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Skein Theory and Algebraic Geometry for the Two-Variable Kauffman Invariant of Links

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SKEIN THEORY AND ALGEBRAIC GEOMETRY FOR THE TWO-VARIABLE KAUFFMAN INVARIANT OF LINKS

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DEDICATION

To Nikki, of course.

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I am greatly indebted to my advisor, Alexei Oblomkov, for his guidance, support, and generosity. From him I have learned how to be a mathematician. For the many, many hours he has spent teaching me mathematics, and for the opportunities he has given me to become a better researcher and teacher, I am extremely grateful.

Without the love and support of my wife and partner in everything, Nikki, this thesis would not have been possible. She believes in me always, and inspires me to do my best in all of my endeavors.

ABSTRACT

SKEIN THEORY AND ALGEBRAIC GEOMETRY FOR THE TWO-VARIABLE KAUFFMAN INVARIANT OF LINKS

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We conjecture a relationship between the Hilbert schemes of points on a singular plane curve and the Kauffman invariant of the link associated to the singularity. Specifcally, we conjecture that the generating function of certain weighted Euler characteristics of the Hilbert schemes is given by a normalized specialization of the difference between the Kauffman and HOMFLY polynomials of the link. We prove the conjecture for torus knots. We also develop some skein theory for computing the Kauffman polynomial of links associated to singular points on plane curves.

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CHAPTER 1

INTRODUCTION

In the study of link invariants, two central problems are that of computing the invariants and that of interpreting what the invariants may tell us about the link or about the mathematical objects from which the link arises. Some invariants, such as the fundamental group of the link, are inherently topological, whereas other more combinatorially presented invariants can have a more obscure meaning. Two link invariants which fall into the latter category are the HOMFLY polynomial and Kauffman polynomial. Each of these invariants (for suitable choices of variables) is an element of $\mathbb{Z}[a^{\pm}, q^{\pm}, (q - q^{-1})^{\pm}]$ for which a topological interpretation in the general setting is not currently known. In this thesis, we make a contribution to both the computation and the interpretation of the Kauffman polynomial, with a particular emphasis on trying to understand the Kauffman polynomial of links that arise from singular points on plane curves.

In algebraic geometry, knots and links arise in the neighborhood of singular points on complex hypersurfaces. In particular, one obtains a link by intersecting a curve with a small 3-sphere centered at a singular point of the curve. The topology of such links was studied in deatail by Milnor in [14]. Recent work [3] by Campillo, Delgado, and Gusein-Zade computes the Alexander polynomial of the link of a singular point from the ring of functions on the curve. In [20], Oblomkov and

Shende conjectured a relationship between the HOMFLY polynomial of the link of a singularity and certain topological invariants of algebro-geometric spaces associated to the singularity known as punctual Hilbert Schemes. In [13], Maulik proved the conjecture of Oblomkov and Shende. It is this proof which is the motivation for the present work.

The central problem of this thesis is how to compute the Kauffman polynomial of the link of a singular plane curve from the algebraic geometry of the curve. The approach we take is two-fold, tackling the problem both from the point of view of the geometry of the curve and from the computational skein-theoretic framework of the knot invariants. On the algebraic geometry side of things, the bulk of the effort is in finding the appropriate spaces and the appropriate invariants of those spaces which will give data about the polynomial invariants of the link. To this end, and motivated by the results of [20], we study certain numerical invariants of Hilbert schemes of points on singular plane curves. This is the subject of Chapter 2 of this thesis. In Chapter 3, we move to the skein-theoretic viewpoint and develop for the Kauffman polynomial some of the analogous results from the theory of the HOMFLY polynomial that were needed in Maulik's proof of the conjecture of Oblomkov and Shende.

CHAPTER 2

PLANE CURVE SINGULARITIES AND THE KAUFFMAN POLYNOMIAL

In this chapter we conjecture a relationship between certain topological invariants of punctual Hilbert schemes on a singular plane curve and the Kauffman polynomial of the link associated to the singularity of the curve. We prove the conjecture for a class of knots known as torus knots.

2.1 Punctual Hilbert schemes on singular plane curves

For the remainder we consider an algebraic plane curve C and a point $p \in C$. The point p may be smooth, but the interesting geometry will arise in the case that C is singular at p, in which case we will take p to be the unique singular point on C. We denote by $C^{[l]}$ the Hilbert scheme of l points on C. The Hilbert scheme $C^{[l]}$ is the moduli space of closed, zero-dimensional, length l subschemes of C. For concreteness, we note that we may think of C as the vanishing locus of a single polynomial $f \in \mathbb{C}[x, y]$ with coordinate ring $R_C = \mathbb{C}[x, y]/(f)$, and we have the set-theoretic equality

$$C^{[l]} = \{\text{ideals } I \in R_C \mid \dim_{\mathbb{C}} (R_C/I) = l\}.$$

Inside $C^{[l]}$ is the punctual Hilbert scheme $C_p^{[l]}$, which is the closed subscheme consisting of elements $C^{[l]}$ supported precisely at $\{p\}$. If we let $\mathfrak{m}$ be the maximal ideal in R_C, then we have (again, as a set)

$$C_p^{[l]} = \left\{ [I] \in C^{[l]} \mid I \subset \mathfrak{m}^d \text{ for some } d \right\},$$

where by convention we write $[I]$ when thinking of the ideal I as a point in the Hilbert scheme.

The reason for considering these punctual Hilbert schemes is the following beautiful and surprising conjecture, originally due to A. Oblomkov and V. Shende in [20].

Theorem 1 (Conjecture 1 in [20], Theorem 1.1 in [13]). *Let $P_C(a, q)$ be the HOMFLY polynomial of the link associated to the unique singular point on a plane curve C. Denote by $\chi(C_p^{[l]})$ the topological Euler characteristic of the punctual Hilbert scheme as defined above. Then*

$$\sum_l \chi(\mathcal{C}_p^{[l]}) q^{2l} = \left[\left(\frac{q}{a} \right)^{\mu - 1} P_C(a, q) \right]_{a=0} \tag{2.1}$$

Here, and throughout the rest of this chapter, we denote by μ the Milnor number of the singularity, which we may take to be defined by

$$\mu = \dim_{\mathbb{C}} \left(R_c / (\partial f / \partial x, \partial f / \partial y) \right)$$

The paper [20] in fact contains an elegant conjectured formula for the full two-variable HOMFLY polynomial, and an even more general result for the colored HOMFLY polynomial has since been proved by Maulik in [13]. We have stated Theorem 1 in the form above because it resembles closely the conjecture which we will posit in this thesis (Conjecture 1 in Section 2.3).

Inspired by equation (2.1), we seek both an algebro-geometric quantity to replace the left hand side of (2.1) and a knot-theoretic quantity involving the Kauffman polynomial to replace the right hand side of (2.1). On the algebraic geometry side, we consider a particular weighted Euler characteristic, which we describe now.

Definition 1. Let X be a topological space and $\gamma : X \to G$ be a constructible function from X to an abelian group G. Then the **weighted Euler characteristic of X with respect to** γ is denoted by $\int_X \gamma \, d\chi$ and is defined by

$$\int_X \gamma \, d\chi = \sum_{g \in G} g\chi\left(\gamma^{-1}(g)\right)$$

Note for example that taking $G = \mathbb{Z}$ and $\gamma(G) = \{1\}$ to be constant gives the usual Euler characteristic of X. For convenience we set collect the punctual Hilbert schemes into a disjoint union

$$C_p^{[*]} := \coprod_l C_p^{[l]}.$$

We define two constructible functions on $C_p^{[*]}$.

Definition 2. Define the functions $m, l : C^{[*]} \to \mathbb{Z}$ to be

$$m([I]) = \text{the minimal number of generators of } I$$

and

$$l([I]) = \dim_{\mathbb{C}}(R_C/I)$$

To avoid confusion, we note that by definition, the length of a closed subscheme $Z \hookrightarrow X$ is the $\mathbb{C}$-dimension of its global sections. That is $l(Z) = h^0(Z) = \dim_{\mathbb{C}}(\mathcal{O}_Z(Z))$. In the affine case, if $X = \operatorname{Spec}(R)$, then $Z = \operatorname{Spec}(R/I)$ for some ideal $I \subset R$, and $\mathcal{O}_Z(Z) = R/I$. So we are in the slightly unfortunate situation where the *length* of Z is what is usually be called the *colength* of I. Thus even though we have used l for "length" function, the notation $l(I)$ or $l([I])$ will mean the colength of the ideal I in the coordinate ring of the curve C.

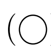
For a formal variable q, we will consider the weighted Euler characteristic corresponding to the function

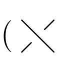
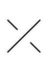
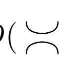
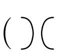
$$mq^{2l} : C^{[*]} \to \mathbb{Z}[q^2].$$

2.2 Conventions for the HOMFLY and Kauffman Polynomials

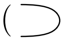
The knot-theoretic quantity which we will relate to the Hilbert schemes defined in Section 2.1 involves both the Kauffman and HOMFLY polynomials of a link. In this section we fix precisely the variables and skein relations we use to define the two invariants.

The Kauffman polynomial can be defined in terms of a regular isotopy invariant D in the variables a and z which satisfies the following skein relations:

(i) D () $= \dfrac{a - a^{-1}}{z} + 1$

(ii) D() $-$ D() $= z\,\Big($ D() $-$ D() $\Big)$

(iii) D() $= a^{-1}D$()

(iv) D() $= aD$()

As is standard in the literature, each of these equations is meant to be a local calculation (in the neighborhood of the crossing small enough to contain no other crossings) on the diagram of a link. For a link L, we write $D(L)(a, z)$ for the invariant computed using the above skein relations.

One definition of the Kauffman polynomial that appears often in the leterature (the so-called "Dubrovnik version") is

$$Y(L)(a, z) = a^{-\omega(L)} D(L)(a, z),$$

where $\omega(L)$ is the writhe of the link. Note that this requires choosing an orientation of the link, which makes the Kauffman invariant an invariant of oriented links. For our purposes, we will change the variables in Y to define the Kauffman polynomial.

Definition 3. For a link L, let $Y(L)(a, z)$ be computed on a diagram of L as above. Define $F(L)(a, q)$ by

$$F(L)(a, q) := Y(L)(-a^{-1}, q - q^{-1})$$

We call $F(L)(a, q)$ the **Kauffman polynomial** of the link L. Wherever convenient we may simply write $F(L)$ or F_L for the Kauffman polynomial of a link L.

With this definition of the Kauffman polynomial, the unknot $\bigcirc$ has value

$$F(\bigcirc) = \frac{a - a^{-1}}{q - q^{-1}} + 1.$$

We use the following definition of the HOMFLY polynomial.

Definition 4. Define the polynomial $P(a, q)$ on the diagram of a link K by the following skein relations.

(a) $aP($  $) - a^{-1}P($  $) = (q - q^{-1})P(\,)(\,)$

(b) $P(\bigcirc) = \dfrac{a - a^{-1}}{q - q^{-1}}$

We call $P(a, q)$ the **HOMFLY polynomial** of K.

The Kauffman and HOMFLY polynomials are known to distinguish different knots, and so are not related to one another by some change of variables. They do however have a common specialization known as the Jones polynomial, J, of a link. If we divide either the Kauffman or HOMFLY polynomial by its value on the unkot, we obtain the so-called normalized Kauffman and HOMFLY polynomials. These normalized versions we denote by $\overline{F}$ and $\overline{P}$. Thus for any link L we have

$$\overline{F}(L) = \frac{F(L)}{F(\bigcirc)}$$

and

$$\overline{P}(L) = \frac{P(L)}{P(\bigcirc)}.$$

With these choices of normalization, one has $\overline{F}(\bigcirc) = \overline{P}(\bigcirc) = 1$. In this notation, the Jones polynomial can be recovered from F as

$$J_L(t) = \overline{F}_L(a = it^{3/4}, q = it^{-1/4})$$

and the Jones polynomial can be obtained from the HOMFLY polynomial as

$$J_L(a) = \overline{P}_L(a, q = a^{-1/2})$$

So in particular, we have that

$$\overline{F}(it^{3/4}, it^{-1/4}) = \overline{P}(t, t^{-1/2})$$

2.3 A Conjecture and Some Examples

We can now state precisely our conjecture, which relates the weighted Euler characteristic from the end of Section 2.1 to the difference of the Kauffman and HOMFLY polynomials of the link using the conventions from Section 2.2.

Conjecture 1. *Let C be a plane curve with unique singular point p, and let $L_{C,p}$ be the link of the singularity. Then*

$$\int_{C_p^{[*]}} mq^{2l}\, d\chi = \left[\frac{1}{1-q^2}\left(\frac{q}{a}\right)^{\mu}\left(F(L_{C,p}) - P(L_{C,p})\right)\right]_{a=0} \tag{2.2}$$

As mentioned in Section 2.1, our conjecture can be viewed as an analog of the known result (2.1). Indeed, observe that

$$\int_{C_p^{[*]}} mq^{2l}\, d\chi = \sum_l \left(\int_{C_p^{[l]}} m\, d\chi \right) q^{2l},$$

which highlights the similarity of the left hand side of (2.2) to the quantity $\sum_l \left(\chi(C_p^{[l]})\right) q^{2l}$ on the left-hand side of (2.1). Note also that the left hand side of (2.2) is fixed solely by the topology of the Hilbert schemes, whereas the quantity $F(L_{C,p}) - P(L_{C,p})$ on the right hand side will look very different depending on the conventions one chooses for either the Kauffman or the HOMFLY polynomial (as there are independent choices for each invariant).

For the remainder of this section we compute some specific examples of Conjecture 1.

Example 1 (Unknot)**.** The conjecture is still true in the case of a smooth curve C (and a necessarily smooth point p). In this case the link L_C is the unknot, which we denote by $\bigcirc$. Choosing a local parameter t at the point p gives an isomorphism $\widehat{\mathcal{O}_{C,p}} \cong \mathbb{C}[[t]]$, which has (t^i) as its unique length i ideal for each i. As each ideal has only one generator, we have that the image of $m : C_p^{[*]} \to \mathbb{Z}$ is $\{1\}$. As remarked after Definition 1 in Section 2.1, this implies that the weighted Euler characteristic with respect to m is the usual topological Euler characteristic χ. Since each Hilbert scheme $C_p^{[l]}$ is a point, have

$$\chi(C_p^{[l]}) = \chi(\{\mathrm{pt}\}) = 1 \qquad \text{for all } l$$

and therefore

$$\int_{C^{[*]}_{\text{smooth } p}} mq^{2l}\, d\chi = \sum_{l=0}^{\infty} \chi\left(C_p^{[l]}\right) q^{2l} = \sum_{l=0}^{\infty} q^{2l} = \frac{1}{1-q^2}$$

The Milnor number at a smooth point is 0, and since $F(\bigcirc) - P(\bigcirc) = 1$ we get

$$\left[\frac{1}{1-q^2}\left(\frac{q}{a}\right)^{\mu}(F(\bigcirc) - P(\bigcirc))\right]_{a=0} = \frac{1}{1-q^2}$$

in agreement with the weighted Euler characteristic computation.

For the next two examples it will be convenient to introduce some notation for the intersection of the fibers of the map $m : C_p^{[*]} \to \mathbb{Z}$ with the fibers of the map $l : C_p^{[*]} \to \mathbb{Z}$. We write

$$m^{-1}(\{i\}) \cap C_p^{[l]} := m_l^{-1}(\{i\})$$

Example 2 (Trefoil Knot)**.** The trefoil knot is the link associated to the cusp curve C defined by $y^2 = x^3$, which has $p = (0,0)$ as its unique singular point. The punctual Hilbert schemes on this curve have been described in [9], where it is shown that $C_p^{[l]} \cong \mathbb{P}^1$ for each $l \geq 2$ and $m(C_p^{[l]}) = \{1, 2\}$. Moreover, if we write $\mathbb{P}^1 = \mathbb{C} \cup \{\text{pt.}\}$, then for each $l \geq 2$ we have

$$m_l^{-1}(\{1\}) = \mathbb{C} \qquad \text{and} \qquad m_l^{-1}(\{2\}) = \{\text{pt.}\},$$

so the calculations we need for the weighted Euler characteristic are

$$1\chi(m_l^{-1}(1)) = 1\chi(\mathbb{C}) = 1 \qquad \text{and} \qquad 2\chi(m_l^{-1}(2)) = 2\chi(\{\text{pt}\}) = 2$$

Thus we have

$$\int_{C_p^{[l]}} m \, d\chi = 1\chi(m_l^{-1}(1)) + 2\chi(m_l^{-1}(2)) = 1 + 2 = 3 \qquad \text{for } l \geq 2,$$

and therefore

$$\int_{C_p^{[*]}} mq^{2l} \, d\chi = 1 + 2q^2 + 3q^4 + 3q^4 + 3q^6 + \cdots = \frac{1 + q^2 + q^4}{1 - q^2} \tag{2.3}$$

The constant term 1 comes from the trivial ideal generated by 1, and the $2q^2$ is due to the maximal ideal which has 2 generators generators and colength 1.

We use the right handed trefoil knot, which has writhe 3, shown in Figure 1

Using the defintions of the Kauffman polynomial F and the HOMFLY polynomial P from Section 2.2, we calculate

$$F_{\text{trefoil}}(a, q) - P_{\text{trefoil}}(a, q) = \frac{1 + q^2 + q^4}{q^2} a^2 - \frac{1 + q^2 + q^4}{q^2} a^4 + a^6 \tag{2.4}$$

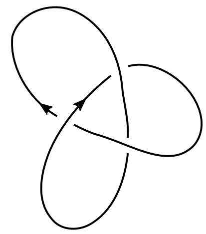


Figure 1. Trefoil knot

The Milnor number of the origin on the curve $y^2 = x^3$ is

$$\mu = \dim_{\mathbb{C}} \left(\mathbb{C}[x,y]/(y^2 - x^3, 2y, -2x^2)\right) = \dim_{\mathbb{C}} \left(\mathbb{C}[x,y]/(y, x^2)\right) = 2$$

Thus from (2.4) we see that the right-hand side of (2.2) in Conjecture 1 is

$$\left[\frac{1}{1-q^2}\left(\frac{q}{a}\right)^2 (F_{\text{trefoil}}(a,q) - P_{\text{trefoil}}(a,q))\right]_{a=0} = \frac{1+q^2+q^4}{1-q^2},$$

matching (2.3).

Example 3 (Hopf link)**.** The Hopf link is the link of the singularity at the origin p of the curve defined by $xy = 0$. The finite colength ideals in the ring $\mathbb{C}[[x,y]]/(xy)$ have been worked out in [22]. The ideals of colength l are

$$I_l^i = (y^l + ax^{l-i}) \text{ for nonzero } a \in \mathbb{C}, 1 \leq i \leq l-1$$

$$J_l^i = (x^{l-i+1}, y^i), 1 \leq i \leq l$$

So again we have $m(C_p^{[*]}) = \{1, 2\}$. Note that $m_l^{-1}(\{1\}) = \{I_l^i\}_i$, which is a disjoint union of $\mathbb{C}^*$'s by the parameter a, and thus

$$\chi\left(m_l^{-1}(\{1\})\right) = 0.$$

Since

$$\chi\left(m_l^{-1}(\{2\})\right) = \chi\left(\left\{(x^l, y), (x^{l-1}, y^2), \ldots, (x, y^l)\right\}\right) = \chi(l \text{ points}) = l$$

we have (accounting for the unique (trivial) ideal of colength zero),

$$\int_{C^{[*]}_{(0,0)}} mq^{2l}\, d\chi = 1 + \sum_{l} 2lq^{2l} = \frac{1+q^4}{(1-q^2)^2}. \tag{2.5}$$

The Hopf link is shown in Figure 2. A calculation using the skein relations

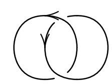


Figure 2. The Hopf link.

shows that

$$F_{\text{Hopf}}(a,q) - P_{\text{Hopf}}(a,q) = a^2 + a^3(a-a^{-1}) + (a^3-a)(q-q^{-1}) + 2a^2\left(\frac{a-a^{-1}}{q-q^{-1}}\right) \tag{2.6}$$

The Milnor number of the singularity of the curve $xy = 0$ is

$$\mu = \dim_{\mathbb{C}}\left((\mathbb{C}[x,y]/(xy,x,y)\right) = \dim_{\mathbb{C}}(\mathbb{C}) = 1.$$

Using this and (2.6) we obtain

$$\left[\frac{1}{1-q^2}\left(\frac{q}{a}\right)\left(F_{\text{Hopf}}(a,q) - P_{\text{Hopf}}(a,q)\right)\right]_{a=0} = 1 + \frac{2q^2}{(1-q^2)^2} = \frac{1+q^4}{(1-q^2)^2},$$

in agreement with the Euler characteristic calculation (2.5).

2.4 Conjecture 1 in context

In the present work, Conjecture 1 arose primarily out of experimentation. But the quantities that appear on each side of equation (2.2) are not without precedent in the literature. The knot-theoretic quantity on the right hand side of (2.2) appears in the physics literature in [6], where the authors claim that there is a well–known relationship between the Kauffman and HOMFLY polynomials given by

$$F(a,q) - P(a,q) = \sum_{g,Q} N^{c=1}_{g,Q}(q-q^{-1})^{2g}a^Q + \sum_{g,Q} N^{c=2}_{g,Q}(q-q^{-1})^{2g+1}a^Q.$$

The integers $N_{g,Q}^{c=i}$ are so-called "BPS degeneracies" which determine Chern-Simons invariants for symplectic and orthogonal guage gruops. Conjecture 1 provides the first mathematical interpretation of some of these BPS degeneracies. If Conjecture 1 can be upgraded to include the full two-variable Kauffman polynomial instead of only the $a \to 0$ limit, then all of the $N_{g,Q}^{c=i}$ would be placed in a mathematical formalism.

The difference between the two link invariants (rather than just, say, the Kauffman polynomial itself) also appears in the physics paper [12], where the author makes an integrality conjecture for link invariants derived from the difference of the Kauffman and HOMFLY polynomials. It is notable that the author of [12] states that part of the inspiration for that work comes from the (mathematical) knot-theoretic work of Morton and Ryder [19], which is in turn related to the work in Chapter 3 of this thesis.

On the mathematical side of things, the reason for considering the difference between the Kauffman and HOMFLY polynomial has its motivations in the form of the integral that appears. In [20], the more general results of [21] are adapted to show that any suitable constructible function ϕ on the Hilbert scheme $C_p^{[*]}$ in place of m on the left hand side of (2.2) will give that the integral has a value of the form

$$\sum_{h=0}^{g} q^{2g-2h}(1-q^2)^{2h-2} n_h(\phi),$$

for some integers $n_h(\phi)$ determined by the function ϕ, where g is the genus of the curve. Data suggests that the Kauffman polynomial does not have this form, but the difference between the Kauffman and HOMFLY polynomial does.

2.5 Torus knots

In this section we will prove Conjecture 1 for the special case of torus knots.

2.5.1 Computation of the Integral

Definition 5. A **torus link** is a link which has an embedding in the standard 2-torus. Torus links are uniquely characterized by an ordered pair (a, b) corresponding to the simple closed curve which wraps around the meridian of the torus a times and around the longitudinal curve b times.

Lemma 1 ([14])**.** *The link of the singular point on the curve $y^k = x^n$ is the (k, n) torus link. If k and n are relatively prime, then the link has only one component and is called the (k, n) torus knot.*

For the remainder of this section, let $C_{k,n}$ be the curve defined by $y^k = x^n$, where $\gcd(k, n) = 1$. These curves have a unique singular point p at the origin. We will omit the subscript p and denote by $C_{k,n}^{[*]}$ the punctual Hilbert schemes. The curve $C_{k,n}$ carries a $\mathbb{C}^*$-action given by $s \cdot y = s^n y$ and $s \cdot x = s^k x$. We can extend this action to ideals, and thus to the Hilbert schemes of points, by acting on the generators of the ideal. The length function l and the minimal number of generators function m are preserved by the $\mathbb{C}^*$ action. We set $R_C = \mathbb{C}[x, y]/(y^k - x^n)$.

Definition 6. Let $I_{k,n} = (y^k - x^n) \subset \mathbb{C}[x, y]$. An ideal $I \in R_C$ is said to be a **monomial ideal** if I is generated by elements of the form $x^a y^b + I_{k,n}$. Equivalently, $I \in R_C$ is monomial if it is generated by the images of monomials in $\mathbb{C}[x, y]$ under the quotient map. The set of monomial ideals is denoted by $\mathcal{I}_{\text{mon}}$.

Lemma 2. *Let $\mathcal{I}$ be the set of ideals in R_C. Then the fixed points of $\mathcal{I}$ under the*

$\mathbb{C}^*$-action are precisely the monomial ideals. That is

$$\mathcal{I}^{\mathbb{C}^*} = \mathcal{I}_{mon}.$$

Proof. Let $\lambda_q : \mathbb{C}^* \to \mathbb{C}^*$ be $\lambda_q(s) = s^q$ for $q \in \mathbb{Z}$, and let

$$V_q = \{f \in \mathbb{C}[x,y]/(y^k - x^n) \mid s \cdot f = s^q f\}$$

be the weight space for λ_q. Suppose that $x^a y^b$ and $x^c y^d$ are both in V_q for some q. Then we have $ka + nb = kc + nd$. Assume without loss of generality that $a < c$ and $d < b$. Since k and n are relatively prime, we have then that $(c-a)/n = (b-d)/k := p \in \mathbb{Z}_{\geq 0}$, and thus

$$\begin{aligned}
x^a y^b - x^c y^d &= x^a y^d (y^{b-d} - x^{c-a}) \\
&= x^a y^d ((y^k)^p - (x^n)^p) \\
&= x^a y^d (y^k - x^n) \left[(y^k)^{p-1} + x^n (y^k)^{p-2} + \cdots + y^k (x^n)^{p-2} + (x^n)^{p-1}\right] \\
&= 0 \mod (y^k - x^n)
\end{aligned}$$

So V_q contains only a single monomial (up to scalar). Since every element in V_q can be written as a sum of monomials, this implies $\dim_{\mathbb{C}} V_q = 1$. Thus any $\mathbb{C}^*$-fixed I has a $\mathbb{C}$-basis of monomials, and so is certainly generated as an ideal by monomials. $\square$

Remark. Lemma 2 is false for general curves which admit a $\mathbb{C}^*$-action. For the curve $y^2 = x^2$, we may take the torus action to simply be $s \cdot x = sx$ and $s \cdot y = sy$, and then both xy and x^2 are fixed points.

Since each space $C_{k,n}^{[l]}$ admits a $\mathbb{C}^*$-action, only the $\mathbb{C}^*$-fixed points contribute to the Euler characteristic. Since we have just shown that the fixed points are the monomial ideals, we have the following result.

Lemma 3. *Let $C_{k,n}$ be the curve defined by $y^k = x^n$ and let $C_{k,n}^{[*]}$ be the (union of) punctual Hilbert schemes as above. Then*

$$\int_{C_{k,n}^{[*]}} mq^{2l}\, d\chi = \sum_{\text{monomial ideals } I} m(I) q^{2l(I)},$$

for monomial ideals I in the ring $\mathbb{C}[x,y]/(y^k - x^n)$.

In light of Lemma 3, we turn our attention to describing monomial ideals in the ring $\mathbb{C}[x,y]/(y^k - x^n)$. This description of monomial ideals, as well as the generating function G in Lemma 4 below enumerating the types of ideals in which we are interested, is essentially that found in [20]. Our analysis differs slightly, as we take a more combinatorial approach.

Monomial ideals of finite length in the ring $\mathbb{C}[x,y]/(y^k - x^n)$ are in one-to-one correspondence with lattice paths in a $k \times \infty$ grid which start at $(i, k-1)$ for some i, and end at $(i+n, 0)$. The correspondence comes from labeling the point (i,j) with the monomial $x^i y^j$. We draw such a diagram by labeling the boxes with the coordinate of their lower left corner. See Figure 3. This allows us to compute the right hand side of Lemma 3 combinatorially.

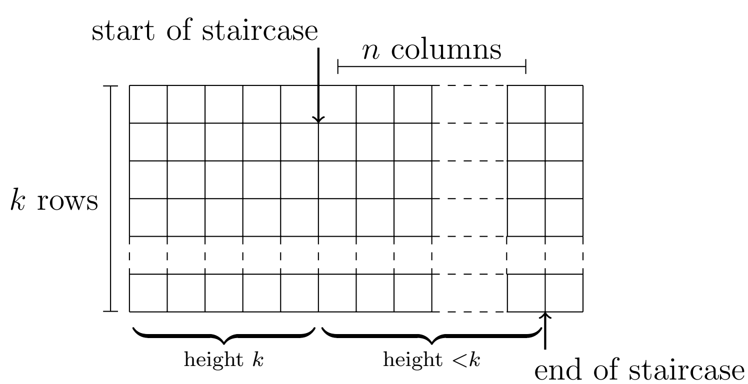


Figure 3. Staircase representing a monomial ideal in the coordinate ring of $C_{k,n}$.

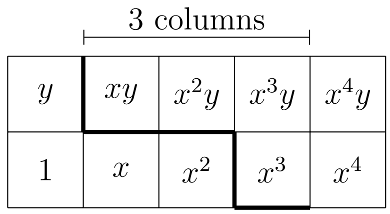


Figure 4. Staircase for the ideal (xy, x^3) in the ring $\mathbb{C}[x, y]/(y^2 - x^3)$.

Example 4. Consider the ring $R_C = \mathbb{C}[x, y]/(y^2 - x^3)$ and the monomial ideal $I \subset R$ given by $I = (xy, x^3)$. We have $\dim_{\mathbb{C}} (R_C/I) = 4$ with a $\mathbb{C}$-basis for R_C/I given by (the classes of) $\{1, x, y, x^2\}$. The ideal I corresponds to the staircase diagram in Figure 4. The generators of I can be read off the diagram as those monomials which sit "above" the staircase on the corners. The basis of the quotient of R_C/I can be read from the diagram as the monomials "under" the staircase.

The situation in Example 4 is completely general. Specifically, the number of generators of a monomial ideal is the number of corners above its staircase, and the colength of a monomial ideal is the number of boxes below the staircase. We keep track of the number of monomial ideals of a given colength and given number of generators via the following generating function.

Lemma 4. *Define the generating function*

$$G(s, t, k, q) = \sum_{\text{staircases}} s^{\#generators} t^{(\#columns\ of\ height\ <k)} (q^2)^{\#squares}.$$

Then

$$G(s, t, k, q) = \left(\frac{1}{1 - q^{2k}}\right) \prod_{i=0}^{k-1} \frac{1 - (1 - s)q^{2i}t}{1 - q^{2i}t}.$$

Proof. We just count the number of columns of each height used in each staircase. A column of height p contributes q^{2p} to G. Each time we use a column of a new height, we get a factor of s in G. Thus

$$G(s,t,k,q) =$$

$$\underbrace{(1+q^{2k}+q^{4k}+\cdots)}_{\text{height } k}\underbrace{(1+stq^{2(k-1)}+st^2q^{4(k-1)}+\cdots)}_{\text{height } k-1}\cdots$$

$$\cdots\underbrace{(1+stq^2+st^2q^4+\cdots)}_{\text{height } 1}\underbrace{(1+st+st^2+\cdots)}_{\text{height } 0}$$

which simplifies to the given equation for G in the lemma. □

Lemma 5. *Define $H(t,k,q)$ by*

$$H(t,k,q) = \frac{\partial G}{\partial s}(1,t,k,q).$$

Then the coefficient of t^n in H precisely the integral $\int_{C_{k,n}^{[*]}} mq^{2l}\,d\chi$.

Proof. The colength of any monomial ideal I, when viewed as a staircase, is the number of boxes below the staircase. This is because those boxes correspond precisely to the monomials not in I, which thus form a basis for the quotient. By construction, a monomial ideal whose staircase uses exactly n columns of height $k-1$ must correspond to a point in $C_{k,n}^{[l]}$ for some l. Therefore

$$G(s,t,k,q) = \sum_{\text{monomial ideals } [I]\in C_{k,n}^{[*]}} s^{m(I)}q^{2l(I)}t^n.$$

Thus

$$H(t,k,q) = \frac{\partial G}{\partial s}(1,t,k,q) = \sum_{\text{monomial ideals } [I]\in C_{k,n}^{[*]}} m(I)q^{2l(I)}t^n.$$

By Lemma 3, the last sum that appears in the previous display is $\int_{C_{k,n}^{[*]}} mq^{2l}\,d\chi$. □

We now give an explicit combinatorial interpretation of the integrals from Lemma 3 in terms of partitions.

Definition 7. A partition is any sequence $\lambda = (\lambda_1, \lambda_2, \dots,)$ with finitely many nonzero terms such that $\lambda_1 \geq \lambda_2 \geq \dots$. Each λ_i is called a part of λ. The number of nonzero parts of λ is denoted by $l(\lambda)$, and we say that λ is a partition of n if $\sum_i \lambda_i = n$.

For any nonnegative integers a, b, let $p_i(a, b)$ be the number of partitions λ of i such that λ has at most b parts and the largest part of λ is at most $a-1$.

Proposition 1. Let C be the curve defined by $y^k = x^n$. Then

$$\int_{C^{[*]}} mq^{2l}\, d\chi = \sum_{l=0}^{\infty} \left(\sum_{j=0}^{l} p_j(k, n-1) \right) q^{2l}$$

Proof. Computing the derivative of G we find that

$$H(t, k, q) = \frac{t}{(1-q^2)} \prod_{i=0}^{k-1} \frac{1}{1-q^{2i}t} \tag{2.7}$$

The product on the right hand side of (2.7) is well known to be the generating function for the $p_i(k, n)$ (see for example [11]). That is,

$$\prod_{i=0}^{k-1} \frac{1}{1-q^{2i}t} = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{\infty} p_i(k, n) q^{2i} \right) t^n.$$

From this it easily follows that

$$H(t, k, q) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} p_j(k, n) \right) q^{2i} \right) t^{n+1}$$

To complete the proof we just extract the coefficient of t^n and use Lemma 5. $\square$

Corollary 1. Let $C_{k,n}^{[\ell]}$ be the punctual Hilbert scheme of ℓ points on the curve $C = (y^k = x^n)$. Then

$$\int_{C_{k,n}^{[\ell]}} m\, d\chi = \sum_{j=0}^{\ell} p_j(k, n-1)$$

Example 5 (Trefoil Knot)**.** Let $k = 2$ and $n = 3$. We have

$$p_0(2,2) = p_1(2,2) = p_2(2,2) = 1,$$

and $p_i(2,2) = 0$ for $i \geq 3$. Thus

$$\int_{C_{2,3}^{[*]}} mq^{2l}\, d\chi = 1 + 2q^2 + 3q^4 + 3q^6 + 3q^8 + \cdots = \frac{1 + q^2 + q^4}{1 - q^2}$$

Remark. The product $\prod_{i=0}^{k-1} \frac{1}{1-q^{2i}t}$ is also known to be the generating function for the q-binomial coefficients (in this case, we are using q^2 instead of q). More precisely,

$$\prod_{i=0}^{k-1} \frac{1}{1 - q^{2i}t} = \sum_{n=0}^{\infty} \begin{bmatrix} k+n-1 \\ n \end{bmatrix}_q t^n$$

where

$$\begin{bmatrix} k+n-1 \\ n \end{bmatrix}_q = \frac{(1-q^{2(k+n-1)})(1-q^{2(k+n-2)})\cdots(1-q^{2k})}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}. \tag{2.8}$$

In particular, we have

$$\sum_{i=0}^{\infty} p_i(k,n) q^{2i} = \begin{bmatrix} k+n-1 \\ n \end{bmatrix}_q$$

2.5.2 The difference of the Kauffman and HOMFLY polynomials for torus knots

To conclude the proof of Conjecture 1 for torus knots, it remains to compute the knot-theoretic quantity on the right-hand side of (2.2). For this we use the following formulas for the Kauffman and HOMFLY polynomials of torus knots, which appear here in versions slightly modified from the originals in order to match our conventions for the knot invariants. Write $T_{k,n}$ for the (k,n) torus knot.

Theorem 2 ([25]).

$$F(T_{k,n}) = a^{kn-n}\left(\sum_{j=0}^{k-1} q^{-n(j-(k-1-j))}(-1)^{k-1-j}\left(\frac{1}{\{k\}}+\frac{1}{\{2j+1-k;1\}}\right)\right. \tag{2.9}$$

$$\left.\times\frac{1}{\{j\}!\{k-1-j\}!}\prod_{i=j+1-k}^{j}\{i;1\}\right)$$

where

$$\{j\} = q^j - q^{-j}$$

$$\{j;1\} = q^j a - q^{-j}a^{-1}$$

and

$$\{j\}! = \{j\}\{j-1\}\cdots\{1\} \qquad \textit{with } \{0\}! = 1$$

Theorem 3 ([8]).

$$P(T_{k,n}) = \frac{(a/q)^{(k-1)(n-1)-1}}{(1-q^{2k})}\sum_{j=0}^{k-1}(-1)^j\frac{(q^2)^{jn+(k-1-j)(k-j)/2}}{[j]![k-1-j]!}\prod_{i=j+1-k}^{j}(q^{2i}-a^2) \tag{2.10}$$

where

$$[j]! = (1-q^{2j})[j-1]! \qquad \textit{with } [0]! = 1$$

The key observation is that to make here is that equation (2.10) is actually contained in equation (2.9) in the following sense. Note that $F(T_{k,n})$ can be written as $F(T_{k,n}) = F_1(T_{k,n}) + F_2(T_{k,n})$, where

$$F_1(T_{k,n}) = a^{kn-n}\left(\sum_{j=0}^{k-1} q^{-n(j-(k-1-j))}(-1)^{k-1-j}\frac{1}{\{k\}}\;\frac{1}{\{j\}!\{k-1-j\}!}\prod_{i=j+1-k}^{j}\{i;1\}\right)$$

$$F_2(T_{k,n}) =$$

$$a^{kn-n}\left(\sum_{j=0}^{k-1} q^{-n(j-(k-1-j))}(-1)^{k-1-j}\frac{1}{\{2j+1-k;1\}}\;\frac{1}{\{j\}!\{k-1-j\}!}\prod_{i=j+1-k}^{j}\{i;1\}\right)$$

With a little algebra, one can show in fact that $F_1(T_{k,n}) = P(T_{k.n})$. Thus we have a formula, namely F_2, for the difference $F - P$ that we are interested in. We record this observation as a lemma.

Lemma 6. *Write $F(T_{k,n}) = F_1(T_{k,n}) + F_2(T_{k,n})$ where F_1 and F_2 are as above. Then $F_1(T_{k,n}) = P(T_{k.n})$, and thus*

$$F(T_{k,n}) - P(T_{k.n}) = a^{kn-n}\left(\sum_{j=0}^{k-1} q^{-n(j-(k-1-j))}(-1)^{k-1-j}\frac{1}{\{2j+1-k;1\}}\right.$$
$$\left.\times\frac{1}{\{j\}!\{k-1-j\}!}\prod_{i=j+1-k}^{j}\{i;1\}\right)$$

As these formulas are quite cumbersome, we give a few remarks about translating between the different notations used. First note that

$$\{j\}! = (-1)^j q^{-j(j+1)/2}[j]!$$

and thus

$$\frac{1}{\{j\}!\{k-1-j\}!} = \frac{(-1)^{k-1}q^{(j(j+1)+(k-j)(k-j-1))/2}}{[j]![k-1-j]!} \tag{2.11}$$

Both $\overline{F_1}(T_{k,n})$ and $\overline{P}(T_{k.n})$ contain a product term. The product in $\overline{F_1}(T_{k,n})$ can be written as

$$\prod_{i=j+1-k}^{j}\{i;1\} = \prod_{i=j+1-k}^{j} q^i a - q^{-i}a^{-1} = \prod_{i=j+1-k}^{j} -q^{-i}a^{-1}(1-q^{2i}a^2)$$
$$= (-1)^k a^{-k} q^{-k(j+1)+k(k+1)/2}\prod_{i=j+1-k}^{j}(1-q^{2i}a^2) \tag{2.12}$$

Similarly, the product term in $\overline{P}(T_{k,n})$ can be written as

$$\prod_{i=j+1-k}^{j}(q^{2i}-a^2) = q^{2k(j+1)-k(k+1)}\prod_{i=j+1-k}^{j}(1-q^{-2i}a^2)$$

Making these substitutions makes the formulas for F and P look much more alike. To make things even nicer, one can make the substitution $q \to -q^{-1}$ in $P(T_{k,n})$, as the HOMFLY polynomial is manifestly invariant under this change of variables.

The Milnor number of the singularity of the curve $y^k - x^n$ is

$$\mu = \dim_{\mathbb{C}} \left(\mathbb{C}[x,y]/(y^k - x^n, ky^{k-1}, -nx^{n-1}) \right)$$
$$= \dim_{\mathbb{C}} \left(\mathbb{C}[x,y]/(x^{n-1}, y^{n-1} \right) = (k-1)(n-1).$$

Thus from Lemma 6, we can compute the following.

Lemma 7.

$$\left[\left(\frac{(q/a)^\mu}{1-q^2} \right) F(T_{k,n}) - P(T_{k,n}) \right]_{a=0} = \frac{q^{2(k-1)(n-1)+k(k+1)/2-k}}{1-q^2} \left(\sum_{j=0}^{k-1} \frac{(-1)^j q^{j(2(1-n)-k)}}{\{j\}!\{k-1-j\}!} \right)$$

Comparing the quantity on the right-hand side of Lemma 7 to the q-binomial coefficients from (2.8), one obtains from Lemma 1 that Conjecture 1 is true for torus knots. We record this as a theorem.

Theorem 4. *Let k and n be positive integers with $\gcd(k,n) = 1$. Conjecture 1 is true for the curve C defined by $y^k - x^n$.*

CHAPTER 3

SKEIN THEORY

The proof in [13] that the Euler characteristic generating function of the spaces $C_p^{[l]}$ corresponds to the HOMFLY polynomial of the singularity's link relies ultimately on results from the skein theory of the HOMFLY polynomial. The essential skein-theoretic results are found in [10], [18], and [17], where the authors establish the computational methods needed to compute the HOMFLY polynomials of the class of links that arise from plane curve singularities. In this chapter, we take the first steps towards developing the analagous skein-theoretic results for the Kauffman polynomial.

The HOMFLY polynomial can be realized as a trace function on a class of algebras known as the Hecke algebras. These Hecke algebras are a quotient of the braid group algebra by a certain quadratic relation amongst its generators. As such, the Hecke algebras can be described in terms of diagrams of strands, and the relations satisfied by the generators are precisely the HOMFLY skein relations from section 2.2.

The Kauffman polynomial can analogously be realized as a trace on a class of algebras known as the BMW algebras (so-named for their discoverers, Birman, Wenzl, and Murakami), which we describe in detail below.

3.1 Definition of the BMW Algebra

The BMW algebras are a family of algebras $\mathcal{B}_n$, one for each natural number n, over a ring of scalars Λ. For our purposes, we take

$$\Lambda = \mathbb{C}[\alpha^{\pm}, s^{\pm}, (s - s^{-1})^{\pm}]$$

Note that Λ contains the quantum integers $[i]$ defined by

$$[i] = \frac{s^i - s^{-i}}{s - s^{-1}} \tag{3.1}$$

Each $\mathcal{B}_n$ can be thought of as the algebra of tangles mod the Kauffman skein relations. The generators for $\mathcal{B}_n$ are

$$\sigma_i \;\text{ for } i = 1, 2, \ldots, n-1 \qquad \text{(usual braid group generators)},$$

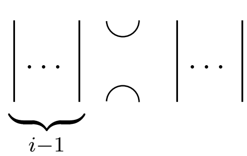
along with an identity $\mathbf{1}_n$. The σ_i satisfy the braid relations, and indeed the algebra $\mathcal{B}_n$ is a quotient of the braid group algebra by relations given below. The algebra $\mathcal{B}_n$ contains elements

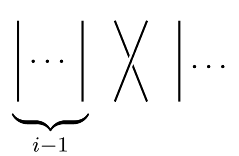
$$h_i \;\text{ for } i = 1, 2, \ldots, n-1$$

which satisfy

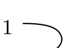
$$\sigma_i - \sigma_i^{-1} = (s - s^{-1})(\mathbf{1}_n - h_i) \qquad \text{(skein relation)} \tag{3.2}$$

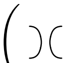
Diagramatically, these elements are given by the following:

$h_i =$ 

$\sigma_i =$ 

The local skein relations are

 $= \alpha^{-1}$  (3.3)

 $-$  $= (s - s^{-1}) \Big($  $-$  $\Big)$ (3.4)

From these we can deduce the factor δ that appears at the removal of a nullhomotopic loop from any diagram L:

$$L \bigcirc = \underbrace{\left(\frac{\alpha - \alpha^{-1}}{s - s^{-1}} + 1\right)}_{:=\delta} L$$

Multiplication in $\mathcal{B}_n$ is done by stacking diagrams. We make the convention that when we multiply diagrams, the diagram on the left goes on bottom. We use $X \otimes Y$ to mean the diagram that consists of diagram X placed to the left of diagram Y, without any of their strands interacting. Thus if X is a diagram on m strands and Y is a diagrams on n strands, then $X \otimes Y$ is a diagram on $m+n$ strands. We may view $\otimes$ as a map (or family of maps)

$$\otimes : \mathcal{B}_m \times \mathcal{B}_n \to \mathcal{B}_{m+n}.$$

Each of these maps is an inclusion. In particular, we have an inclusion $\otimes : \mathcal{B}_n \to \mathcal{B}_{n+1}$ given by

$$X \to X \otimes \mathbf{1}_1 \tag{3.5}$$

which adds a strand to the right of X. We will occasionally refer to "the" BMW algebra $\mathcal{B}$, by which we mean $\mathcal{B} = \bigcup_n \mathcal{B}_n$ using the inclusion (3.5).

The quotient of $\mathcal{B}_n$ by the two-sided ideal generated by $\{h_1, h_2, \ldots, h_{n-1}\}$ gives an algebra $\mathcal{H}_n$ known as the Hecke algebra. There is further a surjective algebra map $\mathcal{H}_n \to \mathcal{TL}_n$ from the Hecke algebra to the Temperley-Lieb algebra given by $\sigma_i \mapsto (s - e_i)$, where the e_i are the generators of $\mathcal{TL}_n$.

3.2 Symmetrizers in the BMW Algebras

In this section, we give a new description of certain idempotents in the BMW algebras known as the symmetrizers. These idempotents are generalizations of the symmetrizers in the Hecke algebras, as well as generalizations of the symmetrizers in the Temperley-Lieb algebras (where they are also known as the Jones-Wenzl projectors).

Unlike the case for the Hecke algebra, there appears to be no known explicit (and non-recursive) formula for the symmetrizers in the BMW algebras. There are however some skein-theoretic descriptions of the symmetrizers in the BMW algebras that appear in the literature. For example, in [4] the coefficients of the symmetrizers in a particular (diagrammatic) basis are calculated. Their main result ([4],Theorem 20) determines these coefficients recursively.

Perhaps the most thorough skein-theoretic description of the symmetrizers in the BMW algebras appears in [2]. There, the authors construct a section of the quotient map $\mathcal{B}_n \to \mathcal{H}_n$, and use this map to carry skein-theoretically constructed idempotents from the Hecke algebra (constructed in [1]) over to the BMW algebras.

We take as our starting point a recursive definition of the symmetrizers in the BMW algebras found in [7]. This recursive definition (our equation (3.10)) is perhaps the most straightforward (albeit computationally cumbersome) way to write down the symmetrizers in the BMW algebra. The launching point for our skein-theoretic calculations is simply draw the diagrams that appear in (3.10) (something not done in [7]). We then adapt some of the skein-theoretic techniques from section 2 of Lukac's thesis [10], where computations are done for the Hecke algebras, for our case of the BMW algebras.

Definition 8. The **symmetrizer** in the BMW algebra $\mathcal{B}_n$ is the unique element f_n satisfying the following:

(i) $f_n^2 = f_n$ (idempotent)

(ii) $f_n\sigma_i = sf_n = \sigma_i f_n$ for $i = 1, \ldots, n-1$.

(iii) $f_n h_i = h_i f_n = 0$ for $i = 1, \ldots, n-1$.

Remark. In the quotient $\mathcal{B}_n \to \mathcal{H}_n$, items (i) and (ii) survive and are the conditions for f_n to be the symmetrizer in the Hecke algebra (see for example [1]). In the quotient $\mathcal{H}_n \to \mathcal{TL}_n$ to the Temperley-Lieb algebras, recall that the map sends σ_i to $(s - e_i)$, and condition (ii) becomes precisely the defining characteristic of the Jones-Wenzl projector f_n. Namely, that $f_n e_i = e_i f_n = 0$ (see [15]).

Instead of working with the symmetrizers directly, we will work with a scalar multiple of f_n, which we will denote by a_n. These a_n are quasi-idempotent, with

$$a_n^2 = \alpha_n a_n,$$

for some scalar α_n which we will record later. Thus we have the relation

$$\left(\frac{1}{\alpha_n}\right) a_n = f_n. \tag{3.6}$$

These quasi-idempotents are sometimes referred to as the symmetrizers in the BMW algebras. Both a_n and f_n generate the same ideal in $\mathcal{B}_n$, which corresponds to the one-dimensional representation $\sigma_i \to s$ of $\mathcal{B}_n$. So the two are equivalent from a representation-theoretic point of view.

For our description of the symmetrizers in the BMW algebras, we will need the following following elements of $\mathcal{B}_{i+1}$ for each i:

$$T_k^i = \sigma_i \sigma_{i-1} \cdots \sigma_{i-k+1} \tag{3.7}$$

$$W_k^i = h_i h_{i-1} \cdots h_{i-k+1} \tag{3.8}$$

$$P_{k,l}^i = \sigma_{i-k} \sigma_{i-k-1} \cdots \sigma_{i-k-l+1} \tag{3.9}$$

The elements $P_{k,l}^i$ are more auxiliary, as what we will be interested in is the T_k^i, the W_k^i, and products $W_k^i P_{k,l}^i$. These elements have nice diagrammatic descriptions

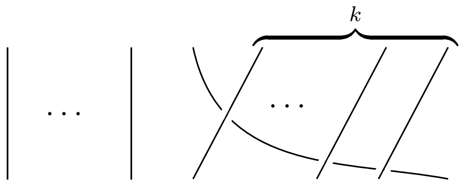


Figure 5. T_k^i

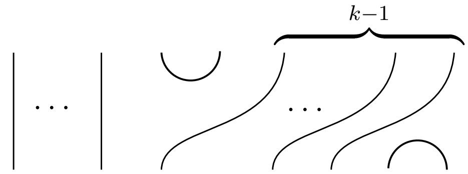


Figure 6. W_k^i

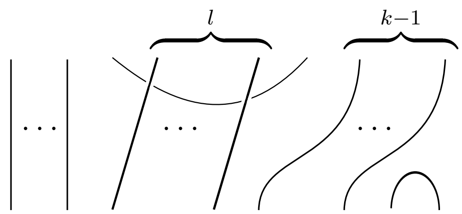


Figure 7. $W_k^i P_{k,l}^i$

shown in Figures 5, 6, and 7.

Despite the ugly notation, these elements are just strings of h's and σ's where the subscript starts with the largest possible and decreases by 1. For example

$$T_4^7 = \sigma_7\sigma_6\sigma_5\sigma_4 \in \mathcal{B}_8$$

$$W_3^5 = h_5h_4h_3 \in \mathcal{B}_6$$

$$W_3^8 P_{3,4}^8 = h_8h_7h_6\sigma_5\sigma_4\sigma_3\sigma_2 \in \mathcal{B}_9$$

We remark that W_k^i and $W_k^i P_{k,l}^i$ are completely determined by a choice of two points on the top row.

In terms of these diagrams just defined, we have the following recursive definition for the quasi-idempotents a_n defined in (3.6).

Proposition 2 ([7])**.** Define elements $a_n \in \mathcal{B}_n$ recursively by $a_1 = \mathbf{1}_1$ and

$$a_{i+1} = (a_i \otimes \mathbf{1}_1)\left(\mathbf{1}_{i+1} + \sum_{k=1}^{i} s^k T_k^i + \beta_i \left[\sum_{k=1}^{i} s^{2(k-1)} W_k^i + \sum_{k=1}^{i-1}\sum_{l=1}^{i-k} s^{2(k-1)+l} W_k^i P_{k,l}^i\right]\right) \tag{3.10}$$

with

$$\beta_i = \frac{1-s^2}{\alpha s^{2i-1}-1}$$

Then the following are true:

(i) $a_n^2 = \alpha_n a_n$, with $\alpha_n = s^{\frac{n(n-1)}{2}}[n]!$.

(ii) $a_n\sigma_i = \sigma_i a_n = s a_n$ for all $i = 1, 2, \ldots, n-1$.

(iii) $a_n h_i = h_i a_n = 0$ for all $i = 1, 2, \ldots, n-1$.

Here, the quantum factorials $[n]!$ are defined by $[n][n-1]\cdots[2][1]$ with $[n]$ as defined in (3.1).

Proof. Since we are working with the quasi-idempotents, we remark how to translate to our result from the result of [7]. Write our equation (3.10) as $a_{i+1} = (a_i \otimes \mathbf{1})J_i$. In [7] they show that the symmetrizers satisfy

$$f_{i+1} = \frac{1}{s^{n-1}[n]}(f_i \otimes \mathbf{1})J_i \tag{3.11}$$

If you assume that $c_n a_n = f_n$ for some constants c_n in (3.11) and compare to (3.10), an inductive argument will show that $c_n = \frac{1}{s^{n(n-1)/2}[n]!}$. $\square$

Example 6. The first nontrivial quasi-idempotent is a_2. Using (3.10) we immediately get that

$$a_2 = \mathbf{1_2} + s\sigma_1 + \beta_1 h_1.$$

In terms of diagrams, this is

Remark. Let $I \subset B_n$ be the ideal generated by the h_i, and let $\widetilde{x}$ denote the image of $x \in B_n$ in the quotient $B_n/I \cong H_n$. If we quotient the relation (3.10), we get elements $\widetilde{a}_i \in H_i$ for each i satisfying

$$\widetilde{a}_{i+1} = \left(\widetilde{a_i \otimes \mathbf{1}_1}\right)\left(\widetilde{\mathbf{1}_{i+1}} + \sum_{k=1}^{i} s^k \widetilde{T_k^i}\right)$$

Of course, since the quotient just kills terms with h_i's, if we think in terms of diagrams we have $\widetilde{a_i \otimes \mathbf{1}_1} = \widetilde{a}_i \otimes \mathbf{1}_1$, $\widetilde{\mathbf{1}_{i+1}} = \mathbf{1}_{i+1}$ and $\widetilde{T_k^i} = T_k^i$. Thus we get elements in the Hecke algebras satisfying the recurrence

$$\widetilde{a}_{i+1} = (\widetilde{a}_i \otimes \mathbf{1}_1)\left(\mathbf{1}_{i+1} + \sum_{k=1}^{i} s^k T_k^i\right) \tag{3.12}$$

Equation (3.12) is known ([16] Lemma 3.2) to describe the quasi-idempotents corresponding to the symmetrizers in the Hecke algebra. That is:

(i) $\widetilde{a}_n^{\,2} = \alpha'_n \widetilde{a}_n$, with $\alpha'_n = s^{\frac{n(n-1)}{2}}[n]!$

(ii) $\widetilde{a}_n \sigma_i = \sigma_i \widetilde{a}_n = s\widetilde{a}_n$ for all $i = 1, 2, \ldots, n-1$.

It is also interesting to note (perhaps more superficially) that all of the diagrams that appear in (3.10) are familiar, in a sense. The diagrams T_k^i appear in a recurrence for the symmetrizers in the Hecke algebras. The diagrams W_k^i are precisely the diagrams used in a well known recurrence for the Jones-Wenzl Projectors in the Temperley-Lieb algebras, originally due to Khovonov and Frenkel in [5] and nicely explained by Morrison in [15]. The remaining diagrams in (3.10) are a "mix" of the two diagrams just mentioned.

In the sequel we will make use of the following involutions on the BMW algebras, with respect to which the symmetrizers behave quite nicely. Let $\rho : \mathcal{B}_n \to \mathcal{B}_n$ be defined on diagrams by

$$\rho(\text{Diagram}) = \text{Diagram with all crossings switched}$$

and on scalars by

$$\rho(s) = s^{-1} \qquad \text{and} \qquad \rho(\alpha) = \alpha^{-1} \tag{3.13}$$

The reason for this "conjugate-linearity" of ρ is so that it preserves the defining relation (3.2), as $\rho(\sigma_i) = \sigma_i^{-1}$ for all i.

Define the Λ-linear operator $\tau : \mathcal{B}_n \to \mathcal{B}_n$ by

$$\tau(\text{Diagram}) = \text{diagram flipped vertically about an axis through its center.} \tag{3.14}$$

The key properties of ρ and τ are summarized in the lemma below.

Lemma 8. *Let ρ and τ be as defined above. Then the following are true.*

(i) For any diagrams X and Y we have $\rho(XY) = \rho(X)\rho(Y)$ and $\tau(XY) = \tau(Y)\tau(X)$.

(ii) The quantum integers $[n]$ are invariant under ρ and τ.

(iii) The symmetrizers f_n are fixed by ρ and τ.

Proof. The only item which is not immediate is item (iii). To show that $\rho(f_n) = f_n$ we need only show that $\rho(f_n)$ satisfies the properties that uniquely characterize f_n. Indeed since $\rho(\sigma_i) = \sigma_i^{-1}$ and $\rho(h_i) = h_i$, we get

$$\sigma_i \rho(f_n) = \rho(\sigma_i^{-1})\rho(f_n) = \rho(\sigma_i^{-1} f_n) = \rho(s^{-1} f_n) = s\rho(f_n)$$

and

$$h_i \rho(f_n) = \rho(h_i)\rho(f_n) = \rho(h_i f_n) = \rho(0) = 0.$$

Since $\rho(f_n)^2 = \rho(f_n^2) = \rho(f_n)$, we have by uniqueness that $\rho(f_n) = f_n$.

The calculation is the same for the map τ, using the fact that that $\tau(\sigma_i) = \sigma_i$. $\square$

Corollary 2. For the quasi-idempotent a_n, we have $\rho(a_n) = s^{n(1-n)}a_n$. The quasi-idempotents are fixed by τ.

Proof. From the equations $\rho(f_n) = f_n$ and $f_n = s^{-\frac{n(n-1)}{2}}([n]!)^{-1}a_n$, we get

$$\rho(a_n) = \rho\left(s^{\frac{n(n-1)}{2}}[n]!f_n\right) = s^{-\frac{n(n-1)}{2}}[n]!f_n = s^{-\frac{n(n-1)}{2}}[n]!s^{-\frac{n(n-1)}{2}}([n]!)^{-1}a_n = s^{n(1-n)}a_n.$$

□

3.3 A New Recurrence for the Symmetrizers

For later work it will be useful to have a recurrence relation for the symmetrizers which is much more compact than (3.10). To this end we need a few technical lemmas. These will be used to give a new recurrence for the symmetrizers in Proposition 3.

Lemma 9. *Let A be a finitely generated algebra with generating set $\{g_1, \ldots, g_n\}$. Let $a \in A$ be a quasi-idempotent element with $a^2 = \epsilon_a a$ for some scalar ϵ_a, and suppose that that $ag_i = g_i a = \alpha a$ for some scalar α and all $1 \leq i \leq n$. Then*

(i) a is central in A

(ii) The element $\widehat{a} := \frac{1}{\epsilon_a}a$ is the unique idempotent satisfying the conditions in the theorem.

(iii) If b is any quasi-idempotent in A, then $ab = ba = \epsilon_b a$.

Proof. Item (i) is true since a commutes with the generators of A. For (ii), suppose that $a' \in A$ is an idempotent satisfying the conditions in the theorem. Then $\widehat{a}a'$ is proportional to both $\widehat{a}$ and a', which means $\widehat{a} = \gamma a'$ for some scalar γ. The idempotent condition then forces $\gamma = 1$, whence $a' = \widehat{a}$.

For item (iii) suppse that $b^2 = \epsilon_b b$. Then $\left(\frac{1}{\epsilon_b} b\right) \widehat{a}$ is an idempotent satisfying the hypotheses of the theorem. So by the uniqueness from (ii) we must have $\left(\frac{1}{\epsilon_b} b\right) \widehat{a} = \widehat{a}$. This is equivalent to (iii). $\square$

Corollary 3. Let $i < k$. For any $0 \leq m \leq k - i$ we have

$$a_k(\mathbf{1}_m \otimes a_i \otimes \mathbf{1}_{k-m-i}) = \alpha_i a_k \tag{3.15}$$

This is shown diagramatically in Figure 8.

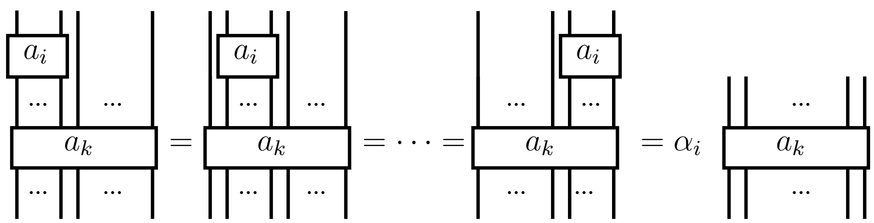


Figure 8. A diagrammiatic view of equation (3.15)

Proof. This follows from Lemma 9. In $\mathcal{B}_k$, the element a_k is a quasi-idempotent satisfying the hypotheses of Lemma 9, and the element $\mathbf{1}_m \otimes a_i \otimes \mathbf{1}_{k-m-i}$ is a quasi-idempotent. $\square$

Lemma 10. *The following equations hold in* $\mathcal{B}_{i+1}$

$$T_k^i(\mathbf{1}_1 \otimes a_i) = s^k(\mathbf{1}_1 \otimes a_i) \qquad \textit{for } k < i \tag{3.16}$$

$$W_k^i(\mathbf{1}_1 \otimes a_i) = 0 \qquad \textit{for } k < i \tag{3.17}$$

$$W_k^i P_{k,l}^i(\mathbf{1}_1 \otimes a_i) = 0 \qquad \textit{for } l < i - k \tag{3.18}$$

$$W_k^i P_{k,i-k}^i(\mathbf{1}_1 \otimes a_i) = s^{i-k} h_i h_{i-1} \cdots h_1(\mathbf{1}_1 \otimes a_i) \tag{3.19}$$

Proof. The subalgebra of $\mathcal{B}_{i+1}$ that fixes the first strand is isomorphic to $\mathcal{B}_i$. Under this identification, each of the elements in equations (3.16)–(3.18) above are elements of $\mathcal{B}_i$. These first three equations then follow from the defining properties of

$a_i \in \mathcal{B}_i$. Since we have shifted over by one strand, these defining properties become

$$\sigma_m(\mathbf{1}_1 \otimes a_i) = s(\mathbf{1}_1 \otimes a_i) \text{ for } 1 < m \leq i$$

and

$$h_m(\mathbf{1}_1 \otimes a_i) = 0 \text{ for } 1 < m \leq i.$$

Thus equation (3.16) follows from the fact that T_k^i contains exactly k terms of the form σ_i:

$$T_k^i(\mathbf{1}_1 \otimes a_i) = \underbrace{\sigma_i \sigma_{i-1} \cdots \sigma_{i-k+1}}_{k\ \sigma\text{'s}}(\mathbf{1}_1 \otimes a_i) = s^k(\mathbf{1}_1 \otimes a_i)$$

Equations (3.17) and (3.18) are zero because they contain $h_m(\mathbf{1}_1 \otimes a_i)$ for some $1 < m \leq i$.

It remains to establish equation (3.19). Note that

$$W_k^i P_{k,i-k}^i(\mathbf{1}_1 \otimes a_i) = h_i h_{i-1} \cdots h_{i-k+1} \sigma_{i-k} \sigma_{k-1} \cdots \sigma_1(\mathbf{1}_1 \otimes a_i),$$

and we know nothing about the product $\sigma_1(\mathbf{1}_1 \otimes a_i)$. To prove equation (3.19) we use the relation

$$h_m \sigma_{m-1} = h_m h_{m-1} \sigma_m^{-1}$$

for any m (shown in Figure 9), and the fact that $\sigma_m^{\pm}$ commutes with $\sigma_n^{\pm}$ so long as $|m - n| \geq 2$ (this is one of the standard braid relations). This gives

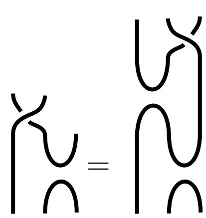


Figure 9. A proof that $h_m \sigma_{m-1} = h_m h_{m-1} \sigma_m^{-1}$.

$$
\begin{aligned}
W_k^i P_{k,i-k}^i &= h_i h_{i-1} \cdots (h_{i-k+1}\sigma_{i-k})\, \sigma_{i-k-1}\sigma_{i-k-2}\cdots\sigma_1 \\
&= h_i h_{i-1} \cdots \left(h_{i-k+1} h_{i-k} \sigma_{i-k+1}^{-1}\right) \sigma_{i-k-1}\cdots\sigma_1 \\
&= h_i h_{i-1} \cdots h_{i-k+1} \left(h_{i-k}\sigma_{i-k-1}\right) \sigma_{i-k-2}\cdots\sigma_1\sigma_{i-k+1}^{-1} \\
&\;\vdots \\
&= h_i h_{i-1} \cdots h_1 \sigma_2^{-1}\sigma_3^{-1}\cdots\sigma_{i-k+1}^{-1}
\end{aligned}
$$

The $i-k$ factors of the form σ_m^{-1} for $m > 1$ account for the factor of s^{i-k} in (3.19). Equations (3.18) and (3.19) are shown in Figures 10 and 11 respectively. $\square$

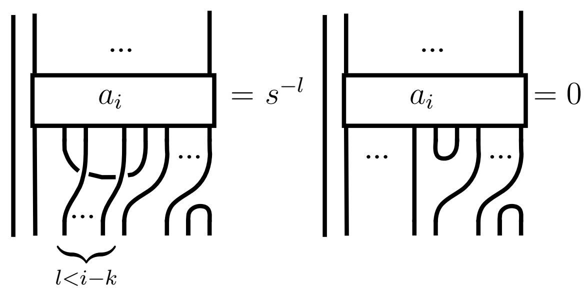


Figure 10. Diagram for equation (3.18).

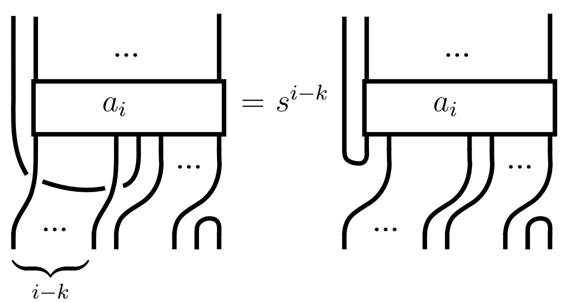


Figure 11. Diagram for equation (3.19).

We can now give a recursive formula for the quasi-idempotents a_i that uses only three terms.

Proposition 3 (New recurrence for symmetrizers)**.** Let $T_i := T_i^i \in \mathcal{B}_{i+1}$ and $W_i := W_i^i \in \mathcal{B}_{i+1}$ be as defined in Section 3.2. That is, $T_i = \sigma_i\sigma_{i-1}\cdots\sigma_1$ and $W_i = h_ih_{i-1}\cdots h_1$. Then the quasi-idempotents a_i satisfy

$$a_{i+1} = \frac{s^{i-1}[i]}{\alpha_i}(a_i\otimes\mathbf{1}_1)(\mathbf{1}_1\otimes a_i)+s^iT_i(\mathbf{1}_1\otimes a_i)+\frac{\beta_is^{i-1}[i]}{\alpha_i}(a_i\otimes\mathbf{1}_1)H_i(\mathbf{1}_1\otimes a_i) \quad (3.20)$$

In terms of diagrams this is is shown in Figure 12.

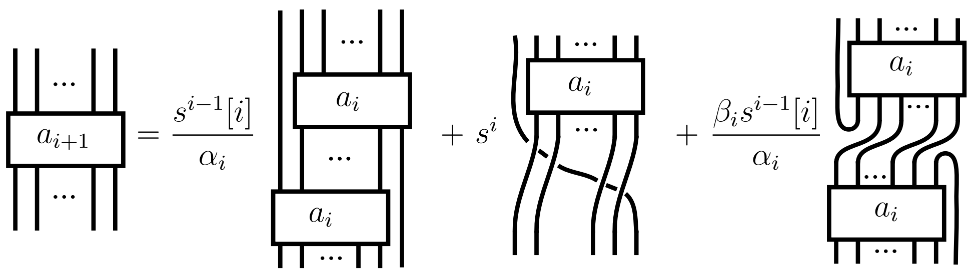


Figure 12. New three-term recurrence for the symmetrizers in the BMW algebras.

Proof. We start with the recurrence (3.10), which we recall here for convenience:

$$a_{i+1} = (a_i\otimes\mathbf{1}_1)\left(\mathbf{1}_{i+1}+\sum_{k=1}^{i}s^kT_k^i+\beta_i\left[\sum_{k=1}^{i}s^{2(k-1)}W_k^i+\sum_{k=1}^{i-1}\sum_{l=1}^{i-k}s^{2(k-1)+l}W_k^iP_{k,l}^i\right]\right) \quad (3.21)$$

Multiplying both sides of 3.21 on the right by $\mathbf{1}_1\otimes a_i$, the left hand side becomes α_ia_{i+1} by Corollary 3 with $m=1$ and $k=i+1$. For the right hand side, we compute each term separately using Lemma 10:

(i) $(a_i\otimes\mathbf{1}_1)\mathbf{1}_{i+1}(\mathbf{1}_1\otimes a_i) = (a_i\otimes\mathbf{1}_1)(\mathbf{1}_1\otimes a_i)$

(ii) By equation (3.16) we have

$$(a_i\otimes\mathbf{1}_1)\left(\sum_{k=1}^{i}s^kT_k^i\right)(\mathbf{1}_1\otimes a_i) = \left(\sum_{k=1}^{i-1}s^{2k}\right)(a_i\otimes\mathbf{1}_1)(\mathbf{1}_1\otimes a_i)+s^i(a_i\otimes\mathbf{1}_1)T_i(\mathbf{1}_1\otimes a_i)$$

Now we claim that T_i has the following property: For any $X\in\mathcal{B}_i$ we have

$$(X\otimes\mathbf{1}_1)T_i = T_i(\mathbf{1}_1\otimes X)$$

To see this, note that $\sigma_m \otimes \mathbf{1}_1 = \sigma_m$ and $\mathbf{1}_1 \otimes \sigma_m = \sigma_{m+1}$ for any $1 \leq m \leq i-1$. So

$$
\begin{aligned}
(\sigma_m \otimes \mathbf{1}_1) T_i = \sigma_m T_i &= \sigma_m(\sigma_i \sigma_{i-1} \cdots \sigma_{m+1} \sigma_m \cdots \sigma_1) \\
&= \sigma_i \sigma_{i-1} \cdots (\sigma_m \sigma_{m+1} \sigma_m) \sigma_{m-1} \cdots \sigma_1 \\
&= \sigma_i \sigma_{i-1} \cdots (\sigma_{m+1} \sigma_m \sigma_{m+1}) \sigma_{m-1} \cdots \sigma_1 \\
&= (\sigma_i \sigma_{i-1} \cdots \sigma_{m+1} \sigma_m \sigma_{m-1} \cdots \sigma_1) \sigma_{m+1} \\
&= T_i(\mathbf{1}_1 \otimes \sigma_m)
\end{aligned}
$$

Therefore

$$
s^i(a_i \otimes \mathbf{1}_1) T_i(\mathbf{1}_1 \otimes a_i) = s^i T_i(\mathbf{1}_1 \otimes a_i)(\mathbf{1}_1 \otimes a_i) = \alpha_i s^i T_i(\mathbf{1}_1 \otimes a_i)
$$

(iii) By equation (3.17) we have

$$
\left(\sum_{k=1}^{i} s^{2(k-1)} W_k^i \right) (\mathbf{1}_1 \otimes a_i) = s^{2(i-1)} W_i (\mathbf{1}_1 \otimes a_i)
$$

(iv) And lastly, by equation (3.18), the only term from the double sum that survives is when $l = i - k$. To compute that term we use equation (3.19):

$$
\begin{aligned}
\left(\sum_{k=1}^{i-1} \sum_{l=1}^{i-k} s^{2(k-1)+l} W_k^i P_{k,l}^i \right) (\mathbf{1}_1 \otimes a_i) &= \left(\sum_{k=1}^{i-1} s^{2(k-1)+(i-k)} W_k^i P_{k,i-k}^i \right) (\mathbf{1}_1 \otimes a_i) \\
&= \left(\sum_{k=1}^{i-1} s^{2(k-1)} \right) h_i h_{i-1} \cdots h_1 (\mathbf{1}_1 \otimes a_i)
\end{aligned}
$$

To complete the proof (3.20), we add up the right hand sides of (i)-(iv), noting that

$$
1 + \sum_{k=1}^{i-1} s^{2k} = s^{2(i-1)} + \sum_{k=1}^{i-1} s^{2(i-1)} = s^{i-1}[i]
$$

$\square$

3.3.1 Symmetrizers in the Hecke Algebras of type A and the Temperley-Lieb Algebra

Recall from Section 3.1 the quotient $\mathcal{B}_n \to \mathcal{H}_n$ which sends $h_i \to 0$. This map eliminates the entire rightmost diagram in Figure 12, giving the following result.

Corollary 4. For $X \in \mathcal{B}_n$, let $\widetilde{X}$ denote the image of X in the quotient $\mathcal{B}_n \to \mathcal{H}_n$. Then the elements $\widetilde{a}_i$ are the (quasi-idempotent) symmetrizers in Hecke algebras $\mathcal{H}_i$ and satisfy the two-term recurrence

$$\widetilde{a}_{i+1} = \frac{s^{i-1}[i]}{\alpha_i}(\widetilde{a}_i \otimes \mathbf{1}_1)(\mathbf{1}_1 \otimes \widetilde{a}_i) + s^i T_i(\mathbf{1}_1 \otimes \widetilde{a}_i). \tag{3.22}$$

Equation (3.22) does not seem to appear in the literature. It's main usefulness to the present work is to allow the work in the later sections to be viewed as direct generalization of work done previously for the Hecke algebra case. This is one reason that the effort was spent deriving the corresponding equation (3.20) in the BMW algebra.

The quotient $\mathcal{H}_n \to \mathcal{TL}_n$ from the Hecke algebra to the Temperley-Lieb algebra given by $\sigma_i \mapsto (s - e_i)$ allows us to also obtain from equation (3.20) the well-known recurrence for the Jones-Wenzl projectors.

3.4 The Two-Pointed Annulus

We now introduce a skein module which, in the setting of the HOMFLY polynomial, has proven to be quite useful for computing knot invariants. The definition below is completely analogous to the corresponding definition found in [16], the only essential difference being that we use here the Kauffman skein relations. This is the first time that this skein module has been studied in relation to the Kauffman polynomial.

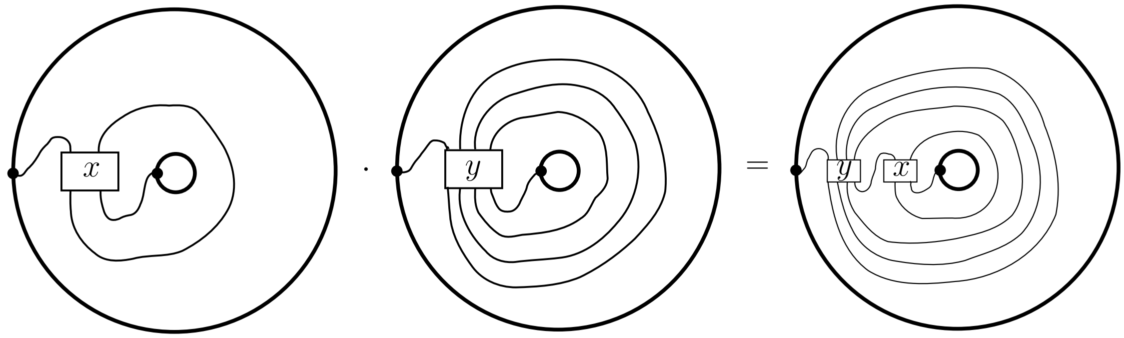


Figure 13. The product xy in $\mathcal{C}'$

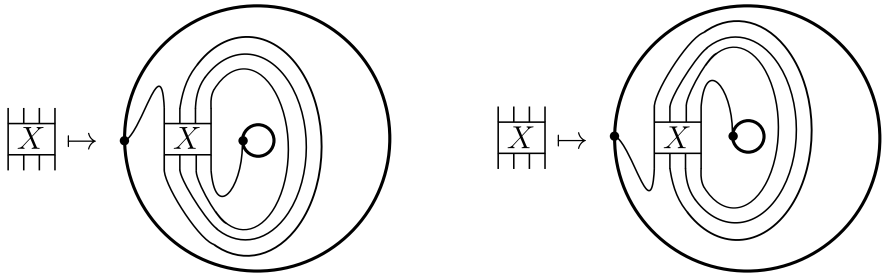
Definition 9. Let A bet the annulus with a distinguished point on each of its boundary components. We define a skein module $\mathcal{C}'$ in A as follows:

- As a set, $\mathcal{C}'$ is the set of diagrams in A with each distinguished point in A as an endpoint of some strand of the diagram, mod the local Kauffman skein relations.

- As an algebra, $\mathcal{C}'$ is formal sums of the diagrams just described, with the product XY in $\mathcal{C}'$ is given by letting the inner boundary of Y become the outer boundary of X in such a way as the distinguished points coincide. This is shown diagrammatically in Figure 13.

We call $\mathcal{C}'$ the **two pointed annulus**.

There are two natural ways to map any element $X \in \mathcal{B}_n$ into the algebra $\mathcal{C}'$ as shown in Figure 14. These maps we call $\Delta'_{l,n}$ and $\Delta'_{u,n}$. Often we will drop the n if it is not important. The is l indicating we attach the inner boundary point of A to a "lower" strand of X, and the u is for "upper". The reason for the $'$ notation is that we will reserve the notation $\mathcal{C}$ for the algebra of diagrams in the annulus (without distinguished boundary points), which we describe in Section 3.5.



The map $\Delta'_l : \mathcal{B}_n \to \mathcal{C}'$ The map $\Delta'_u : \mathcal{B}_n \to \mathcal{C}'$

Figure 14. Maps from $\mathcal{B}$ to $\mathcal{C}'$

The presence of the map Δ'_u is an essential difference between our present situation and that for the Hecke algebras. It is necessitated by the elements $h_i \in \mathcal{B}_n$. However, we will see (Lemma 12) that if one considers all of $\mathcal{B}$, then in fact $\Delta'_l(\mathcal{B}) = \Delta'_u(\mathcal{B})$. Alternatively (Lemma 14), the map Δ'_u can be seen as a consequence of the existence of the involution τ on $\mathcal{B}_n$, as the Hecke algebra does not admit such an involution.

Definition 10. In $\mathcal{C}'$, we define the element t to be the diagram consisting of a single stand that starts at the inner boundary point of A, wraps once clockwise around the core of the annulus, and connects to the outer boundary point of A. With repsect to the multiplication in $\mathcal{C}'$, the element t is invertible with inverse t^{-1} given by a strand that circles the annulus once counter-clockwise. We denote the identity in $\mathcal{C}$ by e. Each of these elements is pictured in Figure 15.

Note that $\mathcal{C}'$ contains a copy of (the additive group) $\mathbb{Z}$ via the map $a \mapsto t^a$. By convention we set $t^0 = e$. We now give some preliminary results on the structure of $\mathcal{C}'$. In Section 3.5 we will describe the structure of $\mathcal{C}'$ in detail.

Lemma 11. *For each n, let $\mathcal{B}_n$ be the BMW algebra on n strands and set $\mathcal{C}'_n = \Delta'_{l,n}(\mathcal{B}_n)$. For $Z' \in \mathcal{C}'_{l,n}$, the map $Z' \mapsto tZ'$ defines an inclusion $\mathcal{C}'_{l,n} \to \mathcal{C}'_{l,n+1}$.*

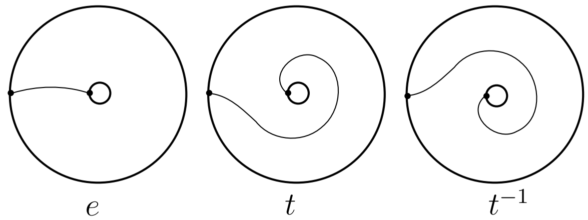


Figure 15. The identity e, and the elements t and t^{-1} of $\mathcal{C}'$.

Moreover, this map is induced by the inclusion $\mathcal{B}_n \to \mathcal{B}'_{n+1}$ given by $X \mapsto X \otimes \mathbf{1}_1$, in the sense that the following diagram commutes:

$$\begin{tikzcd} \mathcal{B}_n \arrow[r, "-\otimes \mathbf{1}_1"] \arrow[d, "\Delta'_{l,n}"'] & \mathcal{B}_{n+1} \arrow[d, "\Delta'_{l,n+1}"] \\ \mathcal{C}_n \arrow[r, "t\cdot"'] & \mathcal{C}_{n+1} \end{tikzcd}$$

Proof. The map $Z' \mapsto Z't$ is injective as t is invertible in $\mathcal{C}'$. That the diagram commutes is shown in Figure 16. □

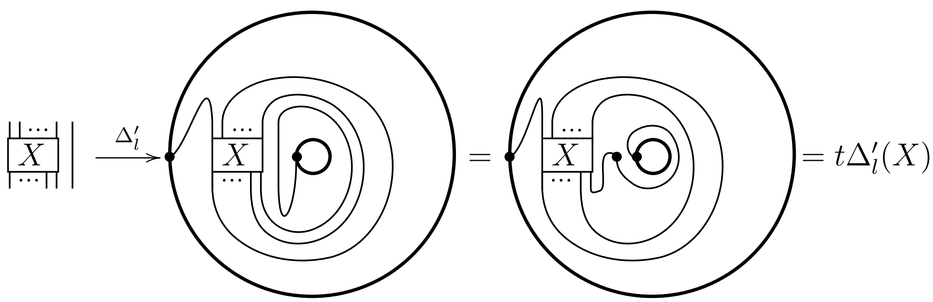


Figure 16. A proof of Lemma 11.

Lemma 12. *Let $X \in \mathcal{B}_n$. Then*

$$\Delta'_{l,n}(X)t = \Delta_{l,n+1}(\mathbf{1}_1 \otimes X) \tag{3.23}$$

$$t\Delta'_{l,n}(X) = \Delta_{l,n+1}(X \otimes \mathbf{1}_1) \text{ (This is Lemma 11)} \tag{3.24}$$

$$\Delta'_{u,n}(X) = \Delta'_{l,n+2}\left((\mathbf{1}_1 \otimes X \otimes \mathbf{1}_1)\, h_{n+1}h_n \cdots h_2\right) \tag{3.25}$$

$$\Delta'_{l,n}(X) = \Delta'_{u,n+2}\left(h_2h_3 \cdots h_{n+1}\,(\mathbf{1}_1 \otimes X \otimes \mathbf{1}_1)\right) \tag{3.26}$$

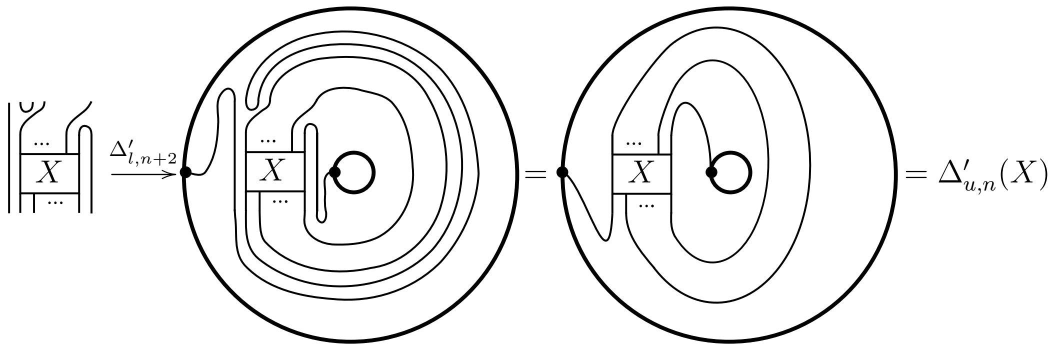


Figure 17. A proof of (3.25) from Lemma 12.

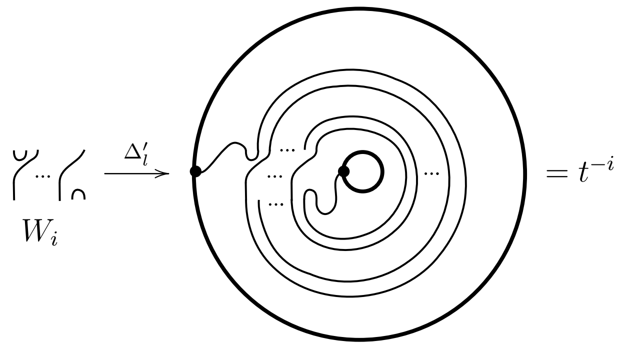


Figure 18. A proof that $\Delta'_l(W_i) = t^{-i}$.

Proof. We provide a diagrammatic proof in Figure 17 for (3.25). The others can be proved similarly.

□

Lemma 13. *Recall that we defined $W_i = h_i \cdots h_{i-1} \in \mathcal{B}_{i+1}$. In $\mathcal{C}'$ we have*

(i) $t^i = \Delta'_l(\mathbf{1}_{i+1})$ *for* $i \geq 0$

(ii) $t^{-i} = \Delta'_l(W_i)$ *for* $i > 0$.

Proof. We prove item (ii) in Figure 18. Note that W_i uses $i+1$ strands. □

We extend the involutions τ and ρ to the algebra $\mathcal{C}'$ in the natural way. We view τ as "global" in the sense that if a diagram X is in the plane or the annulus, we get

$\tau(X)$ by flipping the entire plane or annulus. The following lemma is immediate.

Lemma 14. *Let τ and ρ act on $\mathcal{C}'$ as described above. Then we have the following commutative diagrams.*

$$\begin{tikzcd} \mathcal{B} \arrow[r, "\tau"] \arrow[d, "\Delta'_l"'] & \mathcal{B} \arrow[d, "\Delta'_u"] \\ \mathcal{C}' \arrow[r, "\tau"'] & \mathcal{C}' \end{tikzcd} \qquad \begin{tikzcd} \mathcal{B} \arrow[r, "\rho"] \arrow[d, "\Delta'_l"'] & \mathcal{B} \arrow[d, "\Delta'_l"] \\ \mathcal{C}' \arrow[r, "\rho"'] & \mathcal{C}' \end{tikzcd}$$

Corollary 5. Let a_n be the quasi-idempotents defined in Proposition 2, and let f_n be the symmetrizers in $\mathcal{B}_n$. Then we have

(i) $\rho(\Delta'_l(f_n)) = \Delta'_l(f_n)$

(ii) $\rho(\Delta'_l(a_n)) = s^{2(1-n)}\Delta'_l(a_n)$

(iii) $\tau(\Delta'_l(f_n)) = \Delta'_u(f_n)$

(iv) $\tau(\Delta'_l(a_n)) = \Delta'_u(a_n)$

Proof. From Lemma 8 we know that $\rho(f_n) = f_n$, so from Lemma 14 we have $\rho(\Delta'_l(f_n)) = \Delta'_l(\rho(f_n)) = \Delta'_l(f_n)$. The other items follow similarly. □

3.5 The Structure of $\mathcal{C}'$

In this section, we prove that that the two-pointed annulus skein $\mathcal{C}'$ defined in section 3.4 is a Laurent polynomial algebra over the (usual) algebra of diagrams in the annulus.

Denote by $\mathcal{C}$ the algebra of diagrams in the annulus (not the two-pointed annulus) modulo the Kauffman skein relations (3.3) and (3.4). The algebra $\mathcal{C}$ has a commutative product given simply by drawing both diagrams in the same annulus.

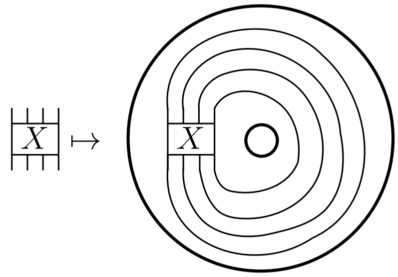


Figure 19. The map $\Delta : B_n \to \mathcal{C}$ taking $X \in B_n$ to its closure $\Delta(X) \in \mathcal{C}$.

Given any diagram $X \in \mathcal{B}_n$, we obtain an element $\Delta(X) \in \mathcal{C}$ by wrapping strand i at the top of X around the core of the annulus and attaching it to strand i at the bottom of X (without introducing any crossings). This is the usual closure operation on braids, and as such we call $\Delta(X)$ the closure of X. We view Δ as a map

$$\Delta : \mathcal{B} \to \mathcal{C},$$

and this is illustrated in Figure 3.6.1. We extend the map Δ by linearity, but note that Δ is not a map of algebras.

The algebra $\mathcal{C}$ acts on the two-pointed annulus $\mathcal{C}'$ as follows. For $X' \in \mathcal{C}'$ and $Y \in \mathcal{C}$, define $X' * Y$ to be the element of $\mathcal{C}'$ given by the diagram consisting of the diagram of Y drawn completely above the diagram of X'. Then $*$ defines a right $\mathcal{C}$-action on $\mathcal{C}'$, turning $\mathcal{C}'$ into a $\mathcal{C}$-algebra. This action was introduced, for the HOMFLY skein, by Morton in [16]. An example of this action is shown in Figure 20.

We define the map $\Theta : \mathcal{C}' \to \mathcal{C}$, by adding a strand behind a diagram in $\mathcal{C}'$ which connects the two distinguished boundary points of the annulus. This is shown in Figure 21.

We record the following Lemma, whose proof is immediate, for later use.

Lemma 15. *Viewing $\mathcal{C}'$ as a $\mathcal{C}$-algebra as above, we have* $\Theta \in \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}', \mathcal{C})$

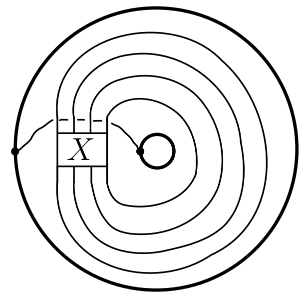


Figure 20. The diagram of $e * \Delta(X)$.

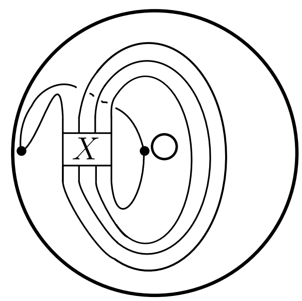


Figure 21. The diagram of $\Theta(\Delta_l'(X))$ in the annulus $\mathcal{C}$

The following fundamental structural result for the algebra $\mathcal{C}$ is due to Turaev.

Theorem 5 ([23])**.** *The algebra $\mathcal{C}$ is isomorphic to the polynomial algebra $\Lambda[X_1, X_2, X_3, \cdots]$, where the elements $X_i \in \mathcal{C}$ are given by*

$$X_i = \Delta(\sigma_i \sigma_{i-1} \cdots \sigma_1)$$

The elements X_i are shown in Figure 22.

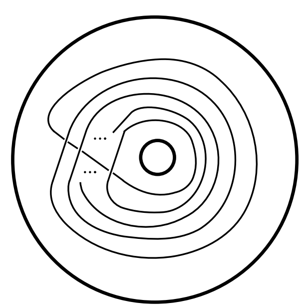


Figure 22. Turaev's generators of $\mathcal{C}$.

The significance of this result for us is the following result.

Lemma 16. *In $\mathcal{C}$, the elements $\Theta(t), \Theta(t^2), \cdots$ are algebraically independent.*

Proof. If we take $X = \mathbf{1}_{i+1} \in \mathcal{B}_{i+1}$ in Figure 3.6.1, we see that

$$\Theta\left(\Delta_l'(\mathbf{1}_{i+1})\right) = \alpha^{-1} X_i \in \mathcal{C}$$

From Lemma 13 (i), we know that $\Delta_l'(\mathbf{1}_{i+1}) = t^i$. Thus the result is immediate fromTheorem 5. $\square$

We now turn our attention to determining the structure of $\mathcal{C}'$ as a $\mathcal{C}$-module. To this end, we introduce a bit of notation. Let $\mathcal{C}_n$ be the closure in the annulus of B_n. That is, we have $\mathcal{C}_n = \Delta(B_n)$. Similarly, let $\mathcal{C}_n' = \Delta_l'(B_n)$. For fixed integers p, q, let $t^p\mathcal{C}_q$ be the Λ-submodule of $\mathcal{C}'$ generated by elements of the form $t^p * C$ for $C \in \mathcal{C}_q$. That is,

$$t^p\mathcal{C}_q = \left\{ \sum_{\text{finite}} \lambda_i (t^p * C_i) \mid \lambda_i \in \Lambda, C_i \in \mathcal{C}_q \right\}$$

We will prove the following result.

Proposition 4. As Λ-modules, we have $\mathcal{C}_n' = \displaystyle\sum_{k=0}^{n-1} \sum_{j=0}^{k} t^{k-2j} \mathcal{C}_{n-k-1}$.

Example 7. As an example of Proposition 4, consider $\sigma_1^{-1} \in B_2$. Here $n = 2$, so the proposition is claiming that

$$\Delta_l'(\sigma_1^{-1}) \in t^{-1}\mathcal{C}_0 + t\mathcal{C}_0 + t^0\mathcal{C}_1$$

Just applying Δ_l' to σ_1^{-1} gives Figure 23. Using the skein relation on the crossing gives Figure 24. The first diagram on the right hand of Figure 24 is in $t^0\mathcal{C}_1$, the second belongs to $t^1\mathcal{C}_0$, and the last is an element of $t^{-1}\mathcal{C}_0$.

We will prove Proposition 4 by induction on the number of crossings. We start with the case of zero crossings in the following lemma.

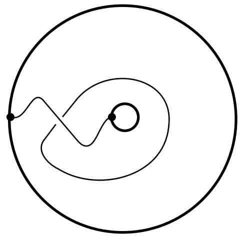
$\Delta'_l(\sigma_1^{-1}) =$ 

Figure 23. $\Delta'_l(\sigma_1)^{-1}$

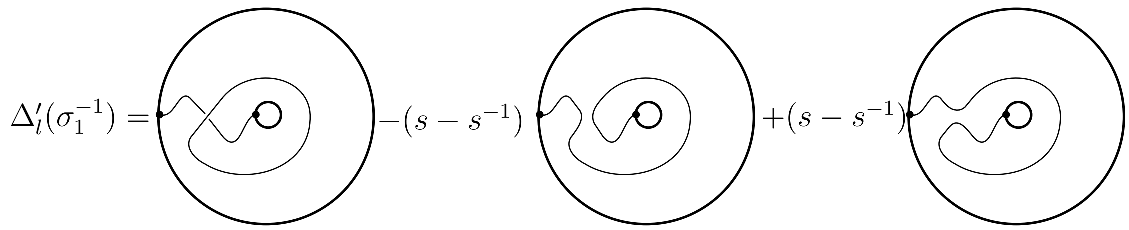


Figure 24. A diagram of $\Delta'_l(\sigma_1)^{-1}$ after applying the skein relation to the crossing.

Lemma 17. *Let $\mathcal{B}_n^0$ be the subset of $\mathcal{B}_n$ consisting of crossingless diagrams. Then*

$$\Delta'_l(\mathcal{B}_n^0) = \sum_{j=0}^{n-1} t^{n-1-2j}\mathcal{C}_0$$

Proof. First note that the word

$$\mathbf{1}_{n-1-j} \otimes W_j = h_{n-1}h_{n-2}\cdots h_{n-j} \in \mathcal{B}_n$$

has no crossings. From equation (3.23) in Lemma 12, we know that $\Delta'_l(\mathbf{1}_{n-1-j} \otimes W_j) = t^{n-1-j}\Delta'_l(W_j)$. From Lemma 13, we have $\Delta'_l(W_j) = t^{-j}$. Thus in $\mathcal{C}'$ we have

$$\Delta'_l(\mathbf{1}_{n-1-j} \otimes W_j) = t^{n-1-j}t^{-j} = t^{n-1-2j} \in t^{n-1-2j}\mathcal{C}_0.$$

Since $0 \leq j \leq n-1$, we get $\sum_{j=0}^{n-1} t^{n-1-2j}\mathcal{C}_0 \subset \Delta'_l(\mathcal{B}_n^0)$.

Next, we claim that for any n-strand crossingless diagram $D \in \mathcal{B}_n$, the element $\Delta'_l(D)$ must be of the form $t^{(n-1)-q}\mathcal{C}_0$ for some $0 \leq q \leq 2(n-1)$. If we follow the strand connected to the inner boundary point, it will wrap around the annulus at most $n-1$ times in any direction and then connect to the outer boundary point.

Moreover, this strand gives a continuous curve connecting the two boundaries of the annulus. Thus the remaining components of $\Delta_l'(D)$ must be nullhomotopic loops, since by assumption they do not cross the strand connecting the two boundaries.

It remains to be shown that q must be even. On the right half of the annulus, there are $n-1$ strands from the diagram $\Delta_l'(D)$. Label them $1,\ldots,n-1$, from left to right. Label the ones used by the boundary arc as by $n_1, n_2, \ldots, n_r$, from left to right. For convenience, label the inner boundary point $n_0 = 0$. Then $n_i - n_{i-1}$ must be odd for all i. To see this, suppose that $n_i - n_{i-1}$ was even for some i. Then the number of strands strictly between strand n_{i-1} and strand n_i is odd. But in D, the strands between n_{i-1} and n_i must all connect to each other since they can't cross the boundary strand. This is impossible since there is an odd number of them.

Now, the number of strands *not* used by the boundary arc is

$$(n_1 - n_0 - 1) + (n_2 - n_1 - 1) + \cdots + (n_r - n_{r-1} - 1) + (n - 1 - n_r - 1) = 2p$$

for some p, since each summand is even. So r, the number of strands used by the boundary arc, is given by

$$r = n - 1 - 2p.$$

Following the boundary arc from the inner boundary, each of these $n-1-2p$ strands will either be traversed going clockwise or counterclockwise. The resulting power of t corresponding to the boundary arc will be

$$\#\{\text{clockwise strands}\} - \#\{\text{counterclockwise strands}\}$$

If there are c counterclockwise strands, then there are $n-1-2p-c$ clockwise strands. So the boundary arc corresponds to t^a where

$$a = (n - 1 - 2p - c) - c = (n - 1) - 2(p + c)$$

This proves our claim with $q = 2(p + c)$. $\square$

We can now finish proving Proposition 4

Proof of Proposition 4. Let $t^{k-2j}\Delta(X) \in t^{k-2j}\mathcal{C}_{n-k-1}$, so that $X \in \mathcal{B}_{n-k-1}$. Then $(X \otimes \mathbf{1}_{k+1}) \in \mathcal{B}_n$. We have that

$$t^{k-2j}\Delta(X) = \Delta_l'((X \otimes \mathbf{1}_{k+1})h_{n-k+j-1}h_{n-k+j-2}\cdots h_{n-k}\sigma_{n-k-1}\sigma_{n-k-2}\cdots\sigma_1) \in \mathcal{C}_{l,n}'$$

(See Figure 25). For the other inclusion, take $\Delta_l'(D) \in \mathcal{C}_{l,n}'$, where D is just a single word in $\mathcal{B}_n$. To show that $\Delta_l'(D) \in \sum_{k=0}^{n-1}\sum_{j=0}^{k} t^{k-2j}\mathcal{C}_{n-k-1}$ we will use induction on the number of crossings in D, with Lemma 17 as the base case. Following the strand from the inner boundary, if this strand ever crosses over another strand, use the skein relation to resolve it. Continuing this until the outer boundary is reached, what remains is

$$\Delta_l'(D) = \Delta_l'(\widetilde{D}) + \Delta_l'(\text{diagrams with fewer crossings than } D), \tag{3.27}$$

where $\Delta_l'(\widetilde{D})$ looks just like $\Delta_l'(D)$, but the boundary strand crosses under every strand it encounters. Thus $\Delta_l'(\widetilde{D}) \in \sum_{k=0}^{n-1} t^k\mathcal{C}_{n-k-1}$ by construction. The other term in (3.27) lies in $\sum_{k=0}^{n-1}\sum_{j=0}^{k} t^{k-2j}\mathcal{C}_{n-k-1}$ by the induction hypothesis since all diagrams have fewer crossings than D.

$\square$

Observe that one corollary of the above discussion is that every element of $\Delta_l'(\mathcal{B})$ can be written as a finite sum

$$\sum_{i\in\mathbb{Z}} t^i C_i,$$

where the C_i are elements of $\mathcal{C}$. It is a somewhat surpising fact, which we prove now, that the powers of t are actually linearly independent over $\mathcal{C}$. First we need a couple of preliminary results.

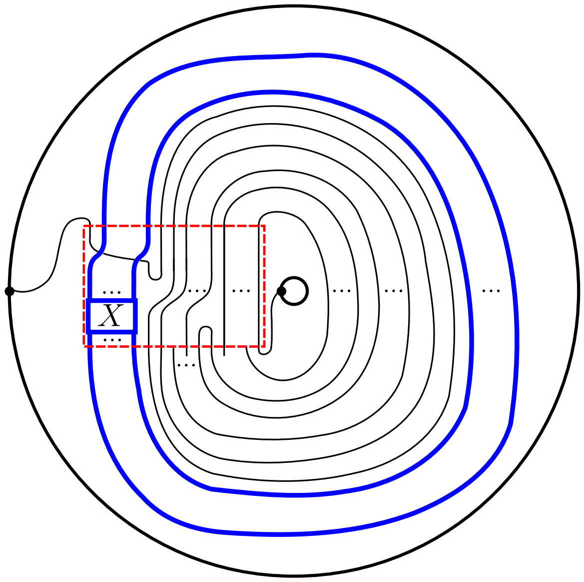


Figure 25. Diagram of $(X \otimes \mathbf{1}_{k+1})h_{n-k+j-1}h_{n-k+j-2}\cdots h_{n-k}\sigma_{n-k-1}\sigma_{n-k-2}\cdots\sigma_1$

Lemma 18. *Let R be an integral domain. If $M \in \mathrm{Mat}_{n\times n}(R)$ with $\det(M) \neq 0$, then the map $R^n \to R^n$ given by $v \mapsto Mv$ is injective.*

Proof. Let $\phi : R \to \widehat{R}$ be the embedding of R into its fraction field $\widehat{R}$. Since $\det(M) \in R$ and $\det(M) \neq 0$, we have that $\phi(\det(M))$ is invertible (nonzero) in $\widehat{R}$. Thus $\phi(M) \in \mathrm{Mat}_{n\times n}(\widehat{R})$ is invertible. If $Mv = 0$, then $\phi(M)\phi(v) = 0$, implying that $\phi(v) = 0$ and thus that $v = 0$. So $\ker(v \mapsto Mv) = \{0\}$. □

Lemma 19. *Let H be an $(m+1)\times(m+1)$ Hankel matrix (a matrix with constant anti-diagonals) with entries $H_{ij} = a_{i+j-2}, 1 \leq i, j \leq m+1$ such that the entries $\{a_k\}_{0\leq k\leq 2m}$ form an algebraically independent set in some algebra. Then the matrix H has nonzero determinant.*

Proof. The determinant of H is

$$\det(H) = \sum_{\sigma \in S_{m+1}} \operatorname{sgn}(\sigma) a_{1+\sigma(1)-2} a_{2+\sigma(2)-2} \cdots a_{m+1+\sigma(m+1)-2}$$

$$= \sum_{\sigma \in S_{m+1}} \operatorname{sgn}(\sigma) a_{\sigma(1)-1} a_{\sigma(2)} \cdots a_{\sigma(m+1)+m-1}$$

Taking σ to be the identity gives the term $a_0 a_2 a_4 \cdots a_{2m}$. We claim that this is the only permutation giving that term in the determinant. Indeed, the only solution to $\sigma(i) + i - 2 = 0$ is $\sigma(i) = i = 1$. If $\sigma(i) = i$ for all $i < k$, then the only solution to $\sigma(i) + i - 2 = 2k$ is $\sigma(i) = i = k + 1$. Indeed, the equation $\sigma(i) + i - 2 = 2k$ implies that $\sigma(i) + i = 2(k + 1)$. Since i and $\sigma(i)$ are both at least $k + 1$, then we must have $\sigma(i) = i = k + 1$.

If the determinant of H was zero, it would give $a_0 a_2 \cdots a_{2m}$ as a polynomial in the other a_i, contradicting the algebraic independence. Thus $\det(H) \neq 0$. $\square$

Proposition 5. Let x be an indeterminate and let $\mathcal{C}[x, x^{-1}]$ be the Laurent polynomial algebra in x over $\mathcal{C}$. Let $\mathcal{C}[t, t^{-1}]$ be the algebra over $\mathcal{C}$ generated by t and t^{-1} with the right action of $\mathcal{C}$ as above. Then the map $\mathcal{C}[x, x^{-1}] \to \mathcal{C}[t, t^{-1}]$ defined by $x \to t$ is an isomorphism of algebras.

Proof. The only thing in question is the injectivity of the map $x \to t$. To establish this we need to show that the powers of t are linearly independent over $\mathcal{C}$. Suppose first that

$$\sum_{i=0}^{m} t^i C_i = 0$$

for some m. Then by repeatedly multiplying through by t, we get $m + 1$ equations

$$\sum_{i=0}^{m} t^{i+k} C_i = 0, \qquad 0 \leq k \leq m.$$

Write this system of equations as $T\vec{C} = 0$, where T is the $(m + 1) \times (m + 1)$ matrix $T_{ij} = t^{i+j-2}$. From Lemma 15 we know $\Theta \in \operatorname{Hom}_{\mathcal{C}}(\mathcal{C}', \mathcal{C})$, and we can thus apply Θ to this last equation to get $\Theta(T)\vec{C} = 0$, where Θ acts element-wise on T. By Lemma 16, the elements $\Theta(t^{i+j-2})$ of $\Theta(T)$ are algebraically independent in $\mathcal{C}$. Thus by Lemma 19, the matrix $\Theta(T)$ has nonzero determinant. By Lemma 18, as $\mathcal{C}$ is an integral domain, the map given by $v \mapsto \Theta(T)v$ is injective. Thus $\vec{C} = 0$, which is what we needed to show.

Lastly, observe that if $\sum_{i=-s}^{m} t^i C_i = 0$, then we could multiply through by t^s and repeat the above argument. $\square$

From Propositions 4 and 5, we obtain the following structural result for $\mathcal{C}'$.

Theorem 6. *The two pointed annulus $\mathcal{C}'$ is the Laurent polynomial algebra over $\mathcal{C}'$ generated by t and t^{-1}. That is, we have an equality of algebras*

$$\mathcal{C}' = \mathcal{C}[t, t^{-1}].$$

3.6 Applications of the Two-Pointed Annulus Skein

In this section we investigate the behavior of the quasi-idempotents a_i in the BMW algebras under the linear map $\Delta'_l : B \to \mathcal{C}'$. Part of the reason for the effort spent deriving the recursive formula (3.20) for the quasi-idempotents was so that the present work will generalize the computations done in [10] for the symmetrizers in the Hecke algebras. As the map Δ'_l is not a map of algebras, once one passes to $\mathcal{C}'$, one loses the ability to obtain results about the Hecke algebras as a quotient of the BMW algebras.

3.6.1 Closures and Traces of the Symmetrizers

Consider a closure operation $\Diamond$ on $\mathcal{B}_n$ defined exactly as Δ in Figure , but taking place in the plane instead of the annulus. This $\Diamond$ is the usual closure operation on

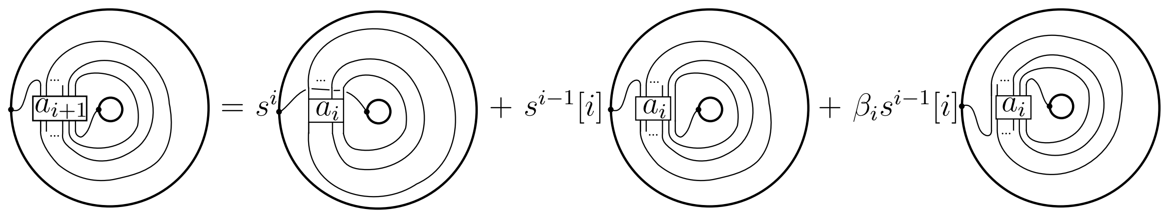


Figure 26. Recurrence relation for the $\Delta_l'(a_i) \in \mathcal{C}'$

diagrams. The image $\Diamond(X)$ is the result of embedding $\Delta(X)$ into the plane. The Kauffman polynomial of $\Diamond(X)$ is the *trace* of X, and we denote this by $\langle X \rangle$.

Using our results about $\mathcal{C}'$ will compute the traces of the symmetrizers a_i in the BMW algebras. We begin by applying Δ_l' to equation (3.20), which gives a nice recurrence in $\mathcal{C}'$ for the elements $\Delta_l'(a_i)$.

Lemma 20. *In $\mathcal{C}'$, the images of the quasi-idempotents a_i satisfy the recurrence*

$$\Delta_l'(a_{i+1}) = s^i\left(e * \Delta(a_i)\right) + s^{i-1}[i]t\Delta_l'(a_i) + s^{i-1}[i]\beta_i t^{-1}\Delta_u'(a_i) \tag{3.28}$$

This is shown in Figure 26.

Proof. Recall equation (3.20):

$$a_{i+1} = \frac{s^{i-1}[i]}{\alpha_i}(a_i \otimes \mathbf{1}_1)(\mathbf{1}_1 \otimes a_i) + s^i T_i(\mathbf{1}_1 \otimes a_i) + \frac{\beta_i s^{i-1}[i]}{\alpha_i}(a_i \otimes \mathbf{1}_1)W_i(\mathbf{1}_1 \otimes a_i),$$

where α_i is the scalar defined by $a_i^2 = \alpha_i a_i$. In Figures 27, 28, and 29, we apply Δ_l' to each term on the right hand side of (3.20). This gives

$$\Delta_l'((a_i \otimes \mathbf{1}_1)(\mathbf{1}_1 \otimes a_i)) = \alpha_i \Delta_l'(a_i \otimes \mathbf{1}_1) \tag{3.29}$$

$$\Delta_l'\left((a_i \otimes \mathbf{1}_1)W_i(\mathbf{1}_1 \otimes a_i)\right) = \alpha_i \Delta_u'(\mathbf{1}_1 \otimes a_i) \tag{3.30}$$

$$\Delta_l'\left(T_i(\mathbf{1}_1 \otimes a_i)\right) = e * \Delta(a_i) \tag{3.31}$$

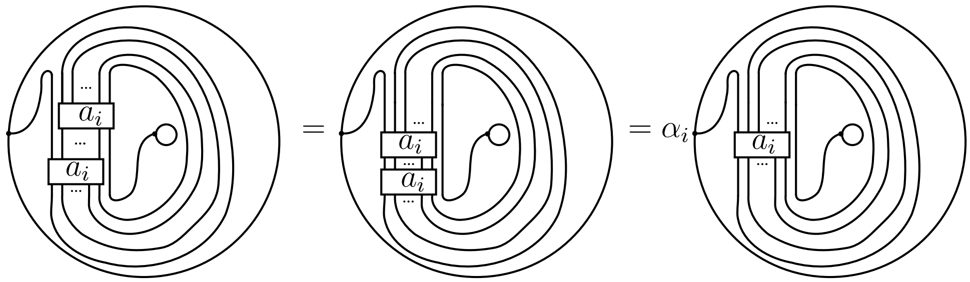


Figure 27. A proof of (3.29).

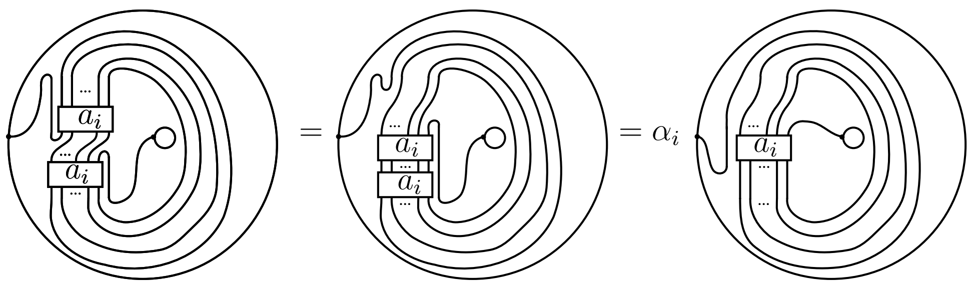


Figure 28. A proof of (3.30).

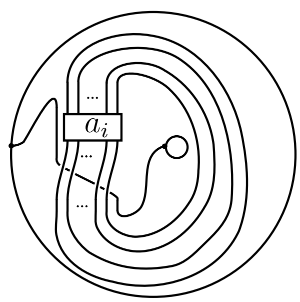


Figure 29. A proof of (3.31).

That $\Delta'_u(\mathbf{1}_1 \otimes a_i) = t^{-1}\Delta'_u(a_i)$ follows from the fact that $\mathcal{C}'$ is commutative, which is consequence of Theorem 6.

□

Remark. It is known ([10]) that if $\widetilde{a}_i$ are the (quasi-idempotent) symmetrizers in the Hecke algebra, then the analogue of equation (3.28) in the corresponding algebra $\mathcal{C}'_{\text{Hecke}}$ is

$$\Delta'(\widetilde{a}_{i+1}) = s^i \left(e * \Delta(\widetilde{a}_i)\right) + s^{i-1}[i]\Delta'(\widetilde{a}_i \otimes \mathbf{1}_1).$$

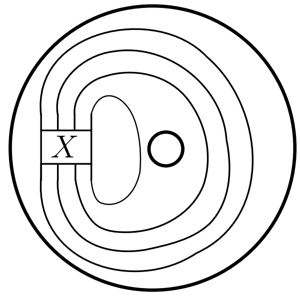


Figure 30. Diagram of $\Delta_-(X)$ in $\mathcal{C}$.

This can be readily obtained from our Corollary 4.

Recall the map Θ from Section 3.5 that adds a strand connecting the distinguished points on the boundary of the annulus. We shall apply Θ to equation (3.28). To do so we need a bit of notation. For a diagram $\Delta(X)$ in the annulus $\mathcal{C}$, let $\Delta_-(X)$ denote the diagram obtained by moving the inner most strand of X across the center boundary of the annulus. The $-$ subscript is indicating that if $X \in B_n$, we have $\Delta(X) \in \mathcal{C}_n$ and $\Delta_-(X) \in \mathcal{C}_{n-1}$. The map Δ_- is illustrated in Figure 30.

Lemma 21. *In $\mathcal{C}$ we have*

$$\Delta_-(a_{i+1}) = \left(\delta + s^{i-1}[i](\alpha s + \beta_i)\right) \Delta(a_i) \tag{3.32}$$

Proof. We compute the result of applyng Θ to each of the terms in equation (3.28).

$$\Theta(\Delta_l'(a_{i+1})) = s^{-i}\Delta_-(a_{i+1}) \qquad \text{(Figure 31)} \tag{3.33}$$

$$\Theta(\Delta_l'(a_i \otimes \mathbf{1}_1)) = \alpha^{-1}s^{-(i-1)}\Delta(a_i) \qquad \text{(Figure 32)} \tag{3.34}$$

$$\Theta(\Delta_u'(\mathbf{1}_1 \otimes a_i)) = \alpha s^{i-1}\Delta(a_i) \qquad \text{(Figure 33)} \tag{3.35}$$

$$\Theta(e * \Delta(a_i)) = \delta\Delta(a_i) \tag{3.36}$$

See the figures next to each equation. Recall that a defining property of the quasi-idempotents a_i is that they absorb a factor of $\sigma_m^{\pm 1}$ at the cost of the scalar

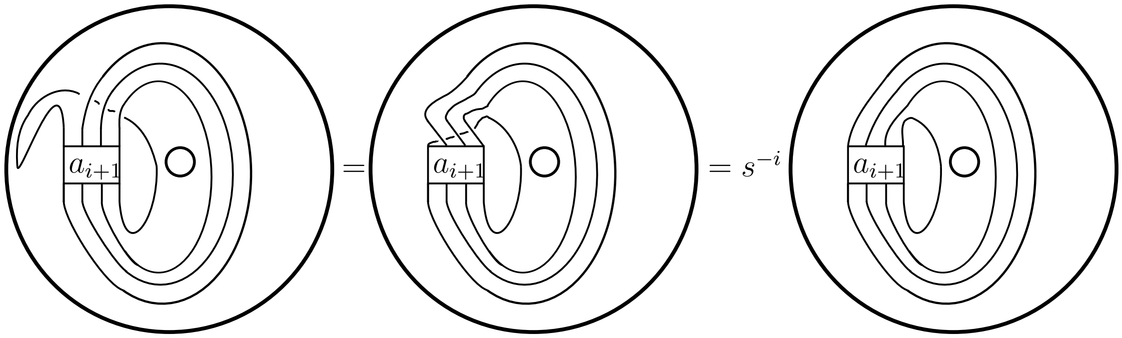


Figure 31. The map Θ applied to a_{i+1}

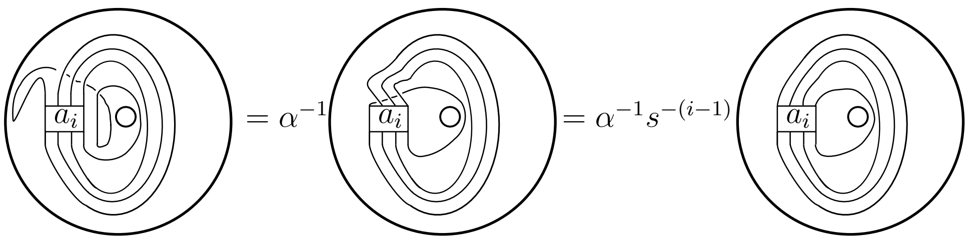


Figure 32. The map Θ applied to $\Delta_l'(a_i \otimes \mathbf{1}_1)$.

factor of $s^{\pm 1}$. Note that $\Theta(e{*}\Delta(a_i))$ is just a nullhomotopic loop behind the diagram $\Delta(a_i)$, hence equation (3.36). Equation (3.32) follows from these calculations. □

If we apply Θ to both sides of (3.28) using Lemma 21, we get

$$\Delta_-(a_{i+1}) = \left(s^{2i}\delta + \alpha^{-1}s^i[i] + \alpha s^{3i-2}[i]\beta_i\right)\Delta(a_i)$$

A little algebra reveals that the scalar factor on the right hand side of the previous

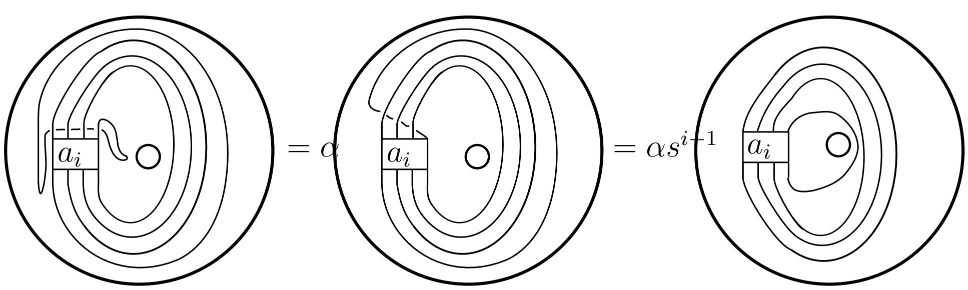


Figure 33. The map Θ applied to $\Delta_u'(\mathbf{1} \otimes a_i)$.

display can be cleaned up:

$$s^{2i}\delta + \alpha^{-1}s^{i}[i] + \alpha s^{3i-2}[i]\beta_i = \delta + s^{i-1}[i](\alpha s + \beta_i),$$

so we get a slightly nicer equation, which we record as a lemma. Equation (3.32) is an equation in the annulus $\mathcal{C}$. Recall that to calculate the trace of a diagram $X \in B_n$ we compute the Kauffman polynomial of the closure $\Diamond(X)$ of X in the plane. There is a "forgetful" map $F : \mathcal{C} \to \{\text{link diagrams in the plane}\}$ which just forgets about the annulus and is the identity on diagrams and scalars, and the diagram $\Delta_{-}(X)$ is in the fiber of F over $\Diamond(X)$. So we can apply F to both sides of (3.32) and we get the same equation in the plane:

$$\Diamond(a_{i+1}) = \left(\delta + s^{i-1}[i](\alpha s + \beta_i)\right) \Diamond(a_i) \tag{3.37}$$

If we now take the trace of the last display, we get the following formula for the Kauffman polynomial of the symmetrizers in the BMW algebra

Proposition 6. The Kauffman polynomial of the quasi-idempotents a_n in the BMW algebras satisfies the recurrence

$$\langle a_{i+1}\rangle = \left(\delta + s^{i-1}[i](\alpha s + \beta_i)\right) \langle a_i\rangle \tag{3.38}$$

Via repeated application, this gives explicit formulas

$$\langle a_n\rangle = \frac{s^{n+1}(\alpha s^{2n-1} - 1)}{\alpha^n(1-s^2)^n(\alpha - s)} \prod_{j=0}^{n-1}(1-\alpha^2 s^{2j-2}), \tag{3.39}$$

and

$$\langle f_n\rangle = [n]! \frac{s^{\frac{n(n+1)}{2}+1}(\alpha s^{2n-1} - 1)}{\alpha^n(1-s^2)^n(\alpha - s)} \prod_{j=0}^{n-1}(1-\alpha^2 s^{2j-2}),$$

where f_n is the symmetrizer in the BMW algebra $\mathcal{B}_n$.

Remark. If the product term in Proposition 6 is taken to be 1 when $n = 0$, then the formulas give $\langle a_0\rangle = \langle f_0\rangle = 1$, which is consistent with our convention that $a_0 = f_0 = 1$.

3.6.2 The Meridian Map and Decorated Hopf Links

In the introduction we remarked that the central motivation for the skein theory developed in this thesis is to obtain BMW analogs of the skein-theoretic results needed in Maulik's proof of Theorem 1. Perhaps the most important such result is a theorem of Lukac and Morton [18], which shows that a certain class of idempotents in the Hecke algebras are eigenvectors for a map known as the meridian map (which we define below). In this section we use the theory we have developed to give the first purely skein-theoretic proof that the symmetrizers in the BMW algebra are eigenvectors of the meridian map, and we compute their eigenvalues.

We shall need the notion of a left $\mathcal{C}$-action on $\mathcal{C}'$, defined analogously to the right action from Section 3.5. For $X' \in \mathcal{C}'$ and $Y \in \mathcal{C}$, we denote by $Y * X'$ the diagram in $\mathcal{C}'$ consisting of Y lying completely below X'.

Consider the map Γ defined by

$$\Gamma = \left(\Delta(B) \subset \mathcal{C} \xrightarrow{*e} \mathcal{C}' \xrightarrow{\Theta} \mathcal{C} \right)$$

Because $*e$ puts a strand connecting the boundary points of the annulus *in front* of a diagram $\Delta(X)$, and Θ connects the boundary points *behind* the boundary points, the effect of Γ is to place a closed loop around all strands of $\Delta(X)$ in the annulus. Thus we may write

$$\Gamma(\Delta(X)) = \Delta(X) \text{ encircled by a meridian loop,}$$

and we call Γ the *meridian map*. The map Γ is demonstrated in Figure 34. The main result of this section is that the closures $\Delta(a_i)$ of the quasi-idempotents are eigenvectors for the meridian map Γ. We compute their eigenvalues.

The key elementary observation is that we obtain $\Delta(a_i) * e$ from $e * \Delta(a_i)$ by

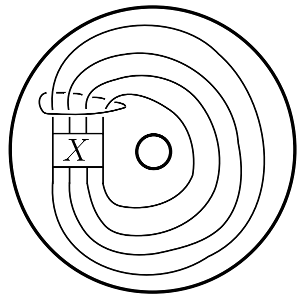


Figure 34. A diagram of $\Gamma(\Delta(X))$ in $\mathcal{C}$.

applying either of the two involutions ρ or τ:

$$\rho(e * \Delta(a_i)) = \Delta(\rho(a_i)) * e = s^{i(1-i)}\left(\Delta(a_i) * e\right) \tag{3.40}$$

and

$$\tau(e * \Delta(a_i)) = \Delta(\tau(a_i)) * e = \Delta(a_i) * e \tag{3.41}$$

We will prove the following result.

Proposition 7. In $\mathcal{C}'$, the closures of the quasi-idempotents a_i satisfy the skein relation

$$\Delta(a_i) * e - e * \Delta(a_i) = (s^i - s^{-i})\left(t^{-1}\Delta'_u(a_i) - t\Delta'_l(a_i)\right) \tag{3.42}$$

Diagramatically, this is shown in Figure 35.

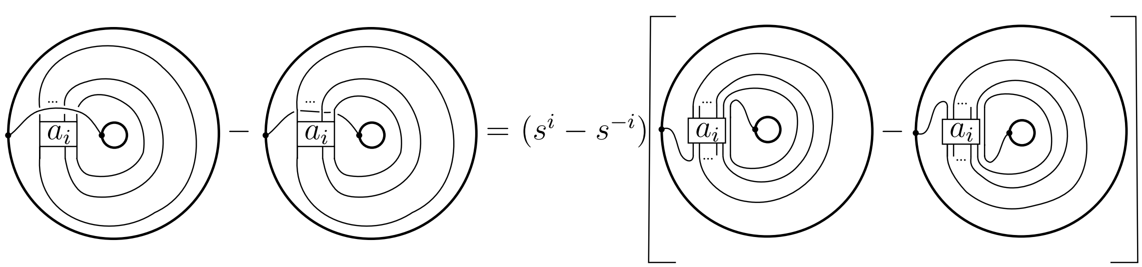


Figure 35. A diagram of the skein relation (3.42).

Before proving Proposition 7, we pause to comment on its form. One of the simplest possible "nontrivial" elements of $\mathcal{C}'$ is a single circle around the annulus that crosses a strand connecting the boundary points of the annulus. The Kauffman skein relations applied to such a diagram give the equation shown in Figure 36. Thus there is a sense in which Figure 36 is a fundamental skein relation in $\mathcal{C}'$. Since

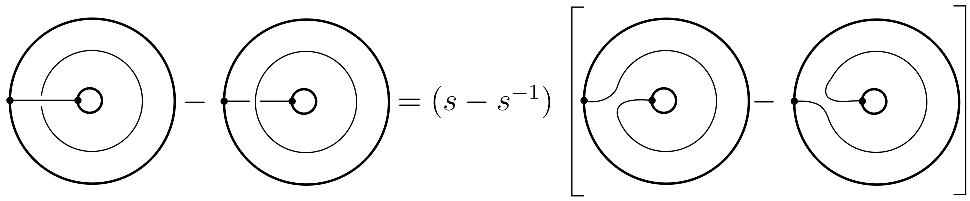


Figure 36. Basic skein relation in $\mathcal{C}'$.

$(s^i - s^{-i}) = [i](s - s^{-1})$, Proposition 7 says that the images of the symmetrizers a_i in $\mathcal{C}'$ satisfy the same fundemtal skein relation as Figure 36, with only the cost of a factor of $[i]$. Of course, Figure 36 is a special case of Proposition 7 with $i = 1$.

proof of Proposition 7. Recall the recurrence 3.28 satisfied by the a_i in $\mathcal{C}'$:

$$\Delta_l'(a_{i+1}) = s^i\,(e * \Delta(a_i)) + s^{i-1}[i]t\Delta_l'(a_i) + s^{i-1}[i]\beta_i t^{-1}\Delta_u'(a_i) \tag{3.43}$$

We apply ρ to equation (3.43). On the left hand side we get

$$\rho\,(\Delta_l'(a_{i+1})) = s^{(i+1)(1-(i+1))}\Delta_l'(a_{i+1}) = s^{-i(i+1)}\Delta_l'(a_{i+1}).$$

For the terms on the right we get

(i) $\rho(s^i(e * \Delta(a_i))) = s^{-i}s^{i(1-i)}(\Delta(a_i) * e) = s^{-i^2}(\Delta(a_i) * e)$

(ii) $\rho(s^{i-1}[i]t\Delta_l'(a_i)) = s^{1-i}s^{i(1-i)}[i]t\Delta_l'(a_i) = s^{1-i^2}[i]t\Delta_l'(a_i)$

(iii) $\rho(s^{i-1}[i]\beta_i t^{-1}\Delta_u'(a_i)) = s^{1-i^2}[i]\rho(\beta_i)t^{-1}\Delta_u'(a_i)$

Putting this all together and solving for $\Delta_l'(a_{i+1})$ gives

$$\Delta_l'(a_{i+1}) = s^i(\Delta(a_i) * e) + s^{i+1}[i]t\Delta_l'(a_i) + s^{i+1}\rho(\beta_i)[i]t^{-1}\Delta_u'(a_i). \tag{3.44}$$

Subtracting equation (3.44) from (3.43) eliminates the $\Delta'_l(a_{i+1})$ term, and we see that

$$\Delta(a_i) * e - e * \Delta(a_i) = [i](s^{-1} - s)t\Delta'_l(a_i) + [i](s^{-1}\beta_i - s\rho(\beta_i))t^{-1}\Delta'_u(a_i) \quad (3.45)$$

Some algebra shows that in fact

$$s^{-1}\beta_i - s\rho(\beta_i) = s - s^{-1},$$

which gives equation (3.42). □

Recall that the main goal of this section was to show that $\Delta(a_i)$ is an eigenvector for the meridian map Γ. This now follows as a corollary to Proposition 7

Theorem 7. *The closures $\Delta(a_i)$ of the quasi-idempotents $a_i \in \mathcal{B}_i$ are eigenvectors of the map Γ, with eigenvalue*

$$\delta + [i](s - s^{-1})(\alpha s^{i-1} - \alpha^{-1}s^{-(i-1)}).$$

That is,

$$\Gamma(\Delta(a_i)) = \left(\delta + [i](s - s^{-1})(\alpha s^{i-1} - \alpha^{-1}s^{-(i-1)})\right)\Delta(a_i)$$

Theorem 7 is shown in Figure 37, with $\gamma_i = \delta + [i](s - s^{-1})(\alpha s^{i-1} - \alpha^{-1}s^{-(i-1)})$.

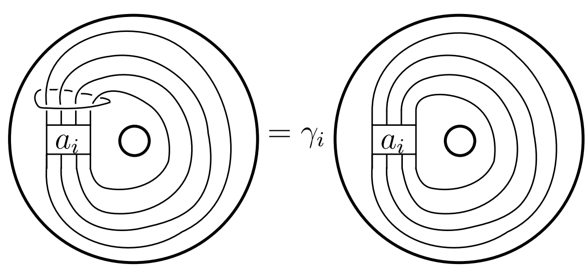


Figure 37. The symmetrizers are eigenvectors of the meridian map.

Proof. Write (3.42) as

$$\Delta(a_i) * e = e * \Delta(a_i) + (s^i - s^{-i})\left(t^{-1}\Delta'_u(a_i) - t\Delta'_l(a_i)\right)$$

Apply Θ to both sides of (3.42) Using Lemma 21. The left hand side becomes $\Gamma(\Delta(a_i))$ by definition of Γ. $\square$

Taking the trace of both sides of Lemma 7 gives the following.

Corollary 6.

$$\langle \Gamma(\Delta(a_i)) \rangle = \left(\delta + [i](s - s^{-1})(\alpha s^{i-1} - \alpha^{-1} s^{-(i-1)})\right) \langle a_i \rangle.$$

Remark. Of course, since all the maps here are linear and the a_i are scalar multiples of the symmetrizers f_i, both Theorem 7 and Corollary 6 hold with a_i replaced by f_i.

Example 8 (unknot)**.** In Corollary 6, take $i = 0$. We have $a_0 = 1$, thus $\Gamma(\Delta(a_0))$ is just a nullhomotopic loop, whose trace in the Kauffman skein is δ. Since $[0] = 0$, the right hand side of Corollary 6 is δ when $i = 0$, as required.

Example 9 (hopf link)**.** The first nontrivial case of Corollary 6 is $i = 1$. Since a_1 is a single strand, we get

$$\Gamma(\Delta(a_1)) = \text{ The Hopf link}$$

So Corollary 6 computes the trace of the Hopf link as

$$\langle \text{Hopf Link} \rangle = \delta \left(\delta + (\alpha - \alpha^{-1})(s - s^{-1})\right).$$

Example 10. We can use (3.38) in conjunction with Corollary 6 to get explicit formulas for $\langle \Gamma(\Delta(a_i)) \rangle$ for larger i. For example,

$$\begin{aligned} \langle \Gamma(\Delta(a_2)) \rangle &= \left(\delta + [2](s - s^{-1})(\alpha s - \alpha^{-1} s^{-1})\right) \langle a_2 \rangle \\ &= \left(\delta + [2](s - s^{-1})(\alpha s - \alpha^{-1} s^{-1})\right) (\delta + \alpha s + \beta_1)\, \delta \end{aligned}$$

Remark. There exists in the BMW algebra B_n an idempotent f_λ corresponding to every partition λ of n. These idempotents were defined and described in detail by Wenzl in [24] using quantum groups. In [2], these idempotents were studied skein-theoretically . In [26], the authors used the results of [2] to show that, in our notation,

$$\Gamma(\Delta(f_\lambda)) = c_\lambda \Delta(f_\lambda),$$

where

$$c_\lambda = \delta + (s - s^1)\left(\alpha \sum_{\Box\in\lambda} s^{2\,\mathrm{cn}(\Box)} - \alpha^{-1}\sum_{\Box\in\lambda} s^{-2\,\mathrm{cn}(\Box)}\right). \tag{3.46}$$

Here, the sums are over all boxes of λ, thought of as a young diagram, and $\mathrm{cn}(\Box)$ is the *content* of the box, given by

$$\mathrm{cn}(\text{box in position } (i,j)) = j - i.$$

(We take the box in the upper left to be in position $(1,1)$, i increases downward, and j increases to the right, like matrix entries). Let (i) be the partition of i whose diagram consists of a single row and i columns. Then the symmetrizers f_i fit into these more general idempotents as

$$f_i = f_{(i)}$$

Using the fact that

$$\sum_{\Box\in(i)} s^{2\,\mathrm{cn}(\Box)} = \sum_{k=0}^{i-1} s^{2k},$$

one can verify with equation (3.46) that

$$c_{(i)} = \delta + [i](s - s^{-1})\left(\alpha s^{i-1} - \alpha^{-1}s^{-(i-1)}\right),$$

which is the eigenvalue we calculated for the symmetrizers in Proposition 7.

One can imagine replacing the single strand around $\Delta(a_i)$ in Figure 37 with, $\Delta(a_j)$ (or more generally $\Delta(X)$ for any $X \in \mathcal{B}_j$ for any j). The resulting link is sometimes referred to as a decorated Hopf link, having "decorated" each component of the link with a nontrivial link. The HOMFLY polynomial of such links is computed in [18]. We believe that the techniques developed in this thesis can be pushed further to compute the Kauffman polynomial of general decorated Hopf links, of which Corollary 6 is a special case (with one component decorated by a symmetrizer). We do not see a path for the techniques used in [26] to be generalized in that direction.

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