambda h^{ab} \tag{1.3}$$

While seemingly innocuous, the fact that the inverse metric truncates after first order in λ is one of the most important facets of the Kerr-Schild ansatz - it will lead to considerable simplifications and cancellations when calculating other quantities that would otherwise be deemed intractable. What was initially viewed as a linear perturbation to the background spacetime, is actually the exact metric. These two properties of the Kerr-Schid ansatz, the simple form of the inverse metric and the "perturbation" being described in terms of a null-vector, are what allow us to investigate these spacetimes in a more nuanced way than would otherwise be possible if using a brute-force method. Further computation shows that, if the null vector is tangent to a geodesic congruence of the background metric, *i.e.* the null-vector is parallel propagated along itself with respect to the background spacetime, then the Ricci tensor $R^a{}_b$ of the KS metric also truncates beyond linear order in λ [15]. These results can also be generalized to non-vacuum cases [16].

[12] The background was taken as flat in the initial formulation by Kerr and Schild, however it was showed by Xanthopolous that this condition could be relaxed to any vacuum solution which would include curved backgrounds.

There are many known exact solutions that can be put into Kerr-Schild form. A few of the notable ones mentioned previously are the static spherically-symmetric neutral Schwarzschild black hole and the Kerr exact solution for stationary axially-symmetric neutral black holes (as well as the Kerr solution's higher-dimensional generalization known as Myers-Perry black holes). The background metric in (1.2) is not restricted to be Minkowski as Kerr-(A)dS black holes are also known to be expressible in Kerr-Schild form and the general higher dimensional (A)dS neutral rotating black holes were similarly found by Gibbons *et. al.* [20] starting from (A)dS background metrics. For charged black holes, the $D = 4$ Kerr-Newman solution for charged rotating black holes can be put in Kerr-Schild form. One can also naturally extend the ansatz to non-vacuum spacetimes such as Einstein-Maxwell theory by taking the vector potential A_a to be proportional to k_a.

On the other hand, there are known exact solutions that cannot be put in to Kerr-Schild form such as the the five dimensional black ring [28]. In addition, Myers-Perry black holes were the higher-dimensional generalization of Kerr black holes and both were expressable in Kerr-Schild form. However, an analogous result for the higher-dimensional generalization of the charged, rotating black holes in Einstein-Maxwell theory, *i.e.* Kerr-Newman black holes, was not as forthcoming. It was shown by Aliev [22] [23] that these solutions were only found in the limit of slow rotation. This meant that although the Kerr-Schild formalism for describing Kerr-Newman black holes was successful in four dimensions, it is restricted in its use for the higher-dimensional analogues. More broadly these counter-examples show that although the Kerr-Schild ansatz has been instrumental in finding exact solutions, its use is constrained in higher dimensional spacetimes, and an analysis of where it is applicable starts with understanding how special a spacetime can be.

1.4 Algebraically special spacetimes and Petrov classification

In $d = 4$ dimensions there are four null vectors, known as "principal null vectors (PNV)" that are associated with the Weyl[13] tensor. They are usually distinct but if two or more happen to coincide, we say the spacetime is algebraically special - essentially, these repeated principal null vectors are eigenvectors of the Weyl tensor. Since spacetimes can be classified by the algebraic type of the Weyl tensor, and since the Kerr-Schild ansatz describes algebraically special spacetimes, this puts restrictions on the form and transformation properties of the null-vector k^a. One can use the Newman-Penrose formalism[14] to express the metric in a null-tetrad basis[15] of the PNV's, and the null-vector k^a which itself is one of the principal null-vectors, will have its direction uniquely defined. Precisely how the null-vector k^a contracts with the Weyl tensor determines how special the particular space is. There are six different possibilities, each one known as a Petrov type. They are I, II, D, N, III, and O. For example, the null-vector for a Type II, Type III, and Type N spacetime would satisfy the following equations, respectively

$$(II) C_{abcd} k^b k^d = \alpha k_a k_c \qquad (III) C_{abcd} k^b k^d = 0 \qquad (N) C_{abcd} k^d = 0. \tag{1.4}$$

Type I is known as algebraically general and Type O describes the situation when the Weyl tensor vanishes and the metric is conformally flat. Each type of algebraically special spacetime represents different physical situations. In four dimensions, with a flat background metric, the null vector k^a in a vacuum KS spacetime is necessarily a repeated principal null vector of the Weyl tensor [15]. In higher dimensions, it was

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[13]The Weyl tensor is the trace-free component of the Riemann curvature tensor which is conformally invariant and vanishes under contraction of any two indices.

[14]The Newman-Penrose formalism is a special type of tetrad formulation of general relativity in four dimensions, specifically with the use of spinors.

[15]In four dimensions, four null-vectors are needed - two being real and two being complex. The complex null-vectors are constructed from two real, orthonormal space-like vectors.

shown in reference [29] that the Weyl tensor of vacuum KS spacetimes is always of Type II, or more, algebraically special, within the classification scheme of reference [30]. The metrics that can be expressed in Kerr-Schild form fall under the category of Type II or more special, whereas the black ring was shown to be of Type I [31] and therefore not expressible in Kerr-Schild form.

Nevertheless, one could make a long list of potentially interesting black hole solutions that have not so far been found via the KS ansatz (or by any other method). Candidates for this list would include the rotating, charged black holes of Einstein-Maxwell theory for $D > 4$, vacuum black holes with non-spherical event horizon topology beyond $D = 5$ (*e.g.* such as those discussed in [21]), as well as black branes and rotating black holes in Lovelock gravity theories (beyond the special cases found in [24, 25] and [26] respectively).

1.5 Initial formalism and the vacuum Einstein equations for Kerr-Schild spacetimes

As stated previously, the null vector k^a leads to simplifications when calculating the Christoffel symbols as well as the Riemann and Ricci tensors. In this section we present the basic formalism that goes into these calculations (see *e.g.* [18, 19] for an extended treatment), assuming the special form of $h_{ab} = k_a k_b$. Letting ∇_a denote the covariant derivative operator compatible with the full Kerr-Schild metric and $\bar{\nabla}_a$ denote the covariant derivative compatible with the background metric $\bar{g}_{ab}$, we see that the two are related when acting upon an arbitrary vector v^b such that $\nabla_a v^b = \bar{\nabla}_a v^b + C^b_{ac} v^c$ with the tensor C^c_{ab} given by

$$\begin{aligned} C^c_{ab} &= \frac{\lambda}{2} g^{cd} \left(\bar{\nabla}_a h_{bd} + \bar{\nabla}_b h_{ad} - \bar{\nabla}_d h_{ab} \right) \\ &= \frac{\lambda}{2} \left(\bar{\nabla}_a h_b{}^c + \bar{\nabla}_b h_a{}^c - \bar{\nabla}^c h_{ab} \right) + \frac{\lambda^2}{2} h^{cd} \bar{\nabla}_d h_{ab} \\ &= \lambda C^{(1)c}_{ab} + \lambda^2 C^{(2)c}_{ab} \end{aligned} \tag{1.5}$$

Note that this truncation only happens for the specific form of $h_{ab} = k_a k_b$, whereas in general there are an infinite number of terms. It is possible to show that the determinant of the full metric reduces to that of the background. Going forward, this will allow us to calculate quantities using solely the background metric and its associated covariant derivative. When h_{ab} is of the form $k_a k_b$, we are able to show that

$$h^{ab} C^c_{ab} = h_c{}^b C^c_{ab} = C^b_{ab} = 0. \tag{1.6}$$

This last property will reduce the number of terms in the Ricci tensor, now given to be

$$R_{ab} = \bar{R}_{ab} + \bar{\nabla}_c C^c_{ab} - C^d_{ac} C^c_{bd} \tag{1.7}$$

where $\bar{R}_{ab}$ is the curvature of the background spacetime. Taking the background to be flat (*i.e.* $\bar{R}_{ab} = 0$) for the time being, we notice that the Ricci tensor contains terms quadratic in the connection coefficients meaning that it is theoretically fourth order in λ with the expansion

$$R_{ab} = \sum_{l=1}^{4} \lambda^l R^{(l)}_{ab}. \tag{1.8}$$

We initially consider the expansion of the Ricci tensor R_{ab} with both of its indices down which goes out to order λ^4. Computation shows that the fourth order contribution $R^{(4)}_{ab}$ vanishes identically. Further progress is facilitated by considering the contracted equation $R_{ab} k^a k^b = 0$. One finds that $R^{(3)}_{ab} k^a k^b$ and $R^{(2)}_{ab} k^a k^b$ vanish identically, while

$$R^{(1)}_{ab} k^a k^b = -(\bar{D} k_a) \bar{D} k^a \tag{1.9}$$

where $\bar{D} = k^c \bar{\nabla}_c$ is the background covariant derivative taken along the null vector k^c. Hence, the vacuum Einstein equation then implies that $\bar{D} k^a$ is a null vector. Since it is also orthogonal to k^a, it follows that the vector $\bar{D} k^a$ must be parallel to the null vector k^a, *i.e.* that $\bar{D} k^a = \phi k^a$ for some function ϕ. This is equivalent to the statement that k^a is tangent to a null geodesic congruence of the background metric. Assuming this to be the case, it then follows that the contribution to the Ricci tensor at order λ^3, which is given by $R^{(3)}_{ab} = -\frac{1}{2}k_a k_b(\bar{D}k_d)\bar{D}k^d$, vanishes as well. The contribution at order λ^2 does not vanish automatically for k^a geodesic. However, one can show the for geodesic k^a, it is related to the order λ^1 according to

$$R^{(2)}_{ab} = k_a k^c R^{(1)}_{cb}. \tag{1.10}$$

Therefore the vacuum field equations will be satisfied if $R^{(1)}_{ab} = 0$. This establishes that for Kerr-Schild metrics with a geodesic null vector k^a, solving the vacuum field equations reduces to solving the linearized equations in h_{ab} around the background metric, namely

$$R^{(1)}_{ab} = \bar{\nabla}_e[\bar{\nabla}_{(a}(k_{b)}k^e) - \frac{1}{2}\bar{\nabla}^e(k_a k_b)] = 0. \tag{1.11}$$

Taking the trace of this equation and defining $\bar{\nabla}_a k^a = \theta$ leaves one with the identity

$$\dot{\theta} + \dot{\phi} + \theta(\theta + \phi) = 0 \tag{1.12}$$

where the 'dot' denotes differentiation with respect to the null vector. It is understood that θ represents the expansion of the null-vector congruence, which is the same with respect to both the metric and its background. Kerr-Schild spacetimes are naturally split into two families; Expanding ($\theta \neq 0$) and Non-expanding ($\theta = 0$). An example of expanding solutions are the familiar Myers-Perry black holes whereas an example of non-expanding solutions are pp-waves belonging to the Kundt class of spacetimes. As usual, the situation is more subtle in $D > 4$ dimensions as all pp-wave spacetimes cannot be put into Kerr-Schild form. This shows once again the difference between four-dimensional spacetimes and those of higher dimensions. PP-waves have many applications and are exact solutions in Brans Dicke theory, some higher curvature theories, and also have appliations in Kaluza-Klein theory as well as modeling gravitational radiation. The fact that they are expressible in Kerr-Schild form in four-dimensions, but not necessarily so in higher dimensions shows that although the Kerr-Schild ansatz has proved invaluable in the search for black hole solutions, its role in higher dimensions is limited, and therefore it makes sense to investigate its possible extensions either through modifications of the ansatz itself, or how it applies in higher derivative theories of gravity.

CHAPTER 2

THE EXTENDED KERR-SCHILD (XKS) ANSATZ

Given[1] how well the Kerr-Schild ansatz has been employed to find exact solutions, it seems reasonable to ask whether it might be possible to extend the ansatz in a way that might allow one to find new black hole solutions not expressable within the Kerr-Schild formalism, *e.g.* the $D = 5$ black ring. One possible extension[2] was suggested recently in reference [32]. The authors showed that the charged, rotating black holes of minimal, gauged $D = 5$ supergravity, originally found in reference [33] and known as the CCLP spacetimes, may be rewritten in a form similar to (1.2), with $g_{ab} = \bar{g}_{ab} + \lambda h_{ab}$ and $\bar{g}_{ab}$ a flat background metric, but now with

$$h_{ab} = H k_a k_b + K(k_a l_b + l_a k_b). \tag{2.1}$$

Here k^a is again a null vector with respect to the background metric $\bar{g}_{ab}$ as well as the full metric g_{ab}. Similar to the Kerr-Schild case, there are now two scalar functions, H and K. The vector l^a is spacelike and orthogonal to k^a with respect to $\bar{g}_{ab}$, such that $\bar{g}_{ab}k^a l^b = 0$, and we are defining $k_a \equiv \bar{g}_{ab}k^b$ and $l_a \equiv \bar{g}_{ab}l^b$. We will call metrics of this general form extended Kerr-Schild or xKS metrics.

Another indication of the usefulness of the xKS ansatz comes from considering higher dimensional pp-waves, which are defined by having a covariantly constant null

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[1]Much of this chapter follows along our previous publication [34]. It has been condensed and streamlined to better fit the thesis format. Some references and discussion have also been added.

[2]see [38] for a closely related extension of the Kerr-Schild ansatz complementary to the work presented here.

vector (and hence a null Killing field). These spacetimes have long been of interest as exact string backgrounds [37]. It is known that in $D = 4$, all pp-wave spacetimes can be cast into Kerr-Schild form (see [19]). However, as discussed in [29], examples of pp-wave spacetimes are known in higher dimensions that have Weyl types [30] that are not compatible with those of Kerr-Schild spacetimes. Therefore, not all higher dimensional pp-waves can be cast in Kerr-Schild form.

On the other hand, the particular example of a non-Kerr-Schild pp-wave given in [29] is of xKS form, and one may speculate that perhaps all higher dimensional pp-waves can be cast in xKS form. Their general properties and classification have since been studied in greater detail by Málek in [35, 36]. Using the Newman-Penrose tetrad formalism to determine the Weyl type, he was able to show that extended Kerr-Schild spacetimes with a geodesic null vector are of Weyl Type I or more special. Depending on the expansion θ of the null vector, the extended Kerr-Schild ansatz can be used to describe Kundt spacetimes in the non-expanding ($\theta = 0$) case[3], as well as the previously discussed CCLP spacetimes in the case of an expanding ($\theta \neq 0$) null congruence. This is similar to the situation in 4 dimensions with some important exceptions. In the non-expanding case in 4D, the geodesic null-vector is *a priori* also non-twisting and non-shearing, belonging to the Kundt class of spacetimes described by the original Kerr-Schild ansatz. This class also exists in higher dimensions, however there is now a second class, one in which the geodesic null-vector may also have shear and twist, and is now described by the extended Kerr-Schild ansatz. For the expanding case in 4D, these spactimes are algebraically special of Type II or D and represent black hole solutions described by the Kerr-Schild ansatz. In 5D this is no longer the case, as we have seen that the CCLP spacetime is of Type I and can be represented by the extended Kerr-Schild ansatz.

[3]With certain assumptions, such as a Ricci-flat condition, higher dimensional pp-waves of Types II, III, and N are described by the xKS. See [35] for further analysis.

The main focus of this section will be an analysis of the vacuum Einstein equations for xKS metrics. As an indication of the simplifications we will find, we first consider the inverse of an xKS metric. Using a similar perturbative expansion as in the Kerr-Schild case, the calculation shows that the inverse metric now truncates beyond second order in λ, being given exactly by

$$g^{ab} = \bar{g}^{ab} - \lambda h^{ab} + \lambda^2 h^{ac} h_c{}^b. \tag{2.2}$$

Recall that the truncation of the inverse metric beyond linear order in the Kerr-Schild case led to a similar truncation of the Ricci tensor $R^a{}_b$ beyond linear order. Our main task below is to discover the degree of simplification of the Ricci tensor that occurs in the xKS case. We will see that for k^a geodesic and l^a also satisfying a certain condition with respect to the background metric, that the Ricci tensor $R^a{}_b$ will truncate beyond second order in λ. The vacuum Einstein equations then reduce to a set of differential equations that are quadratic in h_{ab}.

2.1 The extended Kerr-Schild form of CCLP spacetimes

Aliev and Çiftçi observed [32] that the charged rotating black holes of minimal $D = 5$ supergravity [33], known as the CCLP solutions[4], may be written in the extended Kerr-Schild form (2.1). Their metrics are presented in a type of spheroidal coordinates and following a sequence of steps given in the Appendix, we have transformed them into Cartesian coordinates. This was done in order to find a higher-dimensional generalization and with the hope of comparing them to the odd-dimensional form of the well known Myers-Perry uncharged rotating black holes [17]. After the transformation, the background is then the familiar 5-dimensional

[4]Interestingly, the authors of the paper stated that the CCLP solutions were found using a brute-force method.

Minkowski spacetime $d\bar{s}^2 = -d\tau^2 + dx^2 + dy^2 + dw^2 + dz^2$, while the vector fields k^a and l^a are then given by

$$k_a dx^a = d\tau - \frac{r(xdx + ydy) + a(xdy - ydx)}{r^2 + a^2} - \frac{r(wdw + zdz) + b(wdz - zdw)}{r^2 + b^2} \tag{2.3}$$

$$l_a dx^a = \frac{b\,(a(xdx + ydy) - r(xdy - ydx))}{r(r^2 + a^2)} + \frac{a\,(b(wdw + zdz) - r(wdz - zdw))}{r(r^2 + b^2)} \tag{2.4}$$

where a and b are defined as the rotation parameters in the $x - y$ and $w - z$ planes, respectively. The scalar functions H and K in (2.1) are given by

$$H = \frac{2m}{\Sigma} - \frac{Q^2}{\Sigma^2}, \qquad K = \frac{Q}{\Sigma}. \tag{2.5}$$

Here we take $\Sigma = r^2 + a^2 \cos^2\theta + b^2 \sin^2\theta$, and r is the spheroidal radial coordinate satisfying

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{w^2 + z^2}{r^2 + b^2} = 1 \tag{2.6}$$

and the angle θ is defined in equation (A.4) in the Appendix. The 1-form gauge potential is proportional to the null vector, and given by $A = (\sqrt{3}Q/2\Sigma)k$.

Indeed we see that the vector k^a is identical to that which appears in the $D = 5$ Myers-Perry uncharged rotating black holes [17]. We also see that similar to the null vector k^a, the spacelike vector l^a is independent of the mass m and charge Q of the spacetime. It can also be easily shown that the vector l^a is orthogonal to k^a in both the $x - y$ and $w - z$ rotation planes. We also note that the null vector k^a satisfies $k^a \bar{\nabla}_a k^b = 0$ for CCLP spacetimes, where $\bar{\nabla}_a$ is the covariant derivative operator for the background metric. This indicates that the vector k^a is tangent to a congruence of affinely parameterized null geodesics. This property is central to the Kerr-Schild construction of the Myers-Perry spacetimes [17] and we see that it holds in xKS case as well. In combination, the vectors k^a and l^a can also be shown to satisfy

$$k^b(\bar{\nabla}_b l_a - \bar{\nabla}_a l_b) = 0, \qquad l^b(\bar{\nabla}_b k_a - \bar{\nabla}_a k_b) = 0 \tag{2.7}$$

which implies the relation

$$k^a \bar{\nabla}_a l^b = -l^a \bar{\nabla}_a k^b \tag{2.8}$$

between the covariant derivative of each vector field along the other.

2.2 Curvature calculations and the vacuum Einstein equations for the extended Kerr-Schild ansatz

We will find it useful to consider the expansion for the Ricci tensor with indices in mixed position, such that $R^a{}_b = g^{ac}R_{cb}$, as it organizes the expansion in a convenient way which will simplify the proceeding calculations. However, due to the extra factor of λ in the inverse metric for the xKS case, the Ricci tensor now has an expansion in powers λ^n that goes out to order $n_{max} = 8$. The coefficients $R^{(n)a}{}_b$ in the expansion of $R^a{}_b$ are simply related to the coefficients in the expansion of R_{ab}, with for example

$$R^{(2)a}{}_b = \bar{g}^{ac}R^{(2)}_{cb} - h^{ac}R^{(1)}_{cb} + h^{ad}h_d{}^c\bar{R}_{cb}. \tag{2.9}$$

It is well known that for KS metrics, many properties that hold for the background metric also remain when calculated using the full metric. Moreover one can show that the expansion, shear and twist of k^a are the same in the KS metric as in the background. This turns out to be true in the xKS case as well. One of the main results in the KS case is that if the vector k^a is geodesic with respect to the full metric it is also geodesic with respect to the background metric. This also holds true in xKS case and can be seen explicitly in the expression $k^a\nabla_a k^b = k^a\bar{\nabla}_a k^b + C^b_{ac}k^a k^c$ where one can check that the quantity $C^b_{ac}k^a k^c$ vanishes, thus showing the relationship.

We are now able to present an analysis of the vacuum Einstein equations for extended Kerr-Schild metrics. We are interested in seeing what simplifications will occur and, in particular, whether the expansion of the Ricci tensor will truncate beyond some relatively low order in λ in a similar way to the KS case. We also hope to find a general condition for the vector l^a analogous to the geodesic condition for k^a, which we saw in the previous section was the case for CCLP spacetimes.

We begin by rescaling the vectors k^a and l^a in the xKS ansatz to absorb the functions H and K which will simplify the calculations. Using the same symbols for the rescaled vectors, the xKS ansatz then takes the form

$$g_{ab} = \bar{g}_{ab} + \lambda h_{ab}, \qquad h_{ab} = k_a k_b + k_a l_b + l_a k_b \tag{2.10}$$

with the vectors still assumed to satisfy the null ($k_a k^a = 0$) and orthogonality ($k_a l^a = 0$) conditions. In our computations it will be helpful to use the relations between terms of successive order in the expansion for the connection coefficients

$$C^{(2)c}_{ab} = -h^c{}_d C^{(1)d}_{ab}, \qquad C^{(3)c}_{ab} = h^c{}_d h^d{}_e C^{(1)e}_{ab}, \tag{2.11}$$

where the first order term is simply $C^{(1)c}_{ab} = \frac{1}{2}\left(\bar{\nabla}_a h_b{}^c + \bar{\nabla}_b h_a{}^c - \bar{\nabla}^c h_{ab}\right)$.

Proceeding initially with the expansion for the Ricci tensor with both indices down, it follows that $R^{(6)}_{ab}$ vanishes identically. Unfortunately $R^{(5)}_{ab}$ does not vanish identically and recalling the procedure for the Kerr-Schild case, we consider the contracted equation $R_{ab}k^a k^b = 0$. This should equal zero once again because we are in vacuum gravity - our intuition is that solving the contracted equation should lead to a condition that will help us with the uncontracted equation, similar to the situation found in the KS case. We find that $R^{(n)}_{ab} k^a k^b$ with $n = 5, 4, 3$ vanish identically, while at order λ^2 we have

$$R^{(2)}_{ab}k^ak^b = \frac{1}{2}\psi^2(\bar{D}k_a)\bar{D}k^a - \frac{1}{2}(l_a\bar{D}k^a)(l_b\bar{D}k^b) \quad (2.12)$$
$$= -\frac{1}{4}\alpha_{ab}\alpha^{ab} \quad (2.13)$$

where $\alpha_{ab} = l_a\bar{D}k_b - l_b\bar{D}k_a$ and ψ^2 is defined to be the norm of the spacelike vector l_a. The vacuum equation implies that the anti-symmetric tensor α_{ab} must be null. Together with the identity $k^a\bar{D}k_a = 0$, this implies that $\bar{D}k^a$ must have the form

$$\bar{D}k^a = \phi k^a + \eta l^a \quad (2.14)$$

for some functions ϕ and η. At order λ^1 one finds

$$R^{(1)}_{ab}k^ak^b = -\bar{D}(l_a\bar{D}k^a) - \theta l_a\bar{D}k^a - (\bar{D}k_a)(\bar{D}k^a + \bar{D}l^a + l^b\bar{\nabla}_bk^a). \quad (2.15)$$

Substituting the form (2.14) into this result gives

$$R^{(1)}_{ab}k^ak^b = -\bar{\nabla}_c(\eta k^cl_bl^b) - \eta^2l_bl^b - \eta l^bl^c\bar{\nabla}_bk_c. \quad (2.16)$$

It is clear that taking k^a to be tangent to a geodesic congruence of the background metric (*i.e.* taking $\eta = 0$) solves $R^{(1)}_{ab}k^ak^b = 0$. This is analogous to the condition found in the KS case, however, it is unclear whether null vectors k^a satisfying (2.14) with $\eta \neq 0$ are possible. We will proceed by assuming that k^a is geodesic. As noted in section (2.1) the null vector field in the CCLP spacetimes satisfies $\bar{D}k^a = 0$.

Given the geodesic condition, we return to the uncontracted equation and find that $R^{(5)}_{ab} = 0$. Due to the complicated nature of calculating $R^{(4)}_{ab}$, we will alter our approach in two ways. Firstly, we choose to work with the Ricci tensor with mixed indices, $R^{(n)a}{}_b$, in which the expansion now goes out to order λ^8. We then find $R^{(n)a}{}_b = 0$ for $n = 5, \ldots, 8$. The second alteration is to adopt a simpler, but still equivalent, form for h_{ab}. Given that the null vector k^a is assumed to satisfy the geodesic condition we can rescale it by a scalar function such that the rescaled vector satisfies $\bar{D}k^a = 0$ (*i.e.* so that the geodesic congruence, to which the rescaled vector is tangent to, is now affinely parameterized). Similarly we can rescale the spacelike vector l^a by a scalar function such that the rescaled vector has unit norm with respect to the background metric. The quantity h_{ab} will now have the form given in (2.1) for CCLP spacetimes for some functions H and K, where k^a and l^a now represent the rescaled vectors. Finally, we can define a new vector m^a such that

$$m^a = l^a + (H/2K)k^a. \tag{2.17}$$

Because the vectors k^a and l^a are orthogonal, the vector m^a will also have unit norm. In terms of m^a, the tensor h_{ab} then reads

$$h_{ab} = K(k_a m_b + m_a k_b), \tag{2.18}$$

where now $k_a k^a = 0$, $m_a m^a = 1$, $k_a m^a = 0$ and $\bar{D}k^a = 0$. This new form for h_{ab} simplifies the calculations considerably[5] and we find that $R^{(4)a}{}_b$ vanishes. Moving on to $R^{(3)a}{}_b$ we obtain the following expression

$$R^{(3)a}{}_b = \frac{1}{2}\bar{\nabla}_d \left(K^3 k_b [k^a v^d - v^a k^d]\right) - \frac{1}{2}K^3 k^a v^d \bar{\nabla}_b k_d \tag{2.19}$$

where[6]

$$v_a = k^b \left\{(\bar{\nabla}_b l_a - \bar{\nabla}_a l_b) - l^c(\bar{\nabla}_b l_c - \bar{\nabla}_c l_b) l_a\right\}. \tag{2.20}$$

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[5]However, note that it is now harder to take a Kerr-Schild limit of the extended Kerr-Schild calculations.

[6]In the expression for v_a, the vector l^a may be replaced by the vector m^a without changing the result for $R^{(3)a}{}_b$.

A sufficient condition for the vanishing of $R^{(3)a}{}_b$ is that the vector v^a be proportional to the null vector,

$$v^a = \alpha k^a \tag{2.21}$$

for some scalar function α. This condition on l^a may be viewed as a counterpart to the geodesic condition for k^a. It is independent of the metric functions H and K, depending only on properties of the vectors l^a and k^a with respect to the background metric. It can be shown that the CCLP spacetimes of section (2.1) satisfy (2.21) with $\alpha = 0$. It remains an open question as to whether the condition on l^a is also necessary. Recently, it was shown in [35] that the vectors k^a and l^a are surface forming.

We have now established a set of sufficient conditions, the geodesic condition on k^a and the condition (2.21) relating k^a and l^a, such that the Ricci tensor with indices in mixed position vanishes beyond quadratic order in λ for xKS spacetimes. One is now left to consider only the quantities $R^{(2)a}{}_b$ and $R^{(1)a}{}_b$ with

$$\begin{aligned} R^{(2)a}{}_b = \; & -\tfrac{1}{2}\bar{\nabla}_d\left\{\bar{\nabla}_b(h^{ac}h_c{}^d) + h^d{}_e(\bar{\nabla}^a h_b{}^e - \bar{\nabla}^e h^a{}_b) + h^a{}_c(\bar{\nabla}^c h_b{}^d - \bar{\nabla}^d h_b{}^c)\right\} \\ & -\tfrac{1}{4}(\bar{\nabla}^e h^{ac} + \bar{\nabla}^c h^{ae} - \bar{\nabla}^a h^{ce})(\bar{\nabla}_e h_{bc} - \bar{\nabla}_b h_{ce} - \bar{\nabla}_c h_{be}) \end{aligned} \tag{2.22}$$

$$R^{(1)a}{}_b = \frac{1}{2}\bar{\nabla}_c\left(\bar{\nabla}^a h_b{}^c + \bar{\nabla}_b h^{ac} - \bar{\nabla}^c h^a{}_b\right). \tag{2.23}$$

Although we have not shown it definitively, we believe that no manipulations of the expression for $R^{(2)a}{}_b$ in the xKS case, using the geodesic condition for k^a in combination with (2.21), will make $R^{(2)a}{}_b$ vanish. It is interesting to note that in the KS case, the inverse metric is first order in λ and the vacuum Einstein equations reduce to the equation $R^{(1)a}{}_b = 0$, which is linear in h_{ab}. While in the xKS case, the inverse metric is second order in λ and the vacuum Einstein equations reduce to $R^{(1)a}{}_b = 0$ and $R^{(2)a}{}_b = 0$, which is quadratic in h_{ab}.

2.3 Adding stress-energy

The CCLP spacetimes [33] shown to be of xKS form in [32] and presented above in section (2.1) are non-vacuum spacetimes in Einstein-Maxwell-Chern-Simons theories of gravity. The equations of motion are given by

$$R^a{}_b - 2\left(F^{ac}F_{bc} - \frac{1}{6}g^a{}_bF^2\right) = 0 \tag{2.24}$$

$$\nabla_a F^{ab} - \frac{1}{2\sqrt{3}\sqrt{-g}}\epsilon^{bcedf}F_{cd}F_{ef} = 0 \tag{2.25}$$

where the second equation is the gauge field equation of motion - the first term is from the standard Maxwell Lagrangian[7], and the second term is the Chern-Simons contribution. To see how the xKS ansatz works in the presence of matter fields, we recall that a key step in our analysis of the KS case was considering the equation $R_{ab}k^ak^b = 0$ which led to the geodesic condition on the null vector k^a. This equation is identically true for vacuum spacetimes and it can easily be shown to hold for Einstiein spacetimes possessing a cosmological constant as well. Although it is possible to consider more general cases for the null vector, we focusing our attention on the geodesic case which in turn implies that the stress-energy tensor should satisfy $T_{ab}k^ak^b = 0$. We will further restrict our attention to the electromagnetic case, with the stress-energy tensor given by

$$T_a{}^b = F_{ac}F^{bc} - \frac{1}{4}g_a{}^bF^2 \tag{2.26}$$

and assume that the gauge potential is related to the xKS null vector k^a according to

$$A_a = \sqrt{\lambda}\beta k_a \tag{2.27}$$

[7]The gauge field equation of motion for Einstein-Maxwell theory is $\nabla_a F^{ab} = 0$

where β is a scalar function. This form of the gauge field holds in $D = 4$ Kerr-Newman spacetimes, in the KS form of the Reissner-Nordstrom spacetime in any dimension, and also in the CCLP spacetimes [33] in xKS form [32]. It is easily checked that the condition $T_{ab}k^a k^b = 0$ is satisfied by this ansatz for the gauge potential.

Let us consider the Kerr-Schild case first. Given that it is necessary to raise two indices on the field strength tensor using the KS inverse metric (1.3) in order to compute the components of $T_a{}^b$, there could in principle be contributions out to order λ^3. However, calculation shows that this is not the case. With the ansatz (2.27) for the gauge potential, the only non-vanishing contributions to $T^a{}_b$ are linear in λ. This is consistent with the reduction in order of the Ricci tensor in mixed form $R^a{}_b$. Had there been a contribution to $T^a{}_b$ at *e.g.* order λ^2, this would have been inconsistent with the vanishing of $R^{(2)a}{}_b$.

Now consider the xKS case. Given the form of the xKS inverse metric (2.2), there could in principle be contributions to $T_a{}^b$ out to order λ^5. Computation shows that while the order λ^n terms in $T_a{}^b$ vanish for $n = 3, 4, 5$, they will generally be non-zero for both $n = 1$ and $n = 2$. This is consistent with our findings above in section (2.2), where we found that, in contrast to the KS case, the term $R^{(2)a}{}_b$ does not generally vanish for xKS spacetimes[8].

For the standard Maxwell Lagrangian, the equations of motion are given by $\nabla_a F^{ab} = 0$ and it is straightforward to substitute in the xKS ansatz. One finds that $F^{ab} = \lambda^{1/2}F^{(1/2)ab} + \lambda^{3/2}F^{(3/2)ab}$ with higher order terms vanishing. It is natural, however, to also include the contribution to the equations of motion coming from the Chern-Simons term in the action of minimal $D = 5$ supergravity that is relevant for the CCLP spacetimes[9]. The gauge field equation of motion is then given by

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[8]It is potentially interesting to note that this same truncation of the stress energy tensor holds if a term $\sqrt{\lambda}\gamma\, l_a$ is added to the gauge potential (2.27) for γ an arbitrary function, if l_a is assumed to satisfy condition (2.21) that implies the vanishing of $R^{(3)a}{}_b$.

[9]Note that the Chern-Simons term does not contribute to the stress energy tensor.

$$\nabla_a F^{ab} - \frac{1}{2\sqrt{3}}\epsilon^{bcedf} F_{cd} F_{ef} = 0. \tag{2.28}$$

At this point, however, a conflict arises in the order by order expansion in powers of λ. Because $\sqrt{-g} = \sqrt{-\bar{g}}$ for xKS spacetimes, one can replace the derivative operator in (2.28) with the background derivative operator. The first term in (2.28) thus has contributions at orders $\lambda^{1/2}$ and $\lambda^{3/2}$, while the second term is manifestly of order λ^1.

We expect that a more subtle analysis would be required in order to properly incorporate the gauge field of minimal $D = 5$ supergravity into our analysis. In hindsight, this is evident from the form of the CCLP spacetimes given in section (2.1). The gauge field is proportional to the charge, and we may therefore think of $\lambda^{1/2}$ as being proportional to the charge Q. In Reissner-Nordstrom spacetimes or in the four dimensional Kerr-Newman spacetimes, the metric depends only on the square of the charge. However, the metric function K in (2.5) is linear in Q. The CCLP metric then appears to include terms proportional to $\lambda^{1/2}$ as well as λ^1. The first term in (2.28) is linear in Q, while the second term is quadratic. It can only be solved by virtue of terms in the metric that are linear in Q.

We will not attempt to carry out such a more subtle analysis here. We note that this issue does not affect our main result in section (2.2), the truncation of the Ricci tensor $R^a{}_b$ beyond quadratic order in λ for xKS metrics with k^a geodesic and k^a and l^a jointly satisfying the condition (2.21).

2.4 Conclusions

In summary, we have shown that for a null vector k^a satisfying the geodesic condition $\bar{D}k^a = 0$, and a spacelike vector l^a satisfying equation (2.21), that the terms $R^{(n)a}{}_b$ in the expansion of the Ricci tensor vanish for $n = 3, \ldots, 8$. The vacuum Einstein equations are quadratic in h_{ab} and reduce to the two equations in (2.22). Condition (2.21) depends only on properties of the vectors k^a and l^a with respect to the background metric and can be regarded as a counterpart to the geodesic condition on k^a. When stress-energy was added in the form of an EM field with the gauge potential being chosen proportional to the null vector, the stress-energy tensor was also seen to truncate beyond second order. This was the case when either choosing an aligned Maxwell field (such that $F_{ab}k^b \propto k_a$) or aligned pure radiation (where $T_{ab} \propto k_a k_b$). However, when considering the full field equations an ambiguity arose - the Chern-Simons term was shown to throw off the balance of the perturbative expansion around λ. This is a subtle point to be explored (hopefully) at a later date.

The Kerr-Schild ansatz has shown its applicability in many instances, but its limitations have been known for some time. Recently there have been searches into extensions of this ansatz in order describe the spacetimes that remain out of reach. One such extension is the extended Kerr-Schild ansatz and as we have seen, the vacuum Einstein equations continue to simplify considerably in the xKS case, although not to the full extent that they do in the original KS case. That being said, the extended Kerr-Schild ansatz has shown to be quite useful in the study of certain higher dimenisional spacetimes and warrants further investigation, perhaps most interestingly in its applicability to higher curvature gravity theories such as Lovelock gravity [39].

CHAPTER 3

LOVELOCK GRAVITY

Another[1] possible extension of the Kerr-Schild ansatz involves looking at how it behaves in higher curvature gravity theories, as opposed to directly manipulating the ansatz itself as we saw with xKS example in the last chapter. One such theory is that of Lovelock gravity [40] which is a class of higher curvature gravity models that enjoy a number of desirable properties that are not shared by generic higher curvature theories. Lovelock gravities have been studied in a wide variety of contexts including brane-world models beginning with the work of [43] and the gauge-gravity correspondence [44, 45, 46, 47, 48, 49, 50, 51, 52, 53].

The action in Einstein's four dimensional General Relativity contains a single power of the Ricci scalar, whereas higher curvature gravity theories entail adding higher order curvature terms composed of the Ricci scalar, Ricci and Riemann tensors, to the action. The Lagrangian for Lovelock gravity [40] combines these quantities in a specific fashion and has the form $\mathcal{L} = \sum_{k=0}^{p} c_k \mathcal{L}_k$ where c_k is an arbitrary constant and $\mathcal{L}_k$ is the Euler density of a $2k$-dimensional manifold

$$\mathcal{L}_k = \frac{1}{2^k}\sqrt{-g}\,\delta^{a_1\ldots a_k b_1\ldots b_k}_{c_1\ldots c_k d_1\ldots d_k}\, R_{a_1 b_1}{}^{c_1 d_1}\ldots R_{a_k b_k}{}^{c_k d_k}, \tag{3.1}$$

where the δ symbol denotes the totally anti-symmetrized product of Kronecker delta functions such that

$$\delta^{a_1\ldots a_k b_1\ldots b_k}_{c_1\ldots c_k d_1\ldots d_k} = \frac{1}{n!}\delta^{a_1\ldots a_k b_1\ldots b_k}_{[c_1\ldots c_k d_1\ldots d_k]}. \tag{3.2}$$

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[1] Much of this chapter follows along our previous publication [39]. It has been condensed and streamlined to better fit the thesis format. Some references and discussion have also been added.

At zeroth order one has $\mathcal{L}_0 = \sqrt{-g}$ and therefore c_0 is proportional to the cosmological constant. At linear order $\mathcal{L}_1 = \sqrt{-g}R$ and we recover the familiar Einstein term in the Lagrangian. The first order correction to the Einstein Lagrangian, the term quadratic in the Riemann tensor in (3.1), is known as the Gauss-Bonnet term and is expressed as $\mathcal{L}_2 = \sqrt{-g}(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2)$. After this, the number of terms at subsequent orders increases rapidly - the $\mathcal{L}_3$ and $\mathcal{L}_4$ terms, *i.e.* the second and third order corrections to the Einstein Lagrangian, contain 8 and 25 terms, respectively.

One of the most attractive aspects of Lovelock theories, as opposed to generic higher curvature theories, is that the equations of motion only depend upon the curvature tensor and not on its derivatives, *i.e.* the equations of motion are second order [40]. This is precisely the situation in General Relativity and in fact, Einstein gravity can be seen as a specific case ($k = 1$) of Lovelock gravity. Another interesting property is that the higher order curvature terms in the Lagrangian are essentially "quasi-topological" since in $D = 2k$ dimensions the variation of $\mathcal{L}_k$ is a total derivative whose volume integral is the topologically invariant Euler character, which does not contribute to the equations of motion. Therefore, one can think of this as the kth term in the Lagrangian "turns on" in $2k+1$ dimensions, *e.g.* the Gauss-Bonnet term $\mathcal{L}_2$ only contributes to the equations of motion for $D \geq 5$. Other useful properties of the Lovelock forumulation are the existence of ghost free constant curvature vacua [41] and a reasonably well behaved initial value formulation [42].

The black hole solutions of Lovelock gravity have been the subject of considerable interest. Static black hole solutions were discovered, beginning with the work of [41, 54, 55] (references [56, 57, 58, 67, 68, 69, 70] review these and related developments, including the thermodynamics of Lovelock black holes). Beyond the static case, many investigators have also been interested in stationary solutions, but so far have only been met with partial success. Rotating solutions in Gauss-Bonnet gravity with asymptotically AdS boundary conditions were found in [59] using first order perturbation theory in the limit of small angular momentum. The full solutions in five dimensions were studied using numerical methods in [60, 61].

The applicability of the Kerr-Schild ansatz in five dimensional Gauss-Bonnet gravity[2] is shown to be restricted in [26]. The authors note the expectation that the Kerr-Schild ansatz would not yield a general solution for rotating black holes in Gauss-Bonnet gravity. However, they do show that the Kerr-Schild ansatz leads to a solution in an interesting special case, namely when the coupling constants are such that the theory admits a unique constant curvature vacuum. We restrict ourselves to this special case[3] of the Lovelock unique vacuum which will be explained in further detail in the next section.

3.1 Lovelock Gravity equations of motion

The equations of motion for the Lovelock theory are essentially a slightly modified version of the Einstein tensor. For maximal order p, the curvature tensor has the form $\mathcal{G}^{(p)a}{}_b = 0$, where $\mathcal{G}^{(p)a}{}_b$ may be written in terms of a new set of parameters[4] $\alpha_0, \ldots, \alpha_p$ in the form

$$\mathcal{G}^{(p)a}{}_b = \alpha_0 \, \delta^{ac_1 \ldots c_p d_1 \ldots d_p}_{be_1 \ldots e_p f_1 \ldots f_p} \left(R_{c_1 d_1}{}^{e_1 f_1} - \alpha_1 \delta^{e_1 f_1}_{c_1 d_1} \right) \cdots \left(R_{c_p d_p}{}^{e_p f_p} - \alpha_p \delta^{e_p f_p}_{c_p d_p} \right). \tag{3.3}$$

For convenience in the following calculations, we will without loss of generality assume that $\alpha_0 = 1$. We will also restrict ourselves to the special case mentioned in the

[2]See [71] for a more recent application, where the authors present a new exact solution in Quadratic Curvature Gravity related to the Kerr-Schild ansatz.

[3]See Appendix C for a discussion of the distinct constant curvature case and an explicit example of black hole solutions in the Gauss-Bonnet gravity.

[4]The coefficients c_k in the Lovelock Lagrangian are given by sums of products of the parameters α_k (see [64] for the explicit form of this relation). Inverting this relation to get the α_k's in terms of the c_k's requires solving a polynomial equation of order p.

introduction, referred to as Lovelock unique vacuum[5] (LUV) theories, for which all the α's are real and equal[6]. This leaves (3.3) in the following form

$$\mathcal{G}^{(p)a}{}_b = \delta^{ac_1 \ldots c_p d_1 \ldots d_p}_{be_1 \ldots e_p f_1 \ldots f_p} \left(R_{c_1 d_1}{}^{e_1 f_1} - \alpha \delta^{e_1 f_1}_{c_1 d_1} \right) \cdots \left(R_{c_p d_p}{}^{e_p f_p} - \alpha \delta^{e_p f_p}_{c_p d_p} \right). \tag{3.4}$$

In the next section we show how the null-vector of the Kerr-Schild ansatz reduces the complexity of the Riemann curvature tensor before moving on to how it reduces the equations of motion, of which, the Einstein and Gauss-Bonnet examples are worked out explicitly.

3.2 The Kerr-Schild Ansatz in Einstein and Gauss-Bonnet Gravity

Contrary to the case in Einstein gravity where it is sufficient to work with the Ricci tensor, for Lovelock theories we will find it necessary to work directly with the Riemann curvature tensor - it shows up explicitly in the quadratic term of the Lovelock Lagrangian. Expressed with one index up, the curvature $R_{abc}{}^d$ of the full Kerr-Schild metric is then related to the curvature $\bar{R}_{abc}{}^d$ of the background metric according to

$$R_{abc}{}^d = \bar{R}_{abc}{}^d + \bar{\nabla}_b C_{ac}{}^d - \bar{\nabla}_a C_{bc}{}^d + C_{ac}{}^e C_{be}{}^d - C_{bc}{}^e C_{ae}{}^d. \tag{3.5}$$

The Lovelock equation of motion (3.3) is expressed with mixed indices and we therefore want to solve for the Riemann tensor in the form $R_{ab}{}^{cd} = g^{ce} R_{abe}{}^d$, which in

---

[5]These theories have been discussed in detail in reference [64]. The analysis of the Kerr-Schild ansatz in these unique vacuum theories works much as it does in Einstein gravity. Unique vacuum theories were also shown to have special properties in reference [24], where extended black brane solutions of Lovelock theories were studied. In dimension $D = 2p + 1$ the unique vacuum theory with highest Lovelock interaction $\mathcal{L}_p$ can be rewritten as a Chern-Simons theory (see [64]) much as Einstein gravity can in $D = 3$ [65, 66].

[6]In the full generalization, the α_p's are distinct and can include complex terms.

principle has an expansion going out to fifth order in λ. Computation shows that the third, fourth, and fifth order terms in the expansion vanish identically, and the remaining relation for the curvature tensor is $R_{ab}{}^{cd} = \bar{R}_{ab}{}^{cd} + \lambda\, R^{(1)cd}_{ab} + \lambda^2\, R^{(2)cd}_{ab}$ where

$$
\begin{aligned}
R^{(1)cd}_{ab} &= -2\bar{\nabla}_{[a}\bar{\nabla}^{[c}k_{b]}k^{d]} + \bar{R}_{ab}{}^{l[c}k^{d]}k_l && (3.6)\\
R^{(2)cd}_{ab} &= k_{[a}k^{[c}A_{b]}{}^{|k|}B_k{}^{d]} + k_{[a}(\bar{D}k^{[c})A_{b]}{}^{d]} - k^{[c}\left[(\bar{D}k_{[a})B_{b]}{}^{d]} - 2\bar{\nabla}_{[a}(k_{b]}\bar{D}k^{d]})\right] && (3.7)
\end{aligned}
$$

with $A_a{}^b = \bar{\nabla}_a k^b + \bar{\nabla}^b k_a$ and $B_a{}^b = \bar{\nabla}_a k^b - \bar{\nabla}^b k_a$. It is useful to note that each term in (3.7) contains at least one factor of the null vector with no derivatives acting on it. This feature will aid in greatly reducing the complexity of the expressions when solving for the equations of motion.

3.2.1 Einstein Gravity

For vacuum Einstein gravity, which is Lovelock gravity with maximum order $p = 1$ and $\alpha_1 = \alpha = 0$, the equations of motion in (3.4) reduce to,

$$
\mathcal{G}^{(1)a}{}_b = \delta^{acd}_{bef} R_{cd}{}^{ef}. \tag{3.8}
$$

It follows that for the Kerr-Schild ansatz the Einstein tensor (3.8) will have the expansion $\mathcal{G}^{(1,n)a}{}_b = \lambda \mathcal{G}^{(1,1)a}{}_b + \lambda^2 \mathcal{G}^{(1,2)a}{}_b$ with the individual terms given by

$$
\begin{aligned}
\mathcal{G}^{(1,1)a}{}_b &= \delta^{acd}_{bef}\, R^{(1)ef}_{cd} && (3.9)\\
\mathcal{G}^{(1,2)a}{}_b &= \delta^{acd}_{bef}\, R^{(2)ef}_{cd} && (3.10)
\end{aligned}
$$

We see that with one factor of the Riemann tensor, the Einstein tensor could have been *a priori* fifth order, but reduces to second order due to the null vector in the Kerr-Schild ansatz.

3.2.2 Gauss-Bonnet Gravity

For Gauss-Bonnet gravity, which is Lovelock gravity with maximum order $p = 2$, the equations of motion have the form $\mathcal{G}^{(2)a}{}_b = 0$ with

$$\mathcal{G}^{(2)a}{}_b = \delta^{acdef}_{bghij}\left(R_{cd}{}^{gh} - \alpha\delta^{gh}_{cd}\right)\left(R_{ef}{}^{ij} - \alpha\delta^{ij}_{ef}\right). \tag{3.11}$$

We assume that the background metric $\bar{g}_{ab}$ in the Kerr-Schild ansatz is a constant curvature vacuum of the theory, so that $\bar{R}_{ab}{}^{cd} = \alpha\delta^{cd}_{ab}$. This takes (3.11) to the further reduced form

$$\mathcal{G}^{(2)a}{}_b = \delta^{acdef}_{bghij} R_{cd}{}^{gh} R_{ef}{}^{ij}. \tag{3.12}$$

We can expand the Gauss-Bonnet field equations in powers of λ, expressing them as the sum $\mathcal{G}^{(2)a}{}_b = \sum_n \lambda^n \mathcal{G}^{(2,n)a}{}_b$. Given the assumptions stated above and plugging in the nonzero terms at orders λ^0, λ^1 and λ^2 in the expansion of the curvature tensor for the Kerr-Schild ansatz, one finds contributions to $\mathcal{G}^{(2,n)a}{}_b$ at orders $n = 2, 3, 4$. Calculation, however, shows that $\mathcal{G}^{(2,4)a}{}_b$ vanishes identically and one is left with the two term expansion for the field equations $\mathcal{G}^{(2,n)a}{}_b = \lambda^2\mathcal{G}^{(2,2)a}{}_b + \lambda^3\mathcal{G}^{(2,3)a}{}_b$ with the individual terms given by

$$\begin{aligned}
\mathcal{G}^{(2,2)a}{}_b &= \delta^{acdef}_{bghij}\, R^{(1)gh}_{cd} R^{(1)ij}_{ef} && (3.13)\\
\mathcal{G}^{(2,3)a}{}_b &= 2\delta^{acdef}_{bghij}\, R^{(1)gh}_{cd} R^{(2)ij}_{ef} && (3.14)
\end{aligned}$$

Please note, in a similar fashion to the Einstein case, we see the highest-order non-vanishing term contains one factor of $R^{(2)ij}_{ef}$. We will see shortly that this is the case for all higher orders in the generic Lovelock case, but first we take a slight a detour in the hopes of finding something special to help us solve the equations of motion.

3.3 Contracting with the null-vector

Following the strategy of [14], we first consider the implications of the Einstein and Gauss-Bonnet tensors contracted twice with the null vector so that we have $\mathcal{G}^{(1)a}{}_a k^b k_a = 0$ and $\mathcal{G}^{(2)a}{}_b k_a k^b = 0$ for the Einstein and Gauss-Bonnet cases respectively. With hindsight in mind, this approach will yield a relationship on the null vector that will reduce the complexity of the expressions when solving the equations of motion. In the Einstein case, inspection shows that the quantity $\mathcal{G}^{(1,2)a}{}_b k_b k^a$ vanishes identically because of anti-symmetrization over repeated factors of the null vector. Similarly for the Gauss-Bonnet case, the quantity $\mathcal{G}^{(2,3)a}{}_b k_a k^b$ vanishes identically, also due to anti-symmetrization over repeated factors of the null vector. The full expression for the contracted Einstein equation then reduces to

$$\begin{aligned} k_a k^b \mathcal{G}^{(1,1)a}{}_b &= \frac{1}{2} k_a k^b \, \delta^{acd}_{bef} \left((\bar{\nabla}_c k_d) \bar{\nabla}^e k^f + (\bar{\nabla}_c k^f) \bar{\nabla}^e k_d \right) \\ &= -\frac{1}{2} (\bar{D} k_c)(\bar{D} k^c). \end{aligned} \tag{3.15}$$

For a Kerr-Schild metric to solve the vacuum field equations, the right hand side of (3.15) must vanish. This is the statement that the vector $\bar{D}k^a$ must itself be null, *i.e.* $(\bar{D}k_c)(\bar{D}k^c) = 0$. Since $\bar{D}k^a$ is also orthogonal to the null vector k^a, it follows that $\bar{D}k^a$ must be proportional to k^a, such that $\bar{D}k^a = \phi k^a$ for some function ϕ. This means that the null vector k^a is tangent to a geodesic congruence of the background. If $\phi = 0$ we say that the geodesic is affinely parameterized.

For the Gauss Bonnet case we find something similar

$$\begin{aligned} k_a k^b \mathcal{G}^{(2,2)a}{}_b &= -24 \, (\bar{D} k_a)(\bar{D} k^b) \, \delta^{acd}_{bef} \, \alpha_{cd}{}^{ef} \\ &= 12 \, (\bar{D} k_a)(\bar{D} k^b) \mathcal{G}^{(1,1)a}{}_b \end{aligned} \tag{3.16}$$

with $\alpha_{cd}{}^{ef} \equiv (\bar{\nabla}_c k_d)\bar{\nabla}^e k^f + (\bar{\nabla}_c k^f)\bar{\nabla}^e k_d$. For a Kerr-Schild metric to solve the vacuum Gauss-Bonnet field equations, the right hand side of (3.16) must vanish. If k^a is in fact geodesic, we see that the *right-hand-side* of (3.16) reduces to the *right-hand-side* of (3.15) with an extra factor of ϕ^2. This establishes that the geodesic condition is at least a sufficient condition for solving the contracted Gauss-Bonnet field equations.

Let us now assume that k^a satisfies the geodesic condition, and see how that alters the expressions for the curvature tensor. It follows then, that the expression (3.7) for $R^{(2)cd}_{ab}$ reduces to

$$R^{(2)cd}_{ab} = k_{[a}k^{[c}E_{b]}{}^{d]} \tag{3.17}$$

with $E_b{}^d = A_b{}^e B_e{}^d - 2\phi B_b{}^d$. Note that the expression for $R^{(2)cd}_{ab}$ now includes factors of the null vector with indices both down and up. Also note that $E_a{}^a = 0$ and that contracting the tensor $E_a{}^b$ with the geodesic null vector gives the simple results $k^a E_a{}^b = -\phi^2 k^b$ and $k_b E_a{}^b = +\phi^2 k_a$. With these results it is straightforward to show that the quantity $\mathcal{G}^{(1,2)a}{}_b$ vanishes identically for k^a geodesic. The vacuum Einstein equations then reduce to the requirement that $\mathcal{G}^{(1,1)a}{}_b = 0$ which is linear in h_{ab}. This is then the advertised result, that for a geodesic null vector the vacuum Einstein equations for the Kerr-Schild ansatz reduce to a linear equation.

For the Gauss-Bonnet case, the key result is that with the use of the geodesic condition, calculation now shows that the quantity $\mathcal{G}^{(2,3)a}{}_b$ vanishes identically. The field equations then reduce to the single equation

$$\mathcal{G}^{(2,2)a}{}_b = \delta^{acdef}_{bghij} R^{(1)gh}_{cd} R^{(1)ij}_{ef} = 0, \tag{3.18}$$

which is quadratic in h_{ab}. In both the Einstein and Gauss-Bonnet cases, the geodesic condition led to the vanishing of the highest order term, the one containing one factor of $R^{(2)cd}_{ab}$. Crucial to this process was the step of contracting the Einstein equation with the null-vector, which in turn revealed its geodesic nature. Using this fact we were able to reduce the complexity of the equations of motion even further. We will attempt to extend this process to general Lovelock orders in the next section. Although we considered only vacuum Einstein gravity, the analysis is essentially unchanged for non-vacuum theories if the stress-energy tensor satisfies $T_{ab}k^a k^b = 0$. In particular, this includes Einstein gravity with a non-vanishing cosmological constant, which we may think of as the most general Lovelock gravity theory including terms up to linear order in the curvature, *i.e.* with maximal order $p = 1$ in the Lovelock Lagrangian.

3.4 Kerr-Schild Ansatz in Lovelock Gravity

We next consider Lovelock theories of arbitrary maximum order p in the curvature tensor. We assume that the spacetime dimension $D \geq 2p+1$, so that the maximum order curvature term is dynamically relevant. Similar to the Einstein and Gauss-Bonnet examples, we consider the unique vacuum case such that $\alpha_1 = \cdots = \alpha_p = \alpha$, and the field equations have the form $\mathcal{G}^{(p)a}{}_b = 0$ with

$$\mathcal{G}^{(p)a}{}_b = \delta^{ac_1d_1\ldots c_pd_p}_{be_1f_1\ldots e_pf_p} \left(R_{c_1d_1}{}^{e_1f_1} - \alpha\delta^{e_1f_1}_{c_1d_1} \right) \cdots \left(R_{c_pd_p}{}^{e_pf_p} - \alpha\delta^{e_pf_p}_{c_pd_p} \right). \tag{3.19}$$

Once again we assume that the background spacetime in the Kerr-Schild ansatz is the constant curvature vacuum, so that the background Riemann tensor is $\bar{R}_{ab}{}^{cd} = \alpha\delta^{cd}_{ab}$. The subsequent analysis then proceeds much as it did in the Einstein and Gauss-Bonnet cases.

One can expand $\mathcal{G}^{(p)a}{}_b = \sum_n \lambda^n \mathcal{G}^{(p,n)a}{}_b$ by plugging in the background curvature and the nonzero terms (3.6) and (3.7) in the expansion of the Riemann curvature tensor. One finds immediately that $\mathcal{G}^{(p,n)a}{}_b = 0$ for $n > p+2$ because of anti-symmetrization over repeated factors of the null vector and further computation shows that $\mathcal{G}^{(p,p+2)a}{}_b = 0$ as well. What remains is then once again a two term expansion, with nonzero contributions at orders λ^p and λ^{p+1} given respectively by

$$\mathcal{G}^{(p,p)a}{}_b = \delta^{ac_1d_1\ldots c_pd_p}_{be_1f_1\ldots e_pf_p} R^{(1)}_{c_1d_1}{}^{e_1f_1} \cdots R^{(1)}_{c_pd_p}{}^{e_pf_p} \tag{3.20}$$

$$\mathcal{G}^{(p,p+1)a}{}_b = p\,\delta^{ac_1d_1\ldots c_pd_p}_{be_1f_1\ldots e_pf_p} R^{(2)}_{c_1d_1}{}^{e_1f_1} R^{(1)}_{c_2d_2}{}^{e_2f_2} \cdots R^{(1)}_{c_pd_p}{}^{e_pf_p} \tag{3.21}$$

These results straightforwardly generalize those found in the Einstein and Gauss-Bonnet cases which correspond to maximum Lovelock orders $p = 1$ and $p = 2$.

We proceed once again by considering the field equation contracted with a pair of null vectors, $k_a k^b \mathcal{G}^{(p)a}{}_b = 0$. It follows immediately that the contraction $k_a k^b \mathcal{G}^{(p,p+1)a}{}_b$ vanishes identically, again due to anti-symmetrization over multiple factors of the null vector. For the contraction $k_a k^b G^{(p,p)a}{}_b$, a result generalizing equation (3.16) from the Gauss-Bonnet case can be shown to hold, namely

$$\mathcal{G}^{(p,p)a}{}_b k_a k^b = 2p(p+1)\mathcal{G}^{(p-1,p-1)a}{}_b(\bar{D}k_a)\bar{D}k^b. \tag{3.22}$$

It then follows by induction, based on the result holding in Einstein gravity ($p = 1$), that the geodesic condition is sufficient for the vanishing of the field equations in the more general Lovelock theory.

Assuming that the null vector k^a is geodesic, and in analogy with the Einstein and Gauss-Bonnet cases, we would like to show that the quantity $\mathcal{G}^{(p,p+1)a}{}_b$ vanishes identically. Although we believe that this will very likely turn out to be the case, the calculation (already long in the Gauss-Bonnet case) has so far proven too cumbersome for us to bring to completion. Should $\mathcal{G}^{(p,p+1)a}{}_b$ vanish as a consequence of the geodesic condition, then one would again be left with a single pth order equation, $\mathcal{G}^{(p,p)a}{}_b = 0$, for the quantity h_{ab} in the Kerr-Schild ansatz to solve the general Lovelock unique vacuum case.

3.5 Conclusions

Our study of the Kerr-Schild ansatz follows analysis of its applicability to 5D Gauss-Bonnet gravity [26] and specifically to rotating black hole solutions in the theory [27]. Extending their analysis to the higher orders in Lovelock theory, we have focused on the unique vacuum case [64] and have shown with definiteness for Gauss-Bonnet gravity, and up to plausible expectations in the general case, that the full field equations reduce to a single equation that is purely of order p in the quantity h_{ab} in the Kerr-Schild ansatz. In the general case of distinct constant curvature vacua, a more complicated set of equations emerge, which appear less promising in terms of their compatibility with the Kerr-Schild ansatz. Finally, we have also studied how the known static black hole solutions of Gauss-Bonnet gravity in the unique vacuum theories fit into our framework. A plausible next step would be looking for rotating black hole solutions of Kerr-Schild form in these theories.

CHAPTER 4

THE KERR-SCHILD ANSATZ IN KALUZA-KLEIN GRAVITY

4.1 Introduction to Kaluza-Klein

Theoretical physics in five dimensions has enjoyed a rejuvination of late due to its being seen as the low energy limit for certain higher-dimensional supergravity and string theories [72, 73]. Any study of higher dimensions should begin with five, and one of the simplest extensions of Einstein's General Relativity is that of Kaluza-Klein theory, which seeks to unify general relativity and electromagnetism, introducing scalar fields by way of an extra spatial dimension. Therefore, a five-dimensional vacuum can be viewed as a four-dimensional spacetime imbued with an electromagnetic field.

One of the first attempts at unification [74] was made in 1914 by Gunnar Nordström (of Reissner-Nordström fame) as a way of bringing together his own scalar theory of gravity[1] with electromagnetism under a unified 5-dimensional framework. The one caveat he added was the so-called "cylinder condition" which said that, *a priori*, the metric components did not depend on the fifth dimension. Interestingly there was little to no impact on the surrounding community to Nordström's unification idea, the author of [77] speculating it was because of its being published so close to the onset of WWI.

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[1]Nordström developed a scalar curvature theory of gravitation at the same time Einstein was developing his tensor theory. Nordström's theory was eventually shown to be incorrect when it did not account for the proper precession of the perihilion of Mercury and did not allow for the deflection of light [75, 76].

Consequently, Theodor Kaluza independently investigated the topic of unification through addition of an extra dimension which he described in a letter to Einstein in 1919, who replied enthusiastically about the idea, remarking [2]

> "*A five-dimensional cylinder world never dawned on me...At first glance I like your idea tremendously.*"

By 1921, Kaluza would publish his results [78] which were subsequently presented to the Prussian Academy by Einstein later that year [79]. Even though unaware of the previous results obtained by Nordström, Kaluza's unification scheme was similar and different in important ways. Unlike Nordström, who was developing his theory pre-relativity and used a scalar potential, Kaluza's unification scheme used the correct Einstein tensor potential. However, in a similar fashion as Nordström, Kaluza would also impose the cylinder condition, but offered no mechanism as to why physics would only depend on the first four dimensions and not the fifth [73].

Kaluza's work was extended in 1926 by Oskar Klein [80] who initially relaxed the cylinder condition and suggested that the extra spatial dimension was in fact compactified, curled up and very small. It was calculated by Klein to be on the order of the Planck length. This result also conveniently explained why we hadn't observed this extra dimension. This compactification would lead to a periodicity requirement in the fifth coordinate, thereby naturally restoring the cylinder condition [73]. We begin our investigation by considering the simplest case in the next section.

4.2 Kaluza-Klein theory in 5 dimensions

We begin with the Kaluza-Klein reduction on a circle from vacuum gravity in $D = 5$ to Einstein-Maxwell-dilaton theory in $D = 4$. We assume the cylindrical condition holds such that the $D = 5$ metric is invariant under translations in the compact x^5 direction and can be expressed in the following form

$$ds_5^2 = e^{-4\phi/\sqrt{3}}(dx^5 + 2A_\mu dx^\mu)^2 + e^{2\phi/\sqrt{3}} g^{(4)}_{\mu\nu} dx^\mu dx^\nu. \tag{4.1}$$

Starting from the action for vacuum gravity in $D = 5$, we have

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g^5} R^5. \tag{4.2}$$

After performing the Kaluza-Klein reduction of the fifth dimension, the resulting action for the $D = 4$ metric $g_{\mu\nu}$, gauge field A_μ, and the dilaton scalar potential ϕ is found to be

$$S = \frac{1}{16\pi G_4} \int d^4x \sqrt{-g^4} \left(R^4 - 2(\nabla\phi)^2 - e^{-2\sqrt{3}\phi} F^2 \right) \tag{4.3}$$

with $G_4 = G_5/L$, where the compact x^5 direction has been identified with period L. Varying this action with respect to the $4D$ inverse metric, the equations of motion can be expressed as

$$\begin{aligned}
\nabla_\mu (e^{-2\sqrt{3}\phi} F^{\mu\nu}) &= 0 \qquad \{i\} \\
\nabla^2\phi + \frac{\sqrt{3}}{2} e^{-2\sqrt{3}\phi} F^2 &= 0 \qquad \{ii\} \\
R_{\mu\nu} - 2(\nabla_\mu\phi)(\nabla_\nu\phi) - 2e^{-2\sqrt{3}\phi}(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F^2) &= 0 \qquad \{iii\}
\end{aligned} \tag{4.4}$$

The four-dimensional component, of a five-dimensional spacetime expressed as in (4.1), would solve this set of 15 four-dimensional equations of motion. This shows explicitly that the $4D$ theory contains matter, the electromagnetic tensor along with a scalar dilaton potential, whereas it was initially derived from a five-dimensional vacuum. Here it is instructive to show how to obtain the Kaluza-Klein form of (4.1) when starting from a known solution. Such an example is provided by the boosted black string which is obtained by adding a flat direction z to the previously known 4D Schwarzschild solution[2],

$$ds_5^2 = -(1-\alpha/r)d\hat{t}^2 + (1-\alpha/r)^{-1}dr^2 + r^2 d\Omega^2 + d\hat{z}^2$$

This is now a 5-dimensional metric that solves the 5-dimensional vacuum equations of motion. Next, performing a boost along the z direction, we take the transformations to be

$$\begin{aligned} \hat{z} &= (\cosh\beta)z + (\sinh\beta)t \\ \hat{t} &= (\sinh\beta)z + (\cosh\beta)t. \end{aligned} \tag{4.5}$$

After some algebra, we find

$$ds_5^2 = (1+\frac{\alpha}{r}\sinh^2\beta)[dz+\frac{(\alpha/r)\sinh\beta\cosh\beta}{1+(\alpha/r)\sinh^2\beta}dt]^2+\frac{-1+(\alpha/r)}{1+(\alpha/r)\sinh^2\beta}dt^2+\frac{dr^2}{(1-\alpha/r)}+r^2d\Omega^2$$

We are now in a position to make the identifications with (4.1) where we see

$$e^{-4\phi/\sqrt{3}} = 1 + (\alpha/r)\sinh^2\beta \tag{4.6}$$

$$2A_\mu dx^\mu = \frac{(\alpha/r)\sinh\beta\cosh\beta}{1+(\alpha/r)\sinh^2\beta}dt \tag{4.7}$$

which puts the metric into the following form

$$\begin{aligned} ds_5^2 &= e^{-4\phi/\sqrt{3}}[dz + 2A_t dt]^2 + e^{4\phi/\sqrt{3}}(-1+\frac{\alpha}{r})dt^2 + \frac{dr^2}{(1-\alpha/r)} + r^2 d\Omega^2 \\ &= e^{-4\phi/\sqrt{3}}[dz + 2A_t dt]^2 + e^{2\phi/\sqrt{3}}[e^{2\phi/\sqrt{3}}(-1+\frac{\alpha}{r})dt^2 + e^{-2\phi/\sqrt{3}}(\frac{dr^2}{(1-\frac{\alpha}{r})} + r^2 d\Omega^2)] \end{aligned} \tag{4.8}$$

[2] see Appendix D for a similar construction starting with the Kerr-Schild form of the metric.

The metric is now in the desired form for a generic Kaluza-Klein parameterization. The 4D metric is clearly seen to be

$$\begin{aligned} ds_4^2 &= g^{(4)}_{\mu\nu} dx^\mu dx^\nu = e^{2\phi/\sqrt{3}}(-1 + \frac{\alpha}{r})dt^2 + e^{-2\phi/\sqrt{3}}(\frac{dr^2}{(1-\alpha/r)} + r^2 d\Omega^2) \\ &= -\frac{1-(\alpha/r)}{(1+\frac{\alpha}{r}\sinh^2\beta)^{(1/2)}}dt^2 + \frac{(1+\frac{\alpha}{r}\sinh^2\beta)^{(1/2)}}{1-\alpha/r}dr^2 + r^2(1+\frac{\alpha}{r}\sinh^2\beta)^{(1/2)}d\Omega^2 \end{aligned} \tag{4.9}$$

As the $4D$ Schwarschild plus a flat direction presented in (4.5) is clearly a solution to the action for a $5D$ vacuum given in (4.2), after performing the Kaluza-Klein reduction and subsequent boost along the compact direction, it can be shown that equation (4.9) is a solution to the field equations (4.4) and the action defined in (4.3).

4.3 A Kerr-Schild analysis in Kaluza-Klein dilaton gravity

Given how well the Kerr-Schild ansatz has been successfully employed in other contexts, it is natural to wonder whether it may be usefully extended in some manner to apply to theories with scalar fields such as Kaluza-Klein gravity, which has gained interest due to its interpretation in a string theory context. A recent example of this is seen in [81], where the authors start from a generalized Myers-Perry black hole solution in $D+1$ dimensions and perform a Kaluza-Klein reduction to find higher dimensional, charged, rotating black holes in Einstein-Maxwell-Dilaton gravity. By subsequently adding a cosmological constant, Wu noticed in [82] (also see [83]) that the form of the metric was similar to that of the Kerr-Schild ansatz. Continuing along this line of thinking, we want to understand how the Kerr-Schild ansatz manifests in the dimensionally reduced setting by studying the resulting equations of motion. In order to understand how the Kerr-Schild ansatz acts in this setting, we need to take a closer look at the null vector. Normally, the Kerr-Schild null-vector k^a does not furnish a natural scalar with which to work - however it is worth noting that if one does not assume that the norm of k^a vanishes, the simple form of the inverse metric may also be achieved by starting with the somewhat more complicated form for the metric

$$g_{ab} = \bar{g}_{ab} + \frac{\lambda k_a k_b}{1 - \lambda k_c k^c}. \tag{4.10}$$

The norm of the null vector $f = k_c k^c$ could then provide a natural ansatz for a scalar field. As we will see, Kaluza-Klein theory serves as a kind of bridge between the ordinary Kerr-Schild ansatz, which applies in the higher dimensional vacuum gravity context and a time-like version, which we will call the modified Kerr-Schild ansatz, that emerges in the dimensionally reduced setting. We hope to see how the simplifications of the field equations from the modified Kerr-Schild ansatz may be understood directly in the dimensionally reduced setting.

Starting with vacuum gravity in $D = 5$ with a flat background in Kerr-Schild form, we can find the 4 dimensional fields that are produced by the dimensional reduction,

$$\begin{aligned}
ds_5^2 &= \eta_{\mu\nu} dx^\mu dx^\nu + (dx^5)^2 + \lambda(k_\mu dx^\mu + k_5 dx^5)(k_\nu dx^\nu + k_5 dx^5) \qquad (4.11)\\
&= (\eta_{\mu\nu} + \lambda k_\mu k_\mu) dx^\mu dx^\nu + (1 + \lambda k_5^2)(dx^5)^2 + \lambda k_5 dx^5 (k_\mu dx^\mu + k_\nu dx^\nu)\\
&= (\eta_{\mu\nu} + \lambda k_\mu k_\mu) dx^\mu dx^\nu + (1 + \lambda k_5^2)[(dx^5)^2 + \lambda \frac{k_5 dx^5}{(1 + \lambda k_5^2)}(k_\mu dx^\mu + k_\nu dx^\nu)\\
&= (1 + \lambda k_5^2)\left(dx^5 + \frac{k_5}{(1 + \lambda k_5^2)} k_\mu dx^\mu\right)^2 + \left(\eta_{\mu\nu} + \frac{\lambda k_\mu k_\nu}{1 + \lambda k_5^2}\right) dx^\mu dx^\nu
\end{aligned}$$

where we have completed the square in going from steps 3 to 4. We are now in a position to read off the $D = 4$ metric, gauge field, and dilaton;

$$e^{-4\phi/\sqrt{3}} = 1 + \lambda k_5^2 \tag{4.12}$$

$$2A_\mu dx^\mu = \frac{k_5}{(1 + \lambda k_5^2)} k_\mu dx^\mu \tag{4.13}$$

$$g_{\mu\nu}^{(4)} = (1 + \lambda k_5^2)^{1/2} \left(\eta_{\mu\nu} + \frac{\lambda k_\mu k_\nu}{1 + \lambda k_5^2}\right) \tag{4.14}$$

Focusing on the $D = 4$ metric (4.14), we see that this has the aforementioned modified Kerr-Schild form in terms of the timelike 4-vector k^μ, which satisfies $k_\mu k^\mu = -k_5^2$. This modified Kerr-Schild form involves both an overall conformal rescaling of the metric by a factor related to the dilaton and also a rescaling of the Kerr-Schild contribution to the metric by a further power of the dilaton field. Alternatively, we can write the $D = 4$ metric as

$$g^{(4)}_{\mu\nu} = e^{-2\phi/\sqrt{3}} \left(\eta_{\mu\nu} + \lambda e^{+4\phi/\sqrt{3}} k_\mu k_\nu \right) . \tag{4.15}$$

Similar metrics and their properties have recently been studied with respect to general rotating charged Kaluza-Klein-(A)dS black hole solutions in [84]. However the authors used a different coupling constant for the dilaton charge than we are considering and also assumed that the 4-vector k^μ was a timelike geodesic with respect to the background (A)dS metric. In this case, when doing a perturbative expansion around the free parameter in the Kerr-Schild ansatz, the authors found a condition that led to inconsistencies with their previous work [82]. Since no free parameter could be found in their scheme, it was not possible to have an analysis of the type we completed in the previous chapters, and a more general construction of expanding the full Lagrangian and field equations of motion around the background spacetime was employed.

4.4 Explicit calculations

In this section we want to look explicitly at the five-dimensional equations of motion, and our hope is to gain insight into how pure geometry in a five-dimensional setting manifests physically in four dimensions. Essentially, we want to go from a vacuum in a pure Einstein theory in $D = 5$, to an Einstein-Maxwell-Dilaton theory in $D = 4$. To accomplish this we start with the usual Kaluza-Klein decomposition and attempt to bring it into a more manageable form.

$$
\begin{aligned}
ds_5^2 &= e^{-4\phi/\sqrt{3}}(dx^5+2A_\mu dx^\mu)^2 + e^{2\phi/\sqrt{3}} g^{(4)}_{\mu\nu} dx^\mu dx^\nu \qquad (4.16)\\
&= e^{-4\phi/\sqrt{3}}\left\{(dx^5+2A_\mu dx^\mu)^2 + e^{6\phi/\sqrt{3}} g^{(4)}_{\mu\nu} dx^\mu dx^\nu\right\}\\
&= e^{-4\phi/\sqrt{3}}\left\{(dx^5+\hat{A}_\mu dx^\mu)^2 + \hat{g_{\mu\nu}} dx^\mu dx^\nu\right\}\\
&= e^{-4\phi/\sqrt{3}}\tilde{g_{ab}}
\end{aligned}
$$

In the above equations, we define

$$
\begin{aligned}
g^{(5)}_{ab} &= e^{-4\phi/\sqrt{3}}\tilde{g_{ab}} \qquad (4.17)\\
\tilde{g_{ab}} &= (dx^5+\hat{A}_\mu dx^\mu)^2 + \hat{g_{\mu\nu}} dx^\mu dx^\nu\\
\hat{g_{\mu\nu}} &= e^{6\phi/\sqrt{3}} g^{(4)}_{\mu\nu}\\
\hat{A}_\mu &= 2A_\mu
\end{aligned}
$$

We want to compute the $D = 5$ Einstein tensor $G^{(5)a}{}_b$ solely in terms of the $4D$ fields; the dilaton ϕ, the Maxwell potential A_μ, and the metric $g_{\mu\nu}$. Latin indices (a,b, etc.) represent the $5D$ coordinates whereas greek indices (μ, ν, etc.) represent the $4D$ quantities. From (4.17) we see that we can first solve for $\hat{G}^{(5)a}{}_b$ in terms of $\hat{A}_\mu$ and $\hat{g_{\mu\nu}}$ which are defined in terms of the 4D fields. From there we can solve for $\tilde{G}^{(5)a}{}_b$ in terms of the "hat" quantities and subsequently find the full $G^{(5)a}{}_b$ by using transformation equations for conformally scaled metrics. We find

$$
\begin{aligned}
G^{(5)5}{}_5 &= \sqrt{3}e^{-2\phi/\sqrt{3}}\left\{\nabla^2\phi + \frac{\sqrt{3}}{2}e^{-2\sqrt{3}\phi}F^2\right\} \qquad (4.18)\\
G^{(5)5}{}_\nu &= -4e^{-2\phi/\sqrt{3}}A^\rho[A_\nu\left\{\nabla_\sigma(e^{-2\sqrt{3}\phi}F^\sigma{}_\rho)\right\} + A_\rho\left\{\nabla_\sigma(e^{-2\sqrt{3}\phi}F^\sigma{}_\nu)\right\}] \qquad (4.19)\\
&+ 2\sqrt{3}e^{-2\phi/\sqrt{3}}A_\nu\left\{\nabla^2\phi + \frac{\sqrt{3}}{2}e^{-2\sqrt{3}\phi}F^2\right\}\\
&+ \left\{R^\mu{}_\nu - 2(\nabla^\mu\phi)(\nabla_\nu\phi) - 2e^{-2\sqrt{3}\phi}(F^\mu{}_\rho F_\nu{}^\rho - \frac{1}{4}g^{(4)\mu}{}_\nu F^2)\right\}\\
G^{(5)\mu}{}_\nu &= 2e^{-2\phi/\sqrt{3}}[A^\mu\left\{\nabla_\sigma(e^{-2\sqrt{3}\phi}F^\sigma{}_\nu)\right\} + A_\nu\left\{\nabla_\sigma(e^{-2\sqrt{3}\phi}F^{\sigma\mu})\right\} \qquad (4.20)\\
&+ \left\{R^\mu{}_\nu - 2(\nabla^\mu\phi)(\nabla_\nu\phi) - 2e^{-2\sqrt{3}\phi}(F^\mu{}_\rho F_\nu{}^\rho - \frac{1}{4}g^{(4)\mu}{}_\nu F^2)\right\}]
\end{aligned}
$$

This expression can be put in compact form by comparing with (4.4), and thereby highlighting the relationship between the four and five-dimensional equations of motion.

$$G^{(5)5}{}_5 = \sqrt{3}e^{-2\phi/\sqrt{3}}\{ii\} \tag{4.21}$$

$$G^{(5)5}{}_\nu = -4e^{-2\phi/\sqrt{3}}A^\rho[A_\nu\{i\}_\rho + A_\rho\{i\}_\nu] + 2\sqrt{3}e^{-2\phi/\sqrt{3}}A_\nu\{ii\} + \{iii\} \tag{4.22}$$

$$G^{(5)\mu}{}_\nu = 2e^{-2\phi/\sqrt{3}}[A^\mu\{i\}_\nu + A_\nu\{i\}^\mu + \{iii\}] \tag{4.23}$$

As $\{i\}$, $\{ii\}$, and $\{iii\}$ are all identically zero, we see that $G^{(5)a}{}_b = 0$ is satisfied as required and that the full five-dimensional metric is expressed in terms of purely four-dimensional quantities. At this point a similar analysis, as to what was done in the previous chapters, could be performed. We saw there that taking the $5D$ vacuum equations of motion and contracting them with the $5D$ null-vector would yield equations implying the geodesic nature of the null-vector. In the present context of Kaluza-Klein theories, we would be able to analyze the situation from a purely $4D$ vantage point, having all the $5D$ equations expressed solely in terms of the $4D$ quantities. It would be interesting if in an analogous way, a "geodesic-like" condition was found applying to the $4D$ solution. This research is ongoing.

4.5 Conclusions

Unification has always been one of the driving forces of physics. From electricity and magnetism, to the Standard Model, and hopefully one day to the union of General Relativity and Quantum Mechanics. One of the earliest attempts was between General Relativity and Electromagnetism, and was cleverly done through the addition of an added spatial dimension. These theories would come to be known as Kaluza-Klein theories, which have gained a renewed popularity of late, due to their appearance as the low energy limit of some string theories. In this chapter, the applicability of the Kerr-Schild ansatz in Kaluza-Klein spacetimes has been discussed. Due to its wide range of use, it makes sense to investigate whether the Kerr-Schild ansatz will provide any advantages to theories with scalar fields.

CHAPTER 5

SUMMARY

The Kerr-Schild ansatz has proved an invaluable tool in the search for black hole solutions in higher dimensions. It was initially employed to find the rotating counterpart to the Schwarzschild black hole solution in four dimensions, which now bears the name of its founder, the Kerr black hole. The ansatz was seen to be applicable as well to the higher-dimensional analogue of the Kerr black hole, the so-called Myers-Perry black holes, to spacetimes enjoying the presence of a cosmological constant, and to some forms of higher-dimensional gravitational radiation such as pp-waves. However, the Kerr-Schild ansatz was restricted to certain types of solutions in higher dimensions, meaning that there existed black hole solutions not described by Kerr-Schild. In this thesis, we have looked at three possible extensions of the Kerr-Schild ansatz in order to determine its effectiveness in describing new black hole solutions.

The first of these to be investigated was the extended Kerr-Schild (xKS) ansatz related to CCLP spacetimes which could be described by the usual Kerr-Schild ansatz with the additon of a spacelike vector component. It was found that in this case, the xKS metrics did indeed reduce the complexity of the equations of motion in an analogous fashion to the KS metrics, while also revealing the geodesic nature of the null-vector. These xKS metrics were found to describe black hole solutions in Einstein-Maxwell-Chern Simons theories, as well as describing higher-dimensional pp-waves.

Another possible extension of the Kerr-Schild ansatz is explored through its application in higher curvature gravity theories, specifically that of Lovelock gravity. The

Lovelock theory comibines the Riemann Tensor along with its contractions in specific combinations to attain certain features such as second order equations of motion and ghost-free vacua. It is the most natural extension of General Relativity to which Einstein gravity can be seen as a specific case of the larger encompassing theory. The KS metrics were shown, in the specific case of the unique vacuum, to reduce the complexity of the equations of motion for these higher curvature theories in an anologous way to the Einstein case with its single factor of the Ricci scalar in the action. The known static black hole solutions of Gauss-Bonnet gravity were discussed, and shown to fit into this framework. This provides evidence that the Kerr-Schild ansatz and its extenstions could be employed in the search for higher-curvature rotating black hole solutions.

The last extension of the Kerr-Schild ansatz, its applicability to Kaluza-Klein theories, was also briefly discussed. Starting with the $4D$ Schwarzschild solution and adding an extra spatial direction solves the $5D$ vacuum equations of motion; after performing a Kaluza-Klein reduction and then boosting along the flat direction, the resulting $4D$ metric attained a "Kerr-Schild-like" form which is a solution to the Einstein-Maxwell-Dilaton gravity theory. We also investigated the form of the five-dimensional equations of motion expressed solely in terms of the four-dimensional quantities. One could now analyze the dynamics from a purely four-dimensional perspective, and this analysis is ongoing.

Black holes are the most majestic objects in the universe. At first, it was due to the fact of nature that regardless of the initial internal structure of a body or the mechanism of its eventual collapse, a massive body of sufficient size would end its life as a black hole described by only 3 parameters - the Mass, Charge, and Spin. This was the famous result that black holes in four dimensions "had no hair" and possessed a spherical horizon topology. It was found later that this maxim was no longer true once one started to search for solutions in higher dimensions. Black holes could now come in different flavors and had exotic names such as Black Strings, Black Rings, Black Branes, and my personal favorite, Black Saturns. Unlike the simple case of four dimensions, this freedom greatly complicates the classification scheme of black holes in higher dimensions and more sophisticated techniques of studying black hole solutions will need to be employed for future efforts. However we shouldn't worry too much since this is precisely what makes black hole physics so fun!

APPENDIX A

TRANSFORMING CCLP SPACETIMES TO CARTESIAN COORDINATES

In this appendix we show how to transform the xKS form of the the $\Lambda = 0$ limit of the CCLP metrics given in [32] into the Cartesian coordinates in section (2.1). The xKS form (2.1) of the $\Lambda = 0$ CCLP spacetimes presented in [32] is

$$\begin{aligned} d\bar{s}^2 &= -dt^2 - 2dr(dt - a\sin^2\theta d\phi - b\cos^2\theta d\psi) + \Sigma d\theta^2 \\ &\quad +(r^2+a^2)\sin^2\theta d\phi^2 + (r^2+b^2)\cos^2\theta d\psi^2 \\ k_a\, dx^a &= dt - a\sin^2\theta d\phi - b\cos^2\theta d\psi, \\ l_a\, dx^a &= -b\sin^2\theta d\phi - a\cos^2\theta d\psi \end{aligned} \tag{A.1}$$

with the functions H and K and the 1-form gauge potential $A_a dx^a$ as given in section (2.1). The flat background metric $\bar{g}_{ab}$ can be transformed into more standard spheroidal coordinates via a transformation such that

$$dt = d\tau - dr, \qquad d\phi = d\varphi - \frac{a}{r^2+a^2}dr, \qquad d\psi = d\chi - \frac{b}{r^2+b^2}dr. \tag{A.2}$$

giving

$$\begin{aligned} d\bar{s}^2 &= -d\tau^2 + \frac{r^2\Sigma}{(r^2+a^2)(r^2+b^2)}dr^2 + \Sigma d\theta^2 + (r^2+a^2)\sin^2\theta d\varphi^2 + (r^2+b^2)\cos^2\theta d\chi^2 \\ k_a\, dx^a &= d\tau - \frac{r^2\Sigma}{(r^2+a^2)(r^2+b^2)}dr - a\sin^2\theta d\varphi - b\cos^2\theta d\chi \\ l_a\, dx^a &= \frac{ab\Sigma}{(r^2+a^2)(r^2+b^2)}dr - b\sin^2\theta d\varphi - a\cos^2\theta d\chi \end{aligned} \tag{A.3}$$

A further transformation may now be made to Cartesian spatial coordinates via

$$\begin{aligned} x &= \sqrt{r^2+a^2}\sin\theta\cos\varphi, \qquad y = \sqrt{r^2+a^2}\sin\theta\sin\varphi \\ w &= \sqrt{r^2+b^2}\cos\theta\cos\chi, \quad z = \sqrt{r^2+b^2}\cos\theta\sin\chi. \end{aligned} \tag{A.4}$$

The spheroidal radial coordinate r satisfies the relation (2.6). so that surfaces of large r are approximately spherically symmetric, while as r approaches to zero they degenerate into the product of a disk of radius a in the xy-plane with a disk of radius b in the wz-plane. The background metric and the vectors k^a and l^a are then those given in (2.3).

APPENDIX B

MYERS-PERRY TO SCHWARZSCHILD WITH KERR-SCHILD

Here it is useful to show that by starting from the Kerr-Schild ansatz for Myers-Perry black holes, we can recover the familiar Schwarzschild solution. Assuming a flat background, the full metric takes the form $g_{\mu\nu} = \eta_{\mu\nu} + \lambda h k_\mu k_\nu$ where h is defined to be a scalar function. This is a slightly different form from (1.2). In essence, we are always able to scale the null-vector k_μ in order to obtain an affine parameterization such that $\bar{D}k_\mu = 0$, however the tradeoff is that the ansatz will now include a scalar function, in this case h. To recover the Schwarzschild solution we may define $h = 2M/r$ and $k_\mu dx^\mu = dt + dr$, such that

$$
\begin{aligned}
ds^2 &= -dt^2 + dr^2 + r^2 d\Omega^2 + (2M/r)(dt+dr)^2 \\
&= -(1-2M/r)dt^2 + (1+2M/r)dr^2 + (4M/r)dtdr + r^2 d\Omega^2 \\
&= -(1-\frac{2M}{r})\left\{dt - \frac{2M/r}{(1-\frac{2M}{r})}dr\right\}^2 + \left\{(1+\frac{2M}{r}) + \frac{4M^2/r^2}{(1-\frac{2M}{r})}\right\}dr^2 + r^2 d\Omega^2 \\
&= -(1-\frac{2M}{r})\hat{d}t^2 + \frac{\hat{d}r^2}{(1-\frac{2M}{r})} + r^2 d\Omega^2. \qquad \text{(B.1)}
\end{aligned}
$$

In an analagous fashion we can find the charged static black hole (Reisnner-Nordstrom) by taking $h = 2M/r + Q^2/r^2$.

APPENDIX C

DISTINCT CONSTANT CURVATURE VACUA AND STATIC GAUSS-BONNET BLACK HOLES

In the more general case, the constants α_1 and α_2 in the Gauss-Bonnet field equation (3.3) are unequal (but still assumed to be real), so that the theory now has two distinct constant curvature vacua. We will assume that the background metric has constant curvature $\bar{R}_{ab}{}^{cd} = \alpha_1 \delta_{ab}^{cd}$ corresponding to one of these two vacua. Much of the analysis from the unique vacuum case ($\alpha_1 = \alpha_2$) carries over to this more general case, however, the key modification is now an additional linear term in the expansion of $\mathcal{G}^{(2)a}{}_b$. Assuming that that the null vector k^a is geodesic once again, one is then left with a pair of equations that must be satisfied, equation (3.18) and also the equation linear in h_{ab}

$$\mathcal{G}^{(2,1)a}{}_b = 4(\alpha_1 - \alpha_2)(D-3)(D-4)\, \delta_{bef}^{acd}\, R_{cd}^{(1)ef} = 0. \tag{C.1}$$

This latter condition is simply the vanishing of the linearized Einstein tensor[1]. In order to have a Kerr-Schild solution of the form (1.2) depending on a free parameter λ, as we have required, it is then necessary to have a solution to the linearized Einstein equations that simultaneously solves equation (C.1). Note also that in the analysis of reference [26], which does not utilize the expansion in λ, the two conditions (3.18)

[1]For higher order Lovelock theories with distinct constant curvature vacua, we assume *e.g.* that all the α's in (3.19) are real and distinct and that we choose the background metric to be the constant curvature vacuum having $\bar{R}_{ab}{}^{cd} = \alpha_1 \delta_{ab}^{cd}$. It is then straightforward to show that there will be additional equations to satisfy at orders λ^n with $n = 1, \ldots, p-1$

and (C.1) are combined. We can see how this added complication manifests in the known static black hole solutions of Gauss-Bonnet gravity.

The static black hole solutions of Gauss-Bonnet gravity have been known for some time [41, 54], and in $D \geq 5$ dimensions have the form $ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2_{D-2}$ with

$$f = 1 - \frac{r^2}{2}\left\{(\alpha_1 + \alpha_2) \pm \sqrt{(\alpha_1 - \alpha_2)^2 + \frac{4\sigma}{r^{D-1}}}\right\} \tag{C.2}$$

where σ is a constant proportional to the black hole mass. The metric is asymptotic at spatial infinity to the vacuum with constant curvature α_1 or α_2, depending on which sign is chosen in (C.2). Examining the Kerr-Schild forms of these static black hole solutions will help us appreciate the difference in how the Kerr-Schild ansatz works in the unique vacuum case and in the more general case of distinct constant curvature vacua.

The Kerr-Schild construction of static black hole metrics starting from a background metric with constant curvature α is done by writing

$$\begin{aligned} ds^2 &= -(1-\alpha r^2)dT^2 + \frac{dr^2}{1-\alpha r^2} + r^2 d\Omega^2_{D-2} + F(r)(dT + \frac{dr}{1-\alpha r^2})^2 && \text{(C.3)} \\ &= -(1-\alpha r^2 - F)dt^2 + \frac{dr^2}{1-\alpha r^2 - F} + r^2 d\Omega^2_{D-2}, && \text{(C.4)} \end{aligned}$$

where the vector k^a with covariant components $k_a dx^a = dT + dr/(1-\alpha r^2)$ is null with respect to the background metric and the second line is obtained from the first by transforming to a new time coordinate t such that $dt = dT + dr/(1-\alpha r^2)(1-\alpha r^2 - F)$. If one takes, for example, $F = c/r^{D-3}$ then this gives the (A)dS-Schwarzschild family of spacetimes. By taking $F = (\alpha' - \alpha)r^2$ one can also express a metric with constant curvature α' starting from the background metric with constant curvature α.

$$ds^2 = -(1-\alpha' r^2)dt^2 + \frac{dr^2}{1-\alpha' r^2} + r^2 d\Omega^2_{D-2}. \tag{C.5}$$

This is an example in which the background metric is not taken to solve the field equations, and since there is no free multiplicative parameter in this case, we are not strictly speaking of Kerr-Schild form as we have defined it in (1.2).

The static Gauss-Bonnet black holes with metric functions (C.2) can be written in Kerr-Schild form in different ways. Taking a flat background, $\alpha = 0$ in (C.3), one can simply take the Kerr-Schild function $F = 1 - f$ to obtain

$$\begin{aligned} ds^2 &= -(1-\alpha r^2)dT^2 + \frac{dr^2}{1-\alpha r^2} + r^2 d\Omega_{D-2}^2 + F(r)(dT + \frac{dr}{1-\alpha r^2})^2 \\ &= -(1-F)dt^2 + \frac{dr^2}{1-F} + r^2 d\Omega_{D-2}^2 \\ &= -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_{D-2}^2. \end{aligned} \tag{C.6}$$

However, given the form of f in (C.2), the flat background will only solve the equations of motion if one of α_1 or α_2 is zero, and this does not fit within our scheme. Alternatively, one can start with *e.g.* the constant curvature background metric with $\alpha = \alpha_1$. The Gauss-Bonnet black hole with these asymptotics is then obtained by taking $F = 1 - \alpha_1 r^2 - f$ such that

$$\begin{aligned} ds^2 &= -(1-\alpha_1 r^2 - F)dt^2 + \frac{dr^2}{1-\alpha_1 r^2 - F} + r^2 d\Omega_{D-2}^2 \\ &= -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_{D-2}^2. \end{aligned} \tag{C.7}$$

The function F is more complicated than the one taken for the flat background, and unlike the case of (A)dS-Schwarzschild, there is no overall free multiplicative factor. This reflects the more complicated set of equations that the Kerr-Schild null vector must satisfy in the case of two distinct vacua.

Now consider the static black holes in the unique vacuum case, which are obtained from (C.2) by setting $\alpha_1 = \alpha_2 = \alpha$. In this limit, the metric function simplifies considerably, becoming

$$f = 1 - \alpha r^2 + \frac{\lambda}{r^{\frac{D-5}{2}}} \tag{C.8}$$

with λ a free parameter. One can note that these spacetimes, which are discussed in some detail in reference [64], have slower than usual fall-off at infinity. However, the main thing to observe is that the Kerr-Schild form of these metrics is simple. Taking the background metric to be the unique constant curvature vacuum, the Kerr-Schild representation is simply (C.3) with $F = \lambda/r^{\frac{D-5}{2}}$. The appearance of a free multiplicative parameter in F indicates that these spacetimes fall within the framework of our analysis, and we take this as evidence that looking for rotating black hole solutions via the Kerr-Schild will be simplest in the unique vacuum case. Given the failure to find rotating Kerr-Schild generalizations of the static black holes of general Gauss-Bonnet theories in [26, 27], it may be interesting to investigate generalizations of the Kerr-Schild ansatz such as the one analyzed in [34].

APPENDIX D

KERR-SCHILD FORM IN KALUZA-KLEIN

We start off with the Kerr-Schild form of the Schwarzschild metric and after adding a flat direction, we have

$$ds_5^2 = -dt^2 + dr^2 + r^2 d\Omega^2 + (\alpha/r)(dt + dr)^2 + dz^2. \tag{D.1}$$

We once again boost along the flat direction using the following transformations

$$\begin{aligned} t &= (\cosh\beta)\hat{t} + (\sinh\beta)\hat{z} \\ z &= (\sinh\beta)\hat{t} + (\cosh\beta)\hat{z} \end{aligned} \tag{D.2}$$

and we are left with

$$ds_5^2 = -d\hat{t}^2 + +d\hat{z}^2 + dr^2 + r^2 d\Omega^2 + (\alpha/r)[(\cosh\beta)d\hat{t} + (\sinh\beta)d\hat{z} + dr]^2. \tag{D.3}$$

After a bit of algebra, we are able to express the metric in the form

$$\begin{aligned} ds_5^2 = (1 + \frac{\alpha}{r}\sinh^2\beta)[d\hat{z} + \frac{(\alpha/r)\sinh\beta\cosh\beta}{1 + (\alpha/r)\sinh^2\beta}d\hat{t} + \frac{(\alpha/r)\sinh\beta}{1 + (\alpha/r)\sinh^2\beta}dr]^2 \\ -d\hat{t}^2 + dr^2 + r^2 d\Omega^2 + \frac{(\alpha/r)\sinh\beta}{1 + (\alpha/r)\sinh^2\beta}[(\cosh\beta)d\hat{t} + dr]^2 \end{aligned} \tag{D.4}$$

Making a similar identification as before,

$$
\begin{aligned}
e^{-4\phi/\sqrt{3}} &= 1+\frac{\alpha}{r}\sinh^2\beta \\
e^{-2\phi/\sqrt{3}} &= (1+\frac{\alpha}{r}\sinh^2\beta)^{(1/2)}
\end{aligned}
$$

we are once again able to peel off the four dimensional metric

$$
\begin{aligned}
ds_4^2 = (1+\frac{\alpha}{r}\sinh^2\beta)^{(1/2)}[-d\hat{t}^2 &+ dr^2+r^2d\Omega^2+\frac{(\alpha/r)\sinh\beta}{1+(\alpha/r)\sinh^2\beta}[(\cosh\beta)d\hat{t}+dr]^2] \\
= (1+\frac{\alpha}{r}\sinh^2\beta)^{(1/2)}[(-1 &+ \frac{(\alpha/r)\cosh^2\beta}{(1+(\alpha/r)\sinh^2\beta}))d\hat{t}^2+(1+\frac{\alpha/r}{(1+(\alpha/r)\sinh^2\beta})dr^2 \\
&+ \frac{2(\alpha/r)\cosh\beta}{(1+(\alpha/r)\sinh^2\beta})) d\hat{t}dr+r^2d\Omega^2].
\end{aligned}
\tag{D.5}
$$

After completing the square, we have

$$
\begin{aligned}
ds_4^2 = (1+\frac{\alpha}{r}\sinh^2\beta)^{(1/2)}\{(-\frac{(1-\alpha/r)}{1+(\alpha/r)\sinh^2\beta})[d\hat{t}+\frac{(\alpha/r)\cosh\beta}{1-(\alpha/r)}dr]^2+r^2d\Omega^2 \\
+\frac{1}{1+\frac{\alpha}{r}\sinh^2\beta}[1+\frac{\alpha}{r}\cosh^2\beta+\frac{(\alpha^2/r^2)\cosh^2\beta}{1-(\alpha/r)}]dr^2\}.
\end{aligned}
\tag{D.6}
$$

Now after redefining the $\hat{t}$ coordinates and sprinkling a little algebra on top, we have

$$
ds_4^2 = (1+\frac{\alpha}{r}\sinh^2\beta)^{(1/2)}\{(\frac{-(1-\frac{\alpha}{r})}{1+\frac{\alpha}{r}\sinh^2\beta})dt^2+\frac{[1+\frac{\alpha}{r}(\cosh^2\beta-1)]}{(1+\frac{\alpha}{r}\sinh^2\beta)}\frac{dr^2}{(1-\frac{\alpha}{r})}+r^2d\Omega^2\}.
\tag{D.7}
$$

Finally, we are left with the expression

$$
ds_4^2 = \frac{-1+(\alpha/r)}{(1+\frac{\alpha}{r}\sinh^2\beta)^{(1/2)}}dt^2+\frac{(1+\frac{\alpha}{r}\sinh^2\beta)^{(1/2)}}{1-\alpha/r}dr^2+r^2(1+\frac{\alpha}{r}\sinh^2\beta)^{(1/2)}d\Omega^2
\tag{D.8}
$$

where we see that the final expression using the Kerr-Schild form for the initial metric matches the previously found expression from (4.9).

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