9-2013

Open Books on Contact Three Orbifolds

Daniel Herr
University of Massachusetts Amherst, dzherr@gmail.com

OPEN BOOKS ON CONTACT THREE ORBIFOLDS

A Dissertation Presented

by

DANIEL HERR

Submitted to the Graduate School of the
University of Massachusetts Amherst in partial fulfillment
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

September 2013

Department of Mathematics and Statistics

OPEN BOOKS ON CONTACT THREE ORBIFOLDS

A Dissertation Presented

by

DANIEL HERR

Approved as to style and content by:

Weimin Chen, Chair

Michael Sullivan, Member

Robert Kusner, Member

David Kastor, Outside Member

Michael Levine, Department Head
Mathematics and Statistics

ACKNOWLEDGEMENTS

First, thanks to all those who poor memory and a desire for brevity have prevented me from naming explicitly.

Thanks to the professors from whose teaching I have benefited, especially Mike Sullivan, Tom Weston, Oliver Schirokauer, and most especially Weimin Chen.

Also to the family and friends who provided emotional support, and allowed me to talk at them about whatever piece of Mathematics was occupying me. Your patience was vital.

Finally thanks to the owners of the Foundry: the coffee shop in Northampton Massachusetts where much of this was written. I can think of no better venue for Mathematical research.

This work was partially supported by a Research Assistantship from NSF grant DMS-1065784, FRG:Collaborative Research: The topology and invariants of smooth 4-manifolds.

ABSTRACT

OPEN BOOKS ON CONTACT THREE ORBIFOLDS

SEPTEMBER 2013

DANIEL HERR

B.A., OBERLIN COLLEGE

Ph.D., UNIVERSITY OF MASSACHUSETTS AMHERST

Directed by: Professor Weimin Chen

In 2002, Giroux showed that every contact structure has a corresponding open book decomposition. This was the converse to a previous construction of Thurston and Winkelnkemper, and made open books a vital tool in the study of contact three manifolds. We extend these results to contact orbifolds, i.e. spaces that are locally diffeomorphic to the quotient of a contact manifold and a compatible finite group action. This involves adapting some of the main concepts and constructions of three dimensional contact geometry to the orbifold setting.

TABLE OF CONTENTS

LIST OF FIGURES

INTRODUCTION

The field of contact geometry has its origins in the work of Sophus Lie from the end of the nineteenth century, though hints of the subject may be found in earlier work. Still, it was not until the latter half of the twentieth that it became a significant focus of research. The past quarter century has seen great advances in our understanding of contact manifolds, especially in dimension three.

A central topic in much of geometry is that of the symmetries a space possesses, i. e. the group actions that will preserve its structure. A vital tool to understand this is the quotient of the space by the action, also called the orbit space. If a finite group G acts upon a manifold M without fixing any points, the set M/G will inherit a manifold structure. If some group elements do have fixed points, we can come close to such a structure. This motivates the definition of an orbifold: a space locally modeled on the quotient spaces $\mathbb{R}^n/G$. These can be thought of as manifolds with singular points, arising from the fixed points of some group elements. The orbit spaces of finite actions are naturally orbifolds, and appropriately defined structures on the orbifold M/G will lift to equivariant structures on M. This is a topic contact geometers have not yet explored. We start this exploration by adapting a central result from three dimensional contact geometry to the orbifold setting.

Contact manifolds are smooth manifolds with a maximally non-integrable hyperplane field, called a contact structure. These structures only occur in odd dimensions and are often thought of as the odd-dimensional analogs of symplectic structures. The study of contact structures in dimension three was revolutionized in 1991 by Giroux's introduction of convex surfaces [14]. Eleven years later, these were instrumental in his proof of a correspondence between appropriate equivalence classes of contact structures and open book decompositions [16]. This correspondence, which has become a central result in the field, built upon work of Thurston and Winkelnkemper from 1975 [32]. They had shown that an open book could be used to construct a contact structure on any three manifold. Giroux showed the converse, and described the appropriate notions of equivalence on both sides.

Our main results are extensions of Thurston-Winkelnkemper's, and Giroux's constructions to contact orbifolds. This requires an understanding of what sorts of orbifolds may carry contact structures and open books. In three dimensions the most compatible are those of cyclic type, i. e. orbifolds everywhere modeled on cyclic actions. A fairly straightforward extension of the corresponding proof for manifolds allows us to show:

Theorem 3.4.1 *Every cyclic type three-orbifold has an open book decomposition.*

Similarly, by adapting Thurston and Winkelnkemper's construction in the appropriate way, we prove the analogous result on three-orbifolds.

Theorem 4.1.2 *Every open book on a closed cyclic type three-orbifold supports a contact structure.*

These two results immediately imply that every cyclic type three orbifold has a contact structure. In the special case where our orbifold arises as the quotient of a cyclic type action, we find:

Corollary 4.1.5 *Every cyclic type G-action on a closed orientable three-manifold M preserves some contact structure.*

Our main result, though, is an adaptation of Giroux's direction. The original construction was based upon an appropriate cell decomposition. We design and construct a version of this decomposition adapted to the orbifold structure, which keeps much of the "meat" of Giroux's construction away from the singular set. This allows us to prove:

xf **Theorem 4.3.1** *Every closed, positive contact three-orbifold (Y, ξ) has an open book decomposition supporting ξ.*

As above, if we consider the orbifolds that arise as the quotients of contact group actions, we find:

Corollary 4.3.4 *Every contact manifold* (M, ξ) *with a positive contact* G *action has a strongly preserved open book that supports* ξ*.*

The first chapter is an introduction to some of the basic ideas and results of contact geometry. Much of this consists of a summary of some foundational results from Giroux's theory of convex surfaces. In Chapter two we begin to address issues of symmetry starting with a description of the types of action that may preserve three dimensional contact structures. We then take some time to define orbifolds, and discuss how some concepts from contact geometry can be applied to them. Chapter three introduces open book decompositions. We start with the definition and some basic facts, then, discuss the notion of compatibility between contact structures and open books. From here we turn to the relationship between open books and group actions, and then define open books on cyclic type orbifolds. We then discuss the compatibility of contact structures and open books in the orbifold setting. The final chapter contains our main results. It starts with a discussion of Thurston and Winkelnkemper's result, and its extension to orbifolds. Then we explain Giroux's construction, and finally how it may be adapted to work on contact orbifolds.

CHAPTER 1

CONTACT MANIFOLDS

1.1 Basics

On a $2n+1$ dimensional manifold M, a hyperplane field, ξ, is a codimension one sub-bundle of TM. It can be defined locally as the kernel of a one-form, called a defining form. Let α be such a local defining form for ξ. One consequence of Frobenius' theorem is that if $\alpha \wedge d\alpha \equiv 0$, then ξ is integrable, i.e. every point sits inside a hypersurface Σ such that $T\Sigma = \xi|_{\Sigma}$. A hyperplane field that has this property on the whole of M is a foliation. At the opposite end of the spectrum are the hyperplane fields that are maximally non-integrable: the contact structures.

Definition 1.1.1. *ξ is a contact structure if, any local defining form α is such that $\alpha \wedge (d\alpha)^n \neq 0$. Such forms are called contact forms for ξ.*

This is equivalent to requiring that ξ be a symplectic vector bundle with symplectic form $d\alpha$. A contact manifold is a pair (M, ξ) of a smooth manifold and a contact structure on that manifold. We will be dealing exclusively with contact manifolds of dimension three.

Considering this dimension, we see that $\alpha \wedge d\alpha = (-\alpha) \wedge d(-\alpha)$. We may cover M by regions which have contact forms. On any overlap, two contact forms α_i and α_j share the same kernel, so each is a multiple of the other by a smooth, nonzero function. Then $\alpha_i \wedge d\alpha_i$ and $\alpha_j \wedge d\alpha_j$ are top forms of the same sign, so whether or not there is a globally defined contact form, the manifold M must be orientable. In other words, in dimension three, contact structures are only possible on orientable manifolds. A global contact form exists if and only if ξ is orientable as a plane field.

Contact structures are closely related to symplectic structures, and are viewed as their odd dimensional analog. A hypersurface M in a symplectic manifold, i. e. a submanifold of codimension one, is of contact type if there is some locally defined Liouville vector field L transverse to M. In this case the symplectic form has a local primitive λ, and the one form $\lambda|_M$ is a contact form on M. A symplectic manifold whose boundary is of contact type is said to be a filling of its boundary.[1] More specifically, it is either a strong symplectic filling, or a concave filling, depending on whether the Liouville vector field is pointing outward or inward. We are most interested in contact manifolds that may relate in these ways to symplectic ones. For this reason, we concern ourselves chiefly with orientable contact structures. Hereafter, unless explicitly told otherwise, assume all contact structures are orientable.

Much like in symplectic geometry we define an equivalence relation based upon maps that preserve these structures.

Definition 1.1.2. *A bijection $f : (M_0, \xi_0) \to (M_1, \xi_1)$ is a contactomorphism if it is a diffeomorphism of the manifolds, and if $f_*(\xi_0) = \xi_1$.[2] The second condition is equivalent to the requirement that any defining form for ξ_1 pull back to a defining form for ξ_0. If such a map exists then the manifolds are contactomorphic.*

1.2 Darboux and Moser

Our first example of a contact structure will be on $\mathbb{R}^3$. Define the forms $\alpha_0 = dz - ydx$, and $\alpha_1 = dz + xdy - ydx$, or in cylindrical coordinates: $dz + r^2 d\theta$. Further, define the

[1] This is not a definition, but an example. The notion of symplectic filling comes in several flavors, two of which are described here.

[2] *By $f_* : TM_0 \to TM_1$ we mean the linearization of the map f. This is also sometimes called Df or Tf*

plane fields ξ_{std_0} and ξ_{std_1} to be the kernels of these forms. A quick computation shows that $\alpha_0 \wedge d\alpha_0 = dx \wedge dy \wedge dz$, and that $\alpha_1 \wedge d\alpha_1 = 2rdr \wedge d\theta \wedge dz$, which are both volume forms on $\mathbb{R}^3$, so the corresponding plane fields are contact structures. Note also that the map sending $(x, y, z) \to \left(x - y, x + y, z + \frac{x^2-y^2}{2}\right)$ is a contactomorphism between these spaces. Both are often referred to as the standard contact structure ξ_{std} on $\mathbb{R}^3$. We call ξ_{std_0} the rectangular version, and ξ_{std_1} the radial.

Much like symplectic structures, contact structures are locally identical. In fact the relevant theorems and their proofs are similar enough to have almost the same name.

Theorem 1.2.1 (The Contact Darboux Theorem)**.** *Every point in a three dimensional contact manifold has a neighborhood U that is contactomorphic to a neighborhood of the origin in $(\mathbb{R}^3, \xi_{std})$.*

To prove this, and several stronger results, the standard technique is called "Moser's Trick." Our goal is to produce an isotopy of the relevant contact structures, i. e. a smooth, one parameter family of such structures. Clearly, any ambient isotopy of the manifold would produce such a thing, via pullbacks of contact forms. The important question is when an isotopy of structures may be expressed as an isotopy of the manifold. Moser gives a method for producing a differential equation from a one parameter family of differential forms.[27] The solution to the equation will be a time dependent vector field whose flow produces the desired isotopy.

Given a smooth family of contact structures ξ_t, for $t \in [0, 1]$, we define them by a smooth family of contact forms α_t. Our goal, then, is to find a family of diffeomorphisms ψ_t so that $\psi_t^* \alpha_t = g_t \alpha_0$, for some family of positive functions g_t. We'll assume that our isotopy is the flow of a time dependent vector field V_t. Then if we differentiate the right side of our desired equation with respect to t, note that:

$$
\begin{aligned}
\frac{d}{dt}(g_t\alpha_0) &= \frac{dg_t}{dt}\alpha_0 \\
&= \frac{dg_t}{dt}\frac{1}{g_t}\psi_t^*\alpha_t \\
&= \psi_t^*\left[\left(\frac{dg_t}{dt}\frac{1}{g_t}\circ\psi_t^{-1}\right)\alpha_t\right] \\
&:= \psi_t^*(h_t\alpha_t)
\end{aligned}
$$

Note that the last line is defining the function h_t in terms of g_t and ψ_t. Differentiating the left side gives us:

$$
\begin{aligned}
\frac{d}{dt}\psi_t^*\alpha_t &= \psi_t^*\left(\frac{d\alpha_t}{dt}+\mathcal{L}_{V_t}\alpha_t\right) \\
&= \psi_t^*\left(\frac{d\alpha_t}{dt}+i_{V_t}(d\alpha_t)+d(\alpha_t(V_t))\right)
\end{aligned}
$$

We decide to be even more greedy and pretend not only that V_t exists, but that it is contained in ξ_t. This makes $\alpha(V_t)=0$. Now our goal has become to satisfy:

$$
\psi_t^*\left(\frac{d\alpha_t}{dt}+i_{V_t}(d\alpha_t)\right)=\psi_t^*(h_t\alpha_t)
$$

Which will be the case, if we satisfy the simpler equation:

$$
\frac{d\alpha_t}{dt}+i_{V_t}(d\alpha_t)=h_t\alpha_t \tag{1.1}
$$

We would like to use this to define the vector field V_t, but we must first find a definition of h_t that does not depend on ψ_t. We do this by using the Reeb vector field for α_t.

Definition 1.2.1. *A contact form α defines an associated vector field R by the requirements that $i_R d\alpha = 0$, and $\alpha(R) = 1$. This is called the Reeb field defined by the contact form α.*

Let R_t be the Reeb field for α_t. If we plug this into the above equation, it simplifies to:

$$\frac{d\alpha}{dt}(R_t) = h_t$$

Now we stop pretending that we have an isotopy, and get to work constructing one. The above equation allows us to define the function h_t, so that our desired vector field V_t is now the only unknown in (1.1), which we may rewrite as:

$$d\alpha_t(V_t, \cdot) = h_t\alpha_t - \frac{d\alpha_t}{dt}$$

We have assumed that V_t is contained in ξ_t, where $d\alpha_t$ is non-degenerate. Therefore, the vector field exists, and we only need it to have a well defined flow.

Note that when $\frac{d\alpha_t}{dt}$ is zero then so is V_t.

This argument may be used to prove several foundational theorems in contact geometry. Most immediately:

Theorem 1.2.2 (Gray's Stability Theorem)**.** *Given a one parameter family of contact structures ξ_t on a closed manifold M, there exists an isotopy ψ_t of M so that $\psi_{t*}(\xi_0) = \xi_t$.*

Proof. Gray discovered this result using different methods, but for us, the above discussion of Moser's trick provides most of the proof. All that remains is the observation that since we are working on a closed manifold, the flow of our time dependent vector field V_t is well defined. It provides the desired isotopy. □

The other theorem we will use Moser's trick to prove is a generalized form of Darboux. Tho original Darboux theorem is covered by the case where C is a single point.

Theorem 1.2.3. *Given a compact subset C of an oriented three manifold M, and two contact structures ξ_0 and ξ_1 on M that agree on C, there exist neighborhoods U_0 and U_1 of C, and an isotopy ψ_t of M that fixes C, so that $\psi_1 : (U_0, \xi_0) \to (U_1, \xi_1)$ is a contactomorphism.*

Proof. We start by choosing contact forms α_0 and α_1 for ξ_0 and ξ_1 respectively, and construct the family of one forms that is the straight line between them. In other words we define $\alpha_t = (1-t)\alpha_0 + t\alpha_1$, for $t \in [0,1]$. We would like to find a neighborhood of C on which each of these is a contact form. Our goal, then, is that:

$$\alpha_t \wedge d\alpha_t = (1-t)^2\alpha_0 \wedge d\alpha_0 + t^2\alpha_1 \wedge d\alpha_1 + (t-t^2)(\alpha_0 \wedge d\alpha_1 + \alpha_1 \wedge d\alpha_0)$$

be a volume form. The first two terms already satisfy this, so we only need worry about the terms of the form $\alpha_i \wedge d\alpha_j$. Note, though, that on C, α_0 and α_1 are both contact forms for the same contact structure, so share the same kernel. Therefore $\alpha_0|_C = g\alpha_1|_C$ for some positive function g. On C, then, the forms $\alpha_0 \wedge d\alpha_1$ and $\alpha_1 \wedge d\alpha_0$ are both nonzero. Since C is compact, this must also be true on some neighborhood U of C. On U every α_t is a contact form, and we have an isotopy of contact structures from α_0 to α_1.

At this point, we utilize Moser's method from above. We are working on an open set rather than a closed manifold, so the vector field V_t that we produce may not have a globally defined flow. Still, since $V_t|_C = 0$, there is some small neighborhood U_0 of C so that for points in U_0, flow with respect to V_t is defined for all $t \in [0,1]$, and stays in some neighborhood U_2. Let U_1 be the image of U_0 under the time-one map. By cutting V_t off with a bump function outside of U_2 we arrive at an isotopy ψ_t of M, defined for $t \in [0,1]$, and $\psi_{1*}(\xi_0)|_{U_0} = \xi_1|_{U_1}$. □

1.3 The Characteristic Foliation

A classic method for studying three dimensional contact structures relies on their interaction with embedded surfaces. Let Σ be a surface inside the oriented contact manifold (M,ξ). Then $\xi|_\Sigma$ and $T\Sigma$ are two plane fields on Σ. We call a point p singular if $\xi_p = T_p\Sigma$. Note that because of the non-integrability of ξ, the singular set S is of dimension one or less. In fact, for generic surfaces it will consist of a discrete collection of points. Away from the singular points, the intersection of ξ and $T\Sigma$ define a line field on Σ. Locally, we may find integral curves for this line field, and every point in the surface is either singular, or contained in one of these curves. We use this to define a singular foliation on Σ.

Definition 1.3.1. *The singular line field $T\Sigma \cap \xi$ integrates to a singular foliation on Σ. This is the characteristic foliation on Σ induced by ξ, or Σ_ξ.*

Locally we may find sections of this line bundle, i. e. vector fields tangent to the leaves of Σ_ξ. If Σ and ξ are oriented, then we may do better.

Definition 1.3.2. *At a nonsingular point p, a vector v in $T_p\Sigma \cap \xi_p$ is positively oriented if, whenever $\{v, v_\xi\}$ and $\{v, v_\Sigma\}$ are oriented bases for ξ_p and $T_p\Sigma$ respectively, then $\{v, v_\xi, v_\Sigma\}$ is an oriented basis for T_pM.*

A vector field V on Σ that is tangent to Σ_ξ and zero at singular points is said to direct Σ_ξ. If in addition to this it is positively oriented at all nonsingular points, then it is a positive directing field.

The characteristic foliation gives us a powerful tool for describing the contact structure near a surface. It is enough to determine the contact structure locally.

Theorem 1.3.1 (Giroux, [14])**.** *Let Σ_0 and Σ_1 be surfaces embedded in contact manifolds, and $\varphi : \Sigma_0 \to \Sigma_1$ a diffeomorphism, that sends their characteristic foliations to each other. Then there exists a contactomorphism $N_0 \to N_1$ between normal neighborhoods of the surfaces that extends φ.*

Proof. We work in the space $\Sigma \times \mathbb{R}$, where Σ is a diffeomorphic copy of the Σ_i surfaces. We consider the two contact structures ξ_0 and ξ_1 that are induced via diffeomorphism to neighborhoods of Σ_0 and Σ_1 respectively.

We would like the structures ξ_i to be orientable on $\Sigma \times \mathbb{R}$. If they are not, we may lift to a double cover of Σ where they are. We do the following construction in the covering space, and project the results down to our original surface.

By possibly doing the above, we ensure the existence of contact forms α_i for ξ_i on $\Sigma \times \mathbb{R}$. In these coordinates, we may rewrite any one-form as $\beta_t + udt$, where u is a smooth function on $\Sigma \times \mathbb{R}$ and β_t a one-parameter family of one-forms on Σ. From now on we will drop the t and refer merely to β. Applying the contact condition, we quickly see that this is a contact form if and only if

$$ud\beta + \beta \wedge \left(du + \frac{d\beta}{dt}\right) > 0 \tag{1.2}$$

Along the surface $\Sigma \times \{0\}$, we represent α_i as $\beta_i + u_i dt$, as above. The fact that the α_i have the same characteristic foliation tells us that $\ker(\beta_0) = \ker(\beta_1)$, so there is some smooth, positive function g on Σ such that $\beta_1 = g\beta_0$. We may multiply any one-form by such a function without changing its kernel, so we construct an alternate contact form for ξ_1: $\hat{\alpha}_1 = \frac{1}{g}\alpha_1$, which on Σ is $\beta_0 + \frac{u_1}{g}dt$. Consider then the one form $\alpha_i = i\,(\hat{\alpha}_1) + (1-i)\alpha_0$. One can quickly see that if we set $u_i = \left(i\frac{u_1}{g} + (1-i)u_0\right)$, so that $\alpha_i = \beta_0 + u_i dt$ then $u_i d\beta_0 + \beta_0 \wedge \left(du_i + \frac{d\beta_0}{dt}\right)$ equals:

$$i\left[\frac{u_1}{g}d\beta_0 + \beta_0 \wedge \left(d\frac{u_1}{g} + \frac{d\beta_0}{dt}\right)\right] + (1-i)\left[u_0 d\beta_0 + \beta_0 \wedge \left(du_o + \frac{d\beta_0}{dt}\right)\right]$$

This is positive on Σ, and therefore positive on some neighborhood of Σ. We can then, after restricting to a small enough neighborhood, isotop ξ_0 and ξ_1 to contact structures that agree on Σ. By 1.2.3 we may isotop ξ_0 and ξ_1 to be identical on a neighborhood of Σ, and we may do this via an ambient isotopy of the manifold. We conclude that a characteristic foliation uniquely determines the germ of a contact structure its surface. $\square$

We now turn from the rigidity of characteristic foliations, to their flexibility. The space of singular foliations on a given set may be given a topology such that the map taking a plane field on M to the induced foliation on Σ is an open map. The space of contact structures is open within the space of plane fields. Furthermore, by a Moser argument, an isotopy of contact structures near a compact surface may be realized by ambient isotopy. These comments yield the following lemma of Giroux:

Lemma 1.3.2 (Giroux, [14])**.** *Let $\mathcal{P}$ be some C^∞-generic property of singular foliations on a surface Σ. Then Σ may be perturbed via a C^∞-small isotopy so that Σ_ξ satisfies $\mathcal{P}$.*

This will be very useful in section 1.6, but we here use it to give our characteristic foliations generic singular points. If our contact structure and surface Σ are oriented, then there exist positive vector fields that direct Σ_ξ. Near an isolated critical point, we choose a positive directing vector field V. In coordinates, the linearization of V is a matrix, and we may analyze the local dynamics of the flow of V by examining its eigenvalues.

If it has one with nonzero real part, or two with the same sign then the singularity is called elliptic. If it has two with opposite signs it is called hyperbolic. Furthermore, we call a singularity positive or negative depending on whether the orientations on ξ and $T\Sigma$ are the same or different respectively. In this classification, negative elliptic singularities will be sinks, positive ones sources, and all hyperbolic singularities saddles. A hyperbolic singularity can be seen to be positive or negative depending on whether its positive or negative eigenvalue is larger in absolute value. This classification will be useful in section 1.6 when we construct arguments based upon characteristic foliations. Note now, though, that every oriented surface may be perturbed so that all of its singularities fall into one of these four categories.

1.4 Tight and Overtwisted

The concept of the characteristic foliation allows us to describe the fundamental dichotomy at the heart of three dimensional contact geometry.

Definition 1.4.1. *A disk is overtwisted if its characteristic foliation contains the boundary as a single closed leaf, and has exactly one critical point in its interior.*

The first vital fact about overtwisted disks is that there are none in $(\mathbb{R}^3, \xi_{std})$. Therefore none can be contained in a Darboux ball, so they are a phenomena that can only happen on "large enough" scales. They allow us to classify three dimensional contact structures onto two classes:

Definition 1.4.2. *A contact structure ξ on a three manifold M is overtwisted if (M, ξ) contains an overtwisted disk. If not, the contact structure is tight.*

Of these, the overtwisted are more numerous, simpler to work with, and less interesting. Eliashberg has shown that the classification of overtwisted contact structures on a three-manifold effectively reduces to the classification of homotopy classes of plane fields [6]. Every three-manifold then has many overtwisted structures. Tight structures are much rarer. Not every three manifold has one. Furthermore, tightness is a necessary condition for symplectic fillability [12] [7] [16]. We will later find issues of tightness to be closely related to our notion of compatibility between contact structures and open books.

1.5 Knots

The contact condition prevents the existence of any disk tangent to a contact structure ξ, so the maximal dimensions of such a submanifold is one.

Definition 1.5.1. *An arc, knot, or link L in a contact three manifold is Legendrian if $TL \subset \xi$. It transverse if $TL \cap \xi = \{0\}$.*

Recall the rectangular version of the standard contact structure on $\mathbb{R}^3$: defined as $\ker(dz - ydx)$. At any point, the contact planes are spanned by the vectors $\{\partial_y, y\partial z + \partial x\}$. A curve $\gamma = (x, y, z)$ is Legendrian if and only if $\frac{dz}{dx} = y$. In the ϵ-neighborhood of any curve, there is a range of possible slopes a Legendrian may have, determined by the range of allowable y coordinates. We may then construct a tube around our original curve whose characteristic foliation consists of spirals that spin around it. This allows us to find a Legendrian arc within ϵ of any given arc. If we started with a circle, we may perturb our construction to insure the Legendrian links up into a closed curve.

If we then apply Darboux's theorem to cover the original curve with standard coordinate neighborhoods, we may show that:

Theorem 1.5.1. *Any compact curve in a contact manifold may be C^0-approximated by a Legendrian curve.*

This may be used to prove the following lemma, which will prove useful later:

Lemma 1.5.2. *Every compact graph in M that is Legendrian with respect to the orientable contact structure ξ has a tight neighborhood.*

Proof. Let G be a compact Legendrian graph in the contact manifold (M, ξ). Every finite graph may be embedded in $\mathbb{R}^3$, so there is a copy $\widetilde{G}$ of G in $(\mathbb{R}^3, \xi_{std})$. By the previous theorem, we may perturb $\widetilde{G}$ to make it Legendrian. This is still an isomorphic copy of G. Every edge may be given a tubular neighborhood using its contact framing. This allows us to construct a diffeomorphism from a neighborhood N of G to a neighborhood $\widetilde{N}$ of $\widetilde{G}$, that identifies the contact structures along G and $\widetilde{G}$. By theorem 1.2.3, there exist contactomorphic neighborhoods (U_0, ξ) and (U_1, ξ_{std}) of G and $\widetilde{G}$, so G has a neighborhood that embeds contactomorphically into $(\mathbb{R}^3, \xi_{std})$. This neighborhood must be tight. □

The study of Legendrian knots is a rich sub-field within knot theory. A number of invariants have been developed to distinguish non-equivalent knots from each other. We will later make use of one of the oldest of these: the twisting of ξ relative to some given framing.

Definition 1.5.2. *We are given a Legendrian knot L contained in a surface Σ. Consider the normal direction to L inside $T\Sigma$. This defines a framing of the knot, but so does the normal direction to L in ξ. We denote the twisting of the contact framing relative to the surface framing: $tw(L, \Sigma)$.*

There is two operations on Legendrians known as positive and negative stabilization, which involve adding small loops. These are done in arbitrarily small neighborhoods. The

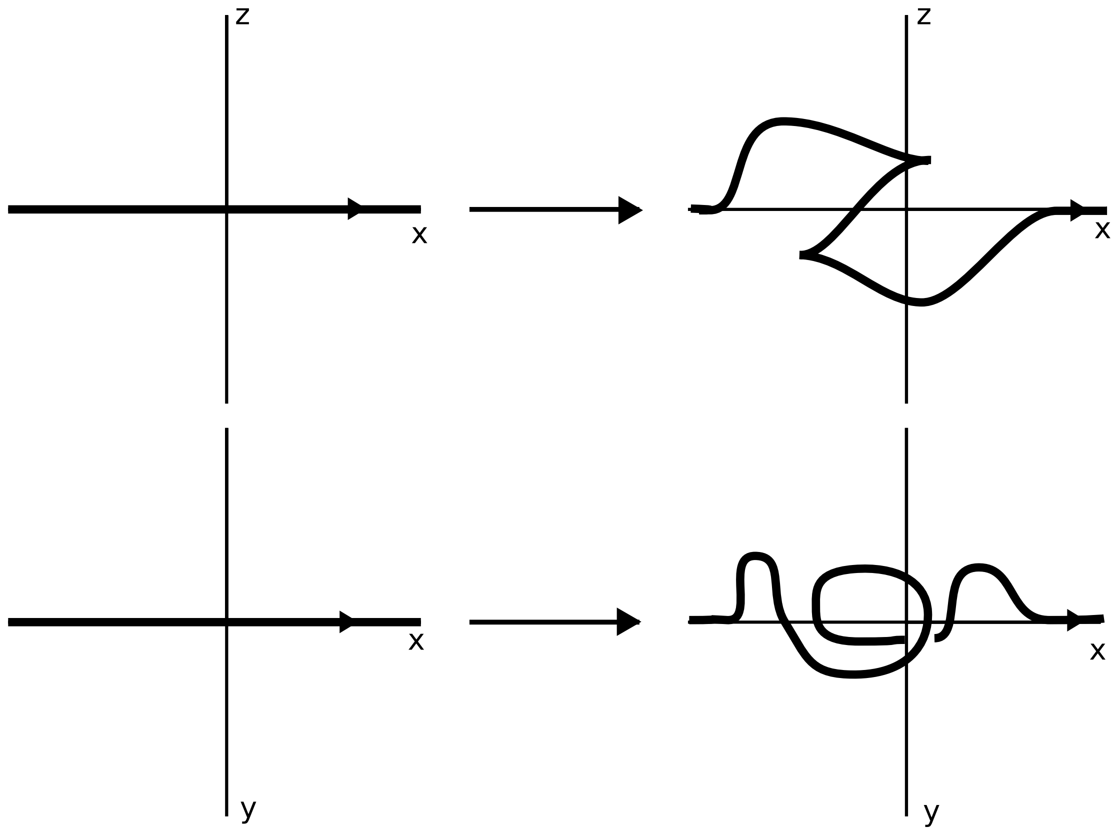

Figure 1. A positive Legendrian stabilization in ξ_{std}

positive version is pictured in figure 1. The important feature of these for our purposes is that if L bounds a surface Σ, then the Legendrian L' which has been stabilized in either way bounds a locally perturbed surface Σ', and $tw(L', \Sigma') = tw(L, \Sigma) - 1$. By preforming this operation, we may decrease the twisting of a Legendrian as much as we want.

There has been much investigation of the twisting, as well as other classical invariants of Legendrian knots. Later we will have use for one of the foundational results from this field:

Theorem 1.5.3 (The Weak Bennequin Inequality)**.** *Assume L is a Legendrian knot contained in, or bounding, the surface Σ in a tight contact three manifold. Then:*

$$tw(L, \Sigma) \leq -\chi(\Sigma)$$

1.6 Convex Surfaces

In [14], Giroux introduced a concept that has become fundamental to three dimensional contact geometry: that of the convex surface. Our summary is mostly drawn from this paper, along with [10], [13], and [21].

The term "convex" arises in reference to convex symplectic structures, i. e. those that are conformally invariant under some Morse flow. The analogous notion on complex manifolds is strict pseudo-convexity.

Definition 1.6.1. *A contact vector field on (M, ξ) is one whose flow preserves ξ. The contact structure is said to be convex if there exists a contact vector field that is the negative gradient of a Morse function.*

Using the standard story in which a Morse function represents height under some embedding, and the Morse flow describes the motion of syrup poured on our manifold, we think of a convex contact structure as being "vertically invariant." We, in a somewhat unfortunate choice of language, call a surface convex if it is horizontal in this metaphor. In other words:

Definition 1.6.2. *A surface $\Sigma \subset M$ is convex if there is a contact vector field transverse to Σ.*

Of course, this definition is slightly different from our above intuitive description. It is more properly described by the following lemma.

Lemma 1.6.1. *A surface $\Sigma \subset (M, \xi)$ is convex if and only if there is an embedding of $\psi : \Sigma \times \mathbb{R} \to M$, sending $\Sigma \times \{0\} \to \Sigma$, so that $(\psi^{-1})_*(\xi)$ is invariant under $\mathbb{R}$-translation.*

Proof. First, if such an embedding exists, then $\psi_*(\partial_t)$ is a contact vector field in M defined near Σ. The embedding sends each $\mathbb{R}$ component to a path in M that will be finite under any choice of metric, so the vector field $\psi_*(\partial_t)$ shrinks to zero near the edges of the normal the image of ψ, so can be extended by zero to the rest of M.

If Σ is convex, then there exists a transverse contact vector field V. We construct a normal neighborhood of Σ by using its flow to define the $\mathbb{R}$ coordinate. Choose some t_0 small enough that the flow is defined for longer in both positive and negative time, and we have constructed a diffeomorphism from a neighborhood of Σ to $\Sigma \times (-t_0, t_0)$, that sends V to ∂_t. Choose a diffeomorphism from the open interval to $\mathbb{R}$, and still V is taken to a vertical field. Use this diffeomorphism to define a contact structure on $\Sigma \times \mathbb{R}$, and it becomes our desired contactomorphism. □

This local picture gives us a useful way to both describe and construct contact vector fields on surfaces. As we have already mentioned, in a neighborhood of the form $\Sigma \times \mathbb{R}$, every one-form may be described as $\alpha = \beta_t + u_t dt$, where β_t is a one-form on $\Sigma \times \{t\}$, and $u_t \in C^\infty(\Sigma \times \{t\})$. In the case of a convex surface, though, we may do much better. Since ξ is vertically invariant, we may describe it via a vertically invariant one-form $\alpha = \beta + udt$, where β and u are a one form and function on Σ.

This formula describes any vertically invariant form. Plugging this in to the contact condition, we arrive at a convex version of equation 1.2:

$$(\beta \wedge du + ud\beta) \wedge dt > 0 \tag{1.3}$$

Such a form will exist in any invariant neighborhood of a convex surface. We do, in fact, have a rough converse to this fact as well.

Lemma 1.6.2. *Let Σ be surface in (M, ξ), and α a contact form near Σ. Define a one form $\beta = \alpha|_{T\Sigma}$ on the surface. Then Σ is a convex surface if and only if there is a smooth function u on Σ such that $\beta \wedge du + ud\beta > 0$.*

Proof. The above discussion allows us to find such a function on any convex surface. Assume instead that we have a function. Then $\hat{\xi} = \ker(\beta + udt)$ is a vertically invariant contact structure on $\Sigma \times \mathbb{R}$. The characteristic foliations on Σ coming from both ξ and $\hat{\xi}$ are defined by the singular line field $\ker(\beta)$, so they are the same. By theorem 1.3.1

then, there is a contactomorphism from $(\Sigma \times (-\epsilon, \epsilon), \hat{\xi})$ to a normal neighborhood of Σ in (M, ξ), so lemma 1.6.1 gives us Σ's convexity. $\square$

This provides our second method for showing that a surface is or is not convex. We may also find a dual version of this criterion. Given an area form ω on Σ, the choice of one-form β is equivalent to the choice of a vector field V via the equation $i_V\omega = \beta$. Then $d\beta$ may be written as $d(i_V\omega) = (\operatorname{div}_\omega V)\omega$. To find the dual version of $\beta \wedge du$, note that $\omega \wedge du = 0$, so $0 = i_V(\omega \wedge du) = i_V\omega \wedge du - \omega \wedge i_V du$, and $i_V\omega \wedge du = -du(V)\omega$. Then $\beta \wedge du + ud\beta = (-du(V) + u \operatorname{div}_\omega V)\omega$. So the above condition on u in terms of $\beta = \alpha|_{T\Sigma}$ may be restated, in terms of an area form ω and a vector field V directing Σ_ξ as the condition that:

$$u \operatorname{div}_\omega V - du(V) > 0 \tag{1.4}$$

If, given an area form and a directing vector field for Σ_ξ, we can find such a function u, then Σ is convex.

Definition 1.6.3. *Given a surface Σ with singular foliation $\mathcal{F}$, a collection of curves $\Gamma \subset \Sigma$ divide the foliation if:*

- *The leaves of $\mathcal{F}$ are transverse to Γ.*
- *$\Sigma \setminus \Gamma$ consists of two (possibly disconnected) surfaces $\Sigma^\pm$ so that $\partial\Sigma^+ = \partial\Sigma^- = \Gamma$.*
- *There exists an area form ω on Σ, and a vector field V that directs $\mathcal{F}$, and points transversely out of Σ^+ along Γ, so that: $\mathcal{L}_V\omega$ is positive on Σ^+ and negative on Σ^-*

The connection to convex surfaces arises from the following:

Definition 1.6.4. *The dividing set Γ induced on a convex surface Σ by the contact vector field V consists of all the points p in Σ at which $V_p \in \xi_p$.*

Note that the vertical invariance of ξ near Σ immediately implies that Γ can contain no Legendrian arcs, as these could be extended vertically to integral surfaces of ξ. Consider further, that if we write $\alpha = \beta + udt$ near our surface, then Γ consists of exactly those points where $\partial_t \in \ker(\alpha)$, i. e. where $u = 0$. From the contact condition we know that $\beta \wedge du > 0$ at these points. Since $\beta \neq 0$, Γ may contain no singular points. Since $du \neq 0$, Γ can contain no isolated points or open sets. It must be a collection of curves. In fact, as one expects from the names:

Lemma 1.6.3. *For any convex surface Σ and choice of transverse contact vector field, the induced set Γ divides the characteristic foliation, Σ_ξ.*

Proof. We take our division into two sub-surfaces from u by setting $\Sigma^+ = u^{-1}(0, \infty)$, and $\Sigma^- = u^{-1}(-\infty, 0)$. Note further, since $du \neq 0$ along $\Gamma = u^{-1}(0)$, every component of Γ is in the boundary of both Σ^+ and Σ^-. Furthermore, $du|_{T\Gamma} = 0$, so $\beta|_{T\Gamma}$ cannot disappear. Thus Σ_ξ is transverse to Γ. All that remains then is the third condition. It will entail slightly more work.

If we remove a small annular neighborhood A of Γ, we may replace α by the contact form $\frac{1}{|u|}\alpha = \frac{\beta}{|u|} \pm dt$ on $\Sigma_0^\pm = \Sigma^\pm \setminus A$. This defines the same contact structure, so our contact condition gives us that $\pm d\frac{\beta}{|u|} > 0$. From now on we will call this area form ω_0. If we use this to define a vector field V by the equation $i_V\omega_0 = \frac{\beta}{|u|}$, we will have that $\mathcal{L}_V\omega_0 = d\frac{\beta}{|u|} = \pm\omega_0$. This satisfies our third condition. We may now restrict our attention solely to A.

Consider a single component of A. We extend V through A, so that it still directs Σ_ξ, and use the flow of V to define coordinates $(x, y) \in (S^1 \times [-1, 1])$. In these coordinates, $V = -\partial_y$, and Σ^+ corresponds to where $y > 0$. We extend these coordinates a little ways outside of A, so that ω_0 is defined for y values outside of $\left(-\frac{2}{3}, \frac{2}{3}\right)$. Finally, we require that these coordinates be oriented so that $dx \wedge dy$ and ω_0 define the same orientation where both are defined.

In these coordinates, define a volume form $\omega_A = f(y)dx \wedge dy$, where f is a positive function we will shortly describe. Now we compute that $\mathcal{L}_V\omega_A = -\frac{df}{dy}dx \wedge dy$. We therefore require that f be decreasing when $y > 0$, and increasing when $y < 0$. This gives the Lie derivative of V our desired signs on $\Sigma^{\pm}$.

We need to interpolate between ω_0 and ω_A to get a single area form. We will do this for Σ^+. A similar construction works on Σ^-. Let $\hat{f}$ be the positive function so that $\omega_A = \hat{f}\omega_0$ where they overlap. Then an interpolation comes in the form of a smooth function $g(x,y)$ so that $g \equiv 1$ when $y \geq 1-\epsilon$, and $g = \hat{f}$ when $y \leq \frac{2}{3}+\epsilon$. Then the form $\omega = g\omega_0$ on the overlap allows us to patch them into a single area form. Note that $\mathcal{L}_V g\omega_0 = dg(V)\omega_0 + g\mathcal{L}_V\omega_0$. Our only condition on g then is that $\frac{dg}{dt} < 0$. This will be possible as long as $\hat{f} > 1$. Recall, though, that our only conditions on the function f which defined ω_A was that it be decreasing on Σ^+ and increasing on Σ^-. We may therefore increase its value so that ω_A is larger than ω_0 on the overlap, and so $\hat{f} > 1$. We may then construct a vector field and area form that allows Γ to satisfy the third condition to divide Σ_ξ □

In fact, this relationship is strong enough to give us another characterization of convex surfaces.

Theorem 1.6.4. *A compact orientable surface $\Sigma \in (M,\xi)$, whose boundary is either empty or Legendrian, is convex if and only if its characteristic foliation is divided by some collection of curves Γ. Furthermore, these curves are defined by some transverse contact vector field.*

Proof. One direction of this is exactly the preceding lemma.

To see the other direction, assume that Σ_ξ is divided by the collection of curves Γ. We let ω and V be the area form and vector field from the third condition in definition 1.6.3. Our goal is to construct a function that satisfies equation 1.4. If we define the function $u \equiv \pm 1$ on $\Sigma^{\pm}$, then $u \operatorname{div}_\omega V - du(V) = \pm \operatorname{div}_\omega V$. This is positive on both of $\Sigma^{\pm}$, so to prove convexity we need only fix things on a neighborhood of Γ.

Since Σ_ξ is transverse to Γ, we may find a an annular neighborhood A of Γ that the leaves of Σ_ξ run transversely through. Much like above, we let $\Sigma_0^\pm$ be the sub-surfaces $\Sigma^\pm \setminus A$.

On a component of A we again define coordinates $(x, y) \in S^1 \times [-1, 1]$, so that $V = -\partial_y$, and $y = 0$ on Γ. On A we let u be a smooth, monotonically increasing function of y such that $u(0) = 0$, and $u \equiv \pm 1$ for large enough, and small enough values of y. Do this on every component of A, and so extend the locally constant function u from $\Sigma_0^\pm$ to the entirety of Σ.

Note that $\operatorname{div}_\omega V$ is positive on Σ^+ and negative on Σ^-. Therefore the term $(u \operatorname{div}_\omega V)$ is positive on all of $\Sigma \setminus \Gamma$. On A, where u is non-constant, $-du(V) = -du(-\partial_y) = \frac{du}{dy} > 0$. Our function u, then, satisfies the equation: $u \operatorname{div}_\omega V - du(V) > 0$ on all of Σ, and so Σ is convex. Furthermore, u corresponds to the germ of a transverse contact vector field tangent to ξ exactly along Γ. □

There is one more characterization of convex surfaces to explain. This will chiefly be useful to prove that convexity is generic for surfaces in contact three-manifolds.

Definition 1.6.5. *A vector field on the surface Σ is almost Morse-Smale if:*

- *Its singular points are nondegenerate.*
- *Its closed orbits all have nondegenerate return map.*
- *There are no flow lines running from a negative singularity to a positive.*

A characteristic foliation is almost Morse-Smale if it is positively directed by an almost Morse-Smale vector field.

The chief use we make of this concept is in the following theorem:

Theorem 1.6.5 (Giroux, [14])**.** *If an orientable surface $\Sigma \subset (M, \xi)$ has almost Morse-Smale characteristic foliation, then it is convex.*

Proof. We will prove this by showing how to construct dividing curves for an almost Morse-Smale singular foliation. We first build two regions around the positive and negative singularities. Around each elliptic singularity we place a disk whose boundary is transverse to Σ_ξ. Around each closed leaf we place a band whose boundary is also transverse to Σ_ξ. This is possible because of the second condition above. Finally we put a band around the stable manifold of every positive hyperbolic point, and a band around the unstable manifold of each negative hyperbolic one, again insuring that the boundary of all such bands are transverse to the foliation. We call the collection containing all of the positive singularities and closed orbits Σ_0^+ and the collection containing negative ones Σ_0^-. Note that the third condition on Σ_ξ ensures that the unstable manifold of a negative hyperbolic point may never intersect the stable manifold of a positive one, so that these two surfaces are disjoint.

We select a volume form on $\Sigma_0^\pm$, whose orientation agrees with that on Σ, and a vector field that directs Σ_ξ. The divergence of V will be positive at positive singularities, and negative at negative ones. By rescaling the vector field, we may ensure that its divergence has the proper sign on Σ_0^+ and Σ_0^-.

Now consider the surface $\Sigma \setminus (\Sigma_0^+ \cup \Sigma_0^-)$. This has a foliation transverse to its boundary, and contains no singular points or closed leaves. It must be composed of annuli. We let Γ consist of curves parallel to the cores of these annuli, and transverse to Σ_ξ. We know that V is pointing out of Σ_0^+ and into Σ_0^-. At this point, we may essentially repeat the second half of the proof of lemma 1.6.3 to show that our Γ divide Σ_ξ. □

A theorem of M. Peixoto [28] [29] shows that vector fields on closed orientable surfaces are C^∞-generically Morse-Smale, which is a stronger condition than the above one. That, together with lemma 1.3.2, yields:

Theorem 1.6.6 (Giroux, [14])**.** *Any closed surface in a contact manifold is C^∞-close to a convex surface.*

In fact, if we allow ourselves to perturb the boundary, any orientable surface may be made convex. Usually, though, when perturbing a surface, we prefer to keep the boundary fixed. Under certain circumstances this is possible, though it takes a little more work.

Theorem 1.6.7 (Kanda, [24])**.** *Let Σ be a compact, orientable surface with Legendrian boundary, and $tw(L, \Sigma) < 0$ for every component $L \subset \partial\Sigma$. This surface may be made convex via the following two perturbations. First a C^0-small isotopy, rel. $\partial\Sigma$, on some neighborhood of the boundary, followed by a C^∞-small isotopy of the rest of the surface.*

Proof. We first make Σ_ξ almost Morse-Smale on a collar around its boundary, via the C^0 perturbation. A Legendrian boundary component L has a neighborhood contactomorphic to a neighborhood of the x axis in $(\mathbb{R}/\mathbb{Z} \times \mathbb{R}^2, \xi_{std})$. On these local coordinates, we will use the standard rectangular form: $dz - ydx$. The contact planes are horizontal along the x axis, so we see the their twisting in the twisting of the annulus from Σ. We may, via C^0 isotopy within our neighborhood, ensure that Σ twists uniformly around the x axis on some smaller neighborhood. The number of rotations, and their direction, is determined by $tw(L, \Sigma)$.

By recalling our conventions for orientation of Σ_ξ, one may see that if ξ is twisting through Σ in a left handed manner, then on L, Σ_ξ will point away from a positive singularity, and toward a negative one. Since we've made this twisting uniform, ξ is rotating in the negative direction at every singularity. Along L, Σ_ξ will always flow from positive singularities to negative. We may further make sure all singularities are nondegenerate, and that there be no closed orbits in our collar.

We do this around every component of $\partial\Sigma$. Now Σ_ξ is almost Morse-Smale on a collar around the boundary. We may make the rest of Σ_ξ almost Morse-Smale, via a C^∞ perturbation that fixes some, possibly smaller, collar. Finally, by 1.6.5 we conclude that our perturbed Σ is convex. □

The above is also possible when $tw(L, \Sigma) = 0$ for some boundary components. We will not need that case, and so have not included its proof. It is known that if $tw(L, \Sigma) > 0$ for any boundary component then the surface can not be made convex.

In the above, we may give Σ a standard form near its boundary.

Definition 1.6.6. *A surface Σ has a standard annular collar if there exists a neighborhood of each boundary component of the form $S^1 \times [-1,1] \times [-1,1]$ with coordinates (x,y,t), where Σ is represented by the annulus $\{t = 0, y \geq 0\}$ and the contact form is the kernel of $\alpha = \sin(2\pi nx)dy + \cos(2\pi nx)dt$, for some positive integer n.*

In proofs of 1.6.7, the C^0 perturbation used is often the one that creates a standard annular collar. We choose not to, as the weaker conditions on our surface near its boundary will make perturbation of surfaces near the singular sets of contact orbifolds easier.

Now we have both methods for detecting convexity of specific surfaces, and some results showing them to be generic. We spend the remainder of the section with results that show them to be useful. To start, recall that a dividing set is induced upon a convex surface by the choice of a transverse contact vector field. We already know that any collection of curves which divide Σ_ξ are arise from some contact vector field. In addition to this, any two dividing sets are related as follows:

Theorem 1.6.8. *Let Σ be a convex surface, and let Γ_0 and Γ_1 be dividing sets corresponding to different transverse contact vector fields. Then these dividing sets are isotopic through curves transverse to Σ_ξ.*

Proof. We work in our local model $\Sigma \times \mathbb{R}$. We construct two such neighborhoods of Σ, each associating ∂_t to the contact vector field of one of our dividing sets. This translates to two contact forms $\beta + u_i dt$, where $i \in \{0, 1\}$. But, if we define $u_i = (1-i)u_0 + iu_1$, this formula still defines a contact form. Then $\Gamma_i = u_i^{-1}(0)$ is a one parameter family of dividing sets for Σ_ξ, starting at Γ_0 and ending at Γ_1. □

Then the isotopy class of dividing sets is a feature of the convex surface itself,i. e. it does not depend on the contact vector field.

A contact structure is locally determined near a surface by it the characteristic foliation it induces. There is a similar relationship between dividing sets and characteristic foliations. By this, we mean:

Theorem 1.6.9 (Giroux's Flexibility Theorem, Giroux [14])**.** *Let Σ be a convex surface, either closed or with Legendrian boundary. Let it have dividing set Γ, and let $\mathcal{F}$ be a singular foliation on Σ divided by Γ, and with standard form near $\partial\Sigma$. Also, let U be some neighborhood of Σ. Then there is an isotopy of embeddings $\varphi_t : \Sigma \to U$, starting with the inclusion, so that $\varphi_t(\Sigma)$ is a convex surface divided by $\varphi_t(\Gamma)$, and $\varphi_1(\Sigma)$ has characteristic foliation $\varphi_1(\mathcal{F})$.*

Proof. We select some vertical neighborhood of Σ inside U, so that we may work on $\Sigma\times\mathbb{R}$. Unlike earlier, we will use z for the $\mathbb{R}$ direction, so as to free up t for a time-parameter. Select some collection of annuli A around Γ small enough that in A both Σ_ξ and $\mathcal{F}$ consist entirely of transverse arcs. Let ω be a volume form and V_0 a vector field directing Σ_ξ, so that $\pm \operatorname{div}_\omega V_0 > 0$ on $\Sigma^\pm$. We also define u to be appropriately ± 1 outside of A, and increasing smoothly from Σ^- to Σ^+ in A, as in previous arguments. We do this such that: $u \operatorname{div}_\omega V_0 - du(V_0) > 0$ on all of Σ, and so $\alpha_0 = i_{V_0}\omega + udz$ is a vertically invariant contact form for our structure. We will use β_0 to refer to the one form $i_{V_0}\omega$.

Now we consider $\mathcal{F}$. We let $\hat{V}_1$ and $\hat{\omega}$ be the directing vector field and the area form respectively from condition three of definition 1.6.3. Since the two omegas are both area forms, there exists a smooth positive function g so that $\hat{\omega} = g\omega$. Then: $g \operatorname{div}_{g\omega}(\hat{V}_1) = \operatorname{div}_\omega(g\hat{V}_1)$. Since g is positive, $\operatorname{div}_\omega(g\hat{V}_1)$ and our original $\operatorname{div}_{g\omega}(\hat{V}_1)$ have the same sign everywhere. Therefore the vector field $V_1 = g\hat{V}_1$ pairs with our first area form ω to satisfy condition three of 1.6.3. We may now forget about $\hat{V}_1$.

The function $u \operatorname{div}_\omega(V_1)$ is greater than zero on all of $\Sigma \setminus \Gamma$. Furthermore, $du(V_1)$ is equal to zero outside of A, and is strictly negative when u is non-constant in A, so strictly negative on all of Γ, and $u \operatorname{div}_\omega(V_1) - du(V_1) > 0$ on all of Σ. Let $\beta_1 = i_{V_1}\omega$, and $\alpha_1 = \beta_1 + udz$. This is a vertically invariant contact form on $\Sigma \times \mathbb{R}$, whose characteristic foliation of $\mathcal{F}$. Note further that if $\beta_t = (1-t)\beta_0 + t\beta_1$, then:

$$u d\beta_t + \beta_t \wedge du = (1-t)[u d\beta_0 + \beta_0 \wedge du] + t[u d\beta_1 + \beta_1 \wedge du]$$

so the kernel of $\alpha_t = \beta_t + u dz$ is a vertically invariant contact structure for every t. Call it ξ_t. Using Moser's trick, we will find a relevant ambient isotopy. To do this, we define the time dependent vector field X_t by requiring that $i_{X_t} d\alpha_t = \left[\frac{d\alpha_t}{dt}(R_{\alpha_t})\right]\alpha_t - \frac{d\alpha_t}{dt}$. Note that, since every one of the forms α_t is vertically invariant, this field X_t must be as well. From this, and the compactness of Σ, we may conclude that if we start with a small enough vertical neighborhood of Σ, X_t will have a well defined flow, ψ_t, all the way from time 0 to time 1.

The vertical invariance of X_t ensures that $\mathcal{L}_{X_t}\partial_z = -\mathcal{L}_{\partial_z} X_t = 0$, so that ψ_t preserves ∂_z. Thus ∂_z is transverse to $\psi_t(\Sigma)$, i. e. each such surface is convex. □

The essential features of Σ_ξ are described by the dividing set. From this, we may deduce conditions sufficient to allow a collection of curves to be included in such a foliation. This will allow us to perturb a convex surface in order to make certain curves Legendrian.

Definition 1.6.7. *A collection of curves C in the convex surface Σ is called non-isolating if C is transverse to Γ_Σ, and if every component of $\Sigma \setminus C$ intersects Γ_Σ*

Theorem 1.6.10 (Legendrian Realization Principle, Honda [21], extending Kanda [23])**.** *Let Σ be a convex surface that is closed or has Legendrian boundary, and C be a non-isolating collection of disjoint embedded curves and arcs in Σ. Then there exists an isotopy of Σ, rel. boundary, through convex surfaces, that takes C to collection of Legendrians.*

Proof. Given theorem 1.6.9, all we need do is to construct a singular foliation $\mathcal{F}$ that is divided by Γ and includes the curves in C as leaves. We will work on a component Σ_0 of $\Sigma \setminus (\Gamma \cup C)$. For the purposes of explanation we will assume $\Sigma_0 \subset \Sigma^+$. The same argument works in Σ^-, but the flows should point in the opposite direction.

Our goal in this is to specify certain forms for Σ_ξ on collars of $\partial\Sigma_0$, so that with these collars removed, the new boundary may be divided into two sets $\gamma^\pm$, such that the flow of Σ_ξ points transversely out of γ^- and into γ^+. We then define the rest of $\mathcal{F}$ by embedding our surface in $\Sigma_0 \times [-1, 1]$, with $\gamma^\pm$ comprising its intersection with $\{z = \pm 1\}$. From a generic embedding, the height function will be Morse-Smale, and have no local extremes. We will use its negative gradient flow to define $\mathcal{F}$ on Σ_0.

Components of $\partial\Sigma_0$ will contribute to γ^- if they contain arcs from Γ, and γ^+ if they do not. Our non-isolating condition will ensure that γ^- will not be empty.

First, we give every component coming from $\partial\Sigma$ a standard collar. We may then treat the interior boundaries of these collars exactly as we will curves in C.

Boundary components coming entirely from circles in C, we make into either closed orbits or circles of singular points. In either case, we may arrange that they repel nearby flow lines. When we remove collar neighborhoods of them, the flow enters the rest of Σ_0 through the boundary of the collar.

Components that come entirely from Γ are also simple. We only need ensure that the flow be exiting Σ_0 transversely through them.

On components containing arcs from Γ and from C, we fist define the flow to be exiting transversely along Γ. If an arc from C is bounded on both sides by arcs from Γ, we place a positive hyperbolic point in its center, such that the unstable manifold points along the arc toward Γ. If we have more than one arc from C adjacent to each other, we place a positive elliptic point at each such vertex, and a hyperbolic point in the center of any arc with elliptic points at each end. This allows our flow to exit through Γ, but has the disadvantage that it now wants to flow into $\Sigma_0$away from Γ. To deal with this, we place a single hyperbolic point to absorb one inward-flow line and direct the others toward Γ.

Now we may cut off a collar containing all the features described, we'll have a smaller surface whose boundary consists of $\gamma^\pm$. Finally, if γ^+ is empty, we place a single positive

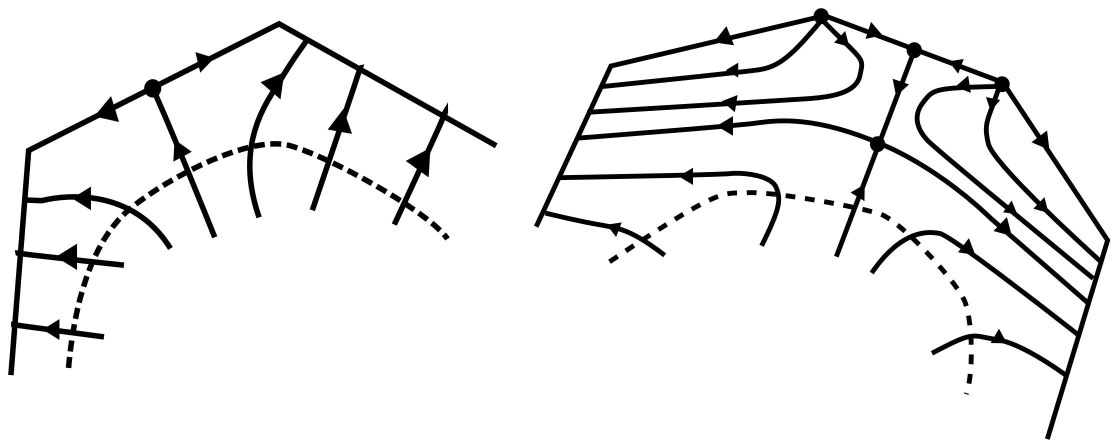

Figure 2. $\mathcal{F}$ near mixed boundary components.

elliptic point in the interior of Σ_0, and cut out a disk around it. This gives us a circle for γ^+.

Our flow on all of Σ_0 is Morse-Smale. When we preform the same operations on the other components of $\Sigma \smallsetminus (\Gamma \cup C)$, we will have a Morse-Smale foliation on all of Σ, and so $\mathcal{F}$ has dividing curves. Note that since we placed all of the positive singularities in Σ^+ and the negative in Σ^-, $\mathcal{F}$ will be divided by our original Γ. Then by Giroux Flexibility, we may perturb Σ to a surface with characteristic foliation $\mathcal{F}$.

□

The Legendrian Realization Principle is sometimes abbreviated as the LeRP. This is usually used when it will be repeatedly applied, and a verb form is desired. Then the verb "to LeRP" may be applied to non-isolating curves.

Theorem 1.6.11 (Edge Rounding, Honda [21])**.** *Assume that Σ_0 and Σ_1 are convex surfaces that intersect transversely along a shared Legendrian boundary component L. After a C^0 isotopy, the dividing sets of Σ_0 and Σ_1 will alternate along L. Also: there exists a smooth convex surface Σ created from $\Sigma_0 \cup \Sigma_1$ by rounding the edge. Each dividing curve of Σ_0 will connect an adjacent curve from Σ_1 in this new surface.*

Proof. We first isotop our surfaces to have the standard annular collars, previously described in definition 1.6.6. We then have coordinates $(x, y, z) \in (\mathbb{R}/\mathbb{Z} \times [-1, 1]^2)$ on some normal neighborhood N of L. Our contact structure on N is the kernel of $\alpha = \sin(2\pi nx)dy + cos(2\pi nx)dz$, where $n = -tw(L, \Sigma_0)$. After some isotopy, we may ensure that $\Sigma_0 \cap N$ is the annulus $\{z = 0, y \geq 0\}$, and $\Sigma_1 \cap N$ the annulus $\{z \geq 0, y = 0\}$. Our transverse contact vector fields for these annuli will be ∂_z and ∂_y respectively.

In these coordinates, the dividing curves for Σ_0 are the lines $x = \frac{2k+1}{4n}$, where k is any integer. For Σ_1, we have the lines $x = \frac{k}{2n}$. This tells us that the dividing curves alternate.

To round the corner, we attach the quarter of the cylinder $(y-\epsilon)^2 + (z-\epsilon)^2 = \epsilon^2$ that is closest to the x axis. The transverse vector field on this surface is the inward pointing radial field for the cylinder. By this, we mean the unit vector field pointing from any (x, y, z) on the surface, toward the center axis (x, ϵ, ϵ). This agrees with ∂_z and ∂_y on Σ_0 and Σ_1 respectively. It also creates spiraling dividing curves that connect the line $x = \frac{k}{2n}$ in Σ_1 to $x = \frac{2k-1}{4n}$ in Σ_0. We must then apply a minor perturbation to make the surface smooth, but this may be done without noticeably affecting the transverse contact fields or dividing curves. □

We end this section with two theorems that connect the structure on convex surfaces to two of our fundamental notions from earlier: twisting and tightness. It should be unsurprising that the twisting of a Legendrian in a convex surface is closely related to its interaction with the dividing set. Γ measures where the contact planes are "vertical" after all, and a plane cannot complete a full revolution without passing through the vertical.

Theorem 1.6.12 (Kanda, [24]). *Let L be a closed Legendrian curve in the convex surface Σ, divided by Γ. Then: $tw(L, \Sigma) = -\frac{1}{2}\#(L \cap \Gamma)$.*

(By # we mean simply the number of such intersections.)

Proof. Let V be a transverse contact vector field that generates Γ. Since it is transverse, the framing that L receives from Σ is the same as the one it receives from V. To prove

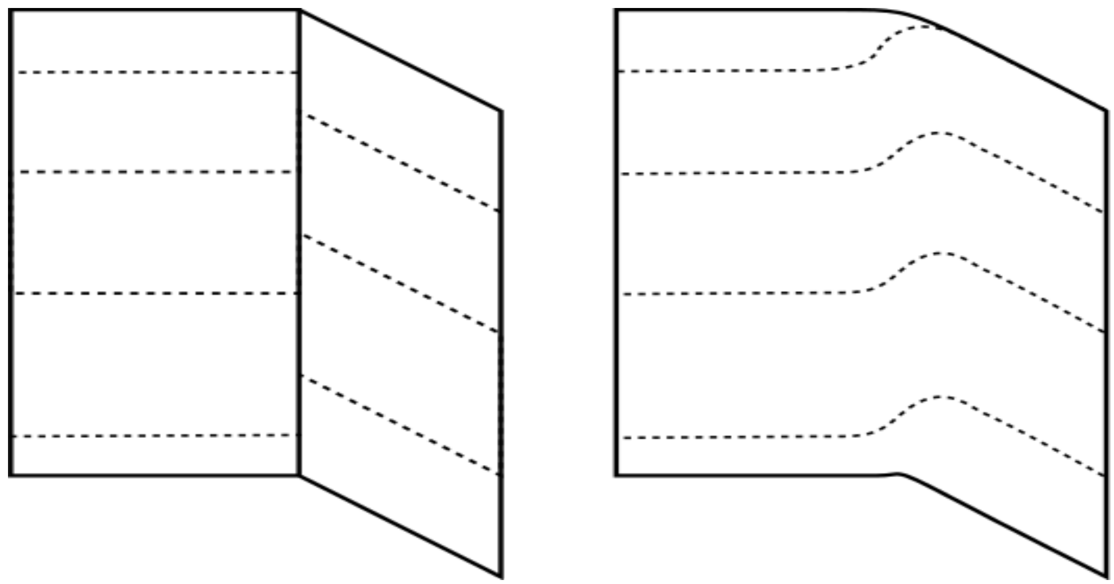

Figure 3. Rounding an edge between convex surfaces.

the theorem, all we need show is that the contact planes always twist past V in a left-handed manner. This will ensure that every two consecutive intersections between L and Γ correspond to one full, left handed twist of ξ. In other words: that every intersection contributes $-\frac{1}{2}$ to the twisting of ξ relative to V.

To see this, let W_1 direct the flow of Σ_ξ, and W_2 be the vector field such that (W_1, W_2) is an oriented basis for $T\Sigma$, and $\beta(W_2) = 1$. Then we let h be the function such that the vector $\widetilde{W} = hW_2 + \partial_t \in \xi$. We are now interested in how the vector field $\widetilde{W}$ rotates past ∂_t.

By definition, $\widetilde{W}$ is in ξ, so $(\beta + udt)(\widetilde{W}) = 0$. By plugging the vector field into our formula, though, we compute that: $(\beta+udt)(hW_2+\partial_t) = h+u$, so $h = -u$. When passing from Σ^+ to Σ^-, then, h passes from negative to positive, which is left handed rotation. When passing from Σ^- to Σ^+, the orientation on L is the opposite of that induced by our rules for Σ_ξ, so the opposite behavior of h still corresponds to left handed rotation. Therefore, each point at which L passes through Γ contributes $-\frac{1}{2}$ to $tw(L, \Sigma)$. $\square$

Our final theorem should also seem reasonable. It concerns what forms a dividing set may have if the contact structure is to be tight near Σ. Some such connection should be expected, as Γ determines Σ_ξ up to isotopy, and Σ_ξ determines ξ on small enough neighborhoods of the surface.

Theorem 1.6.13 (Giroux's Tightness Criterion, Giroux [15])**.** *Let Σ be a convex surface that is either closed or has Legendrian boundary. A vertically invariant neighborhood of $\Sigma \neq S^2$ is tight if and only if Γ_Σ contains no contractible circles. If $\Sigma = S^2$ a vertical neighborhood is tight if and only if Γ_Σ consists of a single circle.*

We will prove the sphere case, and the easier direction of the general case. For the full proof see [15] or [10].

Proof. If Σ is a sphere and Γ is connected, then we may, after some isotopies coming from 1.6.8 and 1.6.9, identify a vertical neighborhood of Σ with one of the unit sphere in $(\mathbb{R}^3, \xi_{std})$, so it is tight. If Γ contains multiple components, we will be able to produce an overtwisted disk using the techniques from the following case.

Assume then that Σ is not a sphere. Let D be a disk bounded by a contractible component of Γ, whose interior contains no other such components. Let D' be a slightly larger disk. If $\Sigma \setminus D'$ contains components of Γ, then $\partial D'$ is a non-isolating curve. Since it does not intersect Γ, we may apply Legendrian realization in such a way as to make it a closed leaf of Σ_ξ. We may then use Giroux Flexibility to perturb Σ_ξ so that D contains exactly one elliptic singular point, and there are no singular points in $D' \setminus D$. In other words: to make D' an overtwisted disk.

Now assume that Γ consists only of the contractible circle. If Σ is a disk, then its boundary is Legendrian, and we may treat it as we did D' above. If not, we again construct D', and seek to perturb Σ so that it has dividing curves outside of D'.

Since Σ is neither a disk nor a sphere, it must contain some homotopically nontrivial curve C. We use Legendrian realization to turn C into a curve of singular points. Then an annulus around C may be identified with one in the $x-y$ plane, around the x axis in

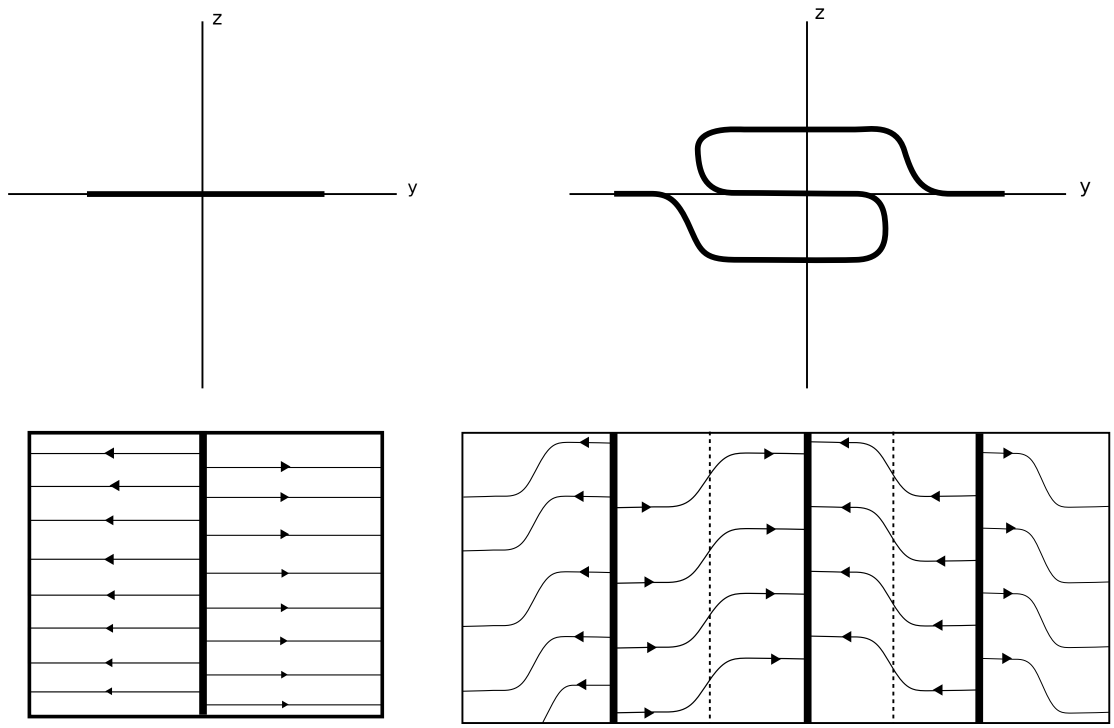

Figure 4. A fold adds two components to Γ_Σ

$S^1 \times \mathbb{R}^2$ with the standard rectangular contact structure. The left of figure 4, below, shows a cross section of this annulus, and a picture of its characteristic foliation. By applying the fold depicted on the right, we change Σ to Σ', and the characteristic foliation on the fold is as pictured. The fold now has two new dividing curves, drawn as dotted lines. Using this model, we may perturb Σ inside any vertical neighborhood, to a surface Σ' with additional dividing curves. Then the argument above allows us to construct an overtwisted disk on Σ'. □

CHAPTER 2

CONTACT ORBIFOLDS

2.1 Contact Actions

We start by considering what sorts of actions are possible on contact manifolds.

Definition 2.1.1. *A contact action of G on (M, ξ) is a smooth action of G on M that preserves the contact structure. In other words every element of G is a contactomorphism. If ξ is oriented and every element of G preserves this orientation, then it is a positive contact action.*

We concern ourselves entirely with positive actions. This is motivated somewhat by the connection to symplectic actions, as these must preserve orientation.

Recall that a contact form induces an orientation on the contact structure it defines. Any two forms for the same contact structure are related by $\alpha_0 = g\alpha_1$ for some nonzero smooth function g. If both induce the same orientation, then $g > 0$. Therefore any positive linear combination of oriented contact forms is another positively oriented contact form. Given any positive contact G-action and any positive contact form α, we may construct an invariant contact form by taking the sum: $\overline{\alpha} = \sum_{g \in G} g^*\alpha$. We will call this "averaging," whether we divide by $|G|$ or not.

Take G to be a finite group acting on (M, ξ), that preserves an orientation on M. Assume furthermore that the action is effective. If not we may instead deal with the action of G/K, where K is the maximal subgroup acting trivially. Every point $p \in M$ has a stabilizer subgroup $\mathrm{Stab}(p)$ in G consisting of all group elements that fix it. The action of Stab(p) then induces a linear action on the tangent space at p. We may, via averaging, construct a metric that this action preserves. With regards to this, the action on T_pM will be orthogonal, so the action may be described by a finite subgroup of $SO(3)$. If we choose coordinates so that ξ_p is the horizontal plane, then we have three options for the action. These are actions of the trivial, cyclic and dihedral groups.

There are two types of cyclic action that preserve the plane. One category consists of the finite rotations around the z axis, the other rotation by π around a horizontal line. The dihedral action is, of course, generated by one of each type. Of these cyclic actions, rotation around the z axis preserves the orientation on the plane, while rotation around a horizontal line does not. From this we may conclude that for positive contact actions, the first type are our only possible local models. Every positive contact action must be of cyclic type.

Definition 2.1.2. *A smooth finite action on a manifold is of cyclic type if the stabilizer of every point is either trivial or cyclic.*

By averaging a metric over our action, we may always find a G-invariant metric m on M. Let p be a point in M with nontrivial stabilizer Γ. There is a neighborhood U_0 around p small enough that $g(U_0) \cap U_0 = \emptyset$ for any g not in Γ, and a Γ-invariant neighborhood V_0 of 0 in T_pM on which the exponential map is a diffeomorphism. We restrict our attention to $V_1 = V_0 \cap exp^{-1}(U_0)$ and its image under exp: U_1. Now, since G-action, and so too Γ-action, preserves m, it must send geodesics to geodesics. From this we quickly see that, for any g in Γ, $exp_p \circ g_* = g \circ exp_p$, so the exponential map yields a diffeomorphism from V_1 to U_1 preserving Γ action. If we assume our G action is of cyclic type, then only the z axis in T_pM is fixed by Γ action, so the set of points in U_1 with nontrivial stabilizer form a curve through p. The restriction to U_0 ensures that this is true for G-action, not merely Γ-action. Therefore, the set of points with nontrivial stabilizer under a cyclic type action form a link in the manifold. We will call this the singular link in M.

Notice that this implies that every point in M has a neighborhood made entirely of points with either the same stabilizer subgroup, or trivial stabilizer. If a point is non-singular, it has a neighborhood disjoint from the singular link. If singular, it has a neighborhood that intersects no other components of that link. To describe an action in a small neighborhood of a point, then, we only need worry about the action of its stabilizer. On a small enough neighborhood, the rest of the group action consists of the permutation of disjoint copies.

To describe the local behavior of a positive contact action, we only need worry about neighborhoods of singular points. First, though, we define the standard contact actions:

Definition 2.1.3. *The standard contact action of order n is an action of $\mathbb{Z}_n$ on the standard contact structure $(\mathbb{R}^3, \ker(dz + r^2 d\theta))$. It is generated by a rotation of $\frac{2\pi}{n}$ around the z axis.*

These are clearly contact actions, as the standard radial contact form is invariant under changes of θ.

Theorem 2.1.1. *(Equivariant Darboux Theorem) Every point in a contact manifold with a positive contact action has a neighborhood that is equivariantly contactomorphic to a neighborhood in $(\mathbb{R}^3, \xi_{std})$ with a standard cyclic action.*

Proof. As stated above, every non-singular point p has a neighborhood containing no singular points, this reduces to the standard Darboux theorem. Assume U is a neighborhood of p small enough to be a Darboux chart, and so that G action permutes contactomorphic copies of it. Then identify it with a neighborhood in $(\mathbb{R}^3, \xi_{std})$, on which we let G act trivially. Then the G action provides maps from every open set in the orbit of U to this same open set. Therefore we only need worry about finding standard neighborhoods on singular points, that are equivariant with respect to their stabilizers.

Let p then be a singular point of a positive contact action. Since we need only worry about the action of $\mathrm{Stab}(p)$, assume that p is, in fact, a fixed point. As noted above,

p has some neighborhood equivariantly diffeomorphic to a neighborhood of the origin in $\mathbb{R}^3$ with a standard cyclic action. On this neighborhood, then, we have two contact structures: $\xi_0 = \xi_{std}$ and ξ_1, which is induced by the chart. Both of these are preserved by our action, so, since the action preserves only horizontal planes along the z axis, they agree there. Let α_1 be an invariant contact form for ξ_1, and α_0 be the standard radial form, which we already know is invariant. As in the proof of 1.2.3, by restricting to a small enough neighborhood of the z axis, we may ensure that each of the forms $\alpha_t = (1-t)\alpha_0 + t\alpha_1$ is a contact form.

Now we use Moser's method to find an isotopy of the manifold that produces this isotopy of contact structures. Recall that to do this we use α_t and its Reeb flow R_t, both of which are invariant under our action, to define a family of functions h_t, and then we use it, together with α_t ,$d\alpha_t$, and $\frac{d\alpha_t}{dt}$ to define a time dependent vector field V_t via the equation:

$$d\alpha_t(V_t, \cdot) = h_t\alpha_t - \frac{d\alpha_t}{dt}$$

Every term is invariant under our action, so at non-singular points V_t must be as well. At every singular point the contact structures agree, so $V_t = 0$. By again restricting to small enough neighborhoods of the z axis, the flow of V_t produces an equivariant isotopy that takes α_0 to α_1. This isotopy allows us to construct the equivariant Darboux chart that was our goal. □

Suppose instead of mapping a neighborhood of a singular point, we instead used a tubular neighborhood of an entire singular knot. Each component has a neighborhood equivariantly diffeomorphic to a tube around the z axis in $\mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$, with the standard cyclic action. An essentially identical argument to the one above shows that:

Corollary 2.1.2. *Every singular link has a tubular neighborhood that is equivariantly contactomorphic to a tube around the z axis in $(\mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z}), \xi_{std})$, with the standard cyclic action.*

2.2 Orbifolds

Orbifolds were first defined by Satake in [30] under the name V-manifold. A few decades later, Thurston independently reintroduced the concept in [31], together with the term Orbifold. These spaces generalize the orbit spaces of $\mathbb{R}^n$ under finite action in much the same way that manifolds generalize $\mathbb{R}^n$.

Definition 2.2.1. *An orbifold chart, or local uniformizing system, on a topological space Y consists of an open set U in Y, an open set V in $\mathbb{R}^n$, an effective action of a finite group Γ on V, and an equivariant continuous surjection $\varphi : V \to U$, that factors through a homeomorphism between U and V/Γ.*

Two local uniformizing systems $(U_i, V_i, \Gamma_i, \varphi_i)$, with i either 1 or 2, are compatible if for every pair of points x_i in $\mathbb{R}^n$, such that $\varphi_1(x_1) = \varphi_2(x_2)$, there are diffeomorphic neighborhoods W_i of x_i so that this diagram commutes.

$$\begin{array}{ccc} W_1 & \longrightarrow & W_2 \\ \downarrow{\scriptstyle \varphi_1} & & {\scriptstyle \varphi_2}\downarrow \\ U_1 \cap U_2 & \xrightarrow{id.} & U_1 \cap U_2 \end{array}$$

We will call the diffeomorphism from W_1 to W_2 a transition map.

A collection of local systems is compatible if any two of them are.

Definition 2.2.2. *A maximal collection of compatible charts is an orbifold atlas. An orbifold Y consists of a topological space and an orbifold atlas covering it. The space itself, without the orbifold structure, is called the underlying space, or $|Y|$.*

A diffeomorphism of orbifolds is a homeomorphism that lifts to an equivariant diffeomorphism in every orbifold chart.

Given any manifold with a discrete group action, the orbit space M/G carries a natural orbifold structure. Each point p can be made the center of an orbifold chart, with $\Gamma = \text{Stab}(p)$. This is an example of, and motivates the definition of, an orbifold covering.

Definition 2.2.3. *Let Y and $\widehat{Y}$ be orbifolds. A covering of Y by $\widehat{Y}$ consists of a continuous map $p : \widehat{Y} \to Y$, such that every point of Y has a neighborhood U with following property: For every component $\widehat{U}$ of $p^{-1}(U)$ there is a chart $\varphi : \widehat{V} \to \widehat{U}$ so that $p \circ \varphi$ is a chart on Y.*

This is a modified version of a topological cover, but it is close enough to the original that orbifolds still have universal covers. In [31] Thurston defined the orbifold fundamental group as the group of deck transforms of this simply connected universal cover. We call an orbifold good if it may be covered by a manifold, which is equivalent to the requirement that its universal cover be a manifold. An orbifold is very good if may be finitely covered by a manifold. In [25] it was proved that if one assumed Thurston's geometrization of Orbifolds, the every good three orbifold is very good. Geometrization was more recently proved in [4].

Definition 2.2.4. *At every point p, we define a group Γ_p, called the local group at p. This is the stabilizer of any point in $\varphi^{-1}(p)$ under Γ action in any chart (U, V, Γ, φ) containing p. A point is singular if and only if it has a non-trivial local group.*

An orbifold chart with the trivial group $\Gamma = \{0\}$ is effectively a manifold chart. It should be clear that around every point p is a chart whose group Γ is Γ_p, so that an orbifold with empty singular set is a manifold.

Definition 2.2.5. *An n-suborbifold $W \in Y$ is a subset of Y that lifts to an n-dimensional submanifold of Y in every local uniformizing system.*

A three dimensional, cyclic-type orbifold is one in which every local group is either trivial or cyclic. It should be clear that every good cyclic-type orbifold is the quotient of a cyclic-type action. The singular set in such an orbifold will be one dimensional. We will often refer to it as the singular link, or, as with the singular sets of foliations, as S. The orbifold coverings of cyclic-type orbifolds have a close relationship to branched coverings.

Definition 2.2.6. *In three dimensions, a branched cover $p : M^3 \to N^3$ is a continuous map between manifolds so that M contains a link L where:*

- *The map $p : (M \smallsetminus L) \to (N \smallsetminus p(L))$ is a covering of manifolds in the usual sense.*
- *Each component of L has a tubular neighborhood U on which there are coordinates that identify p with the map $(r, \theta, z) \to (r^n, n\theta, z)$ on $\mathbb{R}^2 \times S^1$.*

We call L the ramification locus, and $p(L)$ the branch locus.

Later it will also be useful to know that:

Theorem 2.2.1. *Every closed three-manifold M has a branched cover $M \to S^3$.*

This was originally proved by Alexander in [2]. Later: [20], [19] and [26] independently showed that this may be done with a three-fold cover. Essentially, the more recent proof involves constructing a cover of S^2 by a surface of arbitrary genus, that extends to a cover of the three-ball by any handlebody filling the surface. This may then be used to construct a covering map from any manifold with a Heegard decomposition, i. e. any closed three-manifold, to a manifold with a genus zero decomposition, i. e. S^3.

The concept of branched covering allows us to put a smooth manifold structure on $|Y|$, compatible with that on $Y \smallsetminus S$. A chart at a singular point of order n will have the form $(U, V, \mathbb{Z}_n, \varphi \circ \psi)$, where ψ is the map on $\mathbb{R}^3$ sending (r, θ, z) to $(r, n\theta, z)$, and φ is a homeomorphism from $V/\mathbb{Z}_n$ to U.

Consider the map on $\mathbb{R}^3/\mathbb{Z}_n$ taking $(r, n\theta, z)$ to $(r^n, n\theta, z)$, that comes from lifting an orbifold cover, and then applying the relevant branched cover. This is a homeomorphism, defined on tubular neighborhoods of the singular set, which may be extended via the identity to the rest of a cyclic-type orbifold.

The chief difference between the concepts lies in the first coordinate of the covering map. On a surface, one can easily see that the result of the map $(r, \theta) \to (r, n\theta)$ will be to create a cone. This is why cyclic singularities on two orbifolds are often referred to as “cone points.” In the case of the branch cover map $(r,\theta) \to (r^n, n\theta)$, the distortion on the radial direction smooths out the result, making it look more like a paraboloid.

Back in three dimensions, this has the effect that for a branched cover $p : M \to N$, any smooth surface in N transverse to the branch locus, lifts to a smooth surface in M. If the cover is arising from a cyclic-type action, then the surface will be invariant. The same is true for orbifolds, but there it is bound up in the definition of a sub-orbifold. A two-orbifold in Y is a surface in $|Y|$ that has its tangent planes on S determined uniquely by the orbifold structure. It must be transverse to S, and its tangent space must be horizontal in the orbifold chart there.[1]

The inverse of the above map consists of lifting a branched cover, and then projecting with an orbifold cover. This map sends every surface in $|Y|$ that is transverse to the singular set to a two orbifold in Y. Intuitively, it stretches the surface in the radial direction, in order to give it a horizontal tangent plane at S.

2.3 Contact Orbifolds

Definition 2.3.1. *A contact structure ξ on an orbifold Y consists of a Γ-invariant contact structure in each of the local uniformizing systems, which are compatible where they overlap. An orbifold with a contact structure is a contact orbifold.*

A positive contact structure has a consistent orientation in every chart, that is preserved by the local action.

From section 2.1, every positive contact action is locally either trivial, or cyclic. From this we conclude that every positive contact orbifold is of cyclic type. We will hereafter limit our attention to positive contact orbifolds, and refer to them merely as contact orbifolds.

[1] Here we are assuming one of our standard charts, which sends S to the z-axis, and uses a standard rotation to generate its cyclic action.

Recall also that for every positive contact action on an orientable contact structure we were able to find an invariant contact form. We may also find contact forms on orbifolds.

Definition 2.3.2. *A differential form on an orbifold consists of a* Γ*-invariant form in each local uniformizing system, that define the same form on the overlap of any two charts.*

Now recall definition 2.2.3, of orbifold covering. The chief requirement for $\widehat{X}$ to cover X was that locally, charts φ on X could be defined from charts $\widehat{\varphi}$ on $\widehat{X}$ via the equation $\varphi = \widehat{\varphi} \circ p$. A contact structure on $U \subset X$ is exactly an invariant contact structure in $\widehat{V} \subset \mathbb{R}^3$, where $\varphi : \widehat{V} \to U$. Then ξ is also invariant under $\widehat{\varphi} : \widehat{V} \to \widehat{U}$, so this defines a contact structure on $\widehat{U} \subset p^{-1}(U)$. From this we see that a contact structure on X lifts to a contact structure on every cover, $\widehat{X}$, of X. So, given a contact action, we may construct G invariant contact structures on M by working on M/G. We will generally prove results on orbifolds, and take the corresponding equivariant fact as a corollary.

We would like to adapt the fundamental dichotomy of three dimensional contact geometry to the orbifold case. There are, though, some issues in extending the notion of tightness to contact three orbifolds that we leave for later work. For our purposes, it will be enough to define the concept for good orbifolds.

Definition 2.3.3. *A contact structure* ξ *on a good orbifold* Y *is tight if it lifts to a tight contact structure on some manifold* M *that covers* Y*.*

Consider the case of a covering $p : \widetilde{M} \to M$ of contact manifolds. This is an example of an orbifold cover, but one where neither space has a singular set. Choose some disk, D, in M. Since D is simply connected, we may conclude that $p^{-1}(D)$ is a disjoint collection of disks and that if we restrict p to any component of that pre-image, it will become a diffeomorphism. In fact, we may make the same argument concerning a normal neighborhood of D, and each restriction of p becomes a contactomorphism. Then D is an overtwisted disk if and only if every component of $p^{-1}(D)$ is. Therefore, an overtwisted manifold may not be covered by a tight one. We call a manifold universally tight if its universal cover is, in which case so is every other cover.

Definition 2.3.4. *A good contact orbifold is universally tight if its universal cover is tight. Every cover of a universally tight orbifold is tight.*

We also want an appropriate notion of convex surfaces in contact orbifolds.

Definition 2.3.5. *A contact vector field V in (Y, ξ) is a vector field on Y that lifts to a contact vector field in every orbifold chart.*

A two-orbifold Σ in Y is a convex surface if there exists a contact vector field transverse to Σ.

A convex vector field induces a dividing set Γ on Σ, consisting of all points at which $V \in \xi$.

Note that the orbifold structure forces ξ to be tangent to Σ at all singular points. Therefore, Γ does not intersect the singular set of Σ.

Note also that in any orbifold cover, a contact vector field must lift to another contact vector field. From this, we may immediately conclude that a convex surface in a covered orbifold always lifts to a convex surface "upstairs." The covering also identifies the dividing sets of the respective surfaces.

We will generally use convex surfaces when working with good orbifolds, and apply the tools of convex surface theory to the manifolds upstairs. We leave the investigation of the extent to which these results may be extended directly to the orbifolds for later work.

CHAPTER 3

OPEN BOOKS

Giroux's result, together with that of Thurston and Winkelnkemper, has shown the close relationship between contact structures and the more topological notion of open book decompositions.

3.1 Books

Open books come in two varieties.

Definition 3.1.1. *An open book decomposition of the three manifold M consists of a pair (B, π) of a link B in M, called the binding, and a fibration $\pi : (M \setminus B) \to S^1$. Every fibre of π is the interior of a compact, oriented surface in M, whose boundary is B. These surfaces are called the pages of the book.*

Definition 3.1.2. *An abstract open book consists of a pair (Σ, φ) of a compact, orientable surface with boundary Σ, called the page, and a diffeomorphism $\varphi : \Sigma \to \Sigma$, called the monodromy. This map is required to fix a neighborhood of $\partial\Sigma$.*

From an abstract open book, we may construct a three manifold $M_{(\Sigma,\varphi)}$, which comes with an open book decomposition. This construction starts with the mapping torus of φ. This is the three manifold $(\Sigma \times [0,1])/\sim$, where $(x,1) \sim (\varphi(x),0)$ for any $x \in \Sigma$. The mapping torus is fibered over the circle, with the projection map coming from projection onto the second coordinate of $\Sigma \times [0,1]$. After the quotient, this maps to $[0,1]/\{0,1\} = S^1$.

The boundary of our mapping torus comes from that of Σ. Each component of $\partial\Sigma$ becomes a cylinder in $\Sigma \times [0,1]$, and then a torus after the quotient. To each of these we glue a solid torus, in order to produce the closed manifold $M_{(\Sigma,\varphi)}$. We do this by identifying the curves $\{x\} \times S^1$, for $x \in \partial\Sigma$, with the boundaries of the meridional disks $D^2 \times \{\theta\}$ in $D^2 \times S^1$. The boundary of each page, $\partial\Sigma \times \{\theta\}$, corresponds to the longitudes $\{x\} \times S^1$. The cores of these solid tori form the binding of an open book decomposition, which extends the fibration on the mapping torus.

Two abstract open books (Σ_i, φ_i), $i \in \{1,2\}$, are equivalent if there is a diffeomorphism $\sigma : \Sigma_1 \to \Sigma_2$ so that $\sigma \circ \varphi_1$ is isotopic to $\varphi_2 \circ \sigma$. Any diffeomorphism of the pages defines a diffeomorphism from $\Sigma_1 \times [0,1]$ to $\Sigma_2 \times [0,1]$. The condition on φ insures that it is still well defined on the mapping tori. We only need extend it to the relevant bindings to have a diffeomorphism between the closed manifolds that identifies their respective open book decompositions. This can be done, since our map already preserves the framings of the boundary tori. From equivalent abstract books we have constructed diffeomorphic manifolds, with diffeomorphic open book decompositions.

In a similar way, we may construct an abstract book from a decomposition (B, π). We start by removing a tubular neighborhood N of B. This done, we are left with a three manifold that fibres over S^1. The fibre $\Sigma = \pi^{-1}(\theta)$ will be the page of our abstract book.

Since this bundle is over S^1, if we remove one of the pages we are left with a bundle over the interval. Every such bundle is trivial, since the interval contracts to a point. This means that we can encode all of the complexity of our bundle into a gluing map that attaches its two ends.

To produce this map, we take a vector field V in our original bundle that is transverse to each page. This defines a first return map, φ from Σ to itself. To guarantee that this map fixes some neighborhood of $\partial\Sigma$, we require our vector field to point in the meridional direction on a neighborhood of $M \smallsetminus N$. Our manifold is the mapping torus of this return map φ, which will be the monodromy of our abstract book. This abstract book constructs a diffeomorphic copy of M, with a diffeomorphic open book decomposition.

Note that decompositions exist within a given manifold, while abstract books provide data for constructing one. For this reason, the abstract book will be our preferred language for the construction of contact structures. Open book decompositions will be the relevant concept for the "Giroux direction", which builds an open book on a given contact manifold or orbifold.

3.2 Books and Contact Structures

Before we explain these constructions we will need the appropriate concept of compatibility between books and contact structures.

Definition 3.2.1. *An open book (B, π) supports ξ if, after some isotopy, it has a defining form α such that $\alpha > 0$ on TB, and $d\alpha > 0$ on the interior of every page.*

There are several equivalent conditions to this, two of which we will describe. The first of these is almost immediate. Recall from definition 1.2.1 that every contact form has an associated Reeb field.

Theorem 3.2.1. *An open book (B, π) on M supports a contact structure ξ if and only if, after some contact isotopy, ξ has a Reeb field R that is positively tangent to B and positively transverse to every page.*

Proof. First assume we have such an R. Let α be its corresponding contact form. Then α must be positive on TB, since R is positively tangent to B. Also note that $i_R(\alpha \wedge d\alpha)$ is $d\alpha$, since $i_R\alpha = 1$ and $i_R d\alpha = 0$. R is positively transverse to the pages, so this means that $d\alpha > 0$ on them.

Assume then that the book supports ξ, and let α be the contact form from definition 3.2.1. Let R be its Reeb field. The fact that $d\alpha$ is a volume form on pages tells us that

R must be positively transverse to them. All we need then is to ensure that R is tangent to B.

We set up coordinates on a tube around B, sending it to the z axis in $\mathbb{R}^2 \times S^1$. Furthermore, construct these so that every page is represented by a radial half-cylinder.[1] Consider then the two pages represented by the plane $\{x = 0\}$. Our field is positively transverse to both. The orientations on the pages are coming from their place in $\Sigma \times [0, 1]$, so the flow of R is winding around the z axis. This means that along this surface, its x component is positive when $y > 0$, and negative when $y < 0$. On the z axis this component must be zero. By using the two pages which comprise the surface $\{y = 0\}$ we may also argue that the y component of R is zero along the z axis. As a Reeb field, R cannot vanish so it must be tangent to B. □

This result is often useful, but for us it will chiefly be used to prove the following:

Theorem 3.2.2. *Let (B, π) be an open book decomposition of (M, ξ), and let Σ be a smooth surface composed of the union of two opposite pages. We would like it to satisfy:*

- *Σ is convex, with dividing set equal to B.*
- *ξ is tight on both components of $M \setminus \Sigma$.*

If, after some isotopy of ξ, there exists such a surface, then (B, π) supports ξ. If (B, π) supports ξ, then, after some isotopy of ξ, every smooth union of pages satisfies these criteria.

Notice that the asymmetry which makes its statement rather awkward strengthens this theorem quite nicely.

Proof. First, we assume that (B, π) supports ξ. Then, after an isotopy, there exists a Reeb field R that is tangent to B and positively transverse to the interior of every page. Let $\Sigma^{\pm}$ be two pages of our book, whose union is the smooth closed surface Σ. Note

[1]They are cylinders because of the quotient on z. If we were in $\mathbb{R}^3$ they would be half-planes.

that, as in the previous proof, the orientations Σ^+ and Σ^- receive as pages of our book induce opposite orientations on Σ. We will give our closed surface the orientation from Σ^+. Then R is positively transverse to the interior of Σ^+ and $-R$ is positively transverse to Σ^-. Both of these are contact fields. To produce a transverse contact field for Σ, we will perturb R and $-R$ on some neighborhood of B into a unified contact field.

Since R is tangent to B, B must be a transverse to ξ. Therefore, the characteristic foliation Σ_ξ is composed of curves transverse to B in Σ. Let A be a collection of annuli around the components of B, small enough that the curves of Σ_ξ pass transversely through them. On any given component of A, we define coordinates (x, y) from $(S^1 \times [-1, 1])$. In these coordinates, the component of B is the circle $\{y = 0\}$, and the leaves of our foliation are the lines on which x is constant. As in the arguments of section 1.6, let α be the contact form associated to R, and let $\beta = \alpha|_{T\Sigma}$. Then a transverse contact vector field on Σ corresponds to a smooth function u such that $ud\beta + \beta \wedge u > 0$. The Reeb field and its opposite $\pm R$ correspond to the constant functions $u \equiv \pm 1$. These only satisfy our condition on the appropriate interiors of $\Sigma^\pm$. We define a function u on Σ by letting $u(y)$ be a smooth, increasing function on A that is equal to 1 when $y \geq 1 - \epsilon$, equal to zero when $y = 0$, and equal to -1 when $y \leq -1 + \epsilon$. We define $u \equiv \pm 1$ on $\Sigma^\pm \setminus A$.

In our neighborhood A, we take the orientation $(\partial x, \partial y)$. The outward normal to Σ^+ is the vector $-\partial_y$, so the induced orientation on B is given by ∂_x. Therefore $\beta(\partial_x) > 0$ on B. By retroactively shrinking A, we may ensure that this is the case on the entire annulus. Therefore $(\beta \wedge du)(\partial_x, \partial_y) = \beta(\partial_x)du(\partial_y) \geq 0$, since u was increasing, depended only on y, and our y coordinate came from the kernel of β. It is zero, only where u is locally constant. We already know that $ud\beta$ is positive on $\Sigma \setminus B$, and zero on B. Then $ud\beta + \beta \wedge du > 0$ on all of Σ, so the function defines a transverse contact vector field. This field is equal to $\pm R$ outside of A, and it is tangent to ξ when $u = 0$, i. e. along B.

We now need to show that ξ is tight on $M \setminus \Sigma$. Let H be one of its components, and Σ_0 a page in the interior of H. Then, using the flow of R, we can embed H in the space

$\Sigma_0 \times \mathbb{R}$, so that $R = \partial_t$ in these coordinates. This is the Reeb field for the contact form α, which we may then write as $\beta + dt$, where $\beta = \alpha|_{\Sigma_0}$ as usual.

Given any positive function f of t, $f\alpha$ is another contact form for ξ. A quick computation shows that its Reeb field will be $\frac{1}{f}\partial_t$. On any open interval (a, b), we may define a smooth positive function $f : (a, b) \to \mathbb{R}$ that is equal to 1 on some region in the interior, and $\lim\limits_{t \to a^+} f = \lim\limits_{t \to b^-} f = \infty$. The flow of the Reeb field $\frac{1}{f}\partial_t$ for $f\alpha$ allows us to construct a contactomorphism from $\Sigma_0 \times (a, b)$ to $\Sigma_0 \times \mathbb{R}$. Thus if our surface Σ_0 has a tight vertical neighborhood, then all of $\Sigma_0 \times \mathbb{R}$ must be tight. The dividing set of Σ_0 contains no contractible circles,[2] so Giroux's criterion, theorem 1.6.13, ensures that Σ_0 has such a neighborhood. From this we conclude that $H \subset \Sigma_0 \times \mathbb{R}$ must be tight.

Now, assume that we have one pair of pages that together make a smooth surface Σ, satisfying our two criteria. The majority of this direction is contained in a lemma of Torisu. Its proof follows this one.

Lemma 3.2.3 (Torisu, [33])**.** *Assume Σ is a surface arising as the union of two pages of (B, π). Then there is at most one contact structure on M that is tight on $M \setminus \Sigma$ and for which Σ is convex, with dividing set equal to B.*

By the first half of this proof, a contact structure supported by (B, π) will satisfy our criteria. Therefore if (B, π) supports a contact structure, then that contact structure is the unique one from the lemma. It will be enough, then, to show that every open book supports some contact structure. This is the result, due to Thurston and Winkelnkemper, that motivated the definition of compatibility between books and structures. We prove it later, as theorem 4.1.1. □

Because this lemma supplies most of its proof, we refer to the above theorem as the Torisu criterion.

Proof of 3.2.3. As above, let H be one of the two open handlebodies in $M \setminus \Sigma$. We

[2]The dividing set is, in fact, empty.

assume that it has a contact structure that satisfies our criteria, and show that it could not have another. We will work with the closure of H, so it now includes the surface Σ. Let Σ_0 be the page of our book, and $\gamma_1, \dots, \gamma_n$ be arcs that cut Σ_0 into a disk, so that $\{\gamma_i \times [0,1]\}$ in $H = \Sigma_0 \times [0,1]$ are a collection of disks that cut H into a three-ball. The double of each γ_i is a nonisolating curve on $\Sigma = \partial H$, so we may use Legendrian realization on it, and then perturb the disk it bounds to a convex surface. This convex disk is contained in a tight contact manifold, so its dividing set must consist of properly embedded arcs. We cut H along it. The resulting manifold contains two copies of our disk, each of which intersect our previous boundary in a Legendrian. We may use the edge rounding lemma, 1.6.11, to create a new handle-body with smooth convex boundary. The double of γ_i which bounded our disk crossed the dividing set of ∂H exactly twice, so each disk must be divided by a single arc. After the edge rounding, these arcs link up with arcs from B to form the dividing set on the resulting ball.

Our open book, and our choice of curves, $\{\gamma_i\}$, then completely determine the dividing set on the boundary the tight three-ball we end up with. In [8], Eliashberg showed that a tight contact structure on the three-ball is uniquely determined by its characteristic foliation on S^2. By Giroux Flexibility, theorem 1.6.9, this means that the contact structure on our three ball is determined up to isotopy by the dividing set on S^2, so H has at most one structure satisfying our criteria. $\square$

3.3 Books and Actions

Now, consider a manifold M with an open book (B, π). We want to consider the actions on M that preserve this structure.

Definition 3.3.1. *A G-action on M preserves an open book decomposition (B, π) if it holds B invariant, and for every $g \in G$ there is some smooth automorphism of S^1, σ_g, so that $\pi \circ g = \sigma_g \circ \pi$. The action preserves (B, π) strongly, or holds (B, π) strongly* *invariant, if every σ_g is the identity.*

Another way of phrasing this is that an action that sends pages to pages preserves the book. An action that holds pages invariant preserves the book strongly.

We will chiefly be interested in strongly invariant books. These correspond to the orbifold books we construct in the next chapter. In order to do this, though, it will be useful to have an appropriate notion of actions on abstract books as well.

Definition 3.3.2. *A G-action on the abstract open book (Σ, φ) is an action on the page, so that $g \circ \varphi = \varphi \circ g$, for every $g \in G$.*

Lemma 3.3.1. *There is a correspondence, up to conjugation, between finite actions on abstract books, and finite actions that strongly preserve the corresponding open book decompositions.*

Proof. Starting with an abstract book, the action on Σ induces an action on $\Sigma \times [0, 1]$ that is trivial on the interval. Now, since every $g \in G$ satisfies $g \circ \varphi = \varphi \circ g$, this action is well defined upon the mapping torus of φ. All that remains is to extend the action to the binding neighborhoods. The boundary curves of Σ are either permuted, fixed or rotated by elements of G. These same actions may be applied to the solid tori around the binding, so they are permuted, fixed or rotated longitudinally. This defines an action on $M_{(\Sigma,\varphi)}$ that strongly preserves the associated book.

Starting with a strongly invariant open book decomposition, we first wish to remove a neighborhood of the binding. Using an invariant metric we may find an invariant tubular neighborhood of B. Furthermore, we may ensure that this neighborhood is covered by an invariant collection of meridional disks. Removing this, we are left with a Σ bundle over S^1. Our action preserves fibres of this fibration, so by choosing one to cut along, we arrive at an action on $\Sigma \times [0, 1]$. At this point, we use the following lemma, which is a special case of a theorem from [5], the proof of which follows this one.

Lemma 3.3.2. *Let Σ be a surface with boundary. An orientation preserving finite action on $M = \Sigma \times [0,1]$ that preserves each $\Sigma \times \{t\}$ may be straightened. In other words by conjugating with a diffeomorphism, it may be transformed into the product of an action on Σ and the trivial action on the interval.*

Our original action may be written as a one-parameter family of actions on Σ, and since it came from an action on the bundle over S^1, every $g \in G$ must satisfy the equation $g_0 = g_1$. After conjugating with the diffeomorphism from the above lemma, we have a single action on Σ and the above becomes that $g \circ \varphi = \varphi \circ g$. This is an action on the abstract book (Σ, φ).

Now if we apply our above construction to this book, we first recover our Σ-bundle with an action conjugate to that on the original Σ-bundle. Let's call the diffeomorphism we are conjugating by σ. We know that σ preserves the fibration. Note further that we built the neighborhood of B out of an invariant collection of meridional disks. Using the boundaries of these as coordinates on the boundaries of the pages, we see that for the original action $g|_{\Sigma_i} = g|_{\Sigma_j}$ on $\partial\Sigma$. Our conjugation map σ then may be the identity on the boundary of our Σ-bundle. Extending it by the identity over the binding neighborhood, we have constructed a diffeomorphic copy of our original manifold with an equivalent action. These constructions produce an equivalence between conjugacy classes of actions on an abstract book, and strongly preserving actions on a manifold with an open book decomposition. □

Proof of 3.3.2. We may work with connected components of Σ, since any group element that permutes the components merely gives us a way to extend our straightening conjugation from one component to its entire orbit.

We start by straightening our action on the boundary of Σ. Choosing one component, and looking only at the subgroup that preserves it, we have an action on $S^1 \times [0,1]$. All finite orientation preserving actions on the circle are cyclic, so the action on our cylinder must be as well. If we quotient by it, we will be left with another cylinder $S^1 \times [0,k]$.

Foliate this cylinder with curves of the form $\{pt\} \times [0, k]$. These lift to a foliation of our original cylinder, and straightening the action is a matter of isotoping our foliation to be of the form $\{pt\} \times [0, 1]$. We may carry this isotopy out on our original manifold, restricting it to some collar around the boundary component, and apply the same to every boundary component of Σ.

If Σ is not a disk, we will attempt to cut it up until each component is. Our goal is to cut Σ into a collection of disks, so we will first reduce its genus to 0, and then connect up the boundary components. If at some point we cause Σ to become disconnected, we do not need worry, but merely continue working a component at a time. Our notation will assume this does not happen, but it would not cause any problem.

Now choose some properly embedded arc γ in $\Sigma \times \{0\}/G$. Since our original action preserved both the orientation, and our product structure, it must be of cyclic-type. This ensures that the singular set is small enough that γ may avoid it. Then γ lifts to an invariant collection of curves in $\Sigma \times \{0\}$. We may extend γ to a disk D in M/G transverse to the product structure, that may be thought of as a one-parameter family of curves with the same endpoints. This can also be made disjoint from the singular set of M/G so that it lifts to a collection of disks. By ambient isotopy, with which we will conjugate our action, we may make each component of the lift of D into the disk of the form $\gamma_i \times [0, 1]$, where γ_i are the lifts of γ. Then cutting along this collection of disks we change M to the form $\Sigma_1 \times [0, 1]$, where Σ_1 is Σ cut along the γ_i. By proper choice of γ, we may reduce the genus or the number of boundary components of Σ.

After this, we again straighten our action along the boundary of Σ_1, being sure to do so in a way that will allow us to re-glue along our cut. Note that after cutting we have two copies of $\{\gamma_i \times [0, 1]\}$, each of which is preserved by our action. Choose one and foliate by the lines $\{pt\} \times [0, 1]$. Then the action will take this foliation to the other disks, and with isotopy these foliations can be made of the same form. Using the same isotopies in both copies of any given disk will ensure that we may re-glue. Again these isotopies only extend into a small collar of the boundary of M, and fix the rest.

We have now reduced to the case where Σ is a disk, and the action is straight on $\partial\Sigma \times [0,1]$. □

This gives us the appropriate equivalence between the two different types of preserved books.

3.4 Books on Orbifolds

Definition 3.4.1. *An open book decomposition (B, π) of the cyclic-type orbifold Y with singular link S consists of a link B in $Y \smallsetminus S$, and a fibration $\pi : (|Y| \smallsetminus B) \to S^1$, whose level sets are the interiors of compact two-orbifolds $\Sigma_\theta \in Y$ with boundary B.*

Given an orbifold cover $p : \widehat{Y} \to Y$, each page lifts to a two-orbifold in $\widehat{Y}$, with boundary $p^{-1}(B)$. The covering map p cannot take a singular point in $\widehat{Y}$ to a nonsingular point in Y, so $p^{-1}(B)$ is a link in $\widehat{Y}$ disjoint from its singular set. Our book (B, π) then lifts to a book $(p^{-1}(B), \pi \circ p)$ on $\widehat{Y}$. As a special case: if the finite group G has a cyclic type action on the three-manifold M, then a book on M/G lifts to a strongly invariant book on M.

Definition 3.4.2. *An orbifold abstract open book is a pair (Σ, φ). The page Σ is a cyclic type, compact two-orbifold with nonempty boundary, whose boundary contains no singular points. The monodromy φ is a diffeomorphism from $\Sigma \to \Sigma$ that fixes a neighborhood of $\partial\Sigma$.*

Since Σ is two dimensional and cyclic type, its singular set will be a collection of isolated points. These may be given numbers based upon the order of their local groups. Then the monodromy may be thought of as a monodromy on the underlying surface $|\Sigma|$ that preserves each set of singular points with a given label. The singular points of order n may be permuted, and each orbit of this permutation corresponds to one component of the singular link in the orbifold this abstract book constructs.

To produce an orbifold from (Σ, φ), we follow the same procedure as in the manifold case. The space $\Sigma \times [0, 1]$ will be a cyclic type three-orbifold, whose boundary consists of two copies of Σ, and a collection of annuli. We produce a mapping torus by using φ to glue the copies of Σ together. The singular set of this mapping torus is disjoint from its boundary, so we may glue on solid tori exactly as in the manifold case.

Next we address the issue of how common these structures are. Almost a century ago Alexander showed that every three manifold has an open book decomposition. The same is true for orbifolds.

Theorem 3.4.1. *Every cyclic type three-orbifold has an open book decomposition.*

Proof. Recall that in section 2.2 we constructed a homeomorphism between cyclic type orbifolds and their underlying manifolds, based upon the notion of a branched cover. We constructed a map from $|Y|$ to Y, that sent every embedded surface in $|Y|$ transverse to the singular link S to a two-orbifold in Y. Therefore it is enough for us to construct an open book decomposition of $|Y|$, whose pages are transverse to S.

To do this, remember that every three-manifold has a branched cover over the three sphere. We choose one such cover $p : |Y| \to S^3$, constructing it so that its ramification locus is disjoint from S. Since both are links in a three manifold, this may be done with a minor perturbation of the covering map. Now let the link L be the union of $p(S)$ and the branch locus of p. Any open book on S^3 whose pages are transverse to L will lift to a book on $|Y|$ whose pages are transverse to S.

In [1], Alexander showed that every link in $\mathbb{R}^3$ could be coiled around the z axis, or equivalently that every link in S^3 could be coiled around the unknot. By this we mean that a smooth link[3] may be isotoped so that it is disjoint from the axis, and every component has a parametrization γ so that $\frac{d\gamma}{d\theta} > 0$. Such a link is everywhere transverse to pages of the open book defined by $(r, \theta, z) \to \theta$, whose binding is the z axis. Adding in the point at infinity, this is the standard book for S^3 bound by the unknot. We may

[3]Alexander showed this for a PL link, but the classes are equivalent in three dimensions.

then lift this book to $|Y|$, and apply our homeomorphism to have an orbifold open book on Y. □

Note that this implies that every cyclic type action on a three-manifold strongly preserves some open book.

3.5 Books on Contact Orbifolds

The final topic to cover, before we describe the orbifold version of Giroux's correspondence, is our notion of compatibility between open books and contact structures on orbifolds. In fact the definition from the manifold case may be directly applied to this more general setting.

Definition 3.5.1. *An open book (B, π) on a contact three-orbifold (Y, ξ) supports the contact structure if, after some isotopy of ξ through contact structures, there is a contact form α on Y so that $\alpha > 0$ on B, and $d\alpha$ is an area form on every page.*

We may also formulate this in terms of Reeb fields.

Definition 3.5.2. *A vector field on an orbifold, with charts $(U_i, V_i, \Gamma_i, \varphi_i)$, consists of a collection of Γ_i invariant vector fields on each V_i, that are identified by the transition maps.*

A Reeb field on (Y, ξ) is a vector field that lifts to a Reeb field in every V_i.

For a cyclic type orbifold, this corresponds exactly to vector fields on $|Y|$ that are tangent to the singular link. Reeb fields are exactly the Reeb fields on the contact manifold $Y \setminus S$ that may be extended to vector fields tangent to S.

Theorem 3.5.1. *An open book (B, π) on (Y, ξ), supports ξ if and only if, after some isotopy of ξ, there exists a Reeb field positively tangent to B, and positively transverse to the interior of every page.*

The proof of this is essentially identical to that of theorem 3.2.1.

As in the manifold case, we are chiefly interested in this in order to prove a version of theorem 3.2.2.

First though, note that the pages of an orbifold book are compact cyclic two-orbifolds with nonempty boundary. The only bad, closed, cyclic two-orbifolds are the teardrop and the spindle.[31] By these, we mean S^2 with either a single cone point, or with two cone points of differing order. If the two-orbifold Σ has nonempty boundary, then it may be embedded in a closed two-orbifold by gluing disks to its boundary components. By using either the smooth disk, or a disk containing a cone point, we may ensure that Σ embeds into a closed two-orbifold that is neither the teardrop nor the spindle. Thus, the pages of a book are always good two-orbifolds. This also implies that any handlebody of the form $(\Sigma \times [-1, 1])$ must be a good three-orbifold. It therefore makes sense, in the following theorem, to discuss the tightness of ξ on such handlebodies.

Theorem 3.5.2. *There are smooth two-orbifolds in Y composed of the union of opposing pages. We consider such surfaces, Σ, that satisfy:*

- *Σ is convex, with dividing set B.*
- *ξ is tight on both components of $Y \smallsetminus \Sigma$.*

If, after some isotopy of ξ, there exists such a surface, then (B, π) supports ξ. If (B, π) supports ξ, then, after some isotopy of ξ, every smooth union of pages satisfies these conditions.

Proof of 3.5.2. First, we assume that (B, π) supports ξ, and so that we have a Reeb field R tangent to B and transverse to our pages. Recall from the proof of 3.2.2 that to show Σ was convex, we constructed a transverse contact vector field. This construction started with the Reeb field R on Σ^+ and with $-R$ on Σ^-. These only needed to be perturbed in a neighborhood of B, so that they could define a single contact vector field, transverse to Σ. We may again choose a Reeb field, that will give us transverse contact fields on the interior of $\Sigma^{\pm}$. Since our binding is separate from the singular link S, it has a tubular neighborhood which is a contact manifold. We preform the perturbation from 3.2.2 within this neighborhood. Therefore Σ is a convex surface divided by B.

As in the manifold case, we may use the Reeb field to represent each handle-body as $\Sigma_0 \times (-1, 1)$. Let $\widehat{\Sigma_0}$ be a smooth surface that finitely covers Σ_0. We also have the handle-body $\widehat{H} = \widehat{\Sigma_0} \times (-1, 1)$ covering H. Our Reeb field R lifts to an equivariant Reeb field on $\widehat{H}$, so we may apply the argument from 3.2.2 to conclude that $\widehat{\xi}$ is tight on $\widehat{H}$.

Now assume that our surface Σ is convex, divided by B, and splits Y into two tightly-covered handle-bodies. Again, let H be one of these. We will want to make use of our convex boundary, so include it in H. As before, let $\widehat{H}$ be a smooth contact handle-body that finitely covers H, whose interior is equal to $\widehat{\Sigma_0} \times (-1, 1)$. Its boundary is a convex surface, whose dividing set is $\partial(\widehat{\Sigma_0}) \times \{0\}$, and its contact structure is tight. By Torisu's lemma 3.2.3, these two criteria uniquely determine its contact structure, and therefore the contact structure on the orbifold H. These criteria then uniquely determine the contact structure on Y, so if (B, π) supports a contact structure, it must support ξ. In the next section, we will extend Thurston and Winkelnkemper's result to orbifolds as theorem 4.1.2, which tells us that (B, π) does indeed support a contact structure on Y. $\square$

CHAPTER 4

THE GIROUX CORRESPONDENCE

4.1 Building Contact Structures

The first direction of the Giroux correspondence was proved in [32] by Thurston and Winkelnkemper as a way to show that every three-manifold has a contact structure. The starting point for this was the fact that every three-manifold has an open book. We already know that this is also true for cyclic type orbifolds. From this we may construct an orbifold contact structure, using a slightly modified version of their construction. We will start by describing their construction, and then explain how it may be modified for the orbifold case.

Theorem 4.1.1 (Thurston and Winkelnkemper [32])**.** *Let M be a smooth three-manifold, with open book (B, π). There exists a contact structure ξ on M, supported by (B, π).*

Theorem 4.1.2. *Every open book on a closed cyclic type three-orbifold supports a contact structure.*

Proof of 4.1.1. In order to construct our contact structure, we will work with the abstract book (Σ, φ) associated to (B, π). We will first build a contact structure on the mapping torus of φ, and then extend this to the solid tori around the binding.

Start by fixing coordinates (s, ψ) on collar neighborhoods of each component of $\partial\Sigma$. For these, let $s \in \left[1, \frac{3}{2}\right)$ be the inward pointing coordinate, and $\psi \in \mathbb{R}/\mathbb{Z}$ a coordinate on the boundary component. Also, make sure that these are small enough that they stay within the boundary neighborhood that φ fixes. Define the set Λ to consist of all one forms λ on Σ that equal $sd\psi$ on our collar neighborhoods, but such that $d\lambda > 0$. A positive linear combination of area forms will be an area form, and on our collar neighborhoods, every two elements of Λ look identical. For $\lambda_0, \lambda_1 \in \Lambda$, and $i \in [0,1]$, then: $i\lambda_1 + (1-i)\lambda_0 \in \Lambda$. Still, to make use of Λ, we must construct at least one element to ensure that it is not empty.

First, we extend our collars a little further in, so now $s \in \left[1, \frac{7}{4}\right)$. Let λ_0 be any one-form on Σ that is equal to $sd\psi$ in these extended collars. By Stokes theorem, we may conclude that $\int_\Sigma d\lambda_0$ is equal the number of boundary components of Σ. Let ω be an area form for Σ that is equal to $ds \wedge d\psi$ in our collar, and that integrates to the same number as $d\lambda_0$.[1] Then the two-form $\omega - d\lambda_0$ is equal to zero on our collar, and $\int_\Sigma \omega - d\lambda_0 = 0$. Cutting off our original, smaller, collar, we are left with a compactly supported two-form in $\Sigma \setminus \left(\left[1, \frac{3}{2}\right] \times \partial\Sigma\right)$. The compactly supported, second de Rahm cohomology of this open surface is isomorphic to $\mathbb{R}$ via integration of two-forms. Therefore, the above form is exact, i. e. there exists a compactly supported one form, β, on this restricted surface, such that $d\beta = \omega - d\lambda_0$. We may extend this to the entirety of Σ by not expanding its support, so β is a primitive for $\omega - d\lambda_0$, that vanishes on our original collar. The one-form $\lambda = \lambda_0 + \beta$ is then equal to λ_0 on the collar, and $d\lambda = \omega$ on all of Σ. It is our first element of Λ.

We know then that Λ is both convex, and non-empty. Let $\lambda_1 \in \Lambda$, and $\lambda_0 = \varphi^*(\lambda_1)$. Note that, since φ is a diffeomorphism of Σ, and is the identity on our collars, λ_0 is also in Λ. So is each of the forms $\lambda_i = i\lambda_1 + (1-i)\lambda_0$, where $i \in [0,1]$. We define λ to be the one form on $\Sigma \times [0,1]$ that these define, via: $\lambda|_{\Sigma \times i} = \lambda_i$. Due to its construction, λ descends to a well defined form on the mapping torus of φ.

For any number k, we define a one form $\alpha_k = \lambda + kdt$, where t is the coordinate on $S^1 = [0,1]/\{0,1\}$. Note then, that $\alpha_k \wedge d\alpha_k = \lambda \wedge d\lambda + kdt \wedge d\lambda$. The form $dt \wedge d\lambda = d\lambda \wedge dt$

---

[1] The somewhat awkward bounds for s are to ensure that ω integrates to less than one on each collar, so that there's still some area left to spread over the rest of Σ.

is a volume form on the mapping torus, since $d\lambda$ is an area form on each page. Since the mapping torus is a compact manifold, there exists a value of k large enough to overcome any possible interference from the first term, so that $\alpha_k \wedge d\alpha_k$ is a volume form. This means that α_k is a contact form on $\Sigma \times [0,1]/\sim$.

All that remains then is to extend α_k to the solid tori that surround the binding. On these tori, we define coordinates that extend our collar coordinates on Σ, so we have coordinates (s,t,ψ), where $s \in [0,1]$, and $t, \psi \in S^1$. On our solid torus $D^2 \times S^1$, (s,t) are polar coordinates on the disk, and ψ is the coordinate on S^1. This describes only the solid tori we are gluing on, but be extending s to take on values up to $\frac{3}{2}$, we may also include our collars.

Near the cores of our solid tori, we use the standard, radial contact form from $\mathbb{R}^3$, which is the form $-d\psi + s^2 dt$ in our coordinates.[2] Let this be α for small values of s, and for $s \geq 1$, we already have the form $\alpha = \alpha_k = s d\psi + k dt$. We need to find some way to interpolate these.

Both of these forms are invariant under changes in ψ and t, which is a feature that it would be a shame to lose. We will define our form as $\alpha = f(s)d\psi + g(s)dt$, and the previous paragraph tells us that $f = -1$, $g = s^2$ for small values of s, while $f = s$, $g = k$ for larger values. Using this formula, $\alpha \wedge d\alpha = \left(g\frac{df}{ds} - f\frac{dg}{ds}\right) ds \wedge d\psi \wedge dt$. In order for α to be a contact form, we need to choose two functions so that, for every value of s, the vectors (f,g) and $\left(\frac{df}{ds}, \frac{dg}{ds}\right)$ are never parallel. This is a smooth path parametrized by s, that starts as $(-1, s^2)$ and ends as (s^2, k). The contact condition requires that this path have no radial tangencies. This is not hard to accomplish, as we are connecting a vertical line in the second quadrant to a horizontal one in the first. Such paths clearly exist, so we may use one to finish the construction of our contact form.

Notice further that α is positive on the binding, which has the opposite orientation to

[2] In our collar coordinates, $ds \wedge d\psi \wedge dt$ is an orientation form. Translating into the standard coordinates, this is the form $dr \wedge dz \wedge d\theta$, which defines the non-standard orientation on R^3. So, to match up the orientations, we make ψ correspond to $-z$, and $dz + r^2 d\theta = -d\psi + s^2 dt$.

that induced by the coordinate ψ. By construction, $d\alpha$ is an area form on the interiors of the pages. In our collar coordinates, $d\alpha = \frac{df}{ds} ds \wedge d\psi + \frac{dg}{dt} ds \wedge dt$ is a volume form on the pages, which are the level surfaces for t. So the contact structure we have constructed supports our book. □

Proof of 4.1.2. Now we carry out the same construction on a contact orbifold, defined by an abstract open book. Note that the page, Σ of an open book is always a good two-orbifold, so is covered by a smooth surface: $\widehat{\Sigma}$. As above, we define collar coordinates (s, ψ) around each boundary component, keeping the collars thin enough to miss any singular points. These coordinates then lift to collar coordinates near the boundary components of $\widehat{\Sigma}$. On these collars, the covering is of a collection of annuli. If we restrict to one component of the pre-image of one of these annuli, we will get a cover mapping $\left[1, \frac{3}{2}\right) \times (\mathbb{R}/n\mathbb{Z}) \to \left[1, \frac{3}{2}\right) \times (\mathbb{R}/\mathbb{Z})$, via the map sending (x, y) to $(x, [y])$. Here $[y]$ is the element of $\mathbb{R}/\mathbb{Z}$ represented by $y \in [0, n]$. If we allow ψ to take on values in $\mathbb{R}/n\mathbb{Z}$, we may lift our collar coordinates up to $\widehat{\Sigma}$.

From this we see that elements of the deck transformation group of our cover act on the collars in $\widehat{\Sigma}$ by permuting components or by rotation of ψ. Permutation of the components send one set of coordinates to another. Importantly then, deck transforms preserve the one-form $sd\psi$ on our collection of collars.

The arguments from the previous proof ensure that there exists a non-empty, convex collection, Λ, of all two forms, λ, on $\widehat{\Sigma}$ such that $d\lambda > 0$ everywhere, and $\lambda = sd\psi$ on our collars. Every deck transform is a diffeomorphism of $\widehat{\Sigma}$, that preserves the form $sd\psi$ on the collars. Therefore, if g is a deck transform and $\lambda \in \Lambda$, then $g^*(\lambda) \in \Lambda$.

Let G be the group of deck transforms. Then, given any $\lambda \in \Lambda$, we define the average $\overline{\lambda} = \frac{1}{|G|} \sum\limits_{g \in G} g^*(\lambda)$, and $\overline{\lambda}$ is a G invariant element of Λ. We'll call the collection of all such G-invariant forms $\overline{\Lambda}$. Notice that, much like Λ, this new collection of forms is convex.

Every element of $\overline{\Lambda}$ descends to a form on our two-orbifold Σ. The collection of forms on the orbifold is still convex, so we may construct a form $\overline{\lambda}$ on the mapping torus $\Sigma \times [0,1]/\sim$, by letting $\overline{\lambda}_i = i\overline{\lambda}_1 + (1-i)\varphi^*(\overline{\lambda}_1)$ as above. We may also define $\alpha_k = \overline{\lambda_i} + kdt$, which is a contact form for large enough values of k.

This gives us our contact structure on the mapping torus. All that remains is to extend it to the binding. Recall, though, that the binding neighborhood is entirely separate from the singular set, so the solid tori and the collars they glue to are manifolds. The construction proceeds exactly as above on these regions. □

From Theorems 3.4.1 and 4.1.2, we immediately derive the following corollary:

Corollary 4.1.3. *Every closed, cyclic type three-orbifold has a contact structure.*

Or, restricting our attention to good orbifolds:

Corollary 4.1.4. *Let M be a smooth manifold with a cyclic type G-action. Then every strongly invariant open book on M supports an invariant contact structure.*

As well as:

Corollary 4.1.5. *Every cyclic type G-action on a closed orientable three-manifold M preserves some contact structure.*

4.2 Constructing Open Books

Thurston and Winkelnkemper's construction motivated our notion of compatibility between open books and contact structures. The true importance of this relation is due to Giroux's proof that:

Theorem 4.2.1 (Giroux, [16])**.** *Every closed contact three-manifold (M, ξ) has an open book decomposition supporting ξ.*

In this section we will prove Giroux's theorem, and in the next we extend it to contact orbifolds. This is a slightly different version of the proof due to Goodman [17], and described in Etnyre's expository paper [9].

The construction is based upon another structure on (M, ξ): that of a cell decomposition adapted to ξ.

Definition 4.2.1. *A contact cell decomposition on (M, ξ) is a cell decomposition on M that satisfies:*

- *Every one-cell is Legendrian.*
- *Every two-cell D has $tw(\partial D, D) = -1$.*
- *The contact structure is tight on each three-cell.*

Such a decomposition is the starting point for Giroux's construction, so we first prove that it must exist.

Theorem 4.2.2. *Every closed contact three-manifold has a contact cell decomposition.*

Proof. We start by aiming for the third criterion. Cover (M, ξ) by Darboux balls, and create a cell decomposition subordinate to the cover. By this we mean that we require each three-cell to be contained in at least one of our Darboux balls. This ensures that ξ is tight on each one.

To satisfy criterion one, we perturb the one-skeleton to make it Legendrian. Recall that any arc in a contact manifold is C^0 close to a Legendrian arc. We may perturb our one skeleton to make it Legendrian, while keeping the decomposition subordinate to our Darboux cover.

All that remains, then, is to deal with the two cells. Let D be one of these. Like all our two-cells, it is contained in a Darboux ball, and so has a neighborhood on which ξ is tight. This means that we may apply the weak Bennequin inequality, to conclude that $tw(\partial D, D) \leq -\chi(D) = -1$. We only need worry about two cells for which the twisting is too low.

Since $tw(\partial D, D)$ is negative, we can apply theorem 1.6.7 to perturb it to a convex surface. There is a slight bit of complexity to this arising from the vertices. To make the surface convex, we first apply a C^0 perturbation near its boundary that makes the surface twist uniformly with respect to ξ, and ensure that ξ never twists through D in a right-handed manner. With a smooth Legendrian boundary, if we have positive twisting on one section and negative on another, our perturbation allows the negative twists to cancel the positive. In our present case, isotoping a twist through a vertex could break the cell decomposition. We therefore need not only that $tw(\partial D, D)$ be negative overall, but that it be non-positive on each one cell. This may force the twisting to be less than -1 on the boundary of most two cells, but that will be dealt with once these disks are convex. By using the Legendrian stabilization operation shown in figure 1, we may decrease the twisting of any one-cell. This is done within a standard neighborhood of the Legendrian curve, and does not change the cell decomposition outside of this neighborhood. We do this to every arc in our one-skeleton, so that no arc has positive twisting relative to the two-cells it bounds. Then we may apply our perturbations from 1.6.7 without twisting D around any vertices. Do so to every two-cell in the decomposition.

Now that they are convex disks, we may see the twisting of a two-cell's boundary represented in its intersection with Γ_D. Since ξ is tight near D, Γ_D may contain no contractible circles, and must therefore consist entirely of embedded arcs. Furthermore, these arcs intersect with the boundary $-2tw(\partial D, D)$ times, so there must be $-tw(\partial D, D)$ of them. Our goal of twisting -1 for each disk, then, may be interpreted as a goal that each disk have a single dividing curve. If the twisting is less then -1, there will be more than one such arc. Then we choose some collection of arcs disjoint from Γ_D that split D into disks, each containing exactly one dividing curve. This collection of arcs is nonisolating, so we may apply the Legendrian Realization Principle, theorem 1.6.10, to make them Legendrian. Adding them to our one skeleton, we have divided D into a collection of convex disks, each of whom is divided by a single arc. Therefore, each has twisting -1. Following the same procedure with every two-cell, we produce a contact decomposition. □

Notice that in the process of constructing this decomposition, we made all of the two-cells into convex surfaces. This feature will be useful in later arguments, so from now on, we'll assume that our contact cell decompositions also exhibit it. Following procedures from the previous proof, any contact cell decomposition may be refined to one with convex two-cells.

To construct a book on some given contact manifold, we first choose a contact cell decomposition with convex two-cells. We will use the one-skeleton to construct the first page of our book, and then use our understanding of the two and three cells to construct all of the others.

The 1-skeleton of our decomposition is a Legendrian graph, and every Legendrian graph has a Ribbon. This is a surface R that satisfies the following:

- R deformation retracts to the one-skeleton.
- $T_pR = \xi_p$ at every point $p \in R$ in the one-skeleton.
- $T_pR \neq \xi_p$ at every point $p \in R$ not in the one-skeleton.

Intuitively: the Ribbon is our Legendrian graph thickened in the direction of ξ. To construct this explicitly, set up a standard tubular chart about each 1-cell that identifies it with some of the x axis in the rectangular version of $(\mathbb{R}^3, \xi_{std})$. In this chart, our ribbon is a horizontal strip around the x axis.

For a zero cell p, we use a chart sending it to the origin in $\mathbb{R}^3$, with the standard radial contact structure. Furthermore, we send the one cells that p bounds to the rays $\theta = \frac{2i\pi}{k}$, for $i \in \{1, 2, ..., k\}$. Then the ribbon is contained in the surface $z = r^2 sin(\frac{k\theta}{2})$. This gives the ribbon in charts that cover the one-skeleton. We glue these together with minor isotopies. The boundary of this surface is a transverse link that we will call B.

Given this construction, the proof of theorem 4.2.1 boils down to the following two claims:

Claim 4.2.3. *The link $B = \partial R$ is the binding of an open book decomposition on M with pages diffeomorphic to R.*

Claim 4.2.4. *The book constructed above supports the contact structure ξ on M.*

Proof of 4.2.3. We start the construction on a neighborhood of our one-skeleton. Around this Legendrian graph, construct a normal neighborhood with convex boundary. As with the ribbon, we will build this by combining neighborhoods of the one-cells and zero-cells. For our model around a one-cell, take a standard convex cylinder around the x axis. Note that the dividing curves of this surface are, or can be made to be, the two horizontal lines where it intersects the $x - y$ plane. We will require that its dividing set $B = \partial R$, the binding of our eventual book. This only requires that we use the same standard neighborhoods as when we constructed R. A convex copy of S^2 around the origin will have a single dividing curve, close to the equator. We may isotop this so that it is the boundary for our model of R near to a zero-cell.

Consider where these models meet at one end of a given one-cell. Their intersection is transverse to both dividing curves, and is nonisolating, i. e. it divides the convex surface into pieces that all contain some of the dividing set. Therefore we may use Legendrian realization, theorem 1.6.10, to make this intersection Legendrian, which will allow us to smooth the corner, linking up the dividing curves. With minor perturbations of R we may ensure that its boundary is the dividing set of our convex surface. We call this normal neighborhood N.

In addition to N, define T to be a small tubular neighborhood of B, and X to denote $M \setminus N$. The manifolds N and X share their boundary: a convex surface divided by B. We define $\widetilde{X}$ and $\widetilde{N}$ to be $X \setminus T$ and $N \setminus T$ respectively. Note that $\widetilde{N}$ is the product of R and an interval, or rather R with a boundary collar removed: a surface we will also call R.

Now we have a small tubular neighborhood, T, of B, and the rest of M is divided into two pieces. One of these, $\widetilde{N}$, is of the form $R \times [0, 1]$. In order to finish construction of our open book, we must show the same for $\widetilde{X}$.

We will use the two cells of our decomposition to cut $\widetilde{X}$, and X, into a collection of three balls. Let D be one of these two-cells. Recall that $tw(\partial D, D) = -1$, so by retroactively shrinking our ribbon R, as well as N, we may ensure that B passes through D exactly twice, and intersects it nowhere else. From this we conclude that the intersection of D and ∂X consists of a circle that passes through B twice. We are happy to see that this curve is non-separating, and so is a candidate for Legendrian realization.

The boundary of $\widetilde{X}$ may be divided into three pieces. There is a collection of annuli, which we call A, coming from the boundary of T, and there are two copies of our collar-less version of R, which we call R^+ and R^-. From the previous paragraph, we expect the intersection of D and $\partial\widetilde{X}$ to consist of two arcs passing transversely through A, and an arc through each of R^+ and R^-. If this is not the case, we may retroactively shrink T until it is.

Cut X along the disk $D \cap X$, which we will hereafter call D. This creates a new three-manifold whose boundary consists of ∂X, and two copies of D. We use Legendrian Realization on the corners where these pieces meet, which then allows us to smooth them so that we have a smooth convex boundary. The dividing set is a link made by one-surgery on B, the new arcs coming from the two copies of D. We call this new manifold X_1.

While we do this, consider what happens to $\widetilde{X}$. We are cutting along the disk $D \cap \widetilde{X}$, which we call $\widetilde{D}$. The boundary of this consists of two arcs through A and one through each of the $R^{\pm}$. After the cut, the union of A and the two copies of $\widetilde{D}$ still form a collection of annuli, A_1, and the rest of $\partial\widetilde{X}_1$ consists of two copies of R, though not actual copies as each has been cut along a properly embedded curve. Call them $R_1^{\pm}$. Before the cut, we choose a foliation of A by circles, which will correspond to the eventual pages. Then we may similarly foliate $\widetilde{D}$ by intervals connecting the leaves from A. After we cut, A_1 is still foliated by circles, and the foliations match on the two copies of $\widetilde{D}$, so that once these curves bound pages, we will be able to re-glue.

This describes how we preform our first cut along a two cell, but the procedure works in the exact same way for all our other cuts. After dealing with each one, we have two three manifolds X_n and $\widetilde{X}_n$, both of which consist of a collection of three-balls, though $\widetilde{X}_n$ has corners that have been smoothed in X_n.

The boundary of X_n is a convex surface, whose dividing set comes from B and the dividing curves of the two-cells. Note, though, that each component of X_n is contained in a three-cell of our decomposition, where ξ is tight. Therefore ∂X_n is a collection of convex two-spheres, each with a tight neighborhood. By Giroux's tightness criterion we conclude that each has a connected dividing set, i. e. that the dividing set of each is a single circle.

This has implications for $\widetilde{X}_n$, since each dividing curve corresponds to one component of A_n. The dividing curves consist of arcs through the two-cells, and the cores of the solid tori in T, so they are almost the cores of A_n.

We conclude that on any given three-ball U in $\widetilde{X}_n$, the set $A \cap \partial U$ consists of a single annulus, and so that the other two pieces of ∂U, which form its parts of R_n^+ and R_n^-, must be disks. Then U may be described as the product of a disk and an interval.

Now we decompose $\widetilde{X}_n$ as the product of a collection of disks and an interval, i. e. as $R_n^+ \times [0,1]$. Furthermore, we require that each $R_n^+ \times \{pt\}$ have as boundary a leaf of our foliation of A_n. That final requirement allows us to glue along the two cells of our cell decomposition, turning $\widetilde{X}_n$ into $\widetilde{X}_{n-1}$ and so on all the way back to $\widetilde{X}$. At each point along the way, we have a representation of $\widetilde{X}_i$ as $R_i^+ \times [0,1]$. We have thus constructed a decomposition of $\widetilde{X}$ as $R \times [0,1]$, and this extends a foliation of A by circles. These circles are longitudes of T, so each may be extended to have boundary B. This finishes our book on M. □

Proof of 4.2.4. Recall our division above of M into two pieces X and N, along a convex surface with dividing set equal to B. This surface consisted of two pages our eventual book, so this setup is tailor made for an application of 3.2.2. All that remains is to verify that ξ is tight on each of X and N.

The second of these is easier, since N is a normal neighborhood of a Legendrian graph. Nothing in our construction relied upon N having any minimum size. We may retroactively shrink it if necessary, and all of the above still holds. By lemma 1.5.2, our one-skeleton has a neighborhood on which ξ is tight, so we need only shrink N until it is contained within such a neighborhood.

The first, i.e. that ξ is tight on X, follows from this lemma.

Lemma 4.2.5 (Colin [3], Honda [22])**.** *Let (M, ξ) be a contact three-manifold, and D a convex, properly embedded disk with Legendrian boundary whose dividing set consists of boundary-parallel arcs. If ξ is tight on $M \setminus D$, then it is tight on M.*

This allows us to conclude that in gluing along the two-cells we were not overtwisting the contact structure, so our tight three-balls glue together into a tight handle-body X. We conclude by theorem 3.2.2 that our open book supports ξ. □

Proof of a special case of 4.2.5. This version of the lemma is due to Honda from [22]. He attributes it "essentially" to Colin [3], who showed this for disks with transverse boundary. To avoid Honda's concept of bypass attachment, which we would rather not take the time to describe, we will only prove the special case that we need: where the dividing set of D consists of a single arc.

Assume that there exists an overtwisted disk $D^{ot} \subset M$. Clearly, since $M \setminus D$ is tight, D^{ot} must intersect D, possibly in some very complicated way. Note, though, that ∂D has nonzero twisting, because of the form of the dividing set. Furthermore, it is contained in ∂M, which determines a framing on it. Any surface in M containing it must induce the same framing, so if any surface in M contains ∂D, the curve must comprise several leaves and several singular points in the characteristic foliation. This implies that it does not intersect the overtwisted disk D^{ot}. Recall that ∂D^{ot} is a closed leaf of D^{ot}_ξ, and D^{ot}_ξ

contains exactly one elliptic singular point.

Then there exists a convex disk D' in M that is isotopic rel. boundary to D, but that does not intersect D^{ot}. We discretize an isotopy from D to D', by which we mean we create a series of convex disks $D = D_0, D_1, \ldots, D_n = D'$ through which the isotopy passes, and such that $D_i \cap D_{i+1} = \partial D$.

Each successive pair of disks then bound a ball in M. Furthermore, D_i and D_{i+1} are convex surfaces intersecting along their Legendrian boundary. We may use edge rounding together with induction to see that each has a single dividing curve, so each of these three-balls is bounded by a convex sphere with a single dividing curve. Therefore, since the contact structure on a convexly bounded three-ball is determined up to isotopy by its dividing curves, each is a Darboux ball. To make $M \setminus D_i$ into $M \setminus D_{i+l}$ involves cutting a Darboux ball off one part, and gluing it onto another. From this we conclude that $M \setminus D$ is contactomorphic to $M \setminus D'$, and so that D^{ot} is contained in a tight manifold. From this contradiction we conclude that M can contain no overtwisted disks, i. e. that it is tight. □

4.3 The Orbifold Construction

We now turn to the corresponding construction on contact orbifolds:

Theorem 4.3.1. *Every closed, positive contact three-orbifold* (Y, ξ) *has an open book decomposition supporting* ξ*.*

To extend Giroux's result to the orbifold case, we will chiefly be concerned with segregating the meat of the construction from the singular set. We do this by using a specific type of cell decomposition.

Definition 4.3.1. *Given a closed cyclic-type three orbifold* Y*, we define an allowable cell decomposition to be a cell decomposition of the underlying manifold* $|Y|$ *such that:*

- *Every two-cell is a suborbifold of Y.*

- *The singular link S does not intersect the interiors of the one or two cells.*

- *The interior of each three cell intersects at most one component of S, and said intersection consists of a single, unknotted arc transverse to its boundary, connecting two zero cells therein.*

Furthermore, it will be called contact if every one cell is Legendrian, every two cell D has $tw(\partial D, D) = -1$, and every three cell is contactly covered by a smooth three ball on which ξ is tight.

As above, before we use such a decomposition, we will show that every cyclic type contact orbifold has one.

Lemma 4.3.2. *Every contact three orbifold (Y, ξ), has an allowable contact cell decomposition with convex two-cells.*

Proof. We start by specifying the cell decomposition near the singular link S. Choose some component K of S. By corollary 2.1.2, this singular knot has a tubular neighborhood covered by a tube around the z axis in $(\mathbb{R}^2 \times S^1, \xi_{std})$. We are using the radial form of ξ_{std}, and our deck transform group consists of the standard rotations around the z axis. Recall that a sphere around the origin in $(\mathbb{R}^3, \xi_{std})$ is a convex surface, whose dividing curve is the equator, and whose characteristic foliation consists of twisted longitude lines. Each of these features is invariant under rotation, so it covers a convex football-skin[3] in Y, with the same sort of dividing curve and characteristic foliation. We cover the z axis in $\mathbb{R}^2 \times S^1$ with some number of these spheres, with the north pole of one equal to the south pole of the next. Then projecting down, we have made K into the string for a necklace of football-shaped beads.

[3]By this I mean a copy of the two-orbifold $S^2/\mathbb{Z}_n$, which is the sphere with a cone-point at each pole. We visualize this as looking like the surface of an American football

These footballs are the first three-cells in our decomposition. The poles will all be zero-cells. In the surface of each, choose two or more leaves from the characteristic foliation, and declare these to be one cells, as they are already Legendrian. We also require that no one-cell arriving at the north pole of a football be tangent to one of the one-cells arriving from above at the south pole of the next one up. This cuts the surface into some number of convex disks which will be two-cells. Each of these disks has dividing set consisting of a single embedded arc, i. e. whatever part of the equator it contains, so each must have twisting -1. We do this around every component of S. Now, we extend this to the rest of Y.

Cover Y by Darboux charts, by which we mean orbifold Darboux charts. Then, we create a cell decomposition for $|Y|$, starting with our already constructed cells near S, such that all future cells are subordinate to this cover. Furthermore, we will allow no new one-cells to have singular points in their boundaries. Note that any new two-cells that hit a singular point, are bounded locally by two transverse Legendrians, that are tangent to the contact plane at the vertex. Therefore, the new two-cell is also tangent to ξ there, and so is a two-orbifold in Y.

Now, as in the previous section, we apply C^0 and C^∞ perturbations to our one and two-cells to make them respectively Legendrian and convex. This is immediately do-able for the new one-cells, as they are all contained in the contact manifold $Y \setminus S$, and we move on to our perturbations of the surfaces. Each two-cell is contained in an orbifold chart, so this is equivalent to doing the perturbation "upstairs." Recall that when doing this above, we arranged things so that we would not need to isotop twists through any vertices of our decomposition. We may therefore fix our two-cells close to any singular points, so our perturbation will not threaten the orbifold structure.

Finally, as in the manifold case, we know that every new two-cell has $tw(\partial D, D) \leq -1$. Furthermore that the dividing set of each consists of properly embedded arcs, and that the number of these arcs is $-tw(\partial D, D)$. Via Legendrian realization, we may introduce

Legendrian curves that split them up, until each has the appropriate twisting. We do this entirely in $Y \setminus S$. □

Recall how we constructed the ribbon R of our one skeleton in the previous section. Following the same procedure will yield a two orbifold in Y. The construction clearly works in charts that avoid S. At singular points, our standard model of R is invariant under rotations that preserve the one-cells. This allows us to construct a Ribbon as before, and again we let B be its boundary.

Claim 4.3.3. *In the above construction, B is the binding of an open book on Y that supports ξ.*

Proof. Very much like the ribbon we just constructed, our normal neighborhood N with convex boundary is built locally in standard charts. In fact it is built in exactly the same manner as in 4.2.3. Away from singular points in R, we may use the exact same argument. Around each singular point, our model for N was a ball around the origin in $\mathbb{R}^3$, so the relevant convex surface is a sphere. This was perturbed slightly to make sure that its dividing set corresponded to the boundary of R, but this perturbation did not need to change the sphere anywhere near the z axis, that corresponds to our singular set. We may, if we wish, remove a tubular neighborhood of S, and the all of the behavior we care about is exactly as in the manifold version.

As in the previous construction, we cut along two cells until $\widetilde{X}$ has been decomposed into $\widetilde{X}_n$: a disjoint collection of three balls. The interactions between our two cells and the boundary of every $\widetilde{X}_i$ along the way happen entirely in $Y \setminus S$. We are left with a collection of three-balls, each of which may be identified with a ball around the origin in the standard contact $\mathbb{R}^3/\mathbb{Z}_m$, where m will be zero for every ball disjoint from S, and the same for $\widetilde{X}_n$. Each three ball lifts to a ball around the origin in $(\mathbb{R}^3, \xi_{std})$. Those coming from X_n have convex boundary, and those from $\widetilde{X}_n$ have boundary that divides into a collection of annuli, and lifts of the pieces of $R^{\pm}$. Importantly, the annuli from $\widetilde{X}_n$ correspond to the dividing curves from X_n, so like in the previous version of this construction, we conclude that every component of $\widetilde{X}_n$ lifts to a three ball whose boundary decomposes into a single annulus and two disks.

Now if we pass back down to the orbifold level, we have a single annulus from above covering a collection of annuli below. We conclude that the collection also consists of a single annulus, so that the boundary of every component of $\widetilde{X}_n$ splits into an annulus and two disks. Furthermore, every singular arc from S intersects a boundary component twice, transversely passing through the disks. We fill the underlying manifold of each ball with disks transverse in S, and having boundary in the annulus. In fact, as above, we may require that these disks extend a foliation of the annuli that will allow us to reglue along our two-cells. Finally, we perturb the disks to be compatible with the orbifold structure, so that they are two-orbifolds.

We then reglue along the disks of the two skeleton, to get a representation of $\widetilde{X}$ as the product of an interval, and the two-orbifold R. This completes our book on Y. All that remains is to show that it supports ξ.

For this we use the orbifold version of Torisu's criterion: theorem 3.5.2. Again, we already have our Heegard splitting of Y into N and X along a convex surface made of pages, and divided by B. Furthermore, each of these handlebodies is of the form $R\times[0,1]$. Our ribbon R is a good two-orbifold, so each of X and N are good. We need only show that each has a tight cover.

Essentially we reuse the arguments from 4.2.4. Since N is a good orbifold, we may lift the contact structure to a cover $\widehat{N}$. The core of this handlebody is a Legendrian graph that covers our original one-skeleton, and by lemma 1.5.2 it must have a tight neighborhood. We retroactively shrink N until its lift $\widehat{N}$ is contained within the tight neighborhood. This is a tight cover of N.

To deal with X, we observe that our two skeleton stays away from S inside X, so in an n-fold cover of X my the manifold $\widehat{X}$, each two cell lifts to n disjoint copies of itself.

These copies cut $\widehat{X}$ into Darboux balls, and have the properties that allow us to apply lemma 4.2.5. Therefore X is covered by a tight handlebody. □

An open book on an orbifold lifts to a strongly preserved book on any orbifold that covers it. From this we get the immediate corollary:

Corollary 4.3.4. *Every contact manifold (M, ξ) with a positive contact G action has a strongly preserved open book that supports ξ.*

In addition to this, recall how we constructed three cells around each component K of our singular link. When we cover the lift of K with spheres, we may choose to use only one. As a model, think of the unit sphere in $R^2 \times ([-1,1]/\{-1,1\})$. The monodromy of our eventual book comes from vertical translation through the three cells of our decomposition. If a singular link K is contained in only one three-ball, then the monodromy must fix the relevant cone point on R. So:

Corollary 4.3.5. *Every contact orbifold (Y, ξ) has an open book supporting ξ whose monodromy fixes all singular points.*

Put another way, each has an open book that singular knots only traverse once. We expect such books to be useful, since much is known about the mapping class groups of surfaces with fixed marked-points.

Notational Index

BIBLIOGRAPHY

[1] J. W. Alexander, *A lemma on systems of knotted curves*, Proc. Nat. Acad. Sci. USA, 9 (1923), 93-95

[2] J. W. Alexander, *Note on Riemann spaces*, Bull. Amer. Math. Soc. 26 (1920) 370-372

[3] V. Colin, *Recollement de variétés de contact tendues*, Bull. Soc. Math. France, 127 (1999) 43-69

[4] M. Boileau, B. Leeb and J. Porti, *Geometrization of 3-dimensional orbifolds*, Ann. of Math. (2) 162 (2005), no. 1, 195–290.

[5] A. Edmonds, C. Livingston, *Group actions on fibered three-manifolds*, Comment. Math. Helv. 58 (1983), no 4, 529-542

[6] Y. Eliashberg, *Classification of overtwisted contact structures on 3-manifolds*, Invent. Math. 98 (1989), 623-637

[7] Y. Eliashberg, *Filling by holomorphic disks and its applications*, Geometry of Low Dimensional Manifolds, Vol. II, Cambridge (1990)

[8] Y. Eliashberg, *Contact 3-manifolds twenty years since J. Martinet's work*, Ann. Inst. Fourier 42 (1992), 165-192

[9] J. Etnyre, *Lectures on open book decompositions and contact structures*, Floer homology, gauge theory and low dimensional topology, Clay Math. Proc. Series 5 (2006), Amer. Math. Soc., Providence, RI, 103-141

[10] J. Etnyre, Convex Surfaces in Contact Geometry:Class Notes. Available at http://people.math.gatech.edu/ etnyre/preprints/papers/surfaces.pdf

[11] J. Etnyre, K. Honda, *On Symplectic Cobordisms*, Math. Annalen 323 (2002) no. 1, 31-39.

[12] J. Etnyre, K. Honda, *On the nonexistence of tight contact structures*, Annals of Math. 153 (2001), 749-766.

[13] H. Geiges, *An Introduction to Contact Topology*, Cambridge: Cambridge University Press. (2008)

[14] E. Giroux, *Convexité en topologie de contact*, Comment. Math. Helv. 66 (1991) no. 4, 637-677

[15] E. Giroux, *Structures de contact sur les variétés fibrées en cercles audessus d'une surface* Comment. Math. Helv. 76 (2001) 218-262

[16] E. Giroux, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, Proceedings of the International Congress of Mathematicians Vol II (2002), Higher Ed. Press, Beijing, 405-414

[17] N. Goodman, *Contact Structures and Open Books*, Ph. D. Thesis. University of Texas at Austin (2003)

[18] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. 82 (1985), 307-374

[19] H. M. Hilden, *Every closed orientable 3-manifold is a 3-fold branched covering space of* S^3, Bull. Amer. Math. Soc. 80(1974) 1243-1244

[20] U. Hirsch, *Über offene Abbildungen auf die 3-Sphäre*, Math. Z. 140:(1974) 203-230

[21] K. Honda, *On the classification of tight contact structures I*, Geometry and Topology 4 (2000), 309-368

[22] K. Honda, *Gluing tight contact structures*, Duke Math. J., 115 (2002), no 3, 435-478

[23] Y. Kanda, *The classification of tight contact structures on the 3-torus*, Comm. in Analysis and Geometry 5 (1997), 413-438

[24] Y. Kanda, *On the Thurston-Bennequin invariant of Legendrian knots and non exactness of Bennequin's inequality*, Invent. Math. 133 (1998), 227-242

[25] D. McCullough and A. Miller, *Manifold covers of* 3*-orbifolds with geometric pieces*, Topology and its Applications 31 (1989), 169-185.

[26] J. M. Montesinos *A representation of closed orientable 3-manifolds as 3-fold branched coverings of* S^3 Bull. Amer. Math. Soc. 80 (1974), 845-846

[27] J. Moser, *On the Volume Elements on a Manifold*, Trans. Amer. Math. Soc. 120 (1965), 286-294

[28] M. Peixoto, *Structural Stability on two-dimensional manifolds*, Topology 1 (1962) 101-120

[29] M. Peixoto, *Structural Stability on two-dimensional manifolds - a further remark*, Topology 2 (1963), 179-180

[30] I. Satake, *The Gauss-Bonnet theorem for V-manifolds,* J. Math Soc. Japan 9 (1957), 464-492

[31] W. Thurston, ch. 13 from *The Geometry and Topology of Three-Manifolds* Unpublished manuscript. Available at: http://library.msri.org/books/gt3m/

[32] W. Thurston, H. Winkelnkemper, *On the existence of contact forms*, Proc. Amer. Math. Soc. 52 (1975), 345-347

[33] I. Torisu, *Convex contact structures and fibered links in 3-manifolds*, Internat. Math. Res. Notices 2000 no. 9, 441-454