2016

Topology of the Affine Springer Fiber in Type A

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TOPOLOGY OF THE AFFINE SPRINGER FIBER IN TYPE A

A Dissertation Presented

by

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Submitted to the Graduate School of the
University of Massachusetts Amherst in partial fulfillment
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

February 2016

Department of Mathematics and Statistics

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ACKNOWLEDGEMENTS

I would like to acknowledge my indebtedness to my advisor, Alexei Oblomkov, for his patience, guidance, and advice throughout the past 5 years. His assistance, suggestions, explanations, and re-explanations made this work possible. I am extremely grateful to Tom Braden, Julianna Tymoczko, and Andrew McGregor for many helpful conversations about my work.

I have received a very great deal of support from family and friends while in graduate school. Fellow mathematicians Nico, Luke, Jeff, Steve, Jenn, and Tom provided much needed commiseration, advice, and occasional distraction. Laura and Nicky, always integral parts of my educational career, offered regular sanity checks. Cat and Rosie have given me near-daily encouragement and, towards the end, a second home. My parents have been a constant source of support and advice. I am indebted to all of them, and others.

And finally, I owe endless thanks to Andy, who has been my favorite part of the last 6 years.

ABSTRACT

TOPOLOGY OF THE AFFINE SPRINGER FIBER IN TYPE A

FEBRUARY 2016

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We develop algorithms for describing elements of the affine Springer fiber in type A for certain $\gamma \in \mathfrak{g}(\mathbb{C}[[t]])$. For these γ, which are equivalued, integral, and regular, it is known that the affine Springer fiber, X_γ, has a paving by affines resulting from the intersection of Schubert cells with X_γ. Our description of the elements of X_γ allow us to understand these affine spaces and write down explicit dimension formulae. We also explore some closure relations between the affine spaces and begin to describe the moment map for the both the regular and extended torus action.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 Affine Springer Fibers

The study of affine Springer fibers, originally defined by Kazhdan and Lusztig in [5] was motivated by the theory of Springer fibers, which enjoy some particularly useful properties. Let G be a connected semisimple algebraic group over an algebraically closed field and fix T, a maximal torus and B, a Borel subgroup containing T. For $\gamma \in \mathfrak{g}$, the Lie algebra of G, the Springer fiber is $Y_\gamma = \{g \in G/B | \mathrm{Ad}(g^{-1}(\gamma) \in \mathfrak{b}\}$, the fiber of γ in the Grothendiek simultaneous resolution. The connected components of Y_γ are indexed by a quotient of the Weyl group; when γ is regular and semisimple, the connected components are indexed by the Weyl group itself. The Weyl group also acts on the cohomology of Y_γ

Kazhdan and Lusztig extended the study of Springer fibers by considering their analogue over local function fields $F = k((t))$. Working within the loop group, $G(F)$, they defined the affine Springer fiber X_γ as a sub-ind scheme of the affine Grassmannian. Lusztig showed in [7] that there is an action of the affine Weyl group $\widetilde{W} = Hom(k^*, t) \rtimes W$ on the cohomology of X_γ.

The theory of affine Springer fibers was motivated in part by attempts to prove the fundamental lemma, because orbital integrals have a connection to the number of points of affine Springer fibers. The method used by Laumon and Ngô in [6] relied on embedding affine Springer fibers into Hitchin fibers. Despite this, the geometry of the affine

Springer fiber remains less understood than the geometry of the finite Springer fibers.

1.2 Main Results and Organization

We will take $F = \mathbb{C}((t))$ with ring of integers $\mathfrak{o} = \mathbb{C}[[t]]$ and consider the affine Grassmannian $X = G(F)/G(\mathfrak{o})$. For $\gamma = \text{diag}\{\gamma_1 t, \gamma_2 t, \ldots, \gamma_n t\} \in \mathfrak{g}(\mathfrak{o})$, γ_i distinct, we will consider the affine Springer fiber X_γ. This choice of γ was originally motivated by the study of the Hilbert scheme of points for curves of the form $y^m = x^k$. In this case, it was proved in [2] that X_γ has a paving by affine spaces, gotten by intersecting the Schubert cells of X with X_γ. The closure relations of the Schubert cells in X is well known and are given by the Bruhat order, but similar relations have not been found for affine Springer fibers. In this thesis, I attempt to understand this paving by developing methods to compute explicit bases for elements of the affine paving and then computing the closure relationships between the spaces.

In Chapter 2, I present background material on the affine Grassmannian and affine Springer fiber. In Chapter 3, I construct a decomposition of X_γ into affine varieties consisting of sets of lattices, indexed by elements of the lattice of translations. These varieties are easy to describe and have a straightforward dimension formula. In Chapter 4, I develop a method that, given one of the affine varieties, uses a Grassmannian to identify a subset of lattices containing the closure. Conjecturally, this subset of lattices is precisely the closure of the variety. In Chapter 5, using the bases from Chapter 3, I identify all the 0 and 1 dimensional orbits of the extended torus action on X_γ and consider the moment graph of X_γ for $n = 2$ and $n = 3$. Directions for future work appear in Chapter 6 and a few longer computations appear in the appendices for reference.

CHAPTER 2

AFFINE GRASSMANNIANS AND AFFINE SPRINGER FIBERS

In this chapter, we recall the definition and basic properties of affine Grassmannians and affine Springer fibers. We also present the pertinent results about the affine Springer fiber that will be used in the sequel, as well as the definition of equivariant cohomology.

2.1 The Affine Grassmannian

2.1.1 Basic Definitions

Let $F = \mathbb{C}((t))$, with ring of integers $\mathfrak{o} = \mathbb{C}[[t]]$. Let G be a reductive group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. In particular, we will work in type A and take $G = SL_n$ or $G = GL_n$. Let T be a maximal torus in G, with Lie algebra $\mathfrak{t}$ and B be a Borel subgroup containing T. Define the Weyl group W of G to be $N(T)/T$, where $N(T)$ is the normalizer of T. Since we are considering GL_n and SL_n, we will always take T to be the diagonal matrices and B to be the upper triangular matrices. The Weyl group in both cases is S_n, the symmetric group on n elements.

Denote $G(F)$ as $\mathbf{G}$ and define $\mathfrak{g}(F) = \mathfrak{g} \otimes_{\mathbb{C}} F$ and $\mathfrak{g}(\mathfrak{o}) = \mathfrak{g} \otimes_{\mathbb{C}} \mathfrak{o}$. The affine Grassmannian X is the quotient $\mathbf{G}/G(\mathfrak{o})$. Let $L = T(F)/T(\mathfrak{o})$ be the lattice of translations. In the case of $G = GL_n$, this is just the set of matrices of the form $\mathrm{diag}\{t^{a_1}, \dots, t^{a_n}\}$, $a_i \in \mathbb{Z}$. For $G = SL_n$, we have the additional restriction that $\sum a_i = 0$.

X is an ind-scheme, and can thus be written as $\bigcup_{n\geq 0} X_n$, where X_n is a scheme. There are multiple characterizations of the affine Grassmannian; in the case of GL_n and SL_n, we can express the elements of X_n in terms of lattices.

2.1.2 Lattices

We define $\Lambda^{st} = \mathfrak{o}^n$ to be the standard lattice. It is trivially an $\mathfrak{o}$ module; we will view it as a submodule of F^n.

Definition 2.1. *([4]) A lattice $\Lambda \subset F^n$ is an $\mathfrak{o}^n$ submodule so that*

1. *There exists $N \in \mathbb{Z}_{\geq 0}$ so that $t^N \Lambda^{st} \subset \Lambda \subset t^{-N}\Lambda^{st}$*
2. *$t^{-N}\Lambda^{st}/\Lambda$ is locally free of finite rank over $\mathbb{C}$.*

More concretely, a full rank lattice Λ is the $\mathfrak{o}$ span of n vectors $\{v_1, \ldots, v_n\}$ in F^n. For $G = GL_n$, the affine Grassmannian is precisely the set of full rank lattices. In the case of $G = SL_n$, the affine Grassmannian is the set of full rank lattices such that $\bigwedge^k \Lambda = \Lambda^{st}$. Given some $x \in X$, we can associate x to the lattice $x\Lambda^{st}$. For the other direction, given a lattice Λ, choose any basis $\{v_1, \ldots, v_n\}$ and let $x \in X$ be the matrix with columns $\{v_1, \ldots, v_n\}$. Any other choice of basis amounts to right multiplying x by some $g \in G(\mathfrak{o})$, so this is well defined.

These lattices also allow us to see the ind-scheme structure of X. For $N \geq 0$, let X_N be the set of lattices such that $t^N\Lambda^{st} \subset \Lambda \subset t^{-N}\Lambda^{st}$. It is shown [4] that X_N is a projective scheme, with $X = \bigcup X_N$.

Given a lattice Λ, one particularly useful piece of data is the minimum degree tuple.

Definition 2.2. *For a lattice $\Lambda \in X$, let d_i be the minimum degree for which $e_i t^{d_i}$+higher degree terms appears in Λ.*

For Λ with minimum degree tuple $(d_1, \ldots, d_n)$, we say the index of Λ is $N = \sum_i d_i$.

2.1.3 Torus Action

There is an evaluation map from $G(\mathfrak{o}) \to G(\mathbb{C})$ sending g to $g(0)$. The Iwahori subgroup, I, corresponding to the Borel subgroup is the preimage of B under this map. We have the Bruhat decomposition of X into Schubert cells:

$$X = \bigcup_{\ell \in L} I\ell G(\mathfrak{o})$$

Let $\ell \in L$ be the diagonal matrix with entries $\{t^{d_1}, \ldots, t^{d_n}\}$. In terms of lattices, the Schubert cell corresponding to l is the set of all lattices with minimum degrees $(d_1, \ldots, d_n)$.

We can define a torus action on X by $T(\mathbb{C})$, acting on X by left multiplication. We will also consider the action of $\mathbb{C}^*$ on F given by scaling t: $\lambda \cdot t^m = \lambda^m t^m$. On $G(F)$, this action preserves $G(\mathfrak{o})$, so we can define the action of this torus on X. Additionally, the scaling or rotation action commutes with the action of $T(\mathbb{C})$, so we can define the extended torus $\widetilde{T} = T(\mathbb{C}) \times \mathbb{C}^*$, which acts on X by

$$(g, \lambda) \cdot xG(\mathfrak{o}) = \lambda \cdot (gxg^{-1})G(\mathfrak{o}).$$

Lemma 2.3. *([2]) The fixed point set of $T(\mathbb{C})$ is Lx_0, where x_0 is the identity in X.*

2.2 Affine Springer Fiber

For any element $\gamma \in \mathfrak{g}(\mathfrak{o})$, we can define the affine Springer fiber

$$X_\gamma = \{g \in X : \mathrm{Ad}(g^{-1})(\gamma) \in \mathfrak{g}(\mathfrak{o})\}.$$

We wish to use lattices to study the affine Springer fiber; to do so, we redefine X_γ in terms of lattices.

Proposition 2.4. *It is equivalent to define $X_\gamma = \{\Lambda \in X : \gamma\Lambda \subset \Lambda\}$*

Proof. Suppose $g \in X_\gamma$, so $g^{-1}\gamma g = x \in \mathfrak{g}(\mathfrak{o})$. Define $\Lambda = g\Lambda^{st}$ and consider $\gamma\Lambda = \gamma g \Lambda^{st}$. Since $\gamma g = gx$, $x \in \mathfrak{g}(\mathfrak{o})$, the columns of γg are in the $\mathfrak{o}$ span of the columns of g, so

$\gamma\Lambda \subset \Lambda$. Therefore every element of X_γ corresponds to lattice Λ such that $\gamma\Lambda \subset \Lambda$. Similarly, if $\gamma\Lambda \subset \Lambda$, choose any basis for Λ and form $gG(\mathfrak{o}) \in X$. Since $\gamma\Lambda = \Lambda x$ for some $x \in \mathfrak{g}(\mathfrak{o})$, $g^{-1}\gamma g = x \in \mathfrak{g}(\mathfrak{o})$. $\square$

For arbitrary γ, X_γ may be empty, a scheme, or an ind-scheme. We will restrict our consideration to γ that satisfy a number of definitions. We say that $\gamma \in \mathfrak{g}(\mathfrak{o})$ is semisimple if $\mathrm{ad}_\gamma : \mathfrak{g}(\mathfrak{o}) \to \mathfrak{g}(\mathfrak{o})$ is diagonalizable. A semisimple element is regular if its centralizer is a maximal torus. In the case of $G = GL_n$ and SL_n, an element will be regular if it has distinct eigenvalues.

Due to Kazhdan and Lusztig, we have the following result in the regular semisimple case:

Proposition 2.5. *([5]) For $\gamma \in \mathfrak{g}(\mathfrak{o})$, X_γ will be finite dimensional if and only if γ is regular and semisimple.*

Given γ, define T_γ to be the centralizer of γ in $\mathbf{G}$ and let $\mathrm{Hom}_F(\mathbb{C}^*, T_\gamma)$ be the cocharacter lattice.

Theorem 2.6. *([5]) Let $\gamma \in \mathfrak{g}(\mathfrak{o})$ be regular and semisimple. Then $Hom_F(\mathbb{C}^*, T_\gamma)$ acts freely on X_γ and $Hom_F(\mathbb{C}^*, T_\gamma)\backslash X_\gamma$ is a projective scheme.*

We say that $\gamma \in \mathfrak{g}(\mathfrak{o})$ is integral if it is diagonalizable over F and equivalued if, in addition to being diagonalizable, all of the eigenvalues have the same valuation. In [CITE GKM purity], Goresky, Kottwitz, and MacPherson show that for regular integral equivalued γ, X_γ admits a paving by Hessenberg varieties. In the case where T is weakly Coxeter, however, they proved a stronger result.

The $\mathbf{G}$ conjugacy classes of maximal tori are indexed by the conjugacy classes in W; if the conjugacy class associated to T contains only Coxeter elements, we say that T is Coxeter in G. We say that T is weakly Coxeter in G if it is Coxeter in M, the centralizer of the maximal split torus in T. Since all split maximal tori - and all maximal tori in GL_n are weakly Coxeter, we will always be considering weakly Coxeter T.

Theorem 2.7. *([2]) Suppose that $T(F)$ is weakly Coxeter and let γ be a regular integral equivalued element of $\mathfrak{t}(\mathfrak{o})$. Then X_γ admits a paving by affine spaces.*

These affine spaces are the intersection of the Schubert cells from the Bruhat decomposition with X_γ. In the following chapter, we will construct the affine spaces by describing bases of the lattices in each space.

2.3 Equivariant Cohomology

Because X_γ has a torus and extended torus action, we are interested in studying the equivariant cohomology of X_γ. We will briefly recall the definition of equivariant cohomology here, following [10], as well as the relevant results in the case of X_γ.

The equivariant cohomology of any T-variety Y is defined by considering a contractible space ET with a free T action and defining $BT = ET/T$ to be the classifying space of T. Let $Y \times_T ET = (Y \times ET)/T$, where T acts on $Y \times ET$ diagonally. This creates a fiber bundle over BT with fiber Y. We define the T equivariant cohomology $H_T^*(Y) := H^*(Y \times_T ET)$. We will use two facts about equivariant cohomology in particular; first, that as in usual cohomology, $H_T^*(Y)$ is a module over $H_T^*(pt)$ and second, that $H_T^*(pt) = S(\mathfrak{t}^*)$, the symmetric ring of the dual lie algebra of T. This is especially helpful for certain T actions on Y.

Definition 2.8. *([10]) A variety Y is equivariantly formal with respect to the action of T if $E^2 = E^\infty$ in the Leray spectral sequence associated to $Y \longrightarrow Y \times_T ET \longrightarrow BT$.*

If Y is equivariantly formal with respect to T, $H_T^*(Y)$ is in fact a free $H_T^*(pt)$ module. In [1], Goresky, Kottwitz, and MacPherson give an algorithm for computing $H_T^*(Y)$ when Y is equivariantly formal and satisfies two additional conditions.

Proposition 2.9. *([10]) If Y is equivariantly formal with respect to T and Y has finitely many T fixed points and finitely many one-dimensional T orbits, then the closure of each one-dimensional* *orbit contains exactly two fixed points and is isomorphic to $\mathbb{CP}^1$, with the fixed points appearing at the poles. The torus acts on each one-dimensional orbit by rotation. For each one-dimensional orbit, there is a codimension one subtorus that fixes the orbit pointwise.*

In this case, the fixed points and one dimensional orbits give combinatorial data that allow for direct computation of $H_T^*(Y)$. When Y is equivariantly formal and has finitely many fixed points, the inclusion map $Y^T \longrightarrow Y$ of the fixed points into Y induces a map $H_T^*(Y) \longrightarrow \oplus_{T \text{ fixed points}} S(\mathfrak{t}^*)$. This map is actually an injection [10]; GKM give an explicit presentation for the image.

Theorem 2.10. *[10] Given Y and T so that Y is equivariantly formal with respect to the T action and has finitely many T fixed points and one-dimensional orbits, denote the one-dimensional orbits $O_1, \ldots, O_n$, each with poles N_i and S_i and stabilizer T_i in T. Then*

$$H_T^*(Y) \cong \{(f_1, f_2, \ldots, f_m) \in \oplus_{T \textit{ fixed points}} S(\mathfrak{t}^*) \mid f_{N_i}|_{t_i} = f_{S_i}\, |_{t_i} \textit{ for all } 1 \leq i \leq n\},$$

as rings.

Thus, in the equivariantly formal case, the equivariant cohomology can be determined via the T fixed points and one-dimensional orbits of T in Y. This data can be represented by the moment graph, with the T fixed points as vertices and one-dimensional orbits as edges between their poles. This moment graph is a linear graph in the image of the moment map,

$$\mu : Y \to \mathfrak{t}^*.$$

Under the moment map, the fixed points will be sent to points in $\mathfrak{t}^*$. Each one dimensional orbit O_i will be sent to the line segment between the images of its poles. The direction of this line segment is the annihilator of $\mathfrak{t}_i$, the lie algebra of the codimension one subtorus that stabilizes O_i.

2.4 Equivariant Cohomology for the Affine Springer Fiber

Thus far, we have described computing equivariant cohomology for varieties Y that, among other properties, have finitely many T fixed points. In the case of the affine Grassmannian and affine Springer fiber, however, that assumption immediately fails. Taking T and $\widetilde{T}$ as defined above, the T and $\widetilde{T}$ fixed points of X and X_γ are the elements corresponding to $l \in L$, the lattice of translations, or diagonal matrices of the form $\mathrm{diag}(t^{d_1}, t^{d_2}, \ldots, t^{d_n})$. Following [11], we can extend the definition of equivariant formality to an ind-scheme with the following assumptions. Let $X = \bigcup X_N$ be an ind-scheme with an action of T so that X_N is preserved under T and the action of T on X_N is equivariantly formal and further assume that each X_N has finitely many T fixed points and $H_2(X)$ is finite dimensional. It is shown in [2] that for our choice of γ, these assumptions are met, and so X_γ is equivariantly formal.

CHAPTER 3

DESCRIPTION OF THE AFFINE SPACES

We will first write down explicit bases for elements in the affine spaces, which will be the intersection of the Schubert cells with X_γ. We will always take $\gamma \in \mathfrak{g}(\mathfrak{o})$ to be the diagonal matrix with entries $\{\gamma_1 t, \gamma_2 t, \ldots, \gamma_n t\}$, with all γ_i distinct. Since γ is diagonal and has distinct eigenvalues, it is regular and semisimple.

In this chapter, we will construct valid lattices in X_γ, then identify them with the Schubert cells of the affine Grassmannian. We will do all computations for $G = GL_n$, since the lattices for $G = SL_n$ will appear as the subset of lattices with index 0.

3.1 Lattice Notation

Let $\Lambda = \text{span}\{v_1, v_2, \cdots, v_n\}$ be a $\mathbb{C}[[t]]$ lattice, with the v_i ordered by non-decreasing initial degree, $\bar{d}_i$. Elements of Λ are n-dimensional vectors with entries in F or, equivalently, Laurent series with n-dimensional complex vectors as coefficients. We begin by performing row reduction on the v_i until the following two conditions hold:

1. The first non-zero entry in the lowest degree coefficient vector of v_i is 1. Let a_i be the position of this entry. Thus we can write $v_i = e_{a_i} t^{d_i} +$ higher degree terms.

2. The a_i^{th} position is 0 in all the other coefficient vectors with degree greater than or equal to $\bar{d}_i$.

We can collect the data above into two tuples: let $\bar{d} = (\bar{d}_1, \bar{d}_2, \cdots, \bar{d}_n)$ be the initial degrees of the v_i and let $\sigma = (a_1, a_2, \cdots, a_n)$ be the positions of the first non-zero entry in the v_i. We choose the notation σ to reflect that it is an element of the symmetric group S_n, a fact which we will exploit later. In order for σ to be well defined, we need the convention that we order the v_i so that if v_i and v_{i+1} have the same initial degree, then $a_i < a_{i+1}$. For a given lattice Λ, we will refer to $\bar{d}$ as the ordered degree tuple and σ as the initial arrangement.

We will first consider GL_2, which is a short computation. We will then consider the case of GL_3, which is more complex. Based on the computation for GL_3, we can describe the general case.

3.2 Two-Dimensional Case

Consider the lattice $\Lambda = \text{span}\{v_1, v_2\}$. There are two possible cases to consider, $\sigma = (1,2)$ and $\sigma = (2,1)$. First, suppose that, after performing row reduction, we can write the basis as

$$v_1 = \begin{bmatrix} 1 \\ c_0 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ c_1 \end{bmatrix} t^{d+1} + \cdots + \begin{bmatrix} 0 \\ c_{e-d-1} \end{bmatrix} t^{e-1},$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^e,$$

where $e \geq d$. In order to have $\gamma\Lambda \subset \Lambda$, we must have $\gamma v_1 \in \text{span}\{tv_1, v_2\}$. We have

$$\gamma v_1 = \begin{bmatrix} \gamma_1 \\ \gamma_2 c_0 \end{bmatrix} t^{d+1} + \begin{bmatrix} 0 \\ \gamma_2 c_1 \end{bmatrix} t^{d+2} + \cdots + \begin{bmatrix} 0 \\ \gamma_2 c_{e-d-1} \end{bmatrix} t^e,$$

so we can subtract $\gamma_1 t v_1$ yielding

$$\gamma v_1 - \gamma_1 t v_1 = \begin{bmatrix} 0 \\ (\gamma_2 - \gamma_1) c_0 \end{bmatrix} t^{d+1} + \begin{bmatrix} 0 \\ (\gamma_2 - \gamma_1) c_1 \end{bmatrix} t^{d+2} + \cdots + \begin{bmatrix} 0 \\ (\gamma_2 - \gamma_1) c_{e-d-1} \end{bmatrix} t^e.$$

Since this must now be in the span of v_2, we see that all of the c_i must be 0 except for c_{e-d-1}, giving us a general form for the lattice:

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ c \end{bmatrix} t^{e-1}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^e.$$

The only exception to this form occurs when $d = e$. In this case, $\Lambda = t^d \Lambda^{st}$ and there is no constant c.

The second possibility is that after row reduction, the basis can be written as

$$v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^d + \begin{bmatrix} c_1 \\ 0 \end{bmatrix} t^{d+1} + \cdots + \begin{bmatrix} c_{e-d-1} \\ 0 \end{bmatrix} t^{e-1},$$

$$v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^e$$

where $e > d$. This case proceeds exactly as in case 1, except when $e = d + 1$. In the general case, we have lattices of the form

$$v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^d + \begin{bmatrix} c \\ 0 \end{bmatrix} t^{e-1}, v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^e.$$

When $e = d + 1$, however, c would occupy the first position in degree d - which has already been assumed to be 0. Therefore, when $e = d + 1$, there is only one possible lattice, with basis

$$v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^d, v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{d+1}.$$

From this case, we see that in general, the structure of the lattices does not depend on the initial arrangement. The lattices with $\sigma = (2, 1)$ are simply the lattices with $\sigma = (1, 2)$ after permutation by $(12) \in S_2$. When the initial degrees only differ by 1, however, we lose the free variable. To distinguish these cases, we will call a lattice in which at least two of the initial degrees differ by exactly 1 a close case and refer to other lattices as widely spaced. Note that a widely spaced lattice can have the same initial degree occur more than once; e.g, in $n = 3$, we would say a lattice with initial degrees $(-3, -3, 6)$ was widely spaced, while $(-1, -1, 0)$ was a close case.

3.3 Three-Dimensional Case

We will first consider the case where all initial degrees are distinct and widely spaced. For simplicity, we will assume that $\sigma = (1, 2, 3)$. Lattices with other initial arrangements can be formed via the action of S_n.

Let $\Lambda = \text{span}\{v_1, v_2, v_3\}$ and perform row reduction so that

$$v_1 = \begin{bmatrix} 1 \\ a_0 \\ b_0 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ a_1 \\ b_1 \end{bmatrix} t^{d+1} + \begin{bmatrix} 0 \\ a_2 \\ b_2 \end{bmatrix} t^{d+2} + \cdots + \begin{bmatrix} 0 \\ a_{e-d-2} \\ b_{e-d-2} \end{bmatrix} t^{e-2} + \begin{bmatrix} 0 \\ a_{e-d-1} \\ b_{e-d-1} \end{bmatrix} t^{e-1}$$
$$+ \begin{bmatrix} 0 \\ 0 \\ b_{e-d} \end{bmatrix} t^e + \begin{bmatrix} 0 \\ 0 \\ b_{e-d+1} \end{bmatrix} t^{e+1} + \cdots + \begin{bmatrix} 0 \\ 0 \\ b_{f-d-2} \end{bmatrix} t^{f-2} + \begin{bmatrix} 0 \\ 0 \\ b_{f-d-1} \end{bmatrix} t^{f-1} \quad (3.1)$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ c_0 \end{bmatrix} t^e + \begin{bmatrix} 0 \\ 0 \\ c_1 \end{bmatrix} t^{e+1} + \cdots + \begin{bmatrix} 0 \\ 0 \\ c_{f-e-2} \end{bmatrix} t^{f-2} + \begin{bmatrix} 0 \\ 0 \\ c_{f-e-1} \end{bmatrix} t^{f-1}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^f$$

Let

$$\gamma = \begin{bmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{bmatrix} t, \quad \gamma_1, \gamma_2, \gamma_3 \text{ distinct and non-zero}$$

Claim 3.1. *We must have $\gamma v_i \in span_{\mathbb{C}[t]}\{v_i, v_{i+1}, \dots, v_n\}$ because by performing row reduction, we know that v_i has a 0 in all positions where $v_1, \dots, v_{i-1}$ has a pivot.*

Given that, we can work backwards:

$$\gamma v_2 = \begin{bmatrix} 0 \\ \gamma_2 \\ \gamma_3 c_0 \end{bmatrix} t^{e+1} + \begin{bmatrix} 0 \\ 0 \\ \gamma_3 c_1 \end{bmatrix} t^{e+2} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \gamma_3 c_{f-e-2} \end{bmatrix} t^{f-1} + \begin{bmatrix} 0 \\ 0 \\ \gamma_3 c_{f-e-1} \end{bmatrix} t^f$$

$$\gamma v_2 - \gamma_2 t v_2 - c(\gamma_3 - \gamma_2) v_3 = \begin{bmatrix} 0 \\ 0 \\ \gamma_3 c_0 - \gamma_2 c_0 \end{bmatrix} t^{e+1} + \begin{bmatrix} 0 \\ 0 \\ \gamma_3 c_1 - \gamma_2 c_1 \end{bmatrix} t^{e+2} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \gamma_3 c_{f-e-2} - \gamma_2 c_{f-e-2} \end{bmatrix} t^{f-1}$$

We can no longer use any multiples of v_2 without reintroducing a non-zero second coordinate and we can't cancel any of the 3rd coordinates with v_3 because the degrees are too low. So if $\gamma v_2 \in \text{span}\{v_2, v_3\}$ then we must have that

$$c_0(\gamma_3 - \gamma_2) = c_1(\gamma_3 - \gamma_2) = \cdots = c_{f-e-2}(\gamma_3 - \gamma_2) = 0$$

and since γ_3 and γ_2 are distinct, that gives that $c_0 = \cdots = c_{f-e-2} = 0$. Therefore,

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^e + \begin{bmatrix} 0 \\ 0 \\ c_{f-e-1} \end{bmatrix} t^{f-1}$$

Now consider γv_1:

$$\gamma v_1 = \begin{bmatrix} 1 \\ \gamma_2 a_0 \\ \gamma_3 b_0 \end{bmatrix} t^{d+1} + \begin{bmatrix} 0 \\ \gamma_2 a_1 \\ \gamma_3 b_1 \end{bmatrix} t^{d+2} + \begin{bmatrix} 0 \\ \gamma_2 a_2 \\ \gamma_3 b_2 \end{bmatrix} t^{d+3} + \cdots + \begin{bmatrix} 0 \\ \gamma_2 a_{e-d-2} \\ \gamma_3 b_{e-d-2} \end{bmatrix} t^{e-1} + \begin{bmatrix} 0 \\ \gamma_2 a_{e-d-1} \\ \gamma_3 b_{e-d-1} \end{bmatrix} t^e$$
$$+ \begin{bmatrix} 0 \\ 0 \\ \gamma_3 b_{e-d} \end{bmatrix} t^{e+1} + \begin{bmatrix} 0 \\ 0 \\ \gamma_3 b_{e-d+1} \end{bmatrix} t^{e+2} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \gamma_3 b_{f-d-2} \end{bmatrix} t^{f-1} + \begin{bmatrix} 0 \\ 0 \\ \gamma_3 b_{f-d-1} \end{bmatrix} t^f \quad (3.2)$$

$$\gamma v_1 - \gamma_1 t v_1 = \begin{bmatrix} 0 \\ a_0(\gamma_2-\gamma_1) \\ b_0(\gamma_3-\gamma_1) \end{bmatrix} t^{d+1} + \begin{bmatrix} 0 \\ a_1(\gamma_2-\gamma_1) \\ b_1(\gamma_3-\gamma_1) \end{bmatrix} t^{d+2} + \begin{bmatrix} 0 \\ a_2(\gamma_2-\gamma_1) \\ b_2(\gamma_3-\gamma_1) \end{bmatrix} t^{d+3} + \cdots + \begin{bmatrix} 0 \\ a_{e-d-2}(\gamma_2-\gamma_1) \\ b_{e-d-2}(\gamma_3-\gamma_1) \end{bmatrix} t^{e-1}$$
$$+ \begin{bmatrix} 0 \\ a_{e-d-1}(\gamma_2-\gamma_1) \\ b_{e-d-1}(\gamma_3-\gamma_1) \end{bmatrix} t^{e} + \begin{bmatrix} 0 \\ 0 \\ b_{e-d}(\gamma_3-\gamma_1) \end{bmatrix} t^{e+1}$$
$$+ \begin{bmatrix} 0 \\ 0 \\ b_{e-d+1}(\gamma_3-\gamma_1) \end{bmatrix} t^{e+2} + \cdots + \begin{bmatrix} 0 \\ 0 \\ b_{f-d-2}(\gamma_3-\gamma_1) \end{bmatrix} t^{f-1} + \begin{bmatrix} 0 \\ 0 \\ b_{f-d-1}(\gamma_3-\gamma_1) \end{bmatrix} t^{f} \quad (3.3)$$

There are no other terms in degrees $e+1, \ldots, f-2$ that we can use to cancel without reintroducing the first or second coordinate, so $b_{e-d} = \cdots = b_{f-d-3} = 0$.

$$\gamma v_1 - \gamma_1 t v_1 - a_{e-d-1}(\gamma_2-\gamma_1)v_2 - b_{f-d-1}(\gamma_3-\gamma_1)v_3 =$$
$$\begin{bmatrix} 1 \\ a_0(\gamma_2-\gamma_1) \\ b_0(\gamma_3-\gamma_1) \end{bmatrix} t^{d+1} + \begin{bmatrix} 0 \\ a_1(\gamma_2-\gamma_1) \\ b_1(\gamma_3-\gamma_1) \end{bmatrix} t^{d+2} + \begin{bmatrix} 0 \\ a_2(\gamma_2-\gamma_1) \\ b_2(\gamma_3-\gamma_1) \end{bmatrix} t^{d+3} + \cdots + \begin{bmatrix} 0 \\ a_{e-d-2}(\gamma_2-\gamma_1) \\ b_{e-d-2}(\gamma_3-\gamma_1) \end{bmatrix} t^{e-1}$$
$$+ \begin{bmatrix} 0 \\ 0 \\ b_{e-d-1}(\gamma_3-\gamma_1) \end{bmatrix} t^{e} + \begin{bmatrix} 0 \\ 0 \\ b_{f-d-2}(\gamma_3-\gamma_1) - a_{e-d-1}c_{f-e-1}(\gamma_2-\gamma_1) \end{bmatrix} t^{f-1} \quad (3.4)$$

Now there are no vectors we can use in the remaining degrees without reintroducing a first or second coordinate, so all the remaining coordinates must be 0. Thus

$$a_0 = \ldots a_{e-d-2} = b_0 = \cdots = b_{e-d-1} = 0, b_{f-d-2}(\gamma_3-\gamma_1) - a_{e-d-1}c_{f-e-1}(\gamma_2-\gamma_1) = 0$$

This last condition can be rewritten as:

$$b_{f-d-2} = a_{e-d-1}c_{f-e-1}\frac{\gamma_2-\gamma_1}{\gamma_3-\gamma_1}$$

Letting $x = \dfrac{\gamma_2 - \gamma_1}{\gamma_3 - \gamma_1}$ and removing unnecessary subscripts, the basis can now be written:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} t^{e-1} + \begin{bmatrix} 0 \\ 0 \\ acx \end{bmatrix} t^{f-2} + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} t^{f-1}$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^e + \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} t^{f-1}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^f$$

By acting on this basis by $\sigma \in S_3$, we can write down bases for the lattices with the same initial degrees, but other initial arrangements. We need to consider, however, the action of σ on x. We defined x above to be $\dfrac{\gamma_2 - \gamma_1}{\gamma_3 - \gamma_1}$, but for other $\sigma \in S_3$, performing the same row reduction will create a different constant. Therefore, we define an action of σ on x by permuting the subscripts; in general, we will write the constant $\frac{\gamma_j - \gamma_i}{\gamma_k - \gamma_i}$ as x_{ijk} or x_σ.

Using this notation, we can easily write down bases for lattices with other initial arrangements. For example, if $\sigma = (2, 1, 3)$, Λ has basis

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^d + \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} t^{e-1} + \begin{bmatrix} 0 \\ 0 \\ acx_{213} \end{bmatrix} t^{f-2} + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} t^{f-1}$$

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^e + \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} t^{f-1}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^f$$

The bases for all initial arrangements appear in Appendix 1.

Although the close cases are essentially similar, as in $n = 2$, some of the constants may be forced to be zero by assumption. For example, consider the case with initial degrees $(d, d+1, f)$ and initial arrangement $(3, 2, 1)$. As before, we can find

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^d + \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix} t_1^d + \begin{bmatrix} b_2 \\ 0 \\ 0 \end{bmatrix} t_2^d + \cdots + \begin{bmatrix} b_{f-d-1} \\ 0 \\ 0 \end{bmatrix} t^{f-1}$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^{d+1} + \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} t^{f-1}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^f$$

Note that by assumption, the first term of v_1 is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Thus we have lost a constant; there is no a in degree $(d+1)-1$ and therefore also no cax_σ. The final simplified form is:

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^d + \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} t^{f-1}$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^{d+1} + \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} t^{f-1}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^f$$

The cases that can cause this constant loss are $(d, d+1, f)$, $(d, e, e+1)$, and $(d, d+1, d+2)$. The simplified forms of lattices of those types for all initial arrangements appear in Appendix 1.

So far, we have only discussed lattices with distinct initial degrees. We must also consider initial degrees of the form (d, d, e), (d, e, e), and (d, d, d). Note that in these cases, the possible initial arrangements will be only a subset of S_3. The bases in for these cases are listed in Appendix 1, but we will also summarize some of them here. Clearly if Λ has degree tuple (d, d, d), $\Lambda = t^d \Lambda^{st}$. In the other two cases, Λ will have two free variables when the degrees are widely spaced. For example, the widely spaced case with $\bar{d} = (d, d, e)$ and $\sigma = (1, 3, 2)$ has basis

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} t^{e-1}, v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} t^{e-1}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^e.$$

In the close cases, we may lose none, one or both of the free variables, depending on σ. If Λ has $\bar{d} = (d, d, d+1)$ with $\sigma = (1, 3, 2)$, the basis is

$$v_1 = \begin{bmatrix} 1 \\ a \\ 0 \end{bmatrix} t^d, v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^d, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^{d+1}.$$

In general, a free variable is lost any time the higher degree vector also has a higher initial position: if $\bar{d}_j = \bar{d}_i + 1$ and $a_j < a_i$, v_i will lose a free variable.

3.4 General Case

Recall that for any Λ, we choose a basis $\{v_1, \ldots, v_n\}$ and perform row reduction so that

1. The first non-zero entry in the lowest degree coefficient vector of v_i is 1. We denote this position by a_i.

2. The a_i^{th} position is 0 in all the other coefficient vectors with degree greater than or equal to $\bar{d}_i$.

With these conventions, the condition that $\gamma\Lambda \subset \Lambda$ requires that for each v_i, γv_i is in the $\mathbb{C}[[t]]$ span of $\{v_i, v_{i+1}, \cdots, v_n\}$. Since v_i is the only basis element with a non-zero entry in the a_i^{th} position, we can rewrite this condition as $\gamma v_i - \gamma_{a_i} t v_i \in \text{span}_{\mathbb{C}[[t]]}\{v_{i+1}, v_{i+2}, \cdots, v_n\}$, or equivalently,

$$(\gamma - \gamma_{a_i} t)v_i = \sum_{j>i} c_j v_j, \quad c_j \in \mathbb{C}[[t]].$$

In fact, once we have performed the above row reduction, we can make a much stronger claim: in order to have $\gamma\Lambda \subset \Lambda$, we must have $\gamma v_i - \gamma_{a_i} t v_i \in \text{span}_{\mathbb{C}}\{v_{i+1}, v_{i+2}, \cdots, v_n\}$. To verify this, note that $\gamma - \gamma_{a_i} t = \text{diag}\{\gamma_1 - \gamma_{a_i}, \cdots, \gamma_{a_i-1} - \gamma_{a_i}, 0, \gamma_{a_i+1} - \gamma_{a_i}, \cdots, \gamma_n - \gamma_{a_i}\} * t$ acts by scaling each term and increasing the degree by 1. If $\gamma v_i - \gamma_{a_i} t v_i$ contains a non-zero multiple of $t^k v_j$ for some $j > i, k > 0$, then v_i contains a non-zero entry in the a_j^{th} position of its $d_j + k - 1$ coefficient vector. This, however, contradicts our second row-reduction condition. Therefore, we can reduce $\gamma\Lambda \subset \Lambda$ to the condition that

$$(\gamma - \gamma_{a_i} t) v_i = \sum_{j>i} c_j v_j, \ \ c_j \in \mathbb{C}.$$

The row reduction conditions give one further constraint on the v_i: in close cases, v_i may be required to be linearly independent of some of the v_j. Recall that we say $(\bar{d}_1, \bar{d}_2, \cdots, \bar{d}_n)$ is a close case if for some i, $\bar{d}_j = \bar{d}_i + 1$, $j > i$.

Definition 3.2. *We say that v_j is below v_i if:*

1. *$j > i$, and*

2. *$\bar{d}_j > \bar{d}_i$, and*

3. *If $\bar{d}_j = \bar{d}_i + 1$, then $a_i < a_j$*

Lemma 3.3. *To guarantee that $\gamma\Lambda \subset \Lambda$, it is enough to require that for each i,*

$$(\gamma - \gamma_{a_i} t) v_i = \sum_{v_j \text{ below } v_i} c_j v_j, \ \ c_j \in \mathbb{C}.$$

Proof. We have already shown that $\gamma\Lambda \subset \Lambda$ implies that $(\gamma - \gamma_{a_i} t) v_i \in \text{span}_{\mathbb{C}}\{v_{i+1}, \cdots, v_n\}$. Now suppose that v_j is not below v_i. If $\gamma v_i - \gamma_{a_i} t v_i$ contains a multiple of v_j, then v_i has a non-zero entry in the a_j^{th} position of the degree $\bar{d}_j - 1$ coefficient vector. If v_j fails to be below v_i because $\bar{d}_j = \bar{d}_i$, then by assumption, v_i cannot have non-zero entries below degree $\bar{d}_j$. The only other possibility is that $\bar{d}_j = \bar{d}_i + 1$ and $a_i > a_j$. We have reduced v_i so that the first non-zero entry is in the a_i^{th} position of the degree $\bar{d}_i$ coefficient vector.

When $a_i > a_j$, the a_j^{th} position of the $\bar{d}_i$ coefficient vector is 0 by definition, so $\gamma v_i - \gamma_{a_i} t v_i$ cannot contain a multiple of v_j. $\square$

Let e_{a_i} denote the a_i^{th} standard basis vector. Note that the kernel of $\gamma - \gamma_{a_i} t$ is $\text{span}_F\{e_{a_i}\}$. We can decompose v_i into $e_{a_i} t^{d_i} + v_i'$ where $v_i' \in \text{span}_F\{e_{a_1}, \ldots, e_{a_{i-1}}, e_{a_{i+1}}, \ldots, e_{a_n}\} = e_{a_i}^{\perp}$, using the standard pairing. While the diagonal matrix $\gamma - \gamma_{a_i} t$ is obviously not invertible, it is invertible on $e_{a_i}^{\perp}$. Since $(\gamma - \gamma_{a_i} t) v_i = (\gamma - \gamma_{a_i} t) v_i'$, we can consider the action of $(\gamma - \gamma_{a_i} t)^{-1}$ on $e_{a_i}^{\perp}$ to write $v_i' = \sum_{v_j \text{ below } v_i} (\gamma - \gamma_{a_i} t)^{-1} c_j v_j$ This leads to the final expression for the basis of Λ.

Theorem 3.4. *Every element $\Lambda \in X_\gamma$ can be expressed by a basis $\{v_1, \ldots, v_n\}$,*

$$v_i = e_{a_i} t^{\bar{d}_i} + \sum_{v_j \text{ below } v_i} (\gamma - \gamma_{a_i} t)^{-1} c_j v_j$$

where $\sigma = (a_1, \ldots, a_n)$ is an element of S_n and $\bar{d} = (\bar{d}_1, \ldots, \bar{d}_n)$ is a non-decreasing sequence of integers.

3.5 Dimension

We showed in the previous section that the form of a lattice $\Lambda \in X_\gamma$ is determined by two tuples: the initial degrees $\bar{d} = (\bar{d}_1, \bar{d}_2, \cdots, \bar{d}_n)$ and the first non-zero positions $\sigma = (a_1, a_2, \cdots, a_n)$. Let $\Lambda_{\bar{d},\sigma}$ be the subvariety of all lattices in X_γ with degree $\bar{d}$ and initial arrangement σ. We wish to develop combinatorial formulas for the dimension of $\Lambda_{\bar{d},\sigma}$. Although these formulas are trivial given the decomposition in the last section, they introduce notation that will be useful in later sections and make dimension computations easier and more explicit.

Definition 3.5. *We say a lattice in $\Lambda_{\bar{d},\sigma}$ is of type $(k_1, \ldots, k_l)$, where k_i is the frequency of the i^{th} distinct $\bar{d}_j$.*

Example 3.6. *A lattice with degrees $(1, 1, 3, 5, 5, 5, 6)$ is of type $(2, 1, 3, 1)$*

Note that $\sum_{i=1}^{l} k_i = n$. The type does not give us any new information, but it will assist in computing the dimension of $\Lambda_{\bar{d},\sigma}$. When we need to recover the degree of the k_i vectors, we will write $d(k_i)$.

Proposition 3.7. *For $\Lambda_{\bar{d},\sigma}$, define the following:*

1. $s_i = \#\{v_k| \ \bar{d}_k = \bar{d}_i + 1$ *and* $a_i > a_k\}$
2. $S = \sum_{i=1}^{n} s_i$.
3. $b_i = \sum_{j=i+1}^{l} k_j$

Then the dimension of $\Lambda_{\bar{d},\sigma}$ is

$$(\sum_{i=1}^{l-1} b_i k_i) - S$$

Proof. The proof of this follows directly from the decomposition. We showed above that for each i, $v_i = e_{a_i} + v'_i$, where v'_i is any element of the vector space spanned by all v_j below v_i. From this, it is clear that the dimension of $\Lambda_{\bar{d},\sigma}$ is

$$\sum_i \#\{v_j \text{ below } v_i\} = \sum_i \#\{v_j|\bar{d}_j > \bar{d}_i\} - \sum_i \#\{v_j|\bar{d}_j = \bar{d}_i + 1, a_j < a_i\}.$$

Recall that v_j is below v_i if either $\bar{d}_i + 1 < \bar{d}_j$ or $\bar{d}_i + 1 = \bar{d}_j$ and $a_i < a_j$. S is exactly the count of all v_j that fail to be below v_i for some $i < j$ because of the second condition, so all that remains is to check that $\sum_i \#\{v_j|\bar{d}_j > \bar{d}_i\} = \sum_{i=1}^{l-1} b_i k_i$. To verify, note that for each of the k_i vectors of degree $\bar{d}(k_i)$, there are b_i vectors of higher degree. □

Corollary 3.8. *The subvariety $\Lambda_{\bar{d},\sigma}$ is isomorphic to $\mathbb{A}^l$, where $l = (\sum_{i=1}^{l} b_i k_i) - S$.*

One benefit of this formula is that it allows us to quickly compute the dimension of any $\Lambda_{\bar{d},\sigma}$ without writing out the basis. Although clumsy to define, the formula written this way is somewhat more intuitive than counting the number of v_j below each v_i. The dimension formula is particularly useful in the widely spaced case, where S will be 0.

In this case – which is essentially the generic case – the dimension of $\Lambda_{\bar{d},\sigma}$ will depend only on the type of $\bar{d}$, not on the specific degrees or on σ. In later chapters, we will be able to visualize how the different types and dimensions interact.

Corollary 3.9. *The largest cells in X_γ correspond to $\bar{d}$ in which all $\bar{d}_i$ are distinct and widely spaced. In this case, the dimension of $\Lambda_{\bar{d},\sigma}$ is $n(n-1)/2$.*

Proof. In this case, v_j will be below v_i for all $j > i$, so $\Lambda_{\bar{d},\sigma}$ will have the maximal number of free variables. Notice that for $\bar{d}$ of this type, σ does not matter. In the notation of the above formula, $S = 0$, $k_i = 1$ and $b_i = n - i$ for all i. Thus, the dimension is

$$\sum_{i=1}^{n} n - i = \frac{n(n-1)}{2}.$$

□

3.6 Lattices and Schubert Cells

Thus far, we have described types of lattices by using two tuples, $\bar{d} = (\bar{d}_1, \ldots, \bar{d}_n)$ listing the initial degrees in increasing order, and $\sigma = (a_1, \ldots, a_n)$ listing the first non-zero position for each of the v_i. This notation is most convenient for writing down explicit bases and computing dimensions of $\Lambda_{\bar{d},\sigma}$. In order to associate $\Lambda_{\bar{d},\sigma}$ to a Schubert cell, we need to develop slightly different notation.

Definition 3.10. *For a lattice $\Lambda \in X_\gamma$ with basis $\{v_1, v_2, \ldots, v_n\}$, let d_i be the initial degree of the basis vector whose first term is $e_i t^{d_i}$.*

Definition 3.11. *Where appropriate, we will refer to $\Lambda_{\bar{d},\sigma}$ as Λ_d.*

Example 3.12. *For the tuples $\bar{d} = (1, 4, 8, 11, 11, 12, 18)$, $a = (4, 5, 1, 3, 7, 6, 2)$, we have $d = (8, 18, 11, 1, 4, 12, 11)$.*

Note that d is just a permutation of $\bar{d}$ - in fact, $d = \sigma\bar{d}$. In many ways, d is the more natural notation; it contains all the information encoded in $\bar{d}$ and σ and, as we will see, makes the relationships between the Λ_d more clear. It also allows us to see the relationship between Λ_d and the Schubert cells: $\Lambda_d = I\ell G(\mathfrak{o}) \cap X_\gamma$, where $\ell \in L$ is the diagonal matrix with entries $\{t_1^{d_1}, \dots, t_n^{d_n}\}$.

In the following chapters, we will use the d notation almost exclusively, except when writing down concrete bases. It will be helpful to have the following recast definition:

Definition 3.13. *For $d = \{d_1, \dots, d_n\}$, with corresponding basis $\{v_1, \dots, v_n\}$, $v_i = e_i t^{d_i} + v_i'$, where v_i' consists of higher degree terms, we say that v_j is below v_i if $d_j > d_i + 1$ or if $d_j = d_i + 1$ and $j > i$.*

CHAPTER 4

CLOSURE RELATIONSHIPS

Given the description of the Λ_d from the last chapter, we can begin to look for closure relationships between the affine varieties. For $n = 2$, we can compute the closure of Λ_d by hand and show that the elements of the closure also arise as elements of the t and γ stable points of a Grassmannian, $\mathrm{Gr}(K, m)^{t,\gamma}$, where K and m depend on d. Based on this, we compute $\mathrm{Gr}(K, m)^{t,\gamma}$ for generic d in $n = 3$ and prove that for all n, the closure of Λ_d will be contained in the image of $\mathrm{Gr}(K, m)^{t,\gamma}$ in X_γ. The proof of this is very simple, but involves developing a tool, degree tables, which makes it simple to describe the possible $\Lambda_{d'}$ in the closure of Λ_d.

4.1 Basic Definitions

Let X_γ^N be the lattices of index N in X_γ. Consider the variety, Λ_d of lattices in X_γ^N described by the degree tuple $d = (d_1, \ldots, d_n)$. Recall that we must have $\sum d_i = N$. Let $\Lambda \in \Lambda_d$ be a generic element. To determine the closure of Λ_d, we define the following two objects:

Definition 4.1. *Let Λ^{min} be the $\mathbb{C}[[t]]$ lattice with basis $\{e_i t_i^m\}$, where m_i is the smallest degree for which a non-zero multiple of e_i appears in one of the basis vectors of the general form of Λ.*

Definition 4.2. *We define Λ^{max} to be the $\mathbb{C}[[t]]$ lattice with basis $\{e_i t_i^k\}$, where k_i is the smallest degree for which $e_i t^{k_i} \in \Lambda$.*

Note that in general, neither Λ^{min} nor Λ^{max} will be valid lattices in X_γ^N.

Since $\Lambda^{max} \subset \Lambda^{min}$, we can form the quotient space, $K = \Lambda^{min}/\Lambda^{max}$, which we will consider as a $\mathbb{C}$ vector space. Furthermore, we can consider the subspace $\Lambda/\Lambda^{max} \subset K$. Multiplication by t and γ are linear maps on K which preserve Λ/Λ^{max}. Let $m = \dim_{\mathbb{C}}(\Lambda/\Lambda^{max})$. We prove in 4.5 that the subvarieties in the closure of Λ_d will appear as at least a subset of the elements of $\mathrm{Gr}(K, m)$ preserved by t and γ, and conjecturally, all the elements of $\mathrm{Gr}(K, m)^{t,\gamma}$ will correspond to subvarieties in the closure.

4.2 Two Dimensional Case

4.2.1 Lattice Decompositions

Recall that in X_γ^0 we have the following cases:

1. $\bar{d} = (-x, x)$, $\sigma = (1, 2)$:
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{-x} + \begin{bmatrix} 0 \\ a \end{bmatrix} t^{x-1}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^{x}$$
2. $\bar{d} = (-x, x)$, $\sigma = (2, 1)$:
$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} t^{-x} + \begin{bmatrix} a \\ 0 \end{bmatrix} t^{x-1}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{x}$$
3. $\bar{d} = (0, 0)$, $\sigma = (1, 2)$:
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

In the case of X_γ^1, we have the following varieties:

1. $\bar{d} = (-x, x+1)$, $\sigma = (1, 2)$, $x \neq 0$:
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{-x} + \begin{bmatrix} 0 \\ a \end{bmatrix} t^{x}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^{x+1}$$

2. $\bar{d} = (-x, x+1)$, $\sigma = (2,1)$, $x \neq 0$:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} t^{-x} + \begin{bmatrix} a \\ 0 \end{bmatrix} t^{x}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{x+1}$$

3. $\bar{d} = (0,1)$, $\sigma = (1,2)$:

$$\begin{bmatrix} 1 \\ a \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} t$$

4. $\bar{d} = (0,1)$, $\sigma = (2,1)$:

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} t$$

For any other values of N, $X_\gamma^N \cong X_\gamma^{N \bmod 2}$ so all other possible varieties are isomorphic to those above.

4.2.2 Motivation

To compute the closure, we can consider the limit points of curves lying in Λ_d. To do so, we introduce an additional variable, s, and consider taking the limit as $s \to \infty$. That is, we take $F = \mathbb{C}[[t]][s]$ and consider lattices generated by bases of the form

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{-x} + \begin{bmatrix} 0 \\ a(s) \end{bmatrix} t^{x-1}, v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^{x}$$

where $a(s)$ is a non-constant polynomial in s. By factoring out the highest power of s, we can write this basis as

$$\frac{1}{s^k} \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{-x} + \begin{bmatrix} 0 \\ b_0 + b(\frac{1}{s}) \end{bmatrix} t^{x-1}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^{x}.$$

Now, upon taking the limit as $s \to \infty$, we see the basis becomes:

$$\begin{bmatrix} 0 \\ b_0 \end{bmatrix} t^{x-1}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^{x},$$

which is spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix} t^{x-1}$ when $b_0 \neq 0$. This, however, is not a complete lattice. Note that in the original lattice, we also had the element $tv_1 - a(s)v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{-x+1}$. This vector is unaffected by the limit and is the lowest degree element in the lattice. Thus, a complete basis for the lattice is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{-x+1}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^{x-1},$$

indicating that the origin of the variety given by $\bar{d} = (-x+1, x-1)$, $\sigma = (1,2)$ is in the closure of $\bar{d} = (-x, x)$, $\sigma = (1,2)$.

This type of computation informs our definition of Λ^{min} and Λ^{max}. Upon setting up and taking the limit, the smallest degree that can occur in each e_i will be the first time a non-zero multiple of e_i appears.

4.2.3 Forming K

Using the enumeration from above:

1. $\bar{d} = (-x, x)$, $\sigma = (1,2)$:

$$\Lambda^{min} = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{-x}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^{x-1} \}$$

$$\Lambda^{max} = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{-x+1}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^{x} \}$$

$$K = \{ v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{-x}, v_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^{x-1} \}$$

Then $\Lambda/\Lambda^{max} = v_1 + av_2$, a one dimensional subset of K. Multiplying elements of K by t or γ is equivalent to multiplying by 0, so $\mathrm{Gr}(K,1)^{t,\gamma} = \mathrm{Gr}(K,1)$. The elements of $\mathrm{Gr}(K,1)$ can be enumerated as $\{v_1 + av_2 \mid a \in \mathbb{C}\} \cup \{v_2\}$. Any subspace of the first type, when lifted back to X_γ, will form an element of Λ_d:

$$\{\begin{bmatrix}1\\0\end{bmatrix} t^{-x} + \begin{bmatrix}0\\a\end{bmatrix} t^{x-1}\} \oplus \Lambda^{max} = \{\begin{bmatrix}1\\0\end{bmatrix} t^{-x} + \begin{bmatrix}0\\a\end{bmatrix} t^{x-1}, \begin{bmatrix}0\\1\end{bmatrix} t^{x}\} \in \Lambda_d.$$

The final element of $\mathrm{Gr}(K,1)$ lifts to the lattice

$$\{\begin{bmatrix}1\\0\end{bmatrix} t^{-x+1}, \begin{bmatrix}0\\1\end{bmatrix} t^{x-1}\}$$

Thus, we have recovered the results of the computation in the previous section:

$$\overline{\Lambda_d} = \Lambda_d \cup \{\begin{bmatrix}1\\0\end{bmatrix} t^{-x+1}, \begin{bmatrix}0\\1\end{bmatrix} t^{x-1}\}$$

2. $\bar{d} = (-x, x)$, $\sigma = (2,1)$:

   This case proceeds essentially identically to the first. We find that

$$K = \{v_1 = \begin{bmatrix}0\\1\end{bmatrix} t^{-x}, v_2 \begin{bmatrix}1\\0\end{bmatrix} t^{x-1}\}$$

   and by computing that $\mathrm{Gr}(K,1)^{t,\gamma} = \{v_1 + av_2 | a \in \mathbb{C}\} \cup \{v_2\}$, find that

$$\overline{\Lambda_d} = \Lambda_d \cup \{\begin{bmatrix}0\\1\end{bmatrix} t^{-x+1}, \begin{bmatrix}1\\0\end{bmatrix} t^{x-1}\}.$$

3. $d = (0,0)$, $\sigma = (1,2)$: Since in this case Λ_d is a single point, note that it is already closed. We have that $\Lambda^{min} = \Lambda = \Lambda^{max}$, so K is the trivial group.

For lattices in X^1_γ, the computations are largely identical. The only interesting case is $\bar{d} = (0,1)$, where the closure depends on the arrangement, σ.

1. $\bar{d} = (-x, x+1)$, $\sigma = (1,2)$, $x \neq 0$:

$$\overline{\Lambda_d} = \Lambda_d \cup \{\begin{bmatrix}1\\0\end{bmatrix} t^{-x+1}, \begin{bmatrix}0\\1\end{bmatrix} t^{x}\}$$

2. $\bar{d} = (-x, x+1)$, $\sigma = (2,1)$, $x \neq 0$:

$$\overline{\Lambda_d} = \Lambda_d \cup \{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^{-x+1}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{x} \}$$

3. $\bar{d} = (0,1)$, $\sigma = (1,2)$:

$$\Lambda^{min} = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{0}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^{0} \}$$

$$\Lambda^{max} = \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{1}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^{1} \}$$

$$K = \{ v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, v_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$$

Since $\Lambda/\Lambda^{max} = v_1 + av_2$, we want to consider elements of $\mathrm{Gr}(K, 1)$ and this case proceeds as above, giving

$$\overline{\Lambda_d} = \Lambda_d \cup \{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^{0}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{1} \}$$

4. $\bar{d} = (0,1)$, $\sigma = (2,1)$: As for $\bar{d} = (0,0)$ in X^0_γ, Λ_d is a single point and K is the trivial group.

4.2.4 Closure Picture

In order to summarize and visualize the data above, we can draw a diagram of the Λ_d and their closures. We represent lattices of the form

$$\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^{a}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^{b} \}$$

by nodes labeled with d. If Λ_d is one-dimensional, we then represent those lattices with an arrow emanating from the appropriate node, terminating at the additional lattice in

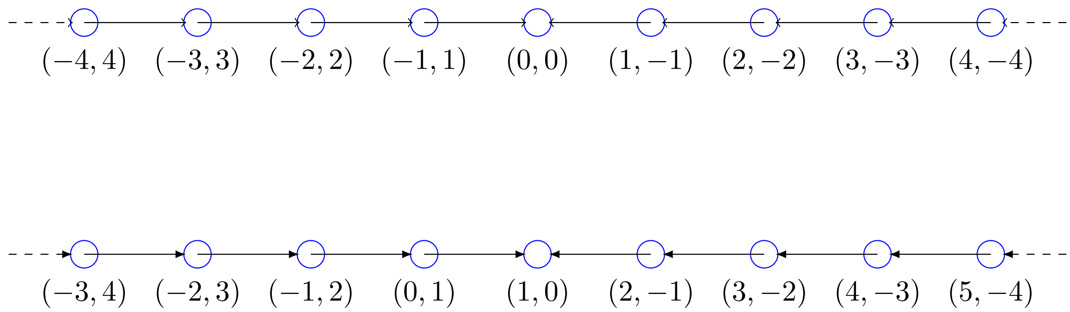


Figure 1. Two-dimensional lattices and closures

the closure.

These nodes and lines are precisely the 0 and 1 dimensional orbits of the torus action discussed in the next chapter.

4.3 Three Dimensional Case

4.3.1 Constructing Lattices From Gr$(K, m)^{t,\gamma}$

We will first provide a completely worked example of computing the elements of $\mathrm{Gr}(K, m)^{t,\gamma}$ for generic d. For our example case, let $\bar{d} = (0, 4, 8)$ and $\sigma = (1, 2, 3)$. This choice is arbitrary as long as d is widely spaced and all the d_i are distinct. In this case, we have $\Lambda = \mathrm{span}\{v_1, v_2, v_3\}$ with

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} t^3 + \begin{bmatrix} 0 \\ 0 \\ bgx_{123} \end{bmatrix} t^6 + \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} t^7$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^4 + \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} t^7, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^8$$

where $x_{123} = \frac{\gamma_2 - \gamma_1}{\gamma_3 - \gamma_1}$. For simplicity, since it is unambiguous, we will omit the subscript for the duration of this section.

Then

$$\Lambda^{min} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^3, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^6 \right\}$$

$$\Lambda^{max} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^2, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^5, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^8 \right\}$$

and

$$K = \text{span}\left\{ u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t, u_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^3, u_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^4, u_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^6, u_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^7 \right\}$$

Since $\Lambda/\Lambda^{max} = \text{span}\{u_1 + bu_3 + bgxu_5 + cu_6, u_2 + bg(x-1)u_6, u_4 + gu_6\}$, we have $m = 3$.

The actions of t and γ on K are given by

$$t = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \gamma = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_3 & 0 \end{bmatrix}$$

Instead of working with t and γ, it is equivalent and computationally simpler to consider $N_1 = (\gamma_2 - \gamma_1)^{-1}(\gamma_2 t - \gamma)$ and $N_2 = (\gamma_2 - \gamma_1)^{-1}(\gamma - \gamma_1 t)$:

$$N_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 - x^{-1} & 0 \end{bmatrix}, N_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x^{-1} & 0 \end{bmatrix}$$

Since $\mathrm{Gr}(K,m)^{t,\gamma} = \mathrm{Gr}(K,m)^{N_1,N_2}$, we can now proceed to describe all possible $\Gamma \in \mathrm{Gr}(K,m)^{N_1,N_2}$ and the lattices they correspond to in X_γ. This is a lengthy computation with many possible cases. It appears in full in the appendix, but to give a sense of the process, we will include the first cases here. We proceed by assuming $\Gamma \in \mathrm{Gr}(K,m)^{N_1,N_2}$ and consider the possible dimensions of $N_1\Gamma$ and $N_2\Gamma$.

1. Suppose that $\dim(N_1\Gamma) = 0$, so $\Gamma \in \mathrm{Gr}(3, \{u_2, u_3, u_4, u_6\})$.

   (a) If $\dim(N_2\Gamma) = 0$, $\Gamma \in \mathrm{Gr}(3, \{u_2, u_4, u_6\})$. Thus $\Gamma = \{u_2, u_4, u_6\}$ By lifting this back to X_γ, we see that Γ corresponds to the subvariety, in this case just a point, of lattices of the form

   $$\mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^4, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^7 \right\}$$

   (b) If $\dim(N_2\Gamma) = 1$, we can choose a basis element

   $$v_1 = a_2u_2 + a_3u_3 + a_4u_4 + a_6u_6, a_3 \neq 0.$$

   Then we must have

   $$v_2 = N_2v_1 = a_3u_4,$$

   which leaves

   $$v_3 = b_2u_2 + b_3u_3 + b_4u_4 + b_6u_6.$$

   We can then perform row reduction on this basis; note we will generally allow the values of the arbitrary constants to change without remarking upon it.

   $$\begin{bmatrix} 0 & a_2 & 1 & 0 & 0 & a_6 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & b_2 & 0 & 0 & 0 & b_6 \end{bmatrix}$$

i. If $b_2 \neq 0$:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & a_6 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & b_6 \end{bmatrix}$$

This corresponds to lattices of the form

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ b_6 \end{bmatrix} t^7, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^3 + \begin{bmatrix} 0 \\ 0 \\ a_6 \end{bmatrix} t^7, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^8 \right\}$$

Note that as a lattice, v_2 is in the span of v_1 and so doesn't appear in the lattice representation- instead, we introduce the last lattice element from the quotient.

ii. If $b_2 = 0$, $b_6 \neq 0$:

$$\begin{bmatrix} 0 & a_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

gives lattice:

$$\operatorname{span}\left\{ \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^3, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^4, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^7 \right\}$$

A. If $a_2 \neq 0$, this becomes:

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ a_2^{-1} \\ 0 \end{bmatrix} t^3, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^4, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^7 \right\}$$

B. If $a_2 = 0$, this is

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^2, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^3, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^7 \right\}$$

where we reintroduce the first basis element from the quotient.

4.3.2 Summarizing Lattices

It is possible to compute the closure of Λ_d by hand, similarly to motivation for the 2 dimensional case, but the computation is much longer and contains many subcases. We have performed the computation and it suggests that in the above variety, the lattices arising from $\mathrm{Gr}(K, m)^{t,\gamma}$ are indeed the closure of Λ_d, so we proceed under that assumption to describe the closure. We can group the lattices in the closure by degree tuple. To generalize this example to any lattice, we consider how each type has changed from the original; the net change vector appears next to each type.

(0, 4, 8): Net change from original type: [0,0,0]

$$\text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}+\begin{bmatrix}0\\a\\0\end{bmatrix}t^3+\begin{bmatrix}0\\0\\acx\end{bmatrix}t^6+\begin{bmatrix}0\\0\\b\end{bmatrix}t^7,\begin{bmatrix}0\\1\\0\end{bmatrix}t^4+\begin{bmatrix}0\\0\\c\end{bmatrix}t^7,\begin{bmatrix}0\\0\\1\end{bmatrix}t^8\right\}$$

(0, 5, 7): Net change from original type: [0, 1, -1]

$$\text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}+\begin{bmatrix}0\\a\\0\end{bmatrix}t^4+\begin{bmatrix}0\\0\\b\end{bmatrix}t^6,\begin{bmatrix}0\\1\\0\end{bmatrix}t^5+\begin{bmatrix}0\\0\\c\end{bmatrix}t^7,\begin{bmatrix}0\\0\\1\end{bmatrix}t^7\right\}$$

(1, 3, 8): Net change from original type: [1, -1, 0]

$$\text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}t+\begin{bmatrix}0\\0\\a\end{bmatrix}t^7,\begin{bmatrix}0\\1\\0\end{bmatrix}t^3+\begin{bmatrix}0\\0\\b\end{bmatrix}t^7,\begin{bmatrix}0\\0\\1\end{bmatrix}t^8\right\}$$

(1, 4, 7): Net change from original type: [1, 0, -1]

$$\text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}t+\begin{bmatrix}0\\a\\0\end{bmatrix}t^3+\begin{bmatrix}0\\0\\b\end{bmatrix}t^6,\begin{bmatrix}0\\1\\0\end{bmatrix}t^4,\begin{bmatrix}0\\0\\1\end{bmatrix}t^7\right\}$$

$$\operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}t+\begin{bmatrix}0\\0\\a\end{bmatrix}t^6,\ \begin{bmatrix}0\\1\\0\end{bmatrix}t^4+\begin{bmatrix}0\\0\\b\end{bmatrix}t^6,\ \begin{bmatrix}0\\0\\1\end{bmatrix}t^7\right\}$$

These two lattices give two planes that intersect in a line.

(1, 5, 6): Net change from original type: [1, 1, -2]

$$\operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}t+\begin{bmatrix}0\\a\\0\end{bmatrix}t^4,\ \begin{bmatrix}0\\1\\0\end{bmatrix}t^5,\ \begin{bmatrix}0\\0\\1\end{bmatrix}t^6\right\}$$

(2, 3 7): Net change from original type [2, -1, -1]

$$\operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}t^2,\ \begin{bmatrix}0\\1\\0\end{bmatrix}t^3+\begin{bmatrix}0\\0\\a\end{bmatrix}t^6,\ \begin{bmatrix}0\\0\\1\end{bmatrix}t^7\right\}$$

(2, 4, 6) : Net change from original type [2, 0, -2]

$$\operatorname{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}t^2,\ \begin{bmatrix}0\\1\\0\end{bmatrix}t^4,\ \begin{bmatrix}0\\0\\1\end{bmatrix}t^6\right\}$$

The net change vectors allow us to generalize this case to other degree tuples. In order to generalize it to lattices with other initial arrangements, we will need to act on the net change vector by σ.

4.4 Summarizing Dimensions and Closure Relations

To determine how these spaces fit together, we need to look at the closures of these new lattices as well. We do so by acting on each tuple by the net change vectors. For example, in the closure of $(0, 5, 7)$, there are lattices of the following types:

$$(1, 5, 6), (1, 4, 7), (2, 4, 6), (0, 6, 6), (1, 6, 5), (2, 5, 5)$$

Of those, the first three are also in the closure of $(0, 4, 8)$, the original lattice. Some work needs to be done to verify that the lattices in the closure match up. For example, it needs to be checked that the lattices of type $(1, 5, 6)$ in the closure of $(0, 5, 7)$ overlap with the lattices of type $(1, 5, 6)$ in the closure of $(0, 4, 8)$. To do so, we see that $(1, 5, 6)$ arises in the closure of $(0, 5, 7)$ via the vector $[1, 0, -1]$. We see that the lattices arising from $[1, 0, -1]$ are of the form:

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} t^4 + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} t^6, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^5, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^7 \right\}$$

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} t^6, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^5 + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} t^6, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^7 \right\}$$

(note the shift in initial degrees to accommodate a $(0, 5, 7)$ as the original lattice). This set of lattices contains

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} t^4, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^5, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^6 \right\},$$

the lattices of type $(1, 5, 6)$ that arise in the closure of $(0, 4, 8)$. It is straightforward to verify that the closure of the subvariety and the closure of the original variety always overlap; containment can go both ways, however. Sometimes, as above, the closure of the subvariety contains the lattices in the closure of the variety. For other cases, the relationship is reversed.

We can summarize the lattices, listed by the net change vector, the dimensions of the subset in the closure and the other lattices contained in their closure as follows:

Table 1. Lattice Subsets In Generic Closure

Type	Dimension	In Closure:
$[0,0,0]$	$\mathbb{C}^3$	
$[0,1,-1]$	$\mathbb{C}^2$	$[1,0,-1], [2,0,-2], [1,1-2]$
$[1,-1,0]$	$\mathbb{C}^2$	$[1,0,-1], [2,0,-2], [2,-1,-1]$
$[1,0,-1]$	$\mathbb{C}^2, \mathbb{C}^2$	$[1,1,-2,], [2,-1,-1], [2,0,-2]$
$[1,1,-2]$	$\mathbb{C}^1$	$[2,0,-2]$
$[2,-1,-1]$	$\mathbb{C}^1$	$[2,0,-2]$
$[2,0,-2]$	point	

We can also visualize the closure relationships in Figure 2, first by net change vector and then by dimension.

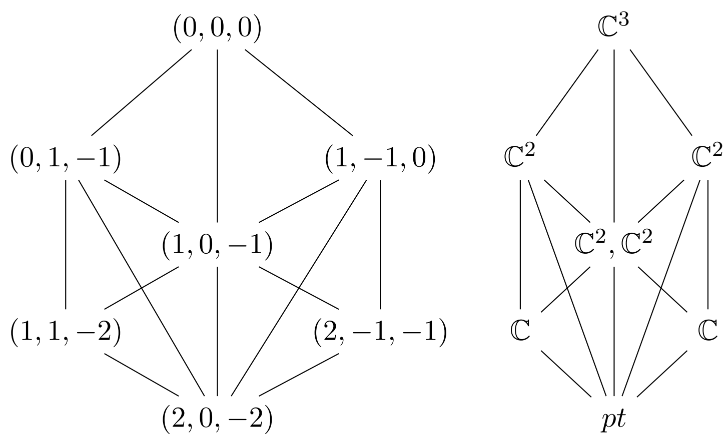


Figure 2. Sub-varieties in the closure of a generic lattice variety

4.5 General Case

Since the computations above relied on using explicit bases for Λ, it was useful to use $\mathbf{d}$ and σ. For the general case, however, we will use the d notation. To define the closure in general, consider the lift of the quotient map $f : \mathrm{Gr}(K, m) \to X$ given by

$f(v) = \pi^{-1}(v)$, where $\pi : F^n \to F^n/\Lambda^{max}$.

Theorem 4.3. *The closure of the variety Λ_d is contained in the image of $Gr(K, m)^{t,\gamma}$ under f*

Since $\mathrm{Gr}(K, m)^{t,\gamma}$ is closed and $\Lambda_d \in f(\mathrm{Gr}(K, m)^{t,\gamma})$, if we can show that f is well defined and that the image lies in X_γ^N, we will have that $\overline{\Lambda_d} \subset f(\mathrm{Gr}(K, m)^{t,\gamma})$. In order to fully prove this claim, we will need to define some additional machinery. It is simple, however, to show that f is well defined and that the image of $\mathrm{Gr}(K, m)^{t,\gamma}$ is in X_γ.

Lemma 4.4. *The image $f(Gr(K, m)^{t,\gamma})$ lies in X_γ.*

Proof. For any $v \in \mathrm{Gr}(K, m)^{t,\gamma}$, consider the image $f(v) = \pi^{-1}(v)$. Any element $u \in f(v)$ can be written as $u' + w$, where $u' \in v$ and $w \in \Lambda^{max}$. To verify that $f(v)$ is a valid lattice, we need to check that $tu \in f(v)$. Clearly, $tw \in \Lambda^{max}$, since Λ^{max} is itself a lattice. It is also clear that $tu' \in \pi^{-1}(v)$, since $tv \subset v$.

Similarly, it is clear that since $\gamma\Lambda^{max}$ is fixed by γ, if $v \in \mathrm{Gr}(K, m)^{t,\gamma}$, then $\pi^{-1}(v)$ will also be preserved by γ. Thus, $\mathrm{Im}(f)$ is indeed a subset of X_γ. $\square$

We have verified that $f(\mathrm{Gr}(K, m)^{t,\gamma}) \subset X_\gamma$, but still need to show that $f(\mathrm{Gr}(K, m)^{t,\gamma}) \subset X_\gamma^N$. This follows from the condition that the subspaces of K be preserved by t. Let d_{min} and d_{max} be the list of initial degrees of Λ^{min} and Λ^{max}, respectively. Note that for each i, $d_{min,i} \leq d_i \leq d_{max,i}$. For any $v \in \mathrm{Gr}(K, m)^{t,\gamma}$, the initial degrees of $f(v)$ must lie between, or possibly equal to, d_{min} and d_{max}. We can organize this data about the possible degrees of $f(v)$ in a grid with n columns, one for each standard basis vector.

$d_{min,1}$	$d_{min,2}$	$\dots$	$d_{min,n}$
$d_{min,1} + 1$	$d_{min,2} + 1$	$\dots$	$d_{min,n} + 1$
$\vdots$	$\vdots$		$\vdots$
$d_{max,1}$	$d_{max,2}$	$\dots$	$d_{max,n}$

For computational simplicity, we then normalize the table by subtracting d_i from each cell in the i^{th} column; thus, the entries will represent possible change in degree from Λ. We want to use this table to visualize what the possible choices of v look like. We can represent the possible v by shading in the boxes corresponding to the lowest degree term of each vector in $f(v)$.

Example 4.5. *We compute the degree table corresponding to Λ_d, $d = (0, 5, 8, 10)$. This is equivalent to $\bar{d} = (0, 5, 8, 10)$ and $\sigma = (1, 2, 3, 4)$; since this is a widely spaced case, any choice of σ would be equivalent. Using the decomposition algorithm, we can write down an explicit basis for Λ.*

$$v_1 = e_1 + \mu_1 e_2 t^4 + \mu_2 \mu_1 x_{123} e_3 t^6 + \mu_3 e_3 t^7 + \mu_1 \mu_2 \mu_4 x_{234} x_{124} e_4 t^7 + (\mu_1 \mu_5 x_{124} + \mu_3 \mu_4 x_{134}) e_4 t^8 + \mu_6 e_4 t^9,$$

$$v_2 = e_2 t^5 + \mu_2 e_3 t^7 + \mu_2 \mu_4 e x_{234} e_4 t^8 + \mu_5 e_4 t^8$$

$$v_3 = e_3 t^8 + \mu_4 e_4 t^9, v_4 = e_4 t^{10}$$

where $\mu_i \in \mathbb{C}$ and recalling the convention that $x_{ijk} = \frac{\gamma_j - \gamma_i}{\gamma_k - \gamma_i}$.

We compute that $d^{min} = (0, 4, 6, 7)$ and $d^{max} = (3, 7, 9, 10)$, giving us the normalized table

0	−1	−2	−3
1	0	−1	−2
2	1	0	−1
3	2	1	0

Note that this table is characteristic of the widely spaced case of type $(1, 1, \cdots, 1)$, where all degrees are distinct. Other table shapes appear in other types. Now, consider $\Lambda/\Lambda^{max} \subset K$. By reading off the table, we see a basis for K is

$$\{e_1 t^0, e_1 t^1, e_1 t^2, e_2 t^4, e_2 t^5, e_2 t^6, e_3 t^6, e_3 t^7, e_3 t^8, e_4 t^7, e_4 t^8, e_4 t^9\},$$

recalling that the final row corresponds to the basis of Λ^{max} and thus does not appear in K. Let $\tilde{v_i}$ be the image of v_i in the quotient. We can then write a concise basis for $\Lambda/\Lambda^{max} \in K$:

$$\{\tilde{v_1}, t\tilde{v_1}, t^2\tilde{v_1}, \tilde{v_2}, t\tilde{v_2}, \tilde{v_3}\}$$

.

We can represent Λ/Λ^{max} on the degree table by shading the boxes corresponding to the lowest degree term of each basis element:

0	-1	-2	-3
1	0	-1	-2
2	1	0	-1
3	2	1	0

In the special case of the degree table of $\Lambda/\Lambda^{max} \subset K$, the degree table will always take the above form: all boxes numbered 0 and greater will be shaded. This table allows us to immediately recover $m = \mathrm{Dim}_{\mathbb{C}}\Lambda/\Lambda^{max}$ by counting the shaded boxes above the bottom row.

Now, consider the degree table for any $v \in \mathrm{Gr}(K, m)^{t,\gamma}$. We can immediately note the following constraints on the table:

1. The condition that $tv \subset v$ means that whenever a box is shaded, every box below in that column must also be shaded.

2. Since $\dim(v) = m$, there must be exactly m boxes above the bottom row.

3. The bottom row, which corresponds to Λ^{max}, will always be shaded.

Definition 4.6. *We call any shaded configuration of a degree table that meets the above conditions t-allowable.*

It is certainly false that any t-allowable diagram corresponds to a unique element of $\mathrm{Gr}(K, m)^{t,\gamma}$. Many possible subspaces will share any given diagram. It is clear, however, that for any $w \subset K$ with a t-allowable diagram, $w \oplus \Lambda^{max}$ will form a valid F^n lattice, since any subspace with a t-allowable diagram will be preserved by t.

Definition 4.7. *For a subspace w with a t-allowable diagram, the index of w is $N + \sum_{i=1}^{n} \delta_i$, where δ_i is the entry of the first shaded box of the i^{th} column.*

Note that the index of w is exactly the index of the lattice $w \oplus \Lambda^{max}$.

Lemma 4.8. *Given $\Lambda_d \in X_\gamma^N$, any subspace corresponding to a t-allowable diagram will have an index of N.*

Proof. First, note that Λ/Λ^{max} will indeed have an index of N. We claim that any other t-allowable configuration will also give an index of N. Consider a diagram with just the bottom row shaded; denote its index M. Note that this is the index of Λ^{max}. Each time a box is placed on the diagram (so that the diagram remains t-allowable), the index will decrease by 1, since the entries of the diagram decrease as up the columns. Thus, after placing m boxes, the index of the diagram must be $M-m$, regardless of the configuration of the boxes. Since Λ/Λ^{max} produces a t-allowable diagram and has an index of N, we see that $M - m = N$ and thus any t-allowable diagram will have an index of N. $\square$

The proof of the above claim also proves that for any $v \in \mathrm{Gr}(K, m)^{t,\gamma}$, $f(v) \in X_\gamma^N$, and thus that $\overline{\Lambda_d} \subset f(\mathrm{Gr}(K, m)^{t,\gamma})$. For $n = 2$, we showed above that for all d, $\overline{\Lambda_d} = f(\mathrm{Gr}(K, m)^{t,\gamma})$. In the case of $n = 3$, computing the closure by hand is more complicated, but suggests that equality holds there as well.

Conjecture 4.9. *The closure of Λ_d is precisely the image of Gr$(K, m)^{t,\gamma}$ under f.*

Computing $\mathrm{Gr}(K, m)^{t,\gamma}$ is simpler than finding the closure of Λ_d by hand, but it is not trivial for larger n. One major benefit to this method is the degree tables used in the proof above. Unlike $\mathrm{Gr}(K, m)^{t,\gamma}$, they are relatively easy to write down and, while they certainly do not encode all the information about $\mathrm{Gr}(K, m)^{t,\gamma}$, one can read all possible degree tuples d' that may show up in the closure of Λ_d by identifying the t-allowable diagrams.

CHAPTER 5

ONE-DIMENSIONAL ORBITS AND MOMENT GRAPHS

One benefit of the explicit bases for Λ is that the one-dimensional orbits immediately become apparent. In this chapter, we identify the one-dimensional orbits of X_γ. In the two dimensional case, the one-dimensional orbits are precisely the Λ_d minus their origin and the moment graph is simply the closure picture in the last chapter. We mimic this approach for the three dimensional case and create pictures of X_γ^N, which we can use to visualize the moment graph as well as to understand how the dimensions of the Λ_d fit together.

5.1 Zero and One-Dimensional Orbits

The action of $T(\mathbb{C}) \times \mathbb{C}^*$ scales the entries of $x \in X_\gamma$ but does not change the degrees or the positions of the coordinates, so $T(\mathbb{C}) \times \mathbb{C}^*$ preserves Λ_d for all $\bar{d}$.

Lemma 5.1. *The one-dimensional orbits of X_γ must lie within Λ_d*

Lemma 5.2. *([2]) The fixed points of the $T(\mathbb{C}) \times \mathbb{C}^*$ action on X_γ are the lattices $\ell\Lambda^{st}$ for $\ell \in L$*

These lattices are the origins of the Λ_d.

Let $\Lambda_d^{i,j}$, $1 \leq i \leq n$ and $j \in \{j \mid v_j \text{ below } v_i\}$ be the set of lattices with basis

$$v_k = e_k t^{d_k}, k \neq i$$

$$v_i = e_i t^{d_i} + c e_j t^{d_j - 1}$$

Proposition 5.3. *The one-dimensional orbits of X_γ are the $\Lambda_d^{i,j}$. The one-dimensional orbits connect the fixed points $\ell_d\Lambda^{st}$ and $\ell_{d+z_i-z_j}\Lambda^{st}$. The extended torus acts on $\Lambda_d^{i,j}$ via the extended weight $(\alpha_{j,i}, d_j - d_i - 1)$, where $\alpha_{j,i}$ is the standard root sending a diagonal matrix M to $M_j M_i^{-1}$.*

Proof. We claim that the $\Lambda_d^{i,j}$ are the only one-dimensional orbits in X_γ. To see this, note that the one-dimensional orbits in X (for our choice of $G = GL_n$ or SL_n) are the orbits of the points $xG(\mathfrak{o})$ where $x = \text{diag}\{d_1, \dots d_n\} + ct^k E_{ji}$. The intersection of the set of $xG(\mathfrak{o})$ of this form with X_γ is precisely the $\Lambda_d^{i,j}$, or the $xG(\mathfrak{o})$ for which the choice of k and $\{i, j\}$ make $xG(\mathfrak{o})$ a valid lattice in X_γ. These are all of the one-dimensional orbits; since $T(\mathbb{C}) \times \mathbb{C}^*$ preserves Λ_d and therefore X_γ, every one-dimensional orbit inside X_γ must also be a one-dimensional orbit of X.

Given the one-dimensional orbit $\Lambda_d^{i,j}$, we can identify the two fixed points in the closure. When $c = 0$, we get the fixed point $\ell_d\Lambda^{st}$, where ℓ_d is the element of L corresponding to d. For the other end point, consider $\lim_{c\to\infty} \Lambda_d^{i,j}$. As $c \to \infty$, v_k will not change for all $k \neq i, j$; $\lim_{c\to\infty} v_i = e_j t^{d_j - 1}$, so in the limit, v_j will shift down a degree. For all c, the vector $tv_i - cv_j = e_i t^{d_i+1}$ is in $\Lambda_d^{i,j}$. In the limit, this is the lowest degree for which e_i appears, so v_i is shifted up a degree to $e_i t^{d_i+1}$. Thus the second fixed point in the closure of Λ_d is the lattice $l\Lambda^{st}$ corresponding to $(d_1, \dots, d_{i-1}, d_i + 1, d_{i+1}, \dots, d_{j-1}, d_j - 1, d_{j+1}, \dots, d_n)$.

We compute the action of the extended torus on $\Lambda_d^{i,j}$ directly. Consider the representation of $\Lambda_d^{i,j}$ as a matrix $\ell_d + ct^{d_j-1}E_{ji}$ and let (M, λ) be an element of the extended torus. Note the reversal of i and j: since the constant ct^{d_j-1} occurs in the j^{th} position of the i^{th} vector, it appears as the $(j, i)^{th}$ entry of the matrix. Denote the diagonal matrix with entries λ^{d_i} by λ^d. The torus action on $\Lambda_d^{i,j}$ is given by

$$M\lambda^d\ell_d + M_i c\lambda^{d_j-1}t^{d_j-1}E_{ji},$$

where M_i is the i^{th} diagonal entry of M.

Since $(M\lambda^d)^{-1} \in G(\mathfrak{o})$, we can right multiply to get

$$\ell_d + M_j M_i^{-1} \lambda^{d_j - d_i - 1} c E_{ji}$$

.

Thus, the weight of the torus action is $(\alpha_{j,i}, d_j - d_i - 1)$.

$\square$

Based on this, we can identify the codimension one subtorus that fixes each one-dimensional orbit pointwise as $T(\mathbb{C}^*) \times (M_i M_j^{-1})^{d_i + 1 - d_j}$. If we restrict to just the standard torus action, the fixed points and one-dimensional orbits are the same, but the codimension one subtorus is the kernel of $\alpha'_{j,i}$.

Note that each one-dimensional orbit corresponds to a coordinate in the basis for elements of Λ_d, so the number of one-dimensional orbits contained in Λ_d is equal to the dimension of the variety.

5.2 Moment Graph for $n = 2$ and $n = 3$

In the two dimensional case, every lattice variety Λ_d is a one-dimensional orbit. Thus the moment graph with the fixed points as vertices and one-dimensional orbits as edges is exactly the closure picture from the last chapter. We wish to emulate this picture in three dimensions. As before, the picture will depend on the index N, modulo 3. First we will begin by visualizing the fixed points. Since each lattice variety corresponds to a single fixed point, this picture is itself helpful in understanding the relative "locations" of the lattices of different type, or degree multiplicity, and dimension. Once we have the pictures of the fixed points, we can identify the one-dimensional orbits.

5.2.1 Index 0 Lattices

First, consider the case X^0_γ. Starting from Λ^{st}, which has $d = (0,0,0)$, we can recover all other possible d by adding multiples of three vectors $\alpha_1 =< 1, 0, -1 >$,

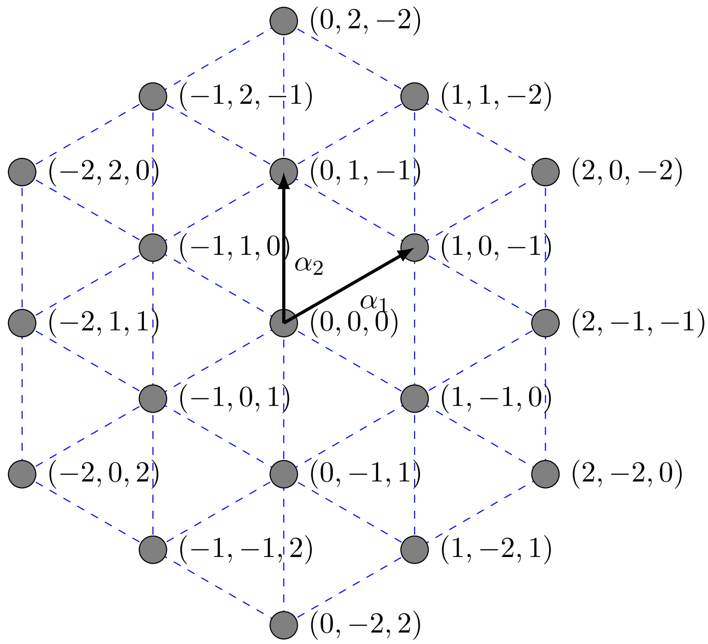


Figure 3. Vertices of lattices with index 0 labeled by minimum degree

$\alpha_2 =< 0, 1, -1 >$, and $\alpha_3 =< -1, 1, 0 >$. Since $\alpha_3 = \alpha_2 - \alpha_1$, we can draw the possible d as the vertices of the lattice generated by α_1 and α_2.

From figure 3, we can see the following:

1. The only lattice of type (3) is $(0, 0, 0)$.

2. The only lattices of type $(1, 2)$ will be positive multiples of $(1, 1, -2)$, $(1, -2, 1)$, or $(-2, 1, 1)$.

3. The only lattices of type $(2, 1)$ will be positive multiples of $(-1, -1, 2)$, $(-1, 2, -1)$, or $(2, -1, -1)$.

In particular, all lattices of type $(2, 1)$ or $(1, 2)$ will lie along 3 lines that intersect at $(0, 0, 0)$ and divide the plane into six sections, each corresponding to an arrangement of initial positions.

By zooming out and coloring the nodes of type $(2, 1)$ and $(1, 2)$ green, we can see this pattern more clearly. Figure 4 shows the possible $\bar{d}$ form the A_2 root system, with the

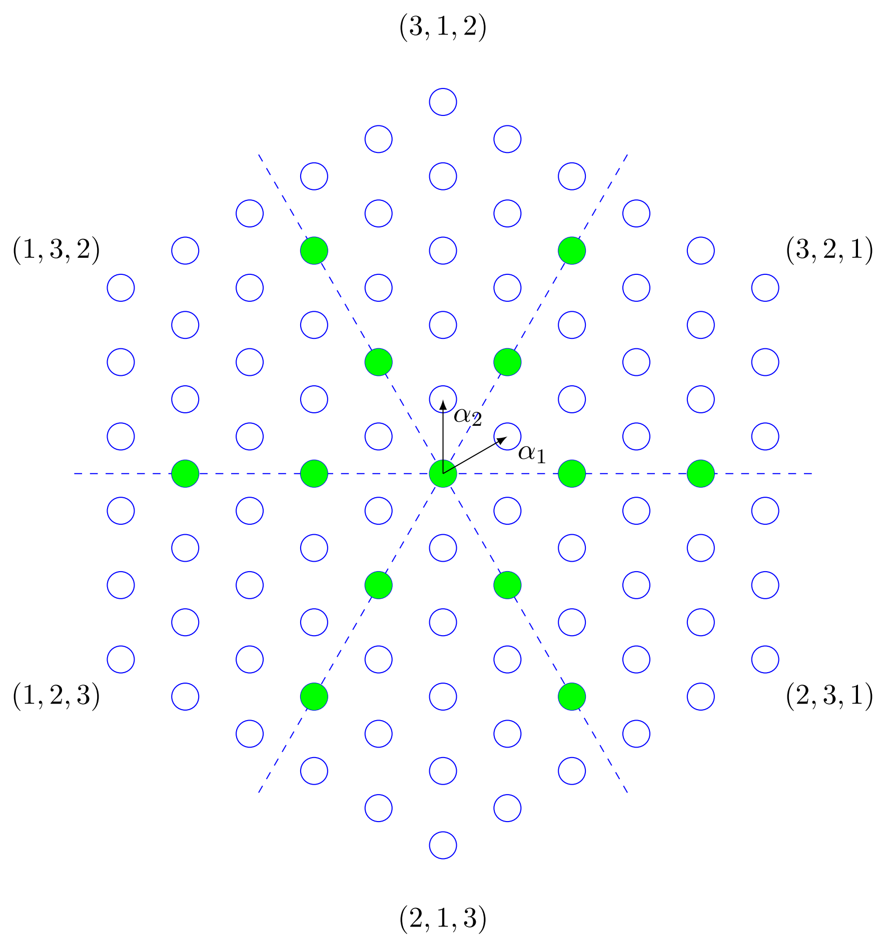


Figure 4. Arrangement of the vertices, colored by type.

lattices of type $(2,1)$ and $(1,2)$ defining the walls and the alcoves corresponding to the possible arrangements.

Representing the lattices of X_γ in this way also allows us to visualize the dimensions of Λ_d.

In figure 5, the black dot is the sole 0-dimensional variety, with $d=(0,0,0)$. The blue dot, $d=(1,0,-1)$ is one-dimensional and the green dots represent all the two dimensional varieties. Every dot along the walls, representing the lattices of types $(2,1)$ and $(1,2)$ will be two dimensional, but we also see three lines of two dimensional varieties along the walls. The dots next to the walls represent lattices for which two degrees differ by only one, say $d_j = d_i + 1$. On one side of the wall, $i < j$ and so there is no loss of

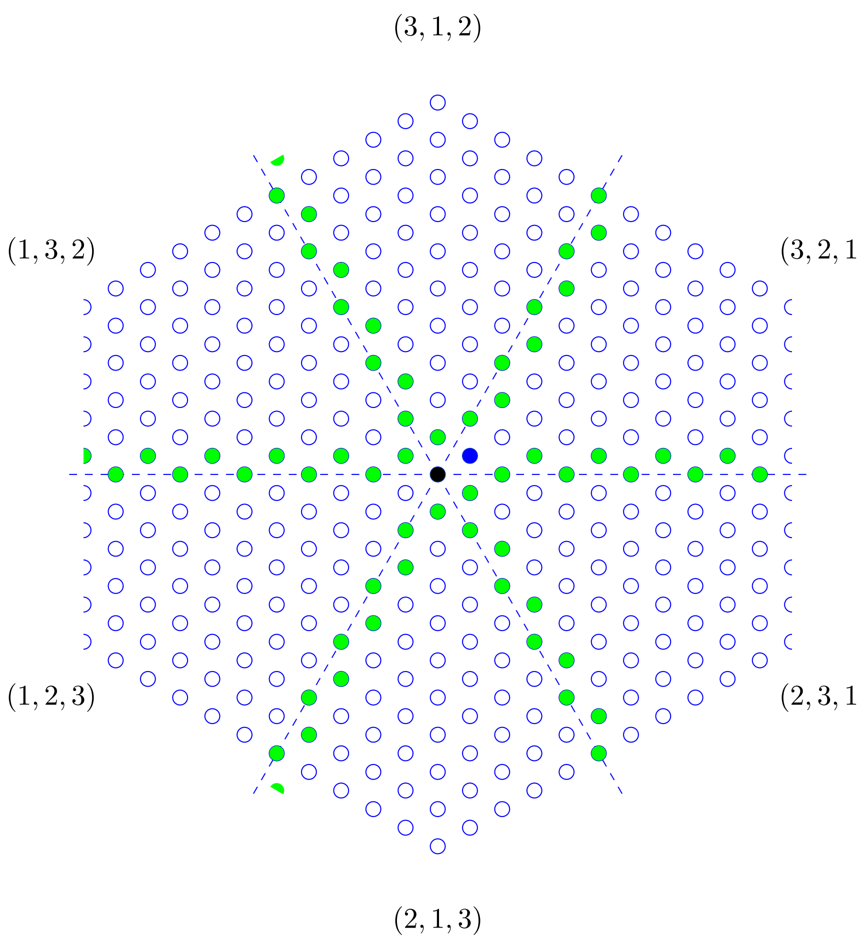
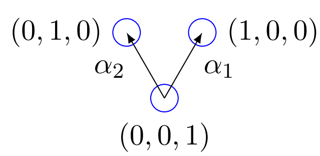


Figure 5. Arrangement of the vertices, colored by dimension.

dimension. Moving towards the wall will add one to d_i (and subtract one from another coordinate), so $d_i = d_j$. Moving one step further in the same direction will result in $d_j + 1 = d_i$, but in this case, since $i < j$, v_i is not below v_j, Λ_d does lose one dimension. The intersection of any of these lines of dimension loss represents losing dimensions in multiple coordinates, thus the number of dimensions lost is equal to the number of lines intersecting at that point.

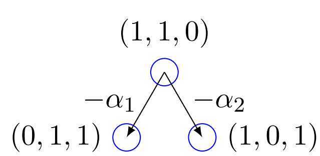
5.2.2 Index 1 and 2 Lattices

For $N = 1$, instead of having a single origin, we consider the three d that are closest to $(0, 0, 0)$: $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Starting from $(0, 0, 1)$ and choosing an orientation of α_1 and α_2, these initial d form a triangle:



We continue tiling in figure 6 to see the walls and alcoves. We can also color by dimensions, using the same key as in the index 0 case. Even though the picture is shifted so the origin lies between three lattices, the layout of the dimensions is essentially the same.

For $N = 2$, we start with $(0, 1, 1)$, $(1, 1, 0)$, and $(1, 0, 1)$ to generate:



As before, in figures 8 and 5.2.2, we extend this tiling to identify the walls and alcoves and color by dimension.

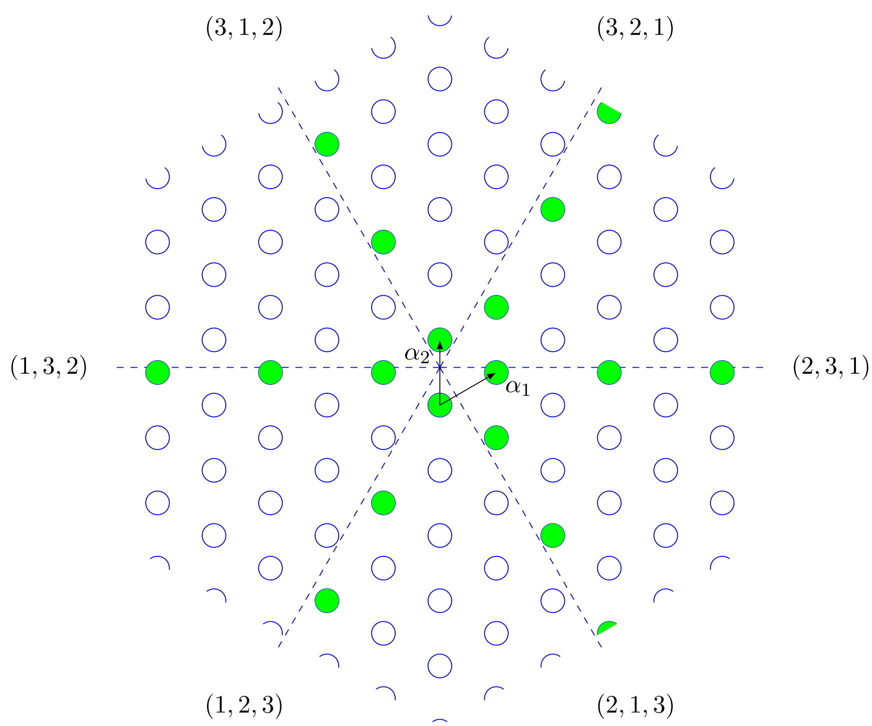


Figure 6. Index 1 lattices, with vertices colored by type.

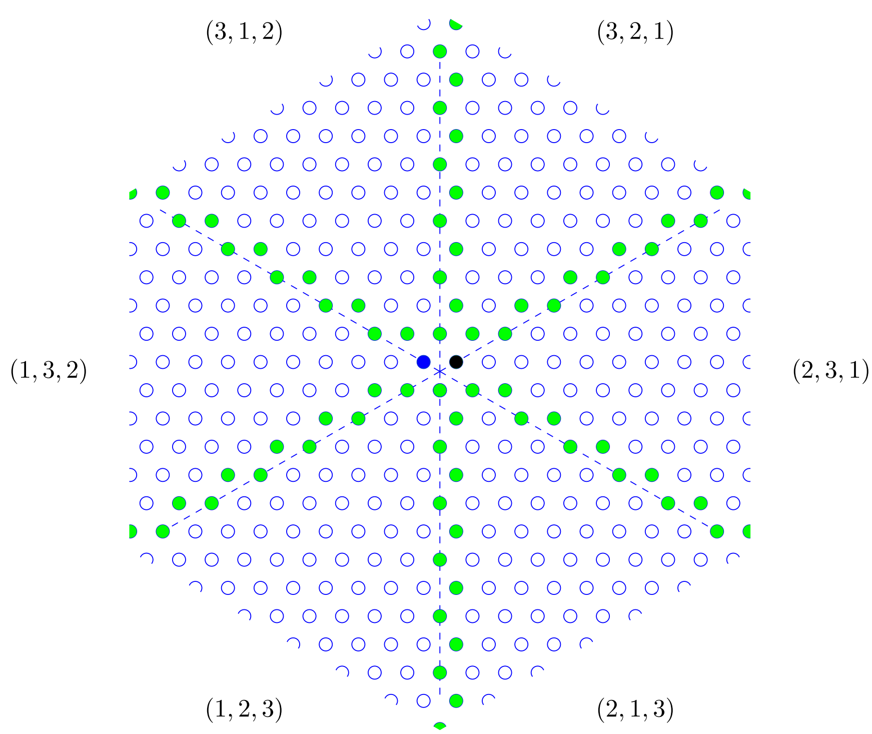


Figure 7. Index 1 lattices, with vertices colored by dimension.

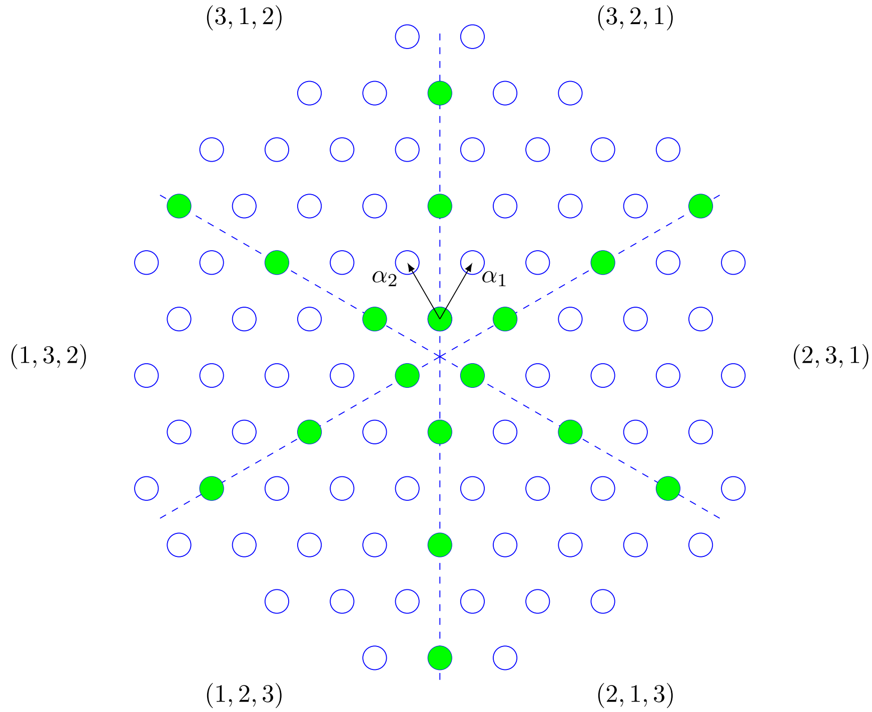


Figure 8. Index 2 vertices, colored by type.

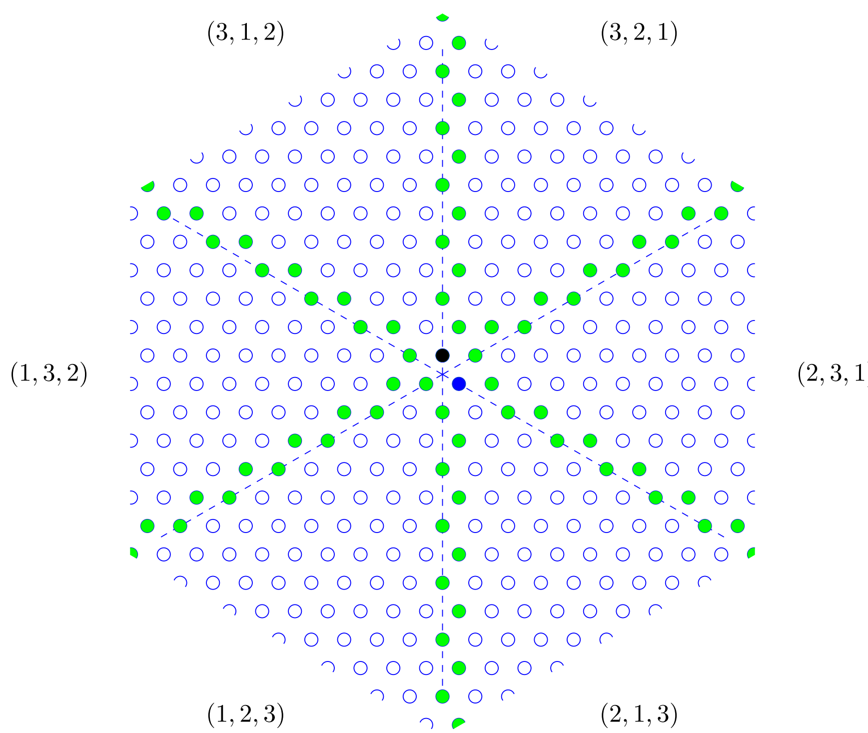


Figure 9. Index 2 vertices, colored by dimension.

5.2.3 One-Dimensional Orbits

We can now draw the one-dimensional orbits on the lattice of fixed points. We will do this for only the index 0 graph, since the other cases are essentially identical. Based on figure5.3the one-dimensional orbits will extend from a vertex d along the vectors $z_i - z_j$ for v_i below v_j, where z_k is the standard basis for $\mathbb{Z}^3$. All one-dimensional orbits are represented by lines of length one; no other lengths occur. In the alcove corresponding to $\sigma = (1, 2, 3)$, the one-dimensional orbits emanating from a widely spaced d are $< 1, 0, -1 >$, $< 0, 1, -1 >$, and $< 1, 0, -1 >$. In the other alcoves, the orbits are rotated so they always point towards the origin. We draw the orbits as arrows emanating from $\lim_{a \to 0} \Lambda_d^{i,j} = d$ and pointing towards $\lim_{a \to \infty} \Lambda_d^{i,j} = d + z_i - z_j$. In figure 5.2.3, we draw just the typical one-dimensional orbits for widely spaced degree tuples. We can fill in all the one-dimensional orbits in the picture to see how the orbits interact with the walls, as in figure11. To make the number of one-dimensional orbits more explicit, the lattice varieties that have some dimension loss and therefore contain fewer one-dimensional orbits are colored gray.

The graphs drawn here are the moment graphs for the T action, not the $\widetilde{T}$ action. Parallel lines correspond to orbits with the same stabilizer, which for the T action depends purely on i and j. In the case of $\widetilde{T}$, the stabilizers depend on d_i and d_j as well as i and j. The moment graph for the $\widetilde{T}$ action is three dimensional and will be the lift of vertices of the T moment graph to a paraboloid.

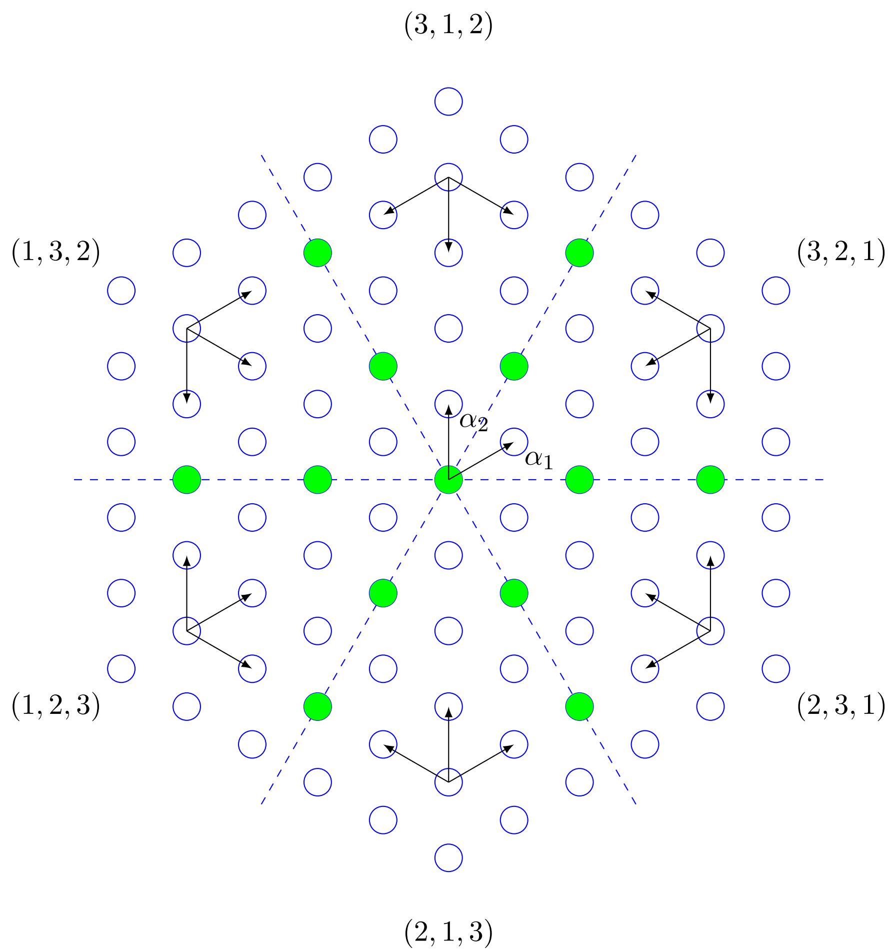


Figure 10. One-dimensional orbits for widely spaced degree tuples

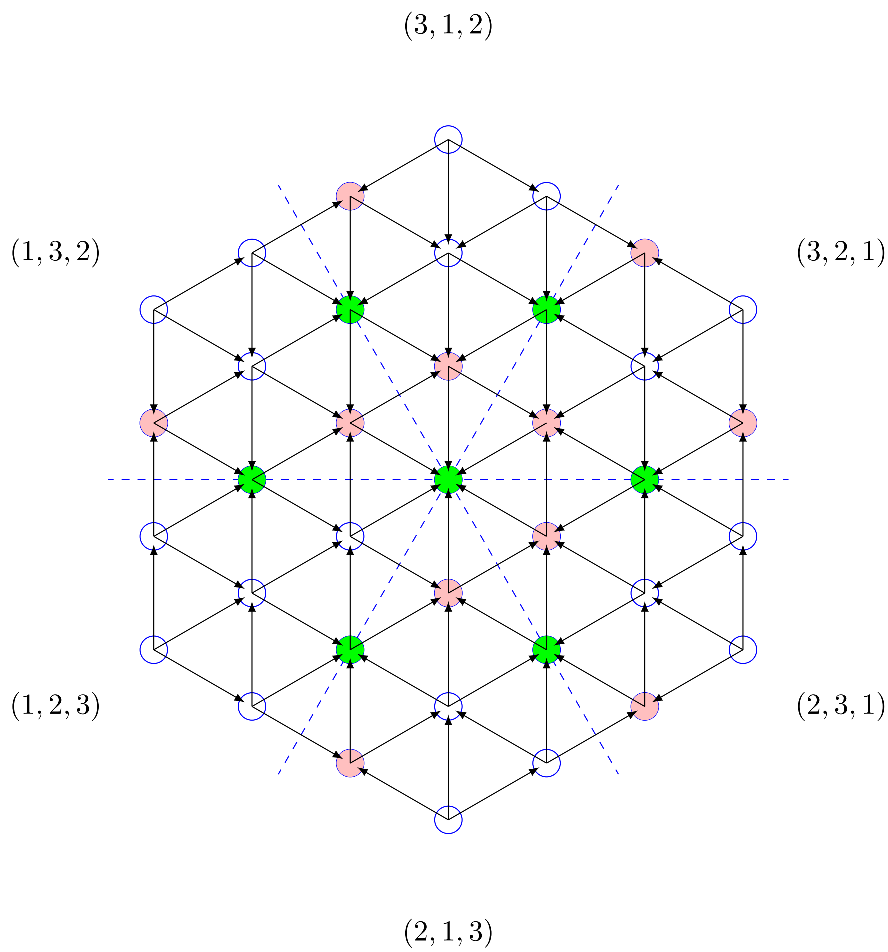


Figure 11. All one-dimensional orbits in index 0

CHAPTER 6

DIRECTIONS FOR FUTURE WORK

The work presented here could be continued in a variety of directions. Most immediate is proving that the closure of Λ_d is indeed the image of $\mathrm{Gr}(K,m)^{t,\gamma}$. The conjecture holds for $n = 2$, as is shown by direct computation in Chapter 4. For $n = 3$, direct computation for the closure appears to support the conjecture, but identifying all elements in the closure quickly becomes difficult. Since the closure relations of the Schubert cells of the affine Grassmannian are given by the Bruhat order, it would be interesting to try to describe the elements of $\mathrm{Gr}(K,m)^{t,\gamma}$ with the Bruhat order. Finally, although using the Grassmannian to compute the closure relations is substantially easier than considering limit points, it is still computationally intensive for $n > 4$. The degree diagrams constructed in the proof of Theorem 4.3 give a simple way to identify all d that may appear in lattices in the closure, but not which subvarieties of Λ_d show up. A better method for listing those subvarieties would be helpful.

In another direction, there is much to do with the equivariant cohomology based on the moment graphs in Chapter 5. The graphs as drawn could be used to compute T equivariant cohomology classes, although doing so with an infinite graph presents some complications. The $\widetilde{T}$ equivariant cohomology classes will be more complicated to describe because the stabilizers are more complicated.

Finally, the results here could be extended for both other γ and other G. The somewhat narrow choice of γ here was inspired largely by a connection of the affine Springer fiber to the Hilbert scheme of points of the algebraic curve $y^k = x^k$ when the characteristic polynomial of γ is $y^k = x^k$ [9]. I believe it would be straightforward to extend the lattice decomposition and other results to a much wider class of γ. The definition of v_j being below v_i would need to be altered to account for γ of other degrees, but the decomposition should be similar to the case considered here. Extending these results to other G is a more complicated endeavor; outside of GL_n and SL_n, we lose the straightforward definition of the affine Grassmannian in terms of lattices. Another possible area of work is considering affine Springer fibers inside affine flag varieties instead of affine Grassmannians.

APPENDIX A

POSSIBLE LATTICE TYPES WHEN $n = 3$

We present here all possible bases for lattices in three dimensions. These bases are organized by type and by the spacing of their degrees. Degrees are assumed to be distinct, increasing, and widely spaced unless otherwise indicated.

A.1 $\bar{d} = (d, e, f)$

$(1, 2, 3)$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} t^{e-1} + \begin{bmatrix} 0 \\ 0 \\ acx \end{bmatrix} t^{f-2} + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} t^{f-1}$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^e + \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} t^{f-1}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^f$$

$(1, 3, 2)$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} t^{e-1} + \begin{bmatrix} 0 \\ acx_{(23)} \\ 0 \end{bmatrix} t^{f-2} + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} t^{f-1}$$

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^e + \begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix} t^{f-1}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^f$$

$(2, 1, 3)$

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^d + \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} t^{e-1} + \begin{bmatrix} 0 \\ 0 \\ acx_{(12)} \end{bmatrix} t^{f-2} + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} t^{f-1}$$

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^e + \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} t^{f-1}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^f$$

$(2, 3, 1)$

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} t^{e-1} + \begin{bmatrix} acx_{(123)} \\ 0 \\ 0 \end{bmatrix} t^{f-2} + \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} t^{f-1}$$

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^e + \begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix} t^{f-1}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^f$$

$(3, 1, 2)$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^d + \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} t^{e-1} + \begin{bmatrix} 0 \\ acx_{(132)} \\ 0 \end{bmatrix} t^{f-2} + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} t^{f-1}$$

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^e + \begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix} t^{f-1}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^f$$

$(3, 2, 1)$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} t^{e-1} + \begin{bmatrix} acx_{(13)} \\ 0 \\ 0 \end{bmatrix} t^{f-2} + \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} t^{f-1}$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^e + \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} t^{f-1}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^f$$

A.2 $\bar{d} = (d, d+1, f)$

$(1, 2, 3)$

$$v_1 = \begin{bmatrix} 1 \\ a \\ 0 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ 0 \\ acx \end{bmatrix} t^{f-2} + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} t^{f-1}$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^{d+1} + \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} t^{f-1}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^f$$

$(1, 3, 2)$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ a \end{bmatrix} t^d + \begin{bmatrix} 0 \\ acx_{(23)} \\ 0 \end{bmatrix} t^{f-2} + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} t^{f-1}$$

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^{d+1} + \begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix} t^{f-1}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^f$$

$(2, 1, 3)$

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} t^{f-1}$$

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^{d+1} + \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} t^{f-1}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^f$$

$(2,3,1)$

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ a \end{bmatrix} t^d + \begin{bmatrix} acx_{(123)} \\ 0 \\ 0 \end{bmatrix} t^{f-2} + \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} t^{f-1}$$

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^{d+1} + \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} t^{f-1}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^f$$

$(3,1,2)$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} t^{f-1}$$

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^{d+1} + \begin{bmatrix} 0 \\ c \\ 0 \end{bmatrix} t^{f-1}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^f$$

$(3,2,1)$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^d + \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} t^{f-1}$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^{d+1} + \begin{bmatrix} c \\ 0 \\ 0 \end{bmatrix} t^{f-1}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^f$$

A.3 $\bar{d} = (d, e, e+1)$

$(1, 2, 3)$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ a \\ acx \end{bmatrix} t^{e-1} + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} t^e$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ c \end{bmatrix} t^e, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^f$$

$(1, 3, 2)$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} t^{e-1} + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} t^e$$

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^e, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^f$$

$(2, 1, 3)$

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^d + \begin{bmatrix} a \\ 0 \\ acx_{(12)} \end{bmatrix} t^{e-1} + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} t^e$$

$$v_2 = \begin{bmatrix} 1 \\ 0 \\ c \end{bmatrix} t^e, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^f$$

$(2, 3, 1)$

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} t^{e-1} + \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} t^e$$

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^e, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^f$$

$(3, 1, 2)$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^d + \begin{bmatrix} a \\ acx_{(312)} \\ 0 \end{bmatrix} t^{e-1} + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} t^e$$

$$v_2 = \begin{bmatrix} 1 \\ c \\ 0 \end{bmatrix} t^e, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^f$$

$(3, 2, 1)$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} t^{e-1} + \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} t^e$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^e, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^f$$

A.4 $\bar{d} = (d, d+1, d+2)$

$(1, 2, 3)$

$$v_1 = \begin{bmatrix} 1 \\ a \\ acx \end{bmatrix} t^d + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} t^{d+1}, v_2 = \begin{bmatrix} 0 \\ 1 \\ c \end{bmatrix} t^{d+1}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^{d+2}$$

$(1,3,2)$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ a \end{bmatrix} t^d + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} t^{d+1}, v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^{d+1}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^{d+2}$$

$(2,1,3)$

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} t^{d+1}, v_2 = \begin{bmatrix} 1 \\ 0 \\ c \end{bmatrix} t^{d+1}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^{d+2}$$

$(2,3,1)$

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ a \end{bmatrix} t^d + \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} t^{d+1}, v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^{d+1}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^{d+2}$$

$(3,1,2)$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} t^{d+1}, v_2 = \begin{bmatrix} 1 \\ c \\ 0 \end{bmatrix} t^{d+1}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^{d+2}$$

$(3,2,1)$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^d + \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} t^{d+1}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^{d+1}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^{d+2}$$

A.5 $\bar{d} = (d,d,e)$

$(1,2,3)$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} t^{e-1}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} t^{e-1}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^e$$

$(1,3,2)$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} t^{e-1}, v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} t^{e-1}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^e$$

$(2,3,1)$

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^d + \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} t^{e-1}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^d + \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} t^{e-1}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^e$$

A.6 $\bar{d} = (d, d, d+1)$

$(1,2,3)$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ a \end{bmatrix} t^d, v_2 = \begin{bmatrix} 0 \\ 1 \\ b \end{bmatrix} t^d, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^{d+1}$$

$(1,3,2)$

$$v_1 = \begin{bmatrix} 1 \\ a \\ 0 \end{bmatrix} t^d, v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^d, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^{d+1}$$

$(2,3,1)$

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^d, v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^d, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^{d+1}$$

A.7 $\bar{d} = (d, e, e)$

$(1, 2, 3)$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^d + \begin{bmatrix} 0 \\ a \\ b \end{bmatrix} t^{e-1}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^e, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^e$$

$(2, 1, 3)$

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^d + \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} t^{e-1}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^e, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^e$$

$(3, 1, 2)$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^d + \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} t^{e-1}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^e, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^e$$

A.8 $\bar{d} = (d, d+1, d+1)$

$(1, 2, 3)$

$$v_1 = \begin{bmatrix} 1 \\ a \\ b \end{bmatrix} t^d, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^{d+1}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^{d+1}$$

$(2, 1, 3)$

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ a \end{bmatrix} t^d, v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^{d+1}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^{d+1}$$

$(3, 1, 2)$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^d, v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^{d+1}, v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^{d+1}$$

A.9 $\bar{d} = (d, d, d)$

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^d, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^d, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^d$$

APPENDIX B

CLOSURE IN 3 DIMENSIONAL CASE

Let

$$\Lambda^{min} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^3, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^6 \right\}$$

$$\Lambda^{max} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^2, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^5, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^8 \right\}$$

and

$$K = \text{span}\left\{ u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t, u_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^3, u_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^4, u_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^6, u_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^7 \right\}$$

Since $\Lambda/\Lambda^{max} = \text{span}\{u_1 + bu_3 + bgxu_5 + cu_6, u_2 + bg(x-1)u_6, u_4 + gu_6\}$, we have $m = 3$.

Recall that

$$N_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 - x^{-1} & 0 \end{bmatrix}, N_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x^{-1} & 0 \end{bmatrix}.$$

We wish to describe the elements of $\text{Gr}(K, m)^{N_1, N_2}$. We proceed in cases:

1. Suppose that dim$(N_1\Gamma) = 0$, so $\Gamma \in \mathrm{Gr}(3, \{u_2, u_3, u_4, u_6\})$.

   (a) If dim$(N_2\Gamma) = 0$, $\Gamma \in \mathrm{Gr}(3, \{u_2, u_4, u_6\})$. Thus $\Gamma = \{u_2, u_4, u_6\}$ By lifting this back to X_γ, we see that Γ corresponds to the subvariety of lattices of the form

$$\mathrm{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix} t, \begin{bmatrix}0\\1\\0\end{bmatrix} t^4, \begin{bmatrix}0\\0\\1\end{bmatrix} t^7\right\}$$

   (b) If dim$(N_2\Gamma) = 1$, we can choose a basis element

$$v_1 = a_2u_2 + a_3u_3 + a_4u_4 + a_6u_6, a_3 \neq 0.$$

   Then we must have

$$v_2 = N_2v_1 = a_3u_4,$$

   which leaves

$$v_3 = b_2u_2 + b_3u_3 + b_4u_4 + b_6u_6.$$

   We can then perform row reduction on this basis by considering the possible cases:

$$\begin{bmatrix}0 & a_2 & 1 & 0 & 0 & a_6\\0 & 0 & 0 & 1 & 0 & 0\\0 & b_2 & 0 & 0 & 0 & b_6\end{bmatrix}$$

      i. If $b_2 \neq 0$:

$$\begin{bmatrix}0 & 0 & 1 & 0 & 0 & a_6\\0 & 0 & 0 & 1 & 0 & 0\\0 & 1 & 0 & 0 & 0 & b_6\end{bmatrix}$$

      This represents the lattice

$$\mathrm{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix} t + \begin{bmatrix}0\\0\\b_6\end{bmatrix} t^7, \begin{bmatrix}0\\1\\0\end{bmatrix} t^3 + \begin{bmatrix}0\\0\\a_6\end{bmatrix} t^7, \begin{bmatrix}0\\0\\1\end{bmatrix} t^8\right\}$$

Note that as a lattice, v_2 is in the span of v_1 and so doesn't appear in the lattice representation- instead, we introduce the last lattice element from the quotient.

ii. If $b_2 = 0$, $b_6 \neq 0$:

$$\begin{bmatrix} 0 & a_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

gives lattice:

$$\text{span}\left\{ \begin{bmatrix} a_2 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^3, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^4, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^7 \right\}$$

A. If $a_2 \neq 0$, this becomes:

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ a_2^{-1} \\ 0 \end{bmatrix} t^3, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^4, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^7 \right\}$$

B. If $a_2 = 0$, this is

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^2, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^3, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^7 \right\}$$

where we reintroduce the first basis element from the quotient.

(c) Note that if $\dim(N_2\Gamma) = 2$, $\{u_4, u_6\} \in \Gamma$, since they span the image of N_2. Since they are both in the kernel of N_2, it is impossible that $\dim N_2\Gamma = 2$ - and by a similar argument, $\dim N_1\Gamma < 2$

2. Suppose $\dim(N_1\Gamma) = 1$:

(a) If $\dim(N_2\Gamma) = 0$, then $\Gamma \in \mathrm{Gr}(3, \{u_1, u_2, u_4, u_6\})$. We choose the following basis:

$$v_1 = a_1u_1 + a_2u_2 + a_4u_4 + a_6u_6, \;\; a_1 \neq 0$$

$$v_2 = N_1 v_1 = u_2$$

$$v_3 = b_1 u_1 + b_2 u_2 + b_4 u_4 + b_6 u_6$$

and perform row reduction, as above.

$$\begin{bmatrix} 1 & 0 & 0 & a_4 & 0 & a_6 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_4 & 0 & b_6 \end{bmatrix}$$

i. If $b_4 \neq 0$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & a_6 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & b_6 \end{bmatrix}$$

This results in the lattice

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_6 \end{bmatrix} t^7, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^4 + \begin{bmatrix} 0 \\ 0 \\ b_6 \end{bmatrix} t^7, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^8 \right\}$$

ii. If $b_4 = 0$, then $b_6 \neq 0$:

$$\begin{bmatrix} 1 & 0 & 0 & a_4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This yields:

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a_4 \\ 0 \end{bmatrix} t^4, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^5, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^7 \right\}$$

(b) If $\dim(N_2\Gamma) = 1$, then $\text{Im}(N_1) = \{u_2, u_6\}$ and $\text{Im}(N_2) = \{u_4, u_6\}$.

i. If $N_1\Gamma = N_2\Gamma = u_6$, then $\Gamma \in \text{Gr}(3, \{u_2, u_4, u_5, u_6\})$, so Γ has a basis of the form

$$v_1 = a_2 u_2 + a_4 u_4 + a_5 u_5 + a_6 u_6, a_5 \neq 0,$$

$$v_2 = N_1 v_1 = N_2 v_2 = u_6,$$

$$v_3 = b_2 u_2 + b_4 u_4 + b_5 u_5 + b_6 u_6.$$

Again, we can perform row reduction:

$$\begin{bmatrix} 0 & a_2 & 0 & a_4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & b_2 & 0 & b_4 & 0 & 0 \end{bmatrix}$$

A. If $b_2 \neq 0$:

$$\begin{bmatrix} 0 & 0 & 0 & a_4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & b_4 & 0 & 0 \end{bmatrix}$$

If $a_4 \neq 0$:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & a_4^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -b_4 a_4^{-1} & 0 \end{bmatrix}$$

We can simplify this expression by renaming $-b_4 a_4^{-1}$ to b_4; this subspace corresponds to lattices of the form

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ b_4 \end{bmatrix} t^6, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^4 + \begin{bmatrix} 0 \\ 0 \\ a_4^{-1} \end{bmatrix} t^6, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^7 \right\}$$

If $a_4 = 0$:

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ b_4 \\ 0 \end{bmatrix} t^4, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^5, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^6 \right\}$$

B. $b_2 = 0, b_4 \neq 0$

$$\begin{bmatrix} 0 & a_2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

If $a_2 \neq 0$:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & a_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ a_2^{-1} \end{bmatrix} t^6, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^4, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^7 \right\}$$

If $a_2 = 0$:

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^2, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^4, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^6 \right\}$$

ii. If $N_1\Gamma \neq N_2\Gamma$, then $N_1\Gamma = \text{span}\{c_2u_2 + c_6u_6\}$, $N_2\Gamma = \text{span}\{d_4u_4 + d_6u_6\}$, where c_2 and d_4 are not both 0. Γ has basis:

$$v_1 = a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 + a_5u_5 + a_6u_6$$

$$v_2 = N_1v_1 = a_1u_2 + a_5(1 - x^{-1})u_6$$

$$v_3 = N_2v_1 = a_3u_4 + a_5x^{-1}u_6$$

A. If $a_1 \neq 0$:

$$\begin{bmatrix} 1 & 0 & a_3 & a_4 & a_5 & a_6 \\ 0 & 1 & 0 & 0 & 0 & a_5(1 - x^{-1}) \\ 0 & 0 & 0 & a_3 & 0 & a_5x^{-1} \end{bmatrix}$$

If $a_3 \neq 0$: (renaming a_6 as necessary)

$$\begin{bmatrix} 1 & 0 & a_3 & 0 & a_5 & a_6 \\ 0 & 1 & 0 & 0 & 0 & a_5(1 - x^{-1}) \\ 0 & 0 & 0 & 1 & 0 & a_5x^{-1}a_3^{-1} \end{bmatrix}$$

Resulting in:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a_3 \\ 0 \end{bmatrix} t^3 + \begin{bmatrix} 0 \\ 0 \\ a_5 \end{bmatrix} t^6 + \begin{bmatrix} 0 \\ 0 \\ a_6 \end{bmatrix} t^7$$

$$u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ a_5(1 - x^{-1}) \end{bmatrix} t^7,$$

$$v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^4 + \begin{bmatrix} 0 \\ 0 \\ a_5 x^{-1} a_3^{-1} \end{bmatrix} t^7$$

$$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^8$$

It's not immediately obvious that this is a valid lattice. To verify it, we need to show that u is in the span of v_1, v_2, v_3 and that it is closed under γ. For the first, note that $u = tv_1 - a_3 v_2 - a_6 v_3$. For the second, if we set $b_6 = a_5 a_3^{-1} x^{-1}$, we get $a_5 = b_6 a_3 x$, so the lattice can be written in standard form as:

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a_3 \\ 0 \end{bmatrix} t^3 + \begin{bmatrix} 0 \\ 0 \\ a_3 b_6 x \end{bmatrix} t^6 + \begin{bmatrix} 0 \\ 0 \\ a_6 \end{bmatrix} t^7, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^4 + \begin{bmatrix} 0 \\ 0 \\ b_6 \end{bmatrix} t^7, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^8 \right\}$$

Furthermore, note that we have just recovered the original lattice.

If $a_3 = 0$, $a_5 \neq 0$:

$$\begin{bmatrix} 1 & 0 & 0 & a_4 & a_5 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Resulting in:

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a_4 \\ 0 \end{bmatrix} t^4 + \begin{bmatrix} 0 \\ 0 \\ a_5 \end{bmatrix} t^6, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^5, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^7 \right\}$$

B. $a_1 = 0$: We must have $a_5 \neq 0$, $a_3 \neq 0$. Otherwise, we are in an earlier case.

$$\begin{bmatrix} 0 & a_2 & a_3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

If $a_2 \neq 0$:

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ a_2^{-1}a_3 \\ 0 \end{bmatrix} t^3 + \begin{bmatrix} 0 \\ 0 \\ a_2^{-1}a_3^{-1} \end{bmatrix} t^6, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^4, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^7 \right\}$$

If $a_2 = 0$:

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t^2, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t^3 + \begin{bmatrix} 0 \\ 0 \\ a_3^{-1} \end{bmatrix} t^6, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^7 \right\}$$

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