

9-2012

New Bilinear Estimates for Quadratic-Derivative Nonlinear Wave Equations in 2+1 Dimensions

Allison J. Tanguay
University of Massachusetts Amherst

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NEW BILINEAR ESTIMATES FOR QUADRATIC-DERIVATIVE
NONLINEAR WAVE EQUATIONS IN 2+1 DIMENSIONS

A Dissertation Presented

by

ALLISON J. TANGUAY

Submitted to the Graduate School of the
University of Massachusetts Amherst in partial fulfillment
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

September 2012

Department of Mathematics and Statistics

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Approved as to style and content by:

Andrea R. Nahmod, Chair

Luc Rey-Bellet, Member

Rodolfo H. Torres, Member

Sridhar Mahadevan, Member

Michael Lavine, Department Head
Mathematics and Statistics

ACKNOWLEDGEMENTS

There are many people who deserve acknowledgement in this work. Greatest recognition belongs to my advisor, Professor Andrea Nahmod, for giving me the motivation and drive to come this far. She has offered me so much help, confidence, advice, and support, and I am eternally grateful to have her as a mentor. This thesis would not be possible without Viktor Grigoryan and I am greatly indebted for all of his time and hard work. In addition to mathematical assistance, he has shown me a great deal of kindness, patience, and understanding, which have been invaluable.

There are several professors whose teaching and guidance have helped me over the years. First, I credit Professor Jeffrey McGowan for giving me the confidence to pursue graduate studies. Second, I would like to thank Professors Sridhar Mahadevan, Luc Rey-Bellet, and Rodolfo Torres for their time serving on my dissertation committee and their helpful comments and suggestions. I am also very grateful to Professor Rey-Bellet for his kindness as my professor, especially during my first year of graduate school. I would also like to thank Professors Tom Braden, Markos Katsoulakis, Franz Pedit, and Tom Weston for their help and encouraging words.

I am very fortunate to have wonderful friends who have provided me with much strength and encouragement as well. Special thanks belongs to several women from the UMass Math Department, whose weekly meetings have helped keep me going: Cat Benincasa, Adena Calden, Elizabeth Drellich, Abby Johnson, Julie Rana, and Caitlin Worth. I also thank Aaron Wolbach for his helpfulness, and for making it through these past six years with me. To my two oldest and best friends, Amy Stankiewicz and Joe Shumsky, thank you for always being there and helping me go the distance.

Finally, my deepest gratitude belongs to my family, whose hard work and unending support have allowed me to reach this point. My most heartfelt appreciation goes to my parents Lorraine and David for giving me everything I ever needed. I am also very grateful to my brother Kyle for being my foundation and making me believe in a limitless sky. I want to thank my grandparents Lillian and Edward Tanguay, Mary Ann and Hugo DiValentino, and my great-grandparents Angelina and Bill Crane for their care and guidance. I am so thankful to my aunt Tricia Thavenius and cousin Heather Beaghen for their loving support and for providing me with strong female role models. This thesis is dedicated to my father and my grandfather Papa, who sparked my interest in mathematics, and inspired me with their greatness, hard work, curiosity, and love of learning.

ABSTRACT

NEW BILINEAR ESTIMATES FOR QUADRATIC-DERIVATIVE NONLINEAR
WAVE EQUATIONS IN 2+1 DIMENSIONS

SEPTEMBER 2012

ALLISON TANGUAY, B.A., CENTRAL CONNECTICUT STATE UNIVERSITY

M.S., UNIVERSITY OF MASSACHUSETTS AMHERST

Ph.D., UNIVERSITY OF MASSACHUSETTS AMHERST

Directed by: Professor Andrea Nahmod

This thesis is concerned with the Cauchy problem for the quadratic derivative nonlinear wave equation in two spatial dimensions. Using standard techniques, we reduce local well-posedness in Fourier Lebesgue spaces to bilinear estimates in associated wave Fourier Lebesgue spaces, for which we prove new product estimates. These estimates then allow us to establish local well-posedness in a parameter range that gives improvement over previously known results on the Sobolev scale.

TABLE OF CONTENTS

	Page
ACKNOWLEDGEMENTS	iv
ABSTRACT	vi
CHAPTER	
1. INTRODUCTION	1
1.1 Local well-posedness	5
1.2 Classical results	8
1.3 The Fourier Lebesgue spaces \widehat{H}_s^r and associated $X_{s,b}^r$ spaces	12
1.4 Statement of the main result	18
1.5 Reduction to bilinear estimates	21
2. BILINEAR FOURIER RESTRICTION ESTIMATES IN L^p SPACES	35
2.1 Reformulation as a trilinear integral estimate	36
2.2 Dyadic decompositions	40
2.3 The L^p bilinear estimates	44
2.3.1 Introduction and preliminaries	44
2.3.2 Sizes of constants	48
3. PRODUCT ESTIMATES FOR WAVE FOURIER LEBESGUE SPACES	60
3.1 Introduction	60
3.2 The case $b_0 = b_1 = 0 < b_2$	61
3.3 The case $b_0 = 0 < b_1, b_2$	63
3.4 The case $b_0 < 0 < b_1, b_2$	72
3.5 Proof of Theorem 1.9	75
3.6 Conclusions	77
APPENDICES	
A. AN ANGULAR WHITNEY DECOMPOSITION	79
B. GEOMETRIC INTEGRAL TECHNIQUES	83
BIBLIOGRAPHY	96

CHAPTER 1

INTRODUCTION

In this dissertation, we prove bilinear space-time estimates related to local existence and uniqueness for the quadratic derivative nonlinear wave equation (DDNLW) in dimension $n = 2$. The problem we study is

$$\begin{cases} \square u = \partial u \partial u \\ u(0, x) = u_0(x) \\ u_t(0, x) = u_1(x) \end{cases} \quad (1.0.1)$$

where $u = u(t, x)$ for $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $\square = \partial_{tt} - \Delta$ is the d'Alembertian, $\Delta = \sum_{i=1}^n \partial_x^2$ is the Laplacian, and $\partial = \partial_t$ or $\partial = \partial_{x_i}$ for $i = 1, \dots, n$. We are interested in finding suitable Banach spaces for the initial data u_0, u_1 for which the problem is locally well-posed in dimension $n = 2$. In particular, we want best results for local well-posedness in terms of regularity of the initial data.

These equations have a natural scaling and for initial data $(u_0, u_1) \in \dot{H}^s \times \dot{H}^{s-1}$ in Sobolev space there is a unique exponent $s = s_c$ for which (1.0.1) is invariant under this scaling. For initial data with regularity $s > s_c$, one expects to have local in time well-posedness. The exponent s_c is called the critical scaling exponent, and for (1.0.1) one can easily show that $s_c = \frac{n}{2}$ (c.f. Section 1.2).

We therefore expect local well-posedness in H^s for $s > 1$ in two dimensions. Using energy estimates alone (c.f. Klainerman and Selberg [12]), one achieves this result only for $s > 2$. The best known results are for initial data $(u_0, u_1) \in H^s \times H^{s-1}$ with $s > 7/4$,

a gap of $3/4$ above the critical scaling s_c , and they follow from Strichartz estimates (c.f. Section 1.2). We are not aware of any well-posedness results for initial data in H^s with $s \leq 7/4$. In this dissertation, we are able to improve upon the previously known well-posedness results by instead considering initial data in the Fourier Lebesgue spaces \widehat{H}_s^r defined in Section 1.3.

In [6], D’Ancona, Foschi, and Selberg prove bilinear estimates in the Fourier L^2 -based $X^{s,b}$ spaces, from which one can also obtain the $s > 7/4$ result by standard arguments. In this dissertation, we thus prove analogous estimates in Fourier $L^{r'}$ -based spaces, $X_{s,b}^r$, where r' is the conjugate exponent of r for $\frac{3}{2} < r \leq 2$. As a consequence, we obtain local solutions in the associated wave-Fourier restriction spaces $X_{s,b}^r$ (see also Section 1.3), which embed into the continuation of the initial data space for $b > 1/r$. When $r = 2$, we of course recover the $7/4^+$ result as a particular case of our local well-posedness theorem. Our main result is the following.

Theorem 1.1 (Main Theorem). *The Cauchy problem (1.0.1) is locally well-posed in $\widehat{H}_s^r(\mathbb{R}^2)$ for $\frac{3}{2} < r \leq 2$ and $s > \frac{3}{2r} + 1$.*

Remark 1. *For a definition of well-posedness see Section 1.1 and for a more precise statement of our theorem see Section 1.4. There is a scaling correspondence between the homogeneous Sobolev and Fourier Lebesgue spaces that allows us to compare results. Specifically, the space \widehat{H}_s^r scales like H^σ for $\sigma = s + n(\frac{1}{2} - \frac{1}{r})$; see Section 1.3 for more details. In this sense, our results correspond to $\sigma > 5/3$ on the Sobolev scale and can therefore be thought of as an improvement of $1/12$ derivative over the previously known results.*

This work is motivated by Grünrock, who in [8] was able to use an $X_{s,b}^r$ approach to circumvent counterexamples of Lindblad [14], in dimension $n = 3$.

If one assumes that the derivative nonlinearity has some additional structure, then

better results are generally possible. For instance, suppose we have

$$\begin{cases} \square u^I = Q(u^J, u^K) \\ u(0, x) = u_0(x) \\ u_t(0, x) = u_1(x) \end{cases} \quad (1.0.2)$$

where Q is a real linear combination of the basic null forms $Q_0(u, v) = \sum_{i=1}^n \partial_i u \partial_i v - \partial_t u \partial_t v$, $Q_{ij}(u, v) = \partial_i u \partial_j v - \partial_j u \partial_i v$ and $Q_{0j}(u, v) = \partial_t u \partial_j v - \partial_j u \partial_t v$. The algebraic structure of the null forms, in particular the cancellation properties they exhibit, allow for better ranges of s . For Q_0 , naturally arising in the wave map problem, Klainerman and Selberg [13] were already able to prove optimal local well-posedness for the full subcritical range, $s > n/2$. For the null forms Q_{ij} and Q_{0j} , the best known local well-posedness results in two dimensions with initial data in $H^s \times H^{s-1}$ are with $s > 5/4$, found in Zhou [25]. Moreover, Zhou proves this is sharp for fixed point methods. Note that this is still $1/4$ above the critical scaling regularity $s_c = 1$. Ultimately, we would like to improve upon these results for the null forms Q_{ij} and Q_{0j} , but first we focus on the more general problem (1.0.1).

The null forms Q_{ij} and Q_{0j} appear naturally in the study of the Monopole equation

$$F_A = *D_A\phi$$

where A denotes a one-form connection on \mathbb{R}^{1+2} , F_A is its curvature, ϕ is the Higgs field, $D_A\phi$ is a covariant derivative, and $*$ is the Hodge star operator with respect to the Minkowski metric on \mathbb{R}^{1+2} . The space-time Monopole Equation is an integrable wave system and an example of a non-abelian gauge theory arising from the anti-self-dual Yang-Mills equation. It has a natural scaling and the critical exponent in the Sobolev scale is $s_c = 0$. Thus, the scaling predicts well-posedness in $L^2(\mathbb{R}^2)$. The system is gauge invariant and takes different forms depending on which gauge one chooses to fix; however, how results in one gauge might translate into results in another is unknown. In [3] and [4], Czubak fixed the Coulomb gauge and derived a system of nonlinear wave equations

coupled with a nonlinear elliptic equation for the gauge transformation. The resulting nonlinearity in the wave part of the system is a combination of terms with different null-form structures and roughly has the form

$$Q(\varphi, \psi) = \partial_t R_i \varphi (\partial_t + iD) \psi - (\partial_t + iD) \varphi \partial_t R_i \psi$$

where $D = (-\Delta)^{1/2}$ and $R_i = (-\Delta)^{-1/2} \partial_i$. Within this equation one has the basic null forms Q_{ij} and Q_{0j} . Using a fixed point argument in suitably adapted variants of the wave Sobolev spaces $H^{s,\theta}$, Czubak was able to prove local in time existence for the monopole equation with data in $H^s(\mathbb{R}^2)$ for $s > 1/4$. Furthermore, the needed bilinear estimates in the $H^{s,\theta}$ spaces are known to be false without this extra $1/4$ -regularity in two dimensions (c.f. Foschi and Klainerman [7]).

In Chapter 1, we review some general theory of local well-posedness and classical results for the system (1.0.1). We then introduce the solution spaces $X_{s,b}^r$ and, following Grünrock's approach in dimension three [8] give an argument that reduces local well-posedness to proving bilinear estimates. In Chapter 2 we prove some technical results that will be used to establish the needed estimates. Using the approach of D'Ancona, Foschi, and Selberg in [6], we reduce our bilinear $X_{s,b}^r$ estimates to trilinear L^p estimates. By suitable dyadic and Whitney type decompositions, we then further reduce the problem to bilinear restriction estimates on thickened subsets of the null cone, as in Selberg [17]. Finally, in Chapter 3 we prove the estimates that allow us to conclude our main local well-posedness theorem.

Our results for the product nonlinearity (1.0.1) give a strong indication that a similar $X_{s,b}^r$ approach would bring corresponding improvements for the null form (1.0.2) as well. This is still unknown, as our methods do not readily exploit the null structure of equations of this form. We suspect that gain may be achieved, as for Grünrock in dimension three, by utilizing results of Foschi and Klainerman in [7]. It seems possible that combining our methods with those described in [7] and employing instead an angular decomposition closer to that in Barcelo et. al. [1] could give further improvements for the null form.

We include a discussion of these techniques in the appendix for future reference.

1.1 Local well-posedness

Classical solutions of (1.0.1) require data with enough regularity so that the equations make sense pointwise. Since we wish to study local existence and uniqueness for systems with lower regularity on the initial data, we must instead look for solutions in a weaker sense. To define this notion, we will reformulate the problem as an integral equation. Suppose that we have the general inhomogeneous equation

$$\square u = F(u) \tag{1.1.1}$$

subject to the initial conditions

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x). \tag{1.1.2}$$

We consider first the corresponding linear homogeneous problem

$$\partial_{tt}u - \Delta u = 0 \tag{1.1.3}$$

with the same initial conditions (1.1.2). Now if u is sufficiently smooth, for example if $u \in C_{t,loc}^2 \mathcal{S}_x$, then taking the Fourier transform in space, we obtain the system

$$\begin{cases} \partial_{tt}\widehat{u}(t, \xi) + |\xi|^2\widehat{u}(t, \xi) = 0 \\ \widehat{u}(0, \xi) = \widehat{u}_0(\xi) \\ \widehat{u}_t(0, \xi) = \widehat{u}_1(\xi) \end{cases}$$

where \widehat{f} denotes the spatial Fourier transform on \mathbb{R}^n , i.e. $\widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx$. This produces an ODE with solution

$$\widehat{u}(t, \xi) = \frac{1}{2} \left(e^{i|\xi|t} + e^{-i|\xi|t} \right) \widehat{u}_0(\xi) + \frac{1}{2i|\xi|} \left(e^{i|\xi|t} - e^{-i|\xi|t} \right) \widehat{u}_1(\xi) \tag{1.1.4}$$

or equivalently,

$$\widehat{u}(t, \xi) = \cos(|\xi|t) \widehat{u}_0(\xi) + \frac{\sin(|\xi|t)}{|\xi|} \widehat{u}_1(\xi).$$

By Duhamel's principle, the solution of the inhomogeneous problem (1.1.1) is then

$$\widehat{u}(t, \xi) = \cos(|\xi|t)\widehat{u}_0(\xi) + \frac{\sin(|\xi|t)}{|\xi|}\widehat{u}_1(\xi) + \int_0^t \frac{\sin(|\xi|(t-s))}{|\xi|}\widehat{F}(u(s, \xi))ds.$$

Now define $|D| = \sqrt{-\Delta}$ to be the Fourier multiplier operator with symbol $|\xi|$. That is,

$$(|D|f)^\wedge(\xi) = |\xi|\widehat{f}(\xi)$$

Taking inverse Fourier transforms, we have

$$u(t, x) = \cos(t\sqrt{-\Delta})u_0(x) + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1(x) + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}F(u(s, x))ds. \quad (1.1.5)$$

This integral equation makes sense also for tempered distributions that are locally L^p in time and space, and so we will say that solutions of (1.1.5) are *distributional solutions* or *weak solutions*. Note that if u is a weak solution and also $u \in C^2(\mathbb{R} \times \mathbb{R}^n)$ then u is also a classical solution. We then define local well-posedness (LWP) as follows.

Definition 1.2. *We say that the Cauchy problem (1.1.1), (1.1.2) is locally well-posed (LWP) in a Banach space X if, given initial data $(u_0, u_1) \in X$, there exists a time $T > 0$ and a solution space $\mathcal{X}_T \subset C([0, T]; X)$ such that the following are true.*

1. *There is a unique $u \in \mathcal{X}_T$ that solves (1.1.1) on $[0, T] \times \mathbb{R}^n$ in the sense of distributions, satisfying the initial conditions (1.1.2).*
2. *The map $(u_0, u_1) \mapsto u$ is locally Lipschitz.*

In general, one tries to establish local well-posedness for a system such as (1.1.1), (1.1.2) by using a fixed point argument in an appropriate Banach space, \mathcal{X}_T . Following [12], we briefly discuss how a contraction mapping argument leads to well-posedness results. We can define a mapping Λ for the equation (1.1.1) using, for instance (1.1.5). We set

$$\Lambda(u)(t, x) = \cos(t\sqrt{-\Delta})u_0(x) + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1(x) + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta})}{\sqrt{-\Delta}}F(u(s, x))ds.$$

Then from distributional theory, finding weak solutions of (1.1.1) is equivalent to finding fixed points for Λ . A common technique is to use a Picard iteration in the Banach space \mathcal{X}_T , as follows. Define $v_{-1} \equiv 0$, and for $j \geq 0$, $v_j = \Lambda(v_{j-1})$. Provided $F(0) = 0$, such as when $F(u) = \partial u \partial u$, it follows that v_0 is a solution of the homogeneous equation (1.1.3) with initial conditions (1.1.2). For subsequent iterates we have $\square v_j = F(v_{j-1})$ with $(v_j, \partial_t v_j)|_{t=0} = (u_0, u_1)$. Equivalently, we can write $v_j = v_0 + \square^{-1} F(v_{j-1})$ where \square^{-1} is the Duhamel operator that assigns to F the solution $v = \square^{-1} F$ of the problem $\square v = F$ with zero initial conditions $v(0, x) = v_t(0, x) = 0$.

Finding the correct Banach space in which to perform these iterations can be rather difficult. Classically, the Cauchy problem has been studied with initial data in Sobolev spaces, $(u_0, u_1) \in H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n)$. If we denote $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$, then the inhomogeneous Sobolev space H^s and the corresponding scale-invariant homogeneous Sobolev space \dot{H}^s are defined to be the completion of the Schwarz class under the norms

$$\|f\|_{H^s} = \left(\int |\hat{f}(\xi)|^2 \langle \xi \rangle^{2s} d\xi \right)^{1/2} \quad \text{and} \quad \|f\|_{\dot{H}^s} = \left(\int |\hat{f}(\xi)|^2 |\xi|^{2s} d\xi \right)^{1/2},$$

respectively. Observe that

$$\left\| e^{\pm it\sqrt{-\Delta}} u_0 \right\|_{\dot{H}^s} = \left\| |\xi|^s e^{\pm it|\xi|} \widetilde{u_0} \right\|_2 = \|u_0\|_{\dot{H}^s}$$

and similarly

$$\left\| \frac{e^{\pm it\sqrt{-\Delta}}}{\sqrt{-\Delta}} u_1 \right\|_{\dot{H}^s} = \left\| |\xi|^s \frac{e^{\pm it|\xi|}}{|\xi|} \widetilde{u_1} \right\|_2 = \|u_1\|_{\dot{H}^{s-1}}.$$

From (1.1.4) we have the representation for the homogeneous solution

$$u(t, x) = \frac{1}{2} \left(e^{it\sqrt{-\Delta}} + e^{-it\sqrt{-\Delta}} \right) u_0(x) + \frac{1}{2i\sqrt{-\Delta}} \left(e^{it\sqrt{-\Delta}} - e^{-it\sqrt{-\Delta}} \right) u_1(x)$$

and it is therefore natural to look for solutions u in subspaces of the continuation space $C_t^0(H^s) \cap C_t^1(H^{s-1})$. However, if we want initial data $(u_0, u_1) \in H^s \times H^{s-1}$, then to remain in the space, each iterate must also satisfy $v_j \in H^s$ and $\partial_t v_j \in H^{s-1}$. Thus, \mathcal{X}_T will depend on the nonlinearity F and the Sobolev exponent s . Consequently, we replace \mathcal{X}_T with \mathcal{X}_T^s and note that in this case we must have the embedding $\mathcal{X}_T^s \hookrightarrow C([0, T], H^s) \cap$

$C^1([0, T], H^{s-1})$, where $C([0, T], X)$ denotes the space of continuous functions from the interval $[0, T]$ into X .

We want to show that the sequence of iterates v_j is Cauchy in the \mathcal{X}_T^s topology, so that there is a limit u . Provided we can show also that $\Lambda(v_j) \rightarrow \Lambda(u)$ or $F(v_j) \rightarrow F(u)$ in the distributional sense, we will have proved local existence in \mathcal{X}_T^s . For the uniqueness, we also need to show that the iteration map is a contraction. Thus, in general we will need to prove linear estimates, such as

$$\|v_0\|_{\mathcal{X}_T^s} \leq C(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}})$$

and nonlinear estimates such as

$$\|\square^{-1}F(u)\|_{\mathcal{X}_T^s} \leq C_T A(\|u\|_{\mathcal{X}_T^s})$$

where A is a continuous function satisfying $A(0) = 0$, to show that the iteration is well defined in \mathcal{X}_T^s . Estimates of the type

$$\|\square^{-1}(F(u) - F(v))\|_{\mathcal{X}_T^s} \leq C_T A'(\|u\|_{\mathcal{X}_T^s}, \|v\|_{\mathcal{X}_T^s})\|u - v\|_{\mathcal{X}_T^s}$$

for A' continuous will show that we have a contraction for T sufficiently small, and also imply the Cauchy property when combined with the estimates above. Later we will see how using the product structure of the nonlinearity $F(u) = \partial u \partial u$ in these estimates gives rise to the bilinear estimates that we prove in Chapter 3.

1.2 Classical results

We now review some classical results concerning existence and uniqueness for the general system (1.1.1) and (1.1.2) where $u : \mathbb{R}^{1+n} \rightarrow \mathbb{R}^N$ and F is a smooth \mathbb{R}^N -valued function satisfying $F(0) = 0$. For certain nonlinearities, such as $F(u) = u^p$ or the derivative nonlinearities considered in this dissertation, these equations have a natural scaling. For the quadratic derivative problem (1.0.1), we see that if $u = u(t, x)$ satisfies

(1.1.1) then so does $u_\lambda(t, x) = u(\lambda t, \lambda x)$. If we take initial data $(u_0, u_1) \in \dot{H}^s \times \dot{H}^{s-1}$, we find that the $\dot{H}^s \times \dot{H}^{s-1}$ norm is invariant under this scaling if and only if $s = \frac{n}{2}$. Indeed, we have

$$\begin{aligned} \|u_\lambda(0, \cdot)\|_{\dot{H}^s} &= \| |\xi|^s \widehat{u}_\lambda(0, \xi) \|_{L^2} \\ &= \lambda^{-n} \| |\xi|^s \widehat{u}(0, \xi/\lambda) \|_{L^2} \\ &= \lambda^{-n/2} \| |\lambda \xi|^s \widehat{u}(0, \xi) \|_{L^2} \\ &= \lambda^{s-n/2} \|u(0, \cdot)\|_{\dot{H}^s}. \end{aligned}$$

Thus, for (1.0.1) the critical scaling exponent is $s_c = n/2$.

With regard to local well-posedness, the scaling suggests a relationship between the size and regularity of the initial data and the time of existence. In the *subcritical case*, $s > s_c$, we expect to be able to extend local well-posedness results for data with small norm to large data by shrinking the interval of existence. This is the best possible case for local well-posedness. On the other hand, in the *supercritical case*, $s < s_c$, we are likely to encounter finite time blow-up, even for small initial data. Finally, in the *critical case*, $s = s_c$, since the norm of the initial data is invariant under scaling, we expect global existence and regularity for sufficiently small data. This can be summarized by the following conjecture, c.f. [12].

Conjecture 1.3 (General Well-posedness Conjecture). *Consider the initial value problem (1.1.1), (1.1.2). The following should hold.*

1. *Local well-posedness for initial data in $H^s \times H^{s-1}$, $s > s_c$.*
2. *Global well-posedness¹ for initial data with small $\dot{H}^{s_c} \times \dot{H}^{s_c-1}$ -norm.*
3. *Ill-posedness for initial data in $H^s \times H^{s-1}$, $s < s_c$.*

One approach to proving local well-posedness is to use energy estimates to close the fixed point argument. For u with initial data $(u_0, u_1) \in H^s \times H^{s-1}$ the equation

¹maybe in a weaker sense

$\square u = F(u)$ satisfies the following energy inequality.

$$\|u(t, \cdot)\|_{H^s} + \|\partial_t u(t, \cdot)\|_{H^{s-1}} \leq C(1+t) \left(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} + \int_0^t \|F(t', \cdot)\|_{H^{s-1}} dt' \right)$$

The same estimate holds with \dot{H}^s and \dot{H}^{s-1} replacing H^s and H^{s-1} , respectively, with constant C instead of $C(1+t)$. In particular,

$$\|Du\|_{L_t^\infty L_x^2} \lesssim \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2} + \|F\|_{L_t^1 L_x^2}$$

where we use the notation \gtrsim, \lesssim , and \simeq to denote the relations \geq, \leq , and $=$ up to multiplicative constants depending on fixed quantities. We write $X \sim Y$ to mean $X \lesssim Y \lesssim X$.

Using the energy estimates, one can obtain the following theorem, see [12].

Theorem 1.4 (Classical Local Existence Theorem). *The equation (1.1.1) is locally well-posed for initial data in $H^s \times H^{s-1}(\mathbb{R}^n)$ for all $s > \frac{n}{2} + 1$.*

In dimension $n = 2$, this is still well above the scaling prediction for (1.0.1). However, one can improve upon this result using Strichartz estimates, which can be found in Sogge [19] or Tao [23]. We state these estimates below and then use them to show (1.0.1) is locally well-posed in $H^s \times H^{s-1}$ for $s > 7/4$. This is a gain of $1/4$ over the energy estimates, but still a gap of $3/4$ above critical scaling.

Theorem 1.5 (Strichartz estimates for the wave equation). *Let $n \geq 2$ and consider the wave equation (1.1.2) with initial data $(u_0, u_1) \in \dot{H}^s \times \dot{H}^{s-1}$. Define the admissible family of pairs to be*

$$\mathcal{A} = \left\{ (q, r) : 2 \leq q, r \leq \infty, \frac{2}{q} + \frac{n-1}{r} \leq \frac{n-1}{2}, (q, r, n) \neq (2, \infty, 3) \right\}.$$

Suppose $(q, r), (\tilde{q}, \tilde{r}) \in \mathcal{A}$ and $s \geq 0$ is such that the following gap condition holds:

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - s = \frac{1}{\tilde{q}} + \frac{n}{\tilde{r}} - 2$$

for $(r, \tilde{r}) < \infty$. Then

$$\|u\|_{L_t^q L_x^r} + \|u\|_{C^0[0, T] \dot{H}^s} + \|\partial_t u\|_{C^0[0, T] \dot{H}^{s-1}} \lesssim \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

Furthermore, for derivatives $D^\gamma u$, we have

$$\|D^\gamma u\|_{L_t^q L_x^r} \lesssim \|u_0\|_{\dot{H}^s} + \|u_1\|_{\dot{H}^{s-1}} + \|D^{\tilde{\gamma}} F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

provided the gap condition with derivatives

$$\frac{1}{q} + \frac{n}{r} - \gamma = \frac{n}{2} - s = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - 2 - \tilde{\gamma} \quad (1.2.1)$$

holds.

With $n = 2$, take $(q, r) = (4, \infty)$ and $(\tilde{q}, \tilde{r}) = (\infty, 2)$ in \mathcal{A} . For $s = 1$, $\gamma = 1/4$, and $\tilde{\gamma} = 0$, (1.2.1) holds, and hence we have

$$\|D^{\frac{1}{4}} u\|_{L_t^4 L_x^\infty} \lesssim \|u_0\|_{\dot{H}^1} + \|u_1\|_{L^2} + \|F\|_{L_t^1 L_x^2}.$$

Applying this to $D^{\frac{3}{4}} u$,

$$\|Du\|_{L_t^4 L_x^\infty} \lesssim \|u_0\|_{\dot{H}^{\frac{7}{4}}} + \|u_1\|_{\dot{H}^{\frac{3}{4}}} + \|F\|_{L_t^1 \dot{H}_x^{\frac{3}{4}}}. \quad (1.2.2)$$

From the energy estimate with $s = 7/4$, for fixed $0 < t < T$ we have

$$\|u(t, \cdot)\|_{\dot{H}^{\frac{7}{4}}} + \|\partial_t u(t, \cdot)\|_{\dot{H}^{\frac{3}{4}}} \lesssim \left(\|u_0\|_{\dot{H}^{\frac{7}{4}}} + \|u_1\|_{\dot{H}^{\frac{3}{4}}} + \int_0^t \|F(t', \cdot)\|_{\dot{H}^{\frac{3}{4}}} dt' \right). \quad (1.2.3)$$

On the other hand, for $F(u) = \partial u \partial u$,

$$\begin{aligned} \int_0^t \|\partial u(t', \cdot) \partial u(t', \cdot)\|_{\dot{H}^{\frac{3}{4}}} dt' &\leq \left(\int_0^t dt' \right)^{3/4} \left(\int_0^t \|\partial u(t', \cdot) \partial u(t', \cdot)\|_{\dot{H}^{\frac{3}{4}}}^4 dt' \right)^{1/4} \\ &\leq T^{3/4} \left(\int_0^t \|\partial u(t', \cdot)\|_{L_x^\infty}^4 \|\partial u(t', \cdot)\|_{\dot{H}^{\frac{3}{4}}}^4 dt' \right)^{1/4} \\ &\leq T^{3/4} \|\partial u\|_{L_t^4 L_x^\infty} \|\partial u\|_{L_t^\infty \dot{H}_x^{\frac{3}{4}}} \end{aligned}$$

whence by (1.2.2) and (1.2.3) we can obtain a contraction on $X_T = C_t^0 H_x^{7/4} \cap C_t^1 H_x^{3/4} \cap L_t^4 W_x^{1, \infty}$ provided T is sufficiently small.

The Strichartz estimates arise from the dispersive qualities of the wave equation. Here the term dispersive means that different frequencies will propagate at different velocities, and hence disperse. Thus, energy cannot concentrate in small spatial regions for a large

period of time. In contrast, frequencies in diffusive equations, such as the heat equation, do not propagate, but rather dissipate over time. For the heat equation, singularities in the initial data weaken as time evolves, giving rise to smooth solutions. However, for the wave equation solutions do not get smoother, but tend to spread and decay in time, which is reflected in the dispersive estimates.

The Strichartz estimates give the best known results for (1.0.1) in dimension $n = 2$ with initial data in Sobolev space. For initial data with lower subcritical regularities, we will instead look at the problem in different spaces. In the work of Klainerman and Machedon [11] and Bourgain [2], the Fourier restriction spaces $X^{s,b}$ have also been useful in studying nonlinear dispersive equations, so we turn our attention now to these spaces.

1.3 The Fourier Lebesgue spaces \widehat{H}_s^r and associated $X_{s,b}^r$ spaces

Using \widehat{f} for the spatial Fourier transform on \mathbb{R}^n , we define the Fourier Lebesgue spaces $\widehat{H}_s^r = \widehat{H}_s^r(\mathbb{R}^n)$ by the norm

$$\|f\|_{\widehat{H}_s^r} = \|\langle \xi \rangle^s \widehat{f}\|_{L^{r'}}$$

and their homogeneous counterparts

$$\|f\|_{\dot{\widehat{H}}_s^r} = \|\ |\xi|^s \widehat{f} \|_{L^{r'}}.$$

When $s = 0$, we shall write $\widehat{H}_0^r = \widehat{L}^r$, i.e.

$$\|f\|_{\widehat{L}^r} = \|\widehat{f}\|_{r'}.$$

We will take initial data in suitable \widehat{H}_s^r spaces, and find solutions in the related $X_{s,b}^r$ spaces described below. To compare our results on the Sobolev scale, we again consider the natural scaling of the equation. Define $f_\lambda(\xi) = f(\lambda\xi)$ for $\lambda > 0$ and recall that

$\|f_\lambda\|_{\dot{H}^s} = \lambda^{s-n/2}\|f\|_{\dot{H}^s}$. Similarly, we obtain

$$\begin{aligned}\|f_\lambda\|_{\hat{H}_s^r} &= \left\| |\xi|^s \widehat{f}_\lambda(\xi) \right\|_{L^{r'}} \\ &= \lambda^{-n} \left\| |\xi|^s \widehat{f}(\xi/\lambda) \right\|_{L^{r'}} \\ &= \lambda^{n/r'-n} \left\| |\lambda\xi|^s \widehat{f}(\xi) \right\|_{L^{r'}} \\ &= \lambda^{s-n/r} \|f\|_{\hat{H}_s^r}.\end{aligned}$$

Thus, from a scaling viewpoint

$$\hat{H}_s^r \sim \dot{H}^\sigma \quad \text{if} \quad \sigma = s + n \left(\frac{1}{2} - \frac{1}{r} \right).$$

For $n = 2$, this gives $\sigma = s + 1 - \frac{2}{r}$.

We now present some theory of the L^2 -based Fourier restriction spaces $X^{s,b}$ following Tao in [23, Chapter 2]. If $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous function, and $s, b \in \mathbb{R}$, we define the Fourier restriction space $X_{\tau=\phi(\xi)}^{s,b}(\mathbb{R}^{1+n})$ to be the completion of the Schwartz class $\mathcal{S}(\mathbb{R}^{1+n})$ with respect to the norm

$$\|u\|_{X_{\tau=\phi(\xi)}^{s,b}} = \|\langle \xi \rangle^s \langle \tau - \phi(\xi) \rangle^b \tilde{u}(\tau, \xi)\|_{L_{\tau, \xi}^2}$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$ and $\tilde{u}(\tau, \xi) = \int \int e^{-i(t\tau + x \cdot \xi)} u(t, x) dt dx$ is the space-time Fourier transform on \mathbb{R}^{1+n} . Recall that for dispersive equations, such as the wave equation, the phase velocity varies with frequency. Here, the function ϕ is the phase velocity associated to the equation, while the group velocity, h , or the overall velocity of wave groups or packets, is $h = \nabla \phi$.

Recall also that for the homogeneous wave equation (1.1.3), we can write the solution in terms of its spatial Fourier transform (1.1.4). In the sense of distributions, the Fourier transform of $e^{i\omega t}$ is the measure $2\pi\delta(\tau - \omega)$ and so we can write the space-time Fourier transform of the homogenous system as

$$\tilde{u}(\tau, \xi) = \pi (\delta(\tau - |\xi|) + \delta(\tau + |\xi|)) \widehat{u}_0(\xi) + \frac{\pi}{i|\xi|} (\delta(\tau - |\xi|) + \delta(\tau + |\xi|)) \widehat{u}_1(\xi).$$

In this way, the Fourier transform of the solution is a measure supported on the light cone $\{(\tau, \xi) : |\tau| = |\xi|\}$. Assuming solutions to (1.1.3) are plane waves $e^{i(t\tau \pm x \cdot \xi)}$, we derive the dispersion relation $\tau = \pm|\xi|$. Then the phase velocity is $\phi(\xi) = \pm|\xi|$ and the group velocity is $h(\xi) = \nabla\phi(\xi) = \pm\frac{\xi}{|\xi|}$. So, the group velocity depends not on speed, but only on direction. In fact, from the dispersion relation, we see that all waves propagate in concentric circles with the same constant speed. For this reason, the wave equation is sometimes referred to as weakly dispersive.

Since we are concerned with the wave equation (1.0.1), we will focus on the $X_{\tau=\phi(\xi)}^{s,b}$ spaces with $\phi(\xi) = \pm|\xi|$, which we will denote simply as $X^{s,b}$. However, since the phase velocity ϕ is multi-valued, we adapt the definition slightly and for $s, b \in \mathbb{R}$, define the wave-Sobolev space $X^{s,b} = X^{s,b}(\mathbb{R}^{1+n})$ to be the completion of the Schwartz class $\mathcal{S}(\mathbb{R}^{1+n})$ with respect to the norm

$$\|u\|_{X^{s,b}} = \|\langle \xi \rangle^s \langle |\tau| - |\xi| \rangle^b \tilde{u}(\tau, \xi)\|_{L_{\tau, \xi}^2}.$$

In much the same way that the elliptic weight $\langle \xi \rangle$ measures regularity in Sobolev space, the wave-Sobolev index s measures elliptic regularity. On the other hand, the index b corresponding to the hyperbolic weight $\langle |\tau| - |\xi| \rangle$ measures the hyperbolic regularity of the solution in L^2 .

From [23, Corollary 2.10], we have $X^{s,b} \hookrightarrow C(\mathbb{R}, H^s) \cap C^1(\mathbb{R}, H^{s-1})$ for $b > 1/2$ and hence one can look for solutions to (1.0.1) with $(u_0, u_1) \in H^s \times H^{s-1}$ in these wave-Sobolev spaces. An $X^{s,b}$ space approach has been used to show improvement over the Strichartz results for (1.0.1) in dimensions $n \geq 3$, such as in Foschi and Klainerman [7], Grünrock [8], and Tataru [24]. In dimension $n = 2$, these methods have produced the $s > 5/4$ results for null forms [25]. For the product (1.0.1) in two dimensions, however, there has been no improvement over the Strichartz results using $X^{s,b}$. Motivated by the $n = 3$ results of Grünrock in [8], we instead turn to the more general Banach spaces $X_{s,b}^r$ for $1 < r \leq 2$ endowed with the norm

$$\|u\|_{X_{s,b}^r} = \|\langle \xi \rangle^s \langle |\tau| - |\xi| \rangle^b \tilde{u}(\tau, \xi)\|_{L_{\tau, \xi}^{r'}}.$$

where $r' \geq 2$ is the conjugate exponent of r , i.e. $\frac{1}{r} + \frac{1}{r'} = 1$. The hope is that lowering the value of r , and hence increasing r' on the Fourier side, will also lower the bound on s . At times we will also wish to restrict our attention to the forward and backward waves, and thus define also the spaces

$$\|u\|_{X_{s,b}^{r,\pm}} = \|\langle \xi \rangle^s \langle \tau \pm |\xi| \rangle^b \tilde{u}(\tau, \xi)\|_{L_{\tau,\xi}^{r'}}.$$

These spaces are particular examples of the more general $X_{s,b}^{r,\phi}$ spaces associated to a phase function $\phi = \phi(\xi)$. As in the $r = 2$ case we define

$$\|u\|_{X_{s,b}^{r,\phi}} = \|\langle \xi \rangle^s \langle \tau - \phi(\xi) \rangle^b \tilde{u}(\tau, \xi)\|_{L_{\tau,\xi}^{r'}}.$$

When a fixed phase function ϕ is clear, we will simply write $X_{s,b}^{r,\phi} = X_{s,b}^r$. We now give some properties of $X_{s,b}^r$ spaces, following Grünrock [9] and [23].

First note that we have the following containment $X_{s_1,b_1}^{r_1} \subset X_{s_0,b_0}^{r_0}$ if $r_1 \leq r_0$, $s_1 - n/r_1 > s_0 - n/r_0$ and $b_1 - 1/r_1 > b_0 - 1/r_0$. Indeed, since $r'_1 \geq r'_0$, we can apply Hölder's inequality to obtain

$$\begin{aligned} \|u\|_{X_{s_0,b_0}^{r_0}} &= \|\langle \xi \rangle^{s_1} \langle |\tau| - \phi(\xi) \rangle^{b_0} \tilde{u}(\tau, \xi)\|_{r'_0} \\ &= \left\| \frac{\langle \xi \rangle^{s_1} \langle |\tau| - \phi(\xi) \rangle^{b_1} \tilde{u}(\tau, \xi)}{\langle \xi \rangle^{s_1 - s_0} \langle |\tau| - \phi(\xi) \rangle^{b_1 - b_0}} \right\|_{r'_0} \\ &\leq \left(\int \left(\langle \xi \rangle^{s_0 - s_1} \langle |\tau| - \phi(\xi) \rangle^{b_0 - b_1} \right)^{\frac{1}{r'_0} - \frac{1}{r'_1}} d(\tau, \xi) \right)^{\frac{1}{r'_0} - \frac{1}{r'_1}} \left\| \langle \xi \rangle^{s_1} \langle |\tau| - \phi(\xi) \rangle^{b_1} \tilde{u}(\tau, \xi) \right\|_{r'_1} \\ &\lesssim \|u\|_{X_{s_1,b_1}^{r_1}} \end{aligned}$$

where in the last step we have used the fact $b_1 - b_0 > \frac{1}{r'_1} - \frac{1}{r'_0}$ and $s_1 - s_0 > n \left(\frac{1}{r'_1} - \frac{1}{r'_0} \right)$.

With $D = \sqrt{-\Delta}$ as above, we write $\mathcal{U}_\phi(t) = e^{it\phi(D)}$. For the operators associated to free wave solutions, $e^{it\sqrt{-\Delta}}$ and $e^{-it\sqrt{-\Delta}}$, we write $\mathcal{U}_\pm(t) = e^{\pm it\sqrt{-\Delta}}$. First note that for any smooth cutoff function $\psi \in \mathcal{S}(\mathbb{R})$, we have

$$\begin{aligned} \mathcal{F}[\psi(t)\mathcal{U}_\phi(\pm t)f](\tau, \xi) &= \int \psi(t)e^{\pm it\phi(\xi)} \widehat{f}(\xi)e^{-it\tau} dt \\ &= \widehat{\psi}(\tau \mp \phi(\xi)) \widehat{f}(\xi). \end{aligned} \tag{1.3.1}$$

Furthermore, this implies

$$\begin{aligned}
\|\psi(t)\mathcal{U}_\phi(-t)f\|_{\widehat{H}_{s,b}^r} &= \left\| \langle \xi \rangle^s \langle \tau \rangle^b \widehat{\psi}(\tau + \phi(\xi)) \widehat{f}(\xi) \right\|_{r'} \\
&= \left\| \langle \xi \rangle^s \langle \tau - \phi(\xi) \rangle^b \widehat{\psi}(\tau) \widehat{f}(\xi) \right\|_{r'} \\
&= \|\psi f\|_{X_{s,b}^r}.
\end{aligned} \tag{1.3.2}$$

On the other hand, we have the following lemma.

Lemma 1.6. *Let $\psi \in \mathcal{S}(\mathbb{R})$ be a smooth time cutoff. Then for any $f \in \widehat{H}_s^r(\mathbb{R}^n)$, we have*

$$\|\psi(t)\mathcal{U}_\phi(t)f\|_{X_{s,b}^r} \lesssim_{\psi,b} \|f\|_{\widehat{H}_s^r}. \tag{1.3.3}$$

Proof. By (1.3.1),

$$\begin{aligned}
\|\psi(t)\mathcal{U}_\phi(t)f\|_{X_{s,b}^r} &= \left\| \langle \xi \rangle^s \langle \tau - \phi(\xi) \rangle^b \widehat{\psi}(\tau - \phi(\xi)) \widehat{f}(\xi) \right\|_{r'} \\
&= \left\| \langle \xi \rangle^s \langle \tau \rangle^b \widehat{\psi}(\tau) \widehat{f}(\xi) \right\|_{r'} \\
&= \|\psi\|_{\widehat{H}_b^r} \|f\|_{\widehat{H}_s^r}.
\end{aligned} \tag{1.3.4}$$

□

To show $X_{s,b}^r \hookrightarrow C(\mathbb{R}, \widehat{H}_s^r)$, we use the following result from [9, Lemma 2.1].

Lemma 1.7. *Let $Y \subset \mathcal{S}'(\mathbb{R}^{1+n})$ be a Banach space stable under multiplication with L_t^∞ , i.e. for all $\psi \in L_t^\infty$ and $u \in Y$,*

$$\|\psi u\|_Y \lesssim \|\psi\|_{L_t^\infty} \|u\|_Y. \tag{1.3.5}$$

Suppose that Y also satisfies the inequality

$$\|e^{it\tau_0}\mathcal{U}_\phi(t)f\|_Y \lesssim \|f\|_{\widehat{H}_s^r} \tag{1.3.6}$$

for all $f \in \widehat{H}_s^r$ and $\tau_0 \in \mathbb{R}$. Then, for all $b > \frac{1}{r}$, we have the estimate

$$\|u\|_Y \lesssim_b \|u\|_{X_{s,b}^r} \tag{1.3.7}$$

with a fixed constant that depends only on b .

Proof. By Fourier inversion,

$$u(t, x) \simeq \int e^{i(t\tau + x \cdot \xi)} \tilde{u}(\tau, \xi) d(\tau, \xi). \quad (1.3.8)$$

Define

$$f_{\tau_0}(x) \simeq \int \tilde{u}(\tau_0 + \phi(\xi), \xi) e^{ix \cdot \xi} d\xi$$

where $\tau_0 = \tau - \phi(\xi)$. Note that if $u \in X_{0,b}^r$ for any $b > 0$ this implies that $f_{\tau_0} \in \widehat{L}^r$. Then

$$\begin{aligned} \mathcal{U}_\phi(t) f_{\tau_0} &\simeq \int e^{ix \cdot y} e^{it\phi(\xi)} \widehat{f_{\tau_0}}(\xi) d\xi \\ &\simeq \int e^{ix \cdot y} e^{it\phi(\xi)} \tilde{u}(\tau_0 + \phi(\xi), \xi) d\xi \end{aligned}$$

and hence

$$\begin{aligned} \int e^{it\tau_0} \mathcal{U}_\phi(t) f_{\tau_0} d\tau_0 &\simeq \int \int e^{ix \cdot y} e^{it\tau_0 + it\phi(\xi)} \tilde{u}(\tau_0 + \phi(\xi), \xi) d\xi d\tau_0 \\ &\simeq \int \int e^{ix \cdot y} e^{it\tau_0} \tilde{u}(\tau_0, \xi) d\tau_0 d\xi. \end{aligned}$$

By Minkowski's inequality,

$$\begin{aligned} \|u\|_Y &\lesssim \int \|e^{it\tau_0} \mathcal{U}_\phi(t) f_{\tau_0}\|_Y d\tau_0 \\ &\lesssim \int \|f_{\tau_0}\|_{\widehat{H}_s^r} d\tau_0 \end{aligned}$$

by hypothesis. Then, by Hölder's inequality,

$$\begin{aligned} \|u\|_Y &\lesssim \int \langle \tau_0 \rangle^{-b} \langle \tau_0 \rangle^b \|f_{\tau_0}\|_{\widehat{H}_s^r} d\tau_0 \\ &\lesssim \left(\int \langle \tau_0 \rangle^{-br} d\tau_0 \right)^{1/r} \left(\int \langle \tau_0 \rangle^{br'} \int \langle \xi \rangle^{sr'} |\widehat{f_{\tau_0}}(\xi)|^{r'} d\xi d\tau_0 \right)^{1/r'}. \end{aligned}$$

Now since $br > 1$, the first term is finite and so

$$\begin{aligned} \|u\|_Y &\lesssim \left(\iint \langle \tau_0 \rangle^{br'} \langle \xi \rangle^{sr'} |\tilde{u}(\tau_0 + \phi(\xi), \xi)|^{r'} d\xi d\tau_0 \right)^{1/r'} \\ &\lesssim \|u\|_{X_{s,b}^r}. \end{aligned}$$

□

Letting $Y = C(\mathbb{R}, \widehat{H}_s^r)$, we see that (1.3.5) clearly holds and

$$\begin{aligned} \left\| \|e^{it\tau_0} \mathcal{U}_\phi(t) f\|_{\widehat{H}_{s,x}^r} \right\|_{L_t^\infty} &= \left\| e^{it\tau_0} \left(\int \langle \xi \rangle^s |e^{it\phi(\xi)} \widehat{f}(\xi)|^{r'} \right)^{1/r'} \right\|_{L_t^\infty} \\ &\leq \left(\int \langle \xi \rangle^s |\widehat{f}(\xi)|^{r'} \right)^{1/r'} \end{aligned}$$

is (1.3.6). Thus, Lemma 1.7 gives the following important result.

Corollary 1.8. *Let $u \in X_{s,b}^r$ for any $b > \frac{1}{r}$. Then we have the inequality*

$$\|u\|_{C_t^0 \widehat{H}_{s,x}^r} \lesssim_b \|u\|_{X_{s,b}^r}. \quad (1.3.9)$$

Now define the spaces $Z_{s,b}^r = Z_{s,b}^r(\mathbb{R}^{1+n})$ by the norm

$$\|u\|_{Z_{s,b}^r} = \|u\|_{X_{s,b}^r} + \|\partial_t u\|_{X_{s-1,b}^r} \quad (1.3.10)$$

and for $\phi(\xi) = \pm|\xi|$ define the associated spaces $Z_{s,b}^{r,\pm}$ accordingly. Analogous to the case when $r = 2$, we have from above that if $b > 1/r$, $Z_{s,b}^r \hookrightarrow C([0, T], \widehat{H}_s^r) \cap C^1([0, T], \widehat{H}_{s-1}^r)$.

As in Sobolev space, for the operators associated to the free solution we see that

$$\left\| e^{\pm it\sqrt{-\Delta}} u_0 \right\|_{\widehat{H}_s^r} = \|u_0\|_{\widehat{H}_s^r} \quad \text{and} \quad \left\| \frac{e^{\pm it\sqrt{-\Delta}}}{\sqrt{-\Delta}} u_1 \right\|_{\widehat{H}_s^r} = \|u_1\|_{\widehat{H}_{s-1}^r}.$$

Thus, we take initial data $(u_0, u_1) \in \widehat{H}_s^r(\mathbb{R}^n) \times \widehat{H}_{s-1}^r(\mathbb{R}^n)$, and look for solutions of (1.0.1) in $X_{s,b}^r$ for appropriate ranges of the exponents s, b , and r .

The main result of this dissertation gives local well-posedness in \widehat{H}_s^r with $s > \frac{3}{2r} + 1$ for $\frac{3}{2} < r \leq 2$. From the scaling, this corresponds to $\sigma > 2 - \frac{1}{2r}$. When $r = 2$, this delivers $\sigma > \frac{7}{4}$, which is the best known result for local well-posedness in H^σ . However, when $r = 3/2$, we obtain $\sigma > \frac{5}{3} = \frac{7}{4} - \frac{1}{12}$. In this way, this can be viewed as an improvement over the Strichartz results, but still leaves a gap of $2/3$ over the scaling prediction.

1.4 Statement of the main result

When studying local well-posedness, it can be useful to reformulate (1.1.1) as a first order system. This may be done in various ways, some of which we include here. For

example, letting $w = (u, u_t)^t$ and $N(w) = (0, F(u))^t$, we can write

$$\frac{dw}{dt} = Lw + N(w)$$

where $L = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$ is a 2×2 matrix. If $\mathcal{U}(t) = e^{tL}$ is well-defined, this generates the solution to the linear equation $\frac{dw}{dt} = Lw$, and by the Duhamel principle or the method of superposition,

$$w(\cdot, t) = e^{tL}w(0) + \int_0^t e^{(t-s)L}N(w(s))ds.$$

Another way is to use the operator $D = (-\Delta)^{1/2}$ from above. Note that $D^{-1} = (-\Delta)^{-1/2}$ is the Fourier multiplier operator with symbol $|\xi|^{-1}$. Defining $u_{\pm} = u \pm iD^{-1}u_t$, we have

$$\begin{aligned} (i\partial_t - D)u_+ &= (i\partial_t - D)(u + iD^{-1}u_t) \\ &= -D^{-1}u_{tt} - Du \\ &= -D^{-1}(u_{tt} - \Delta u) \\ &= -D^{-1}(\square u) \end{aligned}$$

and similarly, $(i\partial_t + D)u_- = D^{-1}(\square u)$. So, we can reformulate (1.1.1) as

$$(i\partial_t \pm D)u_{\mp} = \pm D^{-1}(F(u)).$$

Similarly, using the Bessel potential operator $J = (I - \Delta)^{1/2}$ with inverse $J^{-1} = (I - \Delta)^{-1/2}$, we define $u_{\pm} = u \pm iJ^{-1}u_t$. Observe that

$$\|J^{-1}u_t\|_{\widehat{H}_s^r} = \|\langle \xi \rangle^s (1 + |\xi|^2)^{-1/2} \widetilde{u}_t\|_{r'} = \|u_t\|_{\widehat{H}_{s-1}^r} \quad (1.4.1)$$

and also $\|Ju\|_{\widehat{H}_{s-1}^r} = \|u\|_{\widehat{H}_s^r}$. Then

$$\begin{aligned} (i\partial_t + J)u_- &= (i\partial_t + J)(u - iJ^{-1}u_t) \\ &= J^{-1}u_{tt} + Ju \\ &= J^{-1}(u_{tt} + (I - \Delta)u) \\ &= J^{-1}(\square u) + J^{-1}u \end{aligned}$$

and in the same way, $\pm J^{-1}(\square u) = (i\partial_t \pm J)u_{\mp} \mp J^{-1}u$. Using this, we can rewrite (1.1.1) as

$$(i\partial_t \pm J)u_{\mp} = \pm J^{-1}F(u) \pm J^{-1}u.$$

For the nonlinearity $F(u) = \partial u \partial u$ in (1.0.1), define the corresponding bilinear form $B_2(u, v) = \partial u \partial v$. Then, since $u_+ + u_- = 2u$, we have

$$\begin{aligned} F(u) &= F\left(\frac{1}{2}u_+ + \frac{1}{2}u_-\right) \\ &= \frac{1}{4}B_2(u_+ + u_-, u_+ + u_-). \end{aligned}$$

From this, we write (1.0.1) as the first order system

$$(i\partial_t \pm J)u_{\mp} = \pm \frac{1}{4}J^{-1}B_2(u_+ + u_-, u_+ + u_-) \pm \frac{1}{2}J^{-1}(u_+ + u_-)$$

or dividing by i ,

$$(\partial_t \pm iJ)u_{\pm} = \pm i\frac{1}{4}J^{-1}B_2(u_+ + u_-, u_+ + u_-) \pm i\frac{1}{2}J^{-1}(u_+ + u_-) \quad (1.4.2)$$

with initial conditions

$$f_{\pm} = u_{\pm}(0, \cdot) = u_0 \pm iJ^{-1}u_1 \in \widehat{H}_s^r. \quad (1.4.3)$$

We then understand a solution (u_+, u_-) to be a solution of the corresponding pair of integral equations

$$u_{\pm}(t, \cdot) = e^{\pm itJ}f_{\pm} + \int_0^t e^{\pm i(t-s)J}F_{\pm}(u(s, \cdot))ds. \quad (1.4.4)$$

where

$$F_{\pm}(u) = \pm i\frac{1}{4}J^{-1}B_2(u_+ + u_-, u_+ + u_-) \pm i\frac{1}{2}J^{-1}(u_+ + u_-)$$

(see Section 1.5 for more details).

We are now ready to state our main result concerning local well-posedness for (1.0.1). We give a theorem for the equivalent first order system (1.4.2) and (1.4.3). Our initial data will be in $\widehat{H}_s^r \times \widehat{H}_s^r$ and solutions will lie in time localized $Z_{s,b}^r$ product spaces, denoted $Z_{s,b}^r(T)$, with the norm

$$\|u\|_{Z_{s,b}^r(T)} = \inf\{\|\bar{u}\|_{Z_{s,b}^r} : \bar{u}|_{[-T,T] \times \mathbb{R}^n} = u\}.$$

In particular, the norm on $Z_{s,b}^{r,\pm}(T)$ is given by

$$\|u\|_{Z_{s,b}^{r,\pm}(T)} = \inf\{\|\bar{u}\|_{X_{s,b}^{r,\pm}} + \|\partial_t \bar{u}\|_{X_{s-1,b}^{r,\pm}} : \bar{u}|_{[-T,T] \times \mathbb{R}^n} = u\}$$

We define similarly any time restricted spaces $X_{s,b}^r(T)$. Note that if $b > 1/r$ then $Z_{s,b}^{r,\pm}(T) \hookrightarrow C([0, T], \widehat{H}_s^r) \cap C^1([0, T], \widehat{H}_{s-1}^r)$. Hence, if the initial data are in $\widehat{H}_s^r \times \widehat{H}_s^r$, then the solution embeds into the continuation space $C([0, T], \widehat{H}_s^r) \times C([0, T], \widehat{H}_s^r)$.

Theorem 1.9. *Let $\frac{3}{2} < r \leq 2$, $s > \frac{3}{2r} + 1$, and $\frac{1}{r} < b < 1$. Given initial data $f_{\pm} \in \widehat{H}_s^r$, there exist $T = T(\|f_+\|_{\widehat{H}_s^r}, \|f_-\|_{\widehat{H}_s^r}) > 0$ and a unique solution $(u_+, u_-) \in Z_{s,b}^{r,+}(T) \times Z_{s,b}^{r,-}(T)$ of the system*

$$(\partial_t \pm iJ)u_{\pm} = \pm i\frac{1}{4}J^{-1}B_2(u_+ + u_-, u_+ + u_-) \pm i\frac{1}{2}J^{-1}(u_+ + u_-) \quad (1.4.5)$$

satisfying the initial conditions

$$f_{\pm}(x) = u_{\pm}(0, x) = u_0(x) \pm iJ^{-1}u_1(x). \quad (1.4.6)$$

The solution is persistent and the flow map

$$(f_+, f_-) \mapsto (u_+, u_-), \quad \widehat{H}_s^r \times \widehat{H}_s^r \rightarrow Z_{s,b}^{r,+}(T) \times Z_{s,b}^{r,-}(T)$$

is locally Lipschitz continuous.

In the next section, we reduce the proof of this theorem to bilinear estimates in $X_{s,b}^r$. This relies on a general local well-posedness scheme introduced by Bourgain and adapted to $X_{s,b}^r$ by Grünrock in [9]. We will prove general estimates of this form in Chapter 3 from which we will obtain our result.

1.5 Reduction to bilinear estimates

The general scheme in [9] reduces local well-posedness for the Cauchy problem

$$\partial_t u - i\phi(D)u = N(u), \quad u(0) = u_0 \in \widehat{H}_s^r \quad (1.5.1)$$

to proving sufficient estimates for the fixed point argument. Here $N(u)$ is a nonlinear function of u and its derivatives. As we showed earlier for the wave equation, we can reformulate (1.5.1) as an integral equation by taking Fourier transforms and finding the solution of an ODE. Then $\mathcal{U}_\phi(t)$ is the operator associated to the homogenous linear equation and by Duhamel's principle, we have the integral representation

$$u(t, \cdot) = \Lambda u(t) = \mathcal{U}_\phi(t)u_0 + \int_0^t \mathcal{U}_\phi(t-s)N(u(s, \cdot))ds. \quad (1.5.2)$$

We begin with the following result [9, Lemma 2.2], which gives the estimate needed for the Duhamel piece

$$\mathcal{U}_{\phi^*R}F(t) = \int_0^t \mathcal{U}(t-s)F(u(s, \cdot))ds. \quad (1.5.3)$$

In the following, $\psi \in C_0^\infty$ will be a smooth function compactly supported in the interval $(-2, 2)$. For $0 < T \leq 1$ we will write $\psi_\delta(t) = \psi(\frac{t}{\delta})$. The space $X_{s,b}^{r,\phi}$ with the phase function ϕ corresponding to (1.5.1) will be denoted simply by $X_{s,b}^r$.

Lemma 1.10. *Assume $-1/r' < b' \leq 0 \leq b \leq b' + 1$. Then for the linear inhomogeneous system*

$$\partial_t v - i\phi(D)v = F, \quad v(0) = 0$$

we have the estimate

$$\|\psi_\delta \mathcal{U}_{\phi^*R}F\|_{X_{s,b}^r} \leq c\delta^{b'-b+1}\|F\|_{X_{s,b'}^r}. \quad (1.5.4)$$

Proof. The proof is found in [8, Section 2.2], so we will just give a sketch. The idea is to first prove the estimate for $Kg(t) = \psi_\delta(t) \int_0^t g(s)ds$, a function of time only. The corresponding estimate is then

$$\|Kg\|_{\widehat{H}_b^r} \leq c\delta^{b'-b+1}\|g\|_{\widehat{H}_{b'}^r}. \quad (1.5.5)$$

Once this is established, we can take a function $g = g(t, x)$ of both time and space, and for a fixed ξ apply (1.5.5) to $\widehat{g}(t, \xi)$. This gives

$$\left(\int \langle \tau \rangle^{r'b} |\widetilde{K}g(\tau, \xi)|^{r'} d\tau \right)^{1/r'} \leq c\delta^{b'-b+1} \left(\int \langle \tau \rangle^{r'b'} |\widetilde{g}(\tau, \xi)|^{r'} d\tau \right)^{1/r'}.$$

Multiplying by $\langle \xi \rangle^{r's}$ and then integrating with respect to ξ gives

$$\|Kg\|_{\widehat{H}_{s,b}^r} \leq c\delta^{b'-b+1}\|g\|_{\widehat{H}_{s,b}^r}$$

where we use the notation $\widehat{H}_{s,b}^r = \widehat{H}_{s,b}^r(\mathbb{R}^{1+n})$ for $s, b \in \mathbb{R}$ to denote the Banach space with norm

$$\|f\|_{\widehat{H}_{s,b}^r} = \|\langle \xi \rangle^s \langle \tau \rangle^b \widetilde{f}\|_{r'}.$$

To establish (1.5.4), we then let $g(t, x) = \mathcal{U}_\phi(-t)F(t)$. This yields

$$\left\| \psi_\delta(t) \int_0^t \mathcal{U}_\phi(-s)F(s)ds \right\|_{\widehat{H}_{s,b}^r} \leq c\delta^{b'-b+1} \|\mathcal{U}_\phi(-t)F\|_{\widehat{H}_{s,b'}^r}.$$

Now note that we can write $\mathcal{U}(-s) = \mathcal{U}(-t)\mathcal{U}(t-s)$. The result then follows from the following fact. Analogous to Lemma 1.6, for a general $h = h(t, x)$ we have

$$\begin{aligned} \|\psi(t)\mathcal{U}_\phi(-t)h(t, x)\|_{\widehat{H}_{s,b}^r} &= \left\| \langle \xi \rangle^s \langle \tau \rangle^b \int \widehat{\psi}(s) \int e^{-it\phi(\xi)} \widehat{h}(t, \xi) e^{-it(\tau-s)} dt ds \right\|_{r'} \\ &= \left\| \langle \xi \rangle^s \langle \tau - \phi(\xi) \rangle^b \int \widehat{\psi}(s) \int \widehat{h}(t, \xi) e^{-it(\tau-s)} dt ds \right\|_{r'} \\ &= \|\psi h\|_{X_{s,b}^r} \end{aligned}$$

and similarly,

$$\|\mathcal{U}_\phi(-t)h(t, x)\|_{\widehat{H}_{s,b}^r} = \|h\|_{X_{s,b}^r}.$$

We also remark that if $h = h(x)$ then

$$\|\psi(t)\mathcal{U}_\phi(-t)h(x)\|_{\widehat{H}_{s,b}^r} = \|\psi h\|_{X_{s,b}^r} = \|\psi\|_{\widehat{H}_b^r} \|h\|_{\widehat{H}_b^r}$$

a fact we use in establishing the one dimensional estimate (1.5.5).

This generalization allows us to apply the estimate (1.5.4) to the nonlinearity $N(u)$ below. The remainder of the proof is devoted to proving the one dimensional estimate, by first rewriting g using the Fourier inversion formula and then integrating in time and multiplying by ψ_δ to get the expression

$$Kg(t) = c\psi_\delta(t) \int \frac{e^{it\tau} - 1}{i\tau} \widehat{g}(\tau) d\tau$$

for Kg . The integral is split over the regions $|\tau| \leq 1/\delta$ and $|\tau| \geq 1/\delta$ and is then estimated depending on the size of τ and the exponents b and b' . \square

The following theorem [8, Theorem 2.3], which relies on Lemma 1.10, is the foundation for our local well-posedness argument.

Theorem 1.11. *Assume that for $s \in \mathbb{R}$ and $r \in (1, \infty)$ given, there exist $b > 1/r$ and $b' \in (b - 1, 0]$ such that the estimates*

$$\|N(u)\|_{X_{s,b'}^r} \leq C \|u\|_{X_{s,b}^r}^\alpha \quad (1.5.6)$$

and

$$\|N(u) - N(v)\|_{X_{s,b'}^r} \leq C \left(\|u\|_{X_{s,b}^r}^{\alpha-1} + \|v\|_{X_{s,b}^r}^{\alpha-1} \right) \|u - v\|_{X_{s,b}^r} \quad (1.5.7)$$

are valid with $\alpha \geq 1$. Then there exist $T = T(\|u_0\|_{\widehat{H}_s^r}) > 0$ and a unique solution $u \in X_{s,b}^r(T)$ of (1.5.1). This solution is persistent and the mapping data upon solution $u_0 \mapsto u$, $\widehat{H}_s^r \rightarrow X_{s,b}^r(T_0)$ is locally Lipschitz continuous for any $T_0 \in (0, T)$.

Proof. The proof of this theorem is found in [8, Section 2.3], however we will include some details that will be useful later. We begin by defining an extension of Λ to $X_{s,b}^r(\delta)$ for any $0 < \delta < 1$. Let ψ be a smooth cutoff function compactly supported in $(-2, 2)$ with $\psi \equiv 1$ on $(-1, 1)$. For any $u \in X_{s,b}^r(\delta)$ with an extension $\bar{u} \in X_{s,b}^r$ let

$$\bar{\Lambda}u(t) = \psi(t)\mathcal{U}_\phi(t)u_0 + \psi_\delta(t) \int_0^t \mathcal{U}_\phi(t-s)N(\bar{u}(s, \cdot))ds. \quad (1.5.8)$$

and note that $\bar{\Lambda}u(t) = \Lambda\bar{u}(t)$ is well defined for $t \in (-\delta, \delta)$. Furthermore, for any extension \bar{u} ,

$$\begin{aligned} \|\bar{\Lambda}u\|_{X_{s,b}^r(\delta)} &\leq \|\psi\mathcal{U}_\phi(t)u_0\|_{X_{s,b}^r} + \|\psi_\delta\mathcal{U}_{\phi^*R}N(\bar{u})\|_{X_{s,b}^r} \\ &\leq C\|u_0\|_{\widehat{H}_s^r} + C\delta^{b'-b+1}\|N(\bar{u})\|_{X_{s,b'}^r} \end{aligned}$$

by Lemma 1.6 and Lemma 1.10. Then using the hypothesis (1.5.6), we have

$$\|\bar{\Lambda}u\|_{X_{s,b}^r(\delta)} \leq C\|u_0\|_{\widehat{H}_s^r} + C\delta^{b'-b+1}\|\bar{u}\|_{X_{s,b}^r}^\alpha.$$

Since this holds for any \bar{u} , taking the infimum over all such extensions, we obtain

$$\|\bar{\Lambda}u\|_{X_{s,b}^r(\delta)} \leq C\|u_0\|_{\widehat{H}_s^r} + C\delta^{b'-b+1}\|u\|_{X_{s,b}^r(\delta)}^\alpha.$$

Thus, $\bar{\Lambda} : X_{s,b}^r(\delta) \rightarrow X_{s,b}^r(\delta)$. Now to show that $\bar{\Lambda}$ is a contraction, for $u, v \in X_{s,b}^r(\delta)$, let $\bar{u}, \bar{v} \in X_{s,b}^r$ be any corresponding extension. Then by Lemma 1.10,

$$\begin{aligned} \|\bar{\Lambda}u - \bar{\Lambda}v\|_{X_{s,b}^r(\delta)} &\leq \|\psi_\delta \mathcal{U}_{\phi^*R}(N(\bar{u}) - N(\bar{v}))\|_{X_{s,b}^r} \\ &\leq C\delta^{b'-b+1} \|N(\bar{u}) - N(\bar{v})\|_{X_{s,b'}^r} \\ &\leq C\delta^{b'-b+1} \left(\|\bar{u}\|_{X_{s,b}^r}^{\alpha-1} + \|\bar{v}\|_{X_{s,b}^r}^{\alpha-1} \right) \|\bar{u} - \bar{v}\|_{X_{s,b}^r} \end{aligned}$$

by hypothesis (1.5.7). Since this holds for any extension, we must have

$$\begin{aligned} \|\bar{\Lambda}u - \bar{\Lambda}v\|_{X_{s,b}^r(\delta)} &\leq C\delta^{b'-b+1} \left(\|u\|_{X_{s,b}^r(\delta)}^{\alpha-1} + \|v\|_{X_{s,b}^r(\delta)}^{\alpha-1} \right) \|u - v\|_{X_{s,b}^r(\delta)} \\ &\leq C\delta^{b'-b+1} (2R^{\alpha-1}) \|u - v\|_{X_{s,b}^r(\delta)} \end{aligned}$$

for any u, v in a ball of radius $R = \|u_0\|_{\widehat{H}_s^r}$ in $X_{s,b}^r(\delta)$. Since $b' - b + 1 > 0$, we can choose $0 < \delta < 1$ sufficiently small so that

$$\delta^{b'-b+1} < \frac{1}{2CR^{\alpha-1}}$$

and hence for $T = \frac{1}{2}C\delta^{b'-b+1}R^{\alpha-1}$,

$$\|\bar{\Lambda}u - \bar{\Lambda}v\|_{X_{s,b}^r(T)} \leq \frac{1}{4} \|u - v\|_{X_{s,b}^r(T)}. \quad (1.5.9)$$

Thus, $\bar{\Lambda}$ is a contraction, and by the contraction mapping principle there exists a solution u of the equation $\bar{\Lambda}u(t) = u(t) = \Lambda u(t)$ on for $t \in (-T, T)$. The uniqueness and Lipschitz continuity statements then follow from the contraction (1.5.9) by standard arguments, c.f. [18, Section 4.1]. The requirement $b > 1/r$ ensures persistence of higher regularity, since we then have $X_{s,b}^r \hookrightarrow C([-T, T], \widehat{H}_s^r)$ for any s .

□

Now we apply this scheme to (1.4.5) with initial conditions (1.4.6). Since the symbol of J is $(1 + |\xi|^2)^{1/2}$, we have $\phi(\xi) = \mp \langle \xi \rangle$. Let us write the corresponding $X_{s,b}^r$ and $Z_{s,b}^r$ spaces as $\widetilde{X}_{s,b}^{r,\pm}$ and $\widetilde{Z}_{s,b}^{r,\pm}$ and call

$$N(u) = \frac{1}{4} J^{-1} B_2(u_+ + u_-, u_+ + u_-) = J^{-1} B_2(u, u),$$

$$F(u) = \frac{1}{2}J^{-1}(u_+ + u_-) = J^{-1}(u)$$

and

$$\bar{\Lambda}u_{\pm} = \psi(t)\mathcal{U}_{\phi}(t)f_{\pm} \pm i\psi_{\delta}\mathcal{U}_{\phi^*R}N(\bar{u}) \pm i\psi_{\delta}\mathcal{U}_{\phi^*R}F(\bar{u}) \quad (1.5.10)$$

where $u = \frac{1}{2}(u_+ + u_-)$. For the fixed point argument, we need to show that $\tilde{\Lambda} = (\bar{\Lambda}, \bar{\Lambda})$ is a contraction on the space $\tilde{Z}_{s,b}^{r,+}(T) \times \tilde{Z}_{s,b}^{r,-}(T)$ for some T , which is equivalent to having a contraction for $\bar{\Lambda}$ on each component.

Reduction to estimates for B_2 in $\tilde{Z}_{s,b}^{r,\pm}$.

• **Estimates for $\bar{\Lambda}$.** First we verify that $\tilde{\Lambda} : \tilde{Z}_{s,b}^{r,+}(\delta) \times \tilde{Z}_{s,b}^{r,-}(\delta) \rightarrow \tilde{Z}_{s,b}^{r,+}(\delta) \times \tilde{Z}_{s,b}^{r,-}(\delta)$ for $0 < \delta < 1$. For any extension \bar{u}_{\pm} of u_{\pm} we use Lemma 1.6 and Lemma 1.10 to obtain

$$\begin{aligned} \|\bar{\Lambda}u_{\pm}\|_{\tilde{X}_{s,b}^{r,\pm}(\delta)} &\leq \|\psi\mathcal{U}_{\phi}f_{\pm}\|_{\tilde{X}_{s,b}^{r,\pm}} + \|\psi_{\delta}\mathcal{U}_{\phi^*R}F(\bar{u})\|_{\tilde{X}_{s,b}^{r,\pm}} + \|\psi_{\delta}\mathcal{U}_{\phi^*R}N(\bar{u})\|_{\tilde{X}_{s,b}^{r,\pm}} \\ &\leq C\|f_{\pm}\|_{\hat{H}_s^r} + C\delta^{b'-b+1} \left(\|F(\bar{u})\|_{\tilde{X}_{s,b'}^{r,\pm}} + \|N(\bar{u})\|_{\tilde{X}_{s,b'}^{r,\pm}} \right). \end{aligned}$$

We will use the following facts. First, by the triangle inequality, we have

$$|\tau| \leq |\xi| + \left| |\tau| - |\xi| \right|. \quad (1.5.11)$$

Next note that

$$\left| |\tau| - |\xi| \right| \leq |\tau \pm \xi| \quad (1.5.12)$$

which is easy to check by splitting into the cases $\tau \geq 0$ and $\tau \leq 0$. Finally,

$$\langle \tau \pm \xi \rangle = \left(1 + \left(\tau \pm (1 + |\xi|^2)^{1/2} \right)^2 \right)^{1/2} \sim \left(1 + (\tau \pm |\xi|)^2 \right)^{1/2} = \langle \tau \pm |\xi| \rangle \quad (1.5.13)$$

which can be verified by checking the cases $|\xi| \leq |\tau|$ and $|\xi| \geq |\tau|$ for $|\xi| \geq 1$ and $|\xi| \leq 1$.

Since $b' < 0 < b$,

$$\begin{aligned}
\|F(u)\|_{\tilde{X}_{s,b'}^{r,\pm}} &= \|\langle \xi \rangle^s \langle \tau \pm \langle \xi \rangle \rangle^{b'} \langle \xi \rangle^{-1} \tilde{u}\|_{r'} \\
&\sim \|\langle \xi \rangle^{s-1} \langle \tau \pm |\xi| \rangle^{b'} \tilde{u}\|_{r'} \\
&\lesssim \|\langle \xi \rangle^{s-1} \langle |\tau| - |\xi| \rangle^{b'} \tilde{u}\|_{r'} \\
&\lesssim \|\langle \xi \rangle^{s-1} \langle |\tau| - |\xi| \rangle^b \tilde{u}\|_{r'} \\
&\lesssim \|\langle \xi \rangle^{s-1} \langle |\tau| - |\xi| \rangle^b \tilde{u}_+\|_{r'} + \|\langle \xi \rangle^{s-1} \langle |\tau| - |\xi| \rangle^b \tilde{u}_-\|_{r'} \\
&\lesssim \|\langle \xi \rangle^{s-1} \langle \tau + |\xi| \rangle^b \tilde{u}_+\|_{r'} + \|\langle \xi \rangle^{s-1} \langle \tau - |\xi| \rangle^b \tilde{u}_-\|_{r'} \\
&\lesssim \|u_+\|_{\tilde{X}_{s,b}^{r,+}} + \|u_-\|_{\tilde{X}_{s,b}^{r,-}}
\end{aligned}$$

and similarly

$$\|F(u) - F(v)\|_{\tilde{X}_{s,b'}^{r,\pm}} \leq \|u_+ - v_+\|_{\tilde{X}_{s,b}^{r,+}} + \|u_- - v_-\|_{\tilde{X}_{s,b}^{r,-}}.$$

Thus, we already have the estimates needed for F in the non-derivative part of the norm.

It remains to establish the corresponding estimates for N , which are given by (1.5.6) and (1.5.7). But note that

$$\begin{aligned}
\|N(u)\|_{\tilde{X}_{s,b'}^{r,\pm}} &= \|J^{-1}B_2(u, u)\|_{\tilde{X}_{s,b'}^{r,\pm}} \\
&= \|B_2(u, u)\|_{\tilde{X}_{s-1,b'}^{r,\pm}}
\end{aligned}$$

so it is enough to show

$$\|B_2(u, u)\|_{\tilde{X}_{s-1,b'}^{r,\pm}} \lesssim \left(\|u_+\|_{\tilde{Z}_{s,b}^{r,+}} + \|u_-\|_{\tilde{Z}_{s,b}^{r,-}} \right)^2$$

and similarly for the contraction.

• **Estimates for $\partial_t \bar{\Lambda}$.** Now for the derivative piece of the $\tilde{Z}_{s,b}^{r,\pm}$ norm, we have

$$\|\partial_t \bar{\Lambda} u_{\pm}\|_{\tilde{X}_{s-1,b}^{r,\pm}(\delta)} \leq \|\partial_t(\psi \mathcal{U}_{\phi} f_{\pm})\|_{\tilde{X}_{s-1,b}^{r,\pm}} + \|\partial_t(\psi_{\delta} \mathcal{U}_{\phi^* R} F(\bar{u}))\|_{\tilde{X}_{s-1,b}^{r,\pm}} + \|\partial_t(\psi_{\delta} \mathcal{U}_{\phi^* R} N(\bar{u}))\|_{\tilde{X}_{s,b}^{r,\pm}}.$$

Using the relations (1.5.11) - (1.5.13) from above, for a suitable function h , we see that

$$\begin{aligned}
\|\partial_t h\|_{\tilde{X}_{s-1,b}^{r,\pm}} &= \|\langle \xi \rangle^{s-1} \langle \tau \pm \langle \xi \rangle \rangle^b |\tau| \tilde{h}\|_{r'} \\
&\leq \|\langle \xi \rangle^{s-1} \langle \tau \pm \langle \xi \rangle \rangle^b |\xi| \tilde{h}\|_{r'} + \|\langle \xi \rangle^{s-1} \langle \tau \pm \langle \xi \rangle \rangle^b \|\tau| - |\xi| \tilde{h}\|_{r'} \\
&\lesssim \|\langle \xi \rangle^s \langle \tau \pm \langle \xi \rangle \rangle^b \tilde{h}\|_{r'} + \|\langle \xi \rangle^{s-1} \langle \tau \pm \langle \xi \rangle \rangle^{b+1} \tilde{h}\|_{r'}.
\end{aligned}$$

So, if ψ is a smooth cutoff and $f_{\pm} \in \widehat{H}_s^r$, we can use Lemma 1.6 to obtain

$$\begin{aligned}
\|\partial_t(\psi \mathcal{U}_{\phi} f_{\pm})\|_{\tilde{X}_{s-1,b}^{r,\pm}} &\lesssim \|\psi \mathcal{U}_{\phi} f_{\pm}\|_{\tilde{X}_{s,b}^{r,\pm}} + \|\psi \mathcal{U}_{\phi} f_{\pm}\|_{\tilde{X}_{s-1,b+1}^{r,\pm}} \\
&\lesssim \|f_{\pm}\|_{\widehat{H}_s^r} + \|f_{\pm}\|_{\widehat{H}_{s-1}^r} \\
&\lesssim \|f_{\pm}\|_{\widehat{H}_s^r}.
\end{aligned}$$

Next for F , we have

$$\begin{aligned}
\|\partial_t(\psi_{\delta} \mathcal{U}_{\phi^* R} F(\bar{u}))\|_{\tilde{X}_{s-1,b}^{r,\pm}} &\leq \|\partial_t(\psi_{\delta}) \mathcal{U}_{\phi^* R} F(\bar{u})\|_{\tilde{X}_{s-1,b}^{r,\pm}} + \|\psi_{\delta} \mathcal{U}_{\phi}(t) F(\bar{u}(0, x))\|_{\tilde{X}_{s-1,b}^{r,\pm}} \\
&\quad + \left\| \psi_{\delta} \int_0^t \partial_t(\mathcal{U}_{\phi}(s) F(\bar{u}(t-s, x))) ds \right\|_{\tilde{X}_{s-1,b}^{r,\pm}}.
\end{aligned}$$

The first term can be handled as in Lemma 1.10 to get

$$\|\partial_t(\psi_{\delta}) \mathcal{U}_{\phi^* R} F(\bar{u})\|_{\tilde{X}_{s-1,b}^{r,\pm}} \lesssim C \delta^{b'-b+1} \|F(\bar{u})\|_{\tilde{X}_{s,b'}^{r,\pm}}.$$

Notice also that the second term will disappear inside the norm when doing the contraction argument, so it suffices to prove this is bounded. We have

$$\begin{aligned}
\|\psi_{\delta} \mathcal{U}_{\phi}(t) F(\bar{u}(0, x))\|_{\tilde{X}_{s-1,b}^{r,\pm}} &= \|\langle \xi \rangle^{s-2} \langle \tau \pm \langle \xi \rangle \rangle^b \widehat{f}_{\pm}(\xi) \int \psi_{\delta}(t) e^{-it\tau \mp it\langle \xi \rangle} dt\|_{r'} \\
&= \|\langle \xi \rangle^{s-2} \langle \tau \rangle^b \widehat{f}_{\pm}(\xi) \int \psi_{\delta}(t) e^{-it\tau} dt\|_{r'} \\
&= \|\widehat{f}_{\pm}\|_{\widehat{H}_{s-2}^r} \|\psi_{\delta}\|_{\widehat{H}_b^r}.
\end{aligned}$$

Now write

$$\begin{aligned}
\left\| \psi_\delta \int_0^t \partial_t (\mathcal{U}_\phi(s) F(\bar{u}(t-s, \cdot))) ds \right\|_{\tilde{X}_{s-1,b}^{r,\pm}} &= \left\| \psi_\delta \int_0^t \mathcal{U}_\phi(s) F(\partial_t \bar{u}(t-s, \cdot)) ds \right\|_{\tilde{X}_{s-1,b}^{r,\pm}} \\
&= \|\psi_\delta \mathcal{U}_{\phi^* R} F(\partial_t \bar{u})\|_{\tilde{X}_{s-1,b}^{r,\pm}} \\
&\lesssim \delta^{b'-b+1} \|F(u)\|_{\tilde{X}_{s-1,b'}^{r,\pm}} \\
&\lesssim \delta^{b'-b+1} \left(\|\partial_t u_+\|_{\tilde{X}_{s-1,b}^{r,+}} + \|\partial_t u_-\|_{\tilde{X}_{s-1,b}^{r,-}} \right)
\end{aligned}$$

as above. Thus, we have all the estimates we need for J . Next we examine

$$\|\partial_t(\psi_\delta \mathcal{U}_\phi N(u))\|_{\tilde{X}_{s-1,b}^{r,\pm}},$$

which as above reduces to three estimates, the first of which is handled exactly as in the case for F . For the second,

$$\begin{aligned}
\|\psi_\delta \mathcal{U}_\phi(t) N(\bar{u}(0, x))\|_{\tilde{X}_{s-1,b}^{r,\pm}} &\leq \|\psi_\delta N(f_\pm)\|_{\tilde{X}_{s-1,b}^{r,\pm}} \\
&\leq \|\psi_\delta B_2(f_\pm, f_\pm)\|_{\tilde{X}_{s-2,b}^{r,\pm}} \\
&\leq \|\psi_\delta B_2(f_\pm, f_\pm)\|_{\tilde{X}_{s-1,b}^{r,\pm}}.
\end{aligned}$$

Finally, as for F , we also have for N ,

$$\begin{aligned}
\left\| \psi_\delta \int_0^t \partial_t (\mathcal{U}_\phi(s) N(\bar{u}(t-s, \cdot))) ds \right\|_{\tilde{X}_{s-1,b}^{r,\pm}} &= \left\| \psi_\delta \int_0^t \mathcal{U}_\phi(s) \partial_t (N(\bar{u}(t-s, \cdot))) ds \right\|_{\tilde{X}_{s-1,b}^{r,\pm}} \\
&= \|\psi_\delta \mathcal{U}_{\phi^* R} \partial_t B_2(\bar{u}, \bar{u})\|_{\tilde{X}_{s-2,b}^{r,\pm}} \\
&\lesssim \delta^{b'-b+1} \|\partial_t B_2(\bar{u}, \bar{u})\|_{\tilde{X}_{s-2,b'}^{r,\pm}}.
\end{aligned}$$

All in all, for N it suffices to show

$$\|\partial_t B_2(u, u)\|_{\tilde{X}_{s-2,b'}^{r,\pm}} \lesssim \left(\|u_+\|_{\tilde{Z}_{s,b}^{r,+}} + \|u_-\|_{\tilde{Z}_{s,b}^{r,-}} \right)^2$$

and

$$\|B_2(u, u)\|_{\tilde{X}_{s-1,b'}^{r,\pm}} \lesssim \left(\|u_+\|_{\tilde{Z}_{s,b}^{r,+}} + \|u_-\|_{\tilde{Z}_{s,b}^{r,-}} \right)^2$$

which both follow from

$$\|B_2(u, u)\|_{\tilde{Z}_{s-1, b'}^{r, \pm}} \lesssim \left(\|u_+\|_{\tilde{Z}_{s, b}^{r, +}} + \|u_-\|_{\tilde{Z}_{s, b}^{r, -}} \right)^2. \quad (1.5.14)$$

For the contraction, we will then need to bound $\|B_2(u, u) - B_2(v, v)\|_{\tilde{Z}_{s-1, b'}^{r, \pm}}$ by a constant times

$$\left(\|(u_+, u_-)\|_{\tilde{Z}_{s, b}^{r, +} \times \tilde{Z}_{s, b}^{r, -}} + \|(v_+, v_-)\|_{\tilde{Z}_{s, b}^{r, +} \times \tilde{Z}_{s, b}^{r, -}} \right) \left(\|u_+ - v_+\|_{\tilde{Z}_{s, b}^{r, +}} + \|u_- - v_-\|_{\tilde{Z}_{s, b}^{r, -}} \right).$$

But observe that

$$\begin{aligned} \|B_2(u, u) - B_2(v, v)\|_{\tilde{Z}_{s-1, b'}^{r, \pm}} &= \|(\partial u)^2 - (\partial v)^2\|_{\tilde{Z}_{s-1, b'}^{r, \pm}} \\ &= \|\partial(u+v)\partial(u-v)\|_{\tilde{Z}_{s-1, b'}^{r, \pm}} \end{aligned} \quad (1.5.15)$$

and the last term could be bounded by

$$\|u+v\|_{\tilde{Z}_{s-1, b'}^{r, \pm}} \|u-v\|_{\tilde{Z}_{s-1, b'}^{r, \pm}} \lesssim \left(\|u\|_{\tilde{Z}_{s, b'}^{r, \pm}} + \|v\|_{\tilde{Z}_{s, b'}^{r, \pm}} \right) \|u-v\|_{\tilde{Z}_{s, b'}^{r, \pm}}$$

provided we had

$$\|B_2(u, v)\|_{\tilde{Z}_{s-1, b'}^{r, \pm}} \lesssim \left(\|u\|_{\tilde{Z}_{s, b}^{r, +} \times \tilde{Z}_{s, b}^{r, -}} + \|v\|_{\tilde{Z}_{s, b}^{r, +} \times \tilde{Z}_{s, b}^{r, -}} \right)^2$$

for general functions u and v .

Reduction to $Z_{s, b}^r$ estimates.

We now show that we can reduce these $\tilde{Z}_{s, b}^{r, \pm}$ estimates to $Z_{s, b}^r$ estimates. Since $\langle \tau \pm \xi \rangle \sim \langle \tau \pm |\xi| \rangle$, the estimate (1.5.14) is equivalent to

$$\begin{aligned} &\|\langle \xi \rangle^{s-1} \langle \tau \pm |\xi| \rangle^{b'} \mathcal{F}(B_2(u, u))\|_{r'} + \|\langle \xi \rangle^{s-2} \langle \tau \pm |\xi| \rangle^{b'} \mathcal{F}(\partial_t B_2(u, u))\|_{r'} \\ &\lesssim \left(\|\langle \xi \rangle^s \langle \tau + |\xi| \rangle^b \mathcal{F}(u_+)\|_{r'} + \|\langle \xi \rangle^s \langle \tau - |\xi| \rangle^b \mathcal{F}(u_-)\|_{r'} \right. \\ &\quad \left. + \|\langle \xi \rangle^{s-1} \langle \tau + |\xi| \rangle^b \mathcal{F}(\partial_t u_+)\|_{r'} + \|\langle \xi \rangle^{s-1} \langle \tau - |\xi| \rangle^b \mathcal{F}(\partial_t u_-)\|_{r'} \right)^2. \end{aligned}$$

Next we use the inequality $\|\tau\| - |\xi| \leq |\tau \pm |\xi||$ to recover $X_{s, b}^r$ norms. Since $b' < 0$,

$$\|\langle \xi \rangle^{s-1} \langle \tau \pm |\xi| \rangle^{b'} \mathcal{F}(B_2(u, u))\|_{r'} + \|\langle \xi \rangle^{s-2} \langle \tau \pm |\xi| \rangle^{b'} \mathcal{F}(\partial_t B_2(u, u))\|_{r'} \lesssim \|B_2(u, u)\|_{Z_{s-1, b'}^r}.$$

Similarly, since $b > 0$,

$$\begin{aligned} \|u\|_{Z_{s,b}^r}^2 &\lesssim \left(\|u_+\|_{Z_{s,b}^r} + \|u_-\|_{Z_{s,b}^r} \right)^2 \\ &\lesssim \left(\|\langle \xi \rangle^s \langle \tau + |\xi| \rangle^b \mathcal{F}(u_+)\|_{r'} + \|\langle \xi \rangle^s \langle \tau - |\xi| \rangle^b \mathcal{F}(u_+)\|_{r'} \right. \\ &\quad \left. + \|\langle \xi \rangle^{s-1} \langle \tau + |\xi| \rangle^b \mathcal{F}(\partial_t u_+)\|_{r'} + \|\langle \xi \rangle^{s-1} \langle \tau - |\xi| \rangle^b \mathcal{F}(\partial_t u_+)\|_{r'} \right)^2. \end{aligned}$$

So, (1.5.14) will follow if we can show

$$\|B_2(u, u)\|_{Z_{s-1,b'}^r} \lesssim \|u\|_{Z_{s,b}^r}^2 \quad (1.5.16)$$

for appropriate values of s, b, b' and r .

Reduction to the main estimates.

We can reduce (1.5.16) to several estimates for $B_2(u, u) = \partial u \partial u$, depending on whether $\partial = \partial_t$ or $\partial = \partial_x$ where $x = x_i$ for $i = 1$ or 2 . All of our estimates will be on the Fourier side, and so the trivial inequality $|ab| \leq \frac{1}{2}(a^2 + b^2)$ applied to the symbols of the derivatives reduces any mixed derivative product to the cases $F(u) = (\partial_t u)^2$ and $F(u) = (\partial_x u)^2$. More generally, let us consider estimates of the type

$$\|\partial u \partial v\|_{Z_{s-1,b'}^r} \lesssim \|u\|_{Z_{s,b}^r} \|v\|_{Z_{s,b}^r} \quad (1.5.17)$$

from which the contraction will also follow, as described above.

Since $b' \in (b-1, 0]$, write $b' = b-1 + \epsilon$ where $0 < \epsilon \leq 1$. Then we want

$$\|\partial u \partial v\|_{Z_{s-1,b-1+\epsilon}^r} \lesssim \|u\|_{Z_{s,b}^r} \|v\|_{Z_{s,b}^r}. \quad (1.5.18)$$

Explicitly, we need

$$\begin{aligned} &\|\partial u \partial v\|_{X_{s-1,b-1+\epsilon}^r} + \|\partial_t(\partial u \partial v)\|_{X_{s-2,b-1+\epsilon}^r} \\ &\lesssim \left(\|u\|_{X_{s,b}^r} + \|\partial_t u\|_{X_{s-1,b}^r} \right) \left(\|v\|_{X_{s,b}^r} + \|\partial_t v\|_{X_{s-1,b}^r} \right) \\ &= \|u\|_{X_{s,b}^r} \|v\|_{X_{s,b}^r} + \|\partial_t u\|_{X_{s-1,b}^r} \|\partial_t v\|_{X_{s-1,b}^r} + \|u\|_{X_{s,b}^r} \|\partial_t v\|_{X_{s-1,b}^r} + \|v\|_{X_{s,b}^r} \|\partial_t u\|_{X_{s-1,b}^r}. \end{aligned}$$

Thus, it will be enough to obtain the estimates

$$\|\partial_t u \partial_t v\|_{X_{s-1,b-1+\epsilon}^r} \lesssim \|\partial_t u\|_{X_{s-1,b}^r} \|\partial_t v\|_{X_{s-1,b}^r}, \quad (1.5.19)$$

$$\|\partial_x u \partial_x v\|_{X_{s-1, b-1+\epsilon}^r} \lesssim \|u\|_{X_{s,b}^r} \|v\|_{X_{s,b}^r}, \quad (1.5.20)$$

$$\|\partial_t(\partial_t u \partial_t v)\|_{X_{s-2, b-1+\epsilon}^r} \lesssim \|\partial_t u\|_{X_{s-1, b}^r} \|\partial_t v\|_{X_{s-1, b}^r}, \quad (1.5.21)$$

and

$$\|\partial_t(\partial_x u \partial_x v)\|_{X_{s-2, b-1+\epsilon}^r} \lesssim \|u\|_{X_{s,b}^r} \|v\|_{X_{s,b}^r}. \quad (1.5.22)$$

For (1.5.20), note that

$$\begin{aligned} \|\partial_{x_i} u\|_{X_{s-1, b}^r} &= \left\| \langle \xi \rangle^{s-1} \langle |\tau| - |\xi| \rangle^b \widetilde{\partial_{x_i} u}(\tau, \xi) \right\|_{L_{\tau, \xi}^{r'}} \\ &= \left\| \langle \xi \rangle^{s-1} \langle |\tau| - |\xi| \rangle^b |\xi_i| \widetilde{u}(\tau, \xi) \right\|_{L_{\tau, \xi}^{r'}} \\ &\leq \left\| \langle \xi \rangle^s \langle |\tau| - |\xi| \rangle^b \widetilde{u}(\tau, \xi) \right\|_{L_{\tau, \xi}^{r'}} \\ &= \|u\|_{X_{s,b}^r} \end{aligned}$$

and so it suffices to show

$$\|\partial_x u \partial_x v\|_{X_{s-1, b-1+\epsilon}^r} \lesssim \|\partial_x u\|_{X_{s-1, b}^r} \|\partial_x v\|_{X_{s-1, b}^r}.$$

Furthermore, since we assume $b - 1 + \epsilon < 0$, i.e. $b < 1$, then $X_{s,0}^r \subset X_{s, b-1+\epsilon}^r$ and

$$\|\partial_x u \partial_x v\|_{X_{s-1, b-1+\epsilon}^r} \lesssim \|\partial_x u \partial_x v\|_{X_{s-1, 0}^r}.$$

So (1.5.20) will follow if

$$\|\partial_x u \partial_x v\|_{X_{s-1, 0}^r} \lesssim \|\partial_x u\|_{X_{s-1, b}^r} \|\partial_x v\|_{X_{s-1, b}^r}. \quad (1.5.23)$$

Replacing now $f = \partial_x u$ and $g = \partial_x v$, this becomes

$$\|fg\|_{X_{s-1, 0}^r} \lesssim \|f\|_{X_{s-1, b}^r} \|g\|_{X_{s-1, b}^r}. \quad (1.5.24)$$

For (1.5.21) and (1.5.22) write

$$\begin{aligned} \|\partial_t(\partial u \partial v)\|_{X_{s-2, b-1+\epsilon}^r} &= \left\| \langle \xi \rangle^{s-2} \langle |\tau| - |\xi| \rangle^{b-1+\epsilon} \mathcal{F}(\partial_t(\partial u \partial v))(\tau, \xi) \right\|_{L_{\tau, \xi}^{r'}} \\ &= \left\| \langle \xi \rangle^{s-2} \langle |\tau| - |\xi| \rangle^{b-1+\epsilon} \tau \mathcal{F}(\partial u \partial v)(\tau, \xi) \right\|_{L_{\tau, \xi}^{r'}}. \end{aligned}$$

Now using that $|\tau| - |\xi| \leq \|\tau\| - \|\xi\|$, we have

$$\begin{aligned}
\|\partial_t(\partial u \partial v)\|_{X_{s-2, b-1+\epsilon}^r} &\leq \left\| \langle \xi \rangle^{s-2} \langle |\tau| - |\xi| \rangle^{b-1+\epsilon} (\|\tau\| - \|\xi\| + |\xi|) \mathcal{F}(\partial u \partial v)(\tau, \xi) \right\|_{L_{\tau, \xi}^{r'}} \\
&\leq \left\| \langle \xi \rangle^{s-2} \langle |\tau| - |\xi| \rangle^{b+\epsilon} \mathcal{F}(\partial u \partial v)(\tau, \xi) \right\|_{L_{\tau, \xi}^{r'}} + \left\| \langle \xi \rangle^{s-1} \langle |\tau| - |\xi| \rangle^{b-1+\epsilon} \mathcal{F}(\partial u \partial v)(\tau, \xi) \right\|_{L_{\tau, \xi}^{r'}} \\
&= \|\partial u \partial v\|_{X_{s-2, b+\epsilon}^r} + \|\partial u \partial v\|_{X_{s-1, b-1+\epsilon}^r}.
\end{aligned}$$

If $\partial = \partial_t$, (1.5.21) will follow if

$$\|\partial u \partial v\|_{X_{s-2, b+\epsilon}^r} \lesssim \|\partial_t u\|_{X_{s-1, b}^r} \|\partial_t v\|_{X_{s-1, b}^r} \quad (1.5.25)$$

and

$$\|\partial u \partial v\|_{X_{s-1, b-1+\epsilon}^r} \lesssim \|\partial_t u\|_{X_{s-1, b}^r} \|\partial_t v\|_{X_{s-1, b}^r}. \quad (1.5.26)$$

The latter is exactly (1.5.19) and follows as above from

$$\|\partial_t u \partial_t v\|_{X_{s-1, 0}^r} \lesssim \|\partial_t u\|_{X_{s-1, b}^r} \|\partial_t v\|_{X_{s-1, b}^r}$$

since $b-1+\epsilon < 0$. Then, as with (1.5.23), this reduces to (1.5.24). Also in this case, the first estimate reads

$$\|\partial_t u \partial_t v\|_{X_{s-2, b+\epsilon}^r} \lesssim \|\partial_t u\|_{X_{s-1, b}^r} \|\partial_t v\|_{X_{s-1, b}^r}$$

and so setting $f = \partial_t u$ and $g = \partial_t v$, it is implied by

$$\|fg\|_{X_{s-2, b+\epsilon}^r} \lesssim \|f\|_{X_{s-1, b}^r} \|g\|_{X_{s-1, b}^r}. \quad (1.5.27)$$

If $\partial = \partial_x$, (1.5.22) follows from

$$\|\partial_x u \partial_x v\|_{X_{s-2, b+\epsilon}^r} \lesssim \|u\|_{X_{s, b}^r} \|v\|_{X_{s, b}^r}$$

and

$$\|\partial_x u \partial_x v\|_{X_{s-1, b-1+\epsilon}^r} \lesssim \|u\|_{X_{s, b}^r} \|v\|_{X_{s, b}^r}.$$

This second estimate is already (1.5.20), which reduces to (1.5.24). For the first estimate, we argue as above to replace it with the estimate

$$\|\partial_x u \partial_x v\|_{X_{s-2, b+\epsilon}^r} \lesssim \|\partial_x u\|_{X_{s-1, b}^r} \|\partial_x v\|_{X_{s-1, b}^r}$$

which is implied also by (1.5.27).

All in all, we expect local well-posedness for (1.0.1) with initial data in $\widehat{H}_s^r \times \widehat{H}_{s-1}^r$, provided we have the following estimates

$$\|uv\|_{X_{s-1,0}^r} \lesssim \|u\|_{X_{s-1,b}^r} \|v\|_{X_{s-1,b}^r} \quad (1.5.28)$$

and

$$\|uv\|_{X_{s-2,b+\epsilon}^r} \lesssim \|u\|_{X_{s-1,b}^r} \|v\|_{X_{s-1,b}^r} \quad (1.5.29)$$

for some r, s , and $\frac{1}{r} < b < 1$. In Chapter 3, we show that these hold for $s > \frac{3}{2r} + 1$ and $\frac{3}{2} < r \leq 2$.

CHAPTER 2

BILINEAR FOURIER RESTRICTION ESTIMATES IN L^p SPACES

In this chapter, we develop the tools we will use to prove bilinear estimates of the type

$$\|uv\|_{X_{-s_0, -b_0}^r} \leq C \|u\|_{X_{s_1, b_1}^r} \|v\|_{X_{s_2, b_2}^r} \quad (2.0.1)$$

needed for Theorem 1.9. We begin by reducing to trilinear L^p integral estimates over domains with restricted spatial frequency interactions. Next we decompose dyadically, based on spatial frequency and distance from the light cone, and prove two dyadic summation lemmas. Finally, in Section 2.3, we reduce to proving bilinear estimates restricted in Fourier space to thickened subsets of the light cone. The constants obtained in these estimates allow us to sum the results of our dyadic decompositions.

In the following, we keep our notation consistent with that in [6]. We will use the notation $\|\cdot\|_p$ for the L^p norm on \mathbb{R}^{1+2} , i.e. for $f = f(t, x)$, we write

$$\|f\|_p = \left(\int_{\mathbb{R}^{1+2}} |f(t, x)|^p d(t, x) \right)^{1/p}.$$

Occasionally, when we want to emphasize the variables of integration, we will also write this norm as $\|f\|_p = \|f\|_{L_{t,x}^p}$. For $p \geq 1$, we denote the conjugate exponent by p' , i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. For $1 < r \leq 2$ we write $\|\cdot\|_{X_{s,b}^r}$ for the norm in the wave-Fourier Lebesgue space, $X_{s,b}^r$. That is,

$$\|u\|_{X_{s,b}^r} = \|\langle \xi \rangle^s \langle |\tau| - |\xi| \rangle^b \tilde{u}(\tau, \xi)\|_{L_{\tau,\xi}^r}$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$ and $\tilde{u}(\tau, \xi) = \int \int e^{-i(t\tau + x \cdot \xi)} u(t, x) dt dx$.

2.1 Reformulation as a trilinear integral estimate

First we show that (2.0.1) is equivalent to

$$|I| \lesssim \|F_0\|_r \|F_1\|_{r'} \|F_2\|_{r'} \quad (2.1.1)$$

where

$$I = \iiint \frac{F_0(X_0)F_1(X_1)F_2(X_2)\delta(X_0 + X_1 + X_2)dX_0dX_1dX_2}{\langle \xi_0 \rangle^{s_0} \langle \xi_1 \rangle^{s_1} \langle \xi_2 \rangle^{s_2} \langle |\tau_0| - |\xi_0| \rangle^{b_0} \langle |\tau_1| - |\xi_1| \rangle^{b_1} \langle |\tau_2| - |\xi_2| \rangle^{b_2}}, \quad (2.1.2)$$

$X_j = (\tau_j, \xi_j) \in \mathbb{R}^{1+2}$ for $j = 0, 1, 2$ and δ is point mass at the origin in \mathbb{R}^{1+n} . To see this,

let $w_i(\tau, \xi) = \langle \xi_i \rangle^{s_i} \langle |\tau_i| - |\xi_i| \rangle^{b_i}$. Then

$$\begin{aligned} \|uv\|_{X_{-s_0, -b_0}^r} &= \left(\int w_0(\tau, \xi)^{-r'} |\widetilde{uv}(\tau, \xi)|^{r'} d(\tau, \xi) \right)^{1/r'} \\ &= \left(\int w_0(\tau, \xi)^{-r'} |\widetilde{u} * \widetilde{v}(\tau, \xi)|^{r'} d(\tau, \xi) \right)^{1/r'} \\ &= \left(\int w_0(\tau, \xi)^{-r'} \left| \int \widetilde{u}(t, x) \widetilde{v}(\tau - t, \xi - x) d(t, x) \right|^{r'} d(\tau, \xi) \right)^{1/r'} \\ &= \left\| w_0^{-1} \int \widetilde{u}(t, x) \widetilde{v}(\cdot - t, \cdot - x) d(t, x) \right\|_{L_{\tau\xi}^{r'}}. \end{aligned}$$

Now let $\alpha = \tau - t$ and $\beta = \xi - x$. Then

$$\begin{aligned} \|uv\|_{X_{-s_0, -b_0}^r} &= \left\| w_0^{-1}(\tau, \xi) \iint \widetilde{u}(t, x) \widetilde{v}(\alpha, \beta) \delta(\alpha + t - \tau, \beta + x - \xi) d(\alpha, \beta) d(t, x) \right\|_{L_{\tau\xi}^{r'}} \\ &= \sup_{f \in L^r} \left| \int \frac{f(-\tau, -\xi)}{\|f\|_r w_0(\tau, \xi)} \iint \widetilde{u}(t, x) \widetilde{v}(\alpha, \beta) \delta(\alpha + t - \tau, \beta + x - \xi) d(\alpha, \beta) d(t, x) d(\tau, \xi) \right| \\ &= \sup_{F_0 \in L^r} \iiint \frac{F_0(X_0)F_1(X_1)F_2(X_2)}{\|F_0\|_r w_0(X_0)w_1(X_1)w_2(X_2)} \delta(X_2 + X_1 + X_0) dX_2 dX_1 dX_0 \end{aligned}$$

where $F_0 = f$, $F_1 = \widetilde{u}w_1$, $F_2 = \widetilde{v}w_2$, $X_0 = (-\tau, -\xi)$, $X_1 = (t, x)$, and $X_2 = (\alpha, \beta)$.

Now if (2.1.1) holds, then

$$\begin{aligned} \|uv\|_{X_{-s_0, -b_0}^r} &\lesssim \sup_{F_0 \in L^2} \frac{\|F_0\|_r \|F_1\|_{r'} \|F_2\|_{r'}}{\|F_0\|_r} \\ &\lesssim \|F_1\|_{r'} \|F_2\|_{r'} \\ &\lesssim \|\widetilde{u}w_1\|_{r'} \|\widetilde{v}w_2\|_{r'} \\ &\lesssim \|u\|_{X_{s_1, b_1}^r} \|v\|_{X_{s_2, b_2}^r} \end{aligned}$$

which is (2.0.1). And if (2.0.1) holds, then

$$\begin{aligned} \sup_{F_0 \in L^r} \frac{|I|}{\|F_0\|_r} &\lesssim \|u\|_{X_{s_1, b_1}^r} \|v\|_{X_{s_2, b_2}^r} \\ &\lesssim \|F_1\|_{r'} \|F_2\|_{r'} \end{aligned}$$

which implies (2.1.1). Without loss of generality, we may assume $F_j \geq 0$ for $j = 0, 1, 2$ and so $I \geq 0$.

Notice that $X_0 + X_1 + X_2 = 0$ in I implies $\xi_0 + \xi_1 + \xi_2 = 0$. From this it follows that $\langle \xi_j \rangle \lesssim \langle \xi_k \rangle + \langle \xi_l \rangle$ for any $j, k, l \in \{0, 1, 2\}$. Indeed, we have

$$\langle \xi_j \rangle = (1 + |\xi_j|^2)^{1/2} = (1 + |\xi_k + \xi_l|^2)^{1/2}.$$

Without loss of generality, assume $|\xi_k| = \max(|\xi_k|, |\xi_l|)$. Then

$$\begin{aligned} \langle \xi_j \rangle &= (1 + |\xi_k + \xi_l|^2)^{1/2} \\ &= (1 + |\xi_k|^2 + 2|\xi_k \cdot \xi_l| + |\xi_l|^2)^{1/2} \\ &\leq (1 + |\xi_k|^2 + 2|\xi_k||\xi_l| + |\xi_l|^2)^{1/2} \\ &\leq (1 + 3|\xi_k|^2 + |\xi_l|^2)^{1/2} \\ &\leq (6 + 4|\xi_k|^2 + 4|\xi_l|^2)^{1/2} \\ &\leq 2((1 + |\xi_k|^2) + (1 + |\xi_l|^2))^{1/2} \\ &\leq 2(\langle \xi_k \rangle + \langle \xi_l \rangle) \end{aligned}$$

for any j, k, l . Now suppose that $\langle \xi_l \rangle = \min_i \langle \xi_i \rangle$. Then

$$\begin{aligned} \langle \xi_j \rangle &\lesssim \langle \xi_k \rangle + \langle \xi_l \rangle \\ &\lesssim \langle \xi_k \rangle \\ &\lesssim \langle \xi_j \rangle + \langle \xi_l \rangle \\ &\lesssim \langle \xi_j \rangle. \end{aligned}$$

So, $\langle \xi_j \rangle \sim \langle \xi_k \rangle$. That is, the two largest of the three frequencies are comparable. Then we can split the integral in three pieces

$$I = I_{LHH} + I_{HLH} + I_{HHL}$$

where I_A is the integral I , restricted over the set A ,

$$I_A = \iiint \chi_A \frac{F_0(X_0)F_1(X_1)F_2(X_2)\delta(X_0 + X_1 + X_2)dX_0dX_1dX_2}{\langle \xi_0 \rangle^{s_0} \langle \xi_1 \rangle^{s_1} \langle \xi_2 \rangle^{s_2} \langle |\tau_0| - |\xi_0| \rangle^{b_0} \langle |\tau_1| - |\xi_1| \rangle^{b_1} \langle |\tau_2| - |\xi_2| \rangle^{b_2}}$$

and LHH, HLH, HHL represent the corresponding set of frequency interactions. More explicitly, we have the following frequency regimes.

1. LHH: $\langle \xi_0 \rangle \lesssim \langle \xi_1 \rangle \sim \langle \xi_2 \rangle$
2. HLH: $\langle \xi_1 \rangle \lesssim \langle \xi_0 \rangle \sim \langle \xi_2 \rangle$
3. HHL: $\langle \xi_2 \rangle \lesssim \langle \xi_0 \rangle \sim \langle \xi_1 \rangle$

From the symmetry in the indices 1 and 2, we point out that it is enough to prove results in the LHH and HLH cases.

We will also need the following result from [6, 2.3] for the case, $b_0 < 0 < b_1, b_2$, in Section 3.4.

Hyperbolic Leibniz rule. If $\tau_0 + \tau_1 + \tau_2 = 0$, $\xi_0 + \xi_1 + \xi_2 = 0$, and \pm_1, \pm_2 are the signs of τ_1 and τ_2 , respectively, then

$$\left| |\tau_0| - |\xi_0| \right| \lesssim |-\tau_1 \pm_1 \xi_1| + |-\tau_2 \pm_2 \xi_2| + \mathbf{b}_{(\pm_1, \pm_2)}(\xi_0, \xi_1, \xi_2) \quad (2.1.3)$$

where

$$\mathbf{b}_{(\pm_1, \pm_2)}(\xi_0, \xi_1, \xi_2) = \begin{cases} |\xi_1| + |\xi_2| - |\xi_0| & \text{if } \pm_1 = \pm_2 \\ |\xi_0| - \left| |\xi_1| - |\xi_2| \right| & \text{if } \pm_1 \neq \pm_2. \end{cases}$$

We also have the estimate

$$\mathbf{b}_{(\pm_1, \pm_2)}(\xi_0, \xi_1, \xi_2) \lesssim \begin{cases} \min(|\xi_1|, |\xi_2|) & \text{if } \pm_1 = \pm_2 \\ \min(|\xi_0|, |\xi_1|, |\xi_2|) & \text{if } \pm_1 \neq \pm_2. \end{cases} \quad (2.1.4)$$

Proof. For (2.1.3), first suppose $\pm_1 = \pm_2$. Then

$$\begin{aligned} \left| |\tau_0| - |\xi_0| \right| &= \left| |\tau_1 + \tau_2| - |\xi_0| \right| \\ &= \left| |\tau_1| - |\xi_1| + |\tau_2| - |\xi_2| + |\xi_1| + |\xi_2| - |\xi_0| \right| \\ &\leq \left| |\tau_1| - |\xi_1| \right| + \left| |\tau_2| - |\xi_2| \right| + \left| |\xi_1| + |\xi_2| - |\xi_0| \right| \\ &\lesssim |-\tau_1 \pm_1 \xi_1| + |-\tau_2 \pm_2 \xi_2| + \mathbf{b}_{(\pm_1, \pm_2)}(\xi_0, \xi_1, \xi_2) \end{aligned} \quad (2.1.5)$$

since the two largest frequencies are comparable, which implies $|\xi_1| + |\xi_2| - |\xi_0| \geq 0$. Next if $\pm_1 \neq \pm_2$, without loss of generality, we can assume $\tau_1 \geq 0$ and $\tau_2 \leq 0$. Then

$$\begin{aligned}
||\tau_0| - |\xi_0|| &\leq | -|\tau_0| + ||\xi_1| - |\xi_2|| | + ||\xi_0| - ||\xi_1| - |\xi_2|| | \\
&\lesssim | -\tau_0 + |\xi_1| - |\xi_2| | + |\xi_0| - ||\xi_1| - |\xi_2|| \\
&\lesssim | -\tau_1 + |\xi_1| - \tau_2 - |\xi_2| | + |\xi_0| - ||\xi_1| - |\xi_2|| \\
&\lesssim ||\tau_1| - |\xi_1| | + ||\tau_2| - |\xi_2| | + \mathfrak{b}_{(\pm_1, \pm_2)}(\xi_0, \xi_1, \xi_2). \tag{2.1.6}
\end{aligned}$$

To establish (2.1.4), first suppose $\pm_1 = \pm_2$. Then

$$\begin{aligned}
\mathfrak{b}_{(\pm_1, \pm_2)} &= |\xi_1| + |\xi_2| - |\xi_0| \\
&= |\xi_1| + | -\xi_1 - \xi_0 | - |\xi_0| \\
&\leq |\xi_1| + |\xi_1| + |\xi_0| - |\xi_0| \\
&\leq 2|\xi_1|.
\end{aligned}$$

Similarly, we obtain $\mathfrak{b}_{(\pm_1, \pm_2)} \leq 2|\xi_2|$. Now if $\pm_1 \neq \pm_2$,

$$\begin{aligned}
\mathfrak{b}_{(\pm_1, \pm_2)} &= |\xi_0| - ||\xi_1| - |\xi_2|| \\
&= |\xi_1 + \xi_2| - ||\xi_1| - |\xi_2|| \\
&\lesssim \max |\xi_i| + \min |\xi_i| - (\max |\xi_i| - \min |\xi_i|) \\
&\lesssim 2 \min |\xi_i|.
\end{aligned}$$

□

We also define the associated bilinear operators $\mathfrak{B}_{(\pm_1, \pm_2)}^\alpha$ for $\alpha > 0$ by

$$\mathcal{F} \left\{ \mathfrak{B}_{(\pm_1, \pm_2)}^\alpha(f, g) \right\}(\xi_0) = \iint [\mathfrak{b}_{(\pm_1, \pm_2)}]^\alpha \hat{f}(\xi_1) \hat{g}(\xi_2) \delta(\xi_0 + \xi_1 + \xi_2) d\xi_1 d\xi_2 \tag{2.1.7}$$

for $f, g \in \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{F}f = \hat{f}$ is the Fourier transform and $\mathfrak{b}_{(\pm_1, \pm_2)} = \mathfrak{b}_{(\pm_1, \pm_2)}(\xi_0, \xi_1, \xi_2)$.

2.2 Dyadic decompositions

Throughout, L, M, N , or their indexed counterparts, will denote dyadic numbers of the form 2^j for $j \in \{0, 1, 2, \dots\}$. We will also use L, M, N to denote dyadic pairs or triples, i.e., we write $N = (N_0, N_1, N_2)$ where the N_i are dyadic as above. In this case, we will write $N_{\min}^{012} = \min(N_0, N_1, N_2)$ and similarly for L and other indices.

For a function F , we define the frequency cutoff functions $F^N(X) = \chi_{\langle \xi \rangle \sim N} F(X)$ and $F^{N,L}(X) = \chi_{\langle |\tau| - |\xi| \rangle \sim L} F^N(X)$. Clearly we have

$$\sum_N \|F^N\|_p^p = \sum_N \int_{\langle \xi \rangle \sim N} |F|^p d\xi \geq \|F\|_p^p$$

and there are positive integers α and β such that

$$\begin{aligned} \sum_N \|F^N\|_p^p &= \sum_N \int_{\langle \xi \rangle \sim N} |F|^p d\xi \\ &= \sum_N \int_{2^{-\alpha} N \leq \langle \xi \rangle \leq 2^\beta N} |F|^p d\xi \\ &\leq (\alpha + \beta + 1) \|F\|_p^p. \end{aligned}$$

Thus, we have $\sum_N \|F^N\|_p^p \sim \|F\|_p^p$, and similarly $\sum_L \|F^{N,L}\|_p^p \sim \|F^N\|_p^p$.

Next define the trilinear convolution form

$$J(F_0, F_1, F_2) = \iiint F_0(X_0) F_1(X_1) F_2(X_2) \delta(X_0 + X_1 + X_2) dX_0 dX_1 dX_2. \quad (2.2.1)$$

Then for $N = (N_0, N_1, N_2)$ and $L = (L_0, L_1, L_2)$, we have

$$I \lesssim \sum_{N,L} \frac{J\left(F_0^{N_0, L_0}, F_1^{N_1, L_1}, F_2^{N_2, L_2}\right)}{N_0^{s_0} N_1^{s_1} N_2^{s_2} L_0^{b_0} L_1^{b_1} L_2^{b_2}} \quad (2.2.2)$$

and for the estimates (2.0.1) have reduced to proving

$$\sum_{N,L} \frac{J\left(F_0^{N_0, L_0}, F_1^{N_1, L_1}, F_2^{N_2, L_2}\right)}{N_0^{s_0} N_1^{s_1} N_2^{s_2} L_0^{b_0} L_1^{b_1} L_2^{b_2}} \lesssim \|F_0\|_r \|F_1\|_{r'} \|F_2\|_{r'}. \quad (2.2.3)$$

We will need the following dyadic summation rule for $1 \leq A < B$ and $a \in \mathbb{R}$,

$$\sum_{A \leq L \leq B} L^a \sim \begin{cases} B^a & \text{if } a > 0 \\ \log\left(\frac{B}{A}\right) & \text{if } a = 0 \\ A^a & \text{if } a < 0. \end{cases} \quad (2.2.4)$$

To see this, write

$$\begin{aligned} \sum_{A \leq L \leq B} L^a &= \sum_{A \leq 2^j \leq B} (2^j)^a \\ &\sim \int_{\log(A)}^{\log(B)} (2^x)^a dx \\ &\sim \begin{cases} \log\left(\frac{B}{A}\right) & \text{if } a = 0 \\ \frac{1}{a}(B^a - A^a) & \text{if } a \neq 0. \end{cases} \end{aligned}$$

From this we derive the following lemma.

Lemma 2.1. *Let $A, B \in \mathbb{R}$. Then*

$$\sum_{N_0 \lesssim N_1} N_0^A \lesssim N_1^B \tag{2.2.5}$$

provided

- i. $B \geq A$*
- ii. $B \geq 0$*
- iii. we exclude $A = B = 0$.*

Proof. From (2.2.4) we have

$$\sum_{N_0 \lesssim N_1} N_0^A \sim \begin{cases} N_1^A & \text{if } A > 0 \\ \log(N_1) & \text{if } A = 0 \\ 1 & \text{if } A < 0. \end{cases}$$

Now the result follows from (i)-(iii)

1. If $A > 0$: $N_1^A \leq N_1^B$ since $A \leq B$ and $N_1 \geq 1$.
2. If $A = 0$: From ii and iii, $B > 0$ and hence $\log(N_1) \lesssim N_1^B$.
3. If $A < 0$: $1 \leq N_1^B$ since $B \geq 0$ and $N_1 \geq 1$.

□

Now we prove two lemmas that will be used repeatedly in establishing the estimates (2.2.3). The first lemma is an extension of [6, Lemma 2.1] to $r > 1$. The main difference in the proof is that instead of using Cauchy-Schwartz in ℓ^2 , we use Hölder's inequality in the dual spaces ℓ^r and $\ell^{r'}$. Since ℓ^r is not self-dual for $r \neq 2$, this approach only works in the HLH and HHL cases, due to the structure of the estimates. For the LHH case, we will need the second lemma. The proof is similar, but requires strict inequalities in the hypotheses to use Hölder.

Lemma 2.2. *Let $r > 1$ and $A, B \in \mathbb{R}$. The estimate*

$$\sum_N \chi_{N_1 \leq N_0 \sim N_2} \frac{N_1^A}{N_0^B} \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'} \lesssim \|F_0\|_r \|F_1\|_{r'} \|F_2\|_{r'}$$

holds provided that

(i) $B \geq A$

(ii) $B \geq 0$

(iii) we exclude $A = B = 0$.

Proof. Let $S = \sum_N \chi_{N_1 \leq N_0 \sim N_2} \frac{N_1^A}{N_0^B} \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'}$. Then

$$\begin{aligned} S &\leq \sum_{N_0, N_2} \chi_{N_0 \sim N_2} \sum_{N_1} \chi_{N_1 \leq N_0} \frac{N_1^A}{N_0^B} \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'} \\ &\leq \|F_1\|_{r'} \sum_{N_0, N_2} \chi_{N_0 \sim N_2} \frac{\Sigma_A(N_0)}{N_0^B} \|F_0^{N_0}\|_r \|F_2^{N_2}\|_{r'} \end{aligned}$$

where

$$\Sigma_A(N_0) = \sum_{N_1} \chi_{N_1 \leq N_0} N_1^A \lesssim N_0^B \tag{2.2.6}$$

by Lemma 2.1. Thus, we have $S \lesssim \|F_1\|_{r'} \sum_{N_0, N_2} \chi_{N_0 \sim N_2} \|F_0^{N_0}\|_r \|F_2^{N_2}\|_{r'}$.

Now since $N_0 \sim N_2$, there are positive integers α, β such that for any N_0 and N_2 ,

$$2^{-\alpha} N_0 \leq N_2 \leq 2^\beta N_0.$$

Hence, for each fixed N_0 there are $\sim \alpha + \beta + 1$ terms in the sum over N_2 . Then by Hölder,

$$\begin{aligned}
\sum_{N_0, N_2} \chi_{N_0 \sim N_2} \|F_0^{N_0}\|_r \|F_2^{N_2}\|_{r'} &= \sum_{N_0} \sum_{N_2=2^{-\alpha} N_0}^{2^\beta N_0} \|F_0^{N_0}\|_r \|F_2^{N_2}\|_{r'} \\
&= \sum_{i=-\alpha}^{\beta} \sum_{N_0} \|F_0^{N_0}\|_r \|F_2^{2^i N_0}\|_{r'} \\
&\lesssim (\alpha + \beta + 1) \sum_{N_0} \|F_0^{N_0}\|_r \|F_2^{N_0}\|_{r'} \\
&\lesssim \left(\sum_{N_0} \|F_0^{N_0}\|_r^r \right)^{1/r} \left(\sum_{N_0} \|F_2^{N_0}\|_{r'}^{r'} \right)^{1/r'} \\
&\lesssim \|F_0\|_r \|F_2\|_{r'}.
\end{aligned}$$

□

Lemma 2.3. *Let $A, B \in \mathbb{R}$. The estimate*

$$\sum_N \chi_{N_0 \leq N_1 \sim N_2} \frac{N_0^A}{N_1^B} \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'} \lesssim \|F_0\|_r \|F_1\|_{r'} \|F_2\|_{r'}$$

holds provided that

(i) $B > A$

(ii) $B > 0$.

Proof. Let

$$\begin{aligned}
S &= \sum_N \chi_{N_0 \leq N_1 \sim N_2} \frac{N_0^A}{N_1^B} \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'} \\
&\leq \sum_{N_1, N_2} \chi_{N_1 \sim N_2} \sum_{N_0} \chi_{N_0 \leq N_1} \frac{N_0^A}{N_1^B} \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'} \\
&\leq \|F_0\|_r \sum_{N_1, N_2} \chi_{N_1 \sim N_2} \frac{\Sigma_A(N_1)}{N_1^B} \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'}
\end{aligned}$$

where Σ_A is defined as in (2.2.6). Now from (i) and (ii) and Lemma 2.1, there is $\epsilon > 0$

such that $\Sigma_A(N_0) \lesssim N_1^{B-\epsilon}$. Then

$$\begin{aligned}
S &\leq \|F_0\|_r \sum_{N_1, N_2} \chi_{N_1 \sim N_2} N_1^{-\epsilon} \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'} & (2.2.7) \\
&= \|F_0\|_r \sum_{N_1} \sum_{N_2=2^{-\alpha}N_1}^{2^\beta N_1} N_1^{-\epsilon} \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'} \\
&= \|F_0\|_r \sum_{i=-\alpha}^{\beta} \sum_{N_1} N_1^{-\epsilon} \|F_1^{N_1}\|_{r'} \|F_2^{2^i N_1}\|_{r'} \\
&\lesssim \|F_0\|_r \|F_2\|_{r'} (\alpha + \beta + 1) \sum_{N_1} N_1^{-\epsilon} \|F_1^{N_1}\|_{r'} \\
&\lesssim \|F_0\|_r \|F_2\|_{r'} \left(\sum_{N_1} (N_1^{-\epsilon})^r \right)^{1/r} \left(\sum_{N_1} (\|F_1^{N_1}\|_{r'})^{r'} \right)^{1/r'} \\
&\lesssim \|F_0\|_r \|F_1\|_{r'} \|F_2\|_{r'}.
\end{aligned}$$

□

2.3 The L^p bilinear estimates

2.3.1 Introduction and preliminaries

For the estimates (2.2.3), we require bounds of the form

$$J\left(F_0^{N_0, L_0}, F_1^{N_1, L_1}, F_2^{N_2, L_2}\right) \lesssim C \|F_0^{N_0, L_0}\|_r \|F_1^{N_1, L_1}\|_{r'} \|F_2^{N_2, L_2}\|_{r'}$$

where the constant $C = C(L, N)$ is optimized so that the resulting summation over N and L is finite. To this end, we proceed as in [17], to obtain corresponding bilinear Fourier restriction estimates of the form

$$\|P_{A_0}(P_{A_1}u_1 \cdot P_{A_2}u_2)\|_{\widehat{L}^r} \leq C \|P_{A_1}u_1\|_{\widehat{L}^r} \|P_{A_2}u_2\|_{\widehat{L}^r} \quad (2.3.1)$$

where $\widehat{L}^r = \widehat{H}_0^r$, $A_0, A_1, A_2 \subset \mathbb{R}^{1+2}$ are measurable sets, and P_A is the Fourier multiplier operator defined by $\widetilde{P_A}u = \chi_A \widetilde{u}$. The A_i will be thickened subsets of the light cone

$K = \{(\tau, \xi) \in \mathbb{R}^{1+2} : |\tau| = |\xi|\}$, truncated in the spatial frequency ξ by balls, annuli, and angular sectors. Let

$$K_{N,L}^\pm = \{(\tau, \xi) \in \mathbb{R}^{1+2} : |\xi| \lesssim N, \tau = \pm|\xi| + O(L)\}$$

$$\dot{K}_{N,L}^\pm = \{(\tau, \xi) \in \mathbb{R}^{1+2} : |\xi| \sim N, \tau = \pm|\xi| + O(L)\}$$

$$\dot{K}_{N,L,\gamma}^\pm(\omega) = \left\{(\tau, \xi) \in \dot{K}_{N,L}^\pm : \theta(\pm\xi, \omega) \leq \gamma\right\}$$

where $N, L, \gamma > 0$, $\omega \in S^1$, and $\theta(x, y)$ is the angle between any two $x, y \in \mathbb{R}^2 \setminus \{0\}$.

By duality, (2.3.1) is equivalent to the trilinear estimate

$$J(F_0^{-A_0}, F_1^{A_1}, F_2^{A_2}) \leq C \|F_0^{-A_0}\|_r \|F_1^{A_1}\|_{r'} \|F_2^{A_2}\|_{r'} \quad (2.3.2)$$

where $F^A(X) = \chi_A(X)F(X)$, $X \in \mathbb{R}^{1+2}$, and J is defined as in (2.2.1). To see this, write

$$\begin{aligned} \|P_{A_0}(P_{A_1}u_1 \cdot P_{A_2}u_2)\|_{\widehat{L^r}} &= \|\mathcal{F}(P_{A_0}(P_{A_1}u_1 \cdot P_{A_2}u_2))\|_{r'} \\ &= \|\chi_{A_0}\mathcal{F}((P_{A_1}u_1 \cdot P_{A_2}u_2))\|_{r'} \\ &= \|\chi_{A_0}(\chi_{A_1}\widetilde{u}_1) * (\chi_{A_2}\widetilde{u}_2)\|_{r'} \\ &= \left(\int \left| \chi_{A_0}(\tau, \xi) \int (\chi_{A_1}\widetilde{u}_1)(t, x)(\chi_{A_2}\widetilde{u}_2)(\tau - t, \xi - x) d(t, x) \right|^{r'} d(\tau, \xi) \right)^{1/r'}. \end{aligned}$$

With the change of variables, $X_2 = -X_0 - X_1$ where $X_0 = (-\tau, -\xi)$, $X_1 = (t, x)$, and using the dual formulation, the last line becomes

$$\sup_{f \in L^r} \int \frac{f(X_0)}{\|f\chi_{A_0}\|_r} \chi_{-A_0}(X_0) \iint (\chi_{A_1}\widetilde{u}_1)(X_1)(\chi_{A_2}\widetilde{u}_2)(X_2) \delta(X_0 + X_1 + X_2) dX_2 dX_1 dX_0.$$

Setting $F_j = \widetilde{u}_j$ for $j = 1, 2$ and $F_0 = f$ gives

$$\sup_{F_0 \in L^r} \frac{1}{\|F_0^{A_0}\|_r} \iiint F_0^{-A_0}(X_0) F_1^{A_1}(X_1) F_2^{A_2}(X_2) \delta(X_0 + X_1 + X_2) dX_2 dX_1 dX_0$$

or

$$\|P_{A_0}(P_{A_1}u_1 \cdot P_{A_2}u_2)\|_{\widehat{L^r}} = \sup_{F_0 \in L^r} \frac{1}{\|F_0^{-A_0}\|_r} J(F_0^{-A_0}, F_1^{A_1}, F_2^{A_2}).$$

With this change of variables, for $i = 1, 2$,

$$\|P_{A_i} u_i\|_{\widehat{L}^r} = \|\chi_{A_i} \widetilde{u}_i\|_{r'} = \|F_i^{A_i}\|_{r'}$$

and hence (2.3.1) and (2.3.2) are equivalent with the same constant C .

We will show that admissible values for this constant C depend on sizes of appropriate intersections of the truncated sets K defined above. To estimate the sizes of these sets, we will utilize some of the results found in [17]. For $0 < \delta \ll r$, let $S_\delta^1(r) = \{\xi \in \mathbb{R}^2 : |\xi| = r + O(\delta)\}$ denote a thickened circle centered at the origin in \mathbb{R}^2 . Given another thickened circle, $\xi_0 + S_\Delta^1(R)$, centered at some point $\xi_0 \in \mathbb{R}^2 \setminus \{0\}$, we use the following lemma to bound the size of the intersection of these sets.

Lemma 2.4. *Suppose $0 < \delta \ll r, 0 < \Delta \ll R, \xi_0 \in \mathbb{R}^2 \setminus \{0\}$. Then*

$$|S_\delta^1(r) \cap (\xi_0 + S_\Delta^1(R))| \lesssim \left(\frac{rR\delta\Delta}{|\xi_0|} \min(\delta, \Delta) \right)^{1/2}. \quad (2.3.3)$$

Proof. This lemma and its proof are found in [17, Section 7]. The strategy is to first rotate about the origin so that ξ_0 lies on the positive x -axis. Then for $\xi = (x, y) \in S_\delta^1(r) \cap (\xi_0 + S_\Delta^1(R))$ it follows easily that x must lie in an interval

$$\frac{|\xi_0|^2 + (r - \delta)^2 - (R + \delta)^2}{2|\xi_0|} < x < \frac{|\xi_0|^2 + (r + \delta)^2 - (R - \delta)^2}{2|\xi_0|}$$

of length $\frac{2(r\delta + R\Delta)}{|\xi_0|}$. Noting also that

$$\sqrt{(r - \delta)^2 - x^2} < y < \sqrt{(r + \delta)^2 - x^2},$$

integrating over this region leads to the result. \square

When decomposing in angular frequencies, we will also use the following facts. For $0 < \gamma \leq \pi$ dyadic, let $\Omega(\gamma)$ be a maximal γ -separated subset of S^1 and for $\omega \in S^1$, let

$$\Gamma_\gamma(\omega) = \{\xi \in \mathbb{R}^2 : \theta(\xi, \omega) \leq \gamma\}.$$

By definition of Ω_γ ,

$$\#\{\omega \in \Omega(\gamma)\} \sim \frac{1}{\gamma} \quad (2.3.4)$$

and

$$\#\{\omega' \in \Omega(\gamma) : \theta(\omega', \omega) \leq k\gamma\} \leq 2k + 1 \quad (2.3.5)$$

for any $k \in \mathbb{N}$ and $\omega \in \Omega(\gamma)$. Then for any $\xi \neq 0$,

$$1 \leq \sum_{\omega \in \Omega(\gamma)} \chi_{\Gamma_\gamma(\omega)}(\xi) \leq 5. \quad (2.3.6)$$

Note the following lemmas from [17, Lemmas 2.4, 2.5], which will allow us to use an angular dyadic decomposition when necessary.

Lemma 2.5. *We have*

$$1 \sim \sum_{\substack{0 < \gamma < 1 \\ \gamma \text{ dyadic}}} \sum_{\substack{\omega_1, \omega_2 \in \Omega(\gamma) \\ 3\gamma \leq \theta(\omega_1, \omega_2) \leq 12\gamma}} \chi_{\Gamma_\gamma(\omega_1)}(\xi_1) \chi_{\Gamma_\gamma(\omega_2)}(\xi_2) \quad (2.3.7)$$

for all $\xi_1, \xi_2 \in \mathbb{R}^2 \setminus \{0\}$ with $\theta(\xi_1, \xi_2) > 0$.

Proof. Let $\xi_1, \xi_2 \in \mathbb{R}^2 \setminus \{0\}$. Note that if $\omega_1, \omega_2 \in \Omega(\gamma)$ are such that $3\gamma \leq \theta(\omega_1, \omega_2) \leq 12\gamma$, $\xi_1 \in \Gamma_\gamma(\omega_1)$ and $\xi_2 \in \Gamma_\gamma(\omega_2)$, then we must have $\gamma \leq \theta(\xi_1, \xi_2) \leq 14\gamma$. Then by (2.3.6),

$$\sum_{\substack{0 < \gamma < 1 \\ \gamma \text{ dyadic}}} \sum_{\substack{\omega_1, \omega_2 \in \Omega(\gamma) \\ 3\gamma \leq \theta(\omega_1, \omega_2) \leq 12\gamma}} \chi_{\Gamma_\gamma(\omega_1)}(\xi_1) \chi_{\Gamma_\gamma(\omega_2)}(\xi_2) \lesssim \sum_{\substack{0 < \gamma < 1 \\ \gamma \text{ dyadic}}} \chi_{\gamma \leq \theta(\xi_1, \xi_2) \leq 14\gamma}. \quad (2.3.8)$$

Now if $\theta(\xi_1, \xi_2) \geq 1$ then we must have $\gamma \geq 1/14$ and so the sum is at most 3. So, suppose $\theta(\xi_1, \xi_2) < 1$. Then there is a unique positive integer j such that $2^{-j} \leq \theta(\xi_1, \xi_2) < 2^{-j+1}$. Now if $k > 0$ is such that $\gamma = 2^{-k}$ then the requirement $2^{-k} \leq \theta(\xi_1, \xi_2) \leq 14 \cdot 2^{-k}$ implies that $j - 1 \leq k < j + 4$. Hence, there are at most 5 such γ . Finally, choosing γ such that $5\gamma \leq \theta(\xi_1, \xi_2) \leq 10\gamma$ ensures the sum is at least 1. \square

Lemma 2.6. *For any $0 < \gamma < 1$ and $k \in \mathbb{N}$,*

$$\chi_{\theta(\xi_1, \xi_2) \leq k\gamma} \lesssim \sum_{\substack{\omega_1, \omega_2 \in \Omega(\gamma) \\ \theta(\omega_1, \omega_2) \leq (k+2)\gamma}} \chi_{\Gamma_\gamma(\omega_1)}(\xi_1) \chi_{\Gamma_\gamma(\omega_2)}(\xi_2) \quad (2.3.9)$$

for all $\xi_1, \xi_2 \in \mathbb{R}^2 \setminus \{0\}$.

Proof. If $\chi_{\theta(\xi_1, \xi_2) \leq k\gamma} = 0$, this is trivial, so suppose $\theta(\xi_1, \xi_2) \leq k\gamma$. By (2.3.6), there are $\omega_1, \omega_2 \in \Omega(\gamma)$ such that $\xi_1 \in \Gamma_\gamma(\omega_1)$ and $\xi_2 \in \Gamma_\gamma(\omega_2)$. It follows then that $\theta(\omega_1, \omega_2) \leq (k+2)\gamma$. \square

2.3.2 Sizes of constants

The next lemma is a direct extension of [17, Lemma 1.1]. We use $|E|$ to denote the measure of the set $E \subset \mathbb{R}^{1+2}$.

Lemma 2.7. *Suppose $1 < r \leq 2$ and $\frac{1}{r} + \frac{1}{r'} = 1$. The estimate (2.3.1) holds with*

$$C \sim \min \left\{ \sup_{X_0 \in A_0} |A_1 \cap (X_0 - A_2)|^{1/r}, \sup_{X_1 \in A_1} |A_0 \cap (X_1 + A_2)|^{1/r'} |A_2|^{1/r-1/r'}, \right. \\ \left. \sup_{X_2 \in A_2} |A_0 \cap (X_2 + A_1)|^{1/r'} |A_1|^{1/r-1/r'} \right\}$$

provided this quantity is finite.

Proof. We will use the dual formulation to establish the estimate (2.3.2). For the first bound, write

$$\begin{aligned} J(F_0^{-A_0}, F_1^{A_1}, F_2^{A_2}) &= \iiint F_0^{-A_0}(X_0) F_1^{A_1}(X_1) F_2^{A_2}(X_2) \delta(X_0 + X_1 + X_2) dX_2 dX_1 dX_0 \\ &= \iint F_0^{-A_0}(X_0) F_1^{A_1}(X_1) F_2^{A_2}(-X_0 - X_1) dX_1 dX_0 \\ &= \int F_0^{-A_0}(-X_0) \int \chi_{A_1 \cap (X_0 - A_2)}(X_1) F_1^{A_1}(X_1) F_2^{A_2}(X_0 - X_1) dX_1 dX_0. \end{aligned}$$

Now applying Hölder's inequality first in the X_1 variable and then in X_0 ,

$$\begin{aligned} J(F_0^{-A_0}, F_1^{A_1}, F_2^{A_2}) &\leq \\ &\leq \int F_0^{-A_0}(-X_0) |A_1 \cap (X_0 - A_2)|^{1/r} \left(\int |F_1^{A_1}(X_1) F_2^{A_2}(X_0 - X_1)|^{r'} dX_1 \right)^{1/r'} dX_0 \\ &\leq \sup_{X_0 \in A_0} |A_1 \cap (X_0 - A_2)|^{1/r} \|F_0^{-A_0}\|_r \left(\iint |F_1^{A_1}(X_1) F_2^{A_2}(X_0 - X_1)|^{r'} dX_1 dX_0 \right)^{1/r'} \\ &\leq \sup_{X_0 \in A_0} |A_1 \cap (X_0 - A_2)|^{1/r} \|F_0^{-A_0}\|_r \|F_1^{A_1}\|_{r'} \|F_2^{A_2}\|_{r'}. \end{aligned}$$

The proofs of the second and third bounds are similar, so we will do the second. The proof relies again on successive applications of Hölder's inequality.

$$\begin{aligned}
J(F_0^{-A_0}, F_1^{A_1}, F_2^{A_2}) &= \iiint F_0^{-A_0}(X_0)F_1^{A_1}(X_1)F_2^{A_2}(X_2)\delta(X_0 + X_1 + X_2)dX_2dX_1dX_0 \\
&= \iint F_0^{-A_0}(-X_0)F_1^{A_1}(X_1)F_2^{A_2}(X_0 - X_1)dX_1dX_0 \\
&= \int F_1^{A_1}(X_1) \int \chi_{A_0 \cap (X_1 + A_2)}(X_0)F_0^{-A_0}(-X_0)F_2^{A_2}(X_0 - X_1)dX_0dX_1 \\
&\leq \int F_1^{A_1}(X_1) |A_0 \cap (X_1 + A_2)|^{1/r'} \left(\int |F_0^{-A_0}(-X_0)F_2^{A_2}(X_0 - X_1)|^r dX_0 \right)^{1/r} dX_1 \\
&\leq \sup_{X_1 \in A_1} |A_0 \cap (X_1 + A_2)|^{1/r'} \|F_1^{A_1}\|_{r'} \left(\iint |F_0^{-A_0}(-X_0)F_2^{A_2}(X_0 - X_1)|^r dX_0dX_1 \right)^{1/r}.
\end{aligned}$$

With a change of variables, we obtain the bound

$$\sup_{X_1 \in A_1} |A_0 \cap (X_1 + A_2)|^{1/r'} \|F_1^{A_1}\|_{r'} \|F_0^{-A_0}\|_r \left(\int |F_2^{A_2}(X_2)|^r dX_2 \right)^{1/r}$$

for $J(F_0^{-A_0}, F_1^{A_1}, F_2^{A_2})$. Now we are after $\|F_2^{A_2}\|_{r'}$ in the last line, but since $1 < r \leq 2$, $2 \leq r' < \infty$, we may apply Hölder's inequality to the functions $F_2^{A_2}$ and χ_{A_2} using the exponents r'/r and $r'/(r' - r)$ when $r \neq 2$. This gives

$$\begin{aligned}
\int |F_2^{A_2}(X_2)|^r dX_2 &\leq \left(\int \chi_{A_2}(X_2) dX_2 \right)^{1-r/r'} \left(\int |F_2^{A_2}(X_2)|^{r'} dX_2 \right)^{r/r'} \\
&\leq |A_2|^{1-r/r'} \|F_2^{A_2}\|_{r'}^r.
\end{aligned}$$

All in all, we have

$$J(F_0^{-A_0}, F_1^{A_1}, F_2^{A_2}) \leq \sup_{X_1 \in A_1} |A_0 \cap (X_1 + A_2)|^{1/r'} |A_2|^{1/r-1/r'} \|F_1^{A_1}\|_{r'} \|F_0^{-A_0}\|_r \|F_2^{A_2}\|_{r'}.$$

□

Theorem 2.8. *Suppose that $1 < r \leq 2$ and $\frac{1}{r} + \frac{1}{r'} = 1$. The estimate*

$$\|P_{K_{N_0, L_0}^{\pm 0}} (P_{K_{N_1, L_1}^{\pm 1}} u_1 \cdot P_{K_{N_2, L_2}^{\pm 2}} u_2)\|_{\widehat{L}^r} \leq C \|u_1\|_{\widehat{L}^{r'}} \|u_2\|_{\widehat{L}^r} \quad (2.3.10)$$

holds with

$$C \sim (N_{\min}^{012})^{\frac{2}{r'}} (N_{\min}^{12})^{\frac{2}{r} - \frac{2}{r'}} (L_{\min}^{12})^{\frac{1}{r}} \quad (2.3.11)$$

regardless of the choice of signs \pm_j . If in addition we assume $3/2 < r \leq 2$ then (2.3.10)

holds with

$$C \sim (N_{\min}^{012})^{\frac{1}{r'}} (N_{\min}^{12})^{\frac{3}{2r} - \frac{1}{r'}} (L_{\min}^{12})^{\frac{1}{r}} (L_{\max}^{12})^{\frac{1}{2r}}. \quad (2.3.12)$$

Proof of (2.3.11). We will separate into the HLH and LHH frequency regimes.

The HLH Case: $N_1 \lesssim N_0 \sim N_2$.

For any $X_0 = (\tau_0, \xi_0) \in K_{N_0, L_0}^{\pm 0}$, set $E = K_{N_1, L_1}^{\pm 1} \cap (X_0 - K_{N_2, L_2}^{\pm 2})$. By Lemma 2.7, the estimate (2.3.10) holds with $C \sim |E|^{1/r}$. By definition,

$$E \subset \{(\tau_1, \xi_1) : |\xi_1| \lesssim N_1, \tau_1 = \pm 1|\xi_1| + O(L_1), \tau_0 - \tau_1 = \pm 2|\xi_0 - \xi_1| + O(L_2)\},$$

so integrating first in τ_1 and then in ξ_1 , we obtain

$$\begin{aligned} |E| &\leq \int \chi_{|\xi_1| \lesssim N_1} \int \chi_{\tau_1 = \pm 1|\xi_1| + O(L_1)} \chi_{\tau_0 - \tau_1 = \pm 2|\xi_0 - \xi_1| + O(L_2)} d\tau_1 d\xi_1 \\ &\lesssim N_1^2 (L_{\min}^{12}). \end{aligned}$$

Then $C \sim N_1^{2/r} (L_{\min}^{12})^{1/r}$, which is (2.3.11) for the HLH case.

The LHH Case: $N_0 \lesssim N_1 \sim N_2$.

First assume $L_1 \leq L_2$. For any $X_2 = (\tau_2, \xi_2) \in K_{N_2, L_2}^{\pm 2}$, set $E = K_{N_0, L_0}^{\pm 0} \cap (X_2 + K_{N_1, L_1}^{\pm 1})$. By Lemma 2.7, the estimate (2.3.10) holds with $C \sim |E|^{1/r'} \left| K_{N_1, L_1}^{\pm 1} \right|^{1/r - 1/r'}$.

We have the containment

$$E \subset \{(\tau_0, \xi_0) : |\xi_0| \lesssim N_0, \tau_0 - \tau_2 = \pm 1|\xi_0 - \xi_2| + O(L_1)\}.$$

As above, integrating first in τ_0 and then in ξ_0 , we obtain $|E| \lesssim N_0^2 L_1$. Similarly, $\left| K_{N_1, L_1}^{\pm 1} \right| \lesssim N_1^2 L_1$. Together these yield (2.3.10) with $C \sim N_0^{2/r'} N_1^{2/r - 2/r'} L_1^{1/r}$.

On the other hand, if $L_2 \leq L_1$, we use $C \sim \sup_{X_1 \in A_1} |A_0 \cap (X_1 + A_2)|^{1/r'} |A_2|^{1/r - 1/r'}$ to obtain $C \sim N_0^{2/r'} N_2^{2/r - 2/r'} L_2^{1/r}$. Since $N_1 \sim N_2$, these two bounds imply (2.3.11) for the LHH case. \square

Proof of (2.3.12). First, note that it is enough to prove the estimate for the corresponding annular frequency cutoff sets, $\dot{K}_{N_i, L_i}^{\pm i}$. Indeed, suppose we have the estimate

$$\|P_{\dot{K}_{N_0, L_0}^{\pm 0}}(P_{\dot{K}_{N_1, L_1}^{\pm 1}} u_1 \cdot P_{\dot{K}_{N_2, L_2}^{\pm 2}} u_2)\|_{\widehat{L}^r} \leq C \|u_1\|_{\widehat{L}^r} \|u_2\|_{\widehat{L}^r} \quad (2.3.13)$$

with C as in (2.3.12). We will show that this implies the same estimate for the sets $K_{N_i, L_i}^{\pm i}$.

Decompose the balls $|\xi_j| \lesssim N_j$ into annuli $|\xi_j| \sim M_j$ for M_j dyadic with $0 < M_j \leq N_j$. Summing over these annuli, we have

$$\begin{aligned} \|P_{K_{N_0, L_0}^{\pm 0}}(P_{K_{N_1, L_1}^{\pm 1}} u_1 \cdot P_{K_{N_2, L_2}^{\pm 2}} u_2)\|_{\widehat{L}^r} &\leq \left\| \sum_{M_0} P_{K_{N_0, L_0}^{\pm 0}} \left(\sum_{M_1} P_{K_{N_1, L_1}^{\pm 1}} u_1 \cdot \sum_{M_2} P_{K_{N_2, L_2}^{\pm 2}} u_2 \right) \right\|_{\widehat{L}^r} \\ &\leq \sum_M \left\| P_{K_{N_0, L_0}^{\pm 0}}(P_{K_{N_1, L_1}^{\pm 1}} u_1 \cdot P_{K_{N_2, L_2}^{\pm 2}} u_2) \right\|_{\widehat{L}^r} \\ &\lesssim \sum_{M_j < N_j} (M_{\min}^{012})^{\frac{1}{r'}} (M_{\min}^{12})^{\frac{3}{2r} - \frac{1}{r'}} (L_{\min}^{12})^{\frac{1}{r}} (L_{\max}^{12})^{\frac{3}{2r}} \|\chi_{K_{N_1, L_1}^{\pm 1}} u_1\|_{\widehat{L}^r} \|\chi_{K_{N_2, L_2}^{\pm 2}} u_2\|_{\widehat{L}^r}. \end{aligned}$$

We will break into the cases $M_{\max}^{012} \leq N_{\min}^{012}$ and $M_{\max}^{012} > N_{\min}^{012}$.

- Suppose $M_{\max}^{012} \leq N_{\min}^{012}$. Then we have

$$\sum_{M_j \leq N_j} (M_{\min}^{012})^{\frac{1}{r'}} (M_{\min}^{12})^{\frac{3}{2r} - \frac{1}{r'}} \leq \sum_{M_j \leq N_{\min}^{012}} (M_{\min}^{012})^{\frac{1}{r'}} (M_{\min}^{12})^{\frac{3}{2r} - \frac{1}{r'}}.$$

We split this last sum into the LHH, HLH, and HHL frequencies in M_i : $M_0 \lesssim M_1 \sim M_2 \leq N_{\min}^{012}$, $M_1 \lesssim M_0 \sim M_2 \leq N_{\min}^{012}$, and $M_2 \lesssim M_0 \sim M_1 \leq N_{\min}^{012}$ and apply Lemma 2.1.

Noting that $M_{\min}^{12} \leq N_{\min}^{12}$, the sum is bounded by

$$(N_{\min}^{12})^{\frac{3}{2r} - \frac{1}{r'}} \left(\sum_{M_1 \sim M_2 \leq N_{\min}^{012}} M_1^{\frac{1}{r'}} + \sum_{M_0 \sim M_2 \leq N_{\min}^{012}} M_0^{\frac{1}{r'}} + \sum_{M_0 \sim M_1 \leq N_{\min}^{012}} M_0^{\frac{1}{r'}} \right)$$

since $r > 1$ implies that $\frac{3}{2r} - \frac{1}{r'} > 0$. Using that the two highest frequencies, $M_i \sim M_j$, in each double sum are comparable, another application of Lemma 2.1 gives

$$\sum_{M_j \leq N_j} (M_{\min}^{012})^{\frac{1}{r'}} (M_{\min}^{12})^{\frac{3}{2r} - \frac{1}{r'}} \lesssim (N_{\min}^{12})^{\frac{3}{2r} - \frac{1}{r'}} (N_{\min}^{012})^{\frac{1}{r'}}$$

which gives the desired result.

• Next suppose $M_{\max}^{012} > N_{\min}^{012}$. For simplicity, we will assume that the balls $|\xi_1| \lesssim N_i$ are in the HLH frequency regime so that $N_{\min}^{012} = N_1$. The strategy for the LHH regime is the same, and the proof is an obvious modification of the HLH case. Recall that (2.3.1) is equivalent to the trilinear estimate (2.3.2). To simplify notation, write

$$A_j = K_{N_j, L_j}^{\pm j} \quad \dot{A}_j = \dot{K}_{M_j, L_j}^{\pm j}$$

and recall that for a set $A \subset \mathbb{R}^{1+2}$, we define $F^A(X) = \chi_A(X)F(X)$ and for a dyadic number N , we define $F^N(X) = \chi_{\langle \xi \rangle \sim N} F(X)$. Then (2.3.13) is equivalent to

$$J(F_0^{-A_0}, F_1^{A_1}, F_2^{A_2}) \leq C \|F_0^{-A_0}\|_r \|F_1^{A_1}\|_{r'} \|F_2^{A_2}\|_{r'}.$$

Hence, as above, we have

$$\begin{aligned} J(F_0^{-A_0}, F_1^{A_1}, F_2^{A_2}) &\leq \sum_M J(F_0^{-A_0}, F_1^{A_1}, F_2^{A_2}) \\ &\lesssim \sum_{M_j < N_j} (M_{\min}^{012})^{\frac{1}{r'}} (M_{\min}^{12})^{\frac{3}{2r} - \frac{1}{r'}} (L_{\min}^{12})^{\frac{1}{r}} (L_{\max}^{12})^{\frac{1}{2r}} \|F_0^{-A_0}\|_r \|F_1^{A_1}\|_{r'} \|F_2^{A_2}\|_{r'}. \end{aligned}$$

Again, we split the sum into frequencies LHH, HLH, and HHL in M . In the LHH term we have

$$\sum_{M_0 \lesssim M_1 \sim M_2} M_0^{\frac{1}{r'}} (M_{\min}^{12})^{\frac{3}{2r} - \frac{1}{r'}} \|F_0^{-A_0}\|_r \|F_1^{A_1}\|_{r'} \|F_2^{A_2}\|_{r'}.$$

Since $M_1 > N_1$, the sum over M_1 is now empty, and since $M_2 \sim M_1 \leq N_1$, this term is bounded by

$$\begin{aligned} &\sum_{M_0 \lesssim M_2 \leq N_1} M_0^{\frac{1}{r'}} (M_{\min}^{12})^{\frac{3}{2r} - \frac{1}{r'}} \|F_0^{-A_0}\|_r \|F_1^{A_1}\|_{r'} \|F_2^{A_2}\|_{r'} \\ &\lesssim \sum_{M_2 \leq N_1} M_2^{\frac{1}{r'}} (M_{\min}^{12})^{\frac{3}{2r} - \frac{1}{r'}} \|F_0^{-A_0}\|_r \|F_1^{A_1}\|_{r'} \|F_2^{A_2}\|_{r'} \\ &\lesssim N_1^{\frac{1}{r'}} (N_{\min}^{12})^{\frac{3}{2r} - \frac{1}{r'}} \|F_0^{-A_0}\|_r \|F_1^{A_1}\|_{r'} \|F_2^{A_2}\|_{r'} \\ &\lesssim N_1^{\frac{3}{2r}} \|F_0^{-A_0}\|_r \|F_1^{A_1}\|_{r'} \|F_2^{A_2}\|_{r'}. \end{aligned}$$

In the HLH case, we have the term

$$\begin{aligned}
\sum_{M_1 \leq N_1 < M_0 \sim M_2} M_1^{\frac{3}{2r}} \|F_0^{-A_0}\|_r \|F_1^{A_1}\|_{r'} \|F_2^{A_2}\|_{r'} &\lesssim \sum_{N_1 < M_0 \sim M_2} N_1^{\frac{3}{2r}} \|F_0^{-A_0}\|_r \|F_1^{A_1}\|_{r'} \|F_2^{A_2}\|_{r'} \\
&\lesssim N_1^{\frac{3}{2r}} \|F_1\|_{r'} \sum_{N_1 < M_0 \sim M_2} \|F_0^{M_0}\|_r \|F_2^{M_2}\|_{r'} \\
&\lesssim N_1^{\frac{3}{2r}} \|F_1\|_{r'} \sum_{M_0} \|F_0^{M_0}\|_r \|F_2^{M_0}\|_{r'} \\
&\lesssim N_1^{\frac{3}{2r}} \|F_1\|_{r'} \left(\sum_{M_0} \|F_0^{M_0}\|_r^r \right)^{1/r} \left(\sum_{M_0} \|F_2^{M_0}\|_{r'}^{r'} \right)^{1/r'} \\
&\lesssim N_1^{\frac{3}{2r}} \|F_1\|_{r'} \|F_0\|_r \|F_2\|_{r'}.
\end{aligned}$$

Finally, for the HHL case, we have the term

$$\sum_{M_2 \lesssim M_0 \sim M_1} M_2^{\frac{3}{2r}} \|F_0^{-A_0}\|_r \|F_1^{A_1}\|_{r'} \|F_2^{A_2}\|_{r'}.$$

Again, since $M_1 > N_1$ in this case, the sum over M_1 is empty. We have

$$\begin{aligned}
\sum_{M_2 \lesssim M_0 \sim M_1} M_2^{\frac{3}{2r}} \|F_0^{-A_0}\|_r \|F_1^{A_1}\|_{r'} \|F_2^{A_2}\|_{r'} &\lesssim \sum_{M_2 \lesssim M_0 \leq N_1} M_2^{\frac{3}{2r}} \|F_0^{-A_0}\|_r \|F_1^{A_1}\|_{r'} \|F_2^{A_2}\|_{r'} \\
&\lesssim \sum_{M_0 \leq N_1} M_0^{\frac{3}{2r}} \|F_0^{-A_0}\|_r \|F_1^{A_1}\|_{r'} \|F_2^{A_2}\|_{r'} \\
&\lesssim N_1^{\frac{3}{2r}} \|F_0^{-A_0}\|_r \|F_1^{A_1}\|_{r'} \|F_2^{A_2}\|_{r'}.
\end{aligned}$$

Thus, we get the desired estimate

$$J(F_0^{-A_0}, F_1^{A_1}, F_2^{A_2}) \lesssim (N_{\min}^{012})^{\frac{1}{r'}} (N_{\min}^{12})^{\frac{3}{2r} - \frac{1}{r'}} (L_{\min}^{12})^{\frac{1}{r}} (L_{\max}^{12})^{\frac{1}{2r}} \|F_0^{-A_0}\|_r \|F_1^{A_1}\|_{r'} \|F_2^{A_2}\|_{r'}$$

which implies (2.3.12).

Now we will establish (2.3.13). Notice that if $C_1 = (N_{\min}^{012})^{\frac{2}{r'}} (N_{\min}^{12})^{\frac{2}{r} - \frac{2}{r'}} (L_{\min}^{12})^{\frac{1}{r}}$ and $C_2 = (N_{\min}^{012})^{\frac{1}{r'}} (N_{\min}^{12})^{\frac{3}{2r} - \frac{1}{r'}} (L_{\min}^{12})^{\frac{1}{r}} (L_{\max}^{12})^{\frac{1}{2r}}$ are the constants from (2.3.11) and

(2.3.12), then since $\frac{1}{2r} - \frac{1}{r'} < 0$ for $\frac{3}{2} < r \leq 2$,

$$\begin{aligned} C_1 &= (N_{\min}^{012})^{\frac{1}{r'}} (N_{\min}^{12})^{\frac{1}{2r} - \frac{1}{r'}} (L_{\max}^{12})^{-\frac{1}{2r}} C_2 \\ &\leq (N_{\min}^{012})^{\frac{1}{2r}} (L_{\max}^{12})^{-\frac{1}{2r}} C_2 \\ &le \left(\frac{N_{\min}^{012}}{L_{\max}^{12}} \right)^{\frac{1}{2r}} C_2. \end{aligned} \tag{2.3.14}$$

If $N_{\min}^{012} < L_{\max}^{12}$, the estimate (2.3.11) is already better than (2.3.12) and therefore we may assume that $L_{\max}^{12} \ll N_{\min}^{012}$.

The HLH Case: $N_1 \lesssim N_0 \sim N_2$.

For any $X_0 = (\tau_0, \xi_0) \in \dot{K}_{N_0, L_0}^{\pm_0}$, set $E = \dot{K}_{N_1, L_1}^{\pm_1} \cap (X_0 - \dot{K}_{N_2, L_2}^{\pm_2})$. Recall that we roughly bounded the size of this set by $N_1^2 L_{\min}^{12}$ in the HLH case of (2.3.11). Since now we have $L_{\max}^{12} \ll N_{\min}^{012}$ we will proceed as in [17] and estimate the size more carefully to lower the exponent on N_1 . First note that

$$E \subset \{(\tau_1, \xi_2) : |\xi_1| \sim N_1, \tau_1 = \pm_1 |\xi_1| + O(L_1), |\xi_0 - \xi_1| \sim N_2, \tau_0 - \tau_1 = \pm_2 |\xi_0 - \xi_1| + O(L_2)\}.$$

For τ_1 fixed, let E^{τ_1} denote the slice $\{\xi_1 \in \mathbb{R}^2 : (\tau_1, \xi_1) \in E\}$. Then

$$E^{\tau_1} \subset S_{L_1}^1(|\tau_1|) \cap (\xi_0 + S_{L_2}^1(|\tau_0 - \tau_1|)) \cap \{\xi_1 : |\xi_1| \sim N_1, |\xi_0 - \xi_1| \sim N_2\}$$

for $\tau_1 \in T = \{\tau_1 = \pm_1 |\xi_1| + O(L_1), \tau_0 - \tau_1 = \pm_2 |\xi_0 - \xi_1| + O(L_2)\}$.

Now integrating first in τ_1 , we have $|E| \lesssim L_{\min}^{12} \sup_{\tau_1 \in T} |E^{\tau_1}|$, so by Lemma 2.4,

$$\begin{aligned} |E| &\lesssim L_{\min}^{12} \sup_{\tau_1 \in T} \left(\frac{L_1 L_2 |\tau_1| |\tau_0 - \tau_1|}{|\xi_0|} L_{\min}^{12} \right)^{1/2} \\ &\lesssim N_1 \left(\frac{L_1 L_2 N_1 N_2}{N_0} L_{\min}^{12} \right)^{1/2} \\ &\lesssim N_1^{3/2} L_{\min}^{12} (L_{\max}^{12})^{1/2}. \end{aligned}$$

Then

$$|E|^{1/r} \lesssim N_1^{3/2r} (L_{\min}^{12})^{1/r} (L_{\max}^{12})^{1/2r}$$

which is (2.3.12) for the HLH case. Applying Lemma 2.7 yields the estimate (2.3.10).

The LHH Case: $N_0 \lesssim N_1 \sim N_2$.

By symmetry, we may assume that $L_1 \leq L_2$ so that (2.3.12) becomes

$$C \sim N_0^{\frac{1}{r'}} N_1^{\frac{3}{2r} - \frac{1}{r'}} L_1^{\frac{1}{2r}} L_2^{\frac{1}{2r}}. \quad (2.3.15)$$

Now with C_1 and C_2 as above, from (2.3.14) we have

$$C_1 = N_0^{\frac{1}{r'}} N_1^{\frac{1}{2r} - \frac{1}{r'}} L_2^{-\frac{1}{2r}} C_2.$$

Thus, if $L_2 > N_0 \left(\frac{N_0}{N_1}\right)^{2r/r'-1}$, the estimate (2.3.11) is already better than (2.3.12) and therefore we may assume that $L_2 \ll N_0 \left(\frac{N_0}{N_1}\right)^{2r/r'-1}$. (Since the exponent $2r/r' - 1 > 0$ only when $r > 3/2$, this would not give any further restriction to the assumption $L_2 < N_0$ for values of r below $3/2$.) So, we want to establish (2.3.15) when

$$L_1 \leq L_2 \ll N_0 \cdot \min \left\{ 1, \left(\frac{N_0}{N_1}\right)^{2r/r'-1} \right\} \ll N_1 \sim N_2.$$

Again, we will use the dual formulation

Recall that we showed (2.3.1) is equivalent to (2.3.2). By a similar argument, this trilinear estimate is also equivalent to

$$\|P_{-A_2}(P_{A_1}u_1 \cdot P_{-A_0}u_0)\|_{\widehat{L}^r} \leq C \|u_1\|_{\widehat{L}^r} \|u_0\|_{\widehat{L}^{r'}}.$$

Hence, for the estimate (2.3.10) it is enough to establish

$$\|P_{\dot{K}_{N_2, L_2}^{\pm 2}}(P_{\dot{K}_{N_1, L_1}^{\pm 1}} u_1 \cdot P_{\dot{K}_{N_0, L_0}^{\pm 0}} u_0)\|_{\widehat{L}^{r'}} \leq C \|u_1\|_{\widehat{L}^r} \|u_0\|_{\widehat{L}^{r'}}.$$

Furthermore, by definition of the operators P_A , we may assume that the supports of \tilde{u}_j are restricted to $\dot{K}_{N_j, L_j}^{\pm j}$ for $j = 0, 1$, and without loss of generality, we may take $\tilde{u}_j \geq 0$.

With these assumptions, we will simplify notation by omitting the $\dot{K}_{N_j, L_j}^{\pm j}$ and writing

$$P_{A_2}(u_1 u_0) = P_{\dot{K}_{N_2, L_2}^{\pm 2}}(P_{\dot{K}_{N_1, L_1}^{\pm 1}} u_1 \cdot P_{\dot{K}_{N_0, L_0}^{\pm 0}} u_0).$$

For $\gamma_0 = \left(\frac{L_2}{N_1}\right)^{1/2}$, we will split into the cases $\theta(\pm_0 \xi_0, \pm_1 \xi_1) \lesssim \gamma_0$ and $\theta(\pm_0 \xi_0, \pm_1 \xi_1) \gg \gamma_0$

γ_0 . Define $u^{\gamma,\omega}$ such that $\widetilde{u^{\gamma,\omega}} = \chi_{\theta(\xi,\omega) \leq \gamma} \widetilde{u}$. By Lemma 2.5,

$$\begin{aligned} P_{A_2}(u_1, u_0) &\sim \sum_{0 < \gamma < 1} \sum_{\substack{\omega_1, \omega_0 \in \Omega(\gamma) \\ 3\gamma \leq \theta(\omega_1, \omega_0) \leq 12\gamma}} \chi_{\Gamma_\gamma(\omega_0)}(\xi_0) \chi_{\Gamma_\gamma(\omega_1)}(\xi_1) P_{A_2}(u_1, u_0) \\ &\sim \sum_{0 < \gamma < \gamma_0} \sum_{\substack{\omega_1, \omega_0 \in \Omega(\gamma) \\ 3\gamma \leq \theta(\omega_1, \omega_0) \leq 12\gamma}} \chi_{\Gamma_\gamma(\omega_0)}(\xi_0) \chi_{\Gamma_\gamma(\omega_1)}(\xi_1) P_{A_2}(u_1, u_0) \\ &\quad + \sum_{\gamma_0 < \gamma < 1} \sum_{\substack{\omega_1, \omega_0 \in \Omega(\gamma) \\ 3\gamma \leq \theta(\omega_1, \omega_0) \leq 12\gamma}} \chi_{\Gamma_\gamma(\omega_0)}(\xi_0) \chi_{\Gamma_\gamma(\omega_1)}(\xi_1) P_{A_2}(u_1, u_0). \end{aligned}$$

Now

$$\begin{aligned} &\sum_{0 < \gamma < \gamma_0} \sum_{\substack{\omega_1, \omega_0 \in \Omega(\gamma) \\ 3\gamma \leq \theta(\omega_1, \omega_0) \leq 12\gamma}} \chi_{\Gamma_\gamma(\omega_0)}(\xi_0) \chi_{\Gamma_\gamma(\omega_1)}(\xi_1) P_{A_2}(u_1, u_0) \\ &\lesssim \sum_{0 < \gamma < \gamma_0} \chi_{\gamma \leq \theta(\xi_0, \xi_1) \leq 14\gamma} \chi_{\Gamma_\gamma(\omega_0)}(\xi_0) \chi_{\Gamma_\gamma(\omega_1)}(\xi_1) P_{A_2}(u_1, u_0) \\ &\lesssim \sum_{\substack{\omega_1, \omega_0 \in \Omega(\gamma_0) \\ \theta(\omega_1, \omega_0) \leq 15\gamma_0}} \chi_{\Gamma_\gamma(\omega_0)}(\xi_0) \chi_{\Gamma_\gamma(\omega_1)}(\xi_1) P_{A_2}(u_1, u_0) \end{aligned}$$

by Lemma 2.6. Therefore,

$$\begin{aligned} \|P_{A_2}(u_1, u_0)\|_{\widehat{L}^{r'}} &\lesssim \sum_{\omega_1, \omega_0 \in \Omega(\gamma_0)} \chi_{\theta(\omega_1, \omega_0) \lesssim \gamma_0} \|\mathcal{F}P_{A_2}(u_1^{\gamma_0, \omega_1}, u_0^{\gamma_0, \omega_0})\|_{L^r} \\ &\quad + \sum_{\gamma_0 < \gamma < 1} \sum_{\omega_1, \omega_0 \in \Omega(\gamma)} \chi_{3\gamma \leq \theta(\omega_1, \omega_0) \leq 12\gamma} \|\mathcal{F}P_{A_2}(u_1^{\gamma, \omega_1}, u_0^{\gamma, \omega_0})\|_{L^r}. \end{aligned} \tag{2.3.16}$$

Call these last two summands Σ_1 and Σ_2 , respectively.

For Σ_1 , we will estimate the volume of the set $E = A_0 \cap (X_2 + A_1)$ for any $X_2 \in A_2 = \dot{K}_{N_2, L_2}^{\pm 2}$ where $A_j \subset \dot{K}_{N_j, L_j, \gamma_0}^{\pm j}(\omega_j)$ is the support of $\widetilde{u_j^{\gamma_0, \omega_j}}$ for $j = 0, 1$. Since $X_0 - X_2 \in A_1$, we have

$$E \subset \{(\tau_0, \xi_0) \in \mathbb{R}^{1+2} : |\xi_0| \sim N_0, |\xi_0 - \xi_2| \sim N_1,$$

$$\tau_0 - \tau_2 = \pm 1 |\xi_0 - \xi_2| + O(L_1), \xi_0 \in \Gamma_{\gamma_0}(\omega_0), \pm 1(\xi_0 - \xi_2) \in \Gamma_{\gamma_0}(\omega_1)\}.$$

Integrating first in τ_0 , we have

$$\begin{aligned} |E| &\lesssim L_1 |\{\xi_0 \in \mathbb{R}^2 : |\xi_0| \sim N_0, \theta(\xi_0, \omega_0) \leq \gamma_0\}| \\ &\lesssim L_1 N_0^2 \gamma_0 \end{aligned}$$

and hence $|E| \lesssim N_0 N_1 \gamma_0 L_1$.

Similarly, $|A_1| \lesssim L_1 N_1^2 \gamma_0$, and by Lemma 2.7, (2.3.2) holds with $C \lesssim (N_0 N_1 \gamma_0 L_1)^{1/r'} (N_1^2 L_1 \gamma_0)^{1/r-1/r'}$. Therefore,

$$\begin{aligned} \|\mathcal{F}P_{A_2}(u_1^{\gamma_0, \omega_1}, u_0^{\gamma_0, \omega_0})\|_{L^r} &= \|P_{A_2}(P_{A_1} u_1 \cdot P_{A_0} u_0)\|_{\widetilde{L}^{r'}} \\ &\lesssim N_0^{1/r'} N_1^{2/r-1/r'} \gamma_0^{1/r} L_1^{1/r} \|u_1^{\widetilde{\gamma}, \omega_1}\|_{r'} \|u_0^{\widetilde{\gamma}, \omega_0}\|_{r'}. \end{aligned}$$

Inserting this into Σ_1 , we obtain

$$\Sigma_1 \lesssim \sum_{\omega_1, \omega_0 \in \Omega(\gamma_0)} \chi_{\theta(\omega_1, \omega_0) \lesssim \gamma_0} N_0^{1/r'} N_1^{2/r-1/r'} \gamma_0^{1/r} L_1^{1/r} \|u_1^{\widetilde{\gamma}, \omega_1}\|_{r'} \|u_0^{\widetilde{\gamma}, \omega_0}\|_{r'}.$$

Using now (2.3.5) and (2.3.4), we have

$$\begin{aligned} &\sum_{\omega_1, \omega_0 \in \Omega(\gamma_0)} \chi_{\theta(\omega_1, \omega_0) \lesssim \gamma_0} \|u_1^{\widetilde{\gamma}, \omega_1}\|_{r'} \|u_0^{\widetilde{\gamma}, \omega_0}\|_{r'} \\ &\lesssim \left(\sum_{\omega_1, \omega_0 \in \Omega(\gamma_0)} \chi_{\theta(\omega_1, \omega_0) \lesssim \gamma_0} \|u_1^{\widetilde{\gamma}, \omega_1}\|_{r'}^{r'} \right)^{1/r'} \left(\sum_{\omega_1, \omega_0 \in \Omega(\gamma_0)} \chi_{\theta(\omega_1, \omega_0) \lesssim \gamma_0} \|u_0^{\widetilde{\gamma}, \omega_0}\|_r^r \right)^{1/r} \\ &\lesssim \left(\sum_{\omega_1 \in \Omega(\gamma_0)} \|u_1^{\widetilde{\gamma}, \omega_1}\|_{r'}^{r'} \right)^{1/r'} \left(\sum_{\omega_0 \in \Omega(\gamma_0)} \|u_0^{\widetilde{\gamma}, \omega_0}\|_r^r \right)^{1/r} \\ &\lesssim \|\widetilde{u}_1\|_{r'} \|\widetilde{u}_0\|_r. \end{aligned}$$

Then $\Sigma_1 \lesssim N_0^{1/r'} N_1^{2/r-1/r'} \gamma_0^{1/r} L_1^{1/r} \|\widetilde{u}_1\|_{r'} \|\widetilde{u}_0\|_r$ and setting $\gamma_0 = \left(\frac{L_2}{N_1}\right)^{1/2}$, gives

$$\Sigma_1 \lesssim N_0^{1/r'} N_1^{3/2r-1/r'} L_2^{1/2r} L_1^{1/r} \|\widetilde{u}_1\|_{r'} \|\widetilde{u}_0\|_r$$

as desired.

For Σ_2 , we will again estimate the volume of the set $E = A_0 \cap (X_2 + A_1)$ for any $X_2 \in A_2 = \dot{K}_{N_2, L_2}^{\pm 2}$, but this time using the additional fact that $3\gamma \leq \theta(\omega_1, \omega_0) \leq 12\gamma$.

Note that

$$E \subset \{(\tau_0, \xi_0) \in \mathbb{R}^{1+2} : \xi_0 \in R, \tau_0 - \tau_2 = \pm_1 |\xi_0 - \xi_2| + O(L_1)\}$$

where

$$R = \{\xi_0 \in \mathbb{R}^2 : |\xi_0| \sim N_0, |\xi_0 - \xi_2| \sim N_1, \xi_0 \in \Gamma_\gamma(\omega_0), \pm_1(\xi_0 - \xi_2) \in \Gamma_\gamma(\omega_1), \}.$$

Now for a fixed τ_0 , define

$$f(\xi_0) = \pm_1 |\xi_0 - \xi_2| = \tau_0 - \tau_2 + O(L_1)$$

so that

$$\nabla f(\xi_0) = \pm_1 \frac{\xi_0 - \xi_2}{|\xi_0 - \xi_2|} = e.$$

For $\omega_0, \omega_1 \in \Omega(\gamma)$, choose coordinates (ξ^1, ξ^2) so that $\frac{\omega_1 - \omega_0}{|\omega_1 - \omega_0|} = (1, 0)$. Then for all $\xi_0 \in E$, we have

$$\begin{aligned} \partial_1 f(\xi_0) &= \nabla f(\xi_0) \cdot (1, 0) \\ &= e \cdot \frac{\omega_1 - \omega_0}{|\omega_1 - \omega_0|} \\ &= \frac{e \cdot \omega_1 - e \cdot \omega_0}{|\omega_1 - \omega_0|} \\ &= \frac{\cos(\theta(e, \omega_1)) - \cos(\theta(e, \omega_0))}{|\omega_1 - \omega_0|} \end{aligned}$$

since $|e| = |\omega_i| = 1$. Now since $\pm_1(\xi_0 - \xi_2) \in \Gamma_\gamma(\omega_1)$, it follows that $\theta(e, \omega_1) \leq \gamma$. Combining this with the assumption $3\gamma \leq \theta(\omega_1, \omega_0) \leq 12\gamma$ for Σ_2 , we conclude that $\theta(e, \omega_0) \geq 2\gamma$. Since $\gamma_0 < \gamma < 1$ and $\gamma_0 = \left(\frac{L_2}{N_1}\right)^{1/2} > 0$, the function $\cos(\gamma)$ is decreasing. Hence,

$$\partial_1 f(\xi_0) \geq \frac{\cos(\gamma) - \cos(2\gamma)}{|\omega_1 - \omega_0|}.$$

Using the Taylor expansion for $\cos(\gamma)$, we have $\cos(\gamma) > 1 - \frac{\gamma^2}{2}$ and $\cos(2\gamma) < 1 - \frac{(2\gamma)^2}{2} + \frac{(2\gamma)^4}{4!}$. Then for $0 < \gamma < 1$, it follows that $\cos(\gamma) - \cos(2\gamma) > \frac{1}{2}\gamma^2$. Also, from $3\gamma \leq \theta(\omega_1, \omega_0) \leq 12\gamma$ we have $|\omega_1 - \omega_0| \sim \gamma$. Thus, $\partial_1 f(\xi_0) \gtrsim \gamma$.

Therefore,

$$|E| \lesssim |\{\tau_0 \in T, \xi_0 \in R : f(\xi_0) = \tau_0 - \tau_2 + O(L_1)\}|$$

where

$$T = \{\tau_0 : \tau_0 = \tau_2 \pm_1 |\xi_2| + O(N_0) + O(L_1)\}$$

so that $|T| \lesssim N_0 + L_1 \lesssim N_0$. Now if τ_0 and τ_2 are fixed, then $f(\xi) = \tau_0 - \tau_2 + O(L_1)$ for

all $\xi \in R$ implies that $\sup_{\xi \in R} |f(\xi_0) - f(\xi)| \lesssim L_1$. But since

$$\begin{aligned} \sup_{\xi \in R} |f(\xi_0) - f(\xi)| &\geq |\partial_1 f(\xi_0)| \sup_{\xi \in R} |\xi_0^1 - \xi^1| \\ &\gtrsim \gamma \sup_{\xi \in R} |\xi_0^1 - \xi^1| \end{aligned}$$

we must have $\sup_{\xi \in R} |\xi_0^1 - \xi^1| \lesssim \frac{L_1}{\gamma}$ for all $\xi_0 \in R$. Integrating first in the ξ_0^1 direction, we find

$$\begin{aligned} |E| &\lesssim \frac{L_1}{\gamma} \left\{ \tau_0 \in T, \xi_0^2 : |\xi_0 - \xi_2| = \pm_1(\tau_0 - \tau_2) + O(L_1) \right\} \\ &\lesssim \frac{L_1}{\gamma} \left\{ \tau_0 \in T, \xi_0^2 : |\xi_0| = \pm_1 \tau_0 + O(L_2) + O(L_1) \right\} \\ &\lesssim \frac{L_1}{\gamma} L_2 N_0. \end{aligned}$$

Using the estimates $|E| \lesssim L_1 L_2 N_0 / \gamma$ and $|A_1| \lesssim N_1^2 L_1 \gamma$ as before, by Lemma 2.7,

$$\|\mathcal{F}P_{A_2}(u_1^{\gamma_0, \omega_1}, u_0^{\gamma_0, \omega_0})\|_{L^r} \lesssim (L_1 L_2 N_0 / \gamma)^{1/r'} (N_1^2 L_1 \gamma)^{1/r - 1/r'} \|\widetilde{u_1^{\gamma, \omega_1}}\|_{r'} \|\widetilde{u_0^{\gamma, \omega_0}}\|_r.$$

Inserting this into Σ_2 , we obtain

$$\Sigma_2 \lesssim \sum_{\gamma_0 < \gamma < 1} \sum_{\omega_1, \omega_0 \in \Omega(\gamma)} \chi_{3\gamma \leq \theta(\omega_1, \omega_0) \leq 12\gamma} L_1^{1/r} (L_2 N_0)^{1/r'} N_1^{2/r - 2/r'} \gamma^{1/r - 2/r'} \|\widetilde{u_1^{\gamma, \omega_1}}\|_{r'} \|\widetilde{u_0^{\gamma, \omega_0}}\|_r.$$

For the inner sum, we proceed as in Σ_1 to obtain

$$\sum_{\omega_1, \omega_0 \in \Omega(\gamma)} \chi_{3\gamma \leq \theta(\omega_1, \omega_0) \leq 12\gamma} \|\widetilde{u_1^{\gamma, \omega_1}}\|_{r'} \|\widetilde{u_0^{\gamma, \omega_0}}\|_r \lesssim \|\widetilde{u_1}\|_{r'} \|\widetilde{u_0}\|_r.$$

Then we have

$$\begin{aligned} \Sigma_2 &\lesssim L_1^{1/r} (L_2 N_0)^{1/r'} N_1^{2/r - 2/r'} \|\widetilde{u_1}\|_{r'} \|\widetilde{u_0}\|_r \sum_{\gamma_0 < \gamma < 1} \gamma^{1/r - 2/r'} \\ &\lesssim L_1^{1/r} (L_2 N_0)^{1/r'} N_1^{2/r - 2/r'} \gamma_0^{1/r - 2/r'} \|\widetilde{u_1}\|_{r'} \|\widetilde{u_0}\|_r \gamma_0^{1/r - 2/r'} \end{aligned}$$

since $1/r - 2/r' < 0$ for $3/2 < r \leq 2$. Setting $\gamma_0 = \left(\frac{L_2}{N_1}\right)^{1/2}$, gives

$$\Sigma_2 \lesssim N_0^{1/r'} N_1^{3/2r - 1/r'} L_2^{1/2r} L_1^{1/r} \|\widetilde{u_1}\|_{r'} \|\widetilde{u_0}\|_r$$

as desired. □

CHAPTER 3

PRODUCT ESTIMATES FOR WAVE FOURIER LEBESGUE SPACES

3.1 Introduction

Our approach follows closely the work of D’Ancona, Foschi, and Selberg in [5] and [6], where the authors prove bilinear estimates of the form

$$\|uv\|_{X^{-s_0,-b_0}} \leq C \|u\|_{X^{s_1,b_1}} \|v\|_{X^{s_2,b_2}} \quad (3.1.1)$$

for certain ranges of s_i and b_i in dimensions $n = 1, 2$, and 3 . We wish to generalize these methods to $X_{s,b}^r$ for $1 < r \leq 2$, and find ranges of s_i, b_i such that when $n = 2$, the estimate

$$\|uv\|_{X_{-s_0,-b_0}^r} \leq C \|u\|_{X_{s_1,b_1}^r} \|v\|_{X_{s_2,b_2}^r} \quad (3.1.2)$$

holds for all $u, v \in \mathcal{S}(\mathbb{R}^{1+2})$. If (3.1.2) holds, we will say the matrix

$$\begin{pmatrix} s_0 & s_1 & s_2 \\ b_0 & b_1 & b_2 \end{pmatrix}^r$$

is a *product*.

Letting $b_i = 0$ and $r = 2$, we have the Sobolev space estimate

$$\|uv\|_{H^{-s_0}} \leq C \|u\|_{H^{s_1}} \|v\|_{H^{s_2}}$$

for all $u, v \in \mathcal{S}(\mathbb{R}^n)$. It is known, see [6], that this estimate holds if and only if

i) $s_0 + s_1 + s_2 \geq \frac{n}{2}$

ii) $s_0 + s_1 + s_2 \geq \max(s_0, s_1, s_2)$

iii) we do not allow equality in both of the above simultaneously.

In [5], it is shown that for the $X^{s,b}$ estimates we must also assume $b_j + b_k \geq 0$ for all $j \neq k$ in $\{0, 1, 2\}$. Since $b_j \geq -b_k$ for all j, k this implies that at most one b_j can be negative. We can also exclude the case where all of the b_j are zero, since these are standard Sobolev estimates. The estimate (3.1.2) is symmetric in b_1 and b_2 , and we cannot have $b_j < 0$ if another $b_k = 0$. Thus, there are seven cases:

i) $b_0 = b_1 = 0 < b_2$

ii) $b_0 = 0 < b_1, b_2$

iii) $0 < b_0, b_1, b_2$

iv) $b_0 < 0 < b_1, b_2$

v) $b_1 = b_2 = 0 < b_0$

vi) $b_1 = 0 < b_0, b_2$

vii) $b_1 < 0 < b_0, b_2$.

When $r = 2$ one can use the self-duality of L^2 in the trilinear estimate (2.1.1) to obtain symmetry in b_0, b_1 , and b_2 . Consequently, in $X^{s,b}$, the last three cases are not needed. We do not have this symmetry in the $X_{s,b}^r$ estimates; however, from our local well-posedness argument in Section 1.5, we are only interested in estimates with $b_0 \leq 0$. For this reason, we eliminate the other cases as well and focus only on cases i, ii, and iv.

3.2 The case $b_0 = b_1 = 0 < b_2$

Theorem 3.1. *Suppose $n = 2$ and $1 < r \leq 2$. Set $b_0 = b_1 = 0$ and assume that*

$$b_2 > \frac{1}{r} \tag{3.2.1}$$

$$s_0 + s_1 + s_2 > \frac{2}{r} \quad (3.2.2)$$

$$s_0 + s_1 \geq 0 \quad (3.2.3)$$

$$s_0 + s_2 \geq 0 \quad (3.2.4)$$

$$s_1 + s_2 \geq \frac{2}{r} - \frac{2}{r'} \quad (3.2.5)$$

Then $\begin{pmatrix} s_0 & s_1 & s_2 \\ 0 & 0 & b_2 \end{pmatrix}^r$ is a product.

Proof. From (2.2.2) with $b_0 = b_1 = 0 < b_2$, we obtain

$$I \lesssim \sum_{N,L} \frac{J\left(F_0^{N_0,L_0}, F_1^{N_1,L_1}, F_2^{N_2,L_2}\right)}{N_0^{s_0} N_1^{s_1} N_2^{s_2} L_2^{b_2}}.$$

So, we do not need to separate in L_0 and L_1 and can replace $F_0^{N_0,L_0}$ and $F_1^{N_1,L_1}$ with $F_0^{N_0}$ and $F_1^{N_1}$. We reduce to proving

$$I \lesssim \sum_{N,L} \frac{J\left(F_0^{N_0}, F_1^{N_1}, F_2^{N_2,L_2}\right)}{N_0^{s_0} N_1^{s_1} N_2^{s_2} L_2^{b_2}} \lesssim \|F_0\|_r \|F_1\|_{r'} \|F_2\|_{r'}$$

or

$$\sum_N \frac{S_N}{N_0^{s_0} N_1^{s_1} N_2^{s_2}} \lesssim \|F_0\|_r \|F_1\|_{r'} \|F_2\|_{r'} \quad (3.2.6)$$

where $S_N = \sum_{L_2} L_2^{-b_2} J(F_0^{N_0}, F_1^{N_1}, F_2^{N_2,L_2})$.

By (2.3.11),

$$\begin{aligned} S_N &\leq \sum_{L_2} L_2^{1/r-b_2} (N_{\min}^{012})^{\frac{2}{r'}} (N_{\min}^{12})^{\frac{2}{r}-\frac{2}{r'}} \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'} \\ &\lesssim (N_{\min}^{012})^{\frac{2}{r'}} (N_{\min}^{12})^{\frac{2}{r}-\frac{2}{r'}} \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'} \end{aligned}$$

since $1/r - b_2 < 0$ by (3.2.1). Now we separate into the different frequency cases.

The HLH case: $N_1 \lesssim N_0 \sim N_2$. In this case we have

$$S_N \lesssim N_1^{\frac{2}{r}} \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'}.$$

Applying Lemma 2.2 with $A = 2/r - s_1$ and $B = s_0 + s_2$ in the inequality

$$\sum_N \frac{S_N}{N_0^{s_0} N_1^{s_1} N_2^{s_2}} \lesssim \sum_N \chi_{N_1 \lesssim N_0 \sim N_2} \frac{N_1^{2/r - s_1}}{N_0^{s_0} N_2^{s_2}} \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'}$$

gives the result, since the hypotheses are satisfied by the conditions above as follows.

(i) $B > A$ by (3.2.2).

(ii) $B \geq 0$ by (3.2.4).

(iii) The strict inequality in (i) excludes $A = B = 0$.

The LHH case: $N_0 \lesssim N_1 \sim N_2$. In this case we have

$$S_N \lesssim N_0^{2/r'} N_1^{\frac{2}{r} - \frac{2}{r'}} \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'}.$$

This gives the inequality

$$\sum_N \frac{S_N}{N_0^{s_0} N_1^{s_1} N_2^{s_2}} \lesssim \sum_N \chi_{N_0 \lesssim N_1 \sim N_2} \frac{N_0^{2/r' - s_0}}{N_1^{s_1 + s_2 + 2/r' - 2/r}} \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'}$$

in which we apply Lemma 2.3 with $A = 2/r' - s_0$ and $B = N_1^{s_1 + s_2 + 2/r' - 2/r}$. Checking the hypotheses, we have

(i) $B > A$ by (3.2.2) and

(ii) $B > 0$ by (3.2.5).

and hence (3.2.6) holds by Lemma 2.3. □

3.3 The case $b_0 = 0 < b_1, b_2$

Theorem 3.2. *Suppose $n = 2$ and $3/2 < r \leq 2$. Set $b_0 = 0$ and assume*

$$b_1, b_2 > 0 \tag{3.3.1}$$

$$b_1 + b_2 > \frac{1}{r} \tag{3.3.2}$$

$$s_0 + s_1 + s_2 > \frac{3}{r} - (b_1 + b_2) \quad (3.3.3)$$

$$s_0 + s_1 + s_2 > \frac{2}{r} - b_1 \quad (3.3.4)$$

$$s_0 + s_1 + s_2 > \frac{2}{r} - b_2 \quad (3.3.5)$$

$$s_0 + s_1 + s_2 > \frac{3}{2r} \quad (3.3.6)$$

$$\left(1 - \frac{r'}{2r}\right) s_0 + s_1 + s_2 > \frac{1}{r} \quad (3.3.7)$$

$$s_1 + s_2 > \frac{2}{r} - \frac{2}{r'} \quad (3.3.8)$$

$$s_0 + s_1 \geq 0 \quad (3.3.9)$$

$$s_0 + s_2 \geq 0 \quad (3.3.10)$$

Then $\begin{pmatrix} s_0 & s_1 & s_2 \\ 0 & b_1 & b_2 \end{pmatrix}^r$ is a product.

Proof. By symmetry we can assume $L_1 \leq L_2$ and by the dyadic decomposition we reduce to showing

$$\sum_N \frac{S_N}{N_0^{s_0} N_1^{s_1} N_2^{s_2}} \lesssim \|F_0\|_r \|F_1\|_{r'} \|F_2\|_{r'}$$

where

$$S_N = \sum_L \chi_{L_1 \leq L_2} \frac{J(F_0^{N_0}, F_1^{N_1, L_1}, F_2^{N_2, L_2})}{L_1^{b_1} L_2^{b_2}}.$$

The HLH case: $N_1 \lesssim N_0 \sim N_2$. The constants (2.3.11) and (2.3.12) now become

$$C_1 = (N_{\min}^{012})^{\frac{2}{r'}} (N_{\min}^{12})^{\frac{2}{r} - \frac{2}{r'}} (L_{\min}^{12})^{\frac{1}{r}} = N_1^{2/r} L_1^{1/r} \quad (3.3.11)$$

$$C_2 = (N_{\min}^{012})^{\frac{1}{r'}} (N_{\min}^{12})^{\frac{3}{2r} - \frac{1}{r'}} (L_{\min}^{12})^{\frac{1}{r}} (L_{\max}^{12})^{\frac{1}{2r}} = N_1^{3/2r} L_1^{1/r} L_2^{1/2r}. \quad (3.3.12)$$

Which estimate is better depends on the relative sizes of L_2 and N_1 , so we split into two sub-cases $L_2 \leq N_1$ and $L_2 > N_1$.

Sub-case 1: $L_2 \leq N_1$. We will use Theorem 2.8 with (3.3.12).

$$\begin{aligned} S_N &= \sum_L \chi_{L_1 \leq L_2 \leq N_1} \frac{J(F_0^{N_0}, F_1^{N_1, L_1}, F_2^{N_2, L_2})}{L_1^{b_1} L_2^{b_2}} \\ &\lesssim \sum_L \chi_{L_1 \leq L_2 \leq N_1} N_1^{3/2r} L_1^{1/r - b_1} L_2^{1/2r - b_2} \|F_0^{N_0}\| \|F_1^{N_1, L_1}\| \|F_2^{N_2, L_2}\| \\ &\lesssim \sigma_{1/2r}(N_1) N_1^{3/2r} \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'} \end{aligned} \quad (3.3.13)$$

where

$$\sigma_q(M) = \sum_L \chi_{L_1 \leq L_2 \leq M} L_1^{1/r-b_1} L_2^{q-b_2}. \quad (3.3.14)$$

Using (2.2.4) repeatedly, we obtain

$$\sigma_q(M) \sim \begin{cases} M^{1/r+q-b_1-b_2} & \text{if } b_1 < 1/r, b_1 + b_2 < 1/r + q \\ \log(M) & \text{if } b_1 < 1/r, b_1 + b_2 = 1/r + q \\ 1 & \text{if } b_1 < 1/r, b_1 + b_2 > 1/r + q \\ M^{q-b_2} \log(M) & \text{if } b_1 = 1/r, b_2 < q \\ \log(M)^2 & \text{if } b_1 = 1/r, b_2 = q \\ 1 & \text{if } b_1 = 1/r, b_2 > q \\ M^{q-b_2} & \text{if } b_1 > 1/r, b_2 < q \\ \log(M) & \text{if } b_1 > 1/r, b_2 = q \\ 1 & \text{if } b_1 > 1/r, b_2 > q. \end{cases}$$

Now using that $\log(M) \lesssim C_\epsilon M^\epsilon$ for any $\epsilon > 0$, we have

$$\sigma_q(M) \lesssim \begin{cases} M^{1/r+q-b_1-b_2} & \text{if } b_1 < 1/r, b_1 + b_2 < 1/r + q \\ M^\epsilon & \text{if } b_1 < 1/r, b_1 + b_2 = 1/r + q \\ & \text{or } b_1 = 1/r, b_2 = q \\ & \text{or } b_1 > 1/r, b_2 = q \\ 1 & \text{if } b_1 < 1/r, b_1 + b_2 > 1/r + q \\ & \text{or } b_1 = 1/r, b_2 > q \\ & \text{or } b_1 > 1/r, b_2 > q \\ M^{q-b_2} M^\epsilon & \text{if } b_1 = 1/r, b_2 < q \\ M^{q-b_2} & \text{if } b_1 > 1/r, b_2 < q. \end{cases} \quad (3.3.15)$$

Combining these estimates with (3.3.13), we have

$$\sum_N \chi_{N_1 \lesssim N_0 \sim N_2} \frac{S_N}{N_0^{s_0} N_1^{s_1} N_2^{s_2}} \lesssim \sum_N \chi_{N_1 \lesssim N_0 \sim N_2} \frac{N_1^A}{N_0^B} \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'}$$

where $B = s_0 + s_2$ and $A = \frac{3}{2r} - s_1 + \alpha$ and α is given by

$$\alpha = \begin{cases} \frac{3}{2r} - b_1 - b_2 & \text{if } b_1 < 1/r, b_1 + b_2 < 1/r + q \\ \epsilon & \text{if } b_1 < 1/r, b_1 + b_2 = 1/r + q \\ & \text{or } b_1 = 1/r, b_2 = q \\ & \text{or } b_1 > 1/r, b_2 = q \\ 0 & \text{if } b_1 < 1/r, b_1 + b_2 > 1/r + q \\ & \text{or } b_1 = 1/r, b_2 > q \\ & \text{or } b_1 > 1/r, b_2 > q \\ 1/2r - b_2 + \epsilon & \text{if } b_1 = 1/r, b_2 < q \\ 1/2r - b_2 & \text{if } b_1 > 1/r, b_2 < q. \end{cases} \quad (3.3.16)$$

We will bound the last sum using Lemma 2.2, so we need to check the hypotheses in each of the five cases above. Note that since $B = s_0 + s_2$ in all cases, $B \geq 0$ is satisfied by (3.3.10).

- If $\alpha = \frac{3}{2r} - b_1 - b_2$, then $A = \frac{3}{r} - s_1 - b_1 - b_2$. Then $B > A$ by (3.3.3), which implies the first and third hypotheses.
- If $\alpha = 0$ or $\alpha = \epsilon$, then $A = \frac{3}{2r} - s_1$ or $A = \frac{3}{2r} - s_1 + \epsilon$, respectively. In both cases $B > A$ by (3.3.6), which implies the first and third hypotheses.
- Similarly, if $\alpha = \frac{1}{2r} - b_2$ or $\alpha = \frac{1}{2r} - b_2 + \epsilon$, then $A = \frac{1}{r} - s_1 - b_2$ or $A = \frac{1}{r} - s_1 - b_2 + \epsilon$, respectively. In both cases $B > A$ by (3.3.5).

Thus, by (2.2), we conclude that

$$\sum_N \chi_{N_1 \lesssim N_0 \sim N_2} \frac{S_N}{N_0^{s_0} N_1^{s_1} N_2^{s_2}} \lesssim \|F_0\|_r \|F_1\|_{r'} \|F_2\|_{r'}.$$

Sub-case 2: $L_2 > N_1$. In this case we will use Theorem 2.8 with (3.3.11).

$$\begin{aligned}
S_N &= \sum_L \chi_{L_1 \leq L_2} \frac{J(F_0^{N_0}, F_1^{N_1, L_1}, F_2^{N_2, L_2})}{L_1^{b_1} L_2^{b_2}} \\
&\lesssim \sum_L \chi_{L_1 \leq L_2} N_1^{2/r} L_1^{1/r} \|F_0^{N_0}\|_r \|F_1^{N_1, L_1}\|_{r'} \|F_2^{N_2, L_2}\|_{r'} \\
&\lesssim \gamma(N_1) N_1^{2/r} \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'}
\end{aligned}$$

where

$$\gamma(M) = \sum_L \chi_{L_1 \leq L_2} \chi_{L_2 \geq M} L_1^{1/r - b_1} L_2^{-b_2}. \quad (3.3.17)$$

Again using (2.2.4) and $\log(N) \lesssim N^\epsilon$, we obtain

$$\gamma(M) \lesssim \begin{cases} M^{1/r - b_1 - b_2} & \text{if } b_1 < 1/r \\ M^{\epsilon - b_2} & \text{if } b_1 = 1/r \\ M^{-b_2} & \text{if } b_1 > 1/r. \end{cases} \quad (3.3.18)$$

We then obtain

$$\sum_N \chi_{N_1 \lesssim N_0 \sim N_2} \frac{S_N}{N_0^{s_0} N_1^{s_1} N_2^{s_2}} \lesssim \sum_N \chi_{N_1 \lesssim N_0 \sim N_2} \frac{N_1^A}{N_0^B} \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'}$$

where $B = s_0 + s_2$ and $A = \frac{2}{r} - s_1 + \beta$ and β is given by

$$\beta = \begin{cases} 1/r - b_1 - b_2 & \text{if } b_1 < 1/r \\ \epsilon - b_2 & \text{if } b_1 = 1/r \\ -b_2 & \text{if } b_1 > 1/r. \end{cases} \quad (3.3.19)$$

So, we again verify the hypotheses of (2.2). Recall that the second hypothesis $B \geq 0$ is satisfied by (3.3.10) in all cases.

- If $\beta = \frac{1}{r} - b_1 - b_2$, then $A = \frac{3}{r} - s_1 - b_1 - b_2$. Then $B > A$ by (3.3.3), which implies the first and third hypotheses.
- If $\beta = -b_2$ or $\beta = -b_2 + \epsilon$, then $A = \frac{2}{r} - s_1 - b_2$ or $A = \frac{2}{r} - s_1 - b_2 + \epsilon$, respectively. In both cases $B > A$ by (3.3.5).

As above, this implies the result in the HLH case.

The LHH case: $N_0 \lesssim N_1 \sim N_2$. The constants (2.3.11) and (2.3.12) now become

$$C_1 = (N_{\min}^{012})^{\frac{2}{r'}} (N_{\min}^{12})^{\frac{2}{r} - \frac{2}{r'}} (L_{\min}^{12})^{\frac{1}{r}} = N_0^{2/r'} N_1^{2/r - 2/r'} L_1^{1/r} \quad (3.3.20)$$

$$C_2 = (N_{\min}^{012})^{\frac{1}{r'}} (N_{\min}^{12})^{\frac{3}{2r} - \frac{1}{r'}} (L_{\min}^{12})^{\frac{1}{r}} (L_{\max}^{12})^{\frac{1}{2r}} = N_0^{1/r'} N_1^{3/2r - 1/r'} L_1^{1/r} L_2^{1/2r}. \quad (3.3.21)$$

Combining these two estimates, we have

$$C \sim N_0^{1/r'} N_1^{3/2r - 1/r'} L_1^{1/r} \min \left(N_0^{1/r'} N_1^{1/2r - 1/r'}, L_2^{1/2r} \right)$$

which is (3.3.20) if $N_0^{1/r'} N_1^{1/2r - 1/r'} < L_2^{1/2r}$ and (3.3.21) if $N_0^{1/r'} N_1^{1/2r - 1/r'} \geq L_2^{1/2r}$. So we will again split into sub-cases.

Sub-case 1: $L_2 \leq \frac{N_0^{2r/r'}}{N_1^{2r/r' - 1}}$. In this case we will use (3.3.21). Note that since $L_2 \geq 1$, we must assume $N_0^{2r/r'} \geq N_1^{2r/r' - 1}$ or equivalently, $N_0 \geq N_1^{1 - r'/2r}$. Then we have

$$\begin{aligned} S_N &\lesssim N_0^{1/r'} N_1^{3/2r - 1/r'} \sum_L \chi_{L_1 \leq L_2 \leq \frac{N_0^{2r/r'}}{N_1^{2r/r' - 1}}} L_1^{1/r - b_1} L_2^{1/2r - b_2} \|F_0^{N_0}\|_r \|F_1^{N_1, L_1}\|_{r'} \|F_2^{N_2, L_2}\|_{r'} \\ &\lesssim N_0^{1/r'} N_1^{3/2r - 1/r'} \sigma_{1/2r} \left(\frac{N_0^{2r/r'}}{N_1^{2r/r' - 1}} \right) \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'} \end{aligned}$$

where $\sigma_q(N)$ is given by (3.3.14). From (3.3.15), we get

$$\begin{aligned} \sum_N \chi_{N_1^{1 - r'/2r} \leq N_0 \lesssim N_1 \sim N_2} \frac{S_N}{N_0^{s_0} N_1^{s_1} N_2^{s_2}} \\ \lesssim \sum_N \chi_{N_1^{1 - r'/2r} \leq N_0 \lesssim N_1 \sim N_2} \frac{N_0^A}{N_1^B} \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'} \end{aligned}$$

where

$$A = \frac{1}{r'} - s_0 + \frac{2r}{r'} \alpha \quad (3.3.22)$$

and

$$B = \frac{1}{r'} - \frac{3}{2r} + s_1 + s_2 + \left(\frac{2r}{r'} - 1 \right) \alpha \quad (3.3.23)$$

for α given by (3.3.16). From the dyadic summation rule (2.2.4),

$$\mu_A(N_1) = \sum_{N_0} \chi_{N_1^{1 - r'/2r} \leq N_0 \lesssim N_1} N_0^A \sim \begin{cases} N_1^A & \text{if } A > 0 \\ \log(N_1) & \text{if } A = 0 \\ N_1^{A(1 - r'/2r)} & \text{if } A < 0 \end{cases} \quad (3.3.24)$$

and hence

$$\begin{aligned} \sum_N \chi_{N_1^{1-r'/2r} \leq N_0 \lesssim N_1 \sim N_2} \frac{S_N}{N_0^{s_0} N_1^{s_1} N_2^{s_2}} &\lesssim \|F_0\|_r \sum_{N_1, N_2} \chi_{N_1 \sim N_2} \frac{\mu_A(N_1)}{N_1^B} \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'} \\ &\lesssim \|F_0\|_r \sum_{N_1, N_2} \chi_{N_1 \sim N_2} N_1^{\tilde{A}-B} \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'} \end{aligned} \quad (3.3.25)$$

where

$$\tilde{A} = \begin{cases} A & \text{if } A > 0 \\ \tilde{\epsilon} & \text{if } A = 0 \\ A(1 - r'/2r) & \text{if } A < 0. \end{cases} \quad (3.3.26)$$

Now comparing (3.3.25) with (2.2.7) in the proof of Lemma 2.3, we see that the inequality

$$\sum_N \chi_{N_1^{1-r'/2r} \leq N_0 \lesssim N_1 \sim N_2} \frac{S_N}{N_0^{s_0} N_1^{s_1} N_2^{s_2}} \lesssim \|F_0\|_r \|F_1\|_{r'} \|F_2\|_{r'}$$

follows from this proof, provided $\tilde{A} - B \leq -\epsilon$ for some $\epsilon > 0$. From (3.3.26), this is equivalent to checking the following two conditions:

- i. $B > A$ if $A \geq 0$
- ii. $B > A \left(1 - \frac{r'}{2r}\right)$ if $A < 0$.

From (3.3.22) and (3.3.23), we see that the first condition is equivalent to

$$s_0 + s_1 + s_2 > \frac{3}{2r} + \alpha \quad (3.3.27)$$

while the second is equivalent to

$$\left(1 - \frac{r'}{2r}\right) s_0 + s_1 + s_2 > \frac{1}{r}. \quad (3.3.28)$$

This last requirement is (3.3.7), so it only remains to verify (3.3.27) for all possible α .

- If $\alpha = \frac{3}{2r} - b_1 - b_2$, then (3.3.27) is equivalent to (3.3.3).
- If $\alpha = 0$ or $\alpha = \epsilon$, then (3.3.27) follows from (3.3.6).
- If $\alpha = \frac{1}{2r} - b_2$ or $\alpha = \frac{1}{2r} - b_2 + \epsilon$, then (3.3.27) follows from (3.3.5).

Sub-case 2: $L_2 > \frac{N_0^{2r/r'}}{N_1^{2r/r'-1}}$. In this case we will use (3.3.20). We have

$$\begin{aligned} S_N &\lesssim N_0^{2/r'} N_1^{2/r-2/r'} \sum_L \chi_{L_1 \leq L_2} \chi_{L_2 > \frac{N_0^{2r/r'}}{N_1^{2r/r'-1}}} L_1^{1/r-b_1} L_2^{-b_2} \|F_0^{N_0}\|_r \|F_1^{N_1, L_1}\|_{r'} \|F_2^{N_2, L_2}\|_{r'} \\ &\lesssim N_0^{2/r'} N_1^{2/r-2/r'} \gamma \left(1 + \frac{N_0^{2r/r'}}{N_1^{2r/r'-1}} \right) \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'} \end{aligned}$$

for γ as in (3.3.17).

- First suppose $N_0 \leq N_1^{1-r'/2r}$ or $\frac{N_0^{2r/r'}}{N_1^{2r/r'-1}} \leq 1$. Then we have

$$1 \leq 1 + \frac{N_0^{2r/r'}}{N_1^{2r/r'-1}} \leq 2$$

and so by (3.3.18), we see that $\gamma \left(1 + \frac{N_0^{2r/r'}}{N_1^{2r/r'-1}} \right) \lesssim 1$. Therefore

$$\begin{aligned} \sum_N \chi_{N_0 \lesssim N_1 \sim N_2} \chi_{N_0 \leq N_1^{1-r'/2r}} \frac{S_N}{N_0^{s_0} N_1^{s_1} N_2^{s_2}} \\ \lesssim \sum_N \chi_{N_0 \lesssim N_1 \sim N_2} \chi_{N_0 \leq N_1^{1-r'/2r}} \frac{N_0^{2/r'} N_1^{2/r-2/r'}}{N_0^{s_0} N_1^{s_1} N_2^{s_2}} \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'} \\ \lesssim \|F_0\|_r \sum_{N_1, N_2} \chi_{N_1 \sim N_2} \frac{\Sigma_A \left(N_1^{1-r'/2r} \right)}{N_1^B} \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'} \end{aligned}$$

where

$$\Sigma_A(M) = \sum_{N_0} \chi_{N_0 \leq M} N_0^A \sim \begin{cases} M^A & \text{if } A > 0 \\ \log(M) & \text{if } A = 0 \\ 1 & \text{if } A < 0 \end{cases}$$

for $A = 2/r' - s_0$ and $B = 2/r' - 2/r + s_1 + s_2$. Thus,

$$\sum_N \chi_{N_0 \lesssim N_1 \sim N_2} \chi_{N_0 \leq N_1^{1-r'/2r}} \frac{S_N}{N_0^{s_0} N_1^{s_1} N_2^{s_2}} \lesssim \|F_0\|_r \sum_{N_1, N_2} \chi_{N_1 \sim N_2} \frac{N_1^{\tilde{A}}}{N_1^B} \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'}$$

where

$$\tilde{A} = \begin{cases} (1 - r'/2r)A & \text{if } A > 0 \\ (1 - r'/2r)\tilde{\epsilon} & \text{if } A = 0 \\ 0 & \text{if } A < 0. \end{cases} \quad (3.3.29)$$

As above, we obtain the desired result from the proof of Lemma 2.3, see (2.2.7), provided that $\tilde{A} - B \leq -\epsilon$ for some $\epsilon > 0$. From (3.3.29), this is equivalent to checking the following two conditions.

- i. $B > A \left(1 - \frac{r'}{2r}\right)$ if $A > 0$.
- ii. $B > 0$ if $A \leq 0$.

However, these are (3.3.7) and (3.3.8), respectively.

- Next suppose $N_0 > N_1^{1-r'/2r}$ or $\frac{N_0^{2r/r'}}{N_1^{2r/r'-1}} > 1$. Then we have

$$\frac{N_0^{2r/r'}}{N_1^{2r/r'-1}} \leq 1 + \frac{N_0^{2r/r'}}{N_1^{2r/r'-1}} \leq 2 \frac{N_0^{2r/r'}}{N_1^{2r/r'-1}}$$

and so by (3.3.18), we see that $\gamma \left(1 + \frac{N_0^{2r/r'}}{N_1^{2r/r'-1}}\right) \lesssim \gamma \left(\frac{N_0^{2r/r'}}{N_1^{2r/r'-1}}\right)$. Then

$$\begin{aligned} & \sum_N \chi_{N_1^{1-r'/2r} < N_0 \lesssim N_1 \sim N_2} \frac{S_N}{N_0^{s_0} N_1^{s_1} N_2^{s_2}} \\ & \lesssim \sum_N \chi_{N_1^{1-r'/2r} < N_0 \lesssim N_1 \sim N_2} \gamma \left(\frac{N_0^{2r/r'}}{N_1^{2r/r'-1}}\right) \frac{N_0^{2/r'} N_1^{2/r-2/r'}}{N_0^{s_0} N_1^{s_1} N_2^{s_2}} \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'} \\ & \lesssim \sum_N \chi_{N_1^{1-r'/2r} < N_0 \lesssim N_1 \sim N_2} \frac{N_0^A}{N_1^B} \|F_0^{N_0}\|_r \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'} \end{aligned}$$

by (3.3.18), where

$$A = \frac{2}{r'} - s_0 + \frac{2r}{r'}\beta \quad (3.3.30)$$

and

$$B = \frac{2}{r'} - \frac{2}{r} + s_1 + s_2 + \left(\frac{2r}{r'} - 1\right)\beta \quad (3.3.31)$$

for β as in (3.3.19). Proceeding as in Sub-case 1, we write

$$\begin{aligned} & \sum_N \chi_{N_1^{1-r'/2r} \leq N_0 \lesssim N_1 \sim N_2} \frac{S_N}{N_0^{s_0} N_1^{s_1} N_2^{s_2}} \lesssim \|F_0\|_r \sum_{N_1, N_2} \chi_{N_1 \sim N_2} \frac{\mu_A(N_1)}{N_1^B} \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'} \\ & \lesssim \|F_0\|_r \sum_{N_1, N_2} \chi_{N_1 \sim N_2} N_1^{\tilde{A}-B} \|F_1^{N_1}\|_{r'} \|F_2^{N_2}\|_{r'} \end{aligned}$$

with μ_A defined by (3.3.24), \tilde{A} defined by (3.3.26). Again, we use the proof of Lemma 2.3 to reduce to proving the two conditions

i. $B > A$ if $A \geq 0$

ii. $B > A \left(1 - \frac{r'}{2r}\right)$ if $A < 0$

for A and B as in (3.3.30) and (3.3.31). The first condition is equivalent to

$$s_0 + s_1 + s_2 > \frac{2}{r} + \beta \quad (3.3.32)$$

while the second reduces to (3.3.7). Therefore, we are left to verify (3.3.32) for the various β , as follows.

- If $\beta = \frac{3}{2r} - b_1 - b_2$, then (3.3.32) is equivalent to (3.3.3).
- If $\beta = -b_2$ or $\beta = \epsilon - b_2$, then (3.3.32) follows from (3.3.5).

□

3.4 The case $b_0 < 0 < b_1, b_2$

Theorem 3.3. *Suppose $n = 2$ and $3/2 < r \leq 2$. Assume*

$$b_0 < 0 < b_1, b_2 \quad (3.4.1)$$

$$b_1 + b_2 > \frac{1}{r} \quad (3.4.2)$$

$$s_0 + s_1 + s_2 > \frac{3}{r} - (b_0 + b_1 + b_2) \quad (3.4.3)$$

$$s_0 + s_1 + s_2 > \frac{2}{r} - (b_0 + b_1) \quad (3.4.4)$$

$$s_0 + s_1 + s_2 > \frac{2}{r} - (b_0 + b_2) \quad (3.4.5)$$

$$s_0 + s_1 + s_2 > \frac{3}{2r} - b_0 \quad (3.4.6)$$

$$\left(1 - \frac{r'}{2r}\right) s_0 + s_1 + s_2 > \frac{1}{r} - b_0 \quad (3.4.7)$$

$$s_1 + s_2 > \frac{2}{r} - \frac{2}{r'} - b_0 \quad (3.4.8)$$

$$s_0 + s_1 \geq -b_0 \quad (3.4.9)$$

$$s_0 + s_2 \geq -b_0 \quad (3.4.10)$$

Furthermore, we assume

1. If either $b_0 + b_1 = 0$ or $b_0 + b_2 = 0$, we assume also that

$$b_1, b_2 > \frac{1}{r} \quad (3.4.11)$$

2. If $b_0 + b_1 \neq 0$ or $b_0 + b_2 \neq 0$, we assume also that

$$b_0 + b_1 > \frac{1}{r} \quad (3.4.12)$$

$$b_0 + b_2 > \frac{1}{r} \quad (3.4.13)$$

Then $\begin{pmatrix} s_0 & s_1 & s_2 \\ b_0 & b_1 & b_2 \end{pmatrix}^r$ is a product.

Proof. Using the Hyperbolic Leibniz Rule, (2.1.3), since $-b_0 > 0$, we have

$$\begin{aligned} \|uv\|_{X_{-s_0, -b_0}^r} &= \|\langle |\tau_0| - |\xi_0| \rangle^{-b_0} \langle \xi_0 \rangle^{-s_0} \widetilde{uv}\|_{L_{\tau_0, \xi_0}^{r'}} \\ &\lesssim \left\| \delta(\xi_0 + \xi_1 + \xi_2) (|\tau_1| - |\xi_1| + |\tau_2| - |\xi_2| + \mathbf{b}_{(\pm 1, \pm 2)}(\xi_0, \xi_1, \xi_2))^{-b_0} \langle \xi_0 \rangle^{-s_0} \widetilde{uv} \right\|_{L_{\tau_0, \xi_0}^{r'}} \\ &\lesssim \left\| \langle |\tau_1| - |\xi_1| \rangle^{-b_0} \langle \xi_0 \rangle^{-s_0} \widetilde{uv} \right\|_{L_{\tau_0, \xi_0}^{r'}} + \left\| \langle |\tau_2| - |\xi_2| \rangle^{-b_0} \langle \xi_0 \rangle^{-s_0} \widetilde{uv} \right\|_{L_{\tau_0, \xi_0}^{r'}} \\ &\quad + \left\| \delta(\xi_0 + \xi_1 + \xi_2) (\mathbf{b}_{(\pm 1, \pm 2)}(\xi_0, \xi_1, \xi_2))^{-b_0} \langle \xi_0 \rangle^{-s_0} \widetilde{uv} \right\|_{L_{\tau_0, \xi_0}^{r'}}. \end{aligned}$$

Note that in the last line we have used that for $a, b, n \geq 0$,

$$(a + b)^n \leq (2 \max(a, b))^n \lesssim a^n + b^n$$

since the frequency interactions guarantee that $\mathbf{b}_{\pm 1, \pm 2} \geq 0$.

For the first term, write

$$\begin{aligned} &\|\langle |\tau_1| - |\xi_1| \rangle^{-b_0} \langle \xi_0 \rangle^{-s_0} \widetilde{uv}\|_{L_{\tau_0, \xi_0}^{r'}} \\ &= \left\| \langle \xi_0 \rangle^{-s_0} \int \langle |\tau_1| - |\xi_1| \rangle^{-b_0} \widetilde{u}(\tau_1, \xi_1) \widetilde{v}(\tau_0 - \tau_1, \xi_0 - \xi_1) d\tau_1 d\xi_1 \right\|_{L_{\tau_0, \xi_0}^{r'}} \\ &= \left\| \langle \xi_0 \rangle^{-s_0} \int \widetilde{w}(\tau_1, \xi_1) \widetilde{v}(\tau_0 - \tau_1, \xi_0 - \xi_1) d\tau_1 d\xi_1 \right\|_{L_{\tau_0, \xi_0}^{r'}} \\ &= \|wv\|_{X_{-s_0, 0}^r} \end{aligned}$$

where $\tilde{w}(\tau_1, \xi_1) = \langle |\tau_1| - |\xi_1| \rangle^{-b_0} \tilde{u}(\tau_1, \xi_1)$. Now

$$\begin{aligned} \|w\|_{X_{s_1, b_1}^r} &= \left\| \langle \xi_1 \rangle^{s_1} \langle |\tau_1| - |\xi_1| \rangle^{b_1} \langle |\tau_1| - |\xi_1| \rangle^{-b_0} \tilde{u} \right\|_{r'} \\ &= \|u\|_{X_{s_1, b_1 - b_0}^r} \end{aligned}$$

so the estimate

$$\left\| \langle |\tau_1| - |\xi_1| \rangle^{-b_0} \langle \xi_0 \rangle^{-s_0} \tilde{u} \tilde{v} \right\|_{L_{\tau_0, \xi_0}^{r'}} \lesssim \|u\|_{X_{s_1, b_1}^r} \|v\|_{X_{s_2, b_2}^r}$$

will follow if $\begin{pmatrix} s_0 & s_1 & s_2 \\ 0 & b_1 + b_0 & b_2 \end{pmatrix}^r$ is a product. Similarly,

$$\left\| \langle |\tau_2| - |\xi_2| \rangle^{-b_0} \langle \xi_0 \rangle^{-s_0} \tilde{u} \tilde{v} \right\|_{L_{\tau_0, \xi_0}^{r'}} \lesssim \|u\|_{X_{s_1, b_1}^r} \|v\|_{X_{s_2, b_2}^r}$$

will follow if $\begin{pmatrix} s_0 & s_1 & s_2 \\ 0 & b_1 & b_2 + b_0 \end{pmatrix}^r$ is a product. If $b_0 = -b_i$ for $i = 1$ or $i = 2$, this follows from conditions (3.4.1) through (3.4.11) and Theorem 3.1. On the other hand, if $b_0 \neq -b_i$ for either i , then we use Theorem 3.2 with the additional assumptions (3.4.12) and (3.4.13) to reach the desired conclusion.

For the the third term,

$$\left\| \delta(\xi_0 + \xi_1 + \xi_2) (\mathfrak{b}_{(\pm 1, \pm 2)}(\xi_0, \xi_1, \xi_2))^{-b_0} \langle \xi_0 \rangle^{-s_0} \tilde{u} \tilde{v} \right\|_{L_{\tau_0, \xi_0}^{r'}} = \|\mathfrak{B}_{(\pm 1, \pm 2)}^{-b_0}(u, v)\|_{X_{-s_0, 0}^r}$$

where $\mathfrak{B}_{(\pm 1, \pm 2)}^\alpha(f, g)$ is given by (2.1.7), we will use the estimate (2.1.4) to obtain

$$\|\mathfrak{B}_{(\pm 1, \pm 2)}^{-b_0}(u, v)\|_{X_{-s_0, 0}^r} \lesssim \left\| \delta(\xi_0 + \xi_1 + \xi_2) |\xi_1|^{-b_0} \langle \xi_0 \rangle^{-s_0} \tilde{u} \tilde{v} \right\|_{L_{\tau_0, \xi_0}^{r'}}.$$

Proceeding as above, we find

$$\|\mathfrak{B}_{(\pm 1, \pm 2)}^{-b_0}(u, v)\|_{X_{-s_0, 0}^r} \lesssim \left\| \langle \xi_0 \rangle^{-s_0} \tilde{u} \tilde{v} \right\|_{L_{\tau_0, \xi_0}^{r'}}.$$

where $\tilde{w}_i(\tau_i, \xi_i) = \langle \xi_i \rangle^{-b_0} \tilde{u}(\tau_i, \xi_i)$. Then $\|w_1\|_{X_{s_1, b_1}^r} = \|u\|_{X_{s_1 - b_0, b_1}^r}$ and $\|w_2\|_{X_{s_2, b_2}^r} = \|u\|_{X_{s_2 - b_0, b_2}^r}$. Hence,

$$\|\mathfrak{B}_{(\pm 1, \pm 2)}^{-b_0}(u, v)\|_{X_{-s_0, 0}^r} \lesssim \|u\|_{X_{s_1, b_1}^r} \|v\|_{X_{s_2, b_2}^r}.$$

follows if $\begin{pmatrix} s_0 & s_1 + b_0 & s_2 \\ 0 & b_1 & b_2 \end{pmatrix}^r$ and $\begin{pmatrix} s_0 & s_1 & s_2 + b_0 \\ 0 & b_1 & b_2 \end{pmatrix}^r$ are products. By Theorem 3.1, this is guaranteed by conditions (3.4.1) - (3.4.10). Note that for this estimate we do not need conditions (3.4.11) through (3.4.13). In particular, the last two conditions, (3.4.12) and (3.4.13), are not needed if $b_0 + b_1 = b_0 + b_2 = 0$. \square

3.5 Proof of Theorem 1.9

From the local well-posedness argument in Section 1.5, all that remains is to verify the exponents in the estimates satisfy the conditions of Theorem 3.2 and Theorem 3.3 for the ranges $s > \frac{3}{2r} + 1$, $\frac{1}{r} < b < 1$, and $\frac{3}{2} < r \leq 2$. Explicitly, we need to show

$$\begin{pmatrix} -(s-1) & s-1 & s-1 \\ 0 & b & b \end{pmatrix}^r \quad \text{and} \quad \begin{pmatrix} -(s-2) & s-1 & s-1 \\ -b & b & b \end{pmatrix}^r \quad (3.5.1)$$

are products under these assumptions. We remark that from Section 1.5, we actually only need to take $s_0 = -b - \epsilon$ in the second exponent matrix above. However, due to the nesting property of the $X_{s,b}^r$ spaces, this will follow from the slightly stronger result we verify below.

First we check that the conditions of Theorem 3.2 hold for

$$\begin{pmatrix} s_0 & s_1 & s_2 \\ 0 & b_1 & b_2 \end{pmatrix}^r = \begin{pmatrix} -(s-1) & s-1 & s-1 \\ 0 & b & b \end{pmatrix}^r. \quad (3.5.2)$$

Note that the first two conditions (3.3.1) and (3.3.2) are trivial since we assume $b > \frac{1}{r}$. Next note that $s_0 + s_1 = s_0 + s_2 = 0$ which imply (3.3.9) and (3.3.10). Condition (3.3.6) is our assumption that $s > \frac{3}{2r} + 1$. Condition (3.3.3) is $s > \frac{3}{r} - 2b + 1$ and conditions (3.3.4) and (3.3.5) both become $s > \frac{2}{r} - b + 1$. But $\frac{3}{r} - 2b + 1 < \frac{3}{2r} + 1$ for $b > \frac{3}{4r}$ and $\frac{2}{r} - b + 1 < \frac{3}{2r} + 1$ for $b > \frac{1}{2r}$. Hence, these conditions are also satisfied by our hypotheses. Condition (3.3.8) reduces to $s > \frac{2}{r}$, which is again implied by our assumption

on s provided $r > \frac{1}{2}$. Finally (3.3.7) can be reduced to $\left(1 + \frac{r'}{2r}\right) s > \frac{1}{r} + \frac{r'}{2r} + 1$. Since $s > \frac{3}{2r} + 1$,

$$\left(1 + \frac{r'}{2r}\right) s > \left(1 + \frac{r'}{2r}\right) \frac{3}{2r} + \left(1 + \frac{r'}{2r}\right) \quad (3.5.3)$$

and hence the condition follows provided $\left(1 + \frac{r'}{2r}\right) \frac{3}{2r} > \frac{1}{r}$ or $\left(1 + \frac{r'}{2r}\right) > \frac{2}{3}$ which holds trivially. Therefore, (3.5.2) is a product.

Next we verify the conditions of Theorem 3.3 for

$$\begin{pmatrix} s_0 & s_1 & s_2 \\ b_0 & b_1 & b_2 \end{pmatrix}^r = \begin{pmatrix} -(s-2) & s-1 & s-1 \\ -b & b & b \end{pmatrix}^r. \quad (3.5.4)$$

As above, (3.4.1) and (3.4.2) hold since $b > \frac{1}{r}$. Noting that $s_0 + s_1 = s_0 + s_2 = 1$, conditions (3.4.9) and (3.4.10) are satisfied by the requirement $b < 1$. Next since $b_0 + b_1 = b_0 + b_2 = 0$, we verify that (3.4.11) is just $b > \frac{1}{r}$ and do not need (3.4.12) or (3.4.13). Noting also that $s_0 + s_1 + s_2 = s$, conditions (3.4.4) and (3.4.5) become $s > \frac{2}{r}$, which we verified above. This will also follow from (3.4.8), which in this case becomes $s > \frac{2}{r} + \frac{b}{2}$. But since $b < 1$, we have

$$\frac{2}{r} + \frac{b}{2} < \frac{2}{r} + \frac{1}{2} < \frac{3}{2r} + 1$$

for $r > 1$. Next, condition (3.4.3) is now $s > \frac{3}{r} - b$, which is satisfied since $\frac{3}{r} - b < \frac{3}{r} < \frac{3}{2r} + 1$ for $r > \frac{3}{2}$. Condition (3.4.6) becomes $s > \frac{3}{2r} + b$, which is clear since $b < 1$. Finally, (3.4.7) is now

$$\left(1 + \frac{r'}{2r}\right) s > \frac{1}{r} + \frac{r'}{r} + b = \left(\frac{1}{r} + \frac{r'}{2r} + b - 1\right) + \left(\frac{r'}{2r} + 1\right). \quad (3.5.5)$$

From (3.5.3), it is enough to show

$$\left(1 + \frac{r'}{2r}\right) \frac{3}{2r} > \left(\frac{1}{r} + \frac{r'}{2r} + b - 1\right).$$

But since $b < 1$,

$$\frac{1}{r} + \frac{r'}{2r} + b - 1 < \frac{1}{r} + \frac{r'}{2r} = \frac{3}{2r} + \frac{r' - 1}{2r} \quad (3.5.6)$$

and this follows if $r' - 1 < \frac{3r'}{2r}$, or equivalently, $1 - \frac{1}{r'} < \frac{3}{2r}$, which is clear since $\frac{1}{r} + \frac{1}{r'} = 1$.

Thus, (3.5.4) is also a product, which completes the proof of Theorem 1.9. \square

3.6 Conclusions

The estimates proved in this chapter are sufficient for establishing local well-posedness of (1.0.1) in $\widehat{H}_s^r(\mathbb{R}^2)$ with $s > \frac{3}{2r} + 1$ and $\frac{3}{2} < r \leq 2$. Recalling the scaling correspondence in dimension 2, $\widehat{H}_s^r \sim \dot{H}^\sigma$ for $\sigma = s + 1 - \frac{2}{r}$, on the Sobolev scale our best results correspond to $\sigma > \frac{5}{3}$ when $r = 3/2$. This is a gap of $2/3$ over the scaling conjecture of $\sigma > 1$.

One limitation of our method is the requirement that $\frac{3}{2} < r \leq 2$, which restricts the range for the parameter s . Recall that this arose in our Theorem 2.8, specifically for the second estimate (2.3.12). This theorem is a generalization of the following from Selberg [17, Theorem 2.1].

Theorem 3.4. *The estimate*

$$\|P_{K_{N_0, L_0}^{\pm 0}}(P_{K_{N_1, L_1}^{\pm 1}} u_1 \cdot P_{K_{N_2, L_2}^{\pm 2}} u_2)\|_{L^2} \leq C \|u_1\|_{L^2} \|u_2\|_{L^2}$$

holds with

$$C \sim N_{\min}^{012} (L_{\min}^{012})^{\frac{1}{2}} \tag{3.6.1}$$

$$C \sim (N_{\min}^{012} L_{\min}^{12})^{\frac{1}{2}} (N_{\min}^{12} L_{\max}^{12})^{\frac{1}{4}} \tag{3.6.2}$$

regardless of the choice of signs \pm_j .

Note that for $r = 2$, our estimates (2.3.11) and (2.3.12) agree with Selberg's (3.6.1) and (3.6.2), respectively. To prove our second estimate in the LHH case we utilize the same angular Whitney decomposition found in [17], which requires us to sum over the angle γ ,

$$\sum_{\gamma_0 < \gamma < 1} \gamma^{1/r - 2/r'}.$$

In order to sum this term, it is necessary that the exponent on γ be negative. However, $1/r - 2/r' < 0$ only for $r > 3/2$. One possible way to circumvent this problem, and potentially expand the range for r , is to refine the angular decomposition.

However, even with an improved range for r , the requirement $s > \frac{3}{2r} + 1$ is still an obstacle to achieving the optimal Sobolev result $\sigma > 1$. Considering the scaling relationship between σ and s , with this requirement the best possible outcome (with $r > 1$) would correspond to Sobolev results of $\sigma > \frac{3}{2}$. For the first product in (3.5.1) that we desire, the most restrictive condition in our theorems is

$$s_0 + s_1 + s_2 > \frac{3}{2r}, \quad (3.6.3)$$

which becomes $s > \frac{3}{2r} + 1$. The condition (3.6.3) results from the exponent $\frac{3}{2r} - \frac{1}{r}$ on N_{\min}^{12} in our estimate (2.3.12), and using our dyadic summation techniques there appears to be no way around this without altering the estimate (2.3.12).

In light of these limitations, we include some reference material in the appendices that may be helpful in future work. An alternative angular Whitney decomposition from [1] is outlined in Appendix A. Appendix B contains some geometric integration results from [7] utilized by Grünrock in [8] for the Cauchy problem in three dimensions. We hope that with these additional techniques, we could improve the results of this thesis, and extend them further in the case of the null form.

A P P E N D I X A

AN ANGULAR WHITNEY DECOMPOSITION

In the paper [1], the authors prove boundedness results for the bilinear Fourier multiplier

$$\sigma(\xi, \eta) = \frac{\xi \cdot \eta}{|\xi||\eta|} + i \left(1 - \left(\frac{\xi \cdot \eta}{|\xi||\eta|} \right)^2 \right)^{1/2}$$

where $\xi, \eta \in \mathbb{R}^n$ with $n \geq 2$. Let $\dot{\xi} = (\dot{\xi}_1, \dot{\xi}_2) = \frac{\xi}{|\xi|}$ and $\dot{\eta} = (\dot{\eta}_1, \dot{\eta}_2) = \frac{\eta}{|\eta|}$. Then in dimension $n = 2$, the imaginary part

$$\sigma_I(\xi, \eta) = \left(1 - \left(\frac{\xi \cdot \eta}{|\xi||\eta|} \right)^2 \right)^{1/2} = \frac{|\xi \wedge \eta|}{|\xi||\eta|}$$

becomes

$$\sigma_I(\xi, \eta) = |\dot{\xi} \wedge \dot{\eta}| = |\dot{\xi}_2 \dot{\eta}_1 - \dot{\xi}_1 \dot{\eta}_2|$$

which is the absolute value of the symbol of the null form Q_{12} . Call the associated operator Q_I , i.e.

$$Q_I(q, q)(x) = \int \int e^{i\pi x \cdot (\xi + \eta)} \sigma_I(\xi, \eta) \frac{\hat{q}(\xi)}{|\xi|} \frac{\hat{q}(\eta)}{|\eta|} d\xi d\eta.$$

The derivatives of σ_I have singularities on the set $L = \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n : \xi \wedge \eta = 0\}$, and in [1] the authors decompose $\mathbb{R}^n \times \mathbb{R}^n \setminus L$ into disjoint cones $Q_{i,j,k}$ to write Q_I as the sum

$$Q_I = \sum_{k=1}^{\infty} \sum_{i=1}^{2n2^{k(n-1)}} \sum_{j \in r(i,k)} T_{i,j,k}$$

where $T_{i,j,k}$ are operators supported on $Q_{i,j,k}$. The decomposition is done so that the sum in j is finite and the number of terms in that sum is universally bounded by a constant

that depends only on the dimension n . Furthermore, for $(\xi, \eta) \in Q_{i,j,k}$, we have

$$\left(1 - \left(\frac{\xi \cdot \eta}{|\xi||\eta|}\right)^2\right)^{1/2} \sim 2^{-k}.$$

In dimension $n = 2$, the decomposition is an angular, Whitney type decomposition and using the multiplicative structure of σ_I , the authors prove the following theorem.

Theorem A.1. *Let $0 \leq \alpha < \beta < \alpha + 1$ and $1 < s < \infty$.*

1. *Assume that $1 < r < 2$ and $\beta < \min\{\alpha + 1 + 2(\frac{1}{s} - \frac{1}{r}), 2(1 + \alpha) + 2(\frac{1}{s} - \frac{2}{r})\}$.*

Then we have

$$\|T_{i,j,k}(f, g)\|_{W_{loc}^{\beta,s}} \leq C2^{-k\bar{\omega}} \|f\|_{W^{\alpha,r}} \|g\|_{W^{\alpha,r}}$$

for any $\bar{\omega} < 1/2 + 1/r'$.

2. *If $\max\{r', \frac{2s}{s+1}\} < t < 2 \leq r < \infty$, $\beta \leq \alpha + 2 + 2(\frac{1}{s} - \frac{1}{r} - \frac{1}{t})$, then*

$$\|T_{i,j,k}(f, g)\|_{W_{loc}^{\beta,s}} \leq C2^{-k\bar{\omega}} (\|f\|_{W^{\alpha,r}} + \|f\|_{L^t}) (\|g\|_{W^{\alpha,r}} + \|g\|_{L^t})$$

for any $\bar{\omega} < 1/2 + 1/r$.

We are interested in the angular decomposition as a possible alternative to the one utilized in Section 2.3, which required us to take $\frac{3}{2} < r \leq 2$.

The Angular Whitney Decomposition

For $k \geq 0$, divide the circle S^1 into $4 \cdot 2^k$ intervals $I_{i,k}$, where

$$I_{i,k} = \left\{ \xi \in \mathbb{R}^2 : |\xi| = 1, \arg(\xi) \in \left((i-1)\frac{\pi}{2 \cdot 2^k}, i\frac{\pi}{2 \cdot 2^k} \right] \right\}$$

The k -dyadic decomposition of S^1 is

$$\mathcal{I}_k = \{I_{i,k}\}_{i=1}^{4 \cdot 2^k}$$

Definition A.2. *Given an interval $I \in \mathcal{I}_k$, its **father** is the unique interval $I_F \in \mathcal{I}_{k-1}$ containing I .*

Definition A.3. We say $I_{i,k}, I_{j,k} \in \mathcal{I}_k$, are **neighbors** if $\bar{I}_{i,k} \cap \bar{I}_{j,k} \neq \emptyset$ or $\bar{I}_{i,k} \cap -\bar{I}_{j,k} \neq \emptyset$.

Notice that for each $k \geq 1$, any interval $I_{i,k}$ has six neighbors: itself, the two adjacent intervals and the reflections of these three. For $k = 0$, there are only four intervals in \mathcal{I}_0 and they are all neighbors.

Definition A.4. We say $I_{i,k}, I_{j,k} \in \mathcal{I}_k, k \geq 1$, are **related**, and write $j \in r(i; k)$, if they are not neighbors but their fathers are.

An important feature of this decomposition is that the size of the set of relations $r(i; k)$ is bounded, independent of k . For $k = 1$, there are 8 intervals in \mathcal{I}_1 . Each of these intervals has 6 neighbors, which cannot be relations. Since all of the intervals in \mathcal{I}_0 are neighbors, every interval in \mathcal{I}_1 has 2 relations. For $k \geq 2$, each interval in \mathcal{I}_{k-1} has 6 neighbors and contains two intervals in \mathcal{I}_k . Suppose $I \in \mathcal{I}_k$ and $I_F \in \mathcal{I}_{k-1}$ is its father. Then the six neighbors of I_F contain 12 intervals of \mathcal{I}_k , 6 of which are I 's neighbors. Hence, I has 6 relations. Therefore,

$$|r(i; k)| = \begin{cases} 2 & \text{if } k = 1 \\ 6 & \text{if } k \geq 2 \end{cases}$$

Now we will use this decomposition of the circle to partition $\mathbb{R}^2 \times \mathbb{R}^2 \setminus L$. For each interval $I_{i,k}$, we define a corresponding cone

$$Q_{i,k} = \left\{ \xi \in \mathbb{R}^2 \setminus \{0\} : \frac{\xi}{|\xi|} \in I_{i,k} \right\}$$

and the set of all such cones

$$\mathcal{Q}_k = \bigcup_{i=1}^{4 \cdot 2^k} Q_{i,k}$$

forms the k -dyadic decomposition of \mathbb{R}^2 . We will extend the definitions of father, neighbor and relation naturally to these cones. For $j \in r(i; k)$, let

$$Q_{i,j,k} = Q_{i,k} \times Q_{j,k}$$

Claim A.5. *The products of cones $Q_{i,j,k}$ form a partition of $\mathbb{R}^2 \times \mathbb{R}^2 \setminus L$.*

$$\mathbb{R}^2 \times \mathbb{R}^2 \setminus L = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{4 \cdot 2^k} \bigcup_{j \in r(i;k)} Q_{i,j,k}$$

To see that this union is disjoint., suppose $(\xi, \eta) \in Q_{i,j,k} \cap Q_{l,m,k}$. Then $\xi \in Q_{i,k}$ and $\xi \in Q_{l,k}$ and hence $i = l$. Similarly, we must have $j = m$. Now suppose that $k \neq n$ and $(\xi, \eta) \in Q_{i,j,k} \cap Q_{l,m,n}$. For each s , let Q_s be the unique cone in \mathcal{Q}_s containing ξ and \tilde{Q}_s be the unique cone containing η . First note that if Q_s and \tilde{Q}_s are neighbors, then their fathers Q_{s-1} and \tilde{Q}_{s-1} are also neighbors. By induction, Q_t and \tilde{Q}_t are neighbors for $0 \leq t \leq s$. Without loss of generality, assume $n < k$. Then $\xi \in Q_{i,k} = Q_k \subset Q_{l,n} = Q_n$, $\eta \in Q_{j,k} = \tilde{Q}_k \subset Q_{m,n} = \tilde{Q}_n$. Since the pair Q_k, \tilde{Q}_k are related, their respective fathers, Q_{k-1}, \tilde{Q}_{k-1} must be neighbors. Since $n \leq k - 1$, it follows from above that Q_n and \tilde{Q}_n are neighbors. This contradicts that Q_n and \tilde{Q}_n are related.

APPENDIX B

GEOMETRIC INTEGRAL TECHNIQUES

In [7], the authors prove estimates of the type

$$\begin{aligned} & \|D^{\beta_0} D_+^{\beta_+} D_-^{\beta_-}(\phi\psi)\|_{L^2(\mathbb{R}^{1+n})} \\ & \lesssim (\|D^{\alpha_1}\phi_0\|_{L^2(\mathbb{R}^n)} + \|D^{\alpha_1-1}\phi_1\|_{L^2(\mathbb{R}^n)}) (\|D^{\alpha_2}\psi_0\|_{L^2(\mathbb{R}^n)} + \|D^{\alpha_2-1}\psi_1\|_{L^2(\mathbb{R}^n)}) \end{aligned} \quad (\text{B.0.1})$$

where $\alpha_1, \alpha_2, \beta_0, \beta_+, \beta_- \in \mathbb{R}$, ϕ, ψ are solutions of the homogeneous wave equations

$$\square\phi = 0, \quad \square\psi = 0 \quad (\text{B.0.2})$$

with initial conditions

$$\phi(0, x) = \phi_0(x), \partial_t\phi(0, x) = \phi_1(x), \quad \psi(0, x) = \psi_0(x), \partial_t\psi(0, x) = \psi_1(x) \quad (\text{B.0.3})$$

and

$$\begin{aligned} \widehat{D^\alpha f}(\xi) &= |\xi|^\alpha \hat{f}(\xi) \\ \widetilde{D_+^\alpha F}(\tau, \xi) &= (|\tau| + |\xi|)^\alpha \tilde{F}(\tau, \xi) \\ \widetilde{D_-^\alpha F}(\tau, \xi) &= ||\tau| - |\xi||^\alpha \tilde{F}(\tau, \xi). \end{aligned}$$

The main result is the following theorem.

Theorem B.1. *Let $n \geq 2$. Let ϕ, ψ be solutions of (B.0.2), (B.0.3). Then the estimate (B.0.1) holds if and only if $\alpha_1, \alpha_2, \beta_0, \beta_+, \beta_-$ satisfy the following conditions:*

$$\beta_0 + \beta_+ + \beta_- = \alpha_1 + \alpha_2 - \frac{n-1}{2}, \quad (\text{B.0.4})$$

$$\beta_- \geq -\frac{n-3}{4}, \quad (\text{B.0.5})$$

$$\beta_0 > -\frac{n-1}{2}, \quad (\text{B.0.6})$$

$$\alpha_i \leq \beta_- + \frac{n-1}{2}, \quad i = 1, 2, \quad (\text{B.0.7})$$

$$\alpha_1 + \alpha_2 \geq \frac{1}{2}, \quad (\text{B.0.8})$$

$$(\alpha_i, \beta_-) \neq \left(\frac{n+1}{4}, -\frac{n-3}{4} \right), \quad i = 1, 2, \quad (\text{B.0.9})$$

$$(\alpha_1 + \alpha_2, \beta_-) \neq \left(\frac{1}{2}, -\frac{n-3}{4} \right). \quad (\text{B.0.10})$$

The key ingredients in the proof of this theorem are two geometric lemmas, which we present here. In [8], Grünrock used the transfer principle and these techniques to prove local well-posedness for (1.0.1) in $\widehat{H}_s^r(\mathbb{R}^3)$ for a range of parameters that closed the pre-existing gap on the Sobolev scale. Using this approach, it may be possible to get the full result also in dimension 2.

Preliminaries

We can decompose a solution ϕ of (B.0.2), (B.0.3) into positive and negative parts ϕ^\pm such that $\phi = \phi^+ + \phi^-$ where

$$\widehat{\phi}^\pm(\tau, \xi) \simeq \delta(\tau \mp |\xi|) \widehat{\phi}_0^\pm(\xi)$$

and

$$\widehat{\phi}_0^\pm(\xi) = \widehat{\phi}_0(\xi) \mp i \frac{\widehat{\phi}_1(\xi)}{|\xi|}, \quad \widehat{\psi}_0^\pm(\xi) = \widehat{\psi}_0(\xi) \mp i \frac{\widehat{\psi}_1(\xi)}{|\xi|}.$$

Then the product $\phi\psi$ can be written as the sum of four pieces

$$\phi\psi = \phi^+\psi^+ + \phi^+\psi^- + \phi^-\psi^+ + \phi^-\psi^-.$$

By exchanging ϕ and ψ , the $(-+)$ case becomes the $(+-)$ and by replacing t with $-t$, the $(--)$ becomes the $(++)$. Thus, it is enough to prove the estimate (B.0.1) for the $(++)$ and $(+-)$ cases. Furthermore, since $\widetilde{\phi^+\psi^+} = \widetilde{\phi^+} * \widetilde{\psi^+}$,

$$\begin{aligned}\widetilde{\phi^+\psi^+}(\tau, \xi) &\simeq \int \delta(t - |\eta|) \widehat{\phi_0^+}(\eta) \delta(\tau - t - |\xi - \eta|) \widehat{\psi_0^+}(\xi - \eta) d(t, \eta) \\ &\simeq \int \delta(\tau - |\eta| - |\xi - \eta|) \widehat{\phi_0^+}(\eta) \widehat{\psi_0^+}(\xi - \eta) d\eta\end{aligned}\quad (\text{B.0.11})$$

and similarly,

$$\widetilde{\phi^+\psi^-}(\tau, \xi) \simeq \int \delta(\tau - |\eta| + |\xi - \eta|) \widehat{\phi_0^+}(\eta) \widehat{\psi_0^-}(\xi - \eta) d\eta. \quad (\text{B.0.12})$$

The first integration (B.0.11) is over a compact manifold, the ellipsoid of revolution with foci at 0 and ξ ,

$$\mathcal{E}(\tau, \xi) = \{\eta \in \mathbb{R}^n : |\eta| + |\xi - \eta| = \tau\} \quad (\text{B.0.13})$$

while the second integration (B.0.12) is over an unbounded manifold, the hyperboloid of revolution with foci at 0 and ξ ,

$$\mathcal{H}(\tau, \xi) = \{\eta \in \mathbb{R}^n : |\eta| - |\xi - \eta| = \tau\}. \quad (\text{B.0.14})$$

Also, in (B.0.11), we must have $\tau \geq |\xi|$ since

$$\tau = |\eta| - |\xi - \eta| \geq |\eta + (\xi - \eta)| = |\xi|$$

while in (B.0.12), we need $|\tau| \leq |\xi|$ since

$$|\tau| = ||\eta| - |\xi - \eta|| \leq |\eta - (\eta - \xi)| = |\xi|.$$

Let ϕ be a smooth function and let S be the hypersurface $S = \{x : \phi(x) = 0\}$. If ϕ is such that $\nabla\phi(x) \neq 0$ for $x \in S \cap \text{supp}f$, then

$$\int f(x) \delta(\phi(x)) dx = \int_S f(x) \frac{dS_x}{|\nabla\phi(x)|} \quad (\text{B.0.15})$$

where dS_x is the induced measure on S . Now let $g \geq 0$ be a smooth function that does not vanish on S . Then

$$\begin{aligned} \int f(x)g(x)\delta(g(x)\phi(x))dx &= \int_S f(x)g(x)\frac{dS_x}{|\nabla(g\phi)|} \\ &= \int_S f(x)g(x)\frac{dS_x}{|\phi\nabla g + g\nabla\phi|} \\ &= \int_S f(x)\frac{dS_x}{|\nabla\phi|}. \end{aligned}$$

Thus,

$$\delta(\phi(x)) = g(x)\delta(g(x)\phi(x)). \quad (\text{B.0.16})$$

Now by taking Fourier Transforms and using Plancherel's theorem, the estimate (B.0.1) in the $(++)$ and $(+-)$ cases follow from the estimates

$$\left\| |\xi|^{\beta_0}(|\tau| + |\xi|)^{\beta_+} \|\tau| - |\xi|^{\beta_-} \widetilde{\phi^+ \psi^\pm}(\tau, \xi) \right\|_{L^2_{\tau, \xi}} \lesssim \left\| |\eta|^{\alpha_1} \widehat{\phi_0}(\eta) \right\|_{L^2_\eta} \left\| |\zeta|^{\alpha_2} \widehat{\psi_0}(\zeta) \right\|_{L^2_\zeta}. \quad (\text{B.0.17})$$

These estimates are equivalent to showing that the operators $B_{(+\pm)} : L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^{1+n})$ defined by

$$\tilde{B}_{(++)}(f, g)(\tau, \xi) = \int \delta(\tau - |\eta| - |\xi - \eta|) \frac{|\xi|^{\beta_0} \tau^{\beta_+} (\tau - |\xi|)^{\beta_-}}{|\eta|^{\alpha_1} |\xi - \eta|^{\alpha_2}} f(\eta) g(\xi - \eta) d\eta \quad (\text{B.0.18})$$

and

$$\tilde{B}_{(+-)}(f, g)(\tau, \xi) = \int \delta(\tau - |\eta| + |\xi - \eta|) \frac{|\xi|^{\beta_0 + \beta_+} (|\xi| - |\tau|)^{\beta_-}}{|\eta|^{\alpha_1} |\xi - \eta|^{\alpha_2}} f(\eta) g(\xi - \eta) d\eta \quad (\text{B.0.19})$$

are bounded. To see this, suppose we know $B_{(++)}$ is bounded. Since $\tau \geq |\xi|$ on the ellipsoid,

$$\begin{aligned} & \left\| |\xi|^{\beta_0}(|\tau| + |\xi|)^{\beta_+} \|\tau| - |\xi|^{\beta_-} \widetilde{\phi^+ \psi^+}(\tau, \xi) \right\|_{L^2_{\tau, \xi}} \\ &= \left\| |\xi|^{\beta_0}(|\tau| + |\xi|)^{\beta_+} \|\tau| - |\xi|^{\beta_-} \int \delta(\tau - |\eta| - |\xi - \eta|) \widehat{\phi_0^+}(\eta) \widehat{\psi_0^+}(\xi - \eta) d\eta \right\|_{L^2_{\tau, \xi}} \\ &\lesssim \left\| |\xi|^{\beta_0} |\tau|^{\beta_+} \|\tau| - |\xi|^{\beta_-} \int \delta(\tau - |\eta| - |\xi - \eta|) \frac{\widehat{\phi_0^+}(\eta) |\eta|^{\alpha_1} \widehat{\psi_0^+}(\xi - \eta) |\xi - \eta|^{\alpha_2}}{|\eta|^{\alpha_1} |\xi - \eta|^{\alpha_2}} d\eta \right\|_{L^2_{\tau, \xi}} \\ &\lesssim \left\| |\eta|^{\alpha_1} \widehat{\phi_0^+}(\eta) \right\|_{L^2_\eta} \left\| |\xi - \eta|^{\alpha_2} \widehat{\psi_0^+}(\xi - \eta) \right\|_{L^2_\xi} \end{aligned}$$

which implies (B.0.17) in the $(++)$ case. The $(+-)$ case follows similarly, since then we have $|\tau| \leq |\xi|$.

Integration on ellipsoids and hyperboloids

The first lemma concerns integration over the ellipsoid $\mathcal{E}(\tau, \xi)$.

Lemma B.2. *Consider the integral*

$$I(F)(\tau, \xi) = \int \delta(\tau - |\eta| - |\xi - \eta|) F(|\eta|, |\xi - \eta|) d\eta$$

defined in the space-time region $\tau \geq |\xi|$. Then

$$I(F)(\tau, \xi) \simeq (\tau^2 - |\xi|^2)^{\frac{n-3}{2}} \int_{-1}^1 F\left(\frac{\tau + |\xi|x}{2}, \frac{\tau - |\xi|x}{2}\right) (\tau^2 - |\xi|^2 x^2) (1 - x^2)^{\frac{n-3}{2}} dx. \quad (\text{B.0.20})$$

Proof. Using (B.0.16), with $g(\eta) = \tau - |\eta| + |\xi - \eta|$, we have

$$\begin{aligned} \delta(\tau - |\eta| - |\xi - \eta|) &= (\tau - |\eta| + |\xi - \eta|) \delta((\tau - |\eta|)^2 - |\xi - \eta|^2) \\ &= 2(\tau - |\eta|) \delta\left(\tau^2 - 2\tau|\eta| + |\eta|^2 - \sum_{i=1}^n (\xi_i - \eta_i)^2\right) \\ &= 2(\tau - |\eta|) \delta\left(\tau^2 - 2\tau|\eta| + |\eta|^2 - \sum_{i=1}^n \xi_i^2 + 2 \sum_{i=1}^n \xi_i \eta_i - \sum_{i=1}^n \eta_i^2\right) \\ &= 2(\tau - |\eta|) \delta(\tau^2 - 2\tau|\eta| - |\xi|^2 + 2\xi \cdot \eta). \end{aligned}$$

Next introduce polar coordinates for η ,

$$\rho = |\eta| \quad \omega = \frac{\eta}{|\eta|} \quad \Rightarrow \quad d\eta = \rho^{n-1} dS_\omega d\rho \quad (\text{B.0.21})$$

and set

$$a = \omega \cdot \frac{\xi}{|\xi|} = |\omega| \frac{|\xi|}{|\xi|} \cos(\xi, \omega) = \cos(\xi, \eta), \quad -1 \leq a \leq 1. \quad (\text{B.0.22})$$

We have an isomorphism $[-1, 1] \times S^{n-2} \xrightarrow{\sim} S^{n-1}$ defined by

$$(a, \omega') \mapsto (a, \sqrt{1 - a^2} \omega')$$

for $\omega' \in S^{n-2}$. Computing the volume forms, it then follows that

$$dS_\omega = (1 - a^2)^{\frac{n-3}{2}} dS_{\omega'} da. \quad (\text{B.0.23})$$

Then the integral becomes

$$\begin{aligned} I(F)(\tau, \xi) &= \int \delta(\tau - |\eta| - |\xi - \eta|) F(|\eta|, |\xi - \eta|) d\eta \\ &= 2 \int (\tau - |\eta|) \delta(\tau^2 - 2\tau|\eta| - |\xi|^2 + 2\xi \cdot \eta) F(|\eta|, |\xi - \eta|) d\eta \\ &\simeq \int_0^\infty \int_{\mathbb{S}^{n-1}} \delta(\tau^2 - |\xi|^2 - 2\tau\rho + 2|\xi|\rho \cos(\xi, \omega)) (\tau - \rho) F(\rho, \tau - \rho) \rho^{n-1} dS_\omega d\rho \\ &\simeq \int_0^\infty \int_{-1}^1 \int_{\mathbb{S}^{n-2}} \delta(\tau^2 - |\xi|^2 - 2\tau\rho + 2|\xi|\rho a) (\tau - \rho) F(\rho, \tau - \rho) \rho^{n-1} (1 - a^2)^{\frac{n-3}{2}} dS_{\omega'} da d\rho \\ &\simeq |\mathbb{S}^{n-2}| \int_0^\infty \int_{-1}^1 \delta(\tau^2 - |\xi|^2 - 2\tau\rho + 2|\xi|\rho a) (\tau - \rho) F(\rho, \tau - \rho) \rho^{n-1} (1 - a^2)^{\frac{n-3}{2}} da d\rho. \end{aligned}$$

Now using the delta, we have

$$\tau^2 - |\xi|^2 - 2\tau\rho + 2|\xi|\rho a = 0$$

so

$$a = -\frac{\tau^2 - |\xi|^2 - 2\tau\rho}{2|\xi|\rho}$$

and

$$\rho = \frac{\tau^2 - |\xi|^2}{2\tau - 2a|\xi|}.$$

Since $-1 \leq a \leq 1$ and $\tau \geq |\xi|$, from above we see that

$$\frac{\tau - |\xi|}{2} \leq \rho \leq \frac{\tau + |\xi|}{2}.$$

Using this and (B.0.15), we have

$$\begin{aligned}
I(F)(\tau, \xi) &\simeq \int_0^\infty \int_{-1}^1 \delta(\tau^2 - |\xi|^2 - 2\tau\rho + 2|\xi|\rho a)(\tau - \rho)F(\rho, \tau - \rho)\rho^{n-1}(1 - a^2)^{\frac{n-3}{2}} da d\rho \\
&\simeq \frac{1}{|\xi|} \int_{\frac{\tau-|\xi|}{2}}^{\frac{\tau+|\xi|}{2}} F(\rho, \tau - \rho)(\tau - \rho)\rho^{n-2} \left(1 - \left(\frac{\tau^2 - |\xi|^2 - 2\tau\rho}{2|\xi|\rho}\right)^2\right)^{\frac{n-3}{2}} d\rho \\
&\simeq \frac{1}{|\xi|} \int_{\frac{\tau-|\xi|}{2}}^{\frac{\tau+|\xi|}{2}} F(\rho, \tau - \rho)(\tau - \rho)\rho \left(\rho^2 - \left(\frac{\tau^2 - |\xi|^2 - 2\tau\rho}{2|\xi|}\right)^2\right)^{\frac{n-3}{2}} d\rho \\
&\simeq \frac{1}{|\xi|^{n-2}} \int_{\frac{\tau-|\xi|}{2}}^{\frac{\tau+|\xi|}{2}} F(\rho, \tau - \rho)(\tau - \rho)\rho \left[\left(\rho|\xi| - \frac{\tau^2 - |\xi|^2 - 2\tau\rho}{2}\right) \left(\rho|\xi| + \frac{\tau^2 - |\xi|^2 - 2\tau\rho}{2}\right)\right]^{\frac{n-3}{2}} d\rho \\
&\simeq \frac{1}{|\xi|^{n-2}} \int_{\frac{\tau-|\xi|}{2}}^{\frac{\tau+|\xi|}{2}} F(\rho, \tau - \rho)(\tau - \rho)\rho \left[\left(\rho(\tau + |\xi|) - \frac{\tau^2 - |\xi|^2}{2}\right) \left(\rho(|\xi| - \tau) + \frac{\tau^2 - |\xi|^2}{2}\right)\right]^{\frac{n-3}{2}} d\rho \\
&\simeq \frac{(\tau^2 - |\xi|^2)^{\frac{n-3}{2}}}{|\xi|^{n-2}} \int_{\frac{\tau-|\xi|}{2}}^{\frac{\tau+|\xi|}{2}} F(\rho, \tau - \rho)(\tau - \rho)\rho \left[\left(\rho - \frac{\tau - |\xi|}{2}\right) \left(\rho + \frac{\tau + |\xi|}{2}\right)\right]^{\frac{n-3}{2}} d\rho.
\end{aligned}$$

Changing variables, let

$$x = \frac{2\rho - \tau}{|\xi|} \quad \text{or} \quad \rho = \frac{\tau + |\xi|x}{2}.$$

Then

$$\begin{aligned}
I(F)(\tau, \xi) &\simeq \frac{(\tau^2 - |\xi|^2)^{\frac{n-3}{2}}}{|\xi|^{n-3}} \int_{-1}^1 F\left(\frac{\tau + |\xi|x}{2}, \frac{\tau - |\xi|x}{2}\right) \frac{\tau^2 - |\xi|^2 x^2}{4} \left[\frac{|\xi|^2 - |\xi|^2 x^2}{4}\right]^{\frac{n-3}{2}} d\rho \\
&\simeq (\tau^2 - |\xi|^2)^{\frac{n-3}{2}} \int_{-1}^1 F\left(\frac{\tau + |\xi|x}{2}, \frac{\tau - |\xi|x}{2}\right) (\tau^2 - |\xi|^2 x^2) (1 - x^2)^{\frac{n-3}{2}} d\rho.
\end{aligned}$$

□

To determine the asymptotic behavior of integrals over $\mathcal{E}(\tau, \xi)$, we use the following lemma.

Lemma B.3. *Let $a \in \mathbb{R}$ and $m > -1$. For $\lambda > 0$, define*

$$H_m^a(\lambda) = \int_0^1 (\lambda + t)^a t^m dt = \lambda^{a+m+1} \int_0^{1/\lambda} (1 + s)^a s^m ds.$$

Then

$$H_m^a \sim \begin{cases} \lambda^a & \text{as } \lambda \rightarrow \infty \\ \lambda^{\min(a+m+1, 0)} & \text{as } \lambda \rightarrow 0 \text{ if } a + m + 1 \neq 0 \\ |\log(\lambda)| & \text{as } \lambda \rightarrow 0 \text{ if } a + m + 1 = 0. \end{cases} \quad (\text{B.0.24})$$

In particular, if $a \leq b$ then $H_m^a(\lambda) \gtrsim H_m^b(\lambda)$ as $\lambda \rightarrow 0$.

Proposition B.4. Let $a, b \in \mathbb{R}$ and $\tau > |\xi|$. Define the integral

$$I(\tau, \xi) = \int \frac{\delta(\tau - |\eta| - |\xi - \eta|)}{|\eta|^a |\xi - \eta|^b} d\eta.$$

We have the following estimate for I :

$$I(\tau, \xi) \sim \tau^A (\tau - |\xi|)^B$$

where

$$A = \max\left(a, b, \frac{n+1}{2}\right) - a - b \quad B = n - 1 - \max\left(a, b, \frac{n+1}{2}\right)$$

except when $\max(a, b) = \frac{n+1}{2}$, in which case we have

$$I(\tau, \xi) \sim \tau^{-\min(a,b)} (\tau - |\xi|)^{\frac{n-3}{2}} \log\left(\frac{\tau}{\tau - |\xi|}\right).$$

Proof. First apply Lemma B.2 with $F(s, t) = s^{-a} t^{-b}$. Then

$$\begin{aligned} I(\tau, \xi) &\simeq (\tau^2 - |\xi|^2)^{\frac{n-3}{2}} \int_{-1}^1 \left(\frac{\tau + |\xi|x}{2}\right)^{-a} \left(\frac{\tau - |\xi|x}{2}\right)^{-b} (\tau^2 - |\xi|^2 x^2) (1 - x^2)^{\frac{n-3}{2}} d\rho \\ &\simeq (\tau^2 - |\xi|^2)^{\frac{n-3}{2}} \frac{|\xi|^{2-a-b}}{2^{-a-b}} \int_{-1}^1 \left(\frac{\tau}{|\xi|} + x\right)^{1-a} \left(\frac{\tau}{|\xi|} - x\right)^{1-b} (1 - x^2)^{\frac{n-3}{2}} d\rho. \end{aligned}$$

Next we split the integral at $x = 0$.

For $-1 \leq x \leq 0$, set $t = 1 + x$ and note that $0 \leq t \leq 1$. We will use the following.

1. $\frac{\tau}{|\xi|} + x = \left(\frac{\tau}{|\xi|} - 1\right) + t$.
2. $\frac{\tau}{|\xi|} - x \sim \frac{\tau}{|\xi|}$ since $\tau > |\xi|$ implies $\frac{\tau}{|\xi|} \leq \frac{\tau}{|\xi|} - x \leq \frac{\tau}{|\xi|} + 1 \leq 2\frac{\tau}{|\xi|}$.
3. $1 - x^2 \sim t$ since $1 \leq 1 - x \leq 2$ implies $t = 1 + x \leq 1 - x^2 = (1 - x)t \leq 2t$.

Then

$$\begin{aligned} \int_{-1}^0 \left(\frac{\tau}{|\xi|} + x\right)^{1-a} \left(\frac{\tau}{|\xi|} - x\right)^{1-b} (1 - x^2)^{\frac{n-3}{2}} d\rho &\sim \left(\frac{\tau}{|\xi|}\right)^{1-b} \int_0^1 \left(\left(\frac{\tau}{|\xi|} - 1\right) + t\right)^{1-a} t^{\frac{n-3}{2}} dt \\ &\sim \left(\frac{\tau}{|\xi|}\right)^{1-b} H_{\frac{n-3}{2}}^{1-a} \left(\frac{\tau}{|\xi|} - 1\right). \end{aligned}$$

Similarly, for $0 \leq x \leq 1$, set $t = 1 - x$ and note that $0 \leq t \leq 1$. The properties we use now are as follows.

1. $\frac{\tau}{|\xi|} - x = \left(\frac{\tau}{|\xi|} - 1\right) + t$.
2. $\frac{\tau}{|\xi|} + x \sim \frac{\tau}{|\xi|}$ since $\tau > |\xi|$ implies $\frac{\tau}{|\xi|} \leq \frac{\tau}{|\xi|} + x \leq \frac{\tau}{|\xi|} + 1 \leq 2\frac{\tau}{|\xi|}$.
3. $1 - x^2 \sim t$ since $1 \leq 1 + x \leq 2$ implies $t = 1 - x \leq 1 - x^2 = (1 + x)t \leq 2t$.

As above, we obtain

$$\int_0^1 \left(\frac{\tau}{|\xi|} + x\right)^{1-a} \left(\frac{\tau}{|\xi|} - x\right)^{1-b} (1-x^2)^{\frac{n-3}{2}} d\rho \sim \left(\frac{\tau}{|\xi|}\right)^{1-a} H_{\frac{n-3}{2}}^{1-b} \left(\frac{\tau}{|\xi|} - 1\right).$$

All in all,

$$I(\tau, \xi) \sim (\tau^2 - |\xi|^2)^{\frac{n-3}{2}} |\xi|^{2-a-b} \left(\left(\frac{\tau}{|\xi|}\right)^{1-b} H_{\frac{n-3}{2}}^{1-a} \left(\frac{\tau}{|\xi|} - 1\right) + \left(\frac{\tau}{|\xi|}\right)^{1-a} H_{\frac{n-3}{2}}^{1-b} \left(\frac{\tau}{|\xi|} - 1\right) \right).$$

Now since $\frac{\tau}{|\xi|} - 1 > 0$ and $\frac{n-3}{2} > -1$ for $n > 1$, we can apply Lemma B.3. When $\tau \gg |\xi|$, we have

$$\begin{aligned} I(\tau, \xi) &\sim (\tau^2 - |\xi|^2)^{\frac{n-3}{2}} |\xi|^{2-a-b} \left(\left(\frac{\tau}{|\xi|}\right)^{1-b} \left(\frac{\tau}{|\xi|} - 1\right)^{1-a} + \left(\frac{\tau}{|\xi|}\right)^{1-a} \left(\frac{\tau}{|\xi|} - 1\right)^{1-b} \right) \\ &\sim (\tau^2 - |\xi|^2)^{\frac{n-3}{2}} |\xi|^{2-a-b} \left(\frac{\tau}{|\xi|}\right)^{2-a-b} \\ &\sim \tau^{n-3} \tau^{2-a-b} \\ &\sim \tau^{n-1-a-b} = \tau^A \tau^B \\ &\sim \tau^A (\tau - |\xi|)^B. \end{aligned}$$

Now suppose $1 \leq \frac{\tau}{|\xi|} \leq 2$ and without loss of generality, assume $\max(a, b) = a$. Then since $1 - a \leq 1 - b$,

$$(\tau^2 - |\xi|^2)^{\frac{n-3}{2}} |\xi|^{2-a-b} H_{\frac{n-3}{2}}^{1-b} \left(\frac{\tau}{|\xi|} - 1\right) \lesssim I(\tau, \xi) \quad (\text{B.0.25})$$

and

$$I(\tau, \xi) \lesssim (\tau^2 - |\xi|^2)^{\frac{n-3}{2}} |\xi|^{2-a-b} H_{\frac{n-3}{2}}^{1-a} \left(\frac{\tau}{|\xi|} - 1\right).$$

For the latter we have

$$\begin{aligned}
I(\tau, \xi) &\lesssim (\tau^2 - |\xi|^2)^{\frac{n-3}{2}} |\xi|^{2-a-b} \left(\frac{\tau}{|\xi|} \right)^{1-b} H_{\frac{n-3}{2}}^{1-a} \left(\frac{\tau}{|\xi|} - 1 \right) \\
&\lesssim (\tau - |\xi|)^{\frac{n-3}{2}} \tau^{\frac{n+1}{2}-a-b} H_{\frac{n-3}{2}}^{1-a} \left(\frac{\tau}{|\xi|} - 1 \right) \\
&\lesssim (\tau - |\xi|)^{\frac{n-3}{2}} \tau^{\frac{n+1}{2}-a-b} H_{\frac{n-3}{2}}^{1-a} \left(\frac{\tau - |\xi|}{\tau} \right)
\end{aligned}$$

by Lemma B.3 since $\tau \sim |\xi|$.

If $1 - a + \frac{n-3}{2} + 1 \neq 0$ or, equivalently, $a \neq \frac{n+1}{2}$ then

$$\begin{aligned}
I(\tau, \xi) &\lesssim (\tau - |\xi|)^{\frac{n-3}{2}} \tau^{\frac{n+1}{2}-a-b} \left(\frac{\tau - |\xi|}{\tau} \right)^{\min(\frac{n+1}{2}-a, 0)} \\
&\lesssim (\tau - |\xi|)^{\frac{n-3}{2} + \min(\frac{n+1}{2}-a, 0)} \tau^{\frac{n+1}{2}-a-b - \min(\frac{n+1}{2}-a, 0)} \\
&\lesssim (\tau - |\xi|)^{n-1 - \max(\frac{n+1}{2}, a)} \tau^{\max(a, \frac{n+1}{2}) - a - b} \\
&\lesssim (\tau - |\xi|)^B \tau^A.
\end{aligned}$$

If $1 - a + \frac{n-3}{2} + 1 = 0$, i.e. $a = \frac{n+1}{2}$ then

$$\begin{aligned}
I(\tau, \xi) &\lesssim (\tau - |\xi|)^{\frac{n-3}{2}} \tau^{-b} \left| \log \left(\frac{\tau - |\xi|}{\tau} \right) \right| \\
&\sim (\tau - |\xi|)^{\frac{n-3}{2}} \tau^{-\min(a, b)} \log \left(\frac{\tau}{\tau - |\xi|} \right).
\end{aligned}$$

Using (B.0.25) we obtain the reverse inequality.

□

Next we turn our attention to integration over the hyperbola $\mathcal{H}(\tau, \xi)$.

Lemma B.5. *Consider the integral*

$$I(F)(\tau, \xi) = \int \delta(\tau - |\eta| + |\xi - \eta|) F(|\eta|, |\xi - \eta|) d\eta$$

defined in the space-time region $\tau \leq |\xi|$. Then

$$I(F)(\tau, \xi) \simeq (|\xi|^2 - \tau^2)^{\frac{n-3}{2}} \int_1^\infty F \left(\frac{|\xi|x + \tau}{2}, \frac{|\xi|x - \tau}{2} \right) (|\xi|^2 x^2 - \tau^2) (x^2 - 1)^{\frac{n-3}{2}} dx. \quad (\text{B.0.26})$$

Proof. Following the same steps as in the proof of Lemma B.2, write

$$\begin{aligned}\delta(\tau - |\eta| + |\xi - \eta|) &= (\tau - |\eta| - |\xi - \eta|)\delta((\tau - \eta)^2 - |\xi - \eta|^2) \\ &= 2(\tau - |\eta|)\delta(\tau^2 - 2\tau|\eta| - |\xi|^2 + 2\xi \cdot \eta).\end{aligned}$$

Again using the polar coordinates (B.0.21), with a as in (B.0.22) and (B.0.23), we have

$$\delta(\tau - |\eta| + |\xi - \eta|) = 2(\tau - \rho)\delta(\tau^2 - |\xi|^2 - 2\tau\rho + 2|\xi|\rho a)$$

as before. However, since $|\tau| \leq |\xi|$, now we have

$$a = -\frac{\tau^2 - |\xi|^2 - 2\tau\rho}{2|\xi|\rho} \geq \frac{2\tau\rho}{2|\xi|\rho} \geq \frac{\tau}{|\xi|}$$

so that $\frac{\tau}{|\xi|} \leq a \leq 1$. Furthermore,

$$\frac{-\tau^2 + |\xi|^2 + 2\tau\rho}{2|\xi|\rho} \leq 1$$

implies $\rho \geq \frac{\tau + |\xi|}{2}$. So

$$\begin{aligned}I(F)(\tau, \xi) &\simeq \int_0^\infty \int_{\frac{\tau}{|\xi|}}^1 \delta(\tau^2 - |\xi|^2 - 2\tau\rho + 2|\xi|\rho a)(\tau - \rho)F(\rho, \tau - \rho)\rho^{n-1}(1 - a^2)^{\frac{n-3}{2}} da d\rho \\ &\simeq \frac{1}{|\xi|} \int_{\frac{\tau + |\xi|}{2}}^\infty F(\rho, \tau - \rho)(\tau - \rho)\rho^{n-2} \left(1 - \left(\frac{\tau^2 - |\xi|^2 - 2\tau\rho}{2|\xi|\rho}\right)^2\right)^{\frac{n-3}{2}} d\rho \\ &\simeq \frac{(|\xi|^2 - \tau^2)^{\frac{n-3}{2}}}{|\xi|^{n-2}} \int_{\frac{\tau + |\xi|}{2}}^\infty F(\rho, \tau - \rho)(\tau - \rho)\rho \left[\left(\rho + \frac{|\xi| - \tau}{2}\right)\left(\rho - \frac{|\xi| + \tau}{2}\right)\right]^{\frac{n-3}{2}} d\rho.\end{aligned}$$

Setting $x = \frac{2\rho - \tau}{|\xi|}$ or $\rho = \frac{|\xi|x + \tau}{2}$ then gives

$$I(F)(\tau, \xi) \simeq (|\xi|^2 - \tau^2)^{\frac{n-3}{2}} \int_1^\infty F\left(\frac{|\xi|x + \tau}{2}, \frac{|\xi|x - \tau}{2}\right) (|\xi|^2 x^2 - \tau^2) (x^2 - 1)^{\frac{n-3}{2}} dx.$$

□

Proposition B.6. *Let $a, b \in \mathbb{R}$ and $\tau < |\xi|$. Define the integral*

$$I(\tau, \xi) = \int_{|\eta| + |\xi - \eta| \leq 2|\xi|} \frac{\delta(\tau - |\eta| + |\xi - \eta|)}{|\eta|^a |\xi - \eta|^b} d\eta.$$

We have the following estimate for I :

- In the region where $0 \leq \tau \leq |\xi|$,

$$I(\tau, \xi) \sim |\xi|^A (|\xi| - \tau)^B$$

where

$$A = \max\left(b, \frac{n+1}{2}\right) - a - b \quad B = n - 1 - \max\left(b, \frac{n+1}{2}\right)$$

except when $b = \frac{n+1}{2}$, in which case we have

$$I(\tau, \xi) \sim |\xi|^{-a} (|\xi| - \tau)^{\frac{n-3}{2}} \log\left(\frac{|\xi|}{|\xi| - \tau}\right).$$

- In the region where $-|\xi| \leq \tau \leq 0$,

$$I(\tau, \xi) \sim |\xi|^A (|\xi| + \tau)^B$$

where

$$A = \max\left(a, \frac{n+1}{2}\right) - a - b \quad B = n - 1 - \max\left(a, \frac{n+1}{2}\right)$$

except when $a = \frac{n+1}{2}$, in which case we have

$$I(\tau, \xi) \sim |\xi|^{-b} (|\xi| - \tau)^{\frac{n-3}{2}} \log\left(\frac{|\xi|}{|\xi| - \tau}\right).$$

Proof. Now we will apply Lemma B.5 with $F(s, t) = s^{-a} t^{-b}$. Note that since

$$x = \frac{2\rho - \tau}{|\xi|} = \frac{2|\eta| - (|\eta| - |\xi - \eta|)}{|\xi|} = \frac{|\eta| + |\xi - \eta|}{|\xi|}$$

restricting to the ellipsoid $|\eta| + |\xi - \eta| \leq 2|\xi|$, we have $1 \leq x \leq 2$. Then

$$\begin{aligned} I(F)(\tau, \xi) &\simeq (|\xi|^2 - \tau^2)^{\frac{n-3}{2}} \int_1^2 \left(\frac{|\xi|x + \tau}{2}\right)^{-a} \left(\frac{|\xi|x - \tau}{2}\right)^{-b} (|\xi|^2 x^2 - \tau^2) (x^2 - 1)^{\frac{n-3}{2}} dx \\ &\simeq |\xi|^{2-a-b} (|\xi|^2 - \tau^2)^{\frac{n-3}{2}} \int_1^2 \left(x + \frac{\tau}{|\xi|}\right)^{1-a} \left(x - \frac{\tau}{|\xi|}\right)^{1-b} (x^2 - 1)^{\frac{n-3}{2}} dx. \end{aligned}$$

Assuming $0 \leq \tau \leq |\xi|$, set $t = x - 1$.

1. $x + \frac{\tau}{|\xi|} \sim 1$ since $1 \leq x \leq x + \frac{\tau}{|\xi|} \leq 2 + 1$.
2. $x - \frac{\tau}{|\xi|} = \left(1 - \frac{\tau}{|\xi|}\right) + t$.

3. $x^2 - 1 \sim t$ since $2 \leq x + 1 \leq 3$ implies $t = x - 1 \leq x^2 - 1 = (x + 1)t \leq 3t$.

Then

$$\begin{aligned} I(F)(\tau, \xi) &\sim |\xi|^{2-a-b} (|\xi|^2 - \tau^2)^{\frac{n-3}{2}} \int_0^1 \left(\left(1 - \frac{\tau}{|\xi|}\right) + t \right)^{1-b} t^{\frac{n-3}{2}} dt \\ &\sim |\xi|^{2-a-b} (|\xi|^2 - \tau^2)^{\frac{n-3}{2}} H_{\frac{n-3}{2}}^{1-b} \left(1 - \frac{\tau}{|\xi|}\right). \end{aligned}$$

If $1 - b + \frac{n-3}{2} + 1 \neq 0$ or, equivalently, $b \neq \frac{n+1}{2}$ then

$$\begin{aligned} I(F)(\tau, \xi) &\sim |\xi|^{2-a-b} (|\xi|^2 - \tau^2)^{\frac{n-3}{2}} \left(1 - \frac{\tau}{|\xi|}\right)^{\min(\frac{n+1}{2}-b, 0)} \\ &\sim |\xi|^{2-a-b} (|\xi| - \tau)^{\frac{n-3}{2}} |\xi|^{\frac{n-3}{2}} |\xi|^{-\min(\frac{n+1}{2}-b, 0)} (|\xi| - \tau)^{\min(\frac{n+1}{2}-b, 0)} \\ &\sim |\xi|^{\max(\frac{n+1}{2}, b)-a-b} (|\xi| - \tau)^{n-1-\max(\frac{n+1}{2}, b)}. \end{aligned}$$

If $1 - b + \frac{n-3}{2} + 1 = 0$, i.e. $b = \frac{n+1}{2}$ then

$$\begin{aligned} I(F)(\tau, \xi) &\sim |\xi|^{-a} (|\xi| - \tau)^{\frac{n-3}{2}} \left| \log \left(1 - \frac{\tau}{|\xi|}\right) \right| \\ &\sim |\xi|^{-a} (|\xi| - \tau)^{\frac{n-3}{2}} \log \left(\frac{|\xi|}{|\xi| - \tau} \right). \end{aligned}$$

Assuming $-|\xi| < \tau \leq 0$, also set $t = x - 1$. Then we have the following.

1. $x - \frac{\tau}{|\xi|} \sim 1$ since $1 \leq x \leq x - \frac{\tau}{|\xi|} \leq 2 + 1$.
2. $x + \frac{\tau}{|\xi|} = \left(1 + \frac{\tau}{|\xi|}\right) + t$.
3. $x^2 - 1 \sim t$.

With these observations, we proceed as above to obtain

$$I(F)(\tau, \xi) \sim |\xi|^{\frac{n+1}{2}-a-b} (|\xi| + \tau)^{\frac{n-3}{2}} H_{\frac{n-3}{2}}^{1-a} \left(1 + \frac{\tau}{|\xi|}\right)$$

and the result will follow exactly as before. □

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