Gravitational Wave Production through Decay of the Inflaton into Intermediary Fields During Slow Roll Inflation

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A Dissertation Presented

by

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ABSTRACT

GRAVITATIONAL WAVE PRODUCTION THROUGH DECAY OF THE INFLATON INTO INTERMEDIARY FIELDS DURING SLOW ROLL INFLATION

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This dissertation looks for possible observable signals of tensor metric perturbations sourced during slow roll inflation from decay of the inflaton field into other intermediary fields. We focus on two main scenarios, one of explosive production of intermediary fields for a short period during inflation and another of prolonged production of vectors due to a derivative coupling of the vectors with the inflaton field. We only find a possible observable signal of tensor perturbations in the second case.

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INTRODUCTION

Inflation has becomes the most likely, and well supported theory of the early universe. There is substantial evidence for it, though it is not yet considered proven. It is characterized as a period of rapid, highly uniform expansion, with an almost constant expansion rate given by the Hubble parameter H, which was many orders of magnitude larger than it is today. This lead to exponential growth of the scale factor a which characterizes the growth of physical distances. It requires, similar to our current dark energy, a period of accelerated expansion, in which pressure is negative requiring that the dominate form of energy density behave very different from ordinary matter. This uniformly inflating region encompass at least our entire observable universe, and potentially much more depending on how long inflation lasted.

Inflation is useful in that it produces a very highly homogeneous and isotropic universe with very small perturbations which then grow in time, and source structure in the universe, leading to the anisotropies in the CMB and later large scale clusters of galaxies and dark matter. There are three main problems that inflation was first suggested to address, the horizon, flatness, and unwanted relics problems.

First some basics. I will use the FRW metric throughout, which in its homogeneous form is given by:

$$ds^2 = dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)\,. \tag{1.1}$$

All three spatial dimensions are treated equivalently; this assumes no spatial curvature and describes a universe which is homogeneous and isotropic. The scale factor allows for a universe which can expand or shrink in time. Physical distances which we actually measure are given by $x_p = \sqrt{a^2(dx^2 + dy^2 + dz^2)}$. This is as opposed to comoving distances described by $x_c = \sqrt{dx^2 + dy^2 + dz^2}$. The comoving coordinates stay fixed in time, independent of expansion, while the physical coordinates evolve in time.

1.1 Horizon Problem

One of the original and most compelling problems that inflation was first suggested to address is the horizon problem. The basic idea is that if the universe were only radiation and matter dominated up until now (really up until the transition to dark energy dominance) then the universe would have to be less causally connected in the past, in fact multiple orders of magnitude less. The question is, how come our universe should appear so homogeneous and isotropic if the far reaches of our observable universe are just coming into contact with us,

and still are not yet in contact with each other, and had no opportunity to communicate in the past?

To show this mathematically, first let A equal the physical distance to our current horizon at the time of last scattering. Let B equal the physical distance to the horizon of last scattering at the time of last scattering. Note that these are two different quantities because the size of the horizon has changed; as the type of energy density which dominated the universe changed, the comoving distance to the horizon as well as the physical distance changed. I am using the definition of horizon as the farthest away point which is receding from us at exactly the speed of light; this physical distance is given by $1/H$ where H is the Hubble parameter. To demonstrate the horizon problem, we want to compare the ratio $(A/B)^3$, which is the ratio of the size of the currently casually connected patch shrunk to the size it was at last scattering, compared to the size of a casually connected patch at last scattering. We take the cube so that we have a ratio of the two volumes.

First we calculate A. The physical distance to the horizon now is $1/H_0$, where H_0 is the value of the Hubble parameter now. The comoving distance to the horizon now is $\frac{1}{a_0 H_0}$, where a_0 is similarly defined as the value of the scale factor now. Since this is the comoving distance though, this never changes, so the comoving distance to the current horizon is always $\frac{1}{a_0 H_0}$. Then the physical distance to the horizon at the time of last scattering, using $x_p = x_c a$, is given by

$$A = \frac{a_{LS}}{a_0 H_0} \,. \tag{1.2}$$

Next we calculate B, the physical distance to the horizon of last scattering, H_{LS}; $B = \frac{1}{H_{LS}}$. To calculate H_{LS} one needs to account for the way the universe has expanded between last scattering and now which is determined by the type of matter dominating the energy density. During last scattering the universe was matter dominated. We will ignore the short period of dark energy dominance which followed matter dominance in this calculation. During matter domination $H \propto a^{-\frac{3}{2}}$. This gives $\frac{H_{LS}}{H_0} = (\frac{a_{LS}}{a_0})^{-\frac{3}{2}}$. Putting everything together we have:

$$\left(\frac{A}{B}\right)^3 = \left(\frac{a_{LS} H_{LS}}{a_0 H_0}\right)^3 = \left(\frac{a_{LS}}{a_0}\left(\frac{a_0}{a_{LS}}\right)^{\frac{3}{2}}\right)^3 = \left(\frac{a_0}{a_{LS}}\right)^{\frac{3}{2}} . \tag{1.3}$$

We then use that $a \propto \frac{1}{T}$, where T is the temperature. This holds as long as only adiabatic/entropy-conserving processes are taking place, which is approximately true. We note that $T_{LS} = 3000\text{K}$ and $T_0 = 2.725\text{K}$. This gives:

$$\left(\frac{A}{B}\right)^3 = \left(\frac{T_{LS}}{T_0}\right)^{\frac{3}{2}} = 3.7 \times 10^4 \,. \tag{1.4}$$

In other words, there were 10^4 disconnected regions at the time of last scattering that make up our current horizon. Since this is last scattering, this is the last time these photons had any opportunity to communicate with each other. The problem is that there seems to be no reason why any of these 10^4 separate patches that make up our CMB should look alike, but they do, which fluctuations only on order 10^{-5}. The problem gets worse the further back if time one looks. For example, if one were to suppose that no new physics kicks in all the way back to the Planck scale, that the universe was only radiation and matter dominated from the Planck scale until now, and we work through the same calculation as above taking into account the radiation dominated period, we would find 7.5×10^{89} disconnected patches

at the Planck era making up our current observable universe. This raises the question, how could it be that the universe was so much less causally connected in the past, and yet now we can measure the temperature on one region of our horizon and the temperature at the complete other end, regions that apparently are just coming in contact with us for the first time, and yet they are almost the same temperature, with fluctuations on order 10^{-5}?

The natural answer is to suppose that these far reaches of our observable universe were actually causally connected in the past. But radiation and matter domination only lead to patches getting further out of contact in the past. To allow for a universe which was more causally connected in the past requires a different type of energy density dominating the universe, one which would lead to accelerated expansion, inflation.

One can see how inflation solves the horizon problem by showing that if inflation lasts long enough, the current observable universe will be causally connected again in the past. Let t_i, standing for initial time, be the latest time inflation could have begun to still solve the horizon problem, to make the observable universe just barely causally connected. Let $d_{pH_0}(t_i)$ be the physical distance to the current horizon at t_i. We require that this be less than or equal to $\frac{1}{H_i}$, the Hubble parameter at time t_i, such that the observable universe now fits within the horizon at t_i. To calculate H_i we note that during inflation H is approximately constant. After inflation the universe went through a period of radiation dominance when $H \propto a^{-2}$, and finally a period of matter dominance when $H \propto a^{-\frac{3}{2}}$. Using the subscript *end* to represent the end of inflation/ the start of radiation dominance, and the subscript $MReq$ to represent the time of matter-radiation equality, we obtain:

$$\frac{H_{end}}{H_{MReq}} = \left(\frac{a_{MReq}}{a_{end}}\right)^2 \tag{1.5}$$

and

$$\frac{H_{MReq}}{H_0} = \left(\frac{a_0}{a_{MReq}}\right)^{\frac{3}{2}} , \tag{1.6}$$

giving:

$$H_i = H_{end} = H_0 \left(\frac{a_0}{a_{MReq}}\right)^{\frac{3}{2}} \left(\frac{a_{MReq}}{a_{end}}\right)^2 . \tag{1.7}$$

Plugging this into the inequality, $d_{pH_0}(t_i) \leq \frac{1}{H_i}$ we obtain:

$$\frac{1}{H_0}\left(\frac{a_i}{a_{end}}\right)\left(\frac{a_{end}}{a_0}\right) \leq \frac{1}{H_0}\left(\frac{a_{MReq}}{a_0}\right)^{\frac{3}{2}}\left(\frac{a_{end}}{a_{MReq}}\right)^2 , \tag{1.8}$$

which simplifies to

$$\frac{a_{end}}{a_i} \geq \frac{a_{MReq}^{\frac{1}{2}} a_0^{\frac{1}{2}}}{a_{end}} . \tag{1.9}$$

We use that $T_{MReq} \approx 9000$ K and estimate that $T_{end} \approx 10^{27}$K, which gives

$$\frac{a_{end}}{a_i} \geq 6.4 \times 10^{24} . \tag{1.10}$$

We use that $a = e^{Ht}$ during inflation so that $\frac{a_{end}}{a_i} = e^{H\Delta t}$ with Δt the minimum time that inflation must last to solve the horizon problem. Then defining N as the number of

e-foldings given by $N = H\Delta t$, we find $N \geq 57.1$. Therefore if inflation lasted at least 57 e-folds, than the entire observable universe would have been causally connected in the past, during inflation, and the horizon problem would be satisfied.

1.2 Flatness Problem

Another major problem in the early universe that inflation addresses is the flatness problem. This states that if the universe is close to the critical energy density now, but not exactly critical, then it had to be incredibly close to critical in the past. The energy density parameter is defined by

$$\Omega = \frac{\rho}{\rho_{crit}} = \frac{8\pi G\rho}{3H^2} \,. \tag{1.11}$$

The Friedmann equation then gives:

$$\Omega - 1 = \frac{\kappa}{H^2 a^2} \,, \tag{1.12}$$

where κ is the spatial curvature of the universe, constant in time. During radiation dominance, $H \propto \frac{1}{a^2}$ so $\Omega - 1 \propto a^2$. During matter dominance, $H \propto \frac{1}{a^{3/2}}$ so $\Omega - 1 \propto a$. In either case $\Omega - 1$ is smaller in the past. Current observations put $\Omega - 1$ consistent with 0 today, which means $\Omega - 1$ had to be exceptionally close to 0 in the past. Calculating how close, if we assume the universe was only matter and radiation dominated all the way back to the Planck era, and we take the ratio of $\Omega - 1$ evaluated at the Planck era with $\Omega - 1$ calculated today: $\frac{\Omega - 1_P}{\Omega - 1_0} = 10^{-60}$. In other words, the universe had to be 60 orders or magnitude closer to critical density at the Planck scale than it is today. The flatness problem is also called the fine-tuning problem. The problem is that although this is possible, it seems highly unnatural for the universe to be so incredibly close to critical density in the past unless there was some mechanism making this so.

To solve the problem, we need a period in the past that would naturally bring the universe very close to critical density. Then even if after the end of inflation till now the universe has been moving away from critical density, if it was sufficiently close to critical at the end of inflation, then we would still except it to be reasonably close to critical today. During inflation H is approximately constant and so $\Omega - 1 \propto \frac{1}{a^2}$ with a increasing exponentially. This leads to $\Omega - 1$ shrinking exponentially during inflation.

We can define the flatness problem as fixed when the amount $\Omega - 1$ changed by from the end of inflation till now, $\frac{\Omega - 1_{end}}{\Omega - 1_0} = 10^{-50}$, is the same amount $\Omega - 1$ changed by during inflation. This requires 58 e-foldings, about the same amount required to solve the horizon problem.

There is also an unwanted relics problem which inflation solves. The idea is that one could expect to see various fields in the early universe which we do not see any sign of, for example primordial magnetic monopoles. Inflation can solve this problem by causing the universe to expand exponentially, quickly dropping the energy density of any other fields down to unperceivably small levels. Therefore even if there were other relic fields around before inflation or early on during inflation, unless the inflaton field was sourcing these fields during inflation, or decayed into them at the very end of inflation, then it is very natural that we would not see any sign of them.

This is all to say, the period immediately after inflation has several initial conditions which seem highly unnatural unless there was a period of accelerated expansion early on, and then these initial conditions arise automatically. This was the initial motivation for inflation. Since then, it has been shown that inflation predicts a spectrum of primordial density fluctuations that to high accuracy matches what is observed in the CMB and what is needed to source large scale structure, and this has provided the strongest evidence for inflation. This was especially compelling because the prediction that inflation would leave small perturbations on the CMB was made over a decade before these perturbations were first detected by COBE. Since then, increasingly accelerate measurements of the CMB and LSS, large scale structure, are all in agreement with the predictions of inflation.

1.3 Charecterizing the Inflaton Field

To solve the horizon, flatness, and relics problems one needs a period of accelerated expansion in which $\ddot{a} > 0$. The second Friedmann equation states $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$ where ρ is the energy density and p is pressure. Requiring that $\ddot{a}$ be positive requires that $p < -\frac{\rho}{3}$. Energy density is always positive, so this means that accelerated expansion requires negative pressure. All ordinary matter and radiation has positive pressure, so this requires a new type of matter.

It is most common to assume the inflaton field is a scalar. Since the universe is very nearly homogeneous and isotropic, this is natural since a vector or tensor inflaton would likely create a universe with a preferred direction. We will call ϕ the inflaton field.

One can calculate what requirements are necessary such that the field ϕ will have negative pressure. Using the stress energy tensor of a scalar field and assuming spatial perturbations of ϕ are higher order:

$$T^{\alpha}_{\beta} = g^{\alpha\nu}\partial_{\nu}\phi\partial_{\beta}\phi - g^{\alpha}_{\beta}[\frac{1}{2}g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi + V(\phi)] \tag{1.13}$$

and using the FRW metric, we find

$$-T^{0}_{0} = \rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{1.14}$$

and

$$T^{i}_{i} = p = \frac{1}{2}\dot{\phi}^2 - V(\phi)\,. \tag{1.15}$$

From this, one can see to satisfy that the pressure be negative requires that the potential energy dominate over the kinetic energy. So the first condition we need require for the universe to inflate is that $V(\phi) > \dot{\phi}^2$. This is the first slow roll condition.

To find the inflaton equation of motion, we write the action of a generic scalar field:

$$S = \int d^4x\sqrt{-g}[\frac{1}{2}g^{\mu\nu}\partial_{\mu}\partial_{\nu}\phi - V(\phi)]\,. \tag{1.16}$$

The FRW metric gives $\sqrt{-g} = a^3$. Solving for the equation of motion, we find

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0\,, \tag{1.17}$$

using that $H = \frac{\dot{a}}{a}$ and assuming spatial perturbations are small.

We have seen that the universe will inflate as long as $\dot{\phi}^2 < V$, but to have successful inflation, we can not have that the universe just passes through a period of $\dot{\phi}^2 < V$ too quickly. Inflation must last sufficiently long to solve the horizon problem, etc. For this, it is assumed that $\ddot{\phi} \ll 3H\dot{\phi}, \frac{\partial V}{\partial \phi}$. This is the second slow roll condition.

The slow roll conditions are quantified by the two slow roll parameters ϵ and η to demonstrate under what conditions a scalar field will produce a period of inflation. The first slow roll condition $\dot{\phi}^2 \ll 2V$ can be rewritten by substituting for $\dot{\phi}$ using the inflaton equation of motion, and using that $\ddot{\phi} \ll 3H\dot{\phi}, \frac{\partial V}{\partial \phi}$, so that we are left with $3H\dot{\phi} = -\frac{\partial V}{\partial \phi}$. This gives

$$\dot{\phi}^2 \ll 2V \rightarrow \frac{(\frac{\partial V}{\partial \phi})^2}{9H^2} \ll 2V \tag{1.18}$$

Then we substitute in the Friedmann equation for H^2: $H^2 = \frac{1}{3M_P^2}\rho$, and use that $\rho \approx V(\phi)$ to give $H^2 \approx \frac{V}{3M_P^2}$. Plugging this in:

$$\frac{M_P^2}{2}\left(\frac{\frac{\partial V}{\partial \phi}}{V}\right)^2 \ll 1 \tag{1.19}$$

The slow roll parameter ϵ is defined as $\frac{M_P^2}{2}\left(\frac{\frac{\partial V}{\partial \phi}}{V}\right)^2$. When ϵ becomes of order one, inflation ends.

The second slow roll condition $\ddot{\phi} \ll 3H\dot{\phi}, \frac{\partial V}{\partial \phi}$ is parametrized using the slow roll parameter η. Starting from the inflation equation of motion, $\dot{\phi} = -\frac{V'}{3H}$, we can get an expression for $\ddot{\phi}$ by taking the derivative of both sides with respect to time:

$$\ddot{\phi} = -\frac{1}{3H}V''\dot{\phi}\,. \tag{1.20}$$

We then plug into this expression again the inflaton equation of motion for $\dot{\phi}$, and plug in the Friedmann equation for H: $H = \frac{V}{3M_P^2}$. This gives:

$$\ddot{\phi} = \frac{V''V'}{3V}M_P^2\,. \tag{1.21}$$

We then plug this back into the original inequality, $\ddot{\phi} \ll V'$ to give:

$$\frac{V''M_P^2}{V} \ll 1 \tag{1.22}$$

and we define $\eta = \frac{V''M_P^2}{V}$. Requiring $\eta \ll 1$ satisfies that the inflaton field stays slowly rolling, guaranteeing that the potential of the inflaton is not too steep. A scalar field will be capable of slow-roll inflation as long as ϵ and $\eta \ll 1$.

This characterizes the homogeneous inflaton field, but there are also small perturbations. Even if the inflaton field were maximally uniform during inflation, there had to be small quantum fluctuations seeded at all frequency scales, required by the uncertainty principle.

As the universe expanded these fluctuations became stretched out, pulled by the expansion. While inside the horizon, this stretched out their wavelength and decreased their amplitude, until eventually the wavelengths were stretched out to a point that they reached the size of the horizon. At this point, they captured energy from the inflaton field and become real physical waves, with amplitude which was 'frozen-in' and could not decrease further.

1.4 Current Standing of Field

Although scalar perturbations from inflation have been observed, tensor perturbations have not. Observing these additional perturbations would confer a much better understanding of what was happening during inflation (the mechanism of inflation and the associated energy scales), and there is good reason to believe they might be observed within the next few years. We care about observing the tensor spectrum for two main reasons. 1. It will help pin down cosmological parameters, some of which can not or would be very difficult to pin down using only the scalar spectrum. 2. Observing these perturbations should help pin down the correct model of inflation, or at least rule out some incorrect models.

Foremost, studying the tensor spectrum allows one to put limits on the Hubble parameter during inflation. For most models of inflation, observing the gravitational wave spectrum from inflation will be equivalent to measuring H. Lower limits on H come from nucleosynthesis and put H above about 10^{-25} GeV during inflation, depending on the model of reheating. Upper limits on H come from the non-observation of primordial gravitational waves on the CMB. One can get limits from each of the CMB perturbative modes (these modes will be explained later), but currently the best constraints come from the T modes, giving $H \leq 9 \cdot 10^{13}$ GeV [1]. As the experiments imaging the CMB anisotropies improve, the best limits will likely come from the B modes, or hopefully eventually, there will be an actual observation from the B modes. The primordial tensor power spectrum is given by $P_h(k) = \frac{2H^2}{\pi M_P^2}$, where M_P is the reduced Planck Mass ($M_P^2 = \frac{1}{8\pi G}$ where G is the gravitational constant). This uses the convention of defining the power spectrum by $\langle h_{ij}(\mathbf{k}, \tau) h^{ij}(\mathbf{k}', \tau) \rangle = \delta^{(3)}(\mathbf{k} + \mathbf{k}') \frac{2\pi^2}{k^3} P(k)$. Measuring this tensor power spectrum will be equivalent to measuring the potential energy of the inflaton field, $V(\phi)$, because $V \propto H^2$ ($H \approx \sqrt{\frac{V}{3M_P^2}}$). Both of these quantities are dynamical though slowly varying during inflation, with the time variation proportional to the slow roll parameter ϵ. The scalar spectrum, which has been observed, does not uniquely determine H but the ratio $\frac{H}{\epsilon}$. Therefore, observing the tensor spectrum, along with the already observed scalar spectrum, will be equivalent to measuring H, ϵ, and $V(\phi)$, providing a much clearer understanding of the energy scales, H and $V(\phi)$, and their rate of change, ϵ, during inflation.

Observing the tensor spectrum will not necessarily provide a measurement of H, but it will provide an upper bound. Whenever the universe is going through a period of accelerated expansion there will be quantum production of gravitational waves. How efficiently these gravitational waves are produced is dependent on the rate of expansion, H. The larger H is, the larger the tensor spectrum. The power spectrum produced this way is the $\frac{2H^2}{\pi M_P^2}$ given above. The scalar power spectrum is already known (regardless of what inflationary model is sourcing it, the magnitude is known to about 2.5×10^{-9}) and therefore a limit on r, the ratio of the tensor to scalar power spectrum which is directly observable, translates into an upper bound on H. This quantum production of tensors will be present regardless, but their could

also be other sources contributing to r. Therefore an observation of r will not necessarily be equivalent to an observation of H.

The best current upper bound on H comes from Planck. Planck recently published a limit of $r(k_0) < 0.11$ at the 95% CL from a combination of results from Planck's temperature studies combined with results from ACT (Atacama Cosmology Telescope) and SPT (South Pole Telescope) [1]. Both ACT and SPT are Earth based telescopes sensitive to the CMB. $k_0 = 0.002\text{Mpc}^{-1}$. The power spectra and r are slightly scale dependent so one needs to pick a scale to evaluate them at, and $k_0 = 0.002\text{Mpc}^{-1}$ is the default choice in these experimental papers. This limit on r translates into a limit on H by using that $\frac{P_h(k_0)}{P_\zeta(k_0)} = r \leq 0.11$, using $P_h = \frac{2H^2}{\pi^2 M_P^2}$, and using $P_\zeta = 2.5 \times 10^{-9}$, this gives $H/M_P < 3.7 \times 10^{-5}$.

One way to distinguish between different models of inflation is by their tensor-to-scalar ratio, r. By model of inflation, one usually means the number of inflationary fields and the form of their potentials. In the simplest inflationary models, there is only one field dominating the energy density during inflation and generating the curvature perturbations; once the form of its potential and the strength of the field are specified, then the associated power spectrum is determined. Inflationary models can be broken into two general classes, large field and small field, where large and small reference to the change in $\Delta\phi$ over the last 60 e-folds of inflation. Generally, the simplest potentials one can write, for example polynomial potentials like the popular $V(\phi) = \frac{1}{2}m^2\phi^2$, fall into the class of large field models.

Large field models predict a large Hubble parameter, near the current bound, which produces a large scalar-to-tensor ratio, and a larger value for the slow roll parameter ϵ. These models are characterized by $\frac{\partial^2 V}{\partial \phi^2} > 0$ and have the two slow roll parameters, ϵ and η, of the same magnitude, but have that the inflaton field value start above the Planck mass, (this consequence will be explained more later). Since $\Delta\phi$ is changing by many more orders of magnitude than in the small field case, the change in the potential during ϕ's slow roll will likewise be larger by many orders of magnitude. The flatness of the potential is parametrized by η, and η is of the same order of ϵ in this case, which is much larger than it is in the small field case. The inflaton is slowed in its rolling by a large friction term provided by the Hubble parameter; notice in the inflaton equation of motion, eq. (1.17), that the Hubble parameter appears in a friction term.

Small field inflation models on the other hand assume the inflaton field starts near an unstable equilibrium, and rolls down its potential to a stable minimum. In this case, $\frac{\partial^2 V}{\partial \phi^2} < 0$ such that the inflaton starts on a maximum of the potential, but starts off much closer to its minimum than in the large field case. The inflaton only rolls down a little to reach its minimum, so the change in the potential and in the field value is much smaller. It rolls slowly not because of a large Hubble parameter providing a lot of friction, the Hubble parameter is much smaller in this case, but because the inflaton starts at a maximum of the potential and so the potential starts exceptionally flat causing the inflaton to roll very slowly. Both ϵ and H are much smaller in this case, with ϵ much smaller than the other slow roll parameters. Since it is the ratio $\frac{H}{M_P}$ that determines the efficiency of gravitational wave production, gravitational wave production is much weaker in small field models, and r is therefore much smaller; far too small to be detecting by upcoming experiments. If tensor modes are observed in the near future, that will mean small field inflation is incorrect, unless there is some other non-standard mechanism sourcing the observed tensors. Some say small field inflation is more natural in that it does not require trans-Planckian changes in field values like large-field inflation, while others say large field inflation is more natural because

it does not require a hierarchy between the slow roll parameters like small field inflation [2].

In the category of small field models fit almost all string theory models of inflation which generically (with a few exceptions) predict an undetectably small tensor spectrum. String theory models in particular have a hard time handling trans-Plnacking changes in field values required for detectable tensor spectra [3].

We can make a rough prediction of the tensor-to-scalar ratio in large field models based on the observed scalar power spectrum spectral index. The scalar power spectrum is typically written as $P_s(k) \propto k^{n_s-1}$ where n_s is the scalar spectral index. $n_s = 1$ corresponded to a perfectly scale invariant power spectrum, and the observed one is almost though not perfectly scale invariant. $n_s = 0.960 \pm 0.0073$ is the recent limit from the Planck collaboration using a combination of data from Planck + WMAP [1]. We can estimate r by using the following equation

$$n_s = 1 - 4\epsilon - 2\delta\,, \tag{1.23}$$

where δ is another slow roll parameter defined as $\delta = \epsilon - \eta$, a combination of the other two slow roll parameters [4]. This equation is a straight forward manipulation of the equation for the scalar power spectrum, found by taking the logarithm of the power spectrum and a logarithmic derivative with respect to momentum, and plugging in for ϵ and δ. Next for the purpose of getting a rough estimate on r, we can assume ϵ and δ are equal. In large field inflationary models, ϵ and δ are of the same order of magnitude. If we assume they are equal:

$$n_s = 1 - 6\epsilon. \tag{1.24}$$

When we plug in the Planck measurement for n_s, we get $\epsilon \approx 0.0067$. Next we use that $r = 16\epsilon$, trivial to derive from comparing the two power spectra $P_h = \frac{2H^2}{\pi^2 M_P^2}$ and $P_\zeta = \frac{H^2}{8\pi^2\epsilon M_P^2}$, to estimate $r \approx 0.107$. So it is likely in the simplest inflationary models that r will be around 0.1. Different inflationary model predict different values for r, although in all large field inflationary models r should be near this amount, close to the current observational bound, while r is predicted to be much smaller from small field inflationary models.

It is straightforward to demonstrate that a detectable r should necessitate generically trans-Planckian changes in ϕ. The lower bound on $\Delta\phi$ one obtains is called the Lyth bound. We start with the definition for the number of efoldings:

$$N = \int_{t_i}^{t_f} H dt\,. \tag{1.25}$$

Then we use that dt can be rewritten as $\frac{1}{\frac{d\phi}{dt}} d\phi$ to give:

$$N = \int_{\phi_i}^{\phi_f} \frac{H}{\dot{\phi}} d\phi\,. \tag{1.26}$$

Then we use the inflaton equation of motion: $3H\dot{\phi} + V' = 0$, and the Friedmann equation: $H = \sqrt{\frac{\rho}{3}}\frac{1}{M_P}$ where ρ is dominated by V, to give:

$$N = -\int_{\phi_i}^{\phi_f} \frac{1}{M_P^2}\frac{V}{V'} d\phi\,. \tag{1.27}$$

Next, as long as we only want a rough order of magnitude estimate, and we want to stay general without choosing any particular potential, we can estimate:

$$\Delta N \approx \frac{\Delta\phi}{M_P^2}\frac{V}{V'}\,. \tag{1.28}$$

We can relate $\frac{V}{V'}$ to r by using the definition of ϵ, $\epsilon = \frac{M_P^2}{2}(\frac{V'}{V})^2$, and the fact that $r = 16\epsilon$, which we get from taking the ratio of the two power spectra. Plugging this in we find

$$\Delta N \frac{r^{\frac{1}{2}}}{2^{\frac{3}{2}}} \approx \frac{\Delta\phi}{M_P}\,. \tag{1.29}$$

If we want an observable r, for example $r = 0.1$, and we want to calculate how much $\Delta\phi$ must change by during the last 60 e-folds of inflation that describe our observable universe, we find $\Delta\phi \approx 7M_P$. This means if a tensor spectrum is detected in the near future, and if this spectrum turns out to be the standard $\frac{H^2}{M_P^2}$ tensor power spectrum, than this will mean that large field inflation is correct.

One can infer limits on r using measurements from any of the CMB anisotropy modes (the T, E or B, more on these later). The current best limit on r directly from the B modes comes from BICEP with a limit of $r < 0.7$ (95% CL) [5]. Temperature mode fluctuations currently give the strongest limit on r, with Planck data constraining r to $r < 0.11$ (95 % CL) [1].

Within the next few years, constraints on r should decrease substantially. Planck is expected to eventually constrain r down to around $r < 0.05$ [6]. Although Planck has finished recording data, and the temperature anisotropy results have been released, the polarization studies are more involved, but should be released sometime within the year. A combination of Earth bound studies should push the limit on r down even further than Planck within the next 4 years to around $r < 0.01$. These include polarimeter studies based in Antarctica focusing on only a small portion of the sky, but targeting areas with low background from our galaxy (BICEP2, Keck, and SPIDER) [7, 8, 9] and several experiments from microwave telescopes in the Atacama desert in Chile including POLARBEAR [10]. Further in the future, another space telescope specially designed to measure CMB polarization modes could bring r down to around $r < 0.0001$ [7].

If the previous estimate of $r \approx 0.1$ is correct, then Planck should be on the verge of being precise enough to detect it. The next generation of polarization studies should most likely be precise enough to give the first observation of the tensor spectrum. And so, there is good reason to suppose we might see primordial gravitational waves on the CMB within the next few years.

Another way to measure primordial gravitational waves is directly in a gravitational wave detector like LIGO, or a European LISA-like experiment, or the proposed BBO [11] or DECIGO [12]. For the standard inflationary tensor spectrum though, the amplitude will be too small to be directly detected for many decades.

It is much more likely the primordial gravity spectrum will first be observed indirectly off CMB anisotropies. There are 3 different ways anisotropies on the CMB are being studied called T, E, and B modes. First, if the energy density of gravitational waves differs from point to point in space, this will cause red and blueshifting of photons as they pass through these energy density perturbations. This anisotropy is denoted by the term T mode since

it is a scalar quantity; T stands for temperature, and the photons coming from the more or less dense regions have different energy or temperature, hence the name. Of course the gravitational wave energy density is only one of many fields exhibiting such energy density anisotropies, perturbing the photon density distribution, and it will be the cumulative effect of all fields which will be measured in the T modes. The gravitational wave contribution is expected to only contribute a small amount towards the perturbations of the T modes compared to the contribution from scalars. Also, the gravitational wave contribution has a much more limited range of scales at which it is relevant compared to scalars, dropping off quickly after $l > 100$ (corresponding to angular scales of about $\theta = 1$ degree). This is a problem because at these small l scales cosmic variance becomes an important and unavoidable limiting factor. Cosmic variance is an uncertainty arising from the fact that we only have one observable universe over which we can make measurements. For example, to collect data for scales $l = 2$ ($\theta = 180$ degrees), the lowest l at which one can make a power spectrum measurement, there are only two independent measurements one can make at this scale which provides terrible statistics. This effect causes important statistical limitations at low l scales. Cosmic variance ceases to become too much of a problem after about $l > 10$, and so there is still a range after cosmic variance stops being a problem before the gravitational wave signal drops off, but it is a much more limited range than for scalars for which one can make measurements which extend to $l \approx 3000$.

The E and B modes represent anisotropies in the polarization of the photons. Photon momentum will be largely towards and away from energy density maxima, and therefore when these photons scatter off electrons, they will scatter in an in-homogeneous manner. There will only be a net polarization of scattered photons if there are quadrupole moments in the energy density, and such quadrupole moments are expected but they will be fairly small. This scattering of photons and electrons which causes the photons to become polarized must happen before decoupling, and so at this time the photons were still coupled in a baryon-electron-photon fluid, and the fields were tightly coupled only allowing for small quadrupole moments in the energy density. Polarization modes have two relevant directions, the direction in which the polarization strength is changing, and the direction of the polarization axis. For E modes these are always aligned, or perpendicular; for B modes they are not. Like the T modes, the E modes also get contributions from all the fields creating these energy density maxima. Therefore, like the T modes, gravitational waves will contribute only a small portion to the signal in the E modes, making the tensor contribution difficult to weed out experimentally. On the other hand, the only way to generate B modes is from a tensor field.

So eventually, the strongest constraints on r will likely come from a measurement of the B modes, since any observation of B modes will be equivalent to measuring the tensor spectrum. The reason the B mode constraint on r currently is not stronger is because the CMB studies to date have for the most part been designed to focus on better accuracy of the T modes. Many of the newer CMB studies (BICEP2, Keck, POLARBEAR, and SPIDER) are designed to focus on polarization, and are consequently expected to provide better limits on r from the B modes.

The primordial anisotropies are expected to be adiabatic; the maximums and minimums in energy density of the various fields present should overlap proportionally. It is possible that there could have been non-adiabatic, isocurvature, anisotropies where a minimum in energy density of one field could occur at the location of a maximum in another field, but

one has to use more complicated inflationary models with inflation fields with extra degrees of freedom for such anisotropies to be generated. Planck reported upper bounds on the maximum fractional contribution to the measured scalar power spectrum, β_{iso}, at pivot scale $k_0 = 0.002$ Mpc^{-1} that could be coming from isocurvature modes for the various types of matter as: $\beta_{iso} < 0.075$ from cold dark matter or baryon isocurvature modes, $\beta_{iso} < 0.27$ from neutrino density isocurvature modes, and $\beta_{iso} < 0.18$ from neutrino velocity field isocurvature modes [1]. The current evidence is still consistent with a universe composed of only adiabatic perturbations.

Assuming the basic inflationary paradigm is essentially correct, there could still be other mechanisms producing primordial perturbations during inflation which could be measured in addition to the standard perturbations. Many such mechanisms have been proposed including modulated fluctuations [13, 14], spatial perturbations in inflaton decay rates [15], and curvaton models [16].

In addition, there have been many studies looking at primordial perturbations generated by instances of particle production during slow roll inflation [17, 18] or in the context of trapped inflation, where multiple instances of particle production slow down the rolling of the inflaton field enough to allow inflation even on a steep potential, [19, 20].

Particle production models are somewhat natural because we know inflation had to end. The universe went through a period at the end of inflation called reheating in which the inflaton field decayed away, producing the standard model degrees of freedom. Therefore for inflation to work, the inflaton must have couplings to other fields. It is then natural to assume that the inflaton could generate some quanta of these other fields it is coupling to during slow roll inflation as well as during reheating. There are of course restrictions on how efficiently the inflaton could produce these other fields during slow roll such that inflation still continue. The question we asked is, is it possible for the inflaton field, through its couplings with other fields, to decay into these other fields during slow roll efficiently enough to leave detectable traces, without ruining inflation or disagreeing with current observational constraints.

In particular we investigated what spectrum of gravitational waves would be generated by particle production during slow roll inflation. The standard $\frac{H^2}{M_P^2}$ tensor power spectrum is almost featureless, but slowly loses amplitude at larger momenta (modes which left the horizon later during inflation). It is common for models of particle production during inflation to have a peak, or multiple peaks in the power spectrum, where the amplitude increases with increasing momentum, distinguishing their spectrum from the standard inflationary spectrum.

We first allowed the inflaton field to couple to scalars and then to vector fields, decaying into these fields during slow roll. These newly sourced fields would further decay into gravitational waves which would then be redshifted with amplitude decreasing as $\frac{1}{a}$, where a again is the scale factor, until they reached horizon size, at which point they get 'frozen in' so that once their wavelength is longer then H, their amplitude remains constant. The assumption is that if enough of these gravitational waves are sourced, when they re-enter the horizon later, after inflation has ended, they can leave a detectable trace either indirectly on the CMB though scattering events with photons, or possibly directly as might someday be possible in a gravitational wave detector.

First we calculate the power spectra for 3 different models of decay of the inflation into scalar and vector fields. The first too models involve a very short period of production of

these fields (non-adiabatic period $\ll 1/H$), by an interaction of the inflaton with these fields squared so the inflation field provides a dynamical mass term to the equation of motion of these fields. If the mass of these fields becomes temporarily zero, the inflation can pass energy into generating quanta of these fields. When these fields become massive again, they decay away quickly (with energy density proportional to number density since they are non-relativistic), redshifting as $1/a^3$ as the universe expands. Before they can effectively disappear, they can generate gravitation waves which are massless and will only redshift till they reach horizon size, and then their amplitude will remain constant. In this way, we do not expect to see these primordial vector fields themselves, but we might see the tensor metric perturbations produced by them. We calculate the power spectra of these gravitational waves and compare it to the standard power spectrum which would be there in any case from generation of gravitational waves in a de-sitter spacetime. We find that for either production of scalars or vectors, the resulting power spectrum of gravitational waves is dwarfed by the background spectrum as long as we require that these produced scalars and vectors posses energy densities much smaller then the inflation field – a necessity if we want inflation to continue. So this method of gravitational wave production can not produce an observable signal.

Next we consider allowing a prolonged period of production of a vector field during slow roll by having the inflaton field couple to derivatives of this vector field. In this case we are able to find a region of parameter space where this model could produce an observable tensor power spectrum signal, but only at the high frequency modes of direct detection experiments. There are too large of constraints for this model at CMB scales from the non-observation of non-Gaussianities of the scalar perturbations which should also be produced by this model.

Next we considered a variation of this model proposed by N. Barnaby et. al where instead of having a direct coupling between the inflaton and the vectors, the vectors are instead directly coupled to a different scalar field slowly evolving during inflation, the coupling with which the vectors are being sourced; these vectors are only gravitationally coupled to the inflaton [21]. Assuming, as in most inflationary models, the inflaton is the primary generator of primordial curvature perturbations, than a weaker coupling with the inflaton field will lead to weaker production of scalar curvature perturbations from these vectors relative to their tensor perturbations. The idea behind this model was a best case scenario for generating larger tensor perturbations relative to scalar, with the assumption that the coupling between the vectors and the inflaton could have any of a spectrum of possible values, and this scenario takes the minimum possible coupling. It was shown in the paper by the Barnaby et. al. group that the weaker constraints from the scalar perturbations in this case allows for a possible detection of the tensor perturbations, even at CMB scales.

We expand on this model by calculating the tensor three-point function for this model, and the subsequent contribution to the temperature three-point function from the tensor perturbations. We used these results to constrain the model parameters by using the non-Gaussianity limit as recently reported by Planck of $f_{NL}^{\text{equil}} = -42 \pm 75$ at the 68% CL [22]. We find this gives the best constraint on the model parameters over most of the parameter space, and although providing a stronger constraint on the model, we find a detection still possible at CMB scales. We find the temperature three-point function for the model is dominated by the contribution from tensor rather than scalar perturbations.

This is the first model in which it has been shown that the non-Gaussianities sourced by the tensor spectrum can dominate those produced in the scalar spectrum. Previously work

on non-Gaussianities typically focuses on non-Gaussianities produced in the scalar spectrum, or less commonly, three-point correlators coupling tensors and scalars. This trend is due to the fact that the scalar power spectrum is at least 9 times larger than the tensor power spectrum (one over the current limit on r), and because it is generally easier to produce scalar non-Gaussianities from perturbations of or interactions with a scalar inflaton than tensor non-Gaussianities. When one considers though that most inflationary models produce small non-Gaussianities, although scalar perturbations dominate the temperature two-point function, it is possible that tensor perturbations dominate the three-point function.

1.5 Gravitational Wave Production

We want to calculate the gravitational waves sourced by the produced particles. We assume these new particles only contribute a perturbation to the metric which takes the form:

$$g_{\mu\nu} = -dt^2(1 + \delta g_{00}) + a(t)^2(\delta_{ij} + \delta g_{ij})dx^i dx^j. \tag{1.30}$$

This metric in general includes scalar, vector, and tensor perturbations. We only care to calculate the gravitational waves produced, and since gravitons are spin two particles, this corresponds to tensor perturbations. We apply the transverse traceless decomposition, and the different perturbation modes decouple; the scalar equations are only dependent on scalars and so on. We are only concerned with tensors and are left with one 3x3 spatial tensor which is transverse and traceless and is not influenced by the other mode perturbations at the linear level, and is gauge invariant. We call it h_{ij}. We will ignore all other mode perturbations since they will not influence the tensor spectrum, and the metric simplifies to:

$$g_{\mu\nu} = -dt^2 + a(t)^2(\delta_{ij} + h_{ij})dx^i dx^j. \tag{1.31}$$

Substituting this into Einstein's equation:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}\,, \tag{1.32}$$

we obtain the equation of motion of the tensor perturbations:

$$h''_{ij} + 2\frac{a'}{a}h'_{ij} - \nabla^2 h_{ij} = \frac{2}{M_P^2}\Pi_{ij}^{lm}T_{lm}\,, \tag{1.33}$$

where Π_{ij}^{lm} is a transverse traceless projector, given by

$$\Pi_{ij}^{lm} = \Pi_i^l \Pi_j^m - \frac{1}{2}\Pi_{ij}\Pi^{lm} \tag{1.34}$$

and

$$\Pi_{ij} = \delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}. \tag{1.35}$$

T_{lm} in general will not be transverse and traceless and will produce all types of modes of perturbations. Applying the operator subtracts out the part of T_{lm} which is not transverse and traceless, so the left and right sides of the equation above will match since we know h_{ij} has to be transverse and traceless. This effectively subtracts out all parts of T_{lm}, which do not produce gravitational waves.

We apply a Fourier transform on $h_{ij}(\mathbf{x},\tau)$:

$$h_{ij}(\mathbf{x},\tau) = \int \frac{d^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}} e^{-i\mathbf{q}\cdot\mathbf{x}} \tilde{h}_{ij}(\mathbf{q},\tau). \tag{1.36}$$

We can now simplify eq. (1.33) to obtain:

$$\tilde{h}''_{ij}(\mathbf{k},\tau) + 2\frac{a'}{a}\tilde{h}'_{ij}(\mathbf{k},\tau) + k^2\tilde{h}_{ij}(\mathbf{k},\tau) = \int \frac{d^3\mathbf{x}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{x}\cdot\mathbf{k}} \frac{2}{M_P^2}\Pi^{lm}_{ij} T_{lm}(\mathbf{x},\tau) \tag{1.37}$$

We solve using a retarded Green's function: $G(\tau,\tau',k)$, such that eq. (1.37) becomes:

$$\tilde{h}_{ij}(\mathbf{k},\tau) = \int d\tau' G(\tau,\tau',k) \int \frac{d^3\mathbf{x}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{x}\cdot\mathbf{k}} \frac{2}{M_P^2}\Pi^{lm}_{ij} T_{lm}(\mathbf{x},\tau), \tag{1.38}$$

and $G(\tau,\tau',k)$ solves:

$$G''(k,\tau) + 2\frac{a'}{a}G'(k,\tau) + k^2 G(k,\tau) = \delta(\tau-\tau'). \tag{1.39}$$

Limits on the τ integral in eq. (1.38) are chosen for the time period over which we want to measure the production of gravitational waves.

Solving eq. (1.39) for the Green's function, we obtain:

$$G(\tau,\tau',k) = \frac{1}{k^3\tau'^2}[\cos(k(\tau-\tau'))(k\tau'-k\tau) + \sin(k(\tau-\tau'))(k^2\tau\tau'+1)]\Theta(\tau-\tau'). \tag{1.40}$$

We can further simplify eq. (1.38) after writing T_{lm} in momentum space and then noting that $\Pi^{lm}_{ij}(\mathbf{q})$ is symmetric on $\mathbf{q}\to-\mathbf{q}$. The $\mathbf{x}$ then integral becomes a delta function.

$$\begin{aligned}\tilde{h}_{ij}(\mathbf{k},\tau) =& \frac{2}{M_P^2}\int_{\tau_{end}}^{0} d\tau' G(\tau,\tau',k) \int_{-\infty}^{\infty} \frac{d^3\mathbf{x}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{x}\cdot\mathbf{k}} \Pi^{lm}_{ij}(\mathbf{q}) \int \frac{d^3\mathbf{q}}{(2\pi)^{3/2}} e^{-i\mathbf{q}\cdot\mathbf{x}} \tilde{T}_{lm}(\mathbf{q},\tau') \\ =& \frac{2}{M_P^2}\int_{\tau_{end}}^{0} d\tau' G(\tau,\tau',k)\Pi^{lm}_{ij}(\mathbf{k})\tilde{T}_{lm}(\mathbf{k},\tau')\end{aligned} \tag{1.41}$$

This equation will be used in the following sections to test the effects of various methods of generating particles during inflation, each of which provides a different contribution to the energy momentum tensor.

CHAPTER 2

SUDDEN DECAY OF THE INFLATON INTO SCALARS

First we consider particle production during inflation from decay of the inflaton field, ϕ, to a scalar field, χ, occurring for a short instant in time, less then a Hubble time, (around $\phi = \phi_0$). As we shall see, the quicker the transition through this period of nonadiabaticity, the more efficient the production of quanta of the scalar field χ. If we do not see a significant signal in this case where $\Delta t_{\text{non-adiabatic}} \ll H$, then we definitely will not see a significant signal in the reverse case. We then examine the tensor spectrum generated by these produced particles. One such interaction which has received the most attention and can generate such a decay of ϕ into χ is given by:

$$S = \frac{1}{2}\int d^4x\sqrt{-g}[-\partial_\mu\chi\partial^\mu\chi - \frac{g^2}{2}(\phi-\phi_0)^2\chi^2]\,, \tag{2.1}$$

where I have neglected terms dependent only on ϕ since they will not contribute to the equation of motion of χ. It is apparent that χ has an effective mass term from its interaction with ϕ. If ϕ should pass through ϕ_0 when slowly rolling down its potential, then χ will become temporarily massless. At this point, it becomes energetically cheap to produce quanta of χ; the ϕ field will loose energy to produce these quanta of χ. The above equation uses conformal time, τ, with prime implying $\frac{d}{d\tau}$. Note $\sqrt{(-g)}$ is the determinant of the metric, only getting contributions from 0th order in which case $\sqrt{(-g(\tau))} = a^4(\tau)$, where a is the scale factor. In conformal time, $a = -\frac{1}{H\tau}$. Ultimately, we want to solve eq. (1.41) for a T_{lm} sourced by quanta of χ. Before we can find such a T_{lm}, we need the equation of motion of χ and the associated two-point function for χ.

2.1 χ Equation of Motion

Solving for the equation of motion of χ gives:

$$\chi'' - \frac{2}{\tau}\chi' - \nabla^2\chi + g^2a^2(\phi-\phi_0)^2\chi = 0\,, \tag{2.2}$$

but note this equation is not canonically normalized. We want to calculate how many quanta of the χ field will be produced, but first we need to relate χ to a field which is canonical and

can be quantized. To do this, we define:

$$\psi = a\chi. \tag{2.3}$$

When plugged into eq. (2.2) this gives:

$$\psi'' + (-\nabla^2 + \frac{1}{\tau^2}(-2 + \frac{g^2}{H^2}(\phi - \phi_0)^2))\psi = 0. \tag{2.4}$$

It is helpful to use the equation of motion in momentum space using the Fourier transform:

$$\psi(\mathbf{x}, \tau) = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} e^{-i\mathbf{k}\cdot\mathbf{x}} \tilde{\psi}(\mathbf{k}, \tau) \tag{2.5}$$

to obtain:

$$\tilde{\psi}'' + \tilde{\psi}(k^2 + \frac{1}{\tau^2}(-2 + \frac{g^2}{H^2}(\phi - \phi_0)^2)) = 0. \tag{2.6}$$

Before we can solve eq. (2.6), we need to specify how ϕ depends on time. The homogeneous equation of motion of the inflaton field is given by

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V(\phi)}{\partial \phi} = 0\,. \tag{2.7}$$

We use the slow roll condition: $\ddot{\phi} \ll 3H\dot{\phi}, V'$. Thus to satisfy slow roll for any inflation potential, $\ddot{\phi}$ is suppressed and $\dot{\phi}$ is approximately constant.

Now we can use that ϕ evolves linearly in time to solve the equation of motion of $\tilde{\psi}$, eq. (2.6). First we promote $\tilde{\psi}$ to an operator and decompose into creation and annihilation operators and mode functions. (While it is most conventional to use a for operators, I am going to use b just to make sure there is no confusion with a, the scale factor):

$$\hat{\tilde{\psi}}(\mathbf{k}, \tau) = \hat{b}_{\mathbf{k}} u(k, \tau) + \hat{b}^{\dagger}_{-\mathbf{k}} u^*(k, \tau). \tag{2.8}$$

The mode function $u(k, \tau)$ will solve the equation:

$$u'' + u(k^2 + \frac{1}{\tau^2}(-2 + \frac{g^2}{H^2}(\phi - \phi_0)^2)) = 0. \tag{2.9}$$

This equation is difficult to solve exactly, so we make an approximation. Our goal is to calculate how many quanta of the $\tilde{\psi}$ field are produced. The time scale during which these quanta are created is very short, $\Delta t \ll \frac{1}{H}$, and so the expansion of the universe during this time is negligible. We can therefore take the Minkowski limit of eq. (2.9); we approximate $H\tau \to$ constant and $H \to 0$. Next we show how long the period of particle production lasts to prove that $\Delta t \ll \frac{1}{H}$, and therefore that this approximation is valid. Quanta of $\tilde{\psi}$ will be produced as long as the $\tilde{\psi}$ field is evolving non-adiabatically. Whenever one has an equation of the form: $f''(x) + \omega^2 f(x) = 0$, the system will evolve non-adiabatically when $\frac{\omega'}{\omega^2} > 1$. We have:

$$\omega = (k^2 + \frac{1}{\tau^2}(-2 + \frac{g^2}{H^2}(\phi - \phi_0)^2))^{\frac{1}{2}}. \tag{2.10}$$

We can simplify the expression by first using $\phi = \dot{\phi}t$ where $\dot{\phi}$ is constant. We can then rewrite: $(\phi - \phi_0) = \dot{\phi}(t - \frac{\phi_0}{\dot{\phi}})$. We define t_0 as the time when ϕ passes through ϕ_0, so $t_0 = \frac{\phi_0}{\dot{\phi}}$. Then $(\phi - \phi_0) = \dot{\phi}(t - t_0)$. Next, we want our expression for ω only in conformal time before we can differentiate ω. We can use that t and τ are related by $t = \frac{1}{H}\ln(-\frac{1}{H\tau})$, during inflation. Plugging this into our expression for ω, we obtain:

$$\omega = (k^2 + \frac{1}{\tau^2}(-2 + \frac{g^2\dot{\phi}^2}{H^4}\ln^2(\frac{\tau_0}{\tau})))^{\frac{1}{2}}\,, \tag{2.11}$$

where τ_0, defined in analogy with t_0, is the conformal time at which ϕ passes through ϕ_0. We can simplify further by dropping the $\frac{-2}{\tau}$ term. Even though the $\frac{-2}{\tau}$ term will dominate over the interaction term for a short while right around $t = t_0$, even within the non-adiabatic region, this term will dominate for a much shorter time than the interaction term, that we assume it does not have a chance to have too large of an effect. In the non-adiabatic limit, we can drop the k term because, as will be shown later, there will be exponential suppression of large k values such that there is only a significant contribution from non-relativistic χ, $\omega \gg k$. This leads to an adiabaticity condition:

$$\frac{\omega'}{\omega^2} = \frac{-1}{g\dot{\phi}(t - t_0)^2} \tag{2.12}$$

and therefore

$$\lim_{\text{non-adiabatic}} |t - t_0| \leq \frac{1}{(g\dot{\phi})^{\frac{1}{2}}}. \tag{2.13}$$

To compare $\sqrt{g\dot{\phi}}$ to H, first we use the definition of the slow roll parameter ϵ, $\epsilon = \frac{M_P^2(\frac{\partial V}{\partial \phi})^2}{2V^2}$. Then we use the equation of motion of the inflaton that $\dot{\phi}^2 = \frac{(\frac{\partial V}{\partial \phi})^2}{9H^2}$, eq. (2.7). We then use the Friedmann equation: $H^2 = \frac{8\pi G}{3}\rho$, where ρ is energy density. We can also plug in $\rho = \frac{\dot{\phi}}{2} + V$ which one obtains from solving for the stress energy tensor from the inflaton Lagrangian, assuming that the inflaton field is a perfect fluid. Next, we apply another slow roll condition that $\frac{\dot{\phi}}{2} \ll V$ to the Friedmann equation to obtain:

$$H = \frac{1}{\sqrt{3}M_P}\sqrt{V}\,. \tag{2.14}$$

Thus we find $\dot{\phi} = \sqrt{2\epsilon}HM_P$, and $\frac{\sqrt{g\dot{\phi}}}{H} = (2\epsilon)^{\frac{1}{4}}(\frac{gM_P}{H})^{\frac{1}{2}}$. So $\sqrt{g\dot{\phi}} = (2\epsilon)^{\frac{1}{4}}(\frac{gM_P}{H})^{\frac{1}{2}}H$ which we assume is much greater then H. This is plausible considering that $H < 10^{14}$ GeV (see intro) and $M_P \approx 10^{18}$ GeV/c^2. This will of course depend on our choice of g, the coupling constant, which is arbitrary, but we do not want g to be too small or we will not have efficient particle production. The remainder of our analysis assumes $g \gg \frac{H^2}{\dot{\phi}}$. With this assumption, we find that Δt during the non-adiabatic period $\ll \frac{1}{H}$.

We shall see that the final power spectrum will be proportional to $\frac{\sqrt{g\dot{\phi}}}{H}$. And so the larger $\sqrt{g\dot{\phi}}$ is relative to H, in other words the shorter the period of nonadiabaticity, the larger the final power spectrum will be. So assuming $\sqrt{g\dot{\phi}} \gg H$ now is fine because without this condition, an observable signal will not be possible.

Our next goal is to take the Minkowski limit of eq. (2.9). We define $\tilde{t} = t - \frac{\phi_0}{\dot{\phi}}$ and note $d\tilde{t} = dt$. Switching to using this real time variable, eq. (2.9) becomes:

$$\ddot{u} + u(k^2 H^2 \tau_0^2 + g^2 \dot{\phi}^2 \tilde{t}^2) = 0\,, \tag{2.15}$$

which can now be solved to obtain:

$$u = c_1 D_{-\frac{1}{2} - \frac{ik^2 H^2 \tau_0^2}{2g\dot{\phi}}}((i+1) g^{\frac{1}{2}} \dot{\phi}^{\frac{1}{2}} \tilde{t}) + c_2 D_{-\frac{1}{2} + \frac{ik^2 H^2 \tau_0^2}{2g\dot{\phi}}}((i-1) g^{\frac{1}{2}} \dot{\phi}^{\frac{1}{2}} \tilde{t}). \tag{2.16}$$

Next we want to apply boundary conditions to solve for the integration constant. We require for $t \to -\infty$ the solution be that of a plane wave. Early enough in time, no particles have been produced, and we expect a vacuum solution. First, we take take the asymptotic expansion of eq. (2.16):

$$u(\tilde{t} \to -\infty) = c_1 [e^{-\frac{i}{2} g\dot{\phi}\tilde{t}^2} (-(1+i) g^{\frac{1}{2}} \dot{\phi}^{\frac{1}{2}} |\tilde{t}|)^{-\frac{1}{2} - \frac{ik^2 H^2 \tau_0^2}{2g\dot{\phi}}} - \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(\frac{1}{2} + \frac{ik^2 H^2 \tau_0^2}{2g\dot{\phi}})} \tag{2.17}$$

$$e^{i\pi(\frac{1}{2} + \frac{ik^2 H^2 \tau_0^2}{2g\dot{\phi}})} e^{\frac{i}{2} g\dot{\phi}\tilde{t}^2} (-(i+1) g^{\frac{1}{2}} \dot{\phi}^{\frac{1}{2}} |\tilde{t}|)^{-\frac{1}{2} + \frac{ik^2 H^2 \tau_0^2}{2g\dot{\phi}}}] + c_2 e^{\frac{i}{2} g\dot{\phi}\tilde{t}^2} (-(i-1) g^{\frac{1}{2}} \dot{\phi}^{\frac{1}{2}} |\tilde{t}|)^{-\frac{1}{2} + \frac{ik^2 H^2 \tau_0^2}{2g\dot{\phi}}}].$$

Next we match this onto our expected plane wave solution:

$$u = \frac{(H\tau_0)^{\frac{1}{2}}}{2^{\frac{1}{2}} (k^2 H^2 \tau_0^2 + g^2 \dot{\phi}^2 \tilde{t}^2)^{\frac{1}{4}}} (\alpha e^{-i \int (k^2 H^2 \tau_0^2 + g^2 \dot{\phi}^2 \tilde{t}^2)^{\frac{1}{2}} d\tilde{t}} + \beta e^{i \int (k^2 H^2 \tau_0^2 + g^2 \dot{\phi}^2 \tilde{t}^2)^{\frac{1}{2}} d\tilde{t}})\,, \tag{2.18}$$

which we obtain from solving for the adiabatic limit of $\ddot{u} + u(k^2 H^2 \tau_0^2 + g^2 \dot{\phi}^2 \tilde{t}^2) = 0$ with $\omega = (k^2 H^2 \tau_0^2 + g^2 \dot{\phi}^2 \tilde{t}^2)^{\frac{1}{2}}$. The adiabatic limit of an equation of the form: $u'' + \omega^2 u = 0$ is:

$$u = \frac{(H\tau_0)^{\frac{1}{2}}}{(2\omega)^{\frac{1}{2}}} (\alpha e^{-i \int \omega d\tilde{t}} + \beta e^{i \int \omega d\tilde{t}}) \tag{2.19}$$

with α and β constant. Taking the $t \to -\infty$ limit of eq. (2.18) gives:

$$u(\tilde{t} \to -\infty) = \frac{(H\tau_0)^{\frac{1}{2}}}{(2g\dot{\phi}|\tilde{t}|)^{\frac{1}{2}}} [\alpha e^{\frac{i}{2} g\dot{\phi}\tilde{t}^2} (\frac{-2g\dot{\phi}\tilde{t}}{kH\tau_0})^{\frac{ik^2 H^2 \tau_0^2}{2g\dot{\phi}}} + \beta e^{-\frac{i}{2} \tilde{t}(g^2 \dot{\phi}^2 \tilde{t}^2)^{\frac{1}{2}}} (\frac{-2g\dot{\phi}\tilde{t}}{kH\tau_0})^{-\frac{ik^2 H^2 \tau_0^2}{2g\dot{\phi}}}]. \tag{2.20}$$

We choose for initial conditions $\alpha = 1$, $\beta = 0$, which gives us:

$$u(\tilde{t} \to -\infty) = \frac{(H\tau_0)^{\frac{1}{2}}}{(2g\dot{\phi}|\tilde{t}|)^{\frac{1}{2}}} e^{\frac{i}{2} g\dot{\phi}\tilde{t}^2} (\frac{2g\dot{\phi}|\tilde{t}|}{kH\tau_0})^{\frac{ik^2 H^2 \tau_0^2}{2g\dot{\phi}}} \tag{2.21}$$

Now we can match this onto eq. (2.16) to solve for our integration constants and obtain:

$$u = \frac{(H\tau_0)^{\frac{1}{2}}}{(2g\dot{\phi})^{\frac{1}{4}}} (\frac{2g\dot{\phi}}{k^2 H^2 \tau_0^2})^{\frac{ik^2 H^2 \tau_0^2}{4g\dot{\phi}}} e^{-\frac{i\pi}{8}} e^{-\frac{k^2 H^2 \tau_0^2 \pi}{8g\dot{\phi}}} D_{-\frac{1}{2} + \frac{ik^2 H^2 \tau_0^2}{2g\dot{\phi}}}((i-1) g^{\frac{1}{2}} \dot{\phi}^{\frac{1}{2}} \tilde{t}). \tag{2.22}$$

This is the mode function of ψ. With this we have the equation of motion of ψ, and therefore also for χ using $\chi = \psi/a$, eq. (2.3).

2.2 Bogolubov Coefficients

We could take our equation of motion of χ, using eq. (2.22) and plug this into T_{lm} and integrate to obtain the power spectrum of gravitational waves using eq. (1.41). This would give us the result we want, but involves a very difficult integral. Indeed, we will take this approach in section 2.5 when we approximate the production of gravitational waves in the non-adiabatic region, where we use various techniques to approximate the integral. Here, we instead employ the Bogolubov method.

We want to calculate how many particles have been produced when $t \to \infty$. Far from t_0, particle production will have ended, and we again expect an adiabatic solution of the form eq. (2.18), but with α and β, Bogolubov coefficients, taking on new values to account for the fact that particles have been produced. By solving for the new values of α and β after particle production has ceased, we can calculate how many quanta are produced. This method allows us to ignore the exact dynamics taking place during the non-adiabatic period around $t \approx t_0$. We take the $t \to \infty$ limit of eq. (2.22) and then match onto our plane wave solution again, eq. (2.18), to solve for α and β, obtaining:

$$\beta(k) = -ie^{-\frac{\pi k^2 H^2 \tau_0^2}{2g\dot{\phi}}} \tag{2.23}$$

and

$$\alpha(k) = \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(\frac{1}{2} - \frac{ik^2 H^2 \tau_0^2}{2g\dot{\phi}})} (\frac{4g\dot{\phi}}{k^2 H^2 \tau_0^2})^{\frac{ik^2 H^2 \tau_0^2}{2g\dot{\phi}}} e^{-\frac{\pi k^2 H^2 \tau_0^2}{4g\dot{\phi}}}. \tag{2.24}$$

2.3 Calculating the Two-Point Function of ψ

We have broken the evolution of ψ into three regions. At $t \to -\infty$ no particles have been produced and ψ has a plane wave solution (since there will always be vacuum fluctuations). Then there is the complicated region where ψ particles are being produced. Finally, as $t \to \infty$, particle production of ψ has stopped and the system starts evolving adiabatically again. We again expect a plane wave solution for ψ, but a different plane wave solution because quanta of ψ have been produced. We can quantize ψ in each adiabatic region, but we note that our operator definition will have to be different in each region, because after quanta of ψ have been produced, our annihilation operator from before production will not annihilate the new vacuum which includes the particles produced. We define two sets of vacua and two corresponding sets of creation and annihilation operators: $\psi(t \to -\infty, \mathbf{k}) = b_{\mathbf{k}} u(k,t) + b_{\mathbf{k}}^\dagger u^*(k,t)$ with $b|0\rangle = 0$, and $\psi(t \to \infty, \mathbf{k}) = \bar{b}_{\mathbf{k}} \bar{u}(k,t) + \bar{b}_{\mathbf{k}}^\dagger \bar{u}^*(k,t)$ with $\bar{b}|\bar{0}\rangle = 0$.

α is a measure of how similar the states are at the beginning and the end of particle production, u and $\bar{u}$. If $\alpha = 1$, then ψ is the same state initially as at the end, and no particles have been produced. β tells us how much mixing there has been between positive and negative frequency modes. We can use this to calculate the number of particles produced, given by $\langle\bar{0}|\hat{N}|\bar{0}\rangle = |\beta|^2$, where $\hat{N} = b^\dagger b$ is the number operator defined for the initial region. Applying this number operator defined from our initial vacuum to our vacuum state at end will tell us how many quantum of our initial definition of particle we have at the end.

So now we know how many particles get produced, $|\beta|^2$, with β given by eq. (2.23). To calculate the effect these particles will have on the tensor spectrum, we will need to calculate the two-point function of ψ: $\langle\tilde{\psi}(p,\tau)\tilde{\psi}(p',\tau')\rangle$, which using the Bogolubov method, we are

now in a position to solve. We normal order with respect to the initial vacuum, but apply the vacuum definition at end to take into account the number of particles produced and obtain:

$$\langle : \tilde{\psi}(\mathbf{p},\tau)\tilde{\psi}(\mathbf{p}',\tau') : \rangle = \frac{\delta^{(3)}(\mathbf{p}+\mathbf{p}')}{2(\omega(p,\tau)\omega(p,\tau'))^{1/2}}\Big[|\beta(p)|^2[e^{i\int_\tau^{\tau'}\omega_p d\tau} + e^{i\int_{\tau'}^{\tau}\omega_p d\tau}]$$
$$+ \alpha(p)\beta^*(p)e^{-i\int_{\tau_0}^{\tau}\omega_p d\tau}e^{-i\int_{\tau_0}^{\tau'}\omega_p d\tau} + \beta(p)\alpha^*(p)e^{i\int_{\tau_0}^{\tau}\omega_p d\tau}e^{i\int_{\tau_0}^{\tau'}\omega_p d\tau}\Big] \tag{2.25}$$

2.4 Calculating the Two-Point Function of h_{ij} from the Adiabatic Region

Above we have calculated the spectrum of ψ particles produced. The next step is to calculate how efficient they are at sourcing gravitational waves. As soon as quanta of ψ are produced, they will contribute to T_{lm} and perturb the metric. This means there will be contributions from regions two and three described above: the non-adiabatic period when quanta of ψ are produced, and the subsequent adiabatic period where quanta of ψ are no-longer produced, but the existing ψ can still source gravitational waves. In this section we calculate the gravitational waves produced in the third region. In the subsequent section we look at the waves produced during the non-adiabatic period, which we find to be of a comparable amplitude.

To calculate the tensor spectrum produced by the ψ particles, we want to solve eq. (1.41). We plug in the stress energy tensor of a scalar field:

$$T_{\mu\nu}(spin=0) = \partial_\mu\chi\partial_\nu\chi - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\chi\partial_\sigma\chi + g_{\mu\nu}V(\chi). \tag{2.26}$$

We note that we must apply the transverse, traceless projector to this and to lowest order the g_{ij} terms are proportional to δ_{ij}, which gives 0 when the transverse, traceless projector is applied. Only the first term in T_{ij} remains.

Now we need to calculate $\tilde{T}_{lm}(\mathbf{q},\tau')$ and we start from the transverse, traceless part of $T_{lm}(\mathbf{x},\tau')$:

$$\begin{aligned} T_{lm}(\mathbf{x},\tau') =& \partial_l\chi(\mathbf{x},\tau')\partial_m\chi(\mathbf{x},\tau') \\ =& -\frac{1}{a^2(\tau')(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty} d^3\mathbf{q}\, q_l(k_m - q_m)\tilde{\psi}(\mathbf{q},\tau')\tilde{\psi}(\mathbf{k}-\mathbf{q},\tau')\,. \end{aligned} \tag{2.27}$$

Plugging this into the expression for $\tilde{h}_{ij}$, eq. (1.41), we obtain:

$$\begin{aligned} \tilde{h}_{ij}(\mathbf{k},\tau) =& -\frac{2}{M_P^2 a^2(\tau')(2\pi)^{\frac{3}{2}}}\int_{\tau_{end}}^{0} d\tau' G(\tau,\tau',k)\Pi_{ij}^{lm}(\mathbf{k})\int_{-\infty}^{\infty} d^3\mathbf{q}\, q_l(k_m - q_m) \\ & \tilde{\psi}(\mathbf{q},\tau')\tilde{\psi}(\mathbf{k}-\mathbf{q},\tau')\,, \end{aligned} \tag{2.28}$$

which we can use to solve the two-point function of tensor perturbations:

$$\langle h_{ij}(\mathbf{k},\tau)h^{ij}(\mathbf{k}',\tau)\rangle = \frac{4}{M_P^4(2\pi)^3}\int_{\tau_{end}}^{0} d\tau' \int_{\tau_{end}}^{0} d\tau'' \frac{1}{a^2(\tau')a^2(\tau'')} G(\tau,\tau',k)G(\tau,\tau'',k')\Pi_{ij}^{lm}(\mathbf{k})$$
$$\Pi_{ab}^{np}(\mathbf{k}')g^{ia}g^{jb}\int_{-\infty}^{\infty} d^3\mathbf{p}\int_{-\infty}^{\infty} d^3\mathbf{p}'\, p_l(k_m - p_m)p'_n(k'_p - p'_p)$$
$$\langle\tilde{\psi}(\mathbf{p},\tau')\tilde{\psi}(\mathbf{k}-\mathbf{p},\tau')\tilde{\psi}(\mathbf{p}',\tau'')\tilde{\psi}(\mathbf{k}'-\mathbf{p}',\tau'')\rangle. \tag{2.29}$$

We can simplify using that $\Pi_{ij}^{ab}k_b = 0$. We expand the operator part using Wick's theorem, ignoring the disconnected piece, and plug in the result of eq. (2.25).

$$\langle h_{ij}(\mathbf{k},\tau)h^{ij}(\mathbf{k}',\tau)\rangle = \frac{8\delta^{(3)}(\mathbf{k}+\mathbf{k}')}{M_P^4(2\pi)^3}\int_{\tau_{end}}^{0} d\tau' \int_{\tau_{end}}^{0} d\tau'' \frac{1}{a^2(\tau')a^2(\tau'')} G(\tau,\tau',k)G(\tau,\tau'',k')$$
$$\Pi_{ij}^{lm}(\mathbf{k})\Pi_{ab}^{np}(\mathbf{k}')g^{ia}g^{jb}\int_{-\infty}^{\infty} d^3\mathbf{p}\frac{1}{4(\omega(p,\tau')\omega(p,\tau'')\omega(|\mathbf{k}-\mathbf{p}|,\tau')\omega(|\mathbf{k}-\mathbf{p}|,\tau''))^{\frac{1}{2}}}$$
$$\cdot\, p_l\, p_m\, p_n\, p_p \cdot [|\beta(p)|^2 e^{i\int_{\tau'}^{\tau''}\omega(p)d\tau} + |\beta(p)|^2 e^{i\int_{\tau''}^{\tau'}\omega(p)d\tau} + \alpha(p)\beta^*(p)e^{-i\int_{\tau_0}^{\tau'}\omega_p d\tau}$$
$$e^{-i\int_{\tau_0}^{\tau''}\omega_p d\tau} + \beta(p)\alpha^*(p)e^{i\int_{\tau_0}^{\tau'}\omega_p d\tau}e^{i\int_{\tau_0}^{\tau''}\omega_p d\tau}]\cdot[|\beta(|\mathbf{k}-\mathbf{p}|)|^2 e^{i\int_{\tau'}^{\tau''}\omega(|\mathbf{k}-\mathbf{p}|)d\tau} +$$
$$+ |\beta(|\mathbf{k}-\mathbf{p}|)|^2 e^{i\int_{\tau''}^{\tau'}\omega(|\mathbf{k}-\mathbf{p}|)d\tau} + \alpha(|\mathbf{k}-\mathbf{p}|)\beta^*(|\mathbf{k}-\mathbf{p}|)e^{-i\int_{\tau_0}^{\tau'}\omega(|\mathbf{k}-\mathbf{p}|)d\tau}$$
$$e^{-i\int_{\tau_0}^{\tau''}\omega(|\mathbf{k}-\mathbf{p}|)d\tau} + \beta(|\mathbf{k}-\mathbf{p}|)\alpha^*(|\mathbf{k}-\mathbf{p}|)e^{i\int_{\tau_0}^{\tau'}\omega(|\mathbf{k}-\mathbf{p}|)d\tau}e^{i\int_{\tau_0}^{\tau''}\omega(|\mathbf{k}-\mathbf{p}|)d\tau}] \tag{2.30}$$

Note simplifying just the terms with summed indices, we obtain:

$$\Pi_{ij}^{lm}(\mathbf{k})\Pi_{ab}^{'np}(-\mathbf{k})g^{ia}g^{jb}p_l\, p_m\, p_n\, p_p = \frac{1}{2}(p^2 - \frac{(\mathbf{p}\cdot\mathbf{k})^2}{k^2})^2 \tag{2.31}$$

We take the non-relativistic limit in which $\omega(k) \gg k$ and so $\omega(p) - \omega(|\mathbf{k}-\mathbf{p}|)) \approx 0$. This is valid considering the exponential suppression of large k appearing in β and α. This suppression limits significant contribution to the integral to $k < \frac{\sqrt{g\dot{\phi}}}{H\tau_0}$. During the adiabatic region, $|\tau| < |\tau_0|$, and so comparing to our expression for ω, we only get a significant contribution from non-relativistic ψ particles. Next we drop the terms in eq. (2.30) which oscillate the most, the terms with the largest imaginary exponentials. These are terms proportional to for example, $e^{\pm i\int_{\tau'}^{\tau''}(\omega(p)+\omega(|\mathbf{k}-\mathbf{p}|))d\tau}$ because in this term the two ω's add together, in which case we except much larger oscillations, then in the case where they

subtract and largely cancel each other out.

$$\langle h_{ij}(\mathbf{k},\tau)h^{ij}(\mathbf{k}',\tau)\rangle = \frac{\delta^{(3)}(\mathbf{k}+\mathbf{k}')}{M_P^4(2\pi)^3}\int_{\tau_{end}}^{0} d\tau' \int_{\tau_{end}}^{0} d\tau'' \frac{1}{a^2(\tau')a^2(\tau'')} G(\tau,\tau',k)G(\tau,\tau'',k')$$
$$\int_{-\infty}^{\infty} d^3\mathbf{p}(p^2 - \frac{(\mathbf{p}\cdot\mathbf{k})^2}{k^2})^2 \frac{1}{(\omega(p,\tau')\omega(p,\tau'')\omega(|\mathbf{k}-\mathbf{p}|,\tau')\omega(|\mathbf{k}-\mathbf{p}|,\tau''))^{\frac{1}{2}}}\cdot$$
$$\Big[2|\beta(p)|^2|\beta(|\mathbf{k}-\mathbf{p}|)|^2 + \alpha(p)\alpha^*(|\mathbf{k}-\mathbf{p}|)\beta(|\mathbf{k}-\mathbf{p}|)\beta^*(p) + \alpha(|\mathbf{k}-\mathbf{p}|)\alpha^*(p)$$
$$\cdot\, \beta(p)\beta^*(|\mathbf{k}-\mathbf{p}|) + +|\beta(p)|^2\alpha(|\mathbf{k}-\mathbf{p}|)\beta^*(|\mathbf{k}-\mathbf{p}|)[e^{-2i\int_{\tau_0}^{\tau'}\omega d\tau} + e^{-2i\int_{\tau_0}^{\tau''}\omega d\tau}]+$$
$$+\, |\beta(p)|^2\beta(|\mathbf{k}-\mathbf{p}|)\alpha^*(|\mathbf{k}-\mathbf{p}|)[e^{2i\int_{\tau_0}^{\tau'}\omega d\tau} + e^{2i\int_{\tau_0}^{\tau''}\omega d\tau}]+$$
$$+\, |\beta(|\mathbf{k}-\mathbf{p}|)|^2\alpha(p)\beta^*(p)[e^{-2i\int_{\tau_0}^{\tau'}\omega d\tau} + e^{-2i\int_{\tau_0}^{\tau''}\omega d\tau}]+$$
$$+\, |\beta(|\mathbf{k}-\mathbf{p}|)|^2\beta(p)\alpha^*(|\mathbf{k}-\mathbf{p}|)[e^{2i\int_{\tau_0}^{\tau'}\omega d\tau} + e^{2i\int_{\tau_0}^{\tau''}\omega d\tau}]\Big] \tag{2.32}$$

Now we can also drop the last 4 terms, because they only give a significant contribution to the integral when τ', $\tau'' \approx \tau_0$, which since we are integrating over τ' and τ'', will be true for a bit, but this does not compare to the first three terms which will give a significant contribution to the integral for all τ', τ''. So we can safely drop the last 4 terms because they contribute so much less overall than the first 3 terms.

$$\langle h_{ij}(\mathbf{k},\tau)h^{ij}(\mathbf{k}',\tau)\rangle = \frac{\delta^{(3)}(\mathbf{k}+\mathbf{k}')}{M_P^4(2\pi)^3}\int_{-\infty}^{\infty} d^3\mathbf{p}(p^2 - \frac{(\mathbf{p}\cdot\mathbf{k})^2}{k^2})^2\Big[2|\beta(p)|^2|\beta(|\mathbf{k}-\mathbf{p}|)|^2+$$
$$+\, \alpha(p)\alpha^*(|\mathbf{k}-\mathbf{p}|)\beta(|\mathbf{k}-\mathbf{p}|)\beta^*(p) + \alpha(|\mathbf{k}-\mathbf{p}|)\alpha^*(p)\beta(p)\beta^*(|\mathbf{k}-\mathbf{p}|)\Big]\int_{\tau_{end}}^{0} d\tau'$$
$$\int_{\tau_{end}}^{0} d\tau'' \frac{1}{a^2(\tau')a^2(\tau'')} G(\tau,\tau',k)G(\tau,\tau'',k')$$
$$\frac{1}{(\omega(p,\tau')\omega(p,\tau'')\omega(|\mathbf{k}-\mathbf{p}|,\tau')\omega(|\mathbf{k}-\mathbf{p}|,\tau''))^{\frac{1}{2}}} \tag{2.33}$$

We can approximate our Green's function by taking the limit $\tau \to 0$, in which case the function simplifies from $G(\tau,\tau',k) = \frac{1}{k^3\tau'^2}[\cos(k(\tau-\tau'))(k\tau' - k\tau) + \sin(k(\tau-\tau'))(k^2\tau\tau' + 1)]\Theta(\tau-\tau')$, to $G(0,\tau',k) = \frac{1}{k^3\tau'^2}[\cos(k\tau')k\tau' - \sin(k\tau')]\Theta(0-\tau')$. We can also simplify ω. Recall we have:

$$\omega^2 = k^2 - \frac{2}{\tau^2} + \frac{g^2}{H^2\tau^2}(\phi-\phi_0)^2 \tag{2.34}$$

First we use our earlier approximation that $\omega \gg k$ to take $k \to 0$ in above. We also ignore the term $\frac{2}{\tau^2}$ because during the adiabatic regime $\frac{g^2}{H^2}(\phi-\phi_0)^2 >> 1$, since we have assumed $\frac{g\dot{\phi}}{H^2} \gg 1$. ω then simplifies to:

$$\omega^2 = \frac{g^2}{H^2\tau^2}(\phi-\phi_0)^2 = \frac{g^2\dot{\phi}^2}{H^4\tau^2} ln^2(\frac{\tau_0}{\tau}). \tag{2.35}$$

$$\langle h_{ij}(k,\tau)h^{ij}(k',\tau)\rangle = \frac{2H^8\delta^{(3)}(\mathbf{k}+\mathbf{k}')}{M_P^4(2\pi)^3k^6g^2\dot{\phi}^2}\int_0^\infty d^3p e^{-\frac{\pi(p^2+(\mathbf{k}+\mathbf{p})^2)H^2\tau_0^2}{g\dot{\phi}}}(p^2-\frac{(\mathbf{p}\cdot\mathbf{k})^2}{k^2})^2 \tag{2.36}$$
$$(\int_{\tau_{end}}^0 d\tau'[\cos(k\tau')\tau'k-\sin(k\tau')]\frac{\tau'}{ln(\frac{\tau_0}{\tau'})})^2$$

Let $y = 1-\frac{\tau'}{\tau_0}$.

$$\langle h_{ij}(\mathbf{k},\tau)h^{ij}(\mathbf{k}',\tau)\rangle = \frac{Hg^{3/2}\dot{\phi}^{3/2}\delta^{(3)}(\mathbf{k}+\mathbf{k}')}{2^{9/2}M_P^4\pi^5k^6\tau_0^3}e^{-\frac{k^2\tau_0^2\pi H^2}{2g\dot{\phi}}}(\int_{\frac{H}{(g\dot{\phi})^{1/2}}}^1 dy[\cos(k\tau_0(1-y))(1-y)k\tau_0 \tag{2.37}$$
$$-\sin(k\tau_0(1-y))]\frac{(1-y)}{ln(1-y)})^2$$

We note that the integrand $\to 0$ as $y\to 1$, so we approximate the integral for small y:

$$\langle h_{ij}(\mathbf{k},\tau)h^{ij}(\mathbf{k}',\tau)\rangle = \frac{Hg^{3/2}\dot{\phi}^{3/2}\delta^{(3)}(\mathbf{k}+\mathbf{k}')}{2^{9/2}M_P^4\pi^5k^6\tau_0^3}e^{-\frac{k^2\tau_0^2\pi H^2}{2g\dot{\phi}}}$$
$$\left(\int_{\frac{H}{(g\dot{\phi})^{1/2}}}^1 dy[\cos(k\tau_0)k\tau_0-\sin(k\tau_0)]\frac{1}{-y}\right)^2, \tag{2.38}$$

which can now be solved to obtain:

$$\langle h_{ij}(\mathbf{k},\tau)h^{ij}(\mathbf{k}',\tau)\rangle = \frac{Hg^{3/2}\dot{\phi}^{3/2}\delta^{(3)}(\mathbf{k}+\mathbf{k}')}{2^{9/2}M_P^4\pi^5k^6\tau_0^3}e^{-\frac{k^2\tau_0^2\pi H^2}{2g\dot{\phi}}}[\cos(k\tau_0)k\tau_0-\sin(k\tau_0)]^2$$
$$\ln^2(\frac{(g\dot{\phi})^{1/2}}{H}). \tag{2.39}$$

We add this result to the two point function that would be there in any case from production of tensor modes on a de Sitter background. The first term below is this background term which is also scale invariant, as seen by the lack of k dependence in the associated power spectra. The signal generated by production of the scalars does have k dependence, and will have a peak at a certain frequency range as opposed to this flat background. For there to be any hope of observing the signal, we would need this peak value to rise significantly above the background.

$$\langle h_{ij}(\mathbf{k},\tau)h^{ij}(\mathbf{k}',\tau)\rangle = \frac{4\delta^{(3)}(\mathbf{k}+\mathbf{k}')H^2}{M_P^2k^3}\Big[1+\frac{g^{3/2}\dot{\phi}^{3/2}}{2^{13/2}HM_P^2\pi^5k^3\tau_0^3}e^{-\frac{k^2\tau_0^2\pi H^2}{2g\dot{\phi}}}[\cos(k\tau_0)k\tau_0-\sin(k\tau_0)]^2$$
$$\ln^2\left(\frac{(g\dot{\phi})^{1/2}}{H}\right)\Big] \tag{2.40}$$

Use that the power spectrum is related to the two point function by $\langle h_{ij}(\mathbf{k},\tau)h^{ij}(\mathbf{k}',\tau)\rangle = \delta^{(3)}(\mathbf{k}+\mathbf{k}')\frac{2\pi^2}{k^3}P(k)$ or $P(k) = \frac{k^3}{2\pi^2\delta^{(3)}(\mathbf{k}+\mathbf{k}')}\langle h_{ij}(\mathbf{k},\tau)h^{ij}(\mathbf{k}',\tau)\rangle$.

$$P(\mathbf{k}) = \frac{2H^2}{\pi^2M_P^2}\left[1+\frac{g^{3/2}\dot{\phi}^{3/2}}{2^{13/2}HM_P^2\pi^5k^3\tau_0^3}e^{-\frac{k^2\tau_0^2\pi H^2}{2g\dot{\phi}}}[\cos(k\tau_0)k\tau_0-\sin(k\tau_0)]^2\ln^2\left(\frac{(g\dot{\phi})^{1/2}}{H}\right)\right] \tag{2.41}$$

First note, that the standard power spectrum from vacuum fluctuations of the inflation field is flat, while the spectrum obtained from production of these scalar fields is peaked for particular momentum. If the peak has an amplitude large enough to dominate over the standard signal, and if this occurs at a momentum scale we are able to observe, then we might be able to see this signal. By estimating the values of the constants in power spectrum, we can estimate the max amplitude of the peak and compare it to the background value.

Also note that the power spectrum we obtain for from the production these scalars only depends on the quantity $k\tau_0$, but not on k or τ_0 separately. The time at which ϕ crosses ϕ_0, τ_0, is arbitrary. Whatever time scale it occurs at though, will set the momentum scale at which we observe the peak. Changing τ_0 will not change the max amplitude of the power spectrum, or the shape, but only shift the peak to lower or higher momentum. So max amplitude of signal is independent of 'when' it was produced. 'When' will just determine the size of mode that gives largest signal. In other words, if the production time happens earlier on, then these modes will enter later, on larger scales, and the largest signal will appear on CMB type scales. If the production happened later on towards the end of inflation, then these modes would enter earlier on, at larger k values, and they might produce the largest signal on LIGO/ LISA type frequencies.

This makes intuitive sense. The number of quanta of our scalar field produced is determined by the rate of change of ϕ, which is constant. So it should make sense, that whenever the sourcing of these fields occurs, the power spectrum will have the same shape and amplitude. Whatever wavelength the power spectrum would tend to favor on production will be continually redshifted throughout the end of inflation. What frequency we would need to look for the signal today would be dependent on when they were produced, τ_0, because τ_0 would tell us how long these modes were redshifted for and what wavelength they reached when inflation ended. For example, if production occurs too early during inflation, more then about 60 e-foldings back into inflation, then we would never be able to see the signal because the peak would occur at such long wavelengths as to never have a change to re-enter the horizon after inflation ended.

Also notice the power spectrum gets a larger signal for larger $\frac{\sqrt{g\dot{\phi}}}{H}$ and remember $\frac{1}{\sqrt{g\dot{\phi}}} = \Delta t_{\text{non-adiabatic}}$. So this confirms our earlier assumption that to get a significant signal we need $\Delta t_{\text{non-adiabatic}}$ much less than a Hubble time. Indeed the quicker the transfer through this non-adiabatic period, the larger the signal. $g\dot{\phi}$ is limited though. g is our coupling constant. In order that we not have a strongly coupled theory, what we assumed from the beginning, we need $g < 1$. Also $\dot{\phi}$ is limited if our inflaton field is slowly rolling.

To estimate the power spectrum, we plug in reasonable values for the constants. We substitute, $H \approx 10^{13} GeV/c^2$, $g \approx 1$, and $\epsilon \approx .005$, and use $\dot{\phi} = \sqrt{2\epsilon} H M_P$. Plugging these values into the exponential, we calculate for what value of $k\tau_0$ the spectrum from particle creation is maximized. We find that the spectrum is peaked for $k\tau_0 = 2.25$. Using this value of $k\tau_0$, we calculate the max height of the peak from particle production to be:

$$P(k\tau_0 = 2.25) = \frac{2H^2}{\pi^2 M_P^2}\left[1 + 1.4 \times 10^{-5}\frac{H^2}{M_P^2}\left(\frac{g\dot{\phi}}{H^2}\right)^{\frac{3}{2}} \ln^2\left(\frac{(g\dot{\phi})^{1/2}}{H}\right)\right] \tag{2.42}$$

Or written another way:

$$P(k\tau_0 = 2.25) = \frac{2H^2}{\pi^2 M_P^2}\left[1 + 1.4\times 10^{-5}\left(\frac{H}{M_P}\right)^{\frac{1}{2}}(g\sqrt{2\epsilon})^{\frac{3}{2}}\ln^2\left(g(2\epsilon)^{\frac{1}{2}}\frac{M_P}{H}\right)^{\frac{1}{2}}\right] \tag{2.43}$$

Here is becomes especially clear due to the limits $g < 1$, $\epsilon \ll 1$, $\frac{H}{M_P} \ll 1$ that there is no possible region of parameter space where the above will give an observable signal unless possibly if we considered strongly coupled theories, allowing $g > 1$. And notice too that since there is no explicit k dependence, just dependence on $k\tau_0$, with the max allowed height invariant on k, we conclude that this signal will be undetectable at all frequency ranges, CMB scales, or direct detection, LIGO/ LISA type scales.

Note $k\tau = 1$ is a horizon sized mode. $k\tau > 1$ characterizes modes inside the horizon and $k\tau < 1$ characterizes modes outside the horizon. The fact that the power spectrum from particle production is peaked for $k\tau_0 = 2.25$, means it is peaked for modes which are just barely, but still slightly inside the horizon when particle production happens.

Using again the substitutions for H, g, and ϵ, the power spectrum simplifies to:

$$P(k\tau = 2.25) = \frac{2H^2}{\pi^2 M_P^2}\left[1 + 0.01\sqrt{\frac{H}{M_P}}\right], \tag{2.44}$$

where $\sqrt{\frac{H}{M_P}} < 10^{-4}$. We conclude that the tensor spectrum generated by adiabatically evolving scalars generated during slow roll, will generate a signal so small as to be very difficult to detect. This result agrees with the result of [23] who came to the same conclusion using a different method, and the results of [21], who show that the signal is unobservable even if ϕ is not the inflaton, but another scalar field. The idea of [21] was that if ϕ is not the inflaton, then limits on the model parameters become less stringent. We found above that the signal is maximized for larger $\frac{\sqrt{g}\dot{\phi}}{H}$. $\dot{\phi}$ is constrained when ϕ is the inflaton field because we require slow roll. But even if ϕ is not the inflaton field, then $\dot{\phi}$ has to satisfy the somewhat weaker constraint that the energy density of ϕ needs to be smaller than the energy density in the inflaton field such that the universe continues inflating. The idea of [21] was to test if this weaker constraint on $\dot{\phi}$ could lead to a detectable signal.

To get this new constraint, we use that the kinetic energy of the ϕ field is given by: $-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 \approx \frac{1}{2}\dot{\phi}^2$, because we want this new field to be fairly homogeneous in space. Then we want to compare this to the inflaton energy density which is dominated by the inflaton potential energy, assuming the inflaton field is slowly rolling. The Friedmann equation gives: $H = \sqrt{\frac{8\pi G}{3}\rho}$, and assuming the inflaton field energy is the dominant contribution to ρ we get, $\rho \approx V$, $H^2 \approx \frac{V}{3M_P^2}$, or $V \approx 3M_P^2 H^2$. Plugging this into our inequality, we require:

$$\begin{aligned} &\frac{1}{2}\dot{\phi}^2 \ll 3M_P^2 H^2 \\ &\dot{\phi} \ll \sqrt{6}M_P H\,. \end{aligned} \tag{2.45}$$

But even using this bound, we still find that the power spectrum from particle production is dwarfed by the standard inflationary tensor spectrum. And note that since this bound

above should hold throughout the duration of inflation, it applies at all frequency ranges. We conclude that sudden production of scalars during slow roll will not lead to an observable effect on the tensor spectrum. Just as this mechanisms of sudden production of scalars will generate tensor perturbations it will also produce scalar curvature perturbations. It is interesting to note that the paper [21] claims that even though the tensor spectrum from this mechanism will not be observable, it might be possible that the peak in the scalar curvature spectrum and bispectrum produced from this mechanism might be observable, but only if the ϕ field is not the inflaton field. Otherwise the observational limits in the scalar spectrum for a scale invariant, non-Gaussian spectrum will be too stringent to allow the spectrum from particle production to be observable (of course these limits only apply at CMB scales).

2.5 Calculating the Two-Point Function of h_{ij} from the Non-Adiabatic Region

Next we calculate the contribution to the tensor spectrum generated during the short period of nonadiabaticity. It is possible that the signal there, even though it occurred for a short time, will be larger.

Our new limits on τ' and τ'' are the limits of the non-adiabatic region, calculated in eq. (2.13): $|t - t_0| \leq \frac{1}{\sqrt{g\dot{\phi}}}$, with $t_0 = \frac{\phi_0}{\dot{\phi}}$. We can turn this into a limit on τ by using that $dt^2 = a^2 d\tau^2$, which is how τ is defined. From this we get

$$\tau = \int \frac{1}{a} dt \,, \tag{2.46}$$

where a is given by $a = e^{\int H dt}$ and during inflation, H is constant so

$$\tau = \int e^{-Ht} dt = -\frac{1}{HC} e^{-Ht} \,. \tag{2.47}$$

There is an integration constant C, that represents a multiplicative scaling freedom in how to relate t to τ, but it drops out of the final result.

We use the same T_{lm} as before:

$$\begin{aligned}\langle h_{ij}(\mathbf{k},\tau) h^{ij}(\mathbf{k}',\tau)\rangle = \frac{4}{M_P^4 (2\pi)^3} \int_{-\frac{1}{HC}e^{-H\left(\frac{\phi_0}{\dot{\phi}} - \frac{1}{(g\dot{\phi})^{\frac{1}{2}}}\right)}}^{-\frac{1}{HC}e^{-H\left(\frac{\phi_0}{\dot{\phi}} + \frac{1}{(g\dot{\phi})^{\frac{1}{2}}}\right)}} d\tau' \int_{-\frac{1}{HC}e^{-H\left(\frac{\phi_0}{\dot{\phi}} - \frac{1}{(g\dot{\phi})^{\frac{1}{2}}}\right)}}^{-\frac{1}{HC}e^{-H\left(\frac{\phi_0}{\dot{\phi}} + \frac{1}{(g\dot{\phi})^{\frac{1}{2}}}\right)}} d\tau'' \frac{1}{a^2(\tau') a^2(\tau'')} \\ G(\tau,\tau',k) G(\tau,\tau'',k') \Pi_{ij}^{lm}(\mathbf{k}) \Pi_{ab}^{np}(\mathbf{k}') g^{ia} g^{jb} \int_{-\infty}^{\infty} d^3\mathbf{p} \int_{-\infty}^{\infty} d^3\mathbf{p}' \, p_l (k_m - p_m) p'_n (k'_p - p'_p) \\ \langle \tilde{\psi}(\mathbf{p},\tau') \tilde{\psi}(\mathbf{k}-\mathbf{p},\tau') \tilde{\psi}(\mathbf{p}',\tau'') \tilde{\psi}(\mathbf{k}'-\mathbf{p}',\tau'')\rangle. \end{aligned} \tag{2.48}$$

But the above expression includes the infinite background we eliminated last time by normal ordering. This time we do not use the Bogolubov method to define creation and annihilation operators. Instead, to subtract out the infinite background energy, we can just manually subtract out the expression for the mode function, but in the limit $\tau \to -\infty$, when we know the ψ field is still in its vacuum state. We are doing effectively the same thing as

before, in that we are subtracting out the 'number' of particles of our field that existed before the start of particle production. We used the Bogolubov method in the previous case because it made the calculations a little easier. In this case we do not have that option. The Bogolubov method is used to connect two adiabatic regions separated by a region of non-adiabaticity. We can define particles and their associated creation and annihilation operators in the adiabatic regions, and by calculating how much these definitions changed, we can calculate the number of particles produced. Here we are only integrating over a region of non-adiabaticity and so by definition, we can not define particles and their associated operators.

We use: $\langle\tilde{\psi}(\mathbf{p},\tau')\tilde{\psi}(\mathbf{p}',\tau'')\rangle = \delta^{(3)}(\mathbf{p}+\mathbf{p}')[u(\mathbf{p},\tau')u^*(\mathbf{p}',\tau'') - \lim_{\tau'\to-\infty} u(\mathbf{p},\tau') \lim_{\tau''\to-\infty} u^*(\mathbf{p}',\tau'')]$ where u is the mode function defined in eq. (2.22).

Let $\lim_{\tau'\to-\infty} u(\mathbf{p},\tau') = u_0(\mathbf{p},\tau')$. We use these expressions and Wick's theorem, ignoring disconnected pieces, to simplify eq. (2.48).

$$\begin{aligned}
\langle h_{ij}(\mathbf{k},\tau)h^{ij}(\mathbf{k}',\tau)\rangle = \frac{4}{M_P^4(2\pi)^3}\int_{-\frac{1}{HC}e^{-H\left(\frac{\phi_0}{\dot{\phi}}-\frac{1}{(g\dot{\phi})^{\frac{1}{2}}}\right)}}^{-\frac{1}{HC}e^{-H\left(\frac{\phi_0}{\dot{\phi}}+\frac{1}{(g\dot{\phi})^{\frac{1}{2}}}\right)}} d\tau' \int_{-\frac{1}{HC}e^{-H\left(\frac{\phi_0}{\dot{\phi}}-\frac{1}{(g\dot{\phi})^{\frac{1}{2}}}\right)}}^{-\frac{1}{HC}e^{-H\left(\frac{\phi_0}{\dot{\phi}}+\frac{1}{(g\dot{\phi})^{\frac{1}{2}}}\right)}} d\tau'' \frac{1}{a^2(\tau')a^2(\tau'')} \\
G(\tau,\tau',k)G(\tau,\tau'',k')\Pi^{lm}_{ij}(\mathbf{k})\Pi^{np}_{ab}(\mathbf{k}')g^{ia}g^{jb}\int_{-\infty}^{\infty} d^3\mathbf{p}\int_{-\infty}^{\infty} d^3\mathbf{p}'\, p_l(k_m-p_m)p'_n(k'_p-p'_p) \\
[\delta^{(3)}(\mathbf{p}+\mathbf{p}')\delta^{(3)}(\mathbf{k}-\mathbf{p}+\mathbf{k}'-\mathbf{p}')[u(\mathbf{p},\tau')u^*(\mathbf{p}',\tau'')-u_0(\mathbf{p},\tau')u_0^*(\mathbf{p}',\tau'')]\cdot \\
[u(\mathbf{k}-\mathbf{p},\tau')u^*(\mathbf{k}'-\mathbf{p}',\tau'')-u_0(\mathbf{k}-\mathbf{p},\tau')u_0^*(\mathbf{k}'-\mathbf{p}',\tau'')] \\
+\delta^{(3)}(\mathbf{p}+\mathbf{k}'-\mathbf{p}')\delta^{(3)}(\mathbf{k}-\mathbf{p}+\mathbf{p}')[u(\mathbf{p},\tau')u^*(\mathbf{k}'-\mathbf{p}',\tau'')-u_0(\mathbf{p},\tau')u_0^*(\mathbf{k}'-\mathbf{p}',\tau'')] \\
\cdot[u(\mathbf{k}-\mathbf{p},\tau')u^*(\mathbf{p}',\tau'')-u_0(\mathbf{k}-\mathbf{p},\tau')u_0^*(\mathbf{p}',\tau'')]]. \qquad (2.49)
\end{aligned}$$

We can eliminate one of the time integrals by using fact that the Green's function is real and u only depends on the magnitude of the momentum:

$$\begin{aligned}
\langle\tilde{h}_{ij}(\mathbf{k},\tau)\tilde{h}^{ij}(\mathbf{k}',\tau)\rangle = \frac{4\delta^{(3)}(\mathbf{k}+\mathbf{k}')}{M_P^4(2\pi)^3}\int_{-\infty}^{\infty} d^3\mathbf{p}(p^2-\frac{(\mathbf{p}\cdot\mathbf{k})^2}{k^2})^2[|\int_{-\frac{1}{HC}e^{-H\left(\frac{\phi_0}{\dot{\phi}}-\frac{1}{(g\dot{\phi})^{\frac{1}{2}}}\right)}}^{-\frac{1}{HC}e^{-H\left(\frac{\phi_0}{\dot{\phi}}+\frac{1}{(g\dot{\phi})^{\frac{1}{2}}}\right)}} d\tau' \\
G(\tau,\tau',k)\frac{1}{a^2(\tau')}u(p,\tau')u(|\mathbf{p}+\mathbf{k}|,\tau')|^2 - |\int_{-\frac{1}{HC}e^{-H\left(\frac{\phi_0}{\dot{\phi}}-\frac{1}{(g\dot{\phi})^{\frac{1}{2}}}\right)}}^{-\frac{1}{HC}e^{-H\left(\frac{\phi_0}{\dot{\phi}}+\frac{1}{(g\dot{\phi})^{\frac{1}{2}}}\right)}} d\tau' G(\tau,\tau',k)\frac{1}{a^2(\tau')}u(p,\tau') \\
u_0(|\mathbf{p}+\mathbf{k}|,\tau')|^2 - |\int_{-\frac{1}{HC}e^{-H\left(\frac{\phi_0}{\dot{\phi}}-\frac{1}{(g\dot{\phi})^{\frac{1}{2}}}\right)}}^{-\frac{1}{HC}e^{-H\left(\frac{\phi_0}{\dot{\phi}}+\frac{1}{(g\dot{\phi})^{\frac{1}{2}}}\right)}} d\tau' G(\tau,\tau',k)\frac{1}{a^2(\tau')}u_0(p,\tau')u(|\mathbf{p}+\mathbf{k}|,\tau')|^2 \\
+|\int_{-\frac{1}{HC}e^{-H\left(\frac{\phi_0}{\dot{\phi}}-\frac{1}{(g\dot{\phi})^{\frac{1}{2}}}\right)}}^{-\frac{1}{HC}e^{-H\left(\frac{\phi_0}{\dot{\phi}}+\frac{1}{(g\dot{\phi})^{\frac{1}{2}}}\right)}} d\tau' G(\tau,\tau',k)\frac{1}{a^2(\tau')}u_0(p,\tau')u_0(|\mathbf{p}+\mathbf{k}|,\tau')|^2]. \qquad (2.50)
\end{aligned}$$

We can again approximate the Green's function by taking the late time limit, $\tau \to 0$, meaning that we want to evaluate this two-point function far into the future.

We can further simplify the Green's function by approximating: $\tau' \approx \tau_0$. We have seen above that $\tau = \tau_0 e^{-H\tilde{t}}$ where $\tilde{t} = t - t_0$ and in the non-adiabatic region $|\tilde{t}| \leq \frac{1}{(g\dot{\phi})^{\frac{1}{2}}}$ and we have assumed $\frac{H}{(g\dot{\phi})^{\frac{1}{2}}} \ll 1$ so we can approximate $\tau \approx \tau_0(1 - H\tilde{t})$. Our Green's function simplifies to:

$$G \approx \frac{1}{k^3\tau_0^2}[\cos(k\tau_0)k\tau_0 - \sin(k\tau_0)]. \tag{2.51}$$

Similarly will approximate $a(\tau) \approx a(\tau_0)$. Taking 0th order of G and a is okay because integral will work out to contain first order and higher terms.

We notice that $G \to \infty$ when $k\tau_0 \ll 1$. So when $k\tau_0$ is small, we get the largest contribution to the integral. We approximate $k = 0$. We can next write the integral above in a dimensionless form defining the variables: $\eta = \frac{(g\dot{\phi})^{\frac{1}{2}}}{H}(1 - \frac{\tau}{\tau_0})$ and $\bar{p} = \frac{H}{(g\dot{\phi})}p\tau_0$.

$$\begin{aligned}\langle \tilde{h}_{ij}(\mathbf{k},\tau)\tilde{h}^{ij}(\mathbf{k}',\tau)\rangle =& \frac{16}{15\pi^2}\frac{\delta^{(3)}(\mathbf{k}+\mathbf{k}')}{k^6|\tau_0^3|}\frac{H^4}{M_P^4}(\frac{g\dot{\phi}}{H^2})^{\frac{3}{2}}[\sin(k\tau_0) - k\tau_0\,\cos(k\tau_0)]^2 \\ & \int d\bar{p}\,\bar{p}^6 \int d\eta' \int d\eta'' F(\bar{p},\eta',\eta'')^2\,,\end{aligned} \tag{2.52}$$

where $F(\bar{p},\eta',\eta'')$ is a dimensionless function, the integral of which gives an O(1) contribution to the the above expression. This gives the same order result as the production during the adiabatic regime. We therefore conclude that explosive production of scalars during slow roll will not generate an observable tensor spectrum.

CHAPTER 3

SUDDEN DECAY OF THE INFLATON INTO VECTORS

We next study a very similar interaction where quanta of a vector field are produced over a short period, less then a Hubble time, during slow roll. We assume a gauge-invariant Lagrangian of the form:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (D_\mu\Phi)^*(D^\mu\Phi) - V(\Phi^*\Phi)\ ,, \tag{3.1}$$

with D_μ, the gauge covariant derivative, given by $D_\mu = (\nabla_\mu - ieA_\mu)$. Expanding the Lagrangian, we obtain:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (\nabla^*_\mu\Phi^*)(\nabla^\mu\Phi) + ie(\nabla^*_\mu\Phi^*)A^\mu\Phi - ieA_\mu\Phi^*\nabla^\mu\Phi - e^2A_\mu\Phi^*A^\mu\Phi - V(\Phi^*,\Phi)\,, \tag{3.2}$$

with an effective mass term for the vector field: $-e^2A_\mu\Phi^*A^\mu\Phi$. If Φ passes through 0, then the vector field becomes massless, and we get particle production of the A_β field just like in the scalar case.

Our goal is to calculate the spectrum of tensors sourced by this vector field. First we need the equation of motion of A_β and its associated power spectrum. Solving for the equation of motion, we obtain:

$$-\frac{1}{4}4(\partial_\alpha\partial^\alpha A^\beta - \partial_\alpha\partial^\beta A^\alpha) = ie(\nabla^{*\beta}\Phi^*)\Phi - ie\Phi^*(\nabla^\beta\Phi) - e^2\Phi^*A^\beta\Phi - e^2A^\beta\Phi^*\Phi. \tag{3.3}$$

We define: $j_\mu = ie(\Phi\partial_\mu\Phi^* - \Phi^*\partial_\mu\Phi)$. Note since we assume Φ is homogeneous in space, $j_i = 0$. Our equation of motion simplifies to:

$$\partial_\alpha\partial^\alpha A^\beta - \partial_\alpha\partial^\beta A^\alpha - 2e^2A^\beta\Phi^*\Phi = -j^\beta \tag{3.4}$$

We take $\beta = i$ since, as will be seen, only the spatial component of the A_β field will appear in the stress energy tensor sourcing the tensor spectrum as long as $A_0 = 0$. We work in the Coulomb gauge where $\nabla\cdot A = 0$ and $A_0 = 0$ (allowed as long as there is no net charge), which selects out only the dynamical degrees of freedom, and choose a background value $\bar{A}(\tau) = 0$, which assumes the average current is zero. We redefine the Φ degrees of freedom

to take advantage of the canceling of the imaginary part. $\bar{\Phi} = \frac{e^{i\Theta(\tau)}}{\sqrt{2}}\rho(\tau)$ so $\bar{\Phi}^*\bar{\Phi} = \frac{\rho^2}{2}$. We expand A_β into a homogeneous part and a spatial fluctuation part: $A = \bar{A}(\tau) + \delta A(\tau, x)$.

Our first order equation of motion is:

$$-\frac{1}{a^2}\delta A''^i + \frac{\nabla^2}{a^2}\delta A^i - \partial_0\partial^i\delta A^0 - e^2\delta A^i\rho^2 = 0. \tag{3.5}$$

We can further expand the first order equation by using the fact that we can always decompose a vector into the gradient of a scalar plus a transverse vector. This will be useful because to first order the scalar, vector, and tensor equations all decouple from each other. Let $\delta A_i = \partial_i A_s + a_{ti}$ where A_{ti} is a transverse vector and A_s is a scalar. The vector equation simplifies to

$$A_t''^i - \nabla^2 A_t^i + a^2e^2\rho^2 A_t^i = 0. \tag{3.6}$$

We transform into momentum space and drop the subscript t. The equation of motion then is:

$$\tilde{A}''^i + (k^2 + a^2e^2\rho^2)\tilde{A}^i = 0. \tag{3.7}$$

We proceed in analogy with the scalar case, using the Bogolubov method. We do not calculate production in the non-adiabatic region this time, assuming the result will be comparable to the result from the adiabatic region, as in the scalar case. We promote $\tilde{A}_i(\mathbf{p}, \tau)$ to an operator and define ψ to our mode functions such that, $\tilde{A}_i(\mathbf{p}, \tau) = \epsilon_{\lambda i}(\mathbf{p})\psi_\lambda(p, \tau)\hat{a}_\lambda(\mathbf{p}) + \epsilon^*_{\lambda i}(-\mathbf{p})\psi^*_\lambda(p, \tau)\hat{a}^\dagger_\lambda(-\mathbf{p})$, where $\epsilon_{\lambda i}$ is the polarization vector. We find the new two-point function, this time for the vector field:

$$\begin{aligned}
&\langle : \tilde{A}_i(\mathbf{p}, \tau)\tilde{A}_j(\mathbf{p}', \tau') : \rangle = \\
&\quad = \delta^{(3)}(\mathbf{p} + \mathbf{p}')\sum_{\lambda\pm}\frac{1}{2}\frac{1}{\omega^{\frac{1}{2}}(p, \tau)\omega^{\frac{1}{2}}(p, \tau')}\Big[|\beta(p)|^2\epsilon_{\lambda i}(\mathbf{p})\epsilon^*_{\lambda j}(\mathbf{p})e^{i\int_\tau^{\tau'}\omega(p,\tilde{\tau})d\tilde{\tau}} + \\
&\quad + |\beta(p)|^2\epsilon^*_{\lambda i}(-\mathbf{p})\epsilon_{\lambda j}(-\mathbf{p})e^{i\int_{\tau'}^{\tau}\omega(p,\tilde{\tau})d\tilde{\tau}} + \epsilon_{\lambda i}(\mathbf{p})\epsilon_{\lambda j}(-\mathbf{p})\alpha(p)\beta^*(p)e^{-i\int_{\tau_0}^{\tau}\omega(p,\tilde{\tau})d\tilde{\tau}}\cdot \\
&\quad \cdot e^{-i\int_{\tau_0}^{\tau'}\omega(p,\tilde{\tau})d\tilde{\tau}} + \epsilon^*_{\lambda i}(-\mathbf{p})\epsilon^*_{\lambda j}(\mathbf{p})\beta(p)\alpha^*(p)e^{i\int_{\tau_0}^{\tau}\omega(p,\tilde{\tau})d\tilde{\tau}}e^{i\int_{\tau_0}^{\tau'}\omega(p,\tilde{\tau})d\tilde{\tau}}\Big]
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
&\langle : \tilde{A}'_i(\mathbf{p}, \tau)\tilde{A}'_j(\mathbf{p}', \tau') : \rangle = \\
&\quad = \delta^{(3)}(\mathbf{p} + \mathbf{p}')\sum_{\lambda\pm}\frac{1}{2}\omega^{\frac{1}{2}}(p, \tau)\omega^{\frac{1}{2}}(p, \tau')\Big[|\beta(p)|^2\epsilon_{\lambda i}(\mathbf{p})\epsilon^*_{\lambda j}(\mathbf{p})e^{i\int_\tau^{\tau'}\omega(p,\tilde{\tau})d\tilde{\tau}} + \\
&\quad + |\beta(p)|^2\epsilon^*_{\lambda i}(-\mathbf{p})\epsilon_{\lambda j}(-\mathbf{p})e^{i\int_{\tau'}^{\tau}\omega(p,\tilde{\tau})d\tilde{\tau}} - \epsilon_{\lambda i}(\mathbf{p})\epsilon_{\lambda j}(-\mathbf{p})\alpha(p)\beta^*(p)e^{-i\int_{\tau_0}^{\tau}\omega(p,\tilde{\tau})d\tilde{\tau}}\cdot \\
&\quad \cdot e^{-i\int_{\tau_0}^{\tau'}\omega(p,\tilde{\tau})d\tilde{\tau}} - \epsilon^*_{\lambda i}(-\mathbf{p})\epsilon^*_{\lambda j}(\mathbf{p})\beta(p)\alpha^*(p)e^{i\int_{\tau_0}^{\tau}\omega(p,\tilde{\tau})d\tilde{\tau}}e^{i\int_{\tau_0}^{\tau'}\omega(p,\tilde{\tau})d\tilde{\tau}}\Big]
\end{aligned} \tag{3.9}$$

with $\alpha(k, \tau) = \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(\frac{1}{2} - \frac{ik^2H^2\tau_0^2}{2e\dot{\rho}})}(\frac{4e\dot{\rho}}{k^2H^2\tau_0^2})^{\frac{ik^2H^2\tau_0^2}{2e\dot{\rho}}}e^{-\frac{\pi k^2H^2\tau_0^2}{4e\dot{\rho}}}$ and $\beta(k, \tau) = -ie^{-\frac{\pi k^2H^2\tau_0^2}{2e\dot{\rho}}}$. To calculate the stress energy tensor we use the definition:

$$T_{\mu\nu} = 2\frac{\partial\,\mathcal{L}}{\partial g^{\mu\nu}} - \mathcal{L}\,g_{\mu\nu}\,, \tag{3.10}$$

where we only care about the terms in $\mathcal{L}$ which depend on the vector field.

We can use the above to solve for the stress energy tensor:

$$\begin{aligned} T_{\mu\nu} =& 2[-\frac{1}{4}F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta} - \frac{1}{4}F_{\beta\mu}F_{\alpha\nu}g^{\beta\alpha} + ie(\partial_\mu\Phi^*)A_\nu\Phi - ieA_\mu\Phi^*\partial_\nu\Phi - e^2A_\mu A_\nu\Phi^*\Phi] \\ &- g_{\mu\nu}[-\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + ie(\partial_\alpha\Phi^*)A^\alpha\Phi - ieA_\alpha\Phi^*\partial^\alpha\Phi - e^2A_\alpha A^\alpha\Phi^*\Phi]\,, \end{aligned} \tag{3.11}$$

which we can simplify by dropping the terms multiplied by $g_{\nu\mu}$ because they will drop out when the transverse, traceless projector is applied, as appears in eq. (1.41).

$$T_{\mu\nu} => -F_{\mu\alpha}F^\alpha_\nu + 2ie(\partial_\mu\Phi^*)A_\nu\Phi - 2ieA_\mu\Phi^*\partial_\nu\Phi - 2e^2A_\mu A_\nu\Phi^*\Phi \tag{3.12}$$

Since we want gravitational waves, we only care about T_{ij}. Since we only need T_{ij} to lowest order, the middle terms above drop out since $\partial_i\Phi$ is first order. The stress energy tensor simplifies to:

$$T_{ij} = a^2[-E_iE_j - B_iB_j + \delta_{ij}|\vec{B}|^2] - 2e^2A_iA_j\Phi^*\Phi\,, \tag{3.13}$$

which we plug into eq. (1.41). We use that $\vec{E} = -\frac{1}{a^2}(-\vec{\nabla}A^0 - \frac{\partial\vec{A}}{\partial\tau})$ and $\vec{B} = \frac{1}{a^2}\vec{\nabla}\times\vec{A}$ or $\frac{1}{a^2}\varepsilon_{ijk}\partial^jA^k = B_i$. We also drop the term proportional to δ_{ij} because this will give 0 when the transverse traceless projector acts on it.

$$\begin{aligned} \tilde{h}_{ij}(k,\tau) = -\frac{2}{M_P^2(2\pi)^{3/2}}\int_{\tau_{end}}^0 d\tau' G(\tau,\tau',k)\Pi^{lm}_{ij}(k)\int_{-\infty}^{\infty} d^3p(\frac{1}{a^2(\tau')}(\tilde{A}'_l(p,\tau')\tilde{A}'_m(|\mathbf{k}-\mathbf{p}|,\tau')- \\ \varepsilon_{ljk}p^j\tilde{A}^k(p,\tau')\varepsilon_{mef}(\mathbf{k^e}-\mathbf{p^e})\tilde{A}^f(|\mathbf{k}-\mathbf{p}|,\tau')) - 2e^2\tilde{A}_i(p,\tau')\tilde{A}_j(k-p,\tau')\Phi^*(\tau')\Phi(\tau')) \end{aligned} \tag{3.14}$$

We define $m(\tau) = ea(\tau)\rho(\tau)$. We note $\Phi^*(\tau)\Phi(\tau) = \rho^2(\tau)/2$ where ρ is real. So $e^2a^2(\tau)\Phi^*(\tau)\Phi(\tau) = e^2a^2(\tau)\rho^2/2 = m^2(\tau)/2$ which simplifies our two-point function into:

$$\begin{aligned} \tilde{h}_{ij}(k,\tau) = -\frac{2}{M_P^2(2\pi)^{3/2}}\int_{\tau_{end}}^0 d\tau' G(\tau,\tau',k)\Pi^{lm}_{ij}(k)\int_{-\infty}^{\infty} d^3p\frac{1}{a^2(\tau')}(\tilde{A}'_l(p,\tau')\tilde{A}'_m(|\mathbf{k}-\mathbf{p}|,\tau')- \\ \varepsilon_{ljk}p^j\tilde{A}^k(p,\tau')\varepsilon_{mef}(\mathbf{k^e}-\mathbf{p^e})\tilde{A}^f(|\mathbf{k}-\mathbf{p}|,\tau') - m^2(\tau')\tilde{A}_i(p,\tau')\tilde{A}_j(k-p,\tau')). \end{aligned} \tag{3.15}$$

We drop the term that came from the B field, the middle term above, because in the non-relativistic regime, the E field term contributes much more. The A' is proportional to ωA and the B field term to kA. We use eq. (3.15) to solve for the two point function, just as in the scalar case, taking the same approximations. In particular, we assume $\frac{e\dot{\rho}}{H^2} \gg 1$ in analogy with $\frac{g\dot{\phi}}{H^2} \gg 1$ in the scalar case. Also we can take the limit $k \ll p$ noting that the two-point function goes to zero for $k\tau_0 \gg 1$. There is also suppression for large p from α and β, but this suppression only becomes significant for $p^2\tau_0^2 < \frac{e\dot{\rho}}{H^2}$ and since we know $\frac{e\dot{\rho}}{H^2} \gg 1$, this suppression becomes significant for $p\tau_0$ already $\gg 1$, and therefore much greater then k. It can be seen the integrand of the two-point function is maximized for these intermediate values of p: $1 \ll p\tau_0 \ll \frac{e\dot{\rho}}{H^2}$ and therefore it is safe to assume $p \gg k$ because only for these values is there significant contribution to the two-point function. After making these

approximations, we solve for the two-point function of gravitational waves and obtain twice the result for the scalar case, the extra factor of 2 coming from the fact that in the vector case their are 2 helicity modes, and each one is equally likely to be produced, since this mechanism is parity conserving, and each as efficient as in the scalar case. So as in the scalar case, the signal is dwarfed by the background signal of gravitational wave production in a de Sitter background, and we conclude that this mechanism will not produce an observable signal.

We hypothesized that if we were to carry out a similar calculation for fermion production during slow roll, we would get the same result, an unobservable tensor spectrum. Since we published our results, another group has worked through a very similar calculation for fermions, the same type coupling but with a different field other than the inflaton sourcing the fermions, and found an order one correction to the power spectrum of tensors produced by the same calculation with vector production [21]. The order 1 correction includes a factor of 2 for fermions relative to vectors coming from the fact that Dirac fermions have twice as many degrees of freedom as 2 helicity state vectors, and there is further a small correction due to spin statistics, arising from it being harder for 4 spin particles to produce two spin 2 gravitons than it is for 4 spin 1 particles to do the same. But the end result is the same; fermions generated from a similar coupling during slow roll inflation, just like spin 0 and spin 1 particles, will not be capable of producing an observable tensor spectrum.

We speculate part of the reason these mechanisms give such a small signal is that the fields generating the gravitational waves very quickly pass through 0 mass, in less than a Hubble time, so the vast majority of the gravitational waves produced from this mechanism occur when the vector field is very massive and non-relativistic. Non-relativistic particles produce small quadrupole moments, and theretofore are naturally ill-suited to producing gravitational waves.

CHAPTER 4

PROLONGED DECAY OF THE INFLATON INTO VECTORS

We calculate production of a vector field interacting with the inflaton field though the Lagrangian:

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\phi}{4f}F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{4.1}$$

where $\tilde{F}^{\mu\nu}$ is the dual tensor given by $= \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$. We assume ϕ above is the inflaton field and a pseudoscalar. The vector field has a U(1) gauge symmetry. We simplify by choosing the Coulomb gauge in which case our vector field satisfies: $A_0 = 0$ and $\partial_i A^i = 0$. $A_0 = 0$ is valid as long as there are no charges, and since we are only considering photons and the inflaton field, we have $A_0 = 0$. Note $F^{\mu\nu}$ is parity even and $\tilde{F}^{\mu\nu}$ is parity odd, creating parity violation in vectors produced through the $\frac{\phi}{4f}F_{\mu\nu}\tilde{F}^{\mu\nu}$ interaction. We expand out the Lagrangian,

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}[\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu \partial^\mu - \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu] - \frac{\phi}{8f}[\partial_\mu A_\nu \varepsilon^{\mu\nu ab}\partial_a A_b \\ &- \partial_\mu A_\nu \varepsilon^{\mu\nu ab}\partial_b A_a - \partial_\nu A_\mu \varepsilon^{\mu\nu ab}\partial_a A_b + \partial_\nu A_\mu \varepsilon^{\mu\nu ab}\partial_b A_a]\end{aligned} \tag{4.2}$$

and solve for the equation of motion of the vector field:

$$0 = -\partial_\alpha \partial^\beta A^\alpha + \partial_\alpha \partial^\alpha A^\beta - \partial_\alpha[\frac{\phi}{f}\varepsilon^{\mu\nu\beta\alpha}\partial_\mu A_\nu]\,. \tag{4.3}$$

We have essentially four equations of motion. On one side we have a zero 4 vector, and on the other we have other 4 vectors. We can separate the equations of motion into the spatial and time parts of our zero 4 vector. For $\beta = k$:

$$0^k = -\partial_\alpha \partial^k A^\alpha + \partial_\alpha \partial^\alpha A^k - \frac{1}{f}[\phi'\varepsilon^{ijk0}\partial_i A_j + \phi\varepsilon^{ijk0}\partial_i A_j^{'} + \phi\varepsilon^{0jki}\partial_i A_j^{'} + \phi\varepsilon^{j0ki}\partial_j\partial_i A_0] \tag{4.4}$$

Note the last term $= 0$ because $A_0 = 0$. The second and third to last terms cancel because they are the same except they contain ε's which are one permutation a part. We further simplify our equation of motion using the identity: $\varepsilon^{0ijk} = \varepsilon^{ijk}$ to obtain:

$$\mathbf{0} = (\frac{\partial^2}{\partial\tau^2} - \nabla^{\mathbf{2}} - \frac{1}{f}\phi'\nabla\times)\mathbf{A}\,. \tag{4.5}$$

Solving for the $\beta = 0$ equation of motion just gives $0 = \nabla \cdot \mathbf{A}'$ which had to be true anyway in the Coulomb gauge. Next we define $\tilde{A}(\tau, k)$ as the momentum space Fourier transform of A given by:

$$A(\tau, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} e^{i\mathbf{k}\cdot\mathbf{x}} \tilde{A}(\tau, \mathbf{k}) \tag{4.6}$$

and plug this into the equation of motion of our vector field:

$$0 = \tilde{A}''^i + k^2 \tilde{A}^i - \frac{i\phi'}{f} \varepsilon^{ijk} k_j \tilde{A}_k \tag{4.7}$$

Next we promote $\tilde{A}$ to an operator and decompose:

$$\hat{\tilde{A}}^j(\mathbf{k}, \tau) = \sum_{\lambda=\pm} [\epsilon^j_\lambda(\mathbf{k}) u_\lambda(\tau, \mathbf{k}) \hat{a}_\lambda(\mathbf{k}) + \epsilon^{*j}_\lambda(-\mathbf{k}) u^*_\lambda(\tau, -\mathbf{k}) \hat{a}^\dagger_\lambda(-\mathbf{k})] \,, \tag{4.8}$$

where $u_\pm$ are the mode functions. Plugging this decomposition into the equation of motion, we obtain:

$$0 = \sum_{\lambda=\pm} \epsilon^i_\lambda u''_\lambda \hat{a}_\lambda + \epsilon^{*i}_\lambda u^{*''}_\lambda \hat{a}^\dagger_\lambda + k^2(\epsilon^i_\lambda u_\lambda \hat{a}_\lambda + \epsilon^{*i}_\lambda u^*_\lambda \hat{a}^\dagger_\lambda) - \frac{i\phi'}{f} \varepsilon^{ijk} k_j (\epsilon_{k\lambda} u_\lambda \hat{a}_\lambda + \epsilon^*_{k\lambda} u^*_\lambda \hat{a}^\dagger_\lambda) \,. \tag{4.9}$$

I am using a basis for the polarization vectors defined so for $\mathbf{k}$ along $\hat{\mathbf{z}}$, the vectors are given by:

$$\begin{aligned} \epsilon_+(\hat{\mathbf{z}}) &= -\frac{1}{\sqrt{2}}(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) = Y^1_1 \\ \epsilon_-(\hat{\mathbf{z}}) &= \frac{1}{\sqrt{2}}(\hat{\mathbf{x}} - i\hat{\mathbf{y}}) = Y^{-1}_1 \\ \epsilon_0(\hat{\mathbf{z}}) &= \hat{\mathbf{z}} = Y^0_1 \,, \end{aligned} \tag{4.10}$$

which can be written as:

$$\epsilon_+(\hat{\mathbf{z}}) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ -i \\ 0 \end{pmatrix} \quad \epsilon_0(\hat{\mathbf{z}}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \epsilon_-(\hat{\mathbf{z}}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \,.$$

Our massless vectors have 2 degrees of freedom, 2 helicity states, and so the polarization of these vectors can only be written in terms of ϵ_+ and ϵ_-, where $\epsilon_0(\mathbf{k}) = \mathbf{k}/k$, the direction of propagation.

The polarization vectors have two types of indices:

1. Sums over the Cartesian components of a vector, labeled by a,b,c,...
2. Specifies +,-,0 state of polarization vector, labeled by λ and σ.

The polarization vectors will satisfy an orthogonality condition given by:

$$\epsilon^*_{\lambda a} \epsilon^a_\sigma = \delta_{\lambda\sigma} \,. \tag{4.11}$$

We simplify the equation of motion by using that $\vec{k} \times \epsilon_\pm = \mp ik\epsilon_\pm$, which we prove below.

$$\vec{k} \times \epsilon_\pm = \varepsilon_{abc} k^b \epsilon^c_\pm = \varepsilon_{abc}(k\epsilon^b_0)\epsilon^c_\pm \tag{4.12}$$

For $a = 1$, this simplifies to:

$$\varepsilon_{1bc}(k\epsilon^b_0)\epsilon^c_\pm = k(\varepsilon_{123}\epsilon^2_0\epsilon^3_\pm + \varepsilon_{132}\epsilon^3_0\epsilon^2_\pm) = k(0 - 1 \cdot -\frac{i}{\sqrt{2}}) \tag{4.13}$$

For $a = 2$, this simplifies to:

$$\varepsilon_{2bc}(k\epsilon^b_0)\epsilon^c_\pm = k(\varepsilon_{213}\epsilon^1_0\epsilon^3_\pm + \varepsilon_{231}\epsilon^3_0\epsilon^1_\pm) \tag{4.14}$$

For $a = 3$:, the above simplifies to:

$$\varepsilon_{3bc}(k\epsilon^b_0)\epsilon^c_\pm = k(\varepsilon_{312}\epsilon^1_0\epsilon^2_\pm + \varepsilon_{321}\epsilon^2_0\epsilon^1_\pm) = 0\,. \tag{4.15}$$

Putting these expressions together, we obtain:

$$\varepsilon_{abc}(k\epsilon^b_0)\epsilon^c_+ = \frac{k}{\sqrt{2}}\begin{pmatrix} i \\ -1 \\ 0 \end{pmatrix} = \frac{ki}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} = -ik\epsilon_+$$

$$\varepsilon_{abc}(k\epsilon^b_0)\epsilon^c_- = \frac{k}{\sqrt{2}}\begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} = \frac{ki}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} = ik\epsilon_-$$

$$\vec{k} \times \epsilon_\pm = \varepsilon_{abc} k^b \epsilon^c_\pm = \mp ik\epsilon_\pm \tag{4.16}$$

Since $\vec{k}$ and ε_{abc} are real, we also get:

$$\vec{k} \times \epsilon^*_\pm = \varepsilon_{abc} k^b \epsilon^{c*}_\pm = \pm ik\epsilon^*_\pm\,. \tag{4.17}$$

Using this result in the equation of motion, we obtain:

$$0 = \sum_{\lambda=\pm} \epsilon^i_\lambda u''_\lambda \hat{a}_\lambda + \epsilon^{*i}_\lambda u^{*''}_\lambda \hat{a}^\dagger_\lambda + k^2(\epsilon^i_\lambda u_\lambda \hat{a}_\lambda + \epsilon^{*i}_\lambda u^*_\lambda \hat{a}^\dagger_\lambda) + \frac{\phi'}{f}(-k\epsilon^i_+ u_+ \hat{a}_+ + k\epsilon^i_- u_- \hat{a}_- + k\epsilon^{*i}_+ u^*_+ \hat{a}^\dagger_+ - k\epsilon^{*i}_- u^*_- a^\dagger_-)\,. \tag{4.18}$$

Next we use that u_+ and u_- are linearly independent, so the equation above must be satisfied by each separately, giving us:

$$0 = \epsilon^i_+ u''_+ \hat{a}_+ + \epsilon^{*i}_+ u^{*''}_+ \hat{a}^\dagger_+ + k^2(\epsilon^i_+ u_+ \hat{a}_+ + \epsilon^{*i}_+ u^*_+ \hat{a}^\dagger_+) + \frac{k\phi'}{f}(-\epsilon^i_+ u_+ \hat{a}_+ + \epsilon^{*i}_+ u^*_+ \hat{a}^\dagger_+) \tag{4.19}$$

and

$$0 = \epsilon^i_- u''_- \hat{a}_- + \epsilon^{*i}_- u^{*''}_- \hat{a}^\dagger_- + k^2(\epsilon^i_- u_- \hat{a}_- + \epsilon^{*i}_- u^*_- \hat{a}^\dagger_-) + \frac{k\phi'}{f}(\epsilon^i_- u_- \hat{a}_- - \epsilon^{*i}_- u^*_- a^\dagger_-)\,. \tag{4.20}$$

Then we use that $\hat{a}_\lambda$ commutes with $\hat{a}^\dagger_\lambda$ to simplify further, obtaining:

$$0 = u_+^{''} + k^2 u_+ - \frac{k\phi'}{f} u_+ \tag{4.21}$$

and

$$0 = u_-^{''} + k^2 u_- + \frac{k\phi'}{f} u_- \,. \tag{4.22}$$

We define

$$\xi = \frac{\dot{\phi}}{2fH} \tag{4.23}$$

and use that $\phi' = \dot{\phi}\frac{dt}{d\tau} = \dot{\phi}a = -\frac{\dot{\phi}}{H\tau}$ to rewrite our equation of motion:

$$0 = u_\pm^{''} + (k^2 \pm \frac{2k\xi}{\tau}) u_\pm \,. \tag{4.24}$$

Solving this differential equation gives a solution in terms of Coulomb wave functions

$$u_+ = c_1 F_0(\xi, -k\tau) + c_2 G_0(\xi, -k\tau) \tag{4.25}$$

and

$$u_- = c_3 F_0(-\xi, -k\tau) + c_4 G_0(-\xi, -k\tau) \,. \tag{4.26}$$

We next wish to solve for the integration constants by applying the initial condition that for $k \to \infty$, we want an adiabatic solution. When the system is adiabatic, the solution should take the form:

$$\frac{1}{(2\omega)^{1/2}} (\alpha e^{i\int \omega d\tau} + \beta e^{-i\int \omega dt}) \,, \tag{4.27}$$

and we will assume without loss of generality a positive frequency solution such that $\lim_{k\to\infty} u_\pm = \frac{1}{(2\omega)^{1/2}} e^{i\int \omega d\tau}$ where $\omega_\pm = (k^2 \pm \frac{2k\xi}{\tau})^{\frac{1}{2}}$ gives ω for u_+ and u_- respectively. To match boundary conditions, we then take the asymptotic expansion of the Colomb wave functions for large k, given in general by [24]:

$$\begin{aligned} F_L(\eta, \rho) &= g\cos(\theta_L) + f\sin(\theta_L) \\ G_L(\eta, \rho) &= f\cos(\theta_L) - g\sin(\theta_L) \,, \end{aligned} \tag{4.28}$$

where

$$\theta_L = \rho - \eta\ln(2\rho) - L\frac{\pi}{2} + \arg[\Gamma(L + 1 + i\eta)] \tag{4.29}$$

and

$$g \approx \sum_{\kappa=0}^{\infty} g_k \qquad f \approx \sum_{\kappa=0}^{\infty} f_k \,. \tag{4.30}$$

f_κ and g_κ are given by:

$$\begin{aligned} f_0 &= 1, \;\; g_0 = 0 \\ f_{k+1} &= a_k f_k - b_k g_k \\ g_{k+1} &= a_k g_k + b_k f_k \end{aligned} \tag{4.31}$$

where

$$a_k = \frac{(2k+1)\eta}{(2k+2)\rho} \qquad b_k = \frac{L(L+1) - k(k+1) + \eta^2}{(2k+2)\rho}\,. \tag{4.32}$$

Notice for $\rho \to \infty$, each term f_κ and g_κ becomes smaller for increasing κ. We therefore approximate:

$$f \approx 1 + \frac{\eta}{2\rho} \qquad g \approx \frac{L(L+1) + \eta^2}{2\rho}\,. \tag{4.33}$$

Plugging this into our equation of motion, we find to lowest order:

$$\lim_{k\to\infty} u_+ \approx -c_1 \sin[k\tau] + c_2 \cos[k\tau]\,. \tag{4.34}$$

We want to match onto $\frac{1}{(2\omega)^{1/2}} e^{i\int_\tau^0 \omega(p,\tilde{\tau})d\tilde{\tau}}$. We use that

$$\int \omega d\tau = \tau\sqrt{k^2 + \frac{2k\xi}{\tau}} + \xi \ln\left|\left(\tau\sqrt{k^2 + \frac{2k\xi}{\tau}} + k\tau + \xi\right)\right| \tag{4.35}$$

to find

$$\begin{aligned} &\frac{1}{(2\omega)^{1/2}} e^{i\int_\tau^0 \omega(p,\tilde{\tau})d\tilde{\tau}} = \\ &= \frac{1}{2^{1/2}(k^2 + \frac{2k\xi}{\tau})^{1/4}} e^{-i\tau(k^2+\frac{2\xi k}{\tau})^{1/2} + i\xi \ln\left|\tau\sqrt{k^2+\frac{2\xi k}{\tau}} + k\tau + \xi\right| + i\xi\ln(\xi)} \\ &= \frac{1}{2^{1/2}(k^2 + \frac{2k\xi}{\tau})^{1/4}} [\cos(-\tau(k^2 + \frac{2\xi k}{\tau})^{1/2} + \xi \ln\left|\tau\sqrt{k^2 + \frac{2\xi k}{\tau}} + k\tau + \xi\right| + \xi\ln(\xi)) \\ &\quad + i\sin(-\tau(k^2 + \frac{2\xi k}{\tau})^{1/2} + \xi \ln\left|\tau\sqrt{k^2 + \frac{2\xi k}{\tau}} + k\tau + \xi\right| + \xi\ln(\xi))]\,. \end{aligned} \tag{4.36}$$

We take the limit of the above for $k \to \infty$ to find:

$$\lim_{k\to\infty} \frac{1}{(2\omega)^{1/2}} e^{i\int_\tau^0 \omega(p,\tilde{\tau})d\tilde{\tau}} \approx \frac{1}{(2k)^{1/2}}[\cos(\tau k) - i\sin(\tau k)] \tag{4.37}$$

and match boundary conditions with the asymptotic expansion of the Coulomb wave function expression to find the integration constants: $c_1 = \frac{i}{(2k)^{1/2}}$ and $c_2 = \frac{1}{(2k)^{1/2}}$. This gives for the full solution for the + mode function:

$$u_+(\tau, k) = \frac{1}{(2k)^{1/2}}[iF_0(\xi, -k\tau) + G_0(\xi, -k\tau)]\,. \tag{4.38}$$

u_- is solved for the same way with the only difference $\xi \to -\xi$, which gives:

$$u_-(\tau, k) = \frac{1}{(2k)^{1/2}}[iF_0(-\xi, -k\tau) + G_0(-\xi, -k\tau)]\,. \tag{4.39}$$

Note ξ is unitless and the value of ξ will determine how many quanta of the vectors are excited and of what helicity. If ξ is positive, the positive helicity mode is excited while the negative helicity mode stays essentially in vacuum and vice versa if ξ is negative, but the number of quanta produced and all other such behavior is the same. We assume without loss of generality that ξ is positive. We can then approximate the mode function u_+ for $\frac{1}{8\xi} < |k\tau| < 2\xi$ where we get the largest signal:

$$u_+(\tau, k) \approx \frac{1}{\sqrt{2k}}\left(\frac{k}{2\xi a(\tau)H}\right)^{1/4} e^{\pi\xi - 2\sqrt{\frac{2\xi k}{a(\tau)H}}} \tag{4.40}$$

while u_- is approximately zero, and we ignore u_- in future.

Unlike in the previous calculations, the Bogolubov method can not be applied here. The Bogolubov method acts as a bridge between an initial adiabatic region and a final adiabatic region, covering a complicated intermediate region in which there is particle production. In this production mechanism, we assume for $\tau \to -\infty$, there is an initial adiabatic region, but for $\tau \to 0$, particle production continues. Note the frequency of our mode function becomes imaginary as $\tau \to 0$, so clearly the process is nonadiabatic in this limit. In practice the production rate will eventually fall off either due to the end of inflation, or due to so many particles being produced that backreaction effects on the inflaton will become important. We will only consider results before either scenario. Either way, the Bogolubov method is inappropriate here, and we instead directly plug our results for the mode functions into our expression for our vector field eq. (4.8), and this we plug into our expression for the metric perturbations sourced by this field eq. (1.41).

Note the mode function above is only part of the full mode function. The full mode function would have a UV divergent component that would lead to the infinite energy solution one finds for quantum fields and which one typically eliminates through normal ordering, etc. We have already subtracted out this term, so the mode function above is the only physical term that actually produces an observable signal. Therefore, when solving for the two-point function of gravitational waves, we will not normal order this time, since the role served by normal ordering has already been accounted for.

4.1 Generation of Tensor Modes by the Gauge Field

We want to eventually solve eq. (1.41), and so the next step is to plug the vector field expression we found above into the stress energy tensor for the vector field. This is very similar to the stress energy tensor calculated earlier in Chapter 3 in eq. (3.13), but now we have a different Lagrangian. We use:

$$T_{\mu\nu} = 2\frac{\partial\,\mathcal{L}}{\partial g^{\mu\nu}} - \mathcal{L}\,g_{\mu\nu}\,, \tag{4.41}$$

and we use eq. (4.1) for $\mathcal{L}$, but only use terms dependent on the vector field. Rewriting $\mathcal{L}$ to make it obvious where we need to take derivatives of the metric:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta} - \frac{\phi}{4f}F_{\mu\nu}\tilde{F}_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta} \tag{4.42}$$

This gives us:

$$T_{\mu\nu} = 2[-\frac{1}{2}F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta} - \frac{\phi}{2f}F_{\mu\alpha}\tilde{F}_{\nu\beta}g^{\alpha\beta}] - [-\frac{1}{4}F_{\mu\nu}F_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta} - \frac{\phi}{4f}F_{\mu\nu}\tilde{F}_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta}]g_{\mu\nu}\,, \quad (4.43)$$

where each $g_{\mu\nu}$ above can be expanded as: $g_{\mu\nu} = g_{\mu\nu\,\mathrm{FRW}} + h_{\mu\nu H} + h_{\mu\nu P}$. Remember our metric is written as:

$$g_{\mu\nu} = a^2(\tau)(-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j). \quad (4.44)$$

$g_{\mu\nu\,\mathrm{FRW}}$ is the unperturbed FRW metric. $h_{\mu\nu H}$ is the perturbation to the metric which solves the homogeneous Einstein equation (would be there even if no vector production - this is the more 'standard' result). $h_{\mu\nu P}$ is the perturbation to the metric which solves the particular Einstein equation with vector fields sourcing $T_{\mu\nu}$.

We will only need T_{ij} since the tensor equations will be independent of the scalar and vector equations to first order. We know any terms in T_{ij} proportional to δ_{ij} will drop out when we apply the transverse, traceless projector, so we will drop those terms now.

$$T_{ij\,\mathrm{effective}} = -F_{i\alpha}F_j^{\alpha} - \frac{\phi}{2f}F_i^{\alpha}\epsilon_{j\alpha}^{\mu\nu}F_{\mu\nu} - a^2(h_{ij\,H} + h_{ij\,P})(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\phi}{8f}F_{\mu\nu}\epsilon_{\delta\rho}^{\mu\nu}F^{\delta\rho}) \quad (4.45)$$

The expression eventually simplifies to:

$$T_{ij\,\mathrm{effective}}(\mathbf{k},\tau) = -\frac{1}{a^2(2\pi)^{\frac{3}{2}}}\int d^3\mathbf{p}[A_i^{'}(\mathbf{p},\tau)A_j^{'}(\mathbf{k}-\mathbf{p},\tau) + \frac{1}{2(2\pi)^{\frac{3}{2}}}\int d^3\mathbf{p}^{'}\cdot$$
$$\cdot\,(h_{ij\,H}(\mathbf{p}^{'},\tau) + h_{ij\,P}(\mathbf{p}^{'},\tau))(\mathbf{A}^{'}(\mathbf{p},\tau)\cdot\mathbf{A}^{'}(\mathbf{k}-\mathbf{p}-\mathbf{p}^{'},\tau))] \quad (4.46)$$

There were also other terms proportional to the B field instead of the E field, (aka proportional to k^2AA instead of $A'A'$) but these were dropped. A' is proportional to frequency times A, where the frequency will have contributions from both the momentum and the potential energies. We assume that our vector field is non-relativistic, that the total energy of the vectors is much larger than the kinetic energy. In this regime the $A'A'$ pieces will dominate over the k^2AA pieces. To see why, we use that the largest production of tensor modes happens when $|k\tau| \ll 2\xi$. Then using that $u_+ \propto k^{-\frac{1}{4}}\xi^{-\frac{1}{4}}a^{-\frac{1}{4}}H^{-\frac{1}{4}}$, we wish to compare:

$$u^{'} \propto k^{\frac{1}{4}}\xi^{\frac{1}{2}}a^{-\frac{7}{4}}H^{-\frac{3}{4}}a^{'} \quad \text{then using } a = -\tfrac{1}{H\tau}\ \ldots$$
$$u^{'} \propto k^{\frac{1}{4}}\xi^{\frac{1}{2}}(H\tau)^{\frac{7}{4}}H^{-\frac{3}{4}}\left(\frac{1}{H\tau^2}\right) = u^{'} \propto k^{\frac{1}{4}}\xi^{\frac{1}{2}}\tau^{-\frac{1}{4}}\,, \quad (4.47)$$

and we compare this to:

$$ku \propto k^{\frac{3}{4}}\xi^{-\frac{1}{4}}a^{-\frac{1}{4}}H^{-\frac{1}{4}} = ku \propto k^{\frac{3}{4}}\xi^{-\frac{1}{4}}(H\tau)^{\frac{1}{4}}H^{-\frac{1}{4}} \quad (4.48)$$

Simplifying, we compare: $\xi^{\frac{3}{4}}$ to $|k\tau|^{\frac{1}{2}}$ and so using $|k\tau| \ll 2\xi$, we conclude $A^{'} \gg kA$.

We plug the above into eq. (1.41):

$$\tilde{h}_{ij\,P}(\mathbf{k},\tau) = \frac{2}{M_P^2}\int_{\tau_{end}}^{0} d\tau' G(\tau,\tau',k)\Pi_{ij}^{lm}(\mathbf{k})\tilde{T}_{lm}(\mathbf{k},\tau') \quad (4.49)$$

$$h_{lm\,P}(\mathbf{k},\tau) = -\frac{2H^2}{M_P^2(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\tau} d\tau^{'}\tau^{'2}G(\tau,\tau^{'},k)\Pi^{ij}_{lm}(\mathbf{k})\int d^3\mathbf{p}\,[A_i^{'}(\mathbf{p},\tau^{'})A_j^{'}(\mathbf{k}-\mathbf{p},\tau)+$$
$$+\frac{1}{2(2\pi)^{\frac{3}{2}}}\int d^3\mathbf{p}^{'}(h_{ij\,H}(\mathbf{p}^{'},\tau)+h_{ij\,P}(\mathbf{p}^{'},\tau))(\mathbf{A}^{'}(\mathbf{p},\tau)\cdot\mathbf{A}^{'}(\mathbf{k}-\mathbf{p}-\mathbf{p}^{'},\tau))] \tag{4.50}$$

The 'standard' perturbation is given by:

$$h_{ij\,H}(\mathbf{k},\tau) = \frac{2iH}{M_P k^{\frac{3}{2}}}[-e^{-ik\tau}\epsilon_i(\mathbf{k})\epsilon_j(\mathbf{k})\hat{a}_H(\mathbf{k}) + e^{ik\tau}\epsilon_i^*(-\mathbf{k})\epsilon_j^*(-\mathbf{k})\hat{a}_H^{\dagger}(-\mathbf{k})] \tag{4.51}$$

I put a subscript H under these operators as a reminder that they are uncorrelated with the operators appearing in the vector field equation. Now we can calculate the two-point function of the gravitational waves. This time though, instead of calculating $\langle\tilde{h}_{ij}(\mathbf{k},\tau)\tilde{h}_{ij}(\mathbf{k}',\tau)\rangle$, we will calculate $\langle\tilde{h}_{\pm}(\mathbf{k},\tau)\tilde{h}_{\pm}(\mathbf{k}',\tau)\rangle$ separately, where $\langle\tilde{h}_{ij}(\mathbf{k},\tau)\tilde{h}_{ij}(\mathbf{k}',\tau)\rangle = \langle\tilde{h}_{+}(\mathbf{k},\tau)\tilde{h}_{+}(\mathbf{k}',\tau)\rangle + \langle\tilde{h}_{-}(\mathbf{k},\tau)\tilde{h}_{-}(\mathbf{k}',\tau)\rangle$ since the cross terms will give zero. We will find a parity violating signal where $\langle\tilde{h}_{+}(\mathbf{k},\tau)\tilde{h}_{+}(\mathbf{k}',\tau)\rangle \neq \langle\tilde{h}_{-}(\mathbf{k},\tau)\tilde{h}_{-}(\mathbf{k}',\tau)\rangle$. We define $\tilde{h}_{\pm}(\mathbf{k},\tau)$ by:

$$\tilde{h}_{\pm}(\mathbf{k},\tau) = \Pi^{ij}_{\pm}(\mathbf{k})\tilde{h}_{ij}(\mathbf{k},\tau) \tag{4.52}$$

and

$$\Pi^{ij}_{\pm}(\mathbf{k}) = \frac{1}{\sqrt{2}}\epsilon^i_{\mp}(\mathbf{k})\epsilon^j_{\mp}(\mathbf{k})\,. \tag{4.53}$$

$$h_{\pm\,P}(\mathbf{k},\tau) = -\frac{2H^2}{M_P^2(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\tau} d\tau^{'}\tau^{'2}G(\tau,\tau^{'},k)\Pi^{ij}_{\pm}(\mathbf{k})\int d^3\mathbf{p}\,[A_i^{'}(\mathbf{p},\tau^{'})A_j^{'}(\mathbf{k}-\mathbf{p},\tau)+$$
$$+\frac{1}{2(2\pi)^{\frac{3}{2}}}\int d^3\mathbf{p}^{'}(h_{ij\,H}(\mathbf{p}^{'},\tau)+h_{ij\,P}(\mathbf{p}^{'},\tau))(\mathbf{A}^{'}(\mathbf{p},\tau)\cdot\mathbf{A}^{'}(\mathbf{k}-\mathbf{p}-\mathbf{p}^{'},\tau))] \tag{4.54}$$

We also simplify making use of the identity $\Pi^{ij}_{\pm}\Pi^{lm}_{ij} = \Pi^{lm}_{\pm}$ Then to calculate the two-point function we expand:

$$\begin{aligned}\langle h_{\pm}h_{\pm}\rangle &= \\ &= \langle(h_{\pm H}+h_{\pm P})(h_{\pm H}+h_{\pm P})\rangle \\ &= \langle h_{\pm H}h_{\pm H}\rangle + \langle h_{\pm H}h_{\pm P}\rangle + \langle h_{\pm P}h_{\pm H}\rangle + \langle h_{\pm P}h_{\pm P}\rangle\end{aligned} \tag{4.55}$$

Now we can expand these terms and only keep terms of order $\frac{H^4}{M_P^4}$. Note $\frac{H^4}{M_P^4} \leq (10^{-4})^4$. We get:

$$\begin{aligned}&\langle h_{\pm}(\mathbf{k},\tau)h_{\pm}(\mathbf{k}',\tau)\rangle = \\ &=\langle h_{\pm\,H}(\mathbf{k},\tau)h_{\pm\,H}(\mathbf{k}',\tau)\rangle - \frac{H^2}{M_P^2(2\pi)^3}\int_{-\infty}^{\tau} d\tau'\tau^{'2}G(\tau,\tau',k')\Pi^{ij}_{\pm}(\mathbf{k}')\int d^3\mathbf{p}\int d^3\mathbf{p}'\cdot \\ &\cdot\big[\langle h_{\pm H}(\mathbf{k},\tau)h_{ij\,H}(\mathbf{p}',\tau')(\mathbf{A}'(\mathbf{p},\tau')\cdot\mathbf{A}'(\mathbf{k}'-\mathbf{p}-\mathbf{p}',\tau'))\rangle + \\ &+\langle h_{ijH}(\mathbf{p}',\tau')(\mathbf{A}^{'}(\mathbf{p},\tau')\cdot\mathbf{A}'(\mathbf{k}-\mathbf{p}-\mathbf{p}',\tau'))h(\mathbf{k}',\tau)\rangle\big] + \\ &+\frac{4H^4}{M_P^4(2\pi)^3}\int_{-\infty}^{\tau} d\tau'\int_{-\infty}^{\tau} d\tau^{''}\,\tau^{'2}\tau^{''2}G(\tau,\tau',k)G(\tau,\tau^{''},k')\Pi^{ij}_{\pm}(\mathbf{k})\Pi^{lm}_{\pm}(\mathbf{k}')\int d^3\mathbf{p}\cdot \\ &\cdot\int d^3\mathbf{p}'\langle A_i^{'}(\mathbf{p},\tau')A_j^{'}(\mathbf{k}-\mathbf{p},\tau')A_l^{'}(\mathbf{p}',\tau^{''})A_m^{'}(\mathbf{k}'-\mathbf{p}',\tau^{''})\rangle\,,\end{aligned} \tag{4.56}$$

where it should be understood that each $h_H \propto \frac{H}{M_P}$. As we will see below, $A' \propto e^{\pi\xi}$ where $\xi \simeq 2.6$. And $e^{2\pi\xi} \approx 10^7$. So the two middle terms which have the same scaling with $\frac{H}{M_P}$ as the last term, but are proportional to $\frac{1}{A'^2}$ relative to the last term, we will drop. Then since the first term is the 'standard' well known result, independent of the vector production method, we will ignore it for now and add it in at the end. Then the only new term we need compute is the last one.

Note the limits on the τ' integral should be over the entire region over which we want to measure the gravitational waves produced. In this case, this production mechanism does not have a start and end time. We integrate from $-\infty < \tau < 0$, recognizing that at some point the signal might become so strong that backreaction will need to be taken into account. We will estimate backreaction later. We use eq. (4.40) to obtain:

$$u'_+(\tau,k) \approx \frac{1}{\sqrt{2k}}(\frac{k}{2\xi aH})^{1/4} e^{\pi\xi - 2\sqrt{\frac{2\xi k}{aH}}} a' \cdot [-\frac{1}{4a} + (\frac{2\xi k}{H})^{\frac{1}{2}} \frac{1}{a^{\frac{3}{2}}}] \,. \tag{4.57}$$

Then we use eq. (4.8):

$$\tilde{A}'_j = \sum_{\lambda=\pm} [\epsilon_{j\lambda}(\mathbf{k}) u'_\lambda(\tau,k)\hat{a}_\lambda(\mathbf{k}) + \epsilon^*_{j\lambda}(-\mathbf{k}) u'^*_\lambda(\tau,k)\hat{a}^\dagger_\lambda(-\mathbf{k})] \,, \tag{4.58}$$

and also note that $u_- = 0$, u_+ only depends on $|\mathbf{k}|$, not $\mathbf{k}$, and u_+ is real. We then obtain:

$$\begin{aligned}\tilde{A}'_j(\mathbf{k},\tau) =& \frac{1}{\sqrt{2k}}(\frac{k}{2a(\tau)\xi H})^{\frac{1}{4}} e^{\pi\xi - 2\sqrt{\frac{2\xi k}{a(\tau)H}}} a'(\tau) \cdot [-\frac{1}{4a(\tau)} + (\frac{2\xi k}{H})^{\frac{1}{2}} \frac{1}{a^{\frac{3}{2}}(\tau)}] \cdot \\ & \cdot \left[\epsilon_{j+}(\mathbf{k})\hat{a}_+(\mathbf{k}) + \epsilon^*_{j+}(-\mathbf{k})\hat{a}^\dagger_+(-\mathbf{k})\right] .\end{aligned} \tag{4.59}$$

Now we are assuming $\xi \gtrsim 1$, and so we approximate $\tilde{A}'_j(\mathbf{k},\tau)$:

$$\begin{aligned}\tilde{A}'_j(\mathbf{k},\tau) \approx& \frac{1}{\sqrt{2k}}(\frac{k}{2a(\tau)\xi H})^{\frac{1}{4}} e^{\pi\xi - 2\sqrt{\frac{2\xi k}{a(\tau)H}}} a'(\tau)(\frac{2\xi k}{H})^{\frac{1}{2}} \frac{1}{a^{\frac{3}{2}}(\tau)} \\ & \cdot \left[\epsilon_{j+}(\mathbf{k})\hat{a}_+(\mathbf{k}) + \epsilon^*_{j+}(-\mathbf{k})\hat{a}^\dagger_+(-\mathbf{k})\right] .\end{aligned} \tag{4.60}$$

Applying this and expanding using Wick's theorem, we obtain:

$$\begin{aligned}\langle \tilde{h}_\pm(\mathbf{k},\tau')\tilde{h}_\pm(\mathbf{k}',\tau'')\rangle =& \frac{4H^4}{M_P^4(2\pi)^3}\Pi^{lm}_\pm(\mathbf{k})\Pi^{ab}_\pm(\mathbf{k}') \int_{\tau_{end}}^0 d\tau' \int_{\tau_{end}}^0 d\tau''(\tau')^2(\tau'')^2 G(\tau,\tau',k) \\ & G(\tau,\tau'',k') \int_{-\infty}^{\infty} d^3\mathbf{p} \int_{-\infty}^{\infty} d^3\mathbf{p}' \Big[\langle \tilde{A}'_l(\mathbf{p},\tau')\tilde{A}'_a(\mathbf{p}',\tau'')\rangle \cdot \langle \tilde{A}'_m(\mathbf{k}-\mathbf{p},\tau')\tilde{A}'_b(\mathbf{k}'-\mathbf{p}',\tau'')\rangle \\ & + \langle \tilde{A}'_l(\mathbf{p},\tau')\tilde{A}'_b(\mathbf{k}'-\mathbf{p}',\tau'')\rangle \cdot \langle \tilde{A}'_m(\mathbf{k}-\mathbf{p},\tau')\tilde{A}'_a(\mathbf{p}',\tau'')\rangle\Big] .\end{aligned} \tag{4.61}$$

We then plug in for $\tilde{A}'_j(\mathbf{k},\tau)$ (eq. (4.60)) and simplify using $[a(\mathbf{k})a^\dagger(\mathbf{k}')] = \delta^{(3)}(\mathbf{k}-\mathbf{k}') =$

$a(\mathbf{k})a^\dagger(\mathbf{k}') - a^\dagger(\mathbf{k}')a(\mathbf{k})$, to obtain:

$$\langle \tilde{h}_\pm(\mathbf{k},\tau')\tilde{h}_\pm(\mathbf{k}',\tau'')\rangle = \frac{4H^4}{M_P^4(2\pi)^3}\delta^{(3)}(\mathbf{k}+\mathbf{k}')\Pi_\pm^{lm}(\mathbf{k})\Pi_\pm^{ab}(\mathbf{k}')\int_{\tau_{end}}^0 d\tau'\int_{\tau_{end}}^0 d\tau''(\tau')^2(\tau'')^2$$
$$G(\tau,\tau',k)G(\tau,\tau'',k')\int_{-\infty}^{\infty} d^3\mathbf{p}\frac{\xi}{2H^3}p^{\frac{1}{2}}|\mathbf{k}-\mathbf{p}|^{\frac{1}{2}}a^{-\frac{7}{2}}(\tau')a^{-\frac{7}{2}}(\tau'')(a'(\tau')a'(\tau''))^2 e^{4\xi H}\cdot$$
$$e^{-2\sqrt{2\xi}(\sqrt{|\tau'|}+\sqrt{|\tau''|})\cdot(\sqrt{p}+\sqrt{|\mathbf{k}-\mathbf{p}|})}\Big[\epsilon_{l+}(\mathbf{p})\epsilon^*_{a+}(\mathbf{p})\epsilon_{m+}(\mathbf{k}-\mathbf{p})\epsilon^*_{b+}(\mathbf{k}-\mathbf{p})+$$
$$+\epsilon_{l+}(\mathbf{p})\epsilon^*_{b+}(\mathbf{p})\epsilon_{m+}(\mathbf{k}-\mathbf{p})\epsilon^*_{a+}(\mathbf{k}-\mathbf{p})\Big]. \tag{4.62}$$

We further simplify noting that during inflation $a = -\frac{1}{H\tau}$ and $a' = \frac{1}{H\tau^2}$:

$$\langle \tilde{h}_\pm(\mathbf{k},\tau')\tilde{h}_\pm(\mathbf{k}',\tau'')\rangle = -\frac{H^4\xi}{4\pi^3 M_P^4}\delta^{(3)}(\mathbf{k}+\mathbf{k}')\Pi_\pm^{lm}(\mathbf{k})\Pi_\pm^{ab}(\mathbf{k}')\int_{\tau_{end}}^0 d\tau'\int_{\tau_{end}}^0 d\tau''(\tau')^{\frac{3}{2}}(\tau'')^{\frac{3}{2}}\cdot$$
$$\cdot G(\tau,\tau',k)G(\tau,\tau'',k')\int_{-\infty}^{\infty} d^3\mathbf{p}\, p^{\frac{1}{2}}|\mathbf{k}-\mathbf{p}|^{\frac{1}{2}}\, e^{4\pi\xi}e^{-2\sqrt{2\xi}(\sqrt{|\tau'|}+\sqrt{|\tau''|})\cdot(\sqrt{p}+\sqrt{|\mathbf{k}-\mathbf{p}|})}\cdot$$
$$\cdot\Big[\epsilon_{l+}(\mathbf{p})\epsilon^*_{a+}(\mathbf{p})\epsilon_{m+}(\mathbf{k}-\mathbf{p})\epsilon^*_{b+}(\mathbf{k}-\mathbf{p}) + \epsilon_{l+}(\mathbf{p})\epsilon^*_{b+}(\mathbf{p})\epsilon_{m+}(\mathbf{k}-\mathbf{p})\epsilon^*_{a+}(\mathbf{k}-\mathbf{p})\Big]. \tag{4.63}$$

We next focus on simplifying the terms being summed over. We take $\mathbf{k}$ to point along $\hat{z}$ and define θ and ϕ to be the angles relating $\mathbf{p}$ to $\mathbf{k}$. This gives us the standard polarization vectors for $\mathbf{k}$. The polarization vectors for $\mathbf{p}$ are more complicated accounting for the fact that they are rotated relative to the standard vectors by the angles θ and ϕ. Thus the polarization vectors for $\mathbf{k}$ are:

$$\epsilon_+(\mathbf{k}) = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ -i \\ 0 \end{pmatrix} \qquad \epsilon_0(\mathbf{k}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \epsilon_-(\mathbf{k}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

and the polarization vectors for $\mathbf{p}$ are:

$$\epsilon_+(\mathbf{p}) = \frac{1}{\sqrt{2}}\begin{pmatrix} -\cos\theta\,\cos\phi + i\sin\phi \\ -\cos\theta\,\sin\phi - i\cos\phi \\ \sin\theta \end{pmatrix} \qquad \epsilon_0(\mathbf{p}) = \begin{pmatrix} \cos\phi\,\sin\theta \\ \sin\phi\,\sin\theta \\ \cos\theta \end{pmatrix}$$

$$\epsilon_-(\mathbf{p}) = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta\,\cos\phi + i\sin\phi \\ \cos\theta\,\sin\phi - i\cos\phi \\ -\sin\theta \end{pmatrix}$$

First we use that $\Pi_\pm^{ij}(\mathbf{k}) = \frac{1}{\sqrt{2}}\epsilon_\mp^i(\mathbf{k})\epsilon_\mp^j(\mathbf{k})$ (eq. (4.53)) to expand the projection tensors:

$$\langle \tilde{h}_\pm(\mathbf{k},\tau')\tilde{h}_\pm(\mathbf{k}',\tau'')\rangle = -\frac{H^4\xi}{8\pi^3 M_P^4}\delta^{(3)}(\mathbf{k}+\mathbf{k}')\epsilon_\mp^l(\mathbf{k})\epsilon_\mp^m(\mathbf{k})\epsilon_\mp^a(-\mathbf{k})\epsilon_\mp^b(-\mathbf{k})\int_{\tau_{end}}^0 d\tau'\int_{\tau_{end}}^0 d\tau''(\tau')^{\frac{3}{2}}\cdot$$
$$\cdot(\tau'')^{\frac{3}{2}}G(\tau,\tau',k)G(\tau,\tau'',k')\int_{-\infty}^{\infty} d^3\mathbf{p}\, p^{\frac{1}{2}}|\mathbf{k}-\mathbf{p}|^{\frac{1}{2}}\, e^{4\xi\pi}e^{-2\sqrt{2\xi}(\sqrt{|\tau'|}+\sqrt{|\tau''|})\cdot(\sqrt{p}+\sqrt{|\mathbf{k}-\mathbf{p}|})}\cdot$$
$$\cdot\Big[\epsilon_{l+}(\mathbf{p})\epsilon^*_{a+}(\mathbf{p})\epsilon_{m+}(\mathbf{k}-\mathbf{p})\epsilon^*_{b+}(\mathbf{k}-\mathbf{p}) + \epsilon_{l+}(\mathbf{p})\epsilon^*_{b+}(\mathbf{p})\epsilon_{m+}(\mathbf{k}-\mathbf{p})\epsilon^*_{a+}(\mathbf{k}-\mathbf{p})\Big]. \tag{4.64}$$

Next we simplify using the identities $\epsilon_\pm(-\mathbf{p}) = \epsilon^*_\pm(\mathbf{p})$ and $|\epsilon^i_\sigma(\mathbf{p_1})\epsilon^i_+(\mathbf{p_2})|^2 = \frac{1}{4}(1 - \sigma\frac{\mathbf{p_1}\cdot\mathbf{p_2}}{p_1 p_2})^2$:

$$\langle\tilde{h}_\pm(\mathbf{k},\tau')\tilde{h}_\pm(\mathbf{k}',\tau'')\rangle = \frac{H^4\xi}{2^6\pi^3 M_P^4}\delta^{(3)}(\mathbf{k}+\mathbf{k}')e^{4\pi\xi}\int_{\tau_{end}}^0 d\tau'\int_{\tau_{end}}^0 d\tau''|\tau'|^{\frac{3}{2}}|\tau''|^{\frac{3}{2}}G(\tau,\tau',k)\cdot$$
$$\cdot\, G(\tau,\tau'',k)\int_{-\infty}^{\infty} d^3\mathbf{p}\, p^{\frac{1}{2}}|\mathbf{k}-\mathbf{p}|^{\frac{1}{2}}\, e^{-2\sqrt{2\xi}(\sqrt{|\tau'|}+\sqrt{|\tau''|})\cdot(\sqrt{p}+\sqrt{|\mathbf{k}-\mathbf{p}|})}\cdot(1\pm\frac{\mathbf{k}\cdot\mathbf{p}}{kp})^2\cdot$$
$$(1\pm\frac{(k^2-\mathbf{k}\cdot\mathbf{p})}{k|\mathbf{k}-\mathbf{p}|})^2\,. \tag{4.65}$$

Next we take the limit $-k\tau\to 0$, meaning late time, so the Green's function, G, simplifies from $G(k,\tau,\tau') = \frac{1}{k^3\tau'^2}[(1+k^2\tau\tau')\sin(k(\tau-\tau')) + k(\tau'-\tau)\cos(k(\tau-\tau'))]$ to $G(k,\tau,\tau')\approx \frac{1}{k^3\tau'^2}[k\tau'\cos(k\tau') - \sin(k\tau')]$.

$$\langle\tilde{h}_\pm(\mathbf{k},\tau')\tilde{h}_\pm(\mathbf{k}',\tau'')\rangle = -\frac{H^4\xi}{2^6\pi^3 M_P^4}\delta^{(3)}(\mathbf{k}+\mathbf{k}')e^{4\pi\xi}\int_{\tau_{end}}^0 d\tau'\int_{\tau_{end}}^0 d\tau''(\tau')^{\frac{3}{2}}(\tau'')^{\frac{3}{2}}\frac{1}{k^6\tau'^2\tau''^2}\cdot$$
$$\cdot\,[k\tau'\cos(k\tau')-\sin(k\tau')]\cdot[k\tau''\cos(k\tau'')-\sin(k\tau'')]\cdot\int_{-\infty}^{\infty} d^3\mathbf{p}\, p^{\frac{1}{2}}(k^2+p^2-2kp\cos\theta)^{\frac{1}{4}}\cdot$$
$$\cdot\, e^{-2\sqrt{2\xi}(\sqrt{|\tau'|}+\sqrt{|\tau''|})\cdot(p^{\frac{1}{2}}+(k^2+p^2-2kp\cos\theta)^{\frac{1}{4}})}\cdot(1\pm\cos\theta)^2\cdot\left(1\pm\frac{(k-p\cos\theta)}{(k^2+p^2-2kp\cos\theta)^{\frac{1}{2}}}\right)^2 \tag{4.66}$$

We first solve the τ', τ'' integrals. Note that they are exponentially suppressed for large τ', τ'' because of the term $e^{-2\sqrt{2\xi}(\sqrt{|\tau'|}+\sqrt{|\tau''|})\cdot(p^{\frac{1}{2}}+(k^2+p^2-2kp\cos\theta)^{\frac{1}{4}})}$. Remember we said $\xi\gtrsim 1$, so we will only get a large contribution to the integral for τ', τ'' small (for $k\tau'$, $k\tau''\ll 1$). Therefore, we can expand for small $k\tau'$ and $k\tau''$:

$\cos(k\tau')\approx 1-\frac{k^2\tau'^2}{2}$ and $\sin(k\tau')\approx k\tau'-\frac{k^3\tau'^3}{6}$, which gives us:

$$\langle\tilde{h}_\pm(\mathbf{k},\tau')\tilde{h}_\pm(\mathbf{k}',\tau'')\rangle = -\frac{H^4\xi}{2^6\ 3^2\pi^3 M_P^4}\delta^{(3)}(\mathbf{k}+\mathbf{k}')e^{4\pi\xi}\int_{\tau_{end}}^0 d\tau'\int_{\tau_{end}}^0 d\tau''(\tau')^{\frac{5}{2}}(\tau'')^{\frac{5}{2}}\int_{-\infty}^{\infty} d^3\mathbf{p}\, p^{\frac{1}{2}}\cdot$$
$$\cdot\,(k^2+p^2-2kp\cos\theta)^{\frac{1}{4}}e^{-2\sqrt{2\xi}(\sqrt{|\tau'|}+\sqrt{|\tau''|})\cdot(p^{\frac{1}{2}}+(k^2+p^2-2kp\cos\theta)^{\frac{1}{4}})}\cdot(1\pm\cos\theta)^2\cdot$$
$$\cdot\left(1\pm\frac{(k-p\cos\theta)}{(k^2+p^2-2kp\cos\theta)^{\frac{1}{2}}}\right)^2\,. \tag{4.67}$$

Next, we make a substitution for a positive time variable η defined by: $\eta=-\tau$, and solve the η integrals using: $\int_0^\infty d\eta'\int_0^\infty d\eta''(\eta')^{\frac{5}{2}}(\eta'')^{\frac{5}{2}}e^{-C(\sqrt{\eta'}+\sqrt{\eta''})} = \frac{2^2\Gamma^2(7)}{C^{14}}$, to obtain:

$$\langle\tilde{h}_\pm(\mathbf{k},\tau')\tilde{h}_\pm(\mathbf{k}',\tau'')\rangle = \frac{H^4\xi\Gamma^2(7)}{2^4\ 3^2\pi^3 M_P^4}\delta^{(3)}(\mathbf{k}+\mathbf{k}')e^{4\pi\xi}\int_{-\infty}^{\infty} d^3\mathbf{p}\, p^{\frac{1}{2}}(k^2+p^2-2p\cos\theta)^{\frac{1}{4}}\cdot$$
$$\cdot\,\frac{1}{(2\sqrt{2\xi}(p^{\frac{1}{2}}+(k^2+p^2-2kp\cos\theta)^{\frac{1}{4}}))^{14}}\cdot(1\pm\cos\theta)^2\cdot\left(1\pm\frac{(k-p\cos\theta)}{(k^2+p^2-2kp\cos\theta)^{\frac{1}{2}}}\right)^2 \tag{4.68}$$

Next, we want to solve the $\mathbf{p}$ integral. First, we define a dimensionless momentum variable $\mathbf{q}$, such that $\mathbf{q} = \frac{\mathbf{p}}{k}$. Next we use numerical integration in Mathematica to do the $\mathbf{q}$ integral obtaining:

$$\langle \tilde{h}_+(\mathbf{k}, \tau')\tilde{h}_+(\mathbf{k}', \tau'')\rangle = 8.6 \times 10^{-7} \frac{H^4}{k^3 M_P^4 \xi^6} \delta^{(3)}(\mathbf{k} + \mathbf{k}') e^{4\pi\xi} \tag{4.69}$$

and

$$\langle \tilde{h}_-(\mathbf{k}, \tau')\tilde{h}_-(\mathbf{k}', \tau'')\rangle = 1.8 \times 10^{-9} \frac{H^4}{k^3 M_P^4 \xi^6} \delta^{(3)}(\mathbf{k} + \mathbf{k}') e^{4\pi\xi} \,. \tag{4.70}$$

Next we want to write the power spectrum. Remember the power spectrum is defined by $P = \frac{k^3}{2\pi^2 \delta^{(3)}(\mathbf{k}+\mathbf{k}')} \langle h_{ij} h^{ij} \rangle = \frac{k^3}{2\pi^2 \delta^{(3)}(\mathbf{k}+\mathbf{k}')} [\langle h_+ h_+ \rangle + \langle h_- h_- \rangle]$ since the cross terms $\langle h_+ h_- \rangle$ and $\langle h_- h_+ \rangle$ are 0. We define $P_{h_+} = \frac{k^3}{2\pi^2 \delta^{(3)}(\mathbf{k}+\mathbf{k}')} \langle h_+ h_+ \rangle$ and $P_{h_-} = \frac{k^3}{2\pi^2 \delta^{(3)}(\mathbf{k}+\mathbf{k}')} \langle h_- h_- \rangle$, so $P = P_{h_+} + P_{h_-}$. And note that since the standard spectrum is given by $P = \frac{2H^2}{\pi^2 M_P^2}$ and is scale invariant, this gives $P_{h_+} = P_{h_-} = \frac{H^2}{\pi^2 M_P^2}$ for the standard spectrum. Putting these together we get:

$$P_{h_+} = \frac{H^2}{\pi^2 M_P^2} [1 + 8.6 \times 10^{-7} \frac{H^2}{M_P^2 \xi^6} e^{4\pi\xi}] \tag{4.71}$$

$$P_{h_-} = \frac{H^2}{\pi^2 M_P^2} [1 + 1.8 \times 10^{-9} \frac{H^2}{M_P^2 \xi^6} e^{4\pi\xi}] \,. \tag{4.72}$$

We see strong parity violation, exhibited in the fact that $\langle \tilde{h}_+(\mathbf{k}, \tau')\tilde{h}_+(\mathbf{k}', \tau'')\rangle$ is approximately three orders of magnitude larger than $\langle \tilde{h}_-(\mathbf{k}, \tau')\tilde{h}_-(\mathbf{k}', \tau'')\rangle$. Also, we estimate this signal could potentially be observable, with the signal from particle production dwarfing the standard signal for a range in parameter space. The signal will be larger for large ξ, but there are limits on ξ form the scalar spectrum. Just as this mechanism will source tensor perturbations, it will also source scalar curvature perturbations. These scalar perturbations will be non-scale invariant as they will be peaked just as the tensor spectrum is, and they will be highly non-Gaussian. Each of these conditions will put a limit on ξ, since the observed scalar spectrum is highly scale invariant and no non-Gaussianities have yet to be detected. It turns out the lack of an observation of non-Gaussianities places the stronger constraint on ξ. This applies to the non-observation of non-Gaussianities both in the CMB and in LSS. As was shown in [25], this constrains ξ to be less than 2.6 at CMB and LSS scales. The above tensor power spectrum for $\xi < 2.6$ will not generate an observable signal at these scales. However, when considering the tensor spectrum, we have another range of scales where an observation might be possible which is LIGO/ LISA type scales. These scales occur at many orders of magnitude higher momentum, and since scalar perturbations can not be observed on these scales, there are no similar constraints. We note that $\xi = \frac{\dot{\phi}}{2fH}$ where f is a constant and $\dot{\phi}$ is either constant or slowly increasing depending on the inflationary model. Also note $\frac{1}{H} \approx \frac{1}{\sqrt{V(\phi)}}$ where $V(\phi)$ is slowly decreasing as the inflaton slowly rolls down its potential. This means ξ is slowly increasing throughout inflation. Since LIGO/ LISA type scales occur at much higher momenta, these are scales which exited the horizon much later during inflaton, when ξ would have been much larger. So it is possible that there could be an observable signal at LIGO/LISA type scales, even if we constrain ξ to be less than 2.6 at CMB/LSS scales.

How much ξ will change between CMB scales and LIGO scales will depend on both the form of the inflationary potential and the number of e-foldings that pass between the time CMB scales and direct detection scales leave the horizon. To estimate if an observable signal might be possible at LIGO scales without violating the CMB constraints, we assume a $m^2\phi^2/2$ form of the potential. Then we relate the change in the field value ϕ to the number of e-foldings. A number of e-foldings N is defined as $N = \int H dt$. We start from

$$N = \int_{t_i}^{t_f} H dt \,, \tag{4.73}$$

where t_i stands for initial time and t_f stands for final time, and N will give the number of e-foldings between them. Then we use that we can rewrite $dt = \frac{dt}{d\phi} d\phi = \frac{1}{\frac{d\phi}{dt}} d\phi$.

$$N = \int_{\phi_i}^{\phi_f} \frac{H}{\dot{\phi}} d\phi \tag{4.74}$$

Next we use the inflaton equation of motion: $3H\dot{\phi} + \frac{dV}{d\phi} = 0$ (using that the $\ddot{\phi}$ term is negligible). This gives us $\dot{\phi} = -\frac{\frac{dV}{d\phi}}{3H}$.

$$N = -3 \int_{\phi_i}^{\phi_f} \frac{H^2}{\frac{dV}{d\phi}} d\phi \tag{4.75}$$

Next we use the Friedmann equation: $H = \sqrt{\frac{8\pi G}{3}\rho}$, and we use that ρ is dominated by $V(\phi)$.

$$N = -\int_{\phi_i}^{\phi_f} 8\pi G \frac{V}{\frac{dV}{d\phi}} d\phi \tag{4.76}$$

So far this is true for any potential $V(\phi)$. Now we substitute for $V(\phi) = \frac{m^2\phi^2}{2}$.

$$N = 4\pi G \int_{\phi_f}^{\phi_i} \phi \, d\phi = 2\pi G \Delta\phi^2 \,. \tag{4.77}$$

And then using that $8\pi G = \frac{1}{M_P^2}$, we find $\Delta N = \frac{\Delta\phi^2}{4M_P^2}$. Solving for $\Delta\phi$ we find: $\Delta\phi = 2M_P\sqrt{\Delta N}$. The mass of the inflaton field is observationally constrained. Using COBE data, it is given by:

$$m^2 = \frac{6\pi^2 P_\zeta}{N_C^2} M_P^2 \,, \tag{4.78}$$

where P_ζ (the scalar power spectrum) has been observed to be approximately $P_\zeta = 2.5 \times 10^{-9}$. COBE scales could have exited the horizon anywhere from $47 < N_C < 62$ e-foldings before the end of inflation. Next we obtain an expression for ϵ in terms of N. We use that ϵ can be written as $\epsilon = \frac{(\frac{dV}{d\phi})^2}{16\pi G V^2}$. Then we plug in for $V = m^2\phi^2/2$.

$$\epsilon = \frac{2M_P^2}{\phi^2} \tag{4.79}$$

Then we plug in for $\phi = 2M_P\sqrt{N}$ to obtain: $\epsilon = \frac{1}{2N}$. Next, we want to similarly write ξ in terms of N. We start from our definition of ξ: $\xi = \frac{\dot{\phi}}{2fH}$. Then we use $\dot{\phi}^2 = 2H^2M_P^2\epsilon$, and that we found above that $\epsilon = \frac{1}{2N}$. This way we can substitute in the ξ formula for $\dot{\phi}^2 = \frac{H^2M_P^2}{N}$ to get $\xi = \frac{M_P}{2f\sqrt{N}}$. Note the only time dependent part is the $\frac{1}{\sqrt{N}}$. So now if we want to take a ratio of ξ at some undetermined period N e-foldings before the end of inflation, and ξ at CMB scales, we get:

$$\frac{\xi}{\xi_C} = \sqrt{\frac{N_C}{N}}\,. \tag{4.80}$$

Next we wish to calculate Ω_{GW}. To do this first we start with calculating ρ_{GW}, the energy density of gravitational waves. This should in principle get contributions from the one-point function, the two-point function, the three-point function, etc. Each contains energy. In this case though, the one-point function is zero and the three-point functions and higher are higher order contributions, so we will only consider the energy in the two-point function. This way we get:

$$\rho_{GW} = \frac{M_P^2}{2}\int dk\, k\left(\frac{a_k}{a_0}\right)^2 P_{GW}(k)\,, \tag{4.81}$$

where $P_{GW}(k)$ is the momentum space power spectrum of gravitational waves. If we think of P_{GW} as a distribution function relating the number density of modes for each momentum k, then multiplying by k and integrating over all k will naturally give the total energy. The $\frac{a_k}{a_0}$ is a transfer function. The P_{GW} we plug in is the P_{GW} generated during inflation, but we want the energy density of gravitational waves today. Noting that P_{GW} is proportional to the two point function $\langle hh\rangle$ and to each factor of h we apply the transfer function $\frac{a_k}{a_0}$ which accounts for the fact that the amplitude of the gravitational waves h is redshifted in proportion to the changing scale factor.

We use the definition of $\Omega_{GW} = \frac{1}{\rho_c}\frac{d\rho_{GW}}{d\ln K}$ where ρ_c is the critical energy density. And since we want Ω_{GW} today, we want the critical density today, which assuming the spatial curvature is 0, we can get from the Friedmann equation: $\rho_c = 3H_0^2M_P^2$. And note, $\frac{d}{d\ln k} = k\frac{d}{dk}$.

$$\Omega_{GW} = \frac{1}{3H_0^2M_P^2}k\frac{d\rho_{GW}}{dk} \tag{4.82}$$

Then we plug in for expression of ρ_{GW} above to get:

$$\Omega_{GW} = \frac{k^2}{6H_0^2}\left(\frac{a_k}{a_0}\right)^2 P_{GW}(k) \tag{4.83}$$

We can rewrite our power spectrum in terms of N and N_C. Starting from:

$$P_{h_+} = \frac{H^2}{\pi^2M_P^2}[1 + 8.6064\times 10^{-7}\frac{H^2}{M_P^2\xi^6}e^{4\pi\xi}]\,. \tag{4.84}$$

We use that $\xi = \xi_C\sqrt{\frac{N_C}{N}}$ to obtain:

$$P_{h_+} = \frac{H^2}{\pi^2M_P^2}[1 + 8.6064\times 10^{-7}\frac{H^2}{M_P^2\xi_C^6}\left(\frac{N}{N_C}\right)^3 e^{4\pi\xi_C\sqrt{\frac{N_C}{N}}}]\,. \tag{4.85}$$

Then we can rewrite H using that $H = \sqrt{\frac{8\pi G V}{3}} = \sqrt{\frac{V}{3}}\frac{1}{M_P}$. Using that $V = \frac{\mu^2\phi^2}{2}$, we get $H = \frac{1}{\sqrt{3}M_P}\frac{\mu\phi}{\sqrt{2}}$. Then using that $\mu^2 = \frac{6\pi^2 P_\zeta M_P^2}{N_C^2}$, we get

$$H = 2\pi\sqrt{P_\zeta}\frac{\sqrt{N}M_P}{N_C}\,. \tag{4.86}$$

Plugging this into the power spectrum, we obtain:

$$P_{h_+} = 10^{-8}\frac{N}{N_C^2}\left[1 + (8.5\times 10^{-14})\left(\frac{N^4}{N_C^5}\right)\frac{e^{4\pi\xi_C\sqrt{\frac{N_C}{N}}}}{\xi_C^6}\right]\,. \tag{4.87}$$

The the total power spectrum: $P_h = P_{h_+} + P_{h_-}$ is given by:

$$P_h = 2\cdot 10^{-8}\frac{N}{N_C^2}\left[1 + (4.3\times 10^{-14})\left(\frac{N^4}{N_C^5}\right)\frac{e^{4\pi\xi_C\sqrt{\frac{N_C}{N}}}}{\xi_C^6}\right]\,. \tag{4.88}$$

We next want to plug this into our Ω_{GW} equation. Note that the $\frac{a_k}{a_0}$ is the transfer function of these gravitational waves after they reenter the horizon. We will be concerned with waves which could potentially be observable at direct detection experiments, which will mean fairly high frequency modes which reenter the horizon during radiation domination. For some mode k which reached horizon size during the radiation dominated epoch, then we have:

$$\frac{H_k}{H_0} = \frac{H_{eq}}{H_0}\frac{H_k}{H_{eq}}\,, \tag{4.89}$$

where eq stands for matter-radiation equality, and H_k is the Hubble parameter evaluated when this mode k reached horizon size. Noting that during a radiation dominated epoch $H \propto \frac{1}{a^2}$, and during a matter dominated epoch $H \propto \frac{1}{a^{\frac{3}{2}}}$, we obtain:

$$\frac{H_k}{H_0} = \frac{a_0^{\frac{3}{2}}}{a_{eq}^{\frac{3}{2}}}\frac{a_{eq}^2}{a_k^2} \tag{4.90}$$

or

$$H_k = \frac{H_0 a_0^{\frac{3}{2}} a_{eq}^{\frac{1}{2}}}{a_k^2}\,, \tag{4.91}$$

where again, a_k is the scale factor evaluated when this mode k reached horizon size. Note k is the comoving frequency so $\frac{1}{k}$ is the comoving wavelength. $\frac{1}{H}$ gives the physical distance to the horizon, so $\frac{1}{aH}$ is the comoving distance to the horizon. We are concerned with this mode k when it reaches horizon size, in which case $k = a_k H_k$. Then plugging this into our Ω_{GW} equation:

$$\Omega_{GW} = \frac{a_0 a_{eq}}{6}P_{GW} = \frac{a_0^2}{6}\left(\frac{a_{eq}}{a_0}\right)P_{GW} \tag{4.92}$$

Next we use that $a \propto \frac{1}{T}$, where T is the temperature, to obtain:

$$\Omega_{GW} = \frac{a_0^2}{6}\left(\frac{T_0}{T_{eq}}\right) P_{GW}\,. \tag{4.93}$$

Note a_0, the scale factor now, is defined to be 1. T_{eq} the temperature of matter radiation equality, $\approx 74,000K$. T_0, the temperature now, $= 2.725K$. We then obtain:

$$\Omega_{GW} = 6.137 \times 10^{-6} P_{GW}\,. \tag{4.94}$$

Then plugging in our equation for P_{GW}...

$$\Omega_{GW} = 1.23 \times 10^{-13}\frac{N}{N_C^2}\left[1 + (4.3 \times 10^{-14})\left(\frac{N^4}{N_C^5}\right)\frac{e^{4\pi\xi_C\sqrt{\frac{N_C}{N}}}}{\xi_C^6}\right] \tag{4.95}$$

and

$$\Omega_{GW}h^2 = 6.01 \times 10^{-14}\frac{N}{N_C^2}\left[1 + (4.3 \times 10^{-14})\left(\frac{N^4}{N_C^5}\right)\frac{e^{4\pi\xi_C\sqrt{\frac{N_C}{N}}}}{\xi_C^6}\right]\,, \tag{4.96}$$

using that $h = .7$. We can next get a limit on ξ_C by requiring that $\Omega_{GW}h^2$ be large enough to be detectable at LIGO. To do this we use that LIGO has a sensitivity of about $\Omega_{GW}h^2 = 10^{-9}$ around $f = 100$ Hz. We then require when $f = 100Hz$, that $\Omega_{GW}h^2$ from our model is larger than 10^{-9}:

$$10^{-9} < 2.6 \times 10^{-27}\frac{N^5}{N_C^7}\frac{e^{4\pi\xi_C\sqrt{\frac{N_C}{N}}}}{\xi_C^6}\,. \tag{4.97}$$

As the inflaton field ϕ is decaying into the vector fields, as long as the energy density of ϕ is by far the most dominant energy density in the universe, than slow roll will continue and the small loss of energy of ϕ into producing the vectors will have a negligible effect on the evolution of ϕ. This is the regime we want to consider. The kinetic energy density of the vector field is given by $\frac{\mathbf{E}^2}{2}$ where $\mathbf{E}$ is the electric field, and we use that for this model $E^2 \gg B^2$. The energy density of the inflaton field is dominated by the potential energy, $V(\phi)$. We get an expression for $V(\phi)$ from the Friedmann equation which states $H = \sqrt{\frac{8\pi G}{3}\rho} \approx \sqrt{\frac{8\pi G}{3}V(\phi)}$. From this we find $V(\phi) = 3M_P^2H^2$. So to require that the energy density of the inflaton field dominate the energy density of our vector field, we require:

$$\frac{\mathbf{E}^2}{2} \ll 3M_P^2H^2 \tag{4.98}$$

We use that $\mathbf{E} = -\frac{\mathbf{A}'}{a^2}$. To carry out our energy estimate, we just need the magnitude of the energy in A'. To obtain this, we want to take the square root of the two-point function of A':

$$|A'| = |\langle A'(\mathbf{x},\eta)A'(\mathbf{x},\eta)\rangle|^{\frac{1}{2}} \tag{4.99}$$

We use the Fourier transform of the vector field into momentum space eq. (4.6), and the expression of our field in momentum space eq. (4.8).

$$\langle A_i'(\mathbf{x},\eta)A_j'(\mathbf{x},\eta)\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\int \frac{d^3\mathbf{k}'}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot\mathbf{x}}e^{i\mathbf{k}'\cdot\mathbf{x}}\epsilon_{i+}(\mathbf{k})\epsilon_{j+}^*(-\mathbf{k}')u_+'(k)u_+'^*(k')\delta^{(3)}(\mathbf{k}+\mathbf{k}')(2\pi)^{\frac{3}{2}} \tag{4.100}$$

Since we only want an energy magnitude estimate, we can drop the dimensionless polarization vectors and the above simplifies:

$$|\langle A'(\mathbf{x},\eta)A'(\mathbf{x},\eta)\rangle| = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}|u_+'(k)|^2 \,. \tag{4.101}$$

Next we plug in our expression for the time derivative of our mode function, eq. (4.57):

$$|\langle A'(\mathbf{x},\eta)A'(\mathbf{x},\eta)\rangle| = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\frac{k^{\frac{1}{2}}\xi^{\frac{1}{2}}}{2^{\frac{1}{2}}\eta^{\frac{1}{2}}}e^{4\pi\xi}e^{-4\sqrt{-2\xi\eta}} \,. \tag{4.102}$$

Preforming this integral and taking the magnitude we obtain:

$$|A'(\mathbf{x},\eta)| = |\langle A'(\mathbf{x},\eta)A'(\mathbf{x},\eta)\rangle|^{\frac{1}{2}} = \frac{e^{\pi\xi}}{60\xi^{\frac{3}{2}}\eta^2} \,. \tag{4.103}$$

Our inequality of the energy densities then becomes:

$$\frac{e^{\pi\xi}}{\xi^{\frac{3}{2}}} \ll \frac{60\sqrt{6}M_P}{H} \,. \tag{4.104}$$

We plug in the expression we found earlier for H in terms of N and N_C, eq. (4.86), to obtain the limit:

$$\frac{e^{\pi\xi_C\sqrt{\frac{N_C}{N}}}}{\xi_C^{\frac{3}{2}}} \ll 4.7\times 10^5\frac{N_C^{\frac{7}{4}}}{N^{\frac{5}{4}}} \,. \tag{4.105}$$

We also want to put a limit on the parameter ξ based on requiring that backreaction not become important. As ϕ is passing energy into the vectors these vectors are also having an effect on the evolution of ϕ. We want to require that this effect is insignificant. We solve the equation of motion of ϕ and compare the strength of the term coming from the interaction of the inflaton field with the vectors to the term from the potential energy of the inflaton field, since during slow roll the potential energy should dominate the kinetic energy term. We require:

$$\frac{e^{2\pi\xi}}{\xi^3} \ll 700\frac{(\frac{\partial V(\phi)}{\partial\phi})^2}{H^6} \,. \tag{4.106}$$

Using our conditions above this gives:

$$\frac{e^{2\pi\xi_C\sqrt{\frac{N_C}{N}}}}{\xi_C^3} \ll 6.4\times 10^{10}\left(\frac{N_C}{N}\right)^{\frac{7}{2}} \,. \tag{4.107}$$

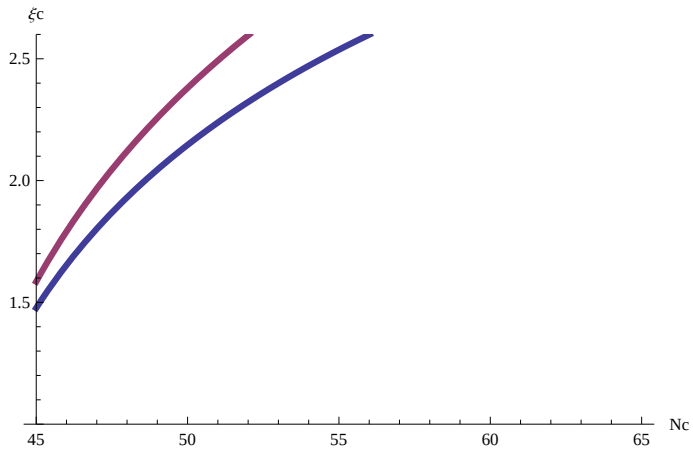

Figure 4.1: The figure shows the minimum value of ξ_C as a function of N_C that is allowed from the two limits of the effect of the vectors on the evolution of ϕ. The upper red line shows the limit from requiring that the energy density of ϕ be larger than the energy density of the vector fields. The bottom blue line is from requiring that backreaction of the vectors on the inflaton field be insignificant. It can be seen that throughout the region of interest, the backreaction limit is the stronger limit.

From this, we can plot this upper bound on ξ as a function of N_C. First we compare the limit from requiring that the energy density of the vectors be less than the energy density of the inflaton field to the limit on backreaction of the vectors on the inflaton field. These two limits are shown in Figure 4.1 which shows the limiting value of ξ_C in each case as a function of N_C. It can be seen that throughout the region of interest, the limit from backreaction is stronger. We want to compare the contribution to the energy density from our production method of tensors to the sensitivity of LIGO. We note that LIGO is most sensitive to frequencies of about 100 Hz. This is related to the wave number variable k, by $f = \frac{kc}{2\pi}$. It is conventional to give k in terms of Mpc^{-1} in which case we use $c = 9.7 \times 10^{-15} \frac{Mpc}{s}$, giving us $k_{LIGO} = 6.5 \times 10^{16} \text{Mpc}^{-1}$.

It can be shown, ex. [26], that the number of e-foldings separating two time two different wave number modes left the horizon is given by $\Delta N = \ln \frac{k_1}{k_2}$. Then the number of e-foldings between when CMB and LIGO scales left the horizon is given by $\Delta N = \ln \frac{6.463 \times 10^{16}}{0.05} = 41.7$, where $k = 0.05$ Mpc^{-1} is a representative scale for which CMB modes exited the horizon.

Therefore using the above, the number of e-foldings between a time a random scale k left the horizon and the end of inflation is given by: $N - N_C = \ln(\frac{k_{CMB}}{k})$. We can expand this into: $N - N_C = \ln(\frac{k_{CMB}}{k_{LIGO}}) + \ln(\frac{k_{LIGO}}{k})$.

$$N - N_C = -41.7 + \ln(\frac{k_{LIGO}}{k}) \tag{4.108}$$

We also note that LIGO has a sensitivity at 100 Hz of about $\Omega_{GW} h^2 \approx 10^{-9}$. We also compare to the Einstein Telescope, a possible future project, which would probe gravitational waves at LIGO frequencies but to better sensitivity reaching $\Omega_{GW} h^2 \approx 10^{-11}$. We can compare this with eq. (4.96), the energy density of the signal from our model, and calculate a lower bound for ξ_C as a function of N_C that would allow detection by LIGO or the Einstein Telescope.

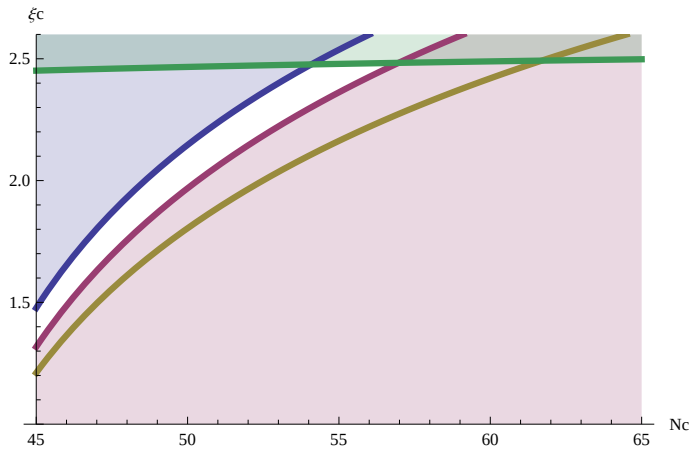

Figure 4.2: The figure shows the possible observable region of parameter space at frequencies of 100 Hz. The almost horizontal green line at the top gives the upper limit on ξ_C from requiring non-observation of non-Gaussianities at LSS scales. The diagonal lines from top to bottom depict: the blue line shows the upper limit on ξ_C from backreaction, the pink line gives a lower limit on ξ_C to allow detection at LIGO, and the yellow line gives a lower bound to allow detection by the Einstein telescope.

Lastly we want to apply a limit on ξ based on the non-observation of non-Gaussianities. Such a limit was reported by [25] and shown to hold at both CMB scales and LSS, large scale structure, scales, requiring that ξ be less than 2.6 at both scales. The strongest bound comes from LSS since these scales exited the horizon a little after CMB scales, and so the constraint from LSS requires that ξ stay below 2.6 a little longer than the constraint from the CMB scales. To be precise, LSS scales left the horizon about 5 e-foldings after CMB scales. We therefore require that ξ evaluated at $N = N_C - 5$ e-foldings before the end of inflation is less than 2.6. We can then translate this into a limit on ξ_C by using eq. (4.80). Then we obtain:

$$\xi_C < 2.6\sqrt{\frac{(N_C - 5)}{N_C}}\,. \tag{4.109}$$

We plot the 4 limits together in Figure 4.2. We do not include the limit from the energy densities of ϕ and the vectors because we found that it gives a weaker limit than the backreaction limit, and since both need to be satisfied, we only include the backreaction limit in Figure 4.2. The white region in the center of the plot shows a section of the parameter space where an observable signal could be detected by LIGO. The two upper shaded regions are excluded by backreaction and the non-observation of non-Gaussianities at LSS scales. The bottom two lines show the observation thresholds corresponding the the sensitivity of LIGO (red) and the Einstein Telescope (yellow). We also note that the bounds from the sensitivity of LIGO and the Einstein Telescope are dependent on what we assumed for the temperature of the universe during matter-radiation equality. Some sources suggest that it could be lower, closer to 30,000 K which would have the effect of widening the region of parameter space where this model could give an observable signal [27].

We produce a similar plot at BBO/ DECIGO frequencies of $f = .1$ Hz shown in Figure 4.3. We use that the estimated sensitivity of BBO would be $\Omega_{GW}h^2 \approx 10^{-13}$. An extended

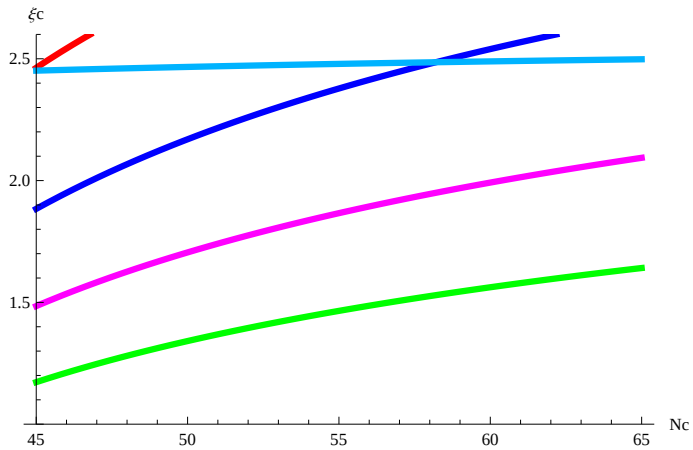

Figure 4.3: The figure shows the possible observable region of parameter space at frequencies of 0.1 Hz. The topmost red line depicts the upper limit on ξ_C from backreaction. The almost horizontal cyan line at the top gives another upper limit on ξ_C from requiring non-observation of non-Gaussianities at LSS scales. The remaining three lines from top to bottom show: the blue line gives a lower limit on ξ_C to allow detection at BBO, the pink line gives the lower limit to allow detection by an extended correlated BBO, and lastly, the green line gives the lower limit to allow detection by DECIGO.

correlated BBO could reach sensitivities of $\Omega_{GW}h^2 \approx 10^{-17}$. A Japanese DECIGO detector would reach sensitivities of $\Omega_{GW}h^2 \approx 10^{-20}$. In the plot, the upper red and cyan lines give the upper bounds on ξ_C from backreaction and the non-observation of non-Gaussianities respectively. The 3 bottom lines give the sensitivity limits for the 3 future detectors that would be sensitive to this frequency. One can see there is a large range of parameter space where this model could give an observable signal. This is largely due to the fact that these 3 detectors are projected to be incredibly sensitive to gravitational waves. Indeed, these three detectors should be capable of measuring the standard inflationary tensor spectrum, at least if large field inflation is correct.

We lastly produce a similar plot for frequencies of $f = .001$ Hz corresponding to a possible future space based LISA like experiment shown in Figure 4.4. There is no backreaction line appearing on this plot. Since LISA scales would be the lowest frequency direct detection scales, LISA would probe modes that exited the horizon earlier than the modes probed by the other detectors. There is less time that passes between the time CMB scales left the horizon and the time LISA scales left the horizon, and the backreaction limit is so weak in this case that it does not fit on the plot. The blue line corresponds to the upper limit on ξ_C from non-observation of non-Gaussianities at LSS scales. The magenta line is a lower limit on ξ_C that would produce a strong enough signal to allow detection by LISA. One can see there is no allowed initial values of ξ_C that would allow detection at LISA without conflicting with the non-Gaussianity limit.

Next, we show an example of how these allowed values of ξ_C would correlate to a measurement of $\Omega_{GW}h^2$ in Figure 4.5. In the figure we use $N_C = 55$ as an example. We use the $f = 100$ Hz upper bound on ξ_C at $N_C = 55$, $\xi_C = 2.5$ as seen in Figure 4.2. We use the bound on ξ_C for $f = 100$ Hz because this frequency gives the strongest bound on ξ_C for all the frequencies we are considering. This is because requiring backreaction to not be

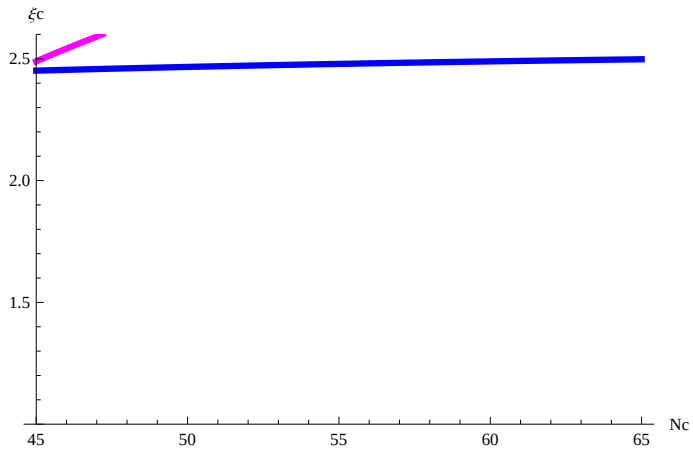

Figure 4.4: The figure attempts to show a possible observable region of parameter space at frequencies of 0.001 Hz. The mostly horizontal blue line gives the upper limit on ξ_C from requiring non-observation of non-Gaussianities at LSS scales. The upper magenta line gives a lower limit on ξ_C to allow detection at LISA. One can see there is no allowed parameter space where a detection at LISA would be possible.

important all the way out to this frequency, will mean backreaction was insignificant at all lower frequencies as well, and the non-Gaussianity limit is independent of the frequency that we wish to make a detection. The red line in Figure 4.5 gives the contribution to $\Omega_{GW}h^2$ from this model for $N_C = 55$ and $\xi_C = 2.5$. The blue line gives the contribution from the standard inflationary tensor spectrum for comparison. Both use the upper bound of $\frac{H}{M_P}$ and so represent their highest possible amplitude. Also in the figure, we compare to the sensitivities of the various gravitational wave detectors. The green line corresponds to the sensitivity of LIGO, magenta is BBO, black is an extended BBO, yellow is DECIGO, and cyan is LISA.

Also note, in the literature it has been suggested that there is an enhancement in this type of signal due to vectors in this case being relativistic, and therefore producing a larger quadrupole moment and a larger signal of gravitational waves then seen in the sudden production mechanisms [21]. In actuality, it does not make sense to call these vectors relativistic or not due to the fact that they are evolving non-adiabatically. Unlike in the sudden production mechanisms, there is no final adiabatic region where we have a precise definition of particle, and can say whether it behaves relativistically or not. The reason this mechanism is capable of being so efficient at sourcing gravitational waves is because of the exponential enhancement of one of the helicities of the mode functions of the vector field, the factor $e^{\pi\xi}$. This leads to exponential production of the vector field over time.

It is worth pointing out that [21] find that if ϕ is not the inflaton field but another scalar field during slow roll, and the above fields are only gravitationally coupled to the inflaton, then the sourcing of scalar perturbations is weaker relative to the tensors, and the limit on ξ from non-observer of non-Gaussianities in the CMB/ LSS becomes a little weaker. They find that in such a case, it is possible that the above mechanism could produce an observable signal at CMB scales.

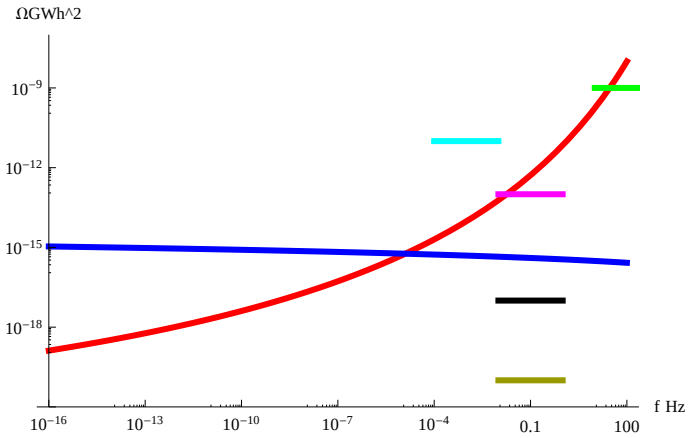

Figure 4.5: The figure shows an example of the contribution to $\Omega_{GW}h^2$ from this model assuming $N_C = 55$ and $\xi_C = 2.5$, the upper bound at $f = 100$ Hz. The upward sloping red line shows the contribution from the model. The downward sloping blue line shows the standard inflationary tensor spectrum for comparison. The other shorter lines slow the sensitivities at various gravitational wave detectors drawn at the frequency range they are sensitive to from top to bottom: green is LIGO, cyan is LISA, magenta is BBO, black is an extended BBO, and yellow is DECIGO.

CHAPTER 5

THREE-POINT FUNCTION OF h_{ij} FROM PROLONGED DECAY OF THE INFLATON INTO VECTORS

Since it is possible to produce an observable tensor signal from the above described $\phi F\tilde{F}$ interaction, and this interaction should be nearly maximally non-Gaussian, it is worth looking into if the non-Gaussianities from this mechanism could be observable. It will be easiest to detect a non-Gaussian signal from the three-point function, although non-Gaussianities could also appear in higher point functions. Since we are discussing small perturbations, higher point functions will be more difficult to detect. Since the three-point function is 0 for a perfectly Gaussian function, measuring a non-zero three-point function is equivalent to measuring non-Gaussianities.

To see why this mechanism should produce an almost maximally non-Gaussian signal, note that the stress energy tensor $T_{ij} \propto \langle A'A' \rangle$ where A is the vector field we are producing. So where as normally one finds the three-point function of some field χ is given by $\langle \chi\chi\chi \rangle$ is clearly 0 if the field χ is Gaussian. For us though, we have our three-point function of tensor perturbations $\propto \langle A'^6 \rangle$ which is non-zero even if the vector field A is perfectly Gaussian. This way it is natural to assume our three-point function will be up to an order one correction given by our two-point function raised to the $\frac{3}{2}$ power, in other words maximally non-Gaussian. We find this is true, with the three-point function evaluating to about 70% of the two-point function raised to the $\frac{3}{2}$. This is a much stronger non-Gaussian signal than is seen from the standard inflationary sources in which the non-Gaussianity parameter is proportional to the slow roll parameters.

Using our equations above for the tensor perturbation $h_\pm$, (eq. (4.49)), and for our vector field A, (eq. (4.60)), we obtain for the three-point function:

$$\begin{aligned}\langle \tilde{h}_\pm(\mathbf{k},\tau)\tilde{h}_\pm(\mathbf{k}',\tau)\tilde{h}_\pm(\mathbf{k}'',\tau)\rangle = &-\frac{2^3 H^6}{M_P^6 (2\pi)^{9/2}} \int_{\tau_{end}}^{0} d\tau' \int_{\tau_{end}}^{0} d\tau'' \int_{\tau_{end}}^{0} d\tau''' G(\tau,\tau',k) \\ & G(\tau,\tau'',k')G(\tau,\tau''',k'')\tau'^2\tau''^2\tau'''^2 \Pi_\pm^{ab}(\mathbf{k})\Pi_\pm^{cd}(\mathbf{k}')\Pi_\pm^{ef}(\mathbf{k}'') \int_{-\infty}^{\infty} d^3\mathbf{p} \int_{-\infty}^{\infty} d^3\mathbf{p}' \int_{-\infty}^{\infty} d^3\mathbf{p}'' \\ & \langle \tilde{A}_a'(\mathbf{p},\tau')\tilde{A}_b'(\mathbf{k}-\mathbf{p},\tau')\tilde{A}_c'(\mathbf{p}',\tau'')\tilde{A}_d'(\mathbf{k}'-\mathbf{p}',\tau'')\tilde{A}_e'(\mathbf{p}'',\tau''')\tilde{A}_f'(\mathbf{k}''-\mathbf{p}'',\tau''')\rangle \end{aligned} \tag{5.1}$$

Using Wick's theorem and ignoring the disconnected pieces we get the following. Note that there is only one term which is completely disconnected, and there are 6 terms which are

partially disconnected. We ignore all of them. The disconnected parts can be visualized as Feynman diagrams with no external lines. These contribute to the vacuum energy. The terms which are partially disconnected represent part of the vacuum energy + part of the two-point function which we are not concerned with here. Note we can tell a disconnected piece is disconnected because, take for example the factor: $\langle \tilde{A}_a^{'}(\mathbf{p},\tau')\tilde{A}_b^{'}(\mathbf{k}-\mathbf{p},\tau')\rangle$, this will be proportional to $\delta^{(3)}(p+k-p)=\delta^{(3)}(k)$. We plug in for the vector fields using eq. (4.60) and plug in for $a=-\frac{1}{H\tau}$:

$$
\begin{aligned}
\langle \tilde{h}_\pm(\mathbf{k},\tau)\tilde{h}_\pm(\mathbf{k}',\tau)\tilde{h}_\pm(\mathbf{k}'',\tau)\rangle = \frac{H^6\xi^{\frac{3}{2}}}{2^3\pi^{\frac{9}{2}}M_P^6}e^{6\pi\xi}\delta^{(3)}(\mathbf{k}+\mathbf{k}'+\mathbf{k}'')\int_{\tau_{end}}^0 d\tau'\int_{\tau_{end}}^0 d\tau''\int_{\tau_{end}}^0 d\tau''' \\
G(\tau,\tau',k)G(\tau,\tau'',k')G(\tau,\tau''',k'')|\tau'|^{\frac{3}{2}}|\tau''|^{\frac{3}{2}}|\tau'''|^{\frac{3}{2}}\Pi_\pm^{ab}(\mathbf{k})\Pi_\pm^{cd}(\mathbf{k}')\Pi_\pm^{ef}(\mathbf{k}'') \\
\int_{-\infty}^{\infty} d^3\mathbf{p}p^{\frac{1}{2}}|\mathbf{k}-\mathbf{p}|^{\frac{1}{2}}e^{-2\sqrt{2\xi|\tau'|}\cdot(p^{\frac{1}{2}}+|\mathbf{k}-\mathbf{p}|^{\frac{1}{2}})}\cdot\Big[|\mathbf{p}+\mathbf{k}'|^{\frac{1}{2}}\cdot \\
\cdot e^{-2\sqrt{2\xi}[|\tau''|^{\frac{1}{2}}\cdot(p^{\frac{1}{2}}+|\mathbf{p}+\mathbf{k}'|^{\frac{1}{2}})+|\tau'''|^{\frac{1}{2}}\cdot(|\mathbf{p}-\mathbf{k}|^{\frac{1}{2}}+|\mathbf{p}+\mathbf{k}'|^{\frac{1}{2}})]}\cdot \\
\cdot\Big[\epsilon_{a+}(\mathbf{p})\epsilon_{d+}^*(\mathbf{p})\epsilon_{b+}(\mathbf{k}-\mathbf{p})\epsilon_{e+}^*(\mathbf{k}-\mathbf{p})\epsilon_{c+}(\mathbf{p}+\mathbf{k}')\epsilon_{f+}^*(\mathbf{p}+\mathbf{k}')+ \\
+\epsilon_{a+}(\mathbf{p})\epsilon_{d+}^*(\mathbf{p})\epsilon_{b+}(\mathbf{k}-\mathbf{p})\epsilon_{f+}^*(\mathbf{k}-\mathbf{p})\epsilon_{c+}(\mathbf{p}+\mathbf{k}')\epsilon_{e+}^*(\mathbf{p}+\mathbf{k}')+ \\
+\epsilon_{a+}(\mathbf{p})\epsilon_{c+}^*(\mathbf{p})\epsilon_{b+}(\mathbf{k}-\mathbf{p})\epsilon_{e+}^*(\mathbf{k}-\mathbf{p})\epsilon_{d+}(\mathbf{p}+\mathbf{k}')\epsilon_{f+}^*(\mathbf{p}+\mathbf{k}')+ \\
+\epsilon_{a+}(\mathbf{p})\epsilon_{c+}^*(\mathbf{p})\epsilon_{b+}(\mathbf{k}-\mathbf{p})\epsilon_{f+}^*(\mathbf{k}-\mathbf{p})\epsilon_{d+}(\mathbf{p}+\mathbf{k}')\epsilon_{e+}^*(\mathbf{p}+\mathbf{k}')\Big]+ \\
+|\mathbf{p}-\mathbf{k}-\mathbf{k}'|^{\frac{1}{2}}e^{-2\sqrt{2\xi}[|\tau''|^{\frac{1}{2}}\cdot(|\mathbf{p}-\mathbf{k}|^{\frac{1}{2}}+|\mathbf{k}''+\mathbf{p}|^{\frac{1}{2}})+|\tau'''|^{\frac{1}{2}}\cdot(p^{\frac{1}{2}}+|\mathbf{k}''+\mathbf{p}|^{\frac{1}{2}})]}\cdot \\
\cdot\Big[\epsilon_{a+}(\mathbf{p})\epsilon_{e+}^*(\mathbf{p})\epsilon_{b+}(\mathbf{k}-\mathbf{p})\epsilon_{c+}^*(\mathbf{k}-\mathbf{p})\epsilon_{d+}(-\mathbf{p}-\mathbf{k}'')\epsilon_{f+}^*(-\mathbf{p}-\mathbf{k}'')+ \\
+\epsilon_{a+}(\mathbf{p})\epsilon_{e+}^*(\mathbf{p})\epsilon_{b+}(\mathbf{k}-\mathbf{p})\epsilon_{d+}^*(\mathbf{k}-\mathbf{p})\epsilon_{c+}(-\mathbf{p}-\mathbf{k}'')\epsilon_{f+}^*(-\mathbf{p}-\mathbf{k}'')+ \\
+\epsilon_{a+}(\mathbf{p})\epsilon_{f+}^*(\mathbf{p})\epsilon_{b+}(\mathbf{k}-\mathbf{p})\epsilon_{c+}^*(\mathbf{k}-\mathbf{p})\epsilon_{d+}(-\mathbf{p}-\mathbf{k}'')\epsilon_{e+}^*(-\mathbf{p}-\mathbf{k}'')+ \\
+\epsilon_{a+}(\mathbf{p})\epsilon_{f+}^*(\mathbf{p})\epsilon_{b+}(\mathbf{k}-\mathbf{p})\epsilon_{d+}^*(\mathbf{k}-\mathbf{p})\epsilon_{c+}(-\mathbf{p}-\mathbf{k}'')\epsilon_{e+}^*(-\mathbf{p}-\mathbf{k}'')\Big]\Big]
\end{aligned}
\tag{5.2}
$$

Now we use the definition of the transverse, traceless projectors: $\Pi_\pm^{ij}(\mathbf{k})=\frac{1}{\sqrt{2}}\epsilon_\mp^i(\mathbf{k})\epsilon_\mp^j(\mathbf{k})$. We notice the above simplifies a lot because the ϵ's appearing in the Π's are interchangeable in a+b and interchangeable in c+d, and interchangeable in e and f. So any line of ϵ's which only differs from another line by an interchange of one or more groups of these variables will

all be equivalent when the sum is taken.

$$\begin{aligned}
\langle \tilde{h}_\pm(\mathbf{k},\tau)\tilde{h}_\pm(\mathbf{k}',\tau)\tilde{h}_\pm(\mathbf{k}'',\tau)\rangle = \frac{H^6\xi^{\frac{3}{2}}}{2^{\frac{9}{2}}\pi^{\frac{9}{2}}M_P^6}e^{6\pi\xi}\delta^{(3)}(\mathbf{k}+\mathbf{k}'+\mathbf{k}'')\int_{\tau_{end}}^0 d\tau'\int_{\tau_{end}}^0 d\tau''\int_{\tau_{end}}^0 d\tau''' \\
G(\tau,\tau',k)G(\tau,\tau'',k')G(\tau,\tau''',k'')|\tau'|^{\frac{3}{2}}|\tau''|^{\frac{3}{2}}|\tau'''|^{\frac{3}{2}}\epsilon^a_\mp(\mathbf{k})\epsilon^b_\mp(\mathbf{k})\epsilon^c_\mp(\mathbf{k}')\epsilon^d_\mp(\mathbf{k}')\epsilon^e_\mp(\mathbf{k}'') \\
\epsilon^f_\mp(\mathbf{k}'')\int_{-\infty}^\infty d^3\mathbf{p}p^{\frac{1}{2}}|\mathbf{k}-\mathbf{p}|^{\frac{1}{2}}e^{-2\sqrt{2\xi|\tau'|}\cdot(p^{\frac{1}{2}}+|\mathbf{k}-\mathbf{p}|^{\frac{1}{2}})}\cdot\Big[4|\mathbf{p}+\mathbf{k}'|^{\frac{1}{2}}\cdot \\
\cdot e^{-2\sqrt{2\xi}[|\tau''|^{\frac{1}{2}}\cdot(p^{\frac{1}{2}}+|\mathbf{p}+\mathbf{k}'|^{\frac{1}{2}})+|\tau'''|^{\frac{1}{2}}\cdot(|\mathbf{p}-\mathbf{k}|^{\frac{1}{2}}+|\mathbf{p}+\mathbf{k}'|^{\frac{1}{2}})]}\cdot \\
\cdot\epsilon_{a+}(\mathbf{p})\epsilon^*_{d+}(\mathbf{p})\epsilon_{b+}(\mathbf{k}-\mathbf{p})\epsilon^*_{e+}(\mathbf{k}-\mathbf{p})\epsilon_{c+}(\mathbf{p}+\mathbf{k}')\epsilon^*_{f+}(\mathbf{p}+\mathbf{k}')+ \\
+4|\mathbf{p}+\mathbf{k}''|^{\frac{1}{2}}e^{-2\sqrt{2\xi}[|\tau''|^{\frac{1}{2}}\cdot(|\mathbf{p}-\mathbf{k}|^{\frac{1}{2}}+|\mathbf{k}''+\mathbf{p}|^{\frac{1}{2}})+|\tau'''|^{\frac{1}{2}}\cdot(p^{\frac{1}{2}}+|\mathbf{k}''+\mathbf{p}|^{\frac{1}{2}})]}\cdot \\
\cdot\epsilon_{a+}(\mathbf{p})\epsilon^*_{e+}(\mathbf{p})\epsilon_{b+}(\mathbf{k}-\mathbf{p})\epsilon^*_{c+}(\mathbf{k}-\mathbf{p})\epsilon_{d+}(-\mathbf{p}-\mathbf{k}'')\epsilon^*_{f+}(-\mathbf{p}-\mathbf{k}'')\Big]
\end{aligned} \tag{5.3}$$

Next we plug in for Green's function, G, given by eq. (1.39) and take the limit $k\tau \to 0$, in other words the late time limit. Then the Green's function simplifies to: $G = \frac{1}{k^3\tau'^2}[k\tau'\cos(k\tau') - \sin(k\tau')]$. Next note that because of the exponential terms, and because we can assume $\xi > 1$, the integrand is strongly suppressed unless $|k\tau'|$, $|k'\tau''|$, $|k''\tau'''| \ll 1$. So we can approximate $G \approx -\frac{\tau'}{3}$. Our three-point function then simplifies to:

$$\begin{aligned}
\langle \tilde{h}_\pm(\mathbf{k},\tau)\tilde{h}_\pm(\mathbf{k}',\tau)\tilde{h}_\pm(\mathbf{k}'',\tau)\rangle = \frac{H^6\xi^{\frac{3}{2}}}{2^{\frac{5}{2}}3^3\pi^{\frac{9}{2}}M_P^6}e^{6\pi\xi}\delta^{(3)}(\mathbf{k}+\mathbf{k}'+\mathbf{k}'')\int_{\tau_{end}}^0 d\tau'\int_{\tau_{end}}^0 d\tau'' \\
\int_{\tau_{end}}^0 d\tau'''|\tau'|^{\frac{5}{2}}|\tau''|^{\frac{5}{2}}|\tau'''|^{\frac{5}{2}}\epsilon^a_\mp(\mathbf{k})\epsilon^b_\mp(\mathbf{k})\epsilon^c_\mp(\mathbf{k}')\epsilon^d_\mp(\mathbf{k}')\epsilon^e_\mp(\mathbf{k}'')\epsilon^f_\mp(\mathbf{k}'') \\
\int_{-\infty}^\infty d^3\mathbf{p}p^{\frac{1}{2}}|\mathbf{k}-\mathbf{p}|^{\frac{1}{2}}e^{-2\sqrt{2\xi|\tau'|}\cdot(p^{\frac{1}{2}}+|\mathbf{k}-\mathbf{p}|^{\frac{1}{2}})}\cdot\Big[|\mathbf{p}+\mathbf{k}'|^{\frac{1}{2}}\cdot \\
\cdot e^{-2\sqrt{2\xi}[|\tau''|^{\frac{1}{2}}\cdot(p^{\frac{1}{2}}+|\mathbf{p}+\mathbf{k}'|^{\frac{1}{2}})+|\tau'''|^{\frac{1}{2}}\cdot(|\mathbf{p}-\mathbf{k}|^{\frac{1}{2}}+|\mathbf{p}+\mathbf{k}'|^{\frac{1}{2}})]}\cdot \\
\cdot\epsilon_{a+}(\mathbf{p})\epsilon^*_{d+}(\mathbf{p})\epsilon_{b+}(\mathbf{k}-\mathbf{p})\epsilon^*_{e+}(\mathbf{k}-\mathbf{p})\epsilon_{c+}(\mathbf{p}+\mathbf{k}')\epsilon^*_{f+}(\mathbf{p}+\mathbf{k}')+ \\
+|\mathbf{p}+\mathbf{k}''|^{\frac{1}{2}}e^{-2\sqrt{2\xi}[|\tau''|^{\frac{1}{2}}\cdot(|\mathbf{p}-\mathbf{k}|^{\frac{1}{2}}+|\mathbf{k}''+\mathbf{p}|^{\frac{1}{2}})+|\tau'''|^{\frac{1}{2}}\cdot(p^{\frac{1}{2}}+|\mathbf{k}''+\mathbf{p}|^{\frac{1}{2}})]}\cdot \\
\cdot\epsilon_{a+}(\mathbf{p})\epsilon^*_{e+}(\mathbf{p})\epsilon_{b+}(\mathbf{k}-\mathbf{p})\epsilon^*_{c+}(\mathbf{k}-\mathbf{p})\epsilon_{d+}(-\mathbf{p}-\mathbf{k}'')\epsilon^*_{f+}(-\mathbf{p}-\mathbf{k}'')\Big].
\end{aligned} \tag{5.4}$$

Next we can solve the time integrals analytically using: $\int_0^\infty |\tau|^{\frac{5}{2}}e^{-c|\tau|^{\frac{1}{2}}}d|\tau| = \frac{1440}{c^7}$. Then we

write the $\mathbf{p}$ integral in spherical coordinates.

$$\begin{aligned}
\langle \tilde{h}_\pm(\mathbf{k},\tau)\tilde{h}_\pm(\mathbf{k}',\tau)\tilde{h}_\pm(\mathbf{k}'',\tau)\rangle = &-\frac{H^6(1440)^3}{2^{34}3^3\pi^{\frac{9}{2}}M_P^6\xi^{\frac{9}{2}}}e^{6\pi\xi}\delta^{(3)}(\mathbf{k}+\mathbf{k}'+\mathbf{k}'')\epsilon^a_\mp(\mathbf{k})\epsilon^b_\mp(\mathbf{k})\epsilon^c_\mp(\mathbf{k}') \\
&\epsilon^d_\mp(\mathbf{k}')\epsilon^e_\mp(\mathbf{k}'')\epsilon^f_\mp(\mathbf{k}'')\int_0^\infty dp\, p^{\frac{5}{2}}\int_0^\pi d\theta\,\sin\theta\int_0^{2\pi}d\phi|\mathbf{k}-\mathbf{p}|^{\frac{1}{2}}(p^{\frac{1}{2}}+|\mathbf{k}-\mathbf{p}|^{\frac{1}{2}})^{-7}\cdot \\
&\cdot\Big[|\mathbf{p}+\mathbf{k}'|^{\frac{1}{2}}(p^{\frac{1}{2}}+|\mathbf{k}'+\mathbf{p}|^{\frac{1}{2}})^{-7}\cdot(|\mathbf{k}-\mathbf{p}|^{\frac{1}{2}}+|\mathbf{k}'+\mathbf{p}|^{\frac{1}{2}})^{-7}\cdot \\
&\cdot\epsilon_{a+}(\mathbf{p})\epsilon^*_{d+}(\mathbf{p})\epsilon_{b+}(\mathbf{k}-\mathbf{p})\epsilon^*_{e+}(\mathbf{k}-\mathbf{p})\epsilon_{c+}(\mathbf{p}+\mathbf{k}')\epsilon^*_{f+}(\mathbf{p}+\mathbf{k}')+ \\
&+|\mathbf{k}''+\mathbf{p}|^{\frac{1}{2}}(|\mathbf{k}-\mathbf{p}|^{\frac{1}{2}}+|\mathbf{k}''+\mathbf{p}|)^{-7}(p^{\frac{1}{2}}+|\mathbf{k}''+\mathbf{p}|^{\frac{1}{2}})^{-7}\cdot \\
&\cdot\epsilon_{a+}(\mathbf{p})\epsilon^*_{e+}(\mathbf{p})\epsilon_{b+}(\mathbf{k}-\mathbf{p})\epsilon^*_{c+}(\mathbf{k}-\mathbf{p})\epsilon_{d+}(-\mathbf{p}-\mathbf{k}'')\epsilon^*_{f+}(-\mathbf{p}-\mathbf{k}'')\Big]
\end{aligned} \tag{5.5}$$

We have the freedom to choose $\mathbf{k} = k\langle 0,0,1\rangle$. Since we are integrating over all $\mathbf{p}$, we can arbitrarily choose $\mathbf{k}$ to be along the z axis without effecting the result of the integral.

Let $\mathbf{k}' = k'\langle \sin\bar{\theta}\cos\bar{\phi}, \sin\bar{\theta}\sin\bar{\phi}, \cos\bar{\theta}\rangle$ and $\mathbf{k}'' = k''\langle \sin\bar{\theta}'\cos\bar{\phi}', \sin\bar{\theta}'\sin\bar{\phi}', \cos\bar{\theta}'\rangle$

We can further choose $\bar{\phi}$ to equal zero. This is another arbitrary degree of freedom since it just chooses the placement of $\mathbf{k}'$ over the $x-y$ plane, and since we are integrating over all $\mathbf{p}$, this will not change the final integral. This also means that to ensure $\delta^{(3)}(\mathbf{k}+\mathbf{k}'+\mathbf{k}'')$, $\bar{\phi}' = \pi$. So this gives us:

$\mathbf{k}' = k'\langle \sin\bar{\theta}, 0, \cos\bar{\theta}\rangle$ and $\mathbf{k}'' = k''\langle -\sin\bar{\theta}', 0, \cos\bar{\theta}'\rangle$.

Now we really only have two free parameters above instead of 4, because once $\bar{\theta}$ and k' are chosen, then $\bar{\theta}'$ and k'' are fixed from requiring $\delta^{(3)}(\mathbf{k}+\mathbf{k}'+\mathbf{k}'')$. But we do not really want our 2 free parameters to be $\bar{\theta}$ and k'. Instead it is conventional to use x_2 and x_3 where $x_2 = \frac{k'}{k}$ and $x_3 = \frac{k''}{k}$. It is also conventional to define k, k', and k'' such that $k \geq k' \geq k''$. This will make it easier to later plot the shape of the three-point function.

So next want to find equations for $\bar{\theta}$ and $\bar{\theta}'$ in terms of x_2 and x_3. With the requirement that $k \geq k' \geq k''$ we can relate the angles to the magnitudes of the momentum vectors using the law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\gamma$ where a, b, and c are sides of a triangle and γ is the angle across from side c. We find:

$$\bar{\theta} = \pi - \cos^{-1}\left[\frac{1+x_2^2-x_3^2}{2x_2}\right] \tag{5.6}$$

and

$$\bar{\theta}' = \pi - \cos^{-1}\left[\frac{1+x_3^2-x_2^2}{2x_3}\right]. \tag{5.7}$$

Next we want to rescale the momenta in the integral above by the magnitude k. This will make the p integral into a dimensionless integral that can be integrated numerically.

Let $\bar{p} = \frac{p}{k}$.

$$\langle \tilde{h}_{\pm}(\mathbf{k},\tau)\tilde{h}_{\pm}(\mathbf{k}',\tau)\tilde{h}_{\pm}(\mathbf{k}'',\tau)\rangle = -\frac{H^6(1440)^3}{2^{34}3^3\pi^{\frac{9}{2}}M_P^6\xi^{\frac{9}{2}}k^6}e^{6\pi\xi}\delta^{(3)}(\mathbf{k}+\mathbf{k}'+\mathbf{k}'')\epsilon^a_{\mp}(\mathbf{k})\epsilon^b_{\mp}(\mathbf{k})\epsilon^c_{\mp}(\mathbf{k}')$$
$$\epsilon^d_{\mp}(\mathbf{k}')\epsilon^e_{\mp}(\mathbf{k}'')\epsilon^f_{\mp}(\mathbf{k}'')\int_0^{\infty}d\bar{p}\,\bar{p}^{\frac{5}{2}}\int_0^{\pi}d\theta\,\sin\theta\int_0^{2\pi}d\phi|\bar{\mathbf{k}}-\bar{\mathbf{p}}|^{\frac{1}{2}}(\bar{p}^{\frac{1}{2}}+|\bar{\mathbf{k}}-\bar{\mathbf{p}}|^{\frac{1}{2}})^{-7}\cdot$$
$$\cdot\Big[|\bar{\mathbf{p}}+\bar{\mathbf{k}}'|^{\frac{1}{2}}(\bar{p}^{\frac{1}{2}}+|\bar{\mathbf{k}}'+\bar{\mathbf{p}}|^{\frac{1}{2}})^{-7}\cdot(|\bar{\mathbf{k}}-\bar{\mathbf{p}}|^{\frac{1}{2}}+|\bar{\mathbf{k}}'+\bar{\mathbf{p}}|^{\frac{1}{2}})^{-7}\cdot$$
$$\cdot\,\epsilon_{a+}(\mathbf{p})\epsilon^*_{d+}(\mathbf{p})\epsilon_{b+}(\mathbf{k}-\mathbf{p})\epsilon^*_{e+}(\mathbf{k}-\mathbf{p})\epsilon_{c+}(\mathbf{p}+\mathbf{k}')\epsilon^*_{f+}(\mathbf{p}+\mathbf{k}')+$$
$$+\,|\bar{\mathbf{k}}''+\bar{\mathbf{p}}|^{\frac{1}{2}}(|\bar{\mathbf{k}}-\bar{\mathbf{p}}|^{\frac{1}{2}}+|\bar{\mathbf{k}}''+\bar{\mathbf{p}}|)^{-7}(\bar{p}^{\frac{1}{2}}+|\bar{\mathbf{k}}''+\bar{\mathbf{p}}|^{\frac{1}{2}})^{-7}\cdot$$
$$\cdot\,\epsilon_{a+}(\mathbf{p})\epsilon^*_{e+}(\mathbf{p})\epsilon_{b+}(\mathbf{k}-\mathbf{p})\epsilon^*_{c+}(\mathbf{k}-\mathbf{p})\epsilon_{d+}(-\mathbf{p}-\mathbf{k}'')\epsilon^*_{f+}(-\mathbf{p}-\mathbf{k}'')\Big] \tag{5.8}$$

where

$\bar{\mathbf{k}} = \langle 0,0,1\rangle$, $\bar{\mathbf{k}}' = x_2\langle \sin\bar{\theta}, 0, \cos\bar{\theta}\rangle$ and $\bar{\mathbf{k}}'' = x_3\langle -\sin\bar{\theta}', 0, \cos\bar{\theta}'\rangle$.

Also note:

$|\bar{\mathbf{k}}-\bar{\mathbf{p}}| = (1+\bar{p}^2-2\bar{p}\cos\theta)^{\frac{1}{2}}$ $|\bar{\mathbf{p}}+\bar{\mathbf{k}}'| = (x_2^2+\bar{p}^2+2\bar{p}x_2(\sin\bar{\theta}\sin\theta\cos\phi+\cos\bar{\theta}\cos\theta))^{\frac{1}{2}}$ $|\bar{\mathbf{p}}+\bar{\mathbf{k}}''| = (x_3^2+\bar{p}^2+2\bar{p}x_3(-\sin\bar{\theta}'\sin\theta\cos\phi+\cos\bar{\theta}'\cos\theta))^{\frac{1}{2}}$

using that $\bar{\mathbf{p}}$ is given by the vector: $\bar{\mathbf{p}} = \bar{p}\langle\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\rangle$.

Next, we need the position angles of the vectors $\mathbf{k}-\mathbf{p}$, $\mathbf{k}'+\mathbf{p}$, and $\mathbf{k}''+\mathbf{p}$ in order that we can calculate the epsilon polarization vectors of these directions. To do this, we use that θ of an arbitrary vector is given by $\theta = \cos^{-1}($ z-component/ magnitude). Similarly, ϕ is given by $\phi = \tan^{-1}$ (y-component/ x-component) $+ \pi * \Theta($- x-component), where Θ is the Heaviside theta function and is added to ensure that the ϕ we obtain is located in the correct quadrant. Using these descriptions we obtain:

$$\theta_{\mathbf{k}-\mathbf{p}} = \cos^{-1}\left(\frac{1-\bar{p}\cos\theta}{(1+\bar{p}^2-2\bar{p}\cos\theta)^{\frac{1}{2}}}\right) \tag{5.9}$$

$$\phi_{\mathbf{k}-\mathbf{p}} = \phi+\pi \tag{5.10}$$

$$\theta_{\mathbf{k}'+\mathbf{p}} = \cos^{-1}\left(\frac{x_2\cos\bar{\theta}+\bar{p}\cos\theta}{(x_2^2+\bar{p}^2+2\bar{p}x_2(\sin\bar{\theta}\sin\theta\cos\phi+\cos\bar{\theta}\cos\theta))^{\frac{1}{2}}}\right) \tag{5.11}$$

$$\phi_{\mathbf{k}'+\mathbf{p}} = \tan^{-1}\left(\frac{\bar{p}\sin\theta\sin\phi}{x_2\sin\bar{\theta}+\bar{p}\sin\theta\cos\phi}\right)+\pi\Theta(-\bar{p}\sin\theta\cos\phi-x_2\sin\bar{\theta}) \tag{5.12}$$

$$\theta_{\mathbf{k}''+\mathbf{p}} = \cos^{-1}\left(\frac{x_3\cos\bar{\theta}'+\bar{p}\cos\theta}{(x_3^2+\bar{p}^2+2\bar{p}x_3(-\sin\bar{\theta}'\sin\theta\cos\phi+\cos\bar{\theta}'\cos\theta))^{\frac{1}{2}}}\right) \tag{5.13}$$

$$\phi_{\mathbf{k}''+\mathbf{p}} = \tan^{-1}\left(\frac{\bar{p}\sin\theta\sin\phi}{-x_3\sin\bar{\theta}'+\bar{p}\sin\theta\cos\phi}\right)+\pi\Theta(-\bar{p}\sin\theta\cos\phi+x_3\sin\bar{\theta}') \tag{5.14}$$

Then we use that the polarization vectors are defined such that for an arbitrary vector $\mathbf{a}$, specified by the position angles θ and ϕ, its polarization vectors are given by:

$$\epsilon_+(\mathbf{a}) = \frac{1}{\sqrt{2}} \begin{pmatrix} -\cos\theta\,\cos\phi + i\sin\phi \\ -\cos\theta\,\sin\phi - i\cos\phi \\ \sin\theta \end{pmatrix}$$

$$\epsilon_-(\mathbf{a}) = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta\,\cos\phi + i\sin\phi \\ \cos\theta\,\sin\phi - i\cos\phi \\ -\sin\theta \end{pmatrix}$$

.

Putting all this together we have a dimensionless integral that can be solved numerically once x_2 and x_3 are chosen. Changing x_2 and x_3 are equivalent to changing the shape of the non-Gaussianity. Below is a plot of the shape of the non-Gaussianity, the amount that the three-point function varies when x_2 and x_3 are changed. The bispectrum is conventionally defined as related to the two point function by:

$$\langle h(\mathbf{k})\, h(\mathbf{k}')\, h(\mathbf{k}'')\rangle = (2\pi)^3 \delta^{(3)}(\mathbf{k}+\mathbf{k}'+\mathbf{k}'') B_h(k,k',k''), \tag{5.15}$$

where the bispectrum, $B_h(k,k',k'')$, only depends on the magnitude of the 3 momentum vectors. Then the shape is defined by:

$$B_h(k,k',k'') = \frac{S(k,k',k'')}{(k\,k'\,k'')^2} F(k) \tag{5.16}$$

So the bispectrum can be broken into a term $F(k)$ which depends on the overall magnitude scale, and a shape function S which is dimensionless and invariant on rescaling of the overall momentum scale. In other words, S only depends on the ratios $x_2 = \frac{k'}{k}$ and $x_3 = \frac{k''}{k}$. The shape function is conventionally normalized to give 1 in the equilateral configuration ($x_2 = x_3 = 1$).

When we plot the shape function for various combinations of x_2 and x_3, we obtain Figure 5.1. We get a clear maximum in the equilateral configuration. The reason for this is because the integrand gets peaked for modes which are horizon sized. In other words, these vectors source tensor perturbations most efficiently when the vectors have wavelengths that are just about to leave the horizon. Therefore, the integrand as a whole is maximized when all three modes reach horizon size at the same time, which means that they are all the same size at the same time, which corresponds to the equilateral limit. Next we calculate the magnitude of the three-point function in the equilateral limit. Taking $x_2 = 1$ and $x_3 = 1$, we compute the integral above,eq. (5.8), numerically and obtain:.

$$\langle \tilde{h}_+(\mathbf{k},0)\tilde{h}_+(\mathbf{k}',0)\tilde{h}_+(\mathbf{k}'',0)\rangle_{eq} = 6.13\times 10^{-10}\frac{H^6}{M_P^6\xi^9 k^6} e^{6\pi\xi}\delta^{(3)}(\mathbf{k}+\mathbf{k}'+\mathbf{k}''). \tag{5.17}$$

For reference we also compute the opposite helicity three-point function and obtain:

$$\langle \tilde{h}_-(\mathbf{k},0)\tilde{h}_-(\mathbf{k}',0)\tilde{h}_-(\mathbf{k}'',0)\rangle_{eq} = 3.5801\times 10^{-14}\frac{H^6}{M_P^6\xi^9 k^6} e^{6\pi\xi}\delta^{(3)}(\mathbf{k}+\mathbf{k}'+\mathbf{k}'') \tag{5.18}$$

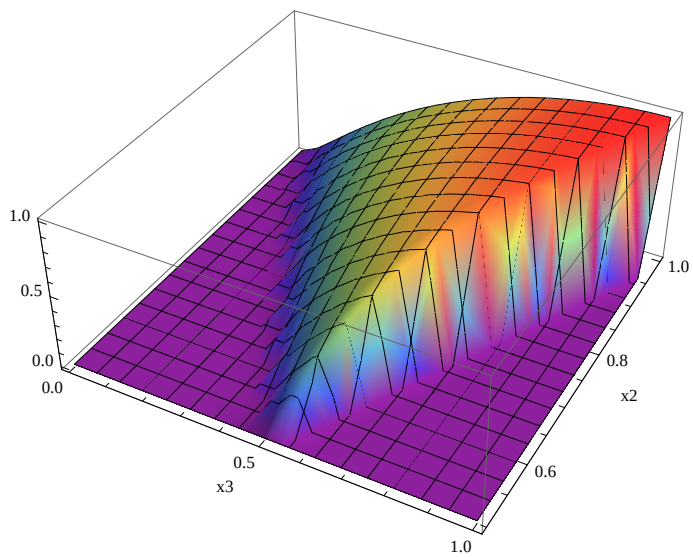

Figure 5.1: The figure shows the shape of $\langle h_+ h_+ h_+ \rangle$. A clear peak in the equilateral limit where $x_2 = x_3 = 1$ can be seen.

This shows as one would expect, the three-point function for the negative helicity mode is greatly suppressed. The cross term three-point functions, for example like $\langle h_+ h_+ h_- \rangle$, are all 0. If one of the three point functions might give an observable signal it will be the $\langle h_+ h_+ h_+ \rangle$ mode in the equilateral limit.

To give a better measure for how strong the non-Gaussianities are from this model, we compute a dimensionless ratio of the three-point function to the two-point function raised to the $\frac{3}{2}$ power. We can relate this to a dimensionless parameter which we call C_{NL} which ranges from 0 to 1, with 1 corresponding to maximal non-Gaussianity. We define this C_{NL} by:

$$C_{NL+eq} = \frac{\langle \tilde{h}_+(\mathbf{k},0)\tilde{h}_+(\mathbf{k}',0)\tilde{h}_+(\mathbf{k}'',0)\rangle_{eq} \frac{k^6}{\delta^{(3)}(\mathbf{k}+\mathbf{k}'+\mathbf{k}'')}}{(\langle \tilde{h}_+(\mathbf{k},0)\tilde{h}_+(\mathbf{k}',0)\rangle \frac{k^3}{\delta^{(3)}(\mathbf{k}+\mathbf{k}')})^{\frac{3}{2}}} \,. \tag{5.19}$$

We then obtain:

$$C_{NL+eq} = 0.77 \,. \tag{5.20}$$

Having a C_{NL} which is order one means this model is maximally non-Gaussian.

The next goal is to relate this non-Gaussianity to observables. Non-Gaussianities in the CMB are conventionally parametrized by a constant f_{NL} given by

$$f_{NL} = \frac{10(2\pi)^{\frac{1}{2}}}{9} \frac{B}{P^2} \,, \tag{5.21}$$

where the bispectrum B is defined by eq. (5.15) and P is the power spectrum. If we were computing scalar curvature perturbations then we would need only use our $\langle \zeta\zeta\zeta \rangle$ three-point function to obtain the bispectrum B, and we would use the standard scalar power spectrum $P_\zeta = 2.5 \times 10^{-9}$ and eq. (5.21) would give us an f_{NL} we could compare directly with Planck

data. This process of comparing with Planck data becomes more complicated when dealing with tensors. What Planck actually measures when computing limits on f_{NL} is $\langle TTT \rangle$ and the temperature power spectrum, where it is known that scalar perturbations dominate the temperature power spectrum. Since the strength of f_{NL} is limited by the strength of the temperature power spectrum, and this power spectrum is dominated by scalar contributions, it is enough to just take the dimensionless ratio of $\langle \zeta\zeta\zeta \rangle$ and $\langle \zeta\zeta \rangle^2$ to compute f_{NL}. If we want to place a limit from tensors though, we need to convert into using the temperature bispectrum and temperature power spectrum. We can not compare a tensor bispectrum directly to a scalar power spectrum, and although we could define an analogous f_{NL} for the dimensionless ratio of $\langle hhh \rangle$ to $\langle hh \rangle^2$, this would be irrelevant for comparison with Planck's temperature f_{NL} since this uses the temperature power spectrum dominated by scalars. We could use such an analogous f_{NL} to compare to limits of non-Gaussianities from the non-observation of a B mode three-point function since only tensors would contribute to this signal, but since the B mode two-point function has yet to be observed and the three-point function will be much weaker still, this will not give us as good of a limit. Instead we want to compare the strength of the non-Gaussianities in this model to Planck's strong f_{NL} limits. To do this, we need to calculate how much our tensor signal from this model will contribution to the bispectrum of $\langle TTT \rangle$, and then take the ratio of this with the total power spectrum from $\langle TT \rangle$ dominated by the scalar perturbations squared. Since the standard inflationary models robustly predict small non-Gaussianities, with f_{NL} proportional to the slow roll parameters, it is possible the signal from our model could dominate the contributions to $\langle TTT \rangle$, even though $\langle TT \rangle$ will be dominated by other signals.

CHAPTER 6

TENSOR CONTRIBUTION TO TEMPERATURE ANISOTROPIES ON THE CMB FROM PROLONGED DECAY OF THE INFLATON INTO VECTORS

6.1 Temperature Power Spectrum from Tensors

As shown in previous sections, a $\phi F\tilde{F}$ interaction between the inflaton field, ϕ, and a vector field during slow roll could produce an observable signal of tensor perturbations at direct detection experiments. The detectability of such a signal will depend on the parameter ξ, defined in eq. (4.23), which is slowly increasing during inflation. The best current limits on ξ come from requiring non-detection of non-Gaussianity from the scalar perturbations sourced by this model at CMB scales, and from requiring that backreaction on the inflaton field to not be significant. As opposed to the standard inflationary models, this model of particle production during inflation produces large non-Gaussianities in both the scalar and the tensor metric perturbations. It would be interesting to calculate the contribution from the tensor perturbations to the temperature three-point function and see if it could produce an observable signal, or a stronger limit than is currently being reported in [25] for the contribution from scalar perturbations.

This is especially relevant in the context proposed in [21] where the field coupled to the vectors ϕ is no longer the inflaton field but another field slowly rolling during inflation. The idea for this slightly different model is that since the strongest limit on ξ comes from the non-observation of non-Gaussianities of scalar perturbations, and these scalar perturbations are generated as a consequence of the vector fields backreacting on the scalar inflaton field sourcing these scalar perturbations, a more lenient limit could be obtained if these vectors did not couple directly to the inflaton field. Not allowing them to couple to the inflaton field at all provides a best case scenario for generating large tensor perturbations while minimizing the contribution to the scalar perturbations. The vector field will always couple to the inflaton field gravitationally, but the strength of a direct coupling, for example in our model above, the strength of the coupling constant f, is arbitrary. So not allowing any direct coupling will minimize the scalar perturbations produced from this model and therefore give a weaker limit on ξ.

Reference [21] reports that with this more lenient limit of ξ at CMB scales from the

non-observation of non-Gaussianities, they are able to get an observable signal of the tensor perturbations even at CMB scales. This is to say, a tensor signal from this model which is larger than the tensor signal from the standard inflationary signal. It would be interesting in this context in which the tensor modes are enhanced relative to the scalar modes, to calculate what the limit on ξ should be from the non-observation of non-Gaussianities from the tensor contribution to $\langle TTT \rangle$. In this context where ϕ is not the inflaton, and the tensor signal is greatly enhanced relative to the scalar perturbation signal; the stronger limit on ξ should be from the lack of an observation of non-Gaussianities from the tensor contribution rather than the scalar.

So we next seek to calculate the tensor contribution from this model to $\langle TTT \rangle$. We start first by calculating $\langle TT \rangle$. We define the temperature perturbations by:

$$\frac{\delta T}{T}(\mathbf{x}_0, \mathbf{l}, \eta_0) = -\frac{1}{2}\int_{\eta_r}^{\eta_0} \frac{\partial h_{ij}(\mathbf{x}(\eta), \eta)}{\partial \eta} l^i l^j d\eta \,, \tag{6.1}$$

where η_0 is the time that we are observing these photons (now), and $\mathbf{x}_0$ is the position we are observing them from (here, Earth). η_r is the time of recombination. $\mathbf{l}$ is the direction in the sky we have to be looking to see the photon, the opposite of the photon momentum direction. $\mathbf{l}$ is a unit vector. Also note that $\mathbf{x}(\eta)$ in $h_{ij}(\mathbf{x}(\eta), \eta)$ is not equal to $\mathbf{x}_0$, the position we are viewing from. We are integrating in time, from last scattering till now. We take h_{ij} at last scattering, and then we evolve it till now by doing this time integral. The position of these gravitons is continually changing with time as they travel towards us (in direction $-\mathbf{l}$) at the speed of light. So we do not want $h_{ij}(\mathbf{x}_0)$, the position they are at now, because this is only true now.

Also note by integrating from η_r to η_0 we include gravitational waves that reentered the horizon any time between re-scattering and now. The longest wavelength modes would enter closer to the present, the shortest mode we consider is the mode that re-enters right at re-scattering. The contribution from h_{ij} to the integral before a mode enters the horizon is zero, so the integral is effectively from time of entry into horizon to the present for each mode. Now we could integrate further back to include modes which entered the horizon before re-scattering, but these really short wavelength modes do not contribute that much to the gravitational wave signal. The amplitude of gravitational waves falls off proportional to $\frac{1}{a}$ once they enter the horizon. The earlier a wave energy the horizon, the shorter the wavelength, the harder it will be to detect.

We will just use h_{ij} from matter domination. The equation changes for radiation domination, and there is a short period of radiation domination after re-scattering before matter domination starts, but as we said, 1. the signal from these waves will be very small and 2. this period between re-scattering and the start of matter domination is very short.

We can now take the temperature two-point function conventionally defined as $C(\theta)$:

$$C(\bar{\theta}) = \left\langle \frac{\delta T}{T}(\mathbf{x}_0, \mathbf{l_1}, \eta_0)\, \frac{\delta T}{T}(\mathbf{x}_0, \mathbf{l_2}, \eta_0) \right\rangle \,, \tag{6.2}$$

where $\bar{\theta}$ is the angle between $\mathbf{l_1}$ and $\mathbf{l_2}$. Note $\bar{\theta}$ is given; I am putting a bar over it to distinguish it from an integration variable θ that will appear later. η_0 and $\mathbf{x}_0$ have to be the same for both terms above because we can only compare photons incident on us at the same spot at the same time. We can compare photons arriving to us from different directions

though, different $\mathbf{l_1}$ and $\mathbf{l_2}$. We expand the temperature two-point function using eq. (6.1):

$$C(\theta) = \frac{1}{4} l_1^i l_1^j l_2^l l_2^m \int_{\eta_r}^{\eta_0} d\eta \int_{\eta_r}^{\eta_0} d\tilde{\eta} \langle \frac{\partial h_{ij}(\mathbf{x}(\eta), \eta)}{\partial \eta} \frac{\partial h_{lm}(\mathbf{x}(\tilde{\eta}), \tilde{\eta})}{\partial \tilde{\eta}} \rangle \,. \tag{6.3}$$

We next Fourier transform our metric perturbations into momentum space using: $h_{ij}(\mathbf{x}, \eta) = \int_{-\infty}^{\infty} \frac{d^3\mathbf{q}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{q}\cdot\mathbf{x}} h_{ij}(\mathbf{q}, \eta)$ to obtain:

$$C(\bar{\theta}) = \frac{1}{4} l_1^i l_1^j l_2^l l_2^m \int_{\eta_r}^{\eta_0} d\eta \int_{\eta_r}^{\eta_0} d\tilde{\eta} \int_{-\infty}^{\infty} \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} \frac{d^3\tilde{\mathbf{k}}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot\mathbf{x}(\eta)} e^{i\tilde{\mathbf{k}}\cdot\mathbf{x}(\tilde{\eta})} \frac{\partial}{\partial \eta} \frac{\partial}{\partial \tilde{\eta}} \langle h_{ij}(\mathbf{k}, \eta) h_{lm}(\tilde{\mathbf{k}}, \tilde{\eta}) \rangle \,. \tag{6.4}$$

We next use that $x^i(\eta) = x_0^i + l^i(\eta - \eta_0)$, where $\mathbf{x_0}$ is the the position now ($\mathbf{x_0} = \mathbf{x}(\eta_0)$) and therefore $\mathbf{x_0}$ is a constant. We use that the gravitational wave is traveling at the speed of light, and we use units where $c = 1$. The direction of motion is $-\mathbf{l}$. Just using $x = x_0 + v_0 \cdot t$ we obtain $x^i(\eta) = x_0^i + l^i(\eta - \eta_0)$. Plugging this into the above we obtain:

$$\begin{aligned} C(\bar{\theta}) =& \frac{1}{4} l_1^i l_1^j l_2^l l_2^m \int_{\eta_r}^{\eta_0} d\eta \int_{\eta_r}^{\eta_0} d\tilde{\eta} \int_{-\infty}^{\infty} \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} \frac{d^3\tilde{\mathbf{k}}}{(2\pi)^{\frac{3}{2}}} e^{i\mathbf{k}\cdot(\mathbf{x_0} + \mathbf{l_1}(\eta - \eta_0))} e^{i\tilde{\mathbf{k}}\cdot(\mathbf{x_0} + \mathbf{l_2}(\tilde{\eta} - \eta_0))} \frac{\partial}{\partial \eta} \frac{\partial}{\partial \tilde{\eta}} \\ & \langle h_{ij}(\mathbf{k}, \eta) h_{lm}(\tilde{\mathbf{k}}, \tilde{\eta}) \rangle \end{aligned} \tag{6.5}$$

Next we want to plug in for h_{ij} for a matter dominated epoch. We can derive h_{ij} from solving Einstein's equation. In a matter dominated universe with $a \propto \eta^2$ this gives:

$$h_{ij}^{''} + \frac{4}{\eta} h_{ij}^{'} - \nabla^2 h_{ij} = \frac{2}{M_P^2} \Pi_{ij}^{lm} T_{lm} \,. \tag{6.6}$$

We will take the vacuum equation with $T_{lm} = 0$. Nothing is sourcing these tensor perturbations to first order in this epoch. Solving this differential equation gives:

$$h_{mat,\,ij}(k, \eta) = \frac{C_{1\,ij}}{k^2\eta^2}(-\cos(k\eta) + \frac{1}{k\eta}\sin(k\eta)) + \frac{C_{2\,ij}}{k^2\eta^2}(-\sin(k\eta) - \frac{1}{k\eta}\cos(k\eta)) \,. \tag{6.7}$$

We then apply boundary conditions. The first is that the amplitude of h_{ij} is constant, 'frozen in,' for modes well outside the horizon (when $k\eta \ll 1$). This gives $C_{2ij} = 0$. The second boundary condition is our initial condition. The initial condition is the amplitude of these modes during inflation when they first exited the horizon. Since the modes are constant while they are outside the horizon, their amplitude when they exit the horizon during inflation is the same as their amplitude when they first reenter the horizon during matter domination. We can match boundary conditions for the two regions, using the amplitude of metric perturbations during inflation as generated by our $\phi F\tilde{F}$ model, eq. (4.54).

Let η at the end of inflation $= \eta_{ei}$. Matching boundary conditions between eq. (6.7) and eq. (4.54) we find:

$$\begin{aligned} C_{1\,ij} =& -\frac{1}{(-\frac{1}{k^2\eta_{ei}^2}\cos(k\eta_{ei}) + \frac{1}{k^3\eta_{ei}^3}\sin(k\eta_{ei}))} \frac{2}{M_P^2(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\eta_{ei}} d\eta' \frac{1}{a^2(\eta')} G(\eta_{ei}, \eta', k) \Pi_{ij}^{lm}(\mathbf{k}) \\ & \int_{-\infty}^{\infty} d^3\mathbf{q} \hat{A}_l^{'}(\mathbf{q}, \eta') \hat{A}_m^{'}(\mathbf{k} - \mathbf{q}, \eta') \,. \end{aligned} \tag{6.8}$$

We can simplify by noting that $k\eta_{ei} \ll 1$ (since all modes we are considering were definitely outside the horizon at the end of inflation.) We can use this to approximate $-\frac{1}{k^2\eta_{ei}^2}\cos(k\eta_{ei}) + \frac{1}{k^3\eta_{ei}^3}\sin(k\eta_{ei}))^2 \approx \frac{1}{3}$ Then plugging this back into our equation for $h_{mat\,ij}$ we obtain:

$$h_{mat,\,ij}(\mathbf{k},\eta) = -3(-\frac{1}{k^2\eta^2}\cos(k\eta) + \frac{1}{k^3\eta^3}\sin(k\eta))\frac{2}{M_P^2(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\eta_{ei}} d\eta' \frac{1}{a^2(\eta')}G(\eta_{ei},\eta',k)$$
$$\Pi_{ij}^{lm}(\mathbf{k})\int_{-\infty}^{\infty} d^3\mathbf{q}\hat{A}_l^{'}(\mathbf{q},\eta')\hat{A}_m^{'}(\mathbf{k}-\mathbf{q},\eta')\,. \tag{6.9}$$

Then we plug this into our $C(\bar{\theta})$ expression:

$$C(\bar{\theta}) = \frac{3^2}{4(2\pi)^3} l_1^i l_1^j l_2^l l_2^m \int_{\eta_r}^{\eta_0} d\eta \int_{\eta_r}^{\eta_0} d\tilde{\eta} \int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} d^3\mathbf{k} \int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} d^3\tilde{\mathbf{k}}\, e^{i\mathbf{k}\cdot(\mathbf{x_0}+\mathbf{l_1}(\eta-\eta_0))} e^{i\tilde{\mathbf{k}}\cdot(\mathbf{x_0}+\mathbf{l_2}(\tilde{\eta}-\eta_0))}\frac{\partial}{\partial\eta}\frac{\partial}{\partial\tilde{\eta}}$$
$$(-\frac{1}{k^2\eta^2}\cos(k\eta) + \frac{1}{k^3\eta^3}\sin(k\eta))(-\frac{1}{\tilde{k}^2\tilde{\eta}^2}\cos(\tilde{k}\tilde{\eta}) + \frac{1}{\tilde{k}^3\tilde{\eta}^3}\sin(\tilde{k}\tilde{\eta}))\frac{4}{M_P^4(2\pi)^3}$$
$$\int_{-\infty}^{\eta_{ei}} d\eta' \int_{-\infty}^{\eta_{ei}} d\eta'' \frac{1}{a^2(\eta')}\frac{1}{a^2(\eta'')}G(\eta_{ei},\eta',k)G(\eta_{ei},\eta'',\tilde{k})\Pi_{ij}^{ab}(\mathbf{k})\Pi_{lm}^{cd}(\tilde{\mathbf{k}})$$
$$\int_{-\infty}^{\infty} d^3\mathbf{q}\int_{-\infty}^{\infty} d^3\tilde{\mathbf{q}}\langle\hat{A}_a^{'}(\mathbf{q},\eta')\hat{A}_b^{'}(\mathbf{k}-\mathbf{q},\eta')\hat{A}_c^{'}(\tilde{\mathbf{q}},\eta^{''})\hat{A}_d^{'}(\tilde{\mathbf{k}}-\tilde{\mathbf{q}},\eta^{''})\rangle\,. \tag{6.10}$$

To discuss the range of k modes we are considering, we want modes which re-enter the horizon after or at recombination so $k\eta_r \leq 1$ and $k \leq \frac{1}{\eta_r}$. We also want modes inside the horizon now so $k\eta_0 > 1$ and $\frac{1}{\eta_0} < k$. Putting these two requirements together: $\frac{1}{\eta_0} < k \leq \frac{1}{\eta_r}$. Next we want to plug in for $\hat{A}_i(\mathbf{k},\eta)$, and we use the result from earlier, eq. (4.60):

$$\hat{A}_i^{'}(\mathbf{k},\eta) \approx 2^{-\frac{1}{4}}\,k^{\frac{1}{4}}\,a^{-\frac{7}{4}}(\eta)\,\xi^{\frac{1}{4}}(\eta)\,H^{-\frac{3}{4}}(\eta)\,a^{'}(\eta)\,e^{\pi\xi(\eta)-2\sqrt{\frac{2\,k\,\xi(\eta)}{a(\eta)\,H(\eta)}}}[\epsilon_{i+}(\mathbf{k})\hat{a}_+(\mathbf{k}) + \epsilon_{i+}^*(-\mathbf{k})a_+^\dagger(-\mathbf{k})] \tag{6.11}$$

I am explicitly showing here what variables depend on time. But note, these A fields as appear above, are integrated over time from $\eta = -\infty$ to η_{ei}. In other words, these functions are only integrated over inflation and therefore H and ξ are approximately constant during this interval, and a takes on its inflationary form, $a = -\frac{1}{H\eta}$. Since this is the only place H and ξ appear in the equation for the temperature two-point function, I will not write out $H(\eta_{\text{inflation}})$ and $\xi(\eta_{\text{inflation}})$ each time, but just treat H and ξ as constants with the knowledge that each time they appear, they take the value they had during inflation.

After plugging in for $a = -\frac{1}{H\eta}$ in the $A^{'}$ expression, the expression simplifies to:

$$\hat{A}_i^{'}(\mathbf{k},\eta) \approx 2^{-\frac{1}{4}}\,k^{\frac{1}{4}}\,\xi^{\frac{1}{4}}\,(-\eta)^{-\frac{1}{4}}\,e^{\pi\xi-2\sqrt{-2k\xi\eta}}[\epsilon_{i+}(\mathbf{k})\hat{a}_+(\mathbf{k}) + \epsilon_{i+}^*(-\mathbf{k})a_+^\dagger(-\mathbf{k})]\,. \tag{6.12}$$

Note η is negative during inflation, so the term under the square root is actually positive. Then we use Wick's theorem to expand the 4 pt function, ignoring the disconnected piece.

Then plug in for A and simplify:

$$
\begin{aligned}
C(\bar{\theta}) = &- \frac{3^2 \xi H^4}{2(2\pi)^6 M_P^4} e^{4\pi\xi} l_1^i l_1^j l_2^l l_2^m \int_{\eta_r}^{\eta_0} d\eta \int_{\eta_r}^{\eta_0} d\tilde{\eta} \int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} d^3\mathbf{k} \int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} d^3\tilde{\mathbf{k}}\, e^{i\mathbf{k}\cdot(\mathbf{x_0}+\mathbf{l_1}(\eta-\eta_0))} e^{i\tilde{\mathbf{k}}\cdot(\mathbf{x_0}+\mathbf{l_2}(\tilde{\eta}-\eta_0))} \\
&\frac{\partial}{\partial\eta}\frac{\partial}{\partial\tilde{\eta}}(-\frac{1}{k^2\eta^2}\cos(k\eta) + \frac{1}{k^3\eta^3}\sin(k\eta))(-\frac{1}{\tilde{k}^2\tilde{\eta}^2}\cos(\tilde{k}\tilde{\eta}) + \frac{1}{\tilde{k}^3\tilde{\eta}^3}\sin(\tilde{k}\tilde{\eta})) \int_{-\infty}^{\eta_{ei}} d\eta' \\
&\int_{-\infty}^{\eta_{ei}} d\eta'' \eta'^{\frac{3}{2}} \eta''^{\frac{3}{2}} G(\eta_{ei}, \eta', k) G(\eta_{ei}, \eta'', \tilde{k}) \Pi_{ij}^{ab}(\mathbf{k}) \Pi_{lm}^{cd}(\tilde{\mathbf{k}}) \\
&\int_{-\infty}^{\infty} d^3\mathbf{q} \int_{-\infty}^{\infty} d^3\tilde{\mathbf{q}} q^{\frac{1}{4}} |\mathbf{k}-\mathbf{q}|^{\frac{1}{4}} \tilde{q}^{\frac{1}{4}} |\tilde{\mathbf{k}}-\tilde{\mathbf{q}}|^{\frac{1}{4}} e^{-2\sqrt{-2q\xi\eta'}} e^{-2\sqrt{-2|\mathbf{k}-\mathbf{q}|\xi\eta'}} e^{-2\sqrt{-2\tilde{q}\xi\eta''}} \\
&e^{-2\sqrt{-2|\tilde{\mathbf{k}}-\tilde{\mathbf{q}}|\xi\eta''}} \big[\langle[\epsilon_{a+}(\mathbf{q})a_+(\mathbf{q}) + \epsilon_{a+}^*(-\mathbf{q})a_+^\dagger(-\mathbf{q})] \cdot [\epsilon_{c+}(\tilde{\mathbf{q}})a_+(\tilde{\mathbf{q}}) + \epsilon_{c+}^*(-\tilde{\mathbf{q}})a_+^\dagger(-\tilde{\mathbf{q}})]\rangle \cdot \\
&\cdot \langle[\epsilon_{b+}(\mathbf{k}-\mathbf{q})a_+(\mathbf{k}-\mathbf{q}) + \epsilon_{b+}^*(\mathbf{q}-\mathbf{k})a_+^\dagger(\mathbf{q}-\mathbf{k})] \cdot [\epsilon_{d+}(\tilde{\mathbf{k}}-\tilde{\mathbf{q}})a_+(\tilde{\mathbf{k}}-\tilde{\mathbf{q}}) + \\
&+ \epsilon_{d+}^*(\tilde{\mathbf{q}}-\tilde{\mathbf{k}})a_+^\dagger(\dagger\mathbf{q}-\dagger\mathbf{k})]\rangle + \langle[\epsilon_{a+}(\mathbf{q})a_+(\mathbf{q}) + \epsilon_{a+}^*(-\mathbf{q})a_+^\dagger(-\mathbf{q})] \cdot \\
&\cdot [\epsilon_{d+}(\tilde{\mathbf{k}}-\tilde{\mathbf{q}})a_+(\tilde{\mathbf{k}}-\tilde{\mathbf{q}}) + +\epsilon_{d+}^*(\tilde{\mathbf{q}}-\tilde{\mathbf{k}})a_+^\dagger(\tilde{\mathbf{q}}-\tilde{\mathbf{k}})]\rangle \cdot \langle[\epsilon_{b+}(\mathbf{k}-\mathbf{q})a_+(\mathbf{k}-\mathbf{q}) + \\
&+ \epsilon_{b+}^*(\mathbf{q}-\mathbf{k})a_+^\dagger(\mathbf{q}-\mathbf{k})a_+^\dagger(\mathbf{q}-\mathbf{k})] \cdot [\epsilon_{c+}(\tilde{\mathbf{q}})a_+(\tilde{\mathbf{q}}) + \epsilon_{c+}^*(-\tilde{\mathbf{q}})a_+^\dagger(-\tilde{\mathbf{q}})]\rangle\big]
\end{aligned}
\tag{6.13}
$$

We use $[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = \delta^{(3)}(\mathbf{k}-\mathbf{k}')$ to simplify the above to:

$$
\begin{aligned}
C(\bar{\theta}) = &- \frac{3^2 \xi H^4}{2(2\pi)^6 M_P^4} e^{4\pi\xi} l_1^i l_1^j l_2^l l_2^m \int_{\eta_r}^{\eta_0} d\eta \int_{\eta_r}^{\eta_0} d\tilde{\eta} \int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} d^3\mathbf{k}\, e^{i\mathbf{k}\cdot(\mathbf{l_1}(\eta-\eta_0)-\mathbf{l_2}(\tilde{\eta}-\eta_0))} \frac{\partial}{\partial\eta}\frac{\partial}{\partial\tilde{\eta}} \\
&(-\frac{1}{k^2\eta^2}\cos(k\eta) + \frac{1}{k^3\eta^3}\sin(k\eta))(-\frac{1}{k^2\tilde{\eta}^2}\cos(k\tilde{\eta}) + \frac{1}{k^3\tilde{\eta}^3}\sin(k\tilde{\eta})) \int_{-\infty}^{\eta_{ei}} d\eta' \\
&\int_{-\infty}^{\eta_{ei}} d\eta'' \eta'^{\frac{3}{2}} \eta''^{\frac{3}{2}} G(\eta_{ei}, \eta', k) G(\eta_{ei}, \eta'', k) \Pi_{ij}^{ab}(\mathbf{k}) \Pi_{lm}^{cd}(-\mathbf{k}) \\
&\int_{-\infty}^{\infty} d^3\mathbf{q}\, q^{\frac{1}{2}}\, |\mathbf{k}-\mathbf{q}|^{\frac{1}{2}}\, e^{-2\sqrt{2\xi}\cdot(\sqrt{q}+\sqrt{|\mathbf{k}-\mathbf{q}|})\cdot(\sqrt{-\eta'}+\sqrt{-\eta''})} \\
&\big[\epsilon_{a+}(\mathbf{q})\epsilon_{c+}^*(\mathbf{q})\epsilon_{b+}(\mathbf{k}-\mathbf{q})\epsilon_{d+}^*(\mathbf{k}-\mathbf{q}) + \epsilon_{a+}(\mathbf{q})\epsilon_{d+}^*(\mathbf{q})\epsilon_{b+}(\mathbf{k}-\mathbf{q})\epsilon_{c+}^*(\mathbf{k}-\mathbf{q})\big]
\end{aligned}
\tag{6.14}
$$

Now note the two terms in brackets above are actually the same. It is easy to see they are equivalent up to interchange of 'c' and 'd'. It can be shown that $\Pi_{lm}^{cd} = \Pi_{ml}^{dc}$, so we can interchange 'c' and 'd' as long as we can also interchange 'l' and 'm'. We can obviously interchange 'l' and 'm' because of the '$l_2^l l_2^m$' part.

We next use that: $\Pi_{ij}^{lm} = \Pi_i^l \Pi_j^m - \frac{1}{2}\Pi_{ij}\Pi^{lm}$ and that $\Pi^{ij}(\mathbf{k}) = (\epsilon_-^i(\mathbf{k})e_+^j(\mathbf{k})) + (\epsilon_+^i(\mathbf{k})e_-^j(\mathbf{k}))$. First we start with simplifying: $l_1^i l_1^j \Pi_{ij}^{ab}(\mathbf{k})\epsilon_{a+}(\mathbf{q})\epsilon_{b+}(\mathbf{k}-\mathbf{q})$, noticing that $l_1^i l_1^j$ is symmetric on interchange of i and j, so terms that differ by $i <->j$ are really the same. We are then left with:

$$
\begin{aligned}
l_1^i l_1^j \Pi_{ij}^{ab}(\mathbf{k})\epsilon_{a+}(\mathbf{q})\epsilon_{b+}(\mathbf{k}-\mathbf{q}) =& l_1^i l_1^j [\epsilon_-^a(k)\epsilon_-^b(k)\epsilon_+^i(k)\epsilon_+^j(k) + \epsilon_+^a(k)\epsilon_+^b(k)\epsilon_-^i(k)\epsilon_-^j(k)]\epsilon_{a+}(\mathbf{q}) \cdot \\
&\cdot \epsilon_{b+}(\mathbf{k}-\mathbf{q}) \\
=& (\epsilon_-(\mathbf{k})\cdot\epsilon_+(\mathbf{q}))(\epsilon_-(\mathbf{k})\cdot\epsilon_+(\mathbf{k}-\mathbf{q}))(\epsilon_+(\mathbf{k})\cdot\mathbf{l}_1)^2 \\
&+ (\epsilon_+(\mathbf{k})\cdot\epsilon_+(\mathbf{q}))(\epsilon_+(\mathbf{k})\cdot\epsilon_+(\mathbf{k}-\mathbf{q}))(\epsilon_-(\mathbf{k})\cdot\mathbf{l}_1)^2)
\end{aligned}
\tag{6.15}
$$

We do the same simplification for the other term. Next, we multiply the terms out and simplify, using that $-\epsilon_-(\mathbf{q}) = \epsilon_+^*(\mathbf{q})$:

$$\begin{aligned}
&l_1^i l_1^j l_2^l l_2^m \Pi_{ij}^{ab}(\mathbf{k})\Pi_{lm}^{cd}(\mathbf{k})\epsilon_{a+}(\mathbf{q})\epsilon_{b+}(\mathbf{k}-\mathbf{q})\epsilon_{c+}^*(\mathbf{q})\epsilon_{d+}^*(\mathbf{k}-\mathbf{q}) = \\
&\quad =|\epsilon_+^*(\mathbf{k})\cdot\epsilon_+(\mathbf{q})|^2|\epsilon_+^*(\mathbf{k})\cdot\epsilon_+(\mathbf{k}-\mathbf{q})|^2(\epsilon_+(\mathbf{k})\cdot\mathbf{l}_1)^2(\epsilon_+^*(\mathbf{k})\cdot\mathbf{l}_2)^2+ \\
&\quad\quad + (\epsilon_+^*(\mathbf{k})\cdot\epsilon_+(\mathbf{q}))(\epsilon_+^*(\mathbf{k})\cdot\epsilon_+^*(\mathbf{q}))(\epsilon_+^*(\mathbf{k})\cdot\epsilon_+(\mathbf{k}-\mathbf{q}))\cdot \\
&\quad\quad \cdot(\epsilon_+^*(\mathbf{k})\cdot\epsilon_+^*(\mathbf{k}-\mathbf{q}))(\epsilon_+(\mathbf{k})\cdot\mathbf{l}_1)(\epsilon_+(\mathbf{k})\cdot\mathbf{l_2})+ \\
&\quad\quad + (\epsilon_+(\mathbf{k})\cdot\epsilon_+(\mathbf{q}))(\epsilon_+(\mathbf{k})\cdot\epsilon_+^*(\mathbf{q}))(\epsilon_+(\mathbf{k})\cdot\epsilon_+(\mathbf{k}-\mathbf{q}))\cdot \\
&\quad\quad \cdot(\epsilon_+(\mathbf{k})\cdot\epsilon_+^*(\mathbf{k}-\mathbf{q}))(\epsilon_+^*(\mathbf{k})\cdot\mathbf{l}_1)(\epsilon_+^*(\mathbf{k})\cdot\mathbf{l}_2)+ \\
&\quad\quad + |\epsilon_+(\mathbf{k})\cdot\epsilon_+(\mathbf{q})|^2|\epsilon_+(\mathbf{k})\cdot\epsilon_+(\mathbf{k}-\mathbf{q})|^2(\epsilon_+^*(\mathbf{k})\cdot\mathbf{l}_1)^2(\epsilon_+(\mathbf{k})\cdot\mathbf{l}_2)^2\,. \qquad (6.16)
\end{aligned}$$

Then we use the identities: $|\epsilon_\sigma(p_1)\cdot\epsilon_+(p_2)|^2 = \frac{1}{4}(1-\frac{\sigma\mathbf{p_1}\cdot\mathbf{p_2}}{p_1p_2})^2$ $|\epsilon_\sigma(p_1)\cdot\epsilon_-(p_2)|^2 = \frac{1}{4}(1+\frac{\sigma\mathbf{p_1}\cdot\mathbf{p_2}}{p_1p_2})^2$ $|\epsilon_\sigma(p_1)\cdot\epsilon_+^*(p_2)|^2 = \frac{1}{4}(1+\frac{\sigma\mathbf{p_1}\cdot\mathbf{p_2}}{p_1p_2})^2$ $|\epsilon_\sigma(p_1)\cdot\epsilon_-^*(p_2)|^2 = \frac{1}{4}(1-\frac{\sigma\mathbf{p_1}\cdot\mathbf{p_2}}{p_1p_2})^2$ These hold for $\sigma = +$ or $-$. Let $k\cdot q = kq\cos\theta_{kq}$ where θ_{kq} is the angle between k and q. This would also be the polar position angle of q if k were along $\hat{z}$. Let ϕ_{kq} be the ϕ position angle of q if k were along $\hat{z}$.

Then $\left(1+\frac{\mathbf{k}\cdot\mathbf{q}}{kq}\right)^2 = (1+\cos\theta_{kq})^2$ and $\left(1+\frac{(\mathbf{k}\cdot(\mathbf{k}-\mathbf{q}))}{k|\mathbf{k}-\mathbf{q}|}\right)^2 = (1+\frac{k-q\cos\theta_{kq}}{\sqrt{k^2+q^2-2kq\cos\theta_{kq}}})$.

Similarly: $\sin^2\theta_{k,k-q} = (1-\cos^2\theta_{k,k-q}) = (1-(\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{q})}{k|\mathbf{k}-\mathbf{q}|})^2) = (1-(\frac{k-q\cos\theta_{kq}}{k\sqrt{k^2+q^2-2kq\cos\theta_{kq}}})^2)$, where $\theta_{k,k-q}$ and $\phi_{k,k-q}$ are again the position angles of $\mathbf{k}-\mathbf{q}$ relative to $\mathbf{k}$.

Next we use that: $(\epsilon_+^*(\mathbf{k})\cdot\epsilon_+(\mathbf{q}))(\epsilon_+^*(\mathbf{k})\cdot\epsilon_+^*(\mathbf{q})) = -\frac{1}{4}\sin^2\theta_{kq}e^{-2i\phi_{kq}}$ $(\epsilon_+(\mathbf{k})\cdot\epsilon_+(\mathbf{q}))(\epsilon_+(\mathbf{k})\cdot\epsilon_+^*(\mathbf{q})) = -\frac{1}{4}\sin^2\theta_{kq}e^{2i\phi_{kq}}$ The above then reduces to:

$$\begin{aligned}
&l_1^i l_1^j l_2^l l_2^m \Pi_{ij}^{ab}(\mathbf{k})\Pi_{lm}^{cd}(\mathbf{k})\epsilon_{a+}(\mathbf{q})\epsilon_{b+}(\mathbf{k}-\mathbf{q})\epsilon_{c+}^*(\mathbf{q})\epsilon_{d+}^*(\mathbf{k}-\mathbf{q}) = \\
&\quad =\frac{1}{16}\left(1+\frac{\mathbf{k}\cdot\mathbf{q}}{kq}\right)^2\left(1+\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{q})}{k|\mathbf{k}-\mathbf{q}|}\right)^2(\epsilon_+(\mathbf{k})\cdot\mathbf{l}_1)^2(\epsilon_+^*(\mathbf{k})\cdot\mathbf{l}_2)^2+ \\
&\quad\quad + \frac{1}{16}\sin^2\theta_{kq}e^{-2i\phi_{kq}}\sin^2\theta_{k,k-q}e^{-2i\phi_{k,k-q}}(\epsilon_+(\mathbf{k})\cdot\mathbf{l}_1)(\epsilon_+(\mathbf{k})\cdot\mathbf{l_2})+ \\
&\quad\quad + \frac{1}{16}\sin^2\theta_{kq}e^{2i\phi_{kq}}\sin^2\theta_{k,k-q}e^{2i\phi_{k,k-q}}(\epsilon_+^*(\mathbf{k})\cdot\mathbf{l}_1)(\epsilon_+^*(\mathbf{k})\cdot\mathbf{l}_2)+ \\
&\quad\quad + \frac{1}{16}\left(1-\frac{\mathbf{k}\cdot\mathbf{q}}{kq}\right)^2\left(1-\frac{(\mathbf{k}\cdot(\mathbf{k}-\mathbf{q}))}{k|\mathbf{k}-\mathbf{q}|}\right)^2(\epsilon_+^*(\mathbf{k})\cdot\mathbf{l}_1)^2(\epsilon_+(\mathbf{k})\cdot\mathbf{l}_2)^2\,. \qquad (6.17)
\end{aligned}$$

Comparing the polar position angles, we find:

$\cos\phi_{kq} = -\cos\phi_{k,k-q}$, $\sin\phi_{kq} = -\sin\phi_{k,k-q}$, and $\phi_{k,k-q} = \phi_{kq}+\pi$. This leads to the two middle terms from eq. (6.17) giving 0 when we do the ϕ integral:

$\int_0^{2\pi} e^{4i\phi}d\phi = \frac{1}{4i}e^{4i\phi}|_0^{2\pi} = 0$ So eq. (6.17) simplifies to:

$$\begin{aligned}
&l_1^i l_1^j l_2^l l_2^m \Pi_{ij}^{ab}(\mathbf{k})\Pi_{lm}^{cd}(\mathbf{k})\epsilon_{a+}(\mathbf{q})\epsilon_{b+}(\mathbf{k}-\mathbf{q})\epsilon_{c+}^*(\mathbf{q})\epsilon_{d+}^*(\mathbf{k}-\mathbf{q}) = \\
&\quad =\frac{1}{16}\left(1+\frac{\mathbf{k}\cdot\mathbf{q}}{kq}\right)^2\left(1+\frac{\mathbf{k}\cdot(\mathbf{k}-\mathbf{q})}{k|\mathbf{k}-\mathbf{q}|}\right)^2(\epsilon_+(\mathbf{k})\cdot\mathbf{l}_1)^2(\epsilon_+^*(\mathbf{k})\cdot\mathbf{l}_2)^2+ \\
&\quad\quad + \frac{1}{16}\left(1-\frac{\mathbf{k}\cdot\mathbf{q}}{kq}\right)^2\left(1-\frac{(\mathbf{k}\cdot(\mathbf{k}-\mathbf{q}))}{k|\mathbf{k}-\mathbf{q}|}\right)^2(\epsilon_+^*(\mathbf{k})\cdot\mathbf{l}_1)^2(\epsilon_+(\mathbf{k})\cdot\mathbf{l}_2)^2 \qquad (6.18)
\end{aligned}$$

Plugging this in to our temperature two-point function equation:

$$
\begin{aligned}
C(\bar{\theta}) = &- \frac{3^2 \xi H^4}{(2\pi)^6 M_P^4} e^{4\pi\xi} \int_{\eta_r}^{\eta_0} d\eta \int_{\eta_r}^{\eta_0} d\tilde{\eta} \int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} d^3\mathbf{k}\, e^{i\mathbf{k}\cdot(\mathbf{l_1}(\eta-\eta_0)-\mathbf{l_2}(\tilde{\eta}-\eta_0))} \frac{\partial}{\partial\eta}\frac{\partial}{\partial\tilde{\eta}} \\
&(-\frac{1}{k^2\eta^2}\cos(k\eta) + \frac{1}{k^3\eta^3}\sin(k\eta))(-\frac{1}{k^2\tilde{\eta}^2}\cos(k\tilde{\eta}) + \frac{1}{k^3\tilde{\eta}^3}\sin(k\tilde{\eta})) \int_{-\infty}^{\eta_{ei}} d\eta' \\
&\int_{-\infty}^{\eta_{ei}} d\eta'' \eta'^{\frac{3}{2}} \eta''^{\frac{3}{2}} G(\eta_{ei},\eta',k) G(\eta_{ei},\eta'',k) \int_{-\infty}^{\infty} d^3\mathbf{q}\, q^{\frac{1}{2}} |\mathbf{k}-\mathbf{q}|^{\frac{1}{2}} \\
&e^{-2\sqrt{2\xi}\cdot(\sqrt{q}+\sqrt{|\mathbf{k}-\mathbf{q}|})\cdot(\sqrt{-\eta'}+\sqrt{-\eta''})} \frac{1}{2^4}[(1+\cos(\theta_{kq}))^2 \left(1 + \frac{k - q\cos\theta_{kq}}{\sqrt{k^2+q^2-2kq\cos\theta_{kq}}}\right)^2 \\
&(\epsilon_+(\mathbf{k})\cdot\mathbf{l}_1)^2(\epsilon_+^*(\mathbf{k})\cdot\mathbf{l}_2)^2 + (1-\cos\theta_{kq})^2 \left(1 - \frac{(k - q\cos\theta_{kq})}{\sqrt{k^2+q^2-2kq\cos\theta_{kq}}}\right)^2 \\
&(\mathbf{e}_+^*(\mathbf{k})\cdot\mathbf{l}_1)^2(\mathbf{e}_+(\mathbf{k})\cdot\mathbf{l}_2)^2]\,, \qquad (6.19)
\end{aligned}
$$

where again θ_{kq} is the angle between $\mathbf{k}$ and $\mathbf{q}$ so $\cos\theta_{kq} = \frac{\mathbf{k}\cdot\mathbf{q}}{kq}$.

We next want to plug in for the Green's function using eq. (1.40), $G(\eta_{ei},\eta',k) = \frac{1}{k^3\eta'^2}[(1+k^2\eta\eta')\sin(k(\eta-\eta')) + k(\eta'-\eta)\cos(k(\eta-\eta'))]$. We can approximate this. First we use that $|\eta'| \geq |\eta_{ei}|$ (from our integration limits). We use this to approximate $|\eta'| \gg |\eta_{ei}|$ which simplifies the Green's function to:
$G(\eta_{ei},\eta',k) \approx \frac{1}{k^3\eta'^2}[-(1+k^2\eta\eta')\sin(k\eta') + k\eta'\cos(k\eta')]$

Next we use that $k\eta_{ei} \ll 1$ (since all modes we are considering were definitely outside the horizon at the end of inflation.) And we use that we get the largest contribution from $k\eta' \approx 1$. To show why this is true, note the exponential suppression term: $e^{-2\sqrt{2\xi}\cdot(\sqrt{q}+\sqrt{|\mathbf{k}-\mathbf{q}|})\cdot(\sqrt{-\eta'}+\sqrt{-\eta''})}$. Note ξ is order 1 for efficient production of the vectors. So for $q|\eta'|$, $q|\eta''| > 1$, we get large exponential suppression. So the integrand will only get significant contributions from $q|\eta'|$, $q|\eta''| < 1$. Similarly we also get exponential suppression for $k|\eta'|$, $k|\eta''| > 1$, (from the $\sqrt{|\mathbf{k}-\mathbf{q}|}(\sqrt{-\eta'}+\sqrt{-\eta''})$ part) since the large but opposite $\mathbf{k}$ and $\mathbf{q}$ case is ruled out since q is separately confined by $q < \frac{1}{|\eta'|}, \frac{1}{|\eta''|}$. So then k becomes similarly confined to $k|\eta'|$, $k|\eta''| < 1$.

Note $k\eta < 1$ corresponds to modes outside the horizon. If we were a little more rigorous and took the whole $k\eta'$ expression, we would find the integrand is not exactly peaked for modes outside horizon, but actually is peaked for modes just barely inside the horizon. This comes from the fact that the integrand is proportional to $e^{-C\sqrt{-k\eta}}\cdot(-k\eta)^{\frac{5}{2}}$ which is peaked at about $k\eta \approx 2.5$ for $C = 1$. The main point is that the integrand is peaked for modes near, but just inside the horizon, and falls out exponentially for modes further inside the horizon. Conceptually this means $\langle TT\rangle$ does not get any contribution from modes outside the horizon, gets a comparatively huge contribution form modes for a short period following their entry into the horizon, and then very little contribution from modes after they have been in the horizon for awhile.

Back to our approximation on the Green's function, we know $k\eta_{ei} \ll 1$, and we also approximate that the most significant contribution to the integral will come from $k\eta'$, $k\eta'' \approx 1$. Then it follows that we can approximate $k^2\eta_{ei}\eta' \ll 1$. Using this, our Green's function

reduces to:
$G(\eta_{ei},\eta',k) \approx \frac{1}{k^3\eta'^2}[-\sin(k\eta')+k\eta'\cos(k\eta')]$. Plugging this into our temperature two-point function:

$$
\begin{aligned}
C(\bar{\theta}) = &-\frac{3^2\,\xi H^4}{2^4(2\pi)^6 M_P^4} e^{4\pi\xi}\int_{\eta_r}^{\eta_0} d\eta \int_{\eta_r}^{\eta_0} d\tilde{\eta}\int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} d^3\mathbf{k}\,\frac{1}{k^6} e^{i\mathbf{k}\cdot(\mathbf{l_1}(\eta-\eta_0)-\mathbf{l_2}(\tilde{\eta}-\eta_0))}\frac{\partial}{\partial\eta}\frac{\partial}{\partial\tilde{\eta}} \\
&(-\frac{1}{k^2\eta^2}\cos(k\eta)+\frac{1}{k^3\eta^3}\sin(k\eta))(-\frac{1}{k^2\tilde{\eta}^2}\cos(k\tilde{\eta})+\frac{1}{k^3\tilde{\eta}^3}\sin(k\tilde{\eta}))\int_{-\infty}^{\eta_{ei}} d\eta'\int_{-\infty}^{\eta_{ei}} d\eta'' \\
&\eta'^{-\frac{1}{2}}\,\eta''^{-\frac{1}{2}}[-\sin(k\eta')+k\eta'\cos(k\eta')][-\sin(k\eta'')+k\eta''\cos(k\eta'')]\int_{-\infty}^{\infty} d^3\mathbf{q}\,q^{\frac{1}{2}}\,|\mathbf{k}-\mathbf{q}|^{\frac{1}{2}} \\
&e^{-2\sqrt{2\xi}\cdot(\sqrt{q}+\sqrt{|\mathbf{k}-\mathbf{q}|})\cdot(\sqrt{-\eta'}+\sqrt{-\eta''})}[(1+\cos(\theta_{kq}))^2\left(1+\frac{k-q\cos\theta_{kq}}{\sqrt{k^2+q^2-2kq\cos\theta_{kq}}}\right)^2\cdot \\
&\cdot(\epsilon_+(\mathbf{k})\cdot\mathbf{l}_1)^2(\epsilon_+^*(\mathbf{k})\cdot\mathbf{l}_2)^2+(1-\cos\theta_{kq})^2\left(1-\frac{(k-q\cos\theta_{kq})}{\sqrt{k^2+q^2-2kq\cos\theta_{kq}}}\right)^2\cdot \\
&\cdot(\mathbf{e}_+^*(\mathbf{k})\cdot\mathbf{l}_1)^2(\mathbf{e}_+(\mathbf{k})\cdot\mathbf{l}_2)^2]\,. \qquad (6.20)
\end{aligned}
$$

Next we want to do η' and η'' integrals, but if we integrate as is, we will get erfi functions. Instead we approximate the Green's functions further using that we found above that $k|\eta'|$, $k|\eta''|<1$ (from the exponential suppression term) to use the small angle expansion of the Green's functions to get:

$$
-\sin(k\eta')+k\eta'\cos(k\eta') \approx -\frac{k^3\eta'^3}{3} \qquad (6.21)
$$

Next we do the η' and η'' integrals using that:

$$
\begin{aligned}
\int_{\infty}^{|\eta_{ei}|} d|\eta'|\,|\eta|'^{\frac{5}{2}} e^{-C\sqrt{|\eta'|}} = &-2e^{-C\sqrt{|\eta_{ei}|}}[\frac{720}{C^7}+\frac{720}{C^6}\sqrt{|\eta_{ei}|}+\frac{360}{C^5}|\eta_{ei}|+\frac{120}{C^4}|\eta_{ei}|^{\frac{3}{2}}+\frac{30}{C^3}|\eta_{ei}|^2 \\
&+\frac{6}{C^2}|\eta_{ei}|^{\frac{5}{2}}+\frac{1}{C}|\eta_{ei}|^3]\,, \qquad (6.22)
\end{aligned}
$$

where $C=2\sqrt{2\xi}\cdot(\sqrt{q}+\sqrt{|\mathbf{k}-\mathbf{q}|})$ and C is positive. Plugging this in:

$$
\begin{aligned}
C(\bar{\theta}) = &\frac{\xi H^4}{2^2(2\pi)^6 M_P^4} e^{4\pi\xi}\int_{\eta_r}^{\eta_0} d\eta \int_{\eta_r}^{\eta_0} d\tilde{\eta}\int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} d^3\mathbf{k}\,e^{i\mathbf{k}\cdot(\mathbf{l_1}(\eta-\eta_0)-\mathbf{l_2}(\tilde{\eta}-\eta_0))}\frac{\partial}{\partial\eta}\frac{\partial}{\partial\tilde{\eta}} \\
&(-\frac{1}{k^2\eta^2}\cos(k\eta)+\frac{1}{k^3\eta^3}\sin(k\eta))(-\frac{1}{k^2\tilde{\eta}^2}\cos(k\tilde{\eta})+\frac{1}{k^3\tilde{\eta}^3}\sin(k\tilde{\eta})) \\
&\int_{-\infty}^{\infty} d^3\mathbf{q}\,q^{\frac{1}{2}}\,|\mathbf{k}-\mathbf{q}|^{\frac{1}{2}} e^{-4\sqrt{2\xi}\cdot(\sqrt{q}+\sqrt{|\mathbf{k}-\mathbf{q}|})\sqrt{|\eta_{ei}|}}[\frac{720}{C^7}+\frac{720}{C^6}\sqrt{|\eta_{ei}|}+\frac{360}{C^5}|\eta_{ei}|+\frac{120}{C^4}|\eta_{ei}|^{\frac{3}{2}}+ \\
&+\frac{30}{C^3}|\eta_{ei}|^2+\frac{6}{C^2}|\eta_{ei}|^{\frac{5}{2}}+\frac{1}{C}|\eta_{ei}|^3]^2[(1+\cos(\theta_{kq}))^2\left(1+\frac{k-q\cos\theta_{kq}}{\sqrt{k^2+q^2-2kq\cos\theta_{kq}}}\right)^2\cdot \\
&\cdot(\epsilon_+(\mathbf{k})\cdot\mathbf{l}_1)^2(\epsilon_+^*(\mathbf{k})\cdot\mathbf{l}_2)^2+(1-\cos\theta_{kq})^2\left(1-\frac{(k-q\cos\theta_{kq})}{\sqrt{k^2+q^2-2kq\cos\theta_{kq}}}\right)^2\cdot \\
&\cdot(\mathbf{e}_+^*(\mathbf{k})\cdot\mathbf{l}_1)^2(\mathbf{e}_+(\mathbf{k})\cdot\mathbf{l}_2)^2] \qquad (6.23)
\end{aligned}
$$

Next we want to do q integral. For the purpose of doing the q integral, we will assume k is along the z axis, even though we will later integrate over k. For purpose of just doing q integral though, we will be integrating over all q, it will not change the q integral whatever direction k is pointing,; for the purpose of solving the q integral, k is just a free parameter. Then for solving the q integral, $\theta_{kq} = \theta$ the integration variable.

Let $p = \frac{q}{k}$ so $q = kp$ and p is unitless. $dq = kdp$. (Note this is allowed even though k is being integrated over. It is correct to write $dq = kdp$ instead of $dq = kdp + pdk$ as long as we go back and do the reverse transformation for $p = \frac{q}{k}$ after solving the p integral, before doing the k integral.)

Note: $|\mathbf{k} - \mathbf{q}| = (k^2 + q^2 - 2kq\cos\theta_{kq})^{\frac{1}{2}} = k^{\frac{1}{2}}(1 + p^2 - 2p\cos\theta_{kq})^{\frac{1}{4}}$
and $C = 2\sqrt{2\xi}(\sqrt{q} + (k^2 + q^2 - 2kq\cos\theta_{kq})^{\frac{1}{4}}) = 2\sqrt{2\xi k}(p^{\frac{1}{2}} + (1 + p^2 - 2p\cos\theta_{kq})^{\frac{1}{4}})$.

Note if we multiply out the $[\frac{720}{C^7} + \frac{720}{C^6}\sqrt{|\eta_{ei}|} + \frac{360}{C^5}|\eta_{ei}| + \frac{120}{C^4}|\eta_{ei}|^{\frac{3}{2}} + \frac{30}{C^3}|\eta_{ei}|^2 + \frac{6}{C^2}|\eta_{ei}|^{\frac{5}{2}} + \frac{1}{C}|\eta_{ei}|^3]^2$ part, this gives a lot of terms, but one term will dominate over all the others, and we can approximate by just keeping that term. Note the exponential suppression term: $e^{-4\sqrt{2\xi k|\eta_{ei}|}\cdot(\sqrt{p}+(1+p^2-2p\cos\theta_{kq})^{\frac{1}{4}})}$. There is exponential suppression when $4\sqrt{2\xi k|\eta_{ei}|}\cdot(\sqrt{p}+(1+p^2-2p\cos\theta_{kq})^{\frac{1}{4}}) \geq 1$. Therefore, we only care about $2\sqrt{2\xi k|\eta_{ei}|}\cdot(\sqrt{p}+(1+p^2-2p\cos\theta_{kq})^{\frac{1}{4}}) \leq \frac{1}{2}$. Looking back at our series of terms, and multiplying and dividing by appropriate factors of $|\eta_{ei}|$:
$\left[\frac{720|\eta_{ei}|^{\frac{7}{2}}}{(\#\text{ less than}\frac{1}{2})^7} + \frac{720|\eta_{ei}|^{\frac{1}{2}+\frac{7}{2}}}{(\#\text{ less than}\frac{1}{2})^6} + ... + \frac{|\eta_{ei}|^{\frac{7}{2}}}{(\#\text{ less than}\frac{1}{2})}\right]^2$ Clearly the first term will dominate, and we will drop the others. Our two-point function then simplifies to:

$$
\begin{aligned}
C(\bar{\theta}) =& \frac{720^2 H^4}{2^{28}\,\pi^5\xi^6 M_P^4} e^{4\pi\xi} \int_{\eta_r}^{\eta_0} d\eta \int_{\eta_r}^{\eta_0} d\tilde{\eta} \int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} d^3\mathbf{k}\, \frac{1}{k^3} e^{i\mathbf{k}\cdot(\mathbf{l_1}(\eta-\eta_0)-\mathbf{l_2}(\tilde{\eta}-\eta_0))} \frac{\partial}{\partial\eta}\frac{\partial}{\partial\tilde{\eta}} \\
& (-\frac{1}{k^2\eta^2}\cos(k\eta) + \frac{1}{k^3\eta^3}\sin(k\eta))(-\frac{1}{k^2\tilde{\eta}^2}\cos(k\tilde{\eta}) + \frac{1}{k^3\tilde{\eta}^3}\sin(k\tilde{\eta})) \int_0^\infty dp \int_0^\pi d\theta_{kq}\, \sin\theta_{kq} \\
& p^{\frac{5}{2}}(1 + p^2 - 2p\cos\theta_{kq})^{\frac{1}{4}} e^{-4\sqrt{2\xi k|\eta_{ei}|}\cdot(\sqrt{p}+(1+p^2-2p\cos\theta_{kq})^{\frac{1}{4}})} \frac{1}{(p^{\frac{1}{2}} + (1 + p^2 - 2p\cos\theta_{kq})^{\frac{1}{4}})^{14}} \\
& \cdot [(1 + \cos(\theta_{kq}))^2 \left(1 + \frac{1 - p\cos\theta_{kq}}{\sqrt{1 + p^2 - 2p\cos\theta_{kq}}}\right)^2 (\epsilon_+(\mathbf{k})\cdot\mathbf{l}_1)^2(\epsilon_+^*(\mathbf{k})\cdot\mathbf{l}_2)^2 + \\
& + (1 - \cos\theta_{kq})^2 \left(1 - \frac{(1 - p\cos\theta_{kq})}{\sqrt{1 + p^2 - 2p\cos\theta_{kq}}}\right)^2 (\mathbf{e}_+^*(\mathbf{k})\cdot\mathbf{l}_1)^2(\mathbf{e}_+(\mathbf{k})\cdot\mathbf{l}_2)^2]\,. \qquad (6.24)
\end{aligned}
$$

We still can not do the θ_{kq} integral as is so we make another approximation. We know $k|\eta_{ei}| \ll 1$ since we will only see modes outside the horizon at the end of inflation. Further, since ξ is order 1, this means: $\frac{1}{4\sqrt{2\xi k\tau_{ei}}} \gg 1$. Looking at our exponential: $e^{-4\sqrt{2\xi k\tau_{ei}}\cdot(\sqrt{p}+(1+p^2-2p\cos\theta_{kq})^{\frac{1}{4}})}$, we note that the exponential is only significantly different from 1 when $p \gg 1$. So if the contribution to the integrand for large p has already fallen off before this exponential suppression becomes important, then we can just approximate the exponential as 1. Looking at the form of the p integral without the exponential:
$\int_0^\infty dp \frac{p^{\frac{5}{2}}(1+p^2-2p\cos\theta_{kq})^{\frac{1}{4}}}{(p^{\frac{1}{2}}+(1+p^2-2p\cos\theta_{kq})^{\frac{1}{4}})^{14}} \cdot [(1+\cos(\theta_{kq}))^2 \left(1 + \frac{1-p\cos\theta_{kq}}{\sqrt{1+p^2-2p\cos\theta_{kq}}}\right)^2 (\epsilon_+(\mathbf{k})\cdot\mathbf{l}_1)^2(\epsilon_+^*(\mathbf{k})\cdot\mathbf{l}_2)^2 +$

$+ (1-\cos\theta_{kq})^2 \left(1 - \frac{(1-p\cos\theta_{kq})}{\sqrt{1+p^2-2p\cos\theta_{kq}}}\right)^2 (\mathbf{e}_+^*(\mathbf{k})\cdot\mathbf{l}_1)^2(\mathbf{e}_+(\mathbf{k})\cdot\mathbf{l}_2)^2]$
When we take the limit $p \gg 1$, this becomes:
$\int_0^\infty dp \frac{1}{p^4}$
which clearly converges without the help of the exponential. So yes, the integrand peaks for $p \ll 1$, and the contribution from $p > 1$ is comparatively small, so by the time the exponential is notably different from 1, for $p \gg 1$, the integrand is already negligible. So we approximate the exponential ≈ 1.

Now the p and θ_{kq} integrals are the same as we found in the $\langle hh\rangle$ calc, so I will substitute the solutions found before:

$$\begin{aligned} C(\bar\theta) =& \frac{720^2 H^4}{2^{28}\,\pi^5\xi^6 M_P^4} e^{4\pi\xi} \int_{\eta_r}^{\eta_0} d\eta \int_{\eta_r}^{\eta_0} d\tilde\eta \int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} d^3\mathbf{k}\, \frac{1}{k^3} e^{i\mathbf{k}\cdot(\mathbf{l_1}(\eta-\eta_0)-\mathbf{l_2}(\tilde\eta-\eta_0))} \frac{\partial}{\partial\eta}\frac{\partial}{\partial\tilde\eta} \\ & (-\frac{1}{k^2\eta^2}\cos(k\eta) + \frac{1}{k^3\eta^3}\sin(k\eta))(-\frac{1}{k^2\tilde\eta^2}\cos(k\tilde\eta) + \frac{1}{k^3\tilde\eta^3}\sin(k\tilde\eta)) \\ & \cdot [(.0024741)(\epsilon_+(\mathbf{k})\cdot\mathbf{l}_1)^2(\epsilon_+^*(\mathbf{k})\cdot\mathbf{l}_2)^2 + (5.27414\times 10^{-6})(\mathbf{e}_+^*(\mathbf{k})\cdot\mathbf{l}_1)^2(\mathbf{e}_+(\mathbf{k})\cdot\mathbf{l}_2)^2] \end{aligned} \tag{6.25}$$

To simplify we take $\mathbf{l_1} = \hat{\mathbf{z}}$. The $\mathbf{l_1}$ and $\mathbf{l_2}$ vectors are not being integrated over, and we are free to define $\hat{\mathbf{z}}$ to be along $\mathbf{l_1}$. We have the further freedom to define the $\hat{\mathbf{x}}$ direction such that $\mathbf{l_2}$ is above $\hat{\mathbf{x}}$. This is allowed since the two-point function will only depend not on any particular choice of $\mathbf{l_1}$ or $\mathbf{l_2}$, but only on the angle between them $\bar\theta$. This gives us:
$\hat{\mathbf{l}}_\mathbf{1} = \langle 0,0,1\rangle$
$\hat{\mathbf{l}}_\mathbf{2} = \langle \sin\bar\theta, 0, \cos\bar\theta\rangle$
$\mathbf{k} = k\langle \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\rangle$ In the next step, we rewrite the exponential terms as sums of Bessel functions and Legendre polynomials and then use the addition theorem for spherical harmonics to write the two sums as one sum. To use this identity, we need to have all the θ and ϕ dependence in the exponential. As it is now, there are θ's and ϕ's built into the $(\epsilon_+(\mathbf{k})\cdot\mathbf{l}_1)^2$ terms. We want to write these terms first in terms of θ, ϕ, and $\bar\theta$. Then we will write these same terms as derivatives acting on the exponential.

Since $\mathbf{l}_1$ is along the $\hat z$ direction:

$$\epsilon_+(\mathbf{l_1}) = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ -i \\ 0 \end{pmatrix} \qquad \epsilon_0(\mathbf{l_1}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \epsilon_-(\mathbf{l_1}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

For the generic vector $\mathbf{k}$:

$$\epsilon_+(\mathbf{k}) = \frac{1}{\sqrt{2}}\begin{pmatrix} -\cos\theta\,\cos\phi + i\sin\phi \\ -\cos\theta\,\sin\phi - i\cos\phi \\ \sin\theta \end{pmatrix} \qquad \epsilon_0(\mathbf{k}) = \begin{pmatrix} \cos\phi\,\sin\theta \\ \sin\phi\,\sin\theta \\ \cos\theta \end{pmatrix}$$

$$\epsilon_-(\mathbf{k}) = \frac{1}{\sqrt{2}}\begin{pmatrix} \cos\theta\,\cos\phi + i\sin\phi \\ \cos\theta\,\sin\phi - i\cos\phi \\ -\sin\theta \end{pmatrix}$$

Rewriting the two-point function without the ϵ's, we obtain:

$$
\begin{aligned}
C(\bar{\theta}) =& \frac{720^2 H^4}{2^{30}\,\pi^5 \xi^6 M_P^4} e^{4\pi\xi} \int_{\eta_r}^{\eta_0} d\eta \int_{\eta_r}^{\eta_0} d\tilde{\eta} \int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} d^3\mathbf{k}\, \frac{1}{k^3} e^{ik\cos\theta(\eta-\eta_0)} e^{-ik(\tilde{\eta}-\eta_0)(\sin\theta\cos\phi\sin\bar{\theta}+\cos\theta\cos\bar{\theta})} \\
& [\frac{\partial}{\partial\eta}\frac{\partial}{\partial\tilde{\eta}}(-\frac{1}{k^2\eta^2}\cos(k\eta)+\frac{1}{k^3\eta^3}\sin(k\eta))(-\frac{1}{k^2\tilde{\eta}^2}\cos(k\tilde{\eta})+\frac{1}{k^3\tilde{\eta}^3}\sin(k\tilde{\eta}))] \\
& \cdot \sin^2\theta[(.0024741)(\cos^2\theta\cos^2\phi\sin^2\bar{\theta}-\sin^2\phi\sin^2\bar{\theta}+\sin^2\theta\cos^2\bar{\theta}+ \\
& + 2i\cos\theta\cos\phi\sin\phi\sin^2\bar{\theta} - 2\cos\theta\sin\theta\cos\phi\cos\bar{\theta}\sin\bar{\theta} - 2i\sin\theta\sin\phi\cos\bar{\theta}\sin\bar{\theta})+ \\
& + (5.27414\times 10^{-6})(\cos^2\theta\cos^2\phi\sin^2\bar{\theta}-\sin^2\phi\sin^2\bar{\theta}+\sin^2\theta\cos^2\bar{\theta} \\
& - 2i\cos\theta\cos\phi\sin\phi\sin^2\bar{\theta} - 2\cos\theta\sin\theta\cos\phi\cos\bar{\theta}\sin\bar{\theta} + 2i\sin\theta\sin\phi\cos\bar{\theta}\sin\bar{\theta})]\,.
\end{aligned}
\tag{6.26}
$$

Note the real terms are the same and in both the (.0024741) and (5.27414×10^{-6}) terms and the imaginary terms are opposites aside from the two very different prefactors. And note the first prefactor came from the $\langle h_+ h_+\rangle$ result and the second prefactor came from the $\langle h_- h_-\rangle$ result. So, if our calculation were instead parity conserving, we would get a factor of 2 times the $\langle h_+ h_+\rangle$ times the real terms and the imaginary terms would all cancel. The imaginary terms all turn out to be zero anyway, once we do the $\int d\phi$ integral. And so the total result will be proportional to real terms $\cdot(\langle h_+ h_+\rangle + \langle h_- h_-\rangle)$ and we will just drop the $\langle h_- h_-\rangle$ since it is so much less then $\langle h_+ h_+\rangle$. The two-point function then simplifies to:

$$
\begin{aligned}
C(\bar{\theta}) =& \frac{720^2(.0024741) H^4}{2^{30}\,\pi^5 \xi^6 M_P^4} e^{4\pi\xi} \int_{\eta_r}^{\eta_0} d\eta \int_{\eta_r}^{\eta_0} d\tilde{\eta} \int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} d^3\mathbf{k}\, \frac{1}{k^3} e^{ik\cos\theta(\eta-\eta_0)} \\
& e^{-ik(\tilde{\eta}-\eta_0)(\sin\theta\cos\phi\sin\bar{\theta}+\cos\theta\cos\bar{\theta})}[\frac{\partial}{\partial\eta}\frac{\partial}{\partial\tilde{\eta}}(-\frac{1}{k^2\eta^2}\cos(k\eta)+\frac{1}{k^3\eta^3}\sin(k\eta))(-\frac{1}{k^2\tilde{\eta}^2}\cos(k\tilde{\eta}) \\
& + \frac{1}{k^3\tilde{\eta}^3}\sin(k\tilde{\eta}))]\cdot\sin^2\theta[\cos^2\theta\cos^2\phi\sin^2\bar{\theta}-\sin^2\phi\sin^2\bar{\theta}+\sin^2\theta\cos^2\bar{\theta} \\
& - 2\cos\theta\sin\theta\cos\phi\cos\bar{\theta}\sin\bar{\theta}]
\end{aligned}
\tag{6.27}
$$

Let $g = k(\eta-\eta_0)$ and $\tilde{g} = k(\tilde{\eta}-\eta_0)$
so $e^{i\mathbf{k}\cdot\mathbf{l_1}(\eta-\eta_0)} = e^{ig\cos\theta}$
and $e^{-i\mathbf{k}\cdot\mathbf{l_2}(\tilde{\eta}-\eta_0)} = e^{-i\tilde{g}(\sin\theta\cos\phi\sin\bar{\theta}+\cos\theta\cos\bar{\theta})}$

Then we can rewrite:

$$
\begin{aligned}
& \sin^2\theta[\cos^2\theta\cos^2\phi\sin^2\bar{\theta}-\sin^2\phi\sin^2\bar{\theta}+\sin^2\theta\cos^2\bar{\theta}-2\cos\theta\sin\theta\cos\phi\cos\bar{\theta}\sin\bar{\theta}]\cdot \\
& \cdot e^{ig\cos\theta} e^{-i\tilde{g}(\sin\theta\cos\phi\sin\bar{\theta}+\cos\theta\cos\bar{\theta})} = \\
& = [-1+2\cos^2\bar{\theta} - \frac{\partial^2}{\partial g^2} - \frac{\partial^2}{\partial\tilde{g}^2} - 4\cos\bar{\theta}\frac{\partial}{\partial g}\frac{\partial}{\partial\tilde{g}} + \frac{\partial^2}{\partial g^2}\frac{\partial^2}{\partial\tilde{g}^2}] e^{ig\cos\theta} e^{-i\tilde{g}(\sin\theta\cos\phi\sin\bar{\theta}+\cos\theta\cos\bar{\theta})}
\end{aligned}
\tag{6.28}
$$

Plugging this into the two-point function, we obtain:

$$
\begin{aligned}
C(\bar{\theta}) =& \frac{720^2(2.5\times10^{-3})H^4}{2^{30}\,\pi^5\xi^6 M_P^4} e^{4\pi\xi}\int_{\eta_r}^{\eta_0} d\eta \int_{\eta_r}^{\eta_0} d\tilde{\eta} \int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} d^3\mathbf{k}\,\frac{1}{k^3}[\frac{\partial}{\partial\eta}\frac{\partial}{\partial\tilde{\eta}} \\
&(-\frac{1}{k^2\eta^2}\cos(k\eta)+\frac{1}{k^3\eta^3}\sin(k\eta))(-\frac{1}{k^2\tilde{\eta}^2}\cos(k\tilde{\eta})+\frac{1}{k^3\tilde{\eta}^3}\sin(k\tilde{\eta}))] \\
&\cdot[-1+2\cos^2\bar{\theta}-\frac{\partial^2}{\partial g^2}-\frac{\partial^2}{\partial\tilde{g}^2}-4\cos\bar{\theta}\frac{\partial}{\partial g}\frac{\partial}{\partial\tilde{g}}+\frac{\partial^2}{\partial g^2}\frac{\partial^2}{\partial\tilde{g}^2}]e^{ig\cos\theta} \\
&e^{-i\tilde{g}(\sin\theta\cos\phi\sin\bar{\theta}+\cos\theta\cos\bar{\theta})}\,.
\end{aligned}
\tag{6.29}
$$

We next want to expand the exponentials as sums of spherical Bessel functions and Legendre polynomials. We use $e^{i\mathbf{k}\cdot\mathbf{x}}=\sum_{l=0}^{\infty}(2l+1)i^l j_l(|\mathbf{k}||\mathbf{x}|)p_l(\hat{\mathbf{k}}\cdot\hat{\mathbf{x}})$ where j_l is the spherical Bessel function and P_l is a Legendre polynomial to obtain:

$$
e^{i\mathbf{k}\cdot\mathbf{l_1}(\eta-\eta_0)}=\sum_{l=0}^{\infty}(2l+1)i^l j_l(k|\eta-\eta_0|)P_l(\cos\theta) \tag{6.30}
$$

and

$$
e^{-i\mathbf{k}\cdot\mathbf{l_2}(\tilde{\eta}-\eta_0)}=\sum_{l=0}^{\infty}(2l+1)i^l j_l(k|\tilde{\eta}-\eta_0|)P_l(-\sin\theta\cos\phi\sin\bar{\theta}-\cos\theta\cos\bar{\theta})\,. \tag{6.31}
$$

Plugging these in:

$$
\begin{aligned}
C(\bar{\theta}) =& \frac{720^2(2.5\times10^{-3})H^4}{2^{30}\,\pi^5\xi^6 M_P^4} e^{4\pi\xi}\int_{\eta_r}^{\eta_0} d\eta \int_{\eta_r}^{\eta_0} d\tilde{\eta} \int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} d^3\mathbf{k}\,\frac{1}{k^3}[\frac{\partial}{\partial\eta}\frac{\partial}{\partial\tilde{\eta}}(-\frac{1}{k^2\eta^2}\cos(k\eta) \\
&+\frac{1}{k^3\eta^3}\sin(k\eta))\cdot(-\frac{1}{k^2\tilde{\eta}^2}\cos(k\tilde{\eta})+\frac{1}{k^3\tilde{\eta}^3}\sin(k\tilde{\eta}))]\cdot\sum_{l,l'=0}^{\infty}(2l+1)(2l'+1)i^{l+l'} \\
&[(-1+2\cos^2\bar{\theta})\cdot j_l(-g)j_{l'}(-\tilde{g})-j_l(-g)\frac{\partial^2}{\partial\tilde{g}^2}j_{l'}(-\tilde{g})-\frac{\partial^2}{\partial\tilde{g}^2}j_l(-g)j_{l'}(-\tilde{g}) \\
&-4\cos\bar{\theta}\frac{\partial}{\partial g}j_l(-g)\frac{\partial}{\partial\tilde{g}}j_{l'}(-\tilde{g})+\frac{\partial^2}{\partial g^2}j_l(-g)\frac{\partial^2}{\partial\tilde{g}^2}j_{l'}(-\tilde{g})]\cdot P_l(\cos\theta) \\
&P_{l'}(-\sin\theta\cos\phi\sin\bar{\theta}-\cos\theta\cos\bar{\theta})\,.
\end{aligned}
\tag{6.32}
$$

Then we use addition theorem of spherical harmonics to expand the Legendre polynomials, which will help to solve the angular integral:

$$
P_l(\hat{\mathbf{n}}\cdot\hat{\mathbf{k}})=\frac{4\pi}{(2l+1)}\sum_{-l}^{l}Y_{ls}(\hat{\mathbf{n}})Y_{ls}^*(\hat{\mathbf{k}})\,. \tag{6.33}
$$

Next we use the orthogonality of spherical harmonics:

$$
\int_0^{\pi} d\theta\,\sin\theta\int_0^{2\pi} d\phi\,Y_{lm}(\theta,\phi)\,Y_{l'm'}^*(\theta,\phi)\;=\delta_{l,l'}\delta_{m,m'} \tag{6.34}
$$

to solve the angular part of the $\mathbf{k}$ integral.

$$C(\bar{\theta}) = \frac{720^2(2.5\times10^{-3})H^4}{2^{30}\,\pi^5\xi^6 M_P^4} e^{4\pi\xi}\int_{\eta_r}^{\eta_0} d\eta \int_{\eta_r}^{\eta_0} d\tilde{\eta}\int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} dk\,\frac{1}{k}[\frac{\partial}{\partial\eta}\frac{\partial}{\partial\tilde{\eta}}(-\frac{1}{k^2\eta^2}\cos(k\eta)$$
$$+\frac{1}{k^3\eta^3}\sin(k\eta))\cdot(-\frac{1}{k^2\tilde{\eta}^2}\cos(k\tilde{\eta})+\frac{1}{k^3\tilde{\eta}^3}\sin(k\tilde{\eta}))]\cdot\sum_{l,l'=0}^{\infty} i^{l+l'}[(-1+2\cos^2\bar{\theta})$$
$$\cdot j_l(-g)j_{l'}(-\tilde{g}) - j_l(-g)\frac{\partial^2}{\partial\tilde{g}^2}j_{l'}(-\tilde{g}) - \frac{\partial^2}{\partial\tilde{g}^2}j_l(-g)j_{l'}(-\tilde{g}) - 4\cos\bar{\theta}\frac{\partial}{\partial g}j_l(-g)\frac{\partial}{\partial\tilde{g}}j_{l'}(-\tilde{g})$$
$$+\frac{\partial^2}{\partial g^2}j_l(-g)\frac{\partial^2}{\partial\tilde{g}^2}j_{l'}(-\tilde{g})]\cdot(4\pi)^2\sum_{s=-l}^{l}\sum_{s'=-l'}^{l'} Y_{ls}(\hat{\mathbf{l}}_{\mathbf{1}})Y^*_{l's'}(-\hat{\mathbf{l}}_{\mathbf{2}})\delta_{l,l'}\delta_{s,s'} \tag{6.35}$$

Now we can use the reverse of the addition theorem to write:

$$\sum_{s=-l}^{l} Y_{ls}(\hat{\mathbf{l}}_{\mathbf{1}})Y^*_{ls}(-\hat{\mathbf{l}}_{\mathbf{2}}) = \frac{(2l+1)}{4\pi}P_l(-\hat{\mathbf{l}}_{\mathbf{1}}\cdot\hat{\mathbf{l}}_{\mathbf{2}})\,. \tag{6.36}$$

$$C(\bar{\theta}) = \frac{720^2(2.5\times10^{-3})H^4}{2^{30}\,\pi^5\xi^6 M_P^4} e^{4\pi\xi}\int_{\eta_r}^{\eta_0} d\eta \int_{\eta_r}^{\eta_0} d\tilde{\eta}\int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} dk\,\frac{1}{k}[\frac{\partial}{\partial\eta}\frac{\partial}{\partial\tilde{\eta}}(-\frac{1}{k^2\eta^2}\cos(k\eta)+\frac{1}{k^3\eta^3}\sin(k\eta))$$
$$\cdot(-\frac{1}{k^2\tilde{\eta}^2}\cos(k\tilde{\eta})+\frac{1}{k^3\tilde{\eta}^3}\sin(k\tilde{\eta}))]\cdot\sum_{l}^{\infty} i^{2l}[(-1+2\cos^2\bar{\theta})\cdot j_l(-g)j_l(-\tilde{g})$$
$$-j_l(-g)\frac{\partial^2}{\partial\tilde{g}^2}j_l(-\tilde{g}) - \frac{\partial^2}{\partial\tilde{g}^2}j_l(-g)j_l(-\tilde{g}) - 4\cos\bar{\theta}\frac{\partial}{\partial g}j_l(-g)\frac{\partial}{\partial\tilde{g}}j_l(-\tilde{g})$$
$$+\frac{\partial^2}{\partial g^2}j_l(-g)\frac{\partial^2}{\partial\tilde{g}^2}j_l(-\tilde{g})]\cdot\cdot(4\pi)(2l+1)P_l(-\hat{\mathbf{l}}_{\mathbf{1}}\cdot\hat{\mathbf{l}}_{\mathbf{2}}) \tag{6.37}$$

Next we use identity: $xP_l(x) = \frac{(l+1)}{(2l+1)}P_{l+1}(x) + \frac{l}{(2l+1)}P_{l-1}(x)$ where for us, $x = -\cos\bar{\theta}$. Taking the next iteration of this, we get:

$x^2P_l(x) = \frac{(l+1)}{(2l+1)}[\frac{(l+2)}{(2l+3)}P_{l+2}(x) + \frac{(l+1)}{(2l+3)}P_l(x)] + \frac{l}{(2l+1)}[\frac{l}{(2l-1)}P_l(x) + \frac{(l-1)}{(2l-1)}P_{l-2}(x)]$.

Plugging these in:

$$C(\bar{\theta}) = \frac{720^2(2.5\times10^{-3})H^4}{2^{28}\,\pi^4\xi^6 M_P^4} e^{4\pi\xi}\int_{\eta_r}^{\eta_0} d\eta \int_{\eta_r}^{\eta_0} d\tilde{\eta}\int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} dk\,\frac{1}{k}[\frac{\partial}{\partial\eta}\frac{\partial}{\partial\tilde{\eta}}(-\frac{1}{k^2\eta^2}\cos(k\eta)+\frac{1}{k^3\eta^3}\sin(k\eta))$$
$$\cdot(-\frac{1}{k^2\tilde{\eta}^2}\cos(k\tilde{\eta})+\frac{1}{k^3\tilde{\eta}^3}\sin(k\tilde{\eta}))]\cdot\sum_{l}^{\infty} i^{2l}(2l+1)[-j_l(-g)j_l(-\tilde{g})P_l(x) + 2j_l(-g)j_l(\tilde{g})$$
$$[\frac{(l+1)(l+2)}{(2l+1)(2l+3)}P_{l+2}(x) + \frac{(l+1)^2}{(2l+1)(2l+3)}P_l(x) + \frac{l^2}{(2l+1)(2l-1)}P_l(x)$$
$$+\frac{l(l-1)}{(2l+1)(1l-1)}P_{l-2}(x)] - j_l(-g)\frac{\partial^2}{\partial\tilde{g}^2}j_l(-\tilde{g})P_l(x) - \frac{\partial^2}{\partial\tilde{g}^2}j_l(-g)j_l(-\tilde{g})P_l(x)$$
$$+4\frac{\partial}{\partial g}j_l(-g)\frac{\partial}{\partial\tilde{g}}j_l(-\tilde{g})[\frac{(l+1)}{(2l+1)}P_{l+1}(x) + \frac{l}{(2l+1)}P_{l-1}(x)]$$
$$+\frac{\partial^2}{\partial g^2}j_l(-g)\frac{\partial^2}{\partial\tilde{g}^2}j_l(-\tilde{g})P_l(x)] \tag{6.38}$$

Next we use the identities: $\frac{\partial}{\partial x} j_l(x) = -\frac{(l+1)}{(2l+1)} j_{l+1}(x) + \frac{l}{(2l+1)} j_{l-1}$ and the second iteration: $\frac{\partial^2}{\partial x^2} j_l(x) = \frac{(l+1)}{(2l+1)}[\frac{(l+2)}{(2l+3)} j_{l+2}(x) - \frac{(l+1)}{(2l+3)} j_l(x)] + \frac{l}{(2l+1)}[-\frac{l}{(2l-1)} j_l(x) + \frac{(l-1)}{(2l-1)} j_{l-2}(x)]$.

Next we want to Fourier transform into spherical harmonics to use C_l instead of $C(\theta)$. C_l is conventionally defined by: $C_l = 2\pi \int_0^\pi d\theta \, \sin\bar{\theta} C(\bar{\theta}) P_l(\cos\bar{\theta})$ We can then solve the $\bar{\theta}$ integral using the identity: $\int_0^\pi d\theta \, \sin\theta P_l(\cos\theta) P_{l'}(\cos\theta) = \frac{2}{(2l+1)} \delta_{l,l'}$.

Both the arguments in the above equation need to be $\cos\theta$, but as of now our expressions in our C_l equation have terms with $P_l(\cos\bar{\theta})$ and $P_{l'}(-\cos\bar{\theta})$. We use identity: $P_l(-x) = (-1)^l P_l(x)$.

Our two-point function eventually simplifies to:

$$C_l = \frac{720^2 (2.5 \times 10^{-3}) H^4}{2^{26} \pi^3 \xi^6 M_P^4} e^{4\pi\xi} \, l\,(l-1)\,(l+1)\,(l+2) \int_{\eta_r}^{\eta_0} d\eta \int_{\eta_r}^{\eta_0} d\tilde{\eta} \int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} dk \, \frac{1}{k} \frac{j_l^2(k(\eta_0-\eta)}{k^2(\eta_0^2-\eta)^2}$$
$$\frac{j_l^2(k(\eta_0-\tilde{\eta})}{k^2(\eta_0^2-\tilde{\eta})^2} [\frac{\partial}{\partial\eta}\frac{\partial}{\partial\tilde{\eta}}(-\frac{1}{k^2\eta^2}\cos(k\eta) + \frac{1}{k^3\eta^3}\sin(k\eta))(-\frac{1}{k^2\tilde{\eta}^2}\cos(k\tilde{\eta}) + \frac{1}{k^3\tilde{\eta}^3}\sin(k\tilde{\eta}))] \, . \tag{6.39}$$

Next, we want to compute the η and $\tilde{\eta}$ integrals. Doing out the partial derivatives, and then integrating still leads to non-standard functions, so we want to make another approximation. We can take advantage of the fact that $\frac{\partial H_{jmat}}{\partial\eta}$ peaks at about $k\eta \approx$ order 1. Regardless of whether we are concerned with $l \gg 1$ or l of order 1, in both cases the integrand will be maximized for $k\eta \approx 1$.

If we look back at the k integral, we see the k integral is clearly maximized for $\eta \approx \eta_0$ as opposed to $\eta \approx \eta_r$, which means for $k\eta \approx 1$, this favors $\eta \approx \eta_0$. This can be seen from:
$\int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} d^3\mathbf{k} \, \frac{1}{k^3}(-\frac{1}{k^2\eta^2}\cos(k\eta) + \frac{1}{k^3\eta^3}\sin(k\eta)) \cdot (-\frac{1}{k^2\tilde{\eta}^2}\cos(k\tilde{\eta}) + \frac{1}{k^3\tilde{\eta}^3}\sin(k\tilde{\eta})) e^{i\mathbf{k}\cdot(\mathbf{l_1}-\mathbf{l_2})\eta_0}$
Since k is maximized for $k = \frac{1}{\eta_r}$, then $\frac{1}{k}$ is maximized for $k = \frac{1}{\eta_0}$. The integrand $\propto \frac{1}{k^3}$, which is maximized for $k \approx \frac{1}{\eta_0}$ (since the exponential just oscillates, and for $k\eta = 1$, the trig pieces just give $.3^2$, a constant). The next step is to decide if we want to focus on $l \gg 1$ or l of order 1, because the approximation we make will be different in each case. First let us talk about the $l \gg 1$ case. The behavior of $\frac{j_l(x)}{x^2}$ for $l \gg 1$ is that for $x < l$, the function is approximately 0 and then becomes peaked for $l \approx x$, after which point it oscillates with decreasing amplitude. Also note, $\frac{j_l(x)}{x^2}$ gives 0 for $x = 0$, corresponding to $\eta = \eta_0$, (except for $l = 1$ or 2). So $\eta = \eta_0$ gives no contribution to the integral (unless $l = 1$ or 2). As we noted above, the k dependence favors $k \approx \frac{1}{\eta_0}$. Since we said $\eta \approx 1/k$ is separately favored from the $\frac{\partial H_{jmat}}{\partial\eta}$ contribution, combining these two effects, $\eta \approx$ but a little less than η_0 is favored. There is actually an even stronger suppression in this region if l large which comes from the fact that $\frac{j_l(x)}{x^2}$ is approximately 0 unless $x \geq l$. Therefore, for $l \gg 1$ there is almost no contribution except for $x \gg 1$ which corresponds to $k(\eta_0 - \eta) \gg 1$. Since we saw separately that the integrand is maximized for $k\eta \approx 1$, this means we get maximum contribution from $k\eta_0 \gg 1$. Therefore, assuming $l \gg 1$ automatically gives largest contribution for $\eta_0 \gg \eta$.

Assuming $\eta_0 \gg \eta$: $\frac{j_l(k(\eta_0-\eta))}{(k(\eta_0-\eta))^2} \approx \frac{j_l(k\eta_0))}{k^2\eta_0^2}$ The only region we care about is $\eta \ll \eta_0$ and we are integrating over η, so for whole region we get significant contribution from, $\frac{j_l(x)}{x^2} \approx$ constant. We can pull the $\frac{j_l(x)}{x^2}$ out of the η integral, and then the $\frac{\partial}{\partial\eta}$ and the $\int d\eta$ cancel

from the fundamental theorem of calculus:

$$C_l = \frac{720^2(2.5\times10^{-3})H^4}{2^{26}\,\pi^3\xi^6 M_P^4} e^{4\pi\xi}\, l\,(l-1)\,(l+1)\,(l+2) \int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} dk\, \frac{1}{k}\frac{j_l^2(k\eta_0)}{k^4\eta_0^4} \left(-\frac{1}{k^2\eta^2}\cos(k\eta) + \frac{1}{k^3\eta^3}\sin(k\eta)\right)^2 \Big|_{\eta=\eta_r}. \tag{6.40}$$

We should really be applying the limits $\Big|_{\eta=\eta_r}^{\eta=\eta_0}$, but since we found $\eta=\eta_0$ gives 0 for large l, we really just take the $\eta=\eta_r$ limit. C_l then simplifies to:

$$C_l = \frac{720^2(2.5\times10^{-3})H^4}{2^{26}\,\pi^3\xi^6 M_P^4} e^{4\pi\xi}\, l\,(l-1)\,(l+1)\,(l+2) \int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} dk\, \frac{1}{k}\frac{j_l^2(k\eta_0)}{k^4\eta_0^4} \left(-\frac{1}{k^2\eta_r^2}\cos(k\eta_r) + \frac{1}{k^3\eta_r^3}\sin(k\eta_r)\right)^2. \tag{6.41}$$

We still can not solve the k integral as is, and so need to make another approximation. It is helpful to first consider the behavior of the various functions that make up the remaining integrand.

First, we know the $\left(-\frac{1}{k^2\eta_r^2}\cos(k\eta_r) + \frac{1}{k^3\eta_r^3}\sin(k\eta_r)\right)^2$ piece will peak for $k=\frac{1}{\eta_r}$, because, as we saw earlier, it favors $k\eta\approx1$ where the η in the argument now is η_r.

Secondly, we know the $\frac{j_l^2(k\eta_0)}{k^4\eta_0^4}$ piece is peaked for $l=k\eta_0$, which gives $k\eta_0\gg1$ since we are in the $l\gg1$ regime. Therefore $k\gg\frac{1}{\eta_0}$ and k is closer to the $\frac{1}{\eta_r}$ part of its limit bounds. ($\eta_r\ll\eta_0$ so $\frac{1}{\eta_r}\gg\frac{1}{\eta_0}$).

Lastly, the $\frac{1}{k^5}$ piece is peaked instead for $k\approx\frac{1}{\eta_0}$ since $k\approx\frac{1}{\eta_0}\to\frac{1}{k^5}\approx\eta_0^5\gg\eta_r^5$. But if $k\approx\frac{1}{\eta_0}$, then we get $\frac{j_l^2(1)}{1^4}$ and with $l\gg1$, this is 0. Therefore even with the $\frac{1}{k^5}$ term, k closer to $\frac{1}{\eta_r}$ wins out.

Now we use the approximation that since we found the integral favors $k\approx\frac{1}{\eta_r}$ and will be 0 for k much less than $\frac{1}{\eta_r}$ from the Bessel function term, we can approximate $\left(-\frac{1}{k^2\eta_r^2}\cos(k\eta_r) + \frac{1}{k^3\eta_r^3}\sin(k\eta_r)\right)^2$ for $k\approx\frac{1}{\eta_r}$ which gives us $(-\cos(1)+\sin(1))^2=.3^2$. This approximation is allowed because this function comes from the equation of motion of gravitational waves in a matter dominated universe, which is approximately constant for waves outside, or more relevantly, at and just within the horizon. In other words, for $k\eta\approx1$. Therefore, since this function is approximately constant over our region of interest, we can use the value it has at horizon entry, and pull it out of the k integral. Our expression for C_l then simplifies to:

$$C_l = \frac{720^2(2.5\times10^{-3})H^4}{2^{26}\,\pi^3\xi^6 M_P^4\eta_0^4}(.3)^2 e^{4\pi\xi}\, l\,(l-1)\,(l+1)\,(l+2) \int_{\frac{1}{\eta_0}}^{\frac{1}{\eta_r}} dk\, \frac{1}{k^5} j_l^2(k\eta_0)\,. \tag{6.42}$$

We use the integral:

$$\int_0^\infty ds\, s^{m-1} j_l^2(s) = 2^{m-3}\pi\frac{\Gamma(2-m)\,\Gamma(l+\frac{m}{2})}{\Gamma^2(\frac{3-m}{2})\,\Gamma(l+2-\frac{m}{2})} \tag{6.43}$$

Let $s = k\eta_0$. Then $k_{\text{lower bound}} = \frac{1}{\eta_0}$ corresponds to $s_{\text{lower bound}} = 1$ and $k_{\text{upper bound}} = \frac{1}{\eta_r}$ corresponds to $s_{\text{upper bound}} = \frac{\eta_0}{\eta_r} \gg 1$ (since $\eta_0 \gg \eta_r$.)

If $\frac{\eta_0}{\eta_r} \gg l$, then we can approximate $\frac{\eta_0}{\eta_r} \approx \infty$. Note $\frac{j_l(x)}{x^2}$ is 0 when $x \gg l$. If we require $\frac{\eta_0}{\eta_r} \gg l$, then setting this bound at ∞ is valid, since once s reaches its upper bound, the integrand has already decreased to approximately 0, so increasing the bound to ∞ will not contribute anything new to the integral anyway.

Next we want to approximate the lower bound as 0. We need to show that contributions from the region $0 < s < 1$ are negligible. This is true because we know $j_l(s)$ for $l \gg 1$ and $s \lesssim 1$ is approximately 0. Then we are able to make use of eq. (6.43) to obtain:

$$C_l = \frac{720^2(2.5\times10^{-3})H^4}{2^{26}\,\pi^3\xi^6 M_P^4}(.3)^2 e^{4\pi\xi}\, l\,(l-1)\,(l+1)\,(l+2)\,2^{-7}\pi\frac{\Gamma(6)\,\Gamma(l-2)}{\Gamma^2(\frac{7}{2})\,\Gamma(l+4)}\,, \tag{6.44}$$

which can be simplified to

$$l(l+1)C_l = 1.5\times10^{-8}\frac{H^4 e^{4\pi\xi}}{M_P^4\,\xi^6}\,, \tag{6.45}$$

using that $l \gg 1$. Then if we estimate $\xi = 2.6$ (the upper limit of ξ at CMB scales if ϕ is the inflaton field) and $\frac{H}{M_p} = 10^{-4}$:

$$l(l+1)C_l = 7.5\times10^{-13}\,. \tag{6.46}$$

This is many orders of magnitude too small to be detected since it will be dwarfed by other contributions to C_l, most notably from the contribution from the standard scalar inflationary metric perturbations. This is not surprising though since we saw that at CMB scales for this limit of ξ, the production of tensors from this model will not even dominate the tensor spectrum, and the tensor spectrum is a sub-leading contributor to C_l. The interesting part will be to use these same techniques to apply to calculating the temperature three-point function. Since the contributions from the standard inflation spectra are dominating C_l but are very strongly Gaussian, it is possible that the contribution from our $\phi F\tilde{F}$ model could dominate the temperature three-point function. We can then use this temperature three-point function to put limits on ξ from the non-observation of non-Gaussianities in $\langle TTT\rangle$ by Planck.

6.2 Flat Sky Approximation

First I tried solving for the temperature bispectrum in the same manner as for the power spectrum, but was unable to solve the integrals that came out in the end, and so instead switched to solving in the flat sky approximation. This approximation is valid for suitably high multipoles l. Since these multipoles roughly correspond to angular separation θ between the directions we are measuring the temperature, high l means small θ which means the three directions we are looking to measure the temperature are closely clustered together. When computing, $\langle\frac{\delta T(\mathbf{x_0},\eta_0,\mathbf{n}_1)}{T}\frac{\delta T(\mathbf{x_0},\eta_0,\mathbf{n}_2)}{T}\frac{\delta T(\mathbf{x_0},\eta_0,\mathbf{n}_3)}{T}\rangle$, the three $\mathbf{n}$ vectors are approximately parallel. In this small region, the sky looks approximately flat, and we can approximate it as such. We call the direction in which these 3 vectors are clustered $\hat{z}$.

We do a Fourier transform exchanging $\frac{\delta T}{T}$ for the variable a given by:

$$a(\mathbf{x_0}, \eta_0, \mathbf{l}) = \int_{-\infty}^{\infty} \frac{d^2\mathbf{n}}{2\pi} e^{-i\mathbf{l}\cdot\mathbf{n}} \frac{\delta T(\mathbf{x_0}, \eta_0, \mathbf{n})}{T} \tag{6.47}$$

In the process we trade the x and y components of the vector $\mathbf{n}$ for the vector $\mathbf{l}$. $\mathbf{l}$ is a 2 dimensional vector whereas $\mathbf{n}$ exists in 3 dimensions. Note $\mathbf{n}$ still only has 2 degrees of freedom because it is a unit vector, while $\mathbf{l}$ is not.

The integrand above over $\mathbf{n}$ has high frequency oscillations when $\mathbf{n}\cdot\mathbf{l}$ is large; this causes the integral to be highly suppressed for these values of $\mathbf{n}$. In other words, the integral only gets significant contributions when $\mathbf{n}$ is approximately along $\hat{z}$, perpendicular to $\mathbf{l}$, (remember $\mathbf{l}$ lives in the x-y plane). It is therefore safe to approximate, $n_z \gg n_x$, n_y. This is what ensures that the 3 directions that we are measuring the temperature are approximately parallel.

6.3 Temperature Bispectrum from Tensors

In order to compare with Planck's f_{NL} result, we need the temperature bispectrum given by:

$$B_{l_1\, l_2\, l_3} = \langle a(\mathbf{l_1}) a(\mathbf{l_2}) a(\mathbf{l_3}) \rangle \frac{1}{\delta^{(3)}(\mathbf{l_1} + \mathbf{l_2} + \mathbf{l_3})} \,. \tag{6.48}$$

Plugging in eq. (6.47) to expand, this becomes:

$$\begin{aligned} B_{l_1\, l_2\, l_3} = \int \frac{d^2\mathbf{n_1}}{2\pi} \int \frac{d^2\mathbf{n_2}}{2\pi} \int d^2 \frac{\mathbf{n_3}}{2\pi} e^{-i\mathbf{l_1}\cdot\mathbf{n_1}} e^{-i\mathbf{l_2}\cdot\mathbf{n_2}} e^{-i\mathbf{l_3}\cdot\mathbf{n_3}} \frac{1}{\delta^{(3)}(\mathbf{l_1} + \mathbf{l_2} + \mathbf{l_3})} \langle \frac{\delta T(\mathbf{x_0}, \eta_0, \mathbf{n}_1)}{T} \\ \frac{\delta T(\mathbf{x_0}, \eta_0, \mathbf{n}_2)}{T} \frac{\delta T(\mathbf{x_0}, \eta_0, \mathbf{n}_3)}{T} \rangle \,. \end{aligned} \tag{6.49}$$

We plug in the definition for the temperature tensor anisotropies as defined for the calculation of the temperature power spectrum:

$$\frac{\delta T(\mathbf{x_0}, \eta_0, \mathbf{n}_2)}{T} = -\frac{1}{2} n_{2\,i} n_{2\,j} \int d\eta \frac{\partial}{\partial \eta} h_{ij}(x, \eta) \,, \tag{6.50}$$

which we can expand

$$\frac{\delta T(\mathbf{x_0}, \eta_0, \mathbf{n}_2)}{T} = -\frac{1}{2} \int d\eta \frac{\partial}{\partial \eta} [n_{2\,x} n_{2\,x} h_{xx}(x, \eta) + n_{2\,y} n_{2\,y} h_{yy}(x, \eta) + n_{2\,z} n_{2\,z} h_{zz}(x, \eta)] \,. \tag{6.51}$$

Since we have in the flat sky approx that $n_z \gg n_x, n_y$, such that $n_z \simeq 1$, we can approximate this as:

$$\frac{\delta T(\mathbf{x_0}, \eta_0, \mathbf{n}_2)}{T} = -\frac{1}{2} \int d\eta \frac{\partial}{\partial \eta} h_{zz}(x, \eta) \,. \tag{6.52}$$

We use Fourier transform of h_{zz} into momentum space and plug the temperature function into the bispectrum to obtain:

$$B_{l_1\,l_2\,l_3} = -\frac{1}{2^3\,(2\pi)^{\frac{15}{2}}}\int d^2\mathbf{n_1}\int d^2\mathbf{n_2}\int d^2\mathbf{n_3} e^{-i\mathbf{l_1}\cdot\mathbf{n_1}} e^{-i\mathbf{l_2}\cdot\mathbf{n_2}} e^{-i\mathbf{l_3}\cdot\mathbf{n_3}} \frac{1}{\delta^{(3)}(\mathbf{l_1}+\mathbf{l_2}+\mathbf{l_3})}$$
$$\int d\eta_1 \int d\eta_2 \int d\eta_3 \frac{\partial}{\partial\eta_1}\frac{\partial}{\partial\eta_2}\frac{\partial}{\partial\eta_3}\int d^3\mathbf{k}_1 \int d^3\mathbf{k}_2 \int d^3\mathbf{k}_3 e^{i\mathbf{k_1}\cdot\mathbf{x_1}} e^{i\mathbf{k_2}\cdot\mathbf{x_2}} e^{i\mathbf{k_3}\cdot\mathbf{x_3}}$$
$$\langle h_{zz}(\mathbf{k}_1,\eta_1)\, h_{zz}(\mathbf{k}_2,\eta_2)\, h_{zz}(\mathbf{k}_3,\eta_3)\rangle\,. \tag{6.53}$$

As we did when solving for the power spectrum, we plug in the matter dominated universe equation for h_{zz}, matching boundary conditions with the amplitude of the perturbations generated during inflation:

$$h_{ij\,mat}(\mathbf{k},\eta) = 3(-\frac{1}{k^2\eta^2}\cos(k\eta) + \frac{1}{k^3\eta^3}\sin(k\eta))h_{ij}(\mathbf{k},\eta_{ei})\,, \tag{6.54}$$

where $\eta_{ei} = \eta$ evaluated at the end of inflation, and

$$h_{ij}(\mathbf{k},\eta_{ei}) = -\frac{2}{M_P^2(2\pi)^{\frac{3}{2}}}\int d\eta' \frac{1}{a^2(\eta')} G(k,\eta',\eta_{ei})\Pi_{ij}^{lm}(\mathbf{k})\int d^3\mathbf{q} A_l^{'}(\mathbf{q},\eta')A_m^{'}(\mathbf{k}-\mathbf{q},\eta') \tag{6.55}$$

is the amplitude of the metric perturbations at the end of inflation.

The $\mathbf{x}$ that appears in the exponental in the bispectrum in eq. (6.53) is given by $\mathbf{x} = \mathbf{x}_0 + \mathbf{n}(\eta-\eta_0)$ and is the position of the photons at the time at which we are evaluating h_{zz}. The bispectrum expands to:

$$B_{l_1\,l_2\,l_3} = -\frac{3^3}{2^3\,(2\pi)^{\frac{15}{2}}}\int d^2\mathbf{n_1}\int d^2\mathbf{n_2}\int d^2\mathbf{n_3} e^{-i\mathbf{l_1}\cdot\mathbf{n_1}} e^{-i\mathbf{l_2}\cdot\mathbf{n_2}} e^{-i\mathbf{l_3}\cdot\mathbf{n_3}} \frac{1}{\delta^{(3)}(\mathbf{l_1}+\mathbf{l_2}+\mathbf{l_3})}\int d\eta_1 \int d\eta_2$$
$$\int d\eta_3 \frac{\partial}{\partial\eta_1}\frac{\partial}{\partial\eta_2}\frac{\partial}{\partial\eta_3}\int d^3\mathbf{k}_1 \int d^3\mathbf{k}_2 \int d^3\mathbf{k}_3 e^{i\mathbf{k_1}\cdot\mathbf{x_0}} e^{i\mathbf{k_2}\cdot\mathbf{x_0}} e^{i\mathbf{k_3}\cdot\mathbf{x_0}} e^{-i\mathbf{k_1}\cdot\mathbf{n_1}(\eta_0-\eta_1)} e^{-i\mathbf{k_2}\cdot\mathbf{n_2}(\eta_0-\eta_2)}$$
$$e^{-i\mathbf{k_3}\cdot\mathbf{n_e}(\eta_0-\eta_3)}(-\frac{1}{k_1^2\eta_1^2}\cos(k_1\eta_1) + \frac{1}{k_1^3\eta_1^3}\sin(k_1\eta_1))(-\frac{1}{k_2^2\eta_2^2}\cos(k_2\eta_2) + \frac{1}{k_2^3\eta_2^3}\sin(k_2\eta_2))$$
$$(-\frac{1}{k_3^2\eta_3^2}\cos(k_3\eta_3) + \frac{1}{k_3^3\eta_3^3}\sin(k_3\eta_3))\langle h_{zz}(\mathbf{k}_1,\eta_{ei})\, h_{zz}(\mathbf{k}_2,\eta_{ei})\, h_{zz}(\mathbf{k}_3,\eta_{ei})\rangle\,. \tag{6.56}$$

Next we approximate the $d\eta$ integrals:

$$\int d\eta e^{i\mathbf{k}\cdot\mathbf{n}\eta}\frac{\partial}{\partial\eta}(-\frac{1}{k^2\eta^2}\cos(k\eta) + \frac{1}{k^3\eta^3}\sin(k\eta)) \tag{6.57}$$

first rewriting as:

$$\int d\eta e^{i\mathbf{k}\cdot\mathbf{n}\eta}[\frac{3}{k^2\eta^3}\cos(k\eta) + (\frac{1}{k\eta^2} - \frac{3}{k^3\eta^4})\sin(k\eta)]\,. \tag{6.58}$$

Note the trig part, although oscillating, oscillates with decreasing amplitude such that it peaks on its first oscillation centered at $k\eta = 2.5$, and then has sharply decreasing amplitude for larger $k\eta$. Note $k\eta = 1$ corresponds to modes the size of the horizon. Therefore, there is

a peak for modes which are almost horizon sized, but still slightly inside the horizon, and a much smaller amplitude for modes which have not reached horizon size yet. This is why the three-point function is peaked in the equilateral shape, because the integral favors when all three modes reach horizon size at the same time.

Next notice the exponential term oscillates between 1 and -1 where the frequency of the oscillations depends on the value of $\mathbf{k}\cdot\mathbf{n}$.

We approximate by using the fact that the integral will be dominated by the region around $k\eta \approx 2.5$ where it is maximal. We integrate the trig part around this maximum without the exponent. Then we multiply the result with the value of the exponential term evaluated at the maximum. In other words, this is effectively like saying the trig part goes to a prefactor which is the result of its integral all by itself, times a delta function on η:

$$\int d\eta e^{i\mathbf{k}\cdot\mathbf{n}\eta}[\frac{3}{k^2\eta^3}\cos(k\eta)+(\frac{1}{k\eta^2}-\frac{3}{k^3\eta^4})\sin(k\eta)] \approx \\ e^{i\mathbf{k}\cdot\mathbf{n}\eta}\Big|_{k\eta=2.5}\int d\eta[\frac{3}{k^2\eta^3}\cos(k\eta)+(\frac{1}{k\eta^2}-\frac{3}{k^3\eta^4})\sin(k\eta)]\,. \tag{6.59}$$

Note regardless of what k is, there is an η such that $k\eta \approx 1$ and:

$$\int d\eta[\frac{3}{k^2\eta^3}\cos(k\eta)+(\frac{1}{k\eta^2}-\frac{3}{k^3\eta^4})\sin(k\eta)] \approx -0.3255 \approx -\frac{1}{3}\,. \tag{6.60}$$

Putting everything together we get the approximation:

$$\int d\eta e^{i\mathbf{k}\cdot\mathbf{n}\eta}\frac{\partial}{\partial\eta}(-\frac{1}{k^2\eta^2}\cos(k\eta)+\frac{1}{k^3\eta^3}\sin(k\eta)) \approx -\frac{1}{3}e^{i\frac{\mathbf{k}\cdot\mathbf{n}2.5}{k}}\,. \tag{6.61}$$

Note the magnitude of k is irrelevant for purpose of the approximation; one can redefine η to absorb the magnitude of k. What will determine how effective an approximation this is, is $\hat{\mathbf{k}}\cdot\mathbf{n}$. To show that this approximation is valid, compare lines in Figure 6.1, comparing the approximation to the actual integral for set examples of $\hat{\mathbf{k}}\cdot\mathbf{n}$. The pink is the exact expression and the blue is the approximation. The approximation is worse for larger values of $\hat{\mathbf{k}}\cdot\mathbf{n}$ when the two vectors are approximately parallel, but even at its worse, the approximation is still only off by a factor of 1.7.

Using this approximation, the bispectrum reduces to:

$$B_{l_1\,l_2\,l_3} = \frac{1}{2^3\,(2\pi)^{\frac{15}{2}}}\int d^2\mathbf{n_1}\int d^2\mathbf{n_2}\int d^2\mathbf{n_3}e^{-i\mathbf{l_1}\cdot\mathbf{n_1}}e^{-i\mathbf{l_2}\cdot\mathbf{n_2}}e^{-i\mathbf{l_3}\cdot\mathbf{n_3}}\frac{1}{\delta^{(3)}(\mathbf{l_1}+\mathbf{l_2}+\mathbf{l_3})}\int d^3\mathbf{k}_1 \\ \int d^3\mathbf{k}_2\int d^3\mathbf{k}_3 e^{i\mathbf{k_1}\cdot\mathbf{x_0}}e^{i\mathbf{k_2}\cdot\mathbf{x_0}}e^{i\mathbf{k_3}\cdot\mathbf{x_0}}e^{-i\mathbf{k_1}\cdot\mathbf{n_1}\eta_0}e^{-i\mathbf{k_2}\cdot\mathbf{n_2}\eta_0}e^{-i\mathbf{k_3}\cdot\mathbf{n_3}\eta_0}e^{i\frac{\mathbf{k_1}\cdot\mathbf{n_1}2.5}{k_1}}e^{i\frac{\mathbf{k_2}\cdot\mathbf{n_2}2.5}{k_2}}e^{i\frac{\mathbf{k_3}\cdot\mathbf{n_3}2.5}{k_3}} \\ \langle h_{zz}(\mathbf{k}_1,\eta_{ei})\,h_{zz}(\mathbf{k}_2,\eta_{ei})\,h_{zz}(\mathbf{k}_3,\eta_{ei})\rangle\,. \tag{6.62}$$

Now we make another approximation. Note the k's are being integrated from $\frac{1}{\eta_0}$ to $\frac{1}{\eta_r}$. We want to compare the terms in $(\eta_0-\frac{2.5}{k})$. When k minimized... $(\eta_0-2.5\eta_0)$ so the second term is larger, but they are of the same order. When k maximized... $\eta_0-2.5\eta_r$ then the second term is much smaller. And so for the vast majority of the range that we are integrating k over, $\eta_0 \gg \frac{2.5}{k}$. Even when k is minimized and this no longer holds, the correction is only a factor of -2.5. So we approximate: $(\eta_0-\frac{2.5}{k}) \approx \eta_0$.

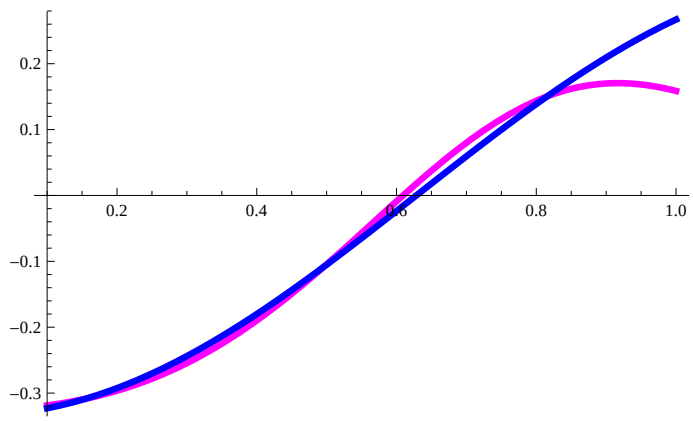

Figure 6.1: The figure tests the validity of the approximation eq. (6.61), plotting on the x-axis $\hat{\mathbf{k}} \cdot \mathbf{n}$. The pink line is the exact expression and the blue is the approximation.

Next, we break $\mathbf{k}$ into parallel and perpendicular components, where parallel is defined to be the x-y plane, the plane in which the $\mathbf{l}$ vectors live, and the z direction is the perpendicular direction.

$$\begin{aligned} B_{l_1\, l_2\, l_3} =& \frac{1}{2^3\,(2\pi)^{\frac{15}{2}}} \int d^2\mathbf{n_1} \int d^2\mathbf{n_2} \int d^2\mathbf{n_3} e^{-i\mathbf{l_1}\cdot\mathbf{n_1}} e^{-i\mathbf{l_2}\cdot\mathbf{n_2}} e^{-i\mathbf{l_3}\cdot\mathbf{n_3}} \frac{1}{\delta^{(3)}(\mathbf{l_1}+\mathbf{l_2}+\mathbf{l_3})} \int d^2\mathbf{k}_{1||} \\ & \int d^2\mathbf{k}_{2||} \int d^2\mathbf{k}_{3||} \int dk_{1z} \int dk_{2z} \int dk_{3z} e^{i\mathbf{k_1}\cdot\mathbf{x_0}} e^{i\mathbf{k_2}\cdot\mathbf{x_0}} e^{i\mathbf{k_3}\cdot\mathbf{x_0}} e^{-i\mathbf{k}_{||\mathbf{1}}\cdot\mathbf{n}_{||\mathbf{1}}\eta_0} e^{-i\mathbf{k}_{||\mathbf{2}}\cdot\mathbf{n}_{||\mathbf{2}}\eta_0} \\ & e^{-i\mathbf{k}_{||\mathbf{3}}\cdot\mathbf{n}_{||\mathbf{3}}\eta_0} e^{-ik_{z1}\eta_0} e^{-ik_{z2}\eta_0} e^{-ik_{z3}\eta_0} \langle h_{zz}(\mathbf{k}_1,\eta_{ei})\, h_{zz}(\mathbf{k}_2,\eta_{ei})\, h_{zz}(\mathbf{k}_3,\eta_{ei})\rangle\,, \end{aligned} \tag{6.63}$$

where I have not expanded the k's in the $e^{i\mathbf{k}\cdot\mathbf{x_0}}$ terms because they will cancel when we eventually get that $\langle h_{zz}(\mathbf{k}_1,\eta_{ei})\, h_{zz}(\mathbf{k}_2,\eta_{ei})\, h_{zz}(\mathbf{k}_3,\eta_{ei})\rangle \propto \delta^3(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})$. Next we use that the $\mathbf{n}$ integrals become δ functions.

$$\begin{aligned} B_{l_1\, l_2\, l_3} =& \frac{1}{2^3\,(2\pi)^{\frac{15}{2}}} \frac{1}{\delta^{(3)}(\mathbf{l_1}+\mathbf{l_2}+\mathbf{l_3})} \int d^2\mathbf{k}_{1||} \int d^2\mathbf{k}_{2||} \int d^2\mathbf{k}_{3||} \int dk_{1z} \int dk_{2z} \int dk_{3z} \\ & \delta^{(2)}(-\mathbf{l_1}-\mathbf{k}_{||\mathbf{1}}\eta_0)(2\pi)^2 \delta^{(2)}(-\mathbf{l_2}-\mathbf{k}_{||\mathbf{2}}\eta_0)(2\pi)^2 \delta^{(2)}(-\mathbf{l_3}-\mathbf{k}_{||\mathbf{3}}\eta_0)(2\pi)^2 e^{i\mathbf{k_1}\cdot\mathbf{x_0}} e^{i\mathbf{k_2}\cdot\mathbf{x_0}} e^{i\mathbf{k_3}\cdot\mathbf{x_0}} \\ & e^{-ik_{z1}\eta_0} e^{-ik_{z2}\eta_0} e^{-ik_{z3}\eta_0} \langle h_{zz}(\mathbf{k}_1,\eta_{ei})\, h_{zz}(\mathbf{k}_2,\eta_{ei})\, h_{zz}(\mathbf{k}_3,\eta_{ei})\rangle \end{aligned} \tag{6.64}$$

Note that: $\delta^2(-l_1 - k_{||1}\eta_0) = \frac{1}{\eta_0^2}\delta^{(2)}(-\frac{l_1}{\eta_0} - k_{||1})$. The k vectors are given by: $\mathbf{k}_1 = \langle -\frac{l_1}{\eta_0}, k_{z1}\rangle$, $\mathbf{k}_2 = \langle -\frac{l_2}{\eta_0}, k_{z2}\rangle$, and $\mathbf{k}_3 = \langle -\frac{l_3}{\eta_0}, k_{z3}\rangle$.

We plug in the definition of the transverse-traceless projector: $\Pi^{lm}_{zz}(k) = \delta_{zl}\delta_{zm} - \frac{\delta_{zl}k_z k_m}{k^2} -$

$\frac{\delta_{zm}k_zk_l}{k^2} + \frac{1}{2k^4}k_z^2k_lk_m - \frac{1}{2}\delta_{lm} + \frac{1}{2k^2}k_lk_m + \frac{1}{2k^2}\delta_{lm}k_z^2.$

$$
\begin{aligned}
B_{l_1\,l_2\,l_3} =& -\frac{H^6}{(2\pi)^6\eta_0^4M_P^6}\frac{1}{\delta^{(3)}(\mathbf{l_1}+\mathbf{l_2}+\mathbf{l_3})}\int d^2\mathbf{k}_{1||}\int d^2\mathbf{k}_{2||}\int d^2\mathbf{k}_{3||}\int dk_{1z}\int dk_{2z}\int dk_{3z}\\
&\delta^{(2)}(-\frac{\mathbf{l_1}}{\eta_0}-\mathbf{k}_{||\mathbf{1}})\delta^{(2)}(-\frac{\mathbf{l_2}}{\eta_0}-\mathbf{k}_{||\mathbf{2}})\delta^{(2)}(-\mathbf{l_3}-\mathbf{k}_{||\mathbf{3}}\eta_0)e^{i\mathbf{k_1}\cdot\mathbf{x_0}}e^{i\mathbf{k_2}\cdot\mathbf{x_0}}e^{i\mathbf{k_3}\cdot\mathbf{x_0}}e^{-ik_{z1}\eta_0}e^{-ik_{z2}\eta_0}\\
&e^{-ik_{z3}\eta_0}\int d\eta_1^{'}\int d\eta_2^{'}\int d\eta_3^{'}G(k_1,\eta_1^{'},\eta_{ei})G(k_2,\eta_2^{'},\eta_{ei})G(k_3,\eta_3^{'},\eta_{ei})[\delta_{za}\delta_{zb}-\frac{\delta_{za}k_zk_b}{k^2}\\
&-\frac{\delta_{zb}k_zk_a}{k^2}+\frac{1}{2k^4}k_z^2k_ak_b-\frac{1}{2}\delta_{ab}+\frac{1}{2k^2}k_ak_b+\frac{1}{2k^2}\delta_{ab}k_z^2][\delta_{zc}\delta_{zd}-\frac{\delta_{zc}k_zk_d}{k^2}-\frac{\delta_{zd}k_zk_c}{k^2}\\
&+\frac{1}{2k^4}k_z^2k_ck_d-\frac{1}{2}\delta_{cd}+\frac{1}{2k^2}k_ck_d+\frac{1}{2k^2}\delta_{cd}k_z^2][\delta_{ze}\delta_{zf}-\frac{\delta_{ze}k_zk_f}{k^2}-\frac{\delta_{zf}k_zk_e}{k^2}+\frac{1}{2k^4}k_z^2k_ek_f\\
&-\frac{1}{2}\delta_{ef}+\frac{1}{2k^2}k_ek_f+\frac{1}{2k^2}\delta_{ef}k_z^2]\eta_1^{'2}\eta_2^{'2}\eta_3^{'2}\int d^3\mathbf{q}_1\int d^3\mathbf{q}_2\int d^3\mathbf{q}_3\\
&\langle A_a^{'}(\mathbf{q}_1,\eta_1^{'})A_b^{'}(\mathbf{k_1}-\mathbf{q_1},\eta_1^{'})A_c^{'}(\mathbf{q}_2,\eta_2^{'})A_d^{'}(\mathbf{k_2}-\mathbf{q}_2,\eta_2^{'})A_3^{'}(\mathbf{q}_3,\eta_3^{'})A_f^{'}(\mathbf{k_3}-\mathbf{q}_3,\eta_3^{'})\rangle
\end{aligned}
\tag{6.65}
$$

We expand using Wick's theorem and then simplify, defining $\tau = -\eta$ such that τ is positive. Then we plug in for the mode function eq. (4.60): $A_+^{'} = (\frac{k\xi}{2\tau})^{\frac{1}{4}}e^{\pi\xi}e^{-2\sqrt{2\xi k\tau}}$.

We use the same approximation for the Green's function we found when calculating the power spectrum for this model, eq. (4.66, 4.67) : $G \approx -\frac{1}{3}\eta' = \frac{\tau'}{3}$.

The η integrals can be solved by: $\int_{-\infty}^{0} d\eta'\tau'^{\frac{5}{2}}e^{-C\sqrt{\tau'}} = \int_{\infty}^{0}(-d\tau')\tau'^{\frac{5}{2}}e^{-C\sqrt{\tau'}} \approx 2e^{-C\sqrt{\tau_{ei}}}\frac{720}{C^7}$. We approximate $\tau_{ei} \to 0$ so η integral $\to 2\frac{720}{C^7}$. We then define the dimensionless integration variables: $\bar{k} = \frac{k\eta_0}{l_1}$ and $\bar{q} = \frac{q\eta_0}{l_1}$.

$$
\begin{aligned}
l_1^4\,B_{l_1\,l_2\,l_3} =& -\frac{720^3H^6e^{6\pi\xi}}{3^3\,2^{28}(2\pi)^6M_P^6\xi^9}\int_{-\infty}^{\infty}d\bar{k}_{z1}\int_{-\infty}^{\infty}\bar{k}_{z2}[\delta_{za}\delta_{zb}-\frac{\delta_{za}\bar{k}_z\bar{k}_b}{\bar{k}^2}-\frac{\delta_{zb}\bar{k}_z\bar{k}_a}{\bar{k}^2}+\frac{1}{2\bar{k}^4}\bar{k}_z^2\bar{k}_a\bar{k}_b\\
&-\frac{1}{2}\delta_{ab}+\frac{1}{2\bar{k}^2}\bar{k}_a\bar{k}_b+\frac{1}{2\bar{k}^2}\delta_{ab}\bar{k}_z^2][\delta_{zc}\delta_{zd}-\frac{\delta_{zc}\bar{k}_z\bar{k}_d}{\bar{k}^2}-\frac{\delta_{zd}\bar{k}_z\bar{k}_c}{\bar{k}^2}+\frac{1}{2\bar{k}^4}\bar{k}_z^2\bar{k}_c\bar{k}_d-\frac{1}{2}\delta_{cd}\\
&+\frac{1}{2\bar{k}^2}\bar{k}_c\bar{k}_d+\frac{1}{2\bar{k}^2}\delta_{cd}\bar{k}_z^2][\delta_{ze}\delta_{zf}-\frac{\delta_{ze}\bar{k}_z\bar{k}_f}{\bar{k}^2}-\frac{\delta_{zf}\bar{k}_z\bar{k}_e}{\bar{k}^2}+\frac{1}{2\bar{k}^4}\bar{k}_z^2\bar{k}_e\bar{k}_f-\frac{1}{2}\delta_{ef}+\frac{1}{2\bar{k}^2}\bar{k}_e\bar{k}_f\\
&+\frac{1}{2\bar{k}^2}\delta_{ef}\bar{k}_z^2]\int_0^{\infty}d\bar{q}_1\int_0^{\pi}d\theta\int_0^{2\pi}d\phi\bar{q}_1^{\frac{5}{2}}|\bar{k}_1-\bar{q}_1|^{\frac{1}{2}}\sin\theta\frac{1}{(\sqrt{\bar{q}_1}+\sqrt{|\bar{k}_1-\bar{q}_1|})^7}\Big[|\bar{k}_2+\bar{q}_1|^{\frac{1}{2}}\\
&\cdot\frac{1}{(\sqrt{\bar{q}_1}+\sqrt{\bar{k}_2+\bar{q}_1})^7}\frac{1}{(\sqrt{|\bar{k}_1-\bar{q}_1|}+\sqrt{\bar{k}_2+\bar{q}_1})^7}\epsilon_{a+}(q_1)\epsilon_{b+}(k_1-q_1)\epsilon_{c+}^*(q_1)\epsilon_{d+}(k_2+q_1)\\
&\cdot\epsilon_{e+}^*(k_1-q_1)\epsilon_{f+}^*(k_2+q_1)+|\bar{k}_1+\bar{k}_2-\bar{q}_1|^{\frac{1}{2}}\frac{1}{(\sqrt{|\bar{k}_1-\bar{q}_1|}+\sqrt{|\bar{k}_1+\bar{k}_2-\bar{q}_1|})^7}\\
&\cdot\frac{1}{(\sqrt{\bar{q}_1}+\sqrt{|\bar{k}_1+\bar{k}_2-\bar{q}_1|})^7}\epsilon_{a+}(q_1)\epsilon_{b+}(k_1-q_1)\epsilon_{c+}^*(k_1-q_1)\epsilon_{d+}(k_1+k_2-q_1)\epsilon_{e+}^*(q_1)\\
&\cdot\epsilon_{f+}^*(k_1+k_2-q_1)\Big]
\end{aligned}
\tag{6.66}
$$

We know the $\mathbf{l}$ vectors exist in the $x-y$ plane. We can choose without loss of generality that $\mathbf{l_1}$ point along the $\hat{x}$ direction. Then we have in general that the $\mathbf{l}$ vectors are given by:

$\mathbf{l}_1 = l_1\langle 1, 0, 0\rangle$
$\mathbf{l}_2 = l_2\langle \cos\bar{\phi}, \sin\bar{\phi}, 0\rangle$
$\mathbf{l}_3 = l_3\langle \cos\bar{\phi}', \sin\bar{\phi}', 0\rangle$
We define $x_2 = \frac{l_2}{l_1}$ and $x_3 = \frac{l_3}{l_1}$, defined such that $1 \geq x_2 \geq x_3$.
This allows us to obtain $\bar{\phi}$ and $\bar{\phi}'$ in terms of x_2 and x_3:

$$\bar{\phi} = \pi - \cos^{-1}(\frac{1}{2x_2}(1 + x_2^2 - x_3^2)) \tag{6.67}$$

$$\bar{\phi}' = \pi + \cos^{-1}(\frac{1}{2x_3}(1 + x_3^2 - x_2^2))\,. \tag{6.68}$$

This form is useful for taking the different shape limits of the bispectrum. For example, in the equilateral limit, where $l_1 = l_2 = l_3$, this gives, $\bar{\phi} = \frac{2\pi}{3}$ and $\bar{\phi}' = \frac{4\pi}{3}$. The l vectors simplify to:
$\mathbf{l}_1 = l_1\langle 1, 0, 0\rangle$, $\mathbf{l}_2 = l_1\langle -\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\rangle$, $\mathbf{l}_3 = l_1\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\rangle$.
We numerically solve the momentum integrals in the equilateral limit in Mathematica:

$$l^4\, B_{l\,eq} = -1.8 \times 10^{-12}\frac{H^6 e^{6\pi\xi}}{M_P^6 \xi^9}\,. \tag{6.69}$$

6.4 The Scalar Bispectrum

We want to use the results of the temperature bispectrum to compare with the non-Gaussianitiy parameter reported by Planck. First we need to compute the non-Gaussianity parameter from the contribution from the scalar metric perturbations. Then we can relate this to compute the contribution from the tensors. So the next goal is to calculate $\langle\zeta\zeta\zeta\rangle$.

We use the definition of the comoving curvature perturbation, ζ, in terms of perturbations of the inflaton field

$$\zeta = -\frac{H}{\dot{\phi}}\delta\phi\,, \tag{6.70}$$

which gives us the bispectrum:

$$\langle\zeta(\eta, \mathbf{k}_1)\zeta(\eta, \mathbf{k}_2)\zeta(\eta, \mathbf{k_3})\rangle = -\frac{H^3}{\dot{\phi}_0^3}\langle\delta\phi(\eta, \mathbf{k}_1)\delta\phi(\eta, \mathbf{k}_2)\delta\phi(\eta, \mathbf{k_3})\rangle\,. \tag{6.71}$$

We need to plug in the integral equation for the inflaton field. This is obtained by first solving for the equation of motion of the inflaton field. The homogeneous part is standard, but the particular solution comes from the interaction with vectors, which we can write as a current J_ϕ:

$$J_\phi(\eta, \mathbf{k}) = \frac{\phi_0' a^3}{4M_P^2 a'}\int\frac{d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\left[-1 + \frac{(p - |\mathbf{k} - \mathbf{p}|)^2}{k^2}\right][\tilde{E}_i(\eta, \mathbf{p})\tilde{E}_i(\eta, \mathbf{k} - \mathbf{p}) + \tilde{B}_j(\eta, \mathbf{p})\tilde{B}_j(\eta, \mathbf{k} - \mathbf{p})]\,. \tag{6.72}$$

The particular solution for the inflaton field, $\tilde{\delta\phi}_P(\mathbf{k},\eta)$, is given by:

$$\tilde{\delta\phi}_P(\mathbf{k},\eta) = \int d\eta' G(\eta,\eta',k) J_\phi(\mathbf{k},\eta')\,, \tag{6.73}$$

where $G(\eta,\eta',k)$ is the same Green's function we obtain in the tensor case, and is the operator which solves the homogeneous equation of motion of the inflaton field:

$$G(\eta,\eta',k) = \frac{1}{k^3\eta'^2}[(k\eta'-k\eta)\cos(k(\eta-\eta'))+(k^2\eta\eta'+1)\sin(k(\eta-\eta'))]\Theta(\eta-\eta')\,. \tag{6.74}$$

As we did in the tensor case, we drop the magnetic field terms for being higher order, and use that the electric field is given by $\mathbf{E} = -\frac{1}{a^2}\mathbf{A}'$. The bispectrum becomes:

$$\begin{aligned}\langle\zeta(\eta,\mathbf{k}_1)\zeta(\eta,\mathbf{k}_2)\zeta(\eta,\mathbf{k}_3)\rangle = &-\frac{H^3}{\dot{\phi}_0^3}\int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta} d\eta_2 \int_{-\infty}^{\eta} d\eta_3 G(\eta,\eta_1,k_1)G(\eta,\eta_2,k_2)G(\eta,\eta_3,k_3)\\ &\frac{\phi_0'(\eta_1)\phi_0'(\eta_2)\phi_0'(\eta_3)a^3(\eta_1)a^3(\eta_2)a^3(\eta_3)}{2^6M_P^6a'(\eta_1)a'(\eta_2)a'(\eta_3)}\int_{-\infty}^{\infty}\frac{d^3\mathbf{p}_1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\frac{d^3\mathbf{p_2}}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\frac{d^3\mathbf{p_3}}{(2\pi)^{\frac{3}{2}}}\\ &\left[-1+\frac{(p_1-|\mathbf{k_1}-\mathbf{p_1}|)^2}{k_1^2}\right]\left[-1+\frac{(p_2-|\mathbf{k_2}-\mathbf{p_2}|)^2}{k_2^2}\right]\left[-1+\frac{(p_3-|\mathbf{k_3}-\mathbf{p_3}|)^2}{k_3^2}\right]\\ &\frac{1}{a^4(\eta_1)a^4(\eta_2)a^4(\eta_3)}\langle\tilde{A}_i'(\eta_1,\mathbf{p}_1)\tilde{A}_i'(\eta_1,\mathbf{k_1}-\mathbf{p_1})\tilde{A}_j'(\eta_2,\mathbf{p}_2)\tilde{A}_j'(\eta_2,\mathbf{k_2}-\mathbf{p}_2)\tilde{A}_k'(\eta_3,\mathbf{p}_3)\\ &\tilde{A}_k'(\eta_3,\mathbf{k_3}-\mathbf{p}_3)\rangle\,.\end{aligned} \tag{6.75}$$

We again plug in for the mode function of the vector field, eq. (4.60):

$$\hat{A}_i'(\mathbf{k},\eta) \approx 2^{-\frac{1}{4}}\,k^{\frac{1}{4}}\,\xi^{\frac{1}{4}}\,(-\eta)^{-\frac{1}{4}}\,e^{\pi\xi-2\sqrt{-2k\xi\eta}}[\epsilon_{i+}(\mathbf{k})\hat{a}_+(\mathbf{k})+\epsilon_{i+}^*(-\mathbf{k})a_+^\dagger(-\mathbf{k})]\,. \tag{6.76}$$

We use the approximation for the Green's function we found earlier,eq. (4.66, 4.67): $G \approx -\frac{1}{3}\eta$. The time integrals are the same as in the tensor case:

$$\int_{-\infty}^{0} d\eta\,\frac{\eta^3}{(-\eta)^{\frac{1}{2}}}e^{-C\sqrt{-\eta}} = -\frac{1440}{C^7}\,. \tag{6.77}$$

We simplify the bispectrum by taking the late time limit in which $\eta \to 0$. We then rewrite

the bispectrum in terms of the dimensionless variables: $q = \frac{p}{k_1}$, $x_2 = \frac{k_2}{k_1}$, and $x_3 = \frac{k_3}{k_1}$:

$$\langle\zeta(0,\mathbf{k}_1)\zeta(0,\mathbf{k}_2)\zeta(0,\mathbf{k}_3)\rangle = -\frac{(1440)^3 H^6 e^{6\pi\xi}\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})}{2^{37}\,3^3\,(2\pi)^{\frac{9}{2}} k_1^6 \xi^9 M_P^6}\int_0^\infty dq\int_0^\pi d\theta\int_0^{2\pi} d\phi\, q^{\frac{5}{2}}$$
$$\sin\theta|\hat{\mathbf{k}}_\mathbf{1}-\mathbf{q}|^{\frac{1}{2}}\left[-1+(q-|\hat{\mathbf{k}}_\mathbf{1}-\mathbf{q}|)^2\right]q^{\frac{1}{2}}|\hat{\mathbf{k}}_\mathbf{1}-\mathbf{q}|^{\frac{1}{2}}\epsilon_{i+}(\mathbf{q})\epsilon_{i+}(\hat{\mathbf{k}}_\mathbf{1}-\mathbf{q})\frac{1}{(q^{\frac{1}{2}}+|\hat{\mathbf{k}}_\mathbf{1}-\mathbf{q}|^{\frac{1}{2}})^7}$$
$$\Big[\left[-1+\frac{(q-|x_2\hat{\mathbf{k}}_\mathbf{2}+\mathbf{q}|)^2}{x_2^2}\right]\left[-1+\frac{(|\hat{\mathbf{k}}_\mathbf{1}-\mathbf{q}|-|x_2\hat{\mathbf{k}}_\mathbf{2}+\mathbf{q}|)^2}{x_3^2}\right]|x_2\hat{\mathbf{k}}_\mathbf{2}+\mathbf{q}|^{\frac{1}{2}}$$
$$\frac{1}{(q^{\frac{1}{2}}+|x_2\hat{\mathbf{k}}_\mathbf{2}+\mathbf{q}|^{\frac{1}{2}})^7}\frac{1}{(|\hat{\mathbf{k}}_\mathbf{1}-\mathbf{q}|^{\frac{1}{2}}+|x_2\hat{\mathbf{k}}_\mathbf{2}+\mathbf{q}|^{\frac{1}{2}})^7})\epsilon^*_{j+}(\mathbf{q})\epsilon_{j+}(x_2\hat{\mathbf{k}}_\mathbf{2}+\mathbf{q})\epsilon^*_{k+}(\hat{\mathbf{k}}_\mathbf{1}-\mathbf{q})$$
$$\epsilon^*_{k+}(x_2\hat{\mathbf{k}}_\mathbf{2}+\mathbf{q})+\left[-1+\frac{(|\hat{\mathbf{k}}_\mathbf{1}-\mathbf{q}|-|x_3\hat{\mathbf{k}}_\mathbf{3}+\mathbf{q}|)^2}{x_2^2}\right]\left[-1+\frac{(q-|x_3\hat{\mathbf{k}}_\mathbf{3}+\mathbf{q}|)^2}{x_3^2}\right]$$
$$|x_3\hat{\mathbf{k}}_\mathbf{3}+\mathbf{q}|^{\frac{1}{2}}\frac{1}{(|\hat{\mathbf{k}}_\mathbf{1}-\mathbf{q}|^{\frac{1}{2}}+|x_3\hat{\mathbf{k}}_\mathbf{3}+\mathbf{q}|^{\frac{1}{2}})^7}\frac{1}{(q^{\frac{1}{2}}+|x_3\hat{\mathbf{k}}_\mathbf{3}+\mathbf{q}|^{\frac{1}{2}})^7}\epsilon^*_{j+}(\hat{\mathbf{k}}_\mathbf{1}-\mathbf{q})\epsilon_{j+}(-x_3\hat{\mathbf{k}}_\mathbf{3}-\mathbf{q})$$
$$\epsilon^*_{k+}(\mathbf{q})\epsilon^*_{k+}(-x_3\hat{\mathbf{k}}_\mathbf{3}-\mathbf{q})\Big]\,. \tag{6.78}$$

Solving the integral numerically in the equilateral limit where $x_2 = x_3 = 1$, we obtain:

$$\langle\zeta(0,\mathbf{k}_1)\zeta(0,\mathbf{k}_2)\zeta(0,\mathbf{k}_3)\rangle_{eq} = 2.3\times10^{-13}\frac{H^6 e^{6\pi\xi}\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})}{k_1^6\xi^9 M_P^6}\,. \tag{6.79}$$

We define the bispectrum in terms of the three-point function as $\langle\zeta\zeta\zeta\rangle = B_\zeta\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})$, to obtain:

$$B_{\zeta\,eq} = 2.3\times10^{-13}\frac{H^6 e^{6\pi\xi}}{k_1^6\xi^9 M_P^6}\,. \tag{6.80}$$

From this we can calculate $f_{NL\,eq\,\zeta}$ using the definition:

$$f_{NL\,eq\,\zeta} = \frac{10k^6}{9(2\pi)^{\frac{5}{2}}}\frac{B_{eq\,\zeta}(k)}{P_\zeta^2(k)}\,, \tag{6.81}$$

and the observed value for the scalar power spectrum, $P_\zeta = 2.5\times10^{-9} = \frac{H^2}{8\pi^2\epsilon M_P^2}[1+4.3\times10^{-10}(8\pi^2)\epsilon\frac{H^2}{M_P^2}\frac{e^{4\pi\xi}}{\xi^6}]$, the total scalar power spectrum. Note there are two unknowns when calculating f_{NL} for this model. One is ξ and for the other, we can use either the ratio $\frac{H}{M_P}$ or the slow roll parameter ϵ. Since both are related through this scalar power spectrum which has been measured we could trade one for the other, either dealing with $\frac{H}{M_P}$ or ϵ. Using $\frac{H}{M_P}$, we obtain for f_{NL}:

$$f_{NL\,eq\,\zeta} = 470\frac{H^6 e^{6\pi\xi}}{M_P^6\xi^9}\,. \tag{6.82}$$

This is the non-Gaussianity parameter from the scalar metric perturbations produced by the model. Note this part of the calculation was previously published by the Barnaby et. al. group [21].

6.5 Temperature Bispectrum from Scalars

Next we calculate the temperature bispectrum sourced by the scalar perturbations. This, in conjunction with the scalar bispectrum, can be used in a ratio to calculate f_{NL} as sourced by the temperature tensor bispectrum. We use the same equation for the bispectrum in terms of the temperature anisotropies as we used in the scalar case, still using the flat sky approximation:

$$B_{l_1\,l_2\,l_3} = \int \frac{d^2\mathbf{n_1}}{2\pi} \int \frac{d^2\mathbf{n_2}}{2\pi} \int \frac{d^2\mathbf{n_3}}{2\pi} e^{-i\mathbf{l_1}\cdot\mathbf{n_1}} e^{-i\mathbf{l_2}\cdot\mathbf{n_2}} e^{-i\mathbf{l_3}\cdot\mathbf{n_3}} \frac{1}{\delta^{(3)}(\mathbf{l_1}+\mathbf{l_2}+\mathbf{l_3})} \langle \frac{\delta T(\mathbf{x_0},\eta_0,\mathbf{n}_1)}{T} \frac{\delta T(\mathbf{x_0},\eta_0,\mathbf{n}_2)}{T} \frac{\delta T(\mathbf{x_0},\eta_0,\mathbf{n}_3)}{T} \rangle \,, \tag{6.83}$$

but now we plug in the equation for contributions to $\frac{\delta T}{T}$ from scalars. We use the definition of the temperature anisotopies sourced by scalar perturbations given in [28]

$$\frac{\delta T}{T}(\eta_0,\mathbf{x}_0,\mathbf{n}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \left[\Phi(\eta_r,\mathbf{k}) + \frac{1}{4}\delta(\eta_r,\mathbf{k}) - \frac{1}{4k^2}\delta'(\eta_r,\mathbf{k}) \frac{\partial}{\partial\eta_0} \right] e^{i\mathbf{k}\cdot(\mathbf{x}_0+\mathbf{n}(\eta_r-\eta_0))} \,, \tag{6.84}$$

where η_r is the time of recombination, η_0 is the time now, Φ is the gravitational potential, and δ is the fractional/ unitless measure of the fluctuations in the energy density of photons defined by $\rho_\gamma(\eta,\mathbf{x}) = \epsilon_\gamma(\eta)(1+\delta(\eta,\mathbf{x}))$ where ρ_γ is the total energy density of photons and ϵ_γ is the average energy density. We use that $\eta_0 \gg \eta_r$ to drop the η_r term in the exponential. We then use an approximation valid for scales where $\theta \gg 1$ degree corresponding to $l \ll 200$, then: $\delta(\eta_r,\mathbf{k}) \approx -\frac{8}{3}\Phi(\eta_r,\mathbf{k})$ and $\delta'(\eta_r,\mathbf{k}) = 0$ [28]. In this limit, the temperature anisotropies reduce to:

$$\frac{\delta T}{T}(\eta_0,\mathbf{x}_0,\mathbf{n})_{l\ll 200} = \frac{1}{3}\int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \Phi(\eta_r,\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{x}_0-\mathbf{n}\eta_0)} \,. \tag{6.85}$$

We are concerned with long wavelength modes, modes which entered the horizon during matter domination; we already assumed this above to get a simplified $\frac{\delta T}{T}$ equation. In a matter dominated universe, Φ is related to ζ, the scalar curvature perturbation, by: $\Phi = -\frac{3}{5}\zeta$. We can plug this relation into our $\frac{\delta T}{T}$ equation:

$$\frac{\delta T}{T}(\eta_0,\mathbf{x}_0,\mathbf{n})_{l\ll 200} = -\frac{1}{5}\int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}} \zeta(\eta_r,\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{x}_0-\mathbf{n}\eta_0)} \,. \tag{6.86}$$

We use this to calculate $B_{l_1,l_2,l_3\,\zeta}$:

$$B_{l_1\,l_2\,l_3\,\zeta} = -\frac{1}{5^3\,(2\pi)^{\frac{15}{2}}} \int d^2\mathbf{n_1} \int d^2\mathbf{n_2} \int d^2\mathbf{n_3} e^{-i\mathbf{l_1}\cdot\mathbf{n_1}} e^{-i\mathbf{l_2}\cdot\mathbf{n_2}} e^{-i\mathbf{l_3}\cdot\mathbf{n_3}} \frac{1}{\delta^{(3)}(\mathbf{l_1}+\mathbf{l_2}+\mathbf{l_3})} \int d^3\mathbf{k}_1 \int d^3\mathbf{k}_2 \int d^3\mathbf{k}_3 e^{i\mathbf{k_1}\cdot(\mathbf{x}_0-\mathbf{n}_1\eta_0)} e^{i\mathbf{k_2}\cdot(\mathbf{x}_0-\mathbf{n}_2\eta_0)} e^{i\mathbf{k_3}\cdot(\mathbf{x}_0-\mathbf{n}_3\eta_0)} \langle \zeta(\eta_r,\mathbf{k}_1)\zeta(\eta_r,\mathbf{k}_2)\zeta(\eta_r,\mathbf{k}_3)\rangle \,. \tag{6.87}$$

Now note the $\mathbf{n}$ integrals become delta functions just as in the tensor case, and we use that in the flat sky approximation the integral only recieves significant contributions from

when the $\mathbf{n}$ vectors are approximately parallel to $\hat{\mathbf{z}}$:

$$B_{l_1\,l_2\,l_3\,\zeta} = \frac{1}{5^3\,(2\pi)^{\frac{15}{2}}}\frac{1}{\delta^{(3)}(\mathbf{l_1}+\mathbf{l_2}+\mathbf{l_3})}\int d^3\mathbf{k}_1\int d^3\mathbf{k}_2\int d^3\mathbf{k}_3\delta^{(2)}(-\mathbf{l_1}-\mathbf{k}_{1||}\eta_0)(2\pi)^2$$
$$\delta^{(2)}(-\mathbf{l_2}-\mathbf{k}_{2||}\eta_0)(2\pi)^2\delta^{(2)}(-\mathbf{l_3}-\mathbf{k}_{3||}\eta_0)(2\pi)^2 e^{i\mathbf{k_1}\cdot\mathbf{x}_0}e^{i\mathbf{k_2}\cdot\mathbf{x}_0}e^{i\mathbf{k_3}\cdot\mathbf{x}_0}e^{-ik_{z1}\eta_0}e^{-ik_{z2}\eta_0}e^{-ik_{z3}\eta_0}$$
$$\langle\zeta(\eta_r,\mathbf{k}_1)\zeta(\eta_r,\mathbf{k}_2)\zeta(\eta_r,\mathbf{k}_3)\rangle\,. \tag{6.88}$$

Next we plug in for the ζ's. We want the ζ's evaluated at recombination, but we match boundary conditions to the amplitude these perturbations had when they exited the horizon during inflation, given by by eq. (6.70): $\zeta = -\frac{H\delta\phi}{\dot{\phi}_0}$. Then we use the equation for $\delta\phi$ used in calculating the scalar bispectrum eq. (6.73):

$$\tilde{\delta\phi}_P(\mathbf{k},\eta) = \int d\eta' G(\eta,\eta',k)J_\phi(\mathbf{k},\eta') \tag{6.89}$$

with current given by eq. (6.72):

$$J_\phi(\eta,\mathbf{k}) = \frac{\phi_0' a^3}{4M_P^2 a'}\int\frac{d^3\mathbf{p}}{(2\pi)^{\frac{3}{2}}}\left[-1+\frac{(p-|\mathbf{k}-\mathbf{p}|)^2}{k^2}\right][\tilde{E}_i(\eta,\mathbf{p})\tilde{E}_i(\eta,\mathbf{k}-\mathbf{p})+\tilde{B}_j(\eta,\mathbf{p})\tilde{B}_j(\eta,\mathbf{k}-\mathbf{p})]\,, \tag{6.90}$$

and solved by the same Green's function $G(\eta,\eta',k)$ given by:

$$G(\eta,\eta',k) = \frac{1}{k^3\eta'^2}[(k\eta'-k\eta)\cos(k(\eta-\eta'))+(k^2\eta\eta'+1)\sin(k(\eta-\eta'))]\Theta(\eta-\eta')\,. \tag{6.91}$$

We use the same approximations as in the previous cases, dropping the B terms and approximating $G\approx -\frac{1}{3}\eta$. We use that $E = -\frac{1}{a^2}A'$ and plug in the expression for the mode functions:

$$\hat{A}_i'(\mathbf{k},\eta)\approx 2^{-\frac{1}{4}}\,k^{\frac{1}{4}}\,\xi^{\frac{1}{4}}\,(-\eta)^{-\frac{1}{4}}\,e^{\pi\xi\,-2\sqrt{-2k\xi\eta}}[\epsilon_{i+}(\mathbf{k})\hat{a}_+(\mathbf{k})+\epsilon_{i+}^*(-\mathbf{k})a_+^\dagger(-\mathbf{k})]\,. \tag{6.92}$$

The time integrals work out the same and the bispectrum simplifies to:

$$B_{l_1\,l_2\,l_3\,\zeta} = \frac{(1440)^3H^6e^{6\pi\xi}}{2^{37}\,3^3\,5^3\,(2\pi)^6\xi^9M_P^6}\frac{1}{\delta^{(3)}(\mathbf{l_1}+\mathbf{l_2}+\mathbf{l_3})}\int d^3\mathbf{k}_1\int d^3\mathbf{k}_2\int d^3\mathbf{k}_3\delta^{(2)}(-\mathbf{l_1}-\mathbf{k}_{||1}\eta_0)$$
$$\delta^{(2)}(-\mathbf{l_2}-\mathbf{k}_{||2}\eta_0)\delta^{(2)}(-\mathbf{l_3}-\mathbf{k}_{||3}\eta_0)\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})e^{i\mathbf{k_1}\cdot\mathbf{x}_0}e^{i\mathbf{k_2}\cdot\mathbf{x}_0}e^{i\mathbf{k_3}\cdot\mathbf{x}_0}e^{-ik_{z1}\eta_0}$$
$$e^{-ik_{z2}\eta_0}e^{-ik_{z3}\eta_0}\int d^3\mathbf{p}\left[-1+\frac{p-|\mathbf{k_1}-\mathbf{p}|}{k_1^2}\right]p^{1/2}|\mathbf{k_1}-\mathbf{p}|^{\frac{1}{2}}\epsilon_{i+}(\mathbf{p})\epsilon_{i+}(\mathbf{k_1}-\mathbf{p})$$
$$\frac{1}{(p^{\frac{1}{2}}+|\mathbf{k_1}-\mathbf{p}|^{\frac{1}{2}})^7}\Bigg[\left[-1+\frac{(p-|\mathbf{k_2}+\mathbf{p}|)^2}{k_2^2}\right]\left[-1+\frac{(|\mathbf{k_1}-\mathbf{p}|-|\mathbf{k_2}+\mathbf{p}|)^2}{k_3^2}\right]|\mathbf{k_2}+\mathbf{p}|^{\frac{1}{2}}$$
$$\frac{1}{(p^{\frac{1}{2}}+|\mathbf{k_2}+\mathbf{p}|^{\frac{1}{2}})^7}\frac{1}{(|\mathbf{k_1}-\mathbf{p}|^{\frac{1}{2}}+|\mathbf{k_2}+\mathbf{p}|^{\frac{1}{2}})^7}\epsilon_{j+}^*(\mathbf{p})\epsilon_{j+}(\mathbf{k_2}+\mathbf{p})\epsilon_{k+}^*(\mathbf{k_1}-\mathbf{p})\epsilon_{k+}^*(\mathbf{k_2}+\mathbf{p})$$
$$+\left[-1+\frac{(|\mathbf{k_1}-\mathbf{p}|-|\mathbf{k_3}+\mathbf{p}|)^2}{k_2^2}\right]\left[-1+\frac{(p-|\mathbf{k_3}+\mathbf{p}|)^2}{k_3^2}\right]|\mathbf{k_3}+\mathbf{p}|^{\frac{1}{2}}$$
$$\frac{1}{(|\mathbf{k_1}-\mathbf{p}|^{\frac{1}{2}}+|\mathbf{k_3}+\mathbf{p}|^{\frac{1}{2}})^7}\frac{1}{(p^{\frac{1}{2}}+|\mathbf{k3}+\mathbf{p}|^{\frac{1}{2}})^7}\epsilon_{j+}^*(\mathbf{k_1}-\mathbf{p})\epsilon_{j+}(-\mathbf{k_3}-\mathbf{p})\epsilon_{k+}^*(\mathbf{p})$$
$$\epsilon_{k+}^*(-\mathbf{k_3}-\mathbf{p})\Bigg]\,. \tag{6.93}$$

Solving using numerical integration in Mathematica in the equilateral limit, we obtain:

$$l^4\, B_{l\,\zeta\,eq} = -4.0 \times 10^{-16} \frac{H^6 e^{6\pi\xi}}{\xi^9 M_P^6} \tag{6.94}$$

the temperature bispectrum sourced by scalar metric perturbations from the model.

6.6 f_{NL}

Non-Gaussianities in the CMB are parametrized by f_{NL} which we can split into contributions from scalars and tensors respectively by: $f_{NL\,total} = f_{NL\,\zeta} + f_{NL\,h}$. One definition of $f_{NL\,\zeta}$ is:

$$f_{NL\,\zeta} = \frac{2^{\frac{1}{2}} 5 \pi^{\frac{3}{2}}}{3^2} \frac{\langle \zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\zeta(\mathbf{k}_3)\rangle}{\langle \zeta(\mathbf{k}_1)\zeta(\mathbf{k}_2)\rangle^2} \frac{\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2})^2}{\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})}\,. \tag{6.95}$$

f_{NL} could equivalently be calculated from

$$f_{NL\,\zeta} \propto \frac{\langle \frac{\delta T}{T}(\mathbf{k}_1)\frac{\delta T}{T}(\mathbf{k}_2)\frac{\delta T}{T}(\mathbf{k}_3)\rangle_\zeta}{\langle \frac{\delta T}{T}(\mathbf{k}_1)\frac{\delta T}{T}(\mathbf{k}_2)\rangle_\zeta^2} \frac{\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2})^2}{\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})}\,. \tag{6.96}$$

$f_{NL\,h}$ can similarly be given by:

$$f_{NL\,h} \propto \frac{\langle \frac{\delta T}{T}(\mathbf{k}_1)\frac{\delta T}{T}(\mathbf{k}_2)\frac{\delta T}{T}(\mathbf{k}_3)\rangle_h}{\langle \frac{\delta T}{T}(\mathbf{k}_1)\frac{\delta T}{T}(\mathbf{k}_2)\rangle_\zeta^2} \frac{\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2})^2}{\delta^{(3)}(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})}\,, \tag{6.97}$$

where the proportionality constant is the same in both for scalars and tensors. Of course what is actually measured by Planck and other such experiments is the total f_{NL} from all sources, where generally it is assumed that the largest cotnribtution will come from the scalars. In this model though, we have seen the tensors produce the largest three-point function. Also note, the denominator in both cases should be $\langle TT\rangle$ total, with contributions again from all sources. The scalars will dominate the temperature two-point function though, and we ignore the tensor contribution to the denominator. We then take the ratio:

$$\frac{f_{NL\,h}}{f_{NL\,\zeta}} = \frac{\langle \frac{\delta T}{T}(\mathbf{k}_1)\frac{\delta T}{T}(\mathbf{k}_2)\frac{\delta T}{T}(\mathbf{k}_3)\rangle_h}{\langle \frac{\delta T}{T}(\mathbf{k}_1)\frac{\delta T}{T}(\mathbf{k}_2)\frac{\delta T}{T}(\mathbf{k}_3)\rangle_\zeta}\,. \tag{6.98}$$

For our model, this ratio comes out to:

$$\frac{f_{NL\,eq\,h}}{f_{NL\,eq\,\zeta}} = \frac{-1.8 \times 10^{-12}}{-4.0 \times 10^{-16}} = 4600\,. \tag{6.99}$$

The contribution to the non-Gaussianities from the tensors is over three orders of magnitude larger than the contribution from the scalars. We then use eq. (6.82), that $f_{NL\,eq\,\zeta} = -470\frac{H^6 e^{6\pi\xi}}{M_P^6 \xi^9}$ to obtain,

$$f_{NL\,eq\,h} = 4600 f_{NL\,eq\,\zeta} = -2.1 \times 10^6 \frac{H^6 e^{6\pi\xi}}{M_P^6 \xi^9}\,. \tag{6.100}$$

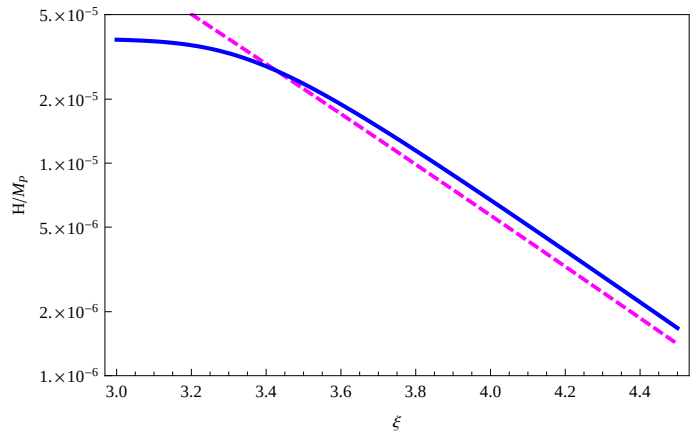

Figure 6.2: The figure shows the maximum allowed value of $\frac{H}{M_P}$ for various values of ξ. The blue line uses the limit from $r < 0.11$, and the dotted pink line uses the limit from $f_{NL}^{\text{equil}} < 150$. The parameter space below both these lines is allowed.

Note the $f_{NL\,eq\,h}$ is scale independent, and only depends on the Hubble parameter H and the model dependent parameter ξ. There is actually some scale dependence to f_{NL}, and it is more correct to say f_{NL} is scale independent on the range of scales over which the approximations we took are valid. This corresponds to scales $1 \ll l < 100$. After about $l \geq 100$, the tensor contribution to the temperature spectrum falls off very quickly, and we therefore expect f_{NL} would fall off very steeply past this scale. In Figure 6.2 we show the limits on the (ξ, H/M_P) plane that originate from the non-observation of tensors and of non-Gaussianities. The pink dashed line in Figure 6.2 is obtained by imposing the limit $f_{NL}^{\text{equil}} < 150$, *i.e.*, twice the 68% uncertainty published by the Planck Collaboration [22]. We compare this to the constraint on the model parameter ξ, from the non-observation of the observable r, the tensor to scalar ratio. r likewise depends only on the parameters ξ and H. The model predicts a tensor to scalar ratio

$$r = \frac{P^{t,+} + P^{t,-}}{P_\zeta} = 8.1 \times 10^7 \, \frac{H^2}{M_P^2} \left(1 + \frac{8.6 \times 10^{-7}}{2} \, \frac{H^2}{M_P^2} \, \frac{e^{4\pi\xi}}{\xi^6} \right) . \tag{6.101}$$

The blue solid line in Figure 6.2 is obtained by applying the limit $r < 0.11$ at the 95% confidence level as published by the Planck Collaboration [1]. In this model r is not in one-to-one correspondence with H/M_P, and one can have detectable tensors for arbitrarily small values of H/M_P. In this case the tensor spectrum would be dominated by the metric perturbations caused by the auxiliary vector fields as opposed to the standard fluctuations caused by the inflationary expansion. We see that for $\xi \lesssim 3.4$, where the contribution of vectors to the tensor power spectrum is weaker, the non-observation of tensors provides the strongest limit on H/M_P, and the the expression for the tensor power spectrum approaches the more standard expression of $P^t = \frac{2\,H^2}{\pi^2\,M_P^2}$. For these small values of ξ, the limit of $r < 0.11$ translates into a limit $\frac{H}{M_P} < 3.7 \times 10^{-5}$.

In Figure 6.3 we show the possibility of a detection of tensors in this model. The solid lines in the figure use the projected sensitivity of r from top to bottom of: Planck, once the polarization results are complete, $r < 0.05$, Spider $r < 0.01$, and a possible future CMBPol experiment $r < 0.0001$ [7]. This is in comparison to the dashed blue line which is

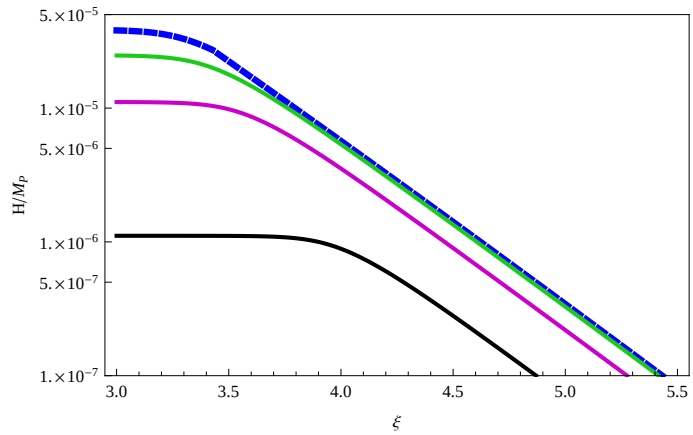

Figure 6.3: The figure compares the possibilities of various experiments of detecting tensor pertubations. The dotted, blue line is the maximum allowed parameters for the model using a combination of the current limits on r and f_{NL}^{equil}. The solid lines correspond to the projected sensitivities to r of top to bottom, Planck with polarization (green), Spider (pink), and CMBPol (black).

the maximum allowed model parameters from applying the current limits on r and f_{NL}^{equil} as displayed in Figure 6.2. We find the possibility of a detection by Spider and CMBPol for arbitrarily small H/M_P for large enough ξ.

Next we look at the likelihood of a detection being made of the chirality of this model through a measurement of non vanishing $\langle TB \rangle$ or $\langle EB \rangle$ correlators, chirality being a more unique, interesting detection signature since it is absent from most inflationary models. We define the chirality of the tensor modes by:

$$\Delta\chi = \left| \frac{P_+ - P_-}{P_+ + P_-} \right| . \tag{6.102}$$

Using our tensor spectrum, we get:

$$\Delta\chi = \frac{8.6 \times 10^{-7} \frac{H^2}{2M_P^2} \frac{e^{6\pi\xi}}{\xi^6}}{1 + 8.6 \times 10^{-7} \frac{H^2}{2\,M_P^2} \frac{e^{6\pi\xi}}{\xi^6}} . \tag{6.103}$$

The results are shown in Figure 6.4 where the pink dotted curve is the best current limit on ξ, using the combined limits of r and f_{NL}^{equil} displayed in Figure 6.2. The solid curves are the 2σ detection sensitivities for various experiments, such that the parameters in the model will need to fall above these curves to allow detection by these various experiments. We use twice the detection limits published in [29], to require a 2σ, rather than a 1σ, detection of primordial parity-violation in the CMB. We find that there is no allowed region of detection from Planck or Spider. On the other hand, a detection by a cosmic variance limited experiment or a CMBPol like experiment is allowed throughout a part of parameter space. The larger ξ is, the more the tensor spectrum is dominated by the auxiliary model fields as opposed to the standard perturbations from expansion, and since it is the contribution from these auxiliary fields that violate parity, the larger ξ, the larger $\Delta\chi$.

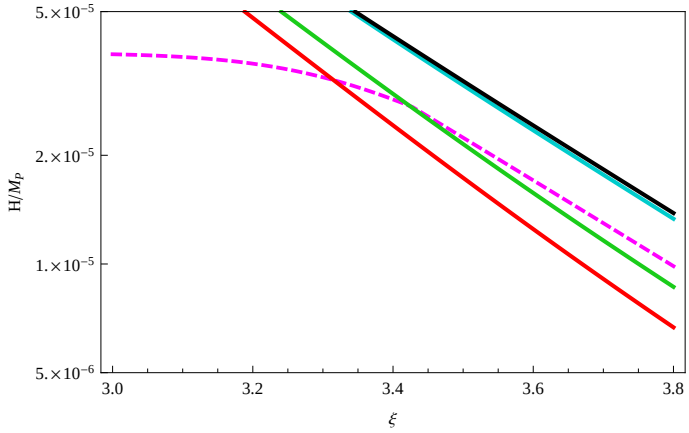

Figure 6.4: The figure shows a comparison of detectability limits of chirality of the primordial tensors for various experiments. The dotted pink line is the maximum allowed H/M_P based on current limits on r and f_{NL}^{equil}. The solid lines are for 2σ detectability for the following experiments listed in order top to bottom: Planck, Spider, CMBPol, and a cosmic variance limited experiment. The experimental lines were derived from Figure 2 of [29].

CHAPTER 7

CONCLUSIONS

So far observational data is consistent with a single inflationary field. As observational data from the CMB continues to provide better evidence on inflation, it is interesting to consider what other scenarios could have been taking place during inflation, and what observational evidence they would leave. The scenario we considered was to allow the production of other non-inflaton fields during slow roll inflation. Since it is assumed that at the end of inflaton, the inflaton field decays into generating the fields which currently make up the universe, it is somewhat natural to assume if the inflaton coupled to other fields at the end of inflation, and had these couplings during inflation as well, maybe enough quanta of these other fields could have been generated to leave an observable signature.

First we considered instances of explosive production in which the inflaton field coupled to other fields in such a way as to provide a time dependent mass term to these other fields. If during slow roll, these other fields passed through a period of zero mass, many quanta of these other fields could be produced. We found though that by requiring that the inflaton field still be the dominate energy density in the universe at the time such that inflation continue, it is not possible to produce a large enough effect of these sourced fields such that the power spectra of their produced metric perturbations can dominate those generated by standard inflation.

We then considered a model with a derivative coupling between a vector field and a slowly rolling scalar field, such that the scalar field provides a time dependent background to the vector field, sourcing quanta of the vectors for a prolonged period of time. We find it is possible to have the spectrum of tensor perturbations sourced by the vectors dominate the standard spectrum of tensor perturbations sourced during inflation. In the case where the vector field is coupled directly to the inflaton though, the spectrum of scalar perturbations also sourced by the vectors produces too large of non-Gaussianities such that to require that these non-Gaussianities have not been detected yet, requires the tensor spectrum to be too small to be detected. One can get around this constraint by allowing detection of the tensor spectrum only at much larger frequency modes probed by direct detection experiments. We find the power spectrum from this model grows exponentially in time, so that even with requiring non-Gaussianities to not be detected at CMB scales, it is possible that the power spectrum could grow enough in the intermediary time after CMB scales left the horizon but before direct detect scales left such that the power spectrum at these direct detection scales could be dominated by the contribution from the model which is highly non-Gaussian.

Reference [21] suggested another possibility of weakening the scalar perturbations relative

to the tensors, if we instead allow for a coupling between the vectors and some other scalar field slowly rolling during inflation, such that the vectors are only gravitationally coupled to the inflaton, then this will also loosen the constraint from the non-observation of scalar non-Gaussianities at CMB scales. We showed for this model the non-Gaussianity parameter $f_{NL}^{\rm equil}$ induced by the tensors sets the strongest constraints on the model for much of the parameter space. We find in this case we can get a tensor spectrum where the contribution from the model dominates the standard spectrum, even at CMB scales. This allows for a possibly detectable tensor spectrum, even for small values of the Hubble parameter.

Aside from allowing a potentially observable signal in the tensor power spectrum, there are several other observable signatures with this model. The chirality of the induced tensor spectrum can produce detectable $\langle TB\rangle$ or $\langle EB\rangle$ correlators, which is rare in inflationary models. We find these detections possible by a future CMBpol-like experiment, but not by Planck or Spider. The chiral tensors should also cause the coefficient B_{ℓ_1,ℓ_2,ℓ_3} to vanish for $\ell_1+\ell_2+\ell_3=$even, differently from the usual case where temperature non-Gaussianities are not generated by a parity-violating source as discussed in [30, 31]. Even more uniquely this model provides the first example, to our knowledge, of a scenario where the three-point function of the tensors is significantly larger than the three-point function of the scalars. We find that the tensors dominate the contribution to $\langle TTT\rangle$, at least at the large angular scales, $l<100$, where the tensor contribution to the temperature CMB anisotropies is not suppressed.

More in general, although clearly $\langle TT\rangle$ is dominated by scalar signals, it is possible that $\langle TTT\rangle$ is dominated by tensor signals. In general this requires a mechanism generating perturbations on CMB scales which is highly non-Gaussian (true generically of particle production models among others), and which produces stronger tensor metric perturbations than scalar ones. This works out in the case of our particular model because 1. there is no direct coupling of produced vectors to the inflaton to enhance the scalar perturbations and 2. this mechanism naturally produces a larger tensor signal than scalar signal because of the phase space available for the decay. The model has relativistic vectors which are decaying into scalar and tensor perturbations, where there is naturally a larger phase space available for the decay into tensors. Since most inflationary models produce very small non-Gaussianities, a mechanism which is swamped by other signals in the temperature two-point function could still dominate the three-point function. The fact that the tensor perturbations could possibly be the dominant source of non-Gaussianities raises hope that maybe non-Gaussianities will be detected in the CMB polarization data even though none have been detected in the temperature data.

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