Reduced order fluid-structure interaction models for thin shells with non-zero Gaussian curvatures to understand the response of aneurysms to flow

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Reduced order fluid-structure interaction models for thin shells with non-zero Gaussian curvatures to understand the response of aneurysms to flow

A Dissertation Presented

By

GARY HAN CHANG

Submitted to the Graduate School of the University of Massachusetts Amherst in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

September 2016

Department of Mechanical and Industrial Engineering
Reduced order fluid-structure interaction models for thin shells with non-zero Gaussian curvatures to understand the response of aneurysms to flow

A Dissertation Presented
By
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I would like to acknowledge the numerous people who helped me in the process of making this thesis and over the course of my graduate study. I would like to start with my advisor, Professor Yahya Modarres-Sadeghi, whose guidance and patience is the foundation of every single piece of my work and my gratitude is beyond my ability in expression. I would also like to express my gratitude to my committee members, Prof. Ian R. Grosse, Prof. Sanjay Raja Arwade and Dr. Clemens M. Schirmer for their advice and support in this project. I would like to thank my lab mates, Banafsheh, Pariya, Dan, Masoud and Suyue for their friendship, encouragement and inspiration over the years. It’s dangerous to go alone! I would also like to thank all the undergraduate researchers for giving me the opportunities to work with them. My friends in Amherst: thanks for your friendship and support, especially Yu-Cheng, who spends more time than he should helping me in the experimental set up. Last but not least, I would like to thank my family and my wife, Yu-Han, without their love and support I would not be able to accomplish anything.
ABSTRACT

REDUCED ORDER FLUID-STRUCTURE INTERACTION MODELS FOR THIN SHELLS WITH NON-ZERO GAUSSIAN CURVATURES TO UNDERSTAND THE RESPONSE OF ANEURYSMS TO FLOW

SEPTEMBER 2016

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In this thesis, a reduced-order model is constructed to study the physiological flow and wall shear stress conditions for aneurysms. The method of local proper orthogonal decomposition (POD) is used to construct the reduced-order modes using a series of CFD results, which are subsequently improved using a QR-factorization technique to satisfy the various boundary conditions in physiological flow problems. This method can effectively construct a computationally efficient physiological model, which allows us to examine the fluid velocities and wall shear stress distributions over a range of different physiological flow parameters.

Aneurysms are the dilation, bulging, or ballooning-out of part of the wall of a human artery. Repetitive forces on an existing aneurysm can lead either to its gradual expansion or to a devastating event of rupture. Traditionally, clinicians use the largest diameter as the sole parameter for standard risk stratification outside of clinical trials. However, there seems to
be no safe small aneurysm size and a more accurate answer depends on the exact distributions of material and geometrical properties of the aneurysms.

Aneurysms can have a very non-uniform distribution of thickness, modulus, and failure tension. We focus on the thins-shell fluid-structures interactions (FSI) with thickness inhomogeneity by studying the influence of one or multiple thin spots on the flow-induced instabilities of flexile shells of revolution with non-zero Gaussian curvatures. The results show that for shells with positive Gaussian curvatures conveying fluid, the existence of a thin spot results in a localized flow-induced buckling response of the shell in the neighborhood of the thin spot, and significantly reduces the critical flow velocity for buckling instability. For shells with negative Gaussian curvatures, the buckling response is extended along the shell’s characteristic lines and the critical flow velocity is only slightly reduced.

Finally, the shell model with non-uniform thickness is combined with the ROM of hemodynamic loads. We show that based on a growth and remodeling (G&R) model, the rate of aneurysmal wall’s elastin degradation can be predicted efficiently with various degrees of thickness inhomogeneity.
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CHAPTER 1
INTRODUCTION

1.1 Background

The goal of the thesis is to develop a reduced order model (ROM) to investigate the dynamic behavior of fluid-structure interactions (FSI) of abdominal aortic aneurysms (AAA) with non-uniform thickness in order to study the mechanisms for their formation, the rate and dynamics of their growth and the events that lead to their rupture and the risk of such a devastating event. Aneurysms are the dilation, bulging, or ballooning-out of part of the wall of a human artery. They can occur at any age, and their formation is multifactorial and results from the interplay between a biological process, likely degenerative and possibly local, in the arterial wall and hemodynamic forces and stimuli, which act on the wall. Repetitive forces on an existing aneurysm can lead either to its gradual expansion or to a sudden rupture, the latter causing a severely disabling and sometimes devastating or fatal situation. Aneurysms are largely asymptomatic and not normally detected at early stages (Nordon et al., 2010; Palombo et al., 2012).

Improvements in noninvasive imaging has led to the routine discovery of unruptured aneurysms and the clinicians are faced with the problem of classifying which aneurysms are likely to rupture and which are not. Traditionally, clinicians use the largest diameter and rate of growth as the sole parameters for standard risk stratification outside of clinical trials (Nader-
Sepahi et al., 2004). However, a more accurate answer depends on the expansion rate and also the critical size at which the aneurysm ruptures. There is consensus amongst clinicians that the risk of rupture increases with size, but there seems to be no safe small aneurysm size (Molyneux, 2002). The main questions in treating aneurysms are (Lasheras, 2007) (i) the exact pathogenesis of the arterial aneurysms, (ii) the factors that determine the growth rate, (iii) the possibility of predicting the risk of rupture quantitatively, and (iv) the patient-specific optimal treatment technique. Currently these questions are unanswered due to a lack of the fundamental knowledge needed to understand the response of the aneurysm’s flexible walls to the blood flow (Lasheras, 2007).

Recently, image based, patient-specific biomechanics is at the edge of becoming clinically practical and applicable in personalized treatments. This is made possible as a result of the advances in medical imaging technologies, computational fluid dynamics (CFD) and biomechanics. However, several obstacles still stand in the way for these techniques to be applied in the clinical treatments of patients with aneurysms because of the uncertainties involved in the medical images and the resulting computational models. A reduced order model (ROM), discussed in this thesis, is most ideal to deal with systems with model uncertainties because of its computational efficiency.

1.2 Reduced order modeling by proper orthogonal decomposition

CFD has become a useful tool in analyzing patient specific aneurysm
models to study the hemodynamic conditions and for planning surgical treatment (Suh et al., 2010; Tse et al., 2011). However, simulating physiological flows remains to be difficult for clinical usage, because of the computational difficulties and the limitations of the image-based models. In order to save computational cost, aneurysms are often treated as an isolated model and the computational domain may include the aneurysms body and several segments of the connected arteries, while in reality an aneurysm is part of the global and complicated circulatory system and the far-field influence can be generated both from upstream and downstream (Quarteroni et al., 2014). Often the analytical solution or the measurement from another site of the patient are used as boundary conditions for CFD (Olufsen et al., 2000; Quarteroni et al., 2014). The computational cost of CFD can increase enormously when FSI and the aneurysm’s material properties are considered, even with commercial software (Z. Li and Kleinstreuer, 2007; Scotti et al., 2008). This makes it even less practical for clinical usage.

This draws our interest to develop a type of reduced order model (ROM) for this problem in order to reduce the computational cost and obtain reliable results. There is always an interest in the reduced order modeling for systems with high nonlinearity and geometrical non-uniformity to replace the full, nonlinear model with a low degree-of-freedom ROM, thus reducing the overall computational cost (Dowell and Hall, 2001). This method provides a possibility to investigate a wide range of different physiological flow boundary conditions without conducting CFD simulations repetitively. Therefore, the feasibility of
using CFD results in practical usage is improved. The ROM can be subsequently used to study the hemodynamics and wall stress of aneurysms for better clinical assessment for the aneurysm’s wall stress and hemodynamic loading.

An ROM for the physiological flow can also greatly contribute to the design and optimization of cardiovascular medical devices. Typically a design process requires several evaluations of performance, cost, stability, fatigue, etc. of the system by varying the system parameters. For a structural analysis, finite-element structural models with thousands of degrees of freedom can often be reduced to a simplified model with a few structural eigenmodes and the performance of the structure can be subsequently evaluated much faster (Dowell and Hall, 2001). This approach has also been used for fluid simulations by applying the method of proper orthogonal decomposition (POD) to CFD results. A typical CFD model with several millions of degrees of freedom can be reduced to a model containing only hundreds of flow modes by POD, which clearly reduces the computational cost in the engineering design process significantly (Liberge and Hamdouni, 2014; Rapún and Vega, 2010). ROM has been utilized in many design and optimization problems involving fluid-structure interactions (Dowell and Hall, 2001; Lassila et al., 2014). As an example, in airfoil design, POD methods have often been utilized with the multi-objective design exploration of supersonic or transonic airfoil shapes based on the surface pressure data from CFD to minimize the drag-to-lift ratio (Oyama et al., 2012). POD model has also been applied to study the 3D flow field around a horizontal axis wind turbine (HAWT) and when
compared to the experiment data, the result from the POD models have shown to be more precise than the 2D inviscid data that is used in traditional HAWT design process (Wang et al., 2012). The POD-based approaches often outperform the traditional optimization algorithms in design and optimization. POD-based models can also be used for determining the dominating flow pattern that is responsible for the loss of flow instability (Rudolf and Štefan, 2012).

1.3 Stress estimation of aneurysmal wall with non-uniform thickness

Computational biomechanics have been used to approximate the wall stress and potential rupture locations of human abdominal aneurysms (Doyle et al., 2010). However, patient-specific properties of the aneurysm's tissues, such as the thickness, strain-stress relations, and tensile and flexural modulus are still difficult to be obtained for biomechanical simulations with the current medical images techniques. It has been found that an aneurysm can have a very non-uniform distribution of thickness, modulus, and failure tension, especially in the neighborhood of the rupture site. It is reported that the measured thickness of the aneurysm's sidewall can be reduced by 75% within several millimeters while the difference between maximum and minimum thickness can be more than ten times (Raghavan et al., 2006). Ruptured aneurysms are found not to be globally weaker than unruptured ones (Raghavan et al., 2011). Therefore, an efficient numerical tool to model the localized inhomogeneity in the thickness of aneurysm wall is necessary.
In practice, constant thickness or a simple thickness-radius relation is often assumed in FEM based simulations to study an aneurysm’s in-plane stress and deformations under hemodynamics loading. These simplified models often result in overly gradual, or sometimes totally erroneous stress distributions when compared to models with non-uniform thickness distribution, which, alas, requires measurement from a harvested aneurysm (Raghavan et al., 2006) or scanning images with ultra high grid resolution (Raut et al., 2015) (Doyle et al., 2010).

It has also been shown that the magnitude of pressure-induced wall stress is closely related to the curvatures of the aneurysmal wall (Lu et al., 2013). With the same thickness and pressure loading, aneurysmal wall with positive Gaussian curvatures has its two in-plane stress components accumulated to resist the loading on the normal direction, and aneurysmal wall with negative Gaussian curvatures has its two in-plane stress components compete with each other. Therefore, high magnitude of tension can often be observed in the locations with negative curvatures. Also, abdominal aneurysms usually have larger thickness to radius ratios, therefore the bending stress, which is proportional to the cubic power of the thickness, is significantly higher than the bending stress in cerebral aneurysms. This further increases the demand of spatial and temporal resolutions for patient-specific scanning images in order to have a better description in the aneurysm’s surface curvature for the aneurysm’s in-plane stress to be accurately estimated.
1.4 Thesis organization

This thesis is organized as follows:

Chapter 2 and 3 discuss the fundamental FSI problem of inhomogeneous thin shell structures. The localized deformation of membrane or thin shell structures induced by geometrical and material imperfections and internal pressure has been widely studied numerically and analytically and is used to model the initiation of aneurysms (Fu et al., 2012) (Alhayani et al., 2013). On the other hand, buckling instabilities can also be induced in thin shell structure conveying fluid, and the large stress generated by the buckling deformation is also anticipated to be related to the initiation of aneurysms (Amabili et al., 2012). Most of the existing studied are limited to cylindrical shells with zero Gaussian curvatures and as mentioned earlier, the critical point of instability and how the deformation propagates on the shell's surface are decided by the value of Gaussian curvature. Therefore, it is to our interest to study the fluid-induced instabilities of shells with non-zero Gaussian curvatures with imperfections.

Chapter 2 discusses the flow-induced instabilities of shells of revolution with a non-zero Gaussian curvature. We consider an FSI model based on an inviscid flow model and a linear thin shell theory. This FSI model is solved by the Galerkin technique. The flow-induced instabilities of shells of revolution with non-uniform cross-sections are investigated and it is shown that their dynamic behavior is closely related to their values of Gaussian curvatures.
Chapter 3 further extends the discussion of flow-induced instabilities of shells with non-zero Gaussian curvatures to the cases with localized imperfections. It is shown that for shells with positive Gaussian curvatures, a thin spot creates a localized deformation and greatly decreases the onset of stability of the structure, while for shells with negative Gaussian curvatures, the influence of a thin spot extends along its characteristics and forms large scale deformations.

In Chapter 4, we construct a ROM model of the physiological flow distributions and loadings inside the abdominal aortic aneurysms based on computational fluid dynamics and proper orthogonal decomposition. Here, we construct ROM models by the POD method to estimate the flow-induced wall shear stress and pressure loading of a simplified abdominal aortic aneurysm. This method allows us to investigate a wide domain of different physiological flow parameters (inflow angle, patient at rest or exercise) without conducting the computationally expensive CFD simulations repetitively, and therefore this method is promising for clinical usage.

Finally, in Chapter 5, the reduced order model for hemodynamic load discussed in Chapter 4 is combined with the computational model for shell structures with non-uniform thickness to model the growth and remodeling (G&R) process of AAAs. The potential locations of high deformation in the G&R process can be recognized efficiently based on the magnitude of wall curvature, the estimated WSS, and the location of elastin degradation.
CHAPTER 2

FLOW-INDUCED INSTABILITIES OF SHELLS OF REVOLUTION WITH NON-ZERO GAUSSIAN CURVATURES CONVEYING FLUID

In this chapter, we study flow-induced instabilities of axis-symmetric shells of revolution with an arbitrary meridian and non-zero Gaussian curvatures. We consider a fluid-structure interaction (FSI) model based on an inviscid flow model and a thin shell theory. This FSI model is solved using a method that combines the Galerkin technique with the boundary element method (BEM). The present method is capable of investigating the dynamic behavior of doubly-curved shells in contact with flow without the need for an analytical solution of the perturbed flow potential. Shells of revolution with different values of non-zero Gaussian curvatures are investigated and their behavior is compared to shells with zero Gaussian curvature. It is found that the added mass natural frequencies of shells of revolution are larger than those of conical shells with the same inlet, outlet and length. Shells of revolution, with both positive and negative Gaussian curvatures, lose their instability by buckling, however, shells with negative Gaussian curvatures buckle at modes similar to those observed in uniform and conical shells, while shells with positive Gaussian curvatures buckle with localized deformations close to the area with higher local flow velocities.

(This chapter is published in Chang, Gary Han, and Yahya Modarres-Sadeghi. "Flow-induced instabilities of shells of revolution with non-zero Gaussian curvatures conveying fluid." Journal of Sound and Vibration 363 (2016): 600-612.)
2.1 Introduction

The dynamics of shell structures in contact with fluid flow have been studied extensively both experimentally and theoretically because of the applications of such structures in engineering and biomechanics systems. The main focus of the existing studies is on the problem of cylindrical shells with a uniform circular cross-section. These studies have been discussed comprehensively in recent books by Paidoussis and Amabili (Amabili, 2009; Païdoussis, 1998). Recently, there has been an increasing interest in understanding the dynamics of shells with non-uniform cross-sections conveying fluid, with a focus on conical shells.

Thin-walled conical shells have several important applications in submarines and offshore drilling rigs. Kurma and Ganesan used a finite element method (FEM) to study the dynamics of conical shells conveying fluid with various semi-vertex angles (Senthil Kumar and Ganesan, 2008). They found that there is a correlation between the shells’ circumferential buckling mode and the circumferential mode with the lowest added mass frequency (the natural frequency of the shell filled with fluid). Kerboua et al. used a semi-analytical FEM to study this system (Kerboua et al., 2010). The displacement functions of the structure were derived from the exact Sander’s thin shell equation for conical shells, while the flow potential solutions were written in polynomial expansions, based on Frobenius method. Bochkarev and Matveenko studied the dynamic behavior of conical shells conveying fluid using the same inviscid fluid model and with different boundary conditions.
for the perturbed flow potential (Bochkarev and Matveenko, 2011). They found that conical shells conveying fluid can undergo flutter or buckling instabilities, depending on the shell’s semi-vertex angle and boundary conditions. The aeroelasticity problem of conical shells subjected to supersonic flow has also been investigated by several researchers (Mason and Blotter, 1986; Pidaparti and Yang, 1993). Usually, the linear piston theory for supersonic flow is utilized in the aeroelastic models.

Shells of revolution are extensively used in different systems, such as pressure vessels and rocket nozzles. The existing studies on shells of revolution are mainly based on FEM (Gould and Leissa, 1986; Srivastava, 1986) or generalized differential quadrature method (Artioli and Viola, 2006; Tornabene et al., 2011). While there is a fair amount of literature on dynamics of shells with cylindrical or conical geometry conveying fluid, the studies on shells of revolution with an arbitrary meridian conveying fluid is quite limited. Ventsel et al. combined the Boundary Element Method (BEM) and FEM to investigate the dynamics of shells of revolution filled with fluid (Ventsel et al., 2010). They studied the effect of added mass of the fluid on the natural frequency of the shell of revolution and their vibration modes. Menaa and Lakis studied the supersonic flutter of a spherical shell with a hybrid FEM method and first-order piston theory (Menaa and Lakis, 2014). They found that by increasing the radius to thickness ratio of a spherical shell, flutter occurs at a higher dynamic pressure.
In this chapter, the flow-induced instabilities of shells of revolution with an arbitrary meridian conveying fluid are studied. The main focus of this study is on doubly-curved shells with non-zero Gaussian curvature. The present algorithm combines the Galerkin’s method with BEM, which is used to determine the induced flow pressure on the shell’s inner wall. Because no analytical solution for the perturbed flow potential is required, the present method is shown to be versatile when studying the instability of shells of revolution with non-uniform cross-sections.

2.2 Equations of motion for a thin shell of revolution

As a first step toward studying the flow-induced instabilities of shells of revolution, we need to calculate the natural frequencies and natural modes of shells of revolution, using their equations of motion. Equations of motion describing the shell structure can be obtained from the doubly-curved thin shell theory, and they have been widely studied (Soedel, 2004; Ventsel et al., 2010). Here we give a short summary of how these equations are derived. The Lagrangian of a shell of revolution is calculated based on the shell’s strain-displacement relation at the shell’s mid-plane by utilizing the thin-shell theory and Love’s hypothesis.
Figure 2-1 Schematic of a shell of revolution.

Figure 2-1 shows a schematic of a shell of revolution and its corresponding coordinate system. In this figure, \( x \) is the axis of revolution, \( a_1 \) is the inlet radius, \( a_2 \) is the outlet radius, \( s \) is the curvilinear coordinate following the shell’s meridian, \( k = k(x) = 1/R_x(x) \) is the principal curvature of the meridian, \( O_x \) is the center of curvature of the meridian, and \( k = k(q) = 1/R_q(q) \) is the principal curvature of the parallel circles. For a shell of revolution with non-uniform cross-sectional area, both principal curvatures are functions of the \( x \)-axis. If the shell is thin and zero strain values are assumed for small rigid body motions, Love’s hypothesis can be applied (Soedel, 2004):

\[
\begin{align*}
\varepsilon_{xx} &= \varepsilon_{x,0} + zk_x, \\
\varepsilon_{qq} &= \varepsilon_{q,0} + zk_q, \\
g_{xq} &= g_{xq,0} + zk_{xq}.
\end{align*}
\]  

(2.1)
where \( \varepsilon_{x,0} \) and \( \varepsilon_{x,0} \) are the strain components in the mid-plane of the shell, and \( \varepsilon_{x,0} \) and \( \varepsilon_{x,0} \) are the strain components in an arbitrary position. The mid-plane strain-displacement relations of shells of revolution are (Soedel, 2004):

\[
\begin{align*}
\varepsilon_{x,0} &= \frac{1}{A_x} \frac{\partial u}{\partial x} + \frac{1}{A_x A} \frac{\partial A}{\partial R_x} v + w, \\
\varepsilon_{q,0} &= \frac{1}{A_q} \frac{\partial v}{\partial q} + \frac{1}{A_q A_x} \frac{\partial A}{\partial R_q} u + w, \\
\gamma_{xq,0} &= \frac{1}{A_x} \frac{\partial v}{\partial x} + \frac{1}{A_q} \frac{\partial u}{\partial q} - \frac{1}{A_x A_q} \frac{\partial A}{\partial R_x} u - \frac{1}{A_x A_q} \frac{\partial A}{\partial R_q} v,
\end{align*}
\]

where \( u = u(s, \theta) \), \( v = v(s, \theta) \) and \( w = w(s, \theta) \) are the axial, circumferential and radial displacements, respectively. The curvature components in each direction are:

\[
\begin{align*}
k_{xx} &= \frac{w}{R_x R_q} + \frac{1}{A_x A_R_q} \frac{\partial A}{\partial R_x} u + \frac{1}{A_x A_R} \frac{\partial A}{\partial R} v + \frac{1}{A_x A_R} \frac{\partial u}{\partial x} + \frac{1}{A_x A_R} \frac{\partial v}{\partial x} \\
&+ \frac{1}{A_x A^2} \frac{\partial A}{\partial x} w + \frac{1}{A_x A^3} \frac{\partial A}{\partial x} \frac{\partial w}{\partial x}, \\
k &= \frac{w}{R_x R_q} + \frac{1}{A_q A_R_x} \frac{\partial A}{\partial R_q} u + \frac{1}{A_q A_R} \frac{\partial A}{\partial R} v + \frac{1}{A_q A_R} \frac{\partial u}{\partial q} + \frac{1}{A_q A_R} \frac{\partial v}{\partial q} \\
&+ \frac{1}{A_q A^2} \frac{\partial A}{\partial q} w + \frac{1}{A_q A^3} \frac{\partial A}{\partial q} \frac{\partial w}{\partial q}, \\
k_x &= \frac{1}{A} \frac{\partial u}{\partial x} \left( \frac{1}{R_x} - \frac{1}{R_q} \right) + \frac{1}{A_q} \frac{\partial v}{\partial x} \left( \frac{1}{R_q} - \frac{1}{R_x} \right) + \frac{2}{A_x A^2} \frac{\partial A}{\partial x} \frac{\partial w}{\partial x} \\
&+ \frac{2}{A_x A^3} \frac{\partial A}{\partial x} \frac{\partial w}{\partial x},
\end{align*}
\]

where \( A_x \) and \( A \) are the Lamé parameters, which can be directly found from
the principal curvature (Soedel, 2004):

$$A_x = \sqrt{1 + \left( \frac{dR_x}{dx} \right)^2}, \quad A = R.$$  \hspace{1cm} (2.4)

Now with the strain-stress relation in hand, the system's Lagrangian, $L = U - T$, where $U$ is the potential energy and $T$ is the kinetic energy, is found to be (Soedel, 2004):

\[
L = \int_s \left( N_x \varepsilon_x + N_\theta \varepsilon_\theta + N_{x\theta} \varepsilon_{x\theta} + M_x k_x + M_\theta k_\theta + M_{x\theta} k_{x\theta} \right) dx d\theta - \frac{h}{s} \int \left( \ddot{u}^2 + \dot{v}^2 + \dot{w}^2 \right) dx d\theta,
\]  \hspace{1cm} (2.5)

where $N_x$, $N$ and $N_x$ are the resultant forces; $M_x$, $M$ and $M_x$ are the resultant moments, and $h$ is the shell's thickness. The resultant forces and moments in different directions are obtained by integrating the strain tensors over the thickness of the shell (Soedel, 2004):

\[
\begin{align*}
N_{xx} &= \int_{h/2}^{h/2} (E / (1 - v^2))(x_{xx} + v), \\
N_x &= \int_{h/2}^{h/2} (E / (1 - v^2))(x_{xx} + v), \\
N_{x\theta} &= \int_{h/2}^{h/2} (E / (1 - v^2))(x_{x\theta} + v), \\
N_\theta &= \int_{h/2}^{h/2} (E / (1 - v^2))(x_{\theta\theta} + v), \\
M_{xx} &= \int_{h/2}^{h/2} x_{xx} dz, \\
M_x &= \int_{h/2}^{h/2} x_{xx} dz, \\
M_{x\theta} &= \int_{h/2}^{h/2} x_{x\theta} dz, \\
M_\theta &= \int_{h/2}^{h/2} x_{\theta\theta} dz, \\
M_{x\theta} &= \int_{h/2}^{h/2} x_{x\theta} dz.
\end{align*}
\]  \hspace{1cm} (2.6)

Finally, the equations of motion of the shells of revolution are found using Hamilton's principle:
\[ \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0, \]  

where \( L \) is the Lagrangian of the system found in Equation (2.5), and \( q \) is the generalized coordinate in the axial, circumferential or radial direction.

### 2.3 The method used to find the natural frequencies and mode shapes

The Rayleigh-Ritz method is popular for calculating the mode shapes and natural frequencies of thin panel or shell structures, especially when the meridians of the shells of revolution can be expressed as a smooth function. In the present study, we consider a shell of revolution with clamped-clamped boundary conditions. The approximate solutions, which are assumed to be a combination of trial functions fitting the appropriate boundary conditions, are inserted into the Lagrangian of Equation (2.5). For a clamped-clamped shell, the boundary conditions are:

\[
\begin{align*}
    u(0, q) &= 0, \quad v(0, q) = 0, \quad w(0, q) = 0, \quad \frac{\partial w(0, q)}{\partial x} = 0, \\
    u(L, q) &= 0, \quad v(L, q) = 0, \quad w(L, q) = 0, \quad \frac{\partial w(L, q)}{\partial x} = 0,
\end{align*}
\]

and the trial solutions can be selected to be (Lam and Hua, 1999):
\[ u = \sum_{m,n} a_{m,n}^u \sin \left( \frac{m}{L} x \right) \cos(n), \]
\[ v = \sum_{m,n} a_{m,n}^v \sin \left( \frac{m}{L} x \right) \sin(n), \]
\[ w = \sum_{m,n} a_{m,n}^w \left[ \sinh \left( \frac{m}{L} x \right) \sin \left( \frac{n}{L} x \right) \right] \cos(n), \]
\[ \frac{\sinh m}{\cos m} = \frac{\sin m}{\cosh m}, \]
\[ \frac{\cos m}{\cosh m} \approx 1. \]

The integers, \( m \) and \( n \), are the axial half-wave and circumferential wave numbers, respectively, and \( a_{m,n}^u, a_{m,n}^v \) are the generalized coordinates of each mode in the axial and circumferential directions, respectively, with different wave numbers (\( m \) and \( n \)).

The coupled mode shapes and eigenfrequencies of the shell can then be found by finding the coefficients to minimize the Lagrangian (F.-M. Li et al., 2009), that is:
\[ \frac{L}{a_{m,n}^u} = 0, \quad \frac{L}{a_{m,n}^v} = 0, \quad \frac{L}{a_{m,n}^w} = 0. \]

This leads to an eigenvalue problem:
\[(M^2 + K)a = 0,\]

(2.12)

where \(a\) is a vector containing the values of \(a_{m,n}^a\) and \(a_{m,n}^v\), and \(M\) and \(K\) are the inertia and stiffness matrices, respectively. The natural frequencies and mode shapes of the shells are subsequently obtained by finding the eigenvectors and eigenvalues in Equation (2.12).

### 2.4 Natural frequencies and mode shapes of conical shells

In order to validate the present method used for shells of revolution, first the Rayleigh-Ritz method is applied to conical shells, which have been studied in the literature (Bochkarev and Matveenko, 2011; Kerboua et al., 2010; Senthil Kumar and Ganesan, 2008). As a special case of shells with axisymmetric geometry, a conical shell has its radius of curvature in the axial direction linearly increasing, while the other radius goes to infinity \((R_x \to \infty)\). The radii of curvature, \(R_x\) and \(R\), and the Lamé parameters, \(A_x\) and \(A\), are related to the semi-vertex cone angle of the conical shell, \(\theta_0\), as (Soedel, 2004):

\[
1/R_x = 0, \quad 1/R = x \tan \theta_0, \quad A_x = 1, \quad A = x \sin \theta_0.
\]

(2.13)

The strain-displacement relations are consequently found to be (Soedel, 2004):
\[
x = \frac{\partial u}{\partial x} z \frac{\partial^2 w}{\partial x^2},
\]
\[
= \frac{1}{x \sin \alpha_0} \frac{\partial v}{\partial \alpha} + \frac{u}{x} + \frac{w}{x \tan \alpha_0} + \frac{1}{x \sin \alpha_0} \frac{1}{\partial \alpha^2} \frac{\partial^2 w}{\partial \alpha^2} + \frac{1}{x \partial \alpha} \frac{\partial w}{\partial \alpha},
\]
\[
\cdot \left( \frac{1}{x^2 \sin \alpha_0 \tan \alpha_0} \frac{\partial v}{\partial \alpha} + \frac{1}{x^2 \sin \alpha_0^2} \frac{\partial^2 w}{\partial \alpha^2} + \frac{1}{x^2 \tan \alpha_0} \frac{\partial w}{\partial \alpha} \right).
\]

In Table 2.1, the dimensionless coupled natural frequencies of a clamped-clamped conical shell with \( h/a_2 = 0.01 \), an aspect ratio of \( (l - l_0) \sin \alpha_0 / a_2 \) of 0.50 and semi-vertex angles, \( \alpha_0 \), of 45° or 60° are calculated using the present model and compared with the results from previous studies. The dimensionless natural frequency here is defined as
\[
f = a_2 \sqrt{\frac{(1 - v^2)}{E}},
\]
where \( f \) is the natural frequency in radians per second, \( a_2 \) is the largest radius of the conical shell, \( \rho \) is the density, \( v \) is the Poisson’s ratio and \( E \) is the Young’s modulus. Three sets of results based on different numerical methods (Lam and Li used Galerkin’s method; Irie et al. used transfer matrix approach; and the present method) are in agreement.
<table>
<thead>
<tr>
<th>n</th>
<th>Lam &amp; Li</th>
<th>Irie et al.</th>
<th>Present</th>
<th>Lam &amp; Li</th>
<th>Irie et al.</th>
<th>Present</th>
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<tr>
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<td>0.4098</td>
<td>0.4202</td>
<td>0.4093</td>
<td>0.4095</td>
</tr>
</tbody>
</table>

Table 2.1 Dimensionless natural frequencies of a clamped-clamped conical shell with \( h/a_2 = 0.01 \) and \( (l - l_0) \sin \alpha_0 /a_2 = 0.5 \), found here and those found in the literature.

2.5 Flow forces inside a shell of revolution

In order to model the flow forces inside a shell of revolution, the flow is assumed to be inviscid, incompressible, and modeled by the potential flow theory. Assuming that there is no separation and the shell’s sidewall is always in contact with flow, the boundary condition can be represented as:

\[
\frac{n}{n_{S_i}} = \frac{W}{t} + U(s) \frac{W}{s}, \tag{2.15}
\]

where \( n \) is the perturbed flow potential, \( S_i \) denotes the shell’s sidewall, \( U = U(s) \) is the mean flow velocity when the shell is not moving, and \( W \) is the shell’s displacement in the normal direction. The first term on the right-hand side of Equation (2.15) is related to the added mass from the flow. The second
The term is associated with the lateral flow velocity which is induced by the slope of the structure \((W/s)\) of the shell’s surface.

In order to solve the Neumann boundary condition problem of Equation (2.15), we take advantage of the superposition principle of Laplace’s equation, and assume that the flow potential can be separated into two parts \((j = j_a + j_b)\) without loss of generality. The shell displacement in the normal direction, \(W\), is separated into the mode shape \((w(s, q))\) times the generalized coordinate \((q(t)):\)

\[
W(s, q, t) = w(s, q)q(t).
\]  

(2.16)

The two parts of the flow potential, \(j_a\) and \(j_b\), are assigned to satisfy the first and second parts of the boundary conditions in Equation (2.15), respectively:

\[
\frac{\partial j_a}{\partial n}_{|_{S_i}} = \frac{W}{t} + w(s, q)\frac{q(t)}{t},
\]  

(2.17a)

\[
\frac{\partial j_b}{\partial n}_{|_{S_i}} = \frac{Q}{A(s)}\frac{\partial W}{\partial s} = Q\left[\frac{1}{A(s)}\frac{\partial w(s, q)}{\partial s}\right]q(t),
\]  

(2.17b)

where \(A = A(s)\) is the shell’s cross-sectional area and \(Q\) is the flow discharge rate. The solutions of Equation (2.17) can be expressed as:
\[ a \big|_{S_a} = A(s, q) \frac{q(t)}{t}, \]  
\[ (2.18a) \]

\[ b \big|_{S_b} = Q_B(s, q)q(t). \]  
\[ (2.18b) \]

The boundary conditions at the inlet and outlet of the shell are chosen to not allow nonzero axial flow velocity:

\[ \frac{a}{n}_{S_0} = 0, \]  
\[ (2.19a) \]

\[ \frac{b}{n}_{S_0} = 0. \]  
\[ (2.19b) \]

The two-dimensional shape functions, \( A(s, q) \) and \( B(s, q) \), are calculated using the boundary element method as discussed in Section 3.6.

### 2.6 Solving the flow potential using the boundary element method

Equation (3.17) and (2.18) separate the original boundary condition problem (Equation (2.15)) into two Neumann problems, and are subsequently solved using BEM formulation separately (Pozrikidis, 2002):

\[
2\pi\phi(\bar{x}_0) + \sum_{i=1}^{\partial} \phi \frac{\partial}{\partial n} \left( \frac{1}{|\bar{x}_0 - \bar{x}|} \right) dS_{ij} - \sum_{j=1}^{\partial} \chi \frac{1}{|\bar{x}_0 - \bar{x}|} dS_{1,j} = 0, \bar{x}_0 \in S_1, \tag{2.20}
\]

\[
\sum_{i=1}^{\partial} \phi \frac{\partial}{\partial n} \left( \frac{1}{|\bar{x}_0 - \bar{x}|} \right) dS_{ij} - \sum_{j=1}^{\partial} \chi \frac{1}{|\bar{x}_0 - \bar{x}|} dS_{1,j} = 0, \bar{x}_0 \in S_0. \tag{2.21}
\]
in which, $\vec{x}_o$ and $\vec{x}$ are 2-Dimensional vectors pointing to a location on the shell’s boundaries ($S_0$, the shell’s inlet and outlet, and $S_1$, the shell’s sidewall);

is the perturbed flow potential to be solved ($j_a$ and $j_b$ in Equation (2.17));

$O$ is the number of surface elements in $S_1$; $X$ represents the boundary conditions in the right-hand side of Equation (2.17), which contains the influence of the shell’s normal displacement, $w$. The single layer potentials (the first summation in Equation (2.20) and (2.21)) and the double layer potentials (the second summation in Equation (2.20) and (2.21)) are calculated for each element. Figure 2-2 demonstrates a sample case for a shell of revolution with $m = 2$ and $n = 4$. The shell’s mode shape (Figure 2-2a) found using Equation (2.16) is taken as the boundary condition for the flow domain and is discretized into small surface elements. Then the flow potentials, $j_a$ and $j_b$, are calculated with their corresponding boundary conditions using Equation (2.20) and (2.21) (Figure 2-2b,c). It can be observed that flow potential has a periodic behavior in both the axial and circumferential directions, corresponding to the given boundary conditions. The same method can be used to calculate the flow potentials for any mode number.
Figure 2-2 (a) The mode shape of a shell of revolution for \( m = 2 \) and \( n = 4 \) and the corresponding fluid potential, (b) \( j_a \) and (c) \( j_b \), calculated by BEM.

Using the present approach to solve the perturbed flow potential as a two-part boundary condition problem rules out the requirement of an analytical or polynomial approximation to the flow potential, which is difficult to be obtained for a shell with an arbitrary meridian.

Figure 2-2 shows a sample of potential flow solutions at \( q = 0 \) along the \( x \)-axis for a conical shell with \( l / a_2 = 1.04 \) and \( \alpha_0 = 30^\circ \). The solid lines in plots depict the profiles of \( j_a \) (left column) and \( j_b \) (right column) for \( n = 11 \), and with different axial half-wave numbers, \( m = 1 - 5 \). The dotted lines depict the corresponding boundary conditions from the shell’s motion used to calculate the potential flows. The dotted lines of the left column correspond to the boundary conditions used to calculate \( j_a \), i.e., \( \frac{W}{t} \), and those of the right column correspond to the boundary conditions used to calculate \( j_b \), i.e., \( U(s) \frac{W}{s} \). One can observe that despite the fact that \( j_a \) and \( j_b \) and their
corresponding boundary conditions have similar periodic behavior, their peaks do not necessarily coincide because of the non-uniformity of the flow domain. This difference increases as the axial half-wave number, \( m \), increases. For higher axial modes, the magnitudes of the calculated flow potentials close to the shell’s inlet and outlet are more dominant than the rest of the shell.

Using Bernoulli’s principle, the steady flow’s dynamic pressure can be approximated as (Kerboua et al., 2010):

\[
P = -r_f \left( \frac{\partial}{\partial t} + \frac{Q}{A(s)} \frac{\partial}{\partial s} \right)_{S_1},
\]  

(2.22)

where \( r_f \) is the fluid density and \( S_1 \) is the sidewall boundary of the fluid’s domain. The flow pressure in terms of the perturbed flow potential can be found by combining Equation (2.18) and (2.22):

\[
P = \int \left( \frac{\partial a}{\partial t} + \frac{Q}{A(s)} \frac{\partial a}{\partial s} + \frac{\partial b}{\partial t} + \frac{Q}{A(s)} \frac{\partial b}{\partial s} \right)
\]

\[
= \int \left( A(s, s_0) \frac{\partial^2 q(t)}{\partial t^2} + \frac{Q}{A(s)} \frac{\partial A(s, s_0)}{\partial s} \frac{\partial q(t)}{\partial t} \right)
\]

\[
+ \int \left( QB(s, s_0) \frac{\partial q(t)}{\partial t} + \frac{Q^2}{A(s)} \frac{\partial B(s, s_0)}{\partial s} q(t) \right).
\]  

(2.23)
Figure 2-3 The potential flow solutions of a conical shell with \( l/a_2 = 1.04 \) and \( \alpha_0 = 30^\circ \) for \( n=11 \) and \( m=1 \) at \( \theta = 0 \): (a) \( W_a \) (solid line) and the corresponding boundary condition \( \frac{W}{t} \), dotted line); (b) \( W_b \) (solid line) and the corresponding boundary condition \( U(s) \frac{W}{s} \), dotted line).

The pressure term is subsequently added to the shell’s equation of motion found in Section 2.2 in order to conduct the stability analysis as discussed in what follows.
2.7 The method used for the stability analysis

In this section, we conduct a stability analysis on shells of revolution conveying fluid. In this analysis, the equation of motion of shells of revolution discussed in Section 2.2 is coupled with the dynamic flow pressure induced by the perturbed fluid potential calculated in Section 2.2. The equations of motion of the coupled system can be written in the form of an eigenvalue problem:

\[ [B] \dot{p} + [E] p = 0, \]  

\[ [B] = \begin{bmatrix} [M + M_f] & [0] \\ [M + M_f] & [C + 2UC_f] \end{bmatrix}, \quad [E] = \begin{bmatrix} -[M + M_f] & [0] \\ [0] & [K + U^2K_f] \end{bmatrix}, \]

\[ p = \begin{bmatrix} \dot{q} \\ q \end{bmatrix}, \]

where \( M, C \) and \( K \) are the mass, damping and stiffness matrices of the structure, respectively, and \( M_f, C_f \), and \( K_f \) are the flow-induced mass, damping and stiffness matrices. The size of these matrices changes with the number of modes used in the discretization. The structural damping is assumed to be zero in the present study.

In order to validate the present numerical method for shells of revolution conveying fluid, first we compare the results of conical shells conveying fluid with the ones existing in the literatures. The instability of conical shells conveying fluid has been investigated mostly by a semi-
analytical finite element method (Senthil Kumar and Ganesan, 2008). A strong correlation has been found between the circumferential mode of the shell which has the lowest frequency and the circumferential buckling mode when the flow loading is applied (Senthil Kumar and Ganesan, 2008).

![Graph](image)

**Figure 2-4 (a)** The natural frequency and **(b)** the added mass frequency of a conical shell with \( l/a = 1.04 \), \( a/h = 584 \) and \( \alpha_0 = 30^\circ \) with clamped-clamped boundary conditions. Dot: present results, line: (Senthil Kumar and Ganesan, 2008).

Figure 2-4 shows the natural frequency (found for a shell with no fluid) and the added mass frequency (found for a shell filled with fluid at rest) of a conical shell with \( l/a = 1.04 \), \( a/h = 584 \) and \( \alpha_0 = 30^\circ \) with clamped-clamped boundary conditions. These parameters are the parameters used by Senthi and Ganesan (Senthil Kumar and Ganesan, 2008). Results from the present
study as well as the results obtained by Senthil and Ganesan using a semi-analytical finite element method with the same geometrical parameters are plotted in Figure 2-4. It can be observed that the calculated natural frequencies and added mass frequencies of the conical shells with the clamped-clamped boundary conditions agree with the results in (Senthil Kumar and Ganesan, 2008) in both the calculated frequencies and the circumferential mode with the lowest frequency \( n = 11 \) for this set of geometrical parameters.

After comparing the natural frequencies of conical shells with those in the literature, we consider the flow-induced instabilities of a conical shell conveying fluid and compare them with the results found in the literatures.

**Figure 2-5** (a) Dimensionless frequency versus flux for a conical shell with \( l/a = 1.04 \), \( a/h = \) and \( \alpha_0 = 30^\circ \). (b) Dimensionless frequency versus flux with different number of axial modes, \( N_m \).
Figure 2-5a shows the dimensionless frequencies, $\Omega$, versus flux, $Q$, for a conical shell with $l/a_2 = 1.04$, $a_2/h = \alpha_a = 30^\circ$ and clamped-clamped boundary conditions found using the method discussed here and by (Bochkarev and Matveenko, 2011) and (Senthil Kumar and Ganesan, 2008). It can be observed that all three models predict a loss of stability by divergence. The dimensionless frequencies and the critical flow velocities are very similar in the present calculation and in the one from (Bochkarev and Matveenko, 2011), while in (Senthil Kumar and Ganesan, 2008) the critical flow velocity is slightly smaller. As mentioned in Section 6, the non-zero axial flow velocity boundary condition for the flow potential at the inlet and outlet ($j = 0$ at $S_0$) is utilized in the present study. As a consequence, the perturbed flow potentials, $ja$ and $jb$, can have nonzero values at the inlet and outlet, where the $\frac{W}{t} + U(s) \frac{W}{s}$ term in Equation (2.15) vanishes. Therefore, the approach of utilizing separation of variables for the flow potential, for example, assuming $= R \left( \frac{\partial W}{\partial t} + U(s) \frac{\partial W}{\partial x} \right)$ where $R$ is a coefficient depending only on the coordinate normal to the shell's surface (as done in (Kerboua et al., 2010)), is not appropriate with the considered boundary conditions because the flow potentials do not necessarily vanish at the shell's inlet and outlet.

In the present method, a converged result can be obtained with
relatively low number of modes. In Figure 2-5b, the dimensionless frequency of the same conical shell is presented for different numbers of axial modes \( (N_m = 5 \text{ to } 8) \), showing a convergence at \( N_m = 7 \) (results based on \( N_m = 7 \) and \( N_m = 8 \) are on top of each other).

### 2.8 Instability of shells of revolution

In this section, the method discussed previously is used to study the flow-induced instabilities of shells of revolution. A family of shells of revolution are chosen with constant ratios of the inlet to outlet radius of \( a_1 / a_2 = 0.69 \) and the length to outlet radius of \( l / a_2 = 3.53 \), and with various thickness ratios \( (h / a_2) \) and non-zero Gaussian curvatures, as shown in Figure 2-6. The radii of the shells’ parallel circles are defined as

\[
R = R^0 + a_{sor} \times a_2 \sin(\pi x),
\]

where \( R^0 \) is the radius of the conical shell with the same length, inlet and outlet area and \( a_{sor} \) is a geometrical parameter to define a shell of revolution’s curved meridians. For the shell of revolution discussed here, \( R_x \) can be found using the description of \( R \) as

\[
1 / R_x = \frac{2 R / s^2}{[1 + (R / s)^2]^{3/2}}.
\]

For such geometries, there is no abrupt change in the shell’s cross-section, and therefore the assumptions of no flow separation and no-slip condition in Equation (2.15) are still valid.
The existence of non-zero Gaussian curvature changes the stress resultant of a doubly-curved shell according to Equation (2.2) and (2.3), and effectively increases the shell’s stiffness. Figure 2-7 shows the dimensionless natural frequency and the added-mass frequency versus $a_{asor}$ and different thickness ratios. The added-mass frequency, $a_{a'}$, is non-dimensionalized similarly to the natural frequency as $f = a_{a'} \Delta \sqrt{(1 - v^2)/E}$. The dimensionless flux is defined as $Q = Q/[p^2l[D/(\rho h)^{1/2}]]$, where $D = Eh^3/[12(1 - v^2)]$. It can be observed that for shells with positive Gaussian curvatures ($a_{asor} > 0$), the natural and added-mass frequencies increase nearly linearly with the increase of $a_{asor}$. For shells with negative Gaussian curvatures ($a_{asor} < 0$), however, the relation between $a_{asor}$ and the shell’s frequencies is more complicated. The shell’s mode shapes switch from one to another with a different circumferential wave number as the magnitude of the negative Gaussian curvature increases. This effect is more obvious in thinner shells than in thicker shells.
Figure 2-6 Schematic of shells of revolution with various Gaussian curvatures and the same length, inlet area and outlet area.

For example, for a shell with \( h/a_2 = 584 \) (Figure 2-7a), the mode shapes switch between \( n = 6 \) and 7, and for a shell with \( h/a_2 = 2 \), the mode shapes switch between \( n = 5 \) and 6, while for a shell with \( h/a_2 = 3 \) (Figure 2-7c), the shells’ mode shape stays at the same circumferential wave number for all the negative Gaussian curvature studied. Because of a thicker shell’s higher bending moments, its natural frequencies of modes with different wave numbers are farther from each other, and therefore it is harder for a thicker shell to switch from one mode shape to another as its geometry is altered.
Figure 2-7 The dimensionless natural frequencies (solid line) and dimensionless added mass frequencies (dashed-dotted line) of shells of revolution with the same inlet and outlet versus various Gaussian curvatures with different thickness: (a) $h/a_2 = 584$, (b) $h/a_2 = 2$, (c) $h/a_2 = 3$.

In Figure 2-8, the dimensionless frequencies are plotted versus the dimensionless flux of shells of revolution with various negative and positive Gaussian curvatures. All the shells with non-zero Gaussian curvatures have higher added mass frequencies (at a flux of 0) than the conical shell. Also, all of these shells lose their stabilities by divergence as the flux reaches a critical value.

Shells with positive Gaussian curvature have higher added mass frequencies and buckle at higher dimensionless flux, $Q_{cri}$, compared to shells with zero Gaussian curvature as can be observed in Figure 2-8 and in Figure 2-9, in which $Q_{cri}$ is plotted for various positive curvatures and for different thickness ratios.
Figure 2-8 Dimensionless frequency versus dimensionless flux of shells of revolution with $a_1/a_2 = 0.69$, $l/a_2 = 3.53$, $h/a_2 =$ and with various values of (a) positive and (b) negative Gaussian curvatures.

Figure 2-9 The dimensionless critical flux, $Q_{crit}$, versus various positive $a_{sar}$ with different thickness ratios. Dotted line: $h/a_2 =$; dotted-dashed line: $h/a_2 = 2$; solid line: $h/a_2 = 3$. 
On the other hand, as can be observed in Figure 2-8b, shells with negative Gaussian curvatures initially have higher added mass frequencies compared with the conical shell with the same length, inlet and outlet area, however they become unstable at lower dimensionless flux and the critical value decreases as the negative Gaussian curvature becomes larger. This suggests that the flow-induced natural frequencies are influenced more dramatically for cases with negative Gaussian curvatures. This influence comes completely from the flow-induced stiffness, since in the flow model used here, the added mass term \( r_f \times A(s, q) \frac{\partial^2 q(t)}{\partial t^2} \) in Equation (2.23) remains constant with changing flow velocity. To further investigate the effect of negative Gaussian curvatures on the instability of a shell conveying fluid, the critical flow velocities at the mid-section of the shell \((\hat{U} \cdot Q_{cri} / A(x = 0.5))\) are plotted versus the geometry parameter, \( a_{sor} \), with various thickness ratios in Figure 2-10a. The fluctuations in the figure are caused by the changes in the axial and circumferential wave numbers involved in the buckling mode shapes of the shells as the magnitude of the geometrical parameters, \( a_{sor} \), changes. Shells with larger magnitudes of negative Gaussian curvature tend to buckle in higher axial modes. For example, in Figure 2-10a, shells buckle in the first axial mode for \( 0.03 < a_{sor} < 0 \), in the second axial mode for \( 0.12 < a_{sor} < 0.04 \) and in the third axial mode for \( 0.3 < a_{sor} < 0.13 \). The circumferential wave number of the buckling mode
shapes also changes. The circumferential wave number decreases for a given axial wave number as the magnitude of the negative Gaussian curvature increases. For example, \( n = 7 \) to \( 6 \) for \( m = 1 \), \( n = 8 \) to \( 7 \) for \( m = 2 \), and \( m = 8 \) to \( 6 \) for \( m = 3 \) in Figure 2-10a.

Figure 2-10 Critical flow velocities at the mid-section of the shell versus the geometry parameter, \( a_{\text{so}} \), with (a) \( h/a_2 = 1 \), (b) \( h/a_2 = 2 \), and (c) \( h/a_2 = 3 \).
As mentioned earlier, with the same length, inlet and outlet areas, a shell of revolution with a smaller thickness ratio has less bending moment and therefore it is easier for its buckling mode shape to switch from one mode to another as its geometry is altered. This, in fact, can be observed in Figure 2-10 as well. For the same range of $a_{sr}$, the thinner shells (Figure 2-10a) have their buckling mode shapes change more frequently than the thicker shells (Figure 2-10b and c).

![Buckling Mode Shapes](image)

**Figure 2-11** The buckling mode shapes of shells of revolution with (a) $a_{sr} = 0.3$, (b) $a_{sr} = 0.1$, (c) $a_{sr} = 0.03$, and (d) $a_{sr} = 0.1$. 
In the case of dry shells (i.e., shells without any fluid inside), the instabilities of shells with negative Gaussian curvatures have been found to be essentially different from shells with positive Gaussian curvatures (Jasion, 2009; Tovstik et al., 2002). Generally, the buckling modes of a shell with negative Gaussian curvature tend to spread along the shell’s entire surface, while for shells with positive Gaussian curvatures, their mode shape’s deformations are more localized. In the present study of the flow-induced instabilities of shells of revolution, similar behavior can be observed in the buckling mode shapes shown in Figure 2-11.
Figure 2-12 The buckling mode shapes of the studied shells of revolution at $q = 0$ with $h/a_2 = b$ and (a) negative and (b) positive curvatures; $h/a_2 = 2b$ and (c) negative and (d) positive curvature; $h/a_2 = 3b$ and (e) negative and (f) positive curvature.

In Figure 2-11(a-c), shells with negative Gaussian curvatures buckle at various axial modes. A shell with a negative curvature and $a_{sor} = -0.3$ buckles in the 3rd axial mode and the 6th circumferential mode (Figure 2-11a). As $a_{sor}$ changes to $-0.1$, the shell buckles in the 2nd axial mode and the 7th circumferential mode (Figure 2-11b), while in Figure 2-11c, a shell with $a_{sor} = -0.03$ buckles in the first axial mode and the 6th circumferential mode.
In Figure 2-11d, the buckling mode shape of a shell with a positive Gaussian curvature is localized in an area close to the shell’s inlet, where the cross-section is smaller and the local flow velocity is higher. The same behavior can be observed in Figure 2-12, where the buckling mode shapes of shells with various thickness ratios and $a_{swr}$ values are plotted at $q = 0$. Shells with larger magnitudes of negative Gaussian curvatures buckle in higher axial modes, while the buckling mode shapes for shells with positive Gaussian curvatures have their largest deformations localized around $x/l = 0.2$.

The finding in the present study suggests that with the existence of a non-zero Gaussian curvature, a shell of revolution conveying fluid can have significantly different behavior compared with a conical shell with similar geometrical parameters (thickness, length, inlet and outlet areas), in terms of the changes in the critical flow velocities and the buckling mode shapes.

2.9 Conclusions

A numerical method to calculate the flow-induced instabilities of shells of revolution with non-zero Gaussian curvatures is discussed in this chapter. The numerical method contains two parts: the first part calculates the coupled eigenmodes of shells of revolution with the Galerkin method, and the second part calculates the contribution of flow forces associated with the shell’s different modes using the boundary elements method (BEM). For the structure part, the model is based on the exact doubly-curved thin shell
equation, thus a precise description of the shell’s dynamic characteristics is obtained for shells of revolution with non-zero Gaussian curvatures. For the flow part, BEM is applied to calculate the perturbed flow potential in the non-uniform fluid domain, thus avoiding the need for analytical solutions for the flow potential. The structure and the fluid are coupled by taking the shell’s coupled-mode shapes, which are calculated by the Rayleigh-Ritz method, as the flow potential’s boundary condition. The boundary condition is separated into two parts, one part is related to the added mass effect and the other part is induced by the structure’s slope as it bends.

Relatively low number of modes can be used to obtain convergence in the present numerical algorithm, which takes advantage of the fact that for a shell of revolution with a circular cross-section, all the circumferential modes with different wave numbers, \( n \), are still uncoupled. Besides, this approach does not require any analytical solution for the perturbed flow potential, which is impossible to be attained for shells with an arbitrary meridian. This makes the present algorithm computationally very efficient and suitable for further developments of nonlinear simulations or reduced-order modeling.

The results are first compared to those for conical shells in the literature, and agreement is found for both the added mass frequencies and the critical flow velocities. Subsequently, the instabilities of shells of revolution with positive and negative Gaussian curvatures are investigated and compared to a conical shell with the same length, inlet and outlet areas. It
is found that shells of revolution with non-zero Gaussian curvatures have higher added mass frequencies compared to the conical shells with the same length, inlet and outlet area. Shells with negative Gaussian curvatures conveying fluid change their buckling mode shapes frequently as the magnitude of the negative curvature increases, while their buckling mode shapes are similar to those observed in uniform and conical shells. However, shells with positive Gaussian curvatures conveying fluid buckle with their deformations localized close to the area with higher local flow velocities.
CHAPTER 3
FLOW-INDUCED BUCKLING OF FLEXIBLE SHELLS WITH NON-ZERO GAUSSIAN CURVATURES AND THIN SPOTS

In this chapter, we study the influence of one or multiple thin spots on the flow-induced instabilities of flexible shells of revolution with non-zero Gaussian curvatures. The shell’s equation of motion is described by a thin doubly-curved shell theory and is coupled with perturbed flow pressure, calculated based on an inviscid flow model. The results show that for shells with positive Gaussian curvatures conveying fluid, the existence of a thin spot results in a localized flow-induced buckling response of the shell in the neighborhood of the thin spot, and significantly reduces the critical flow velocity for buckling instability. For shells with negative Gaussian curvatures, the buckling response is extended along the shell’s characteristic lines and the critical flow velocity is only slightly reduced. We also show that the length scale of the localized deformation generated by a thin spot is proportional to the shell’s global thickness when the stiffness of the thin spot is negligible compared with the rest of the shell. When two thin spots exist at a distance, their influences are independent from each other for shells with positive Gaussian curvatures, but large-scale deformations can be created due to multiple thin spots on shells with negative curvatures, depending on the thin spots’ relative position.
3.1 Introduction

Flow-induced instabilities of shell structures have been studied extensively both from a fundamental point of view and due to their various applications in engineering and biological systems (Païdoussis, 1998). The majority of these studies are for the case of a perfect cylindrical shell conveying fluid, and some studies exist on flow-induced response of conical shells (Amabili and Païdoussis, 2003). In a recent study, the authors have discussed the flow-induced response of a shell with non-zero Gaussian curvature (Chang and Modarres-Sadeghi, 2015). While the majority of studies on flow-induced response of a shell are for perfect shells, some studies on shells with global imperfections do exist. In these studies, modal imperfections are considered in which the geometric imperfections are assumed to have the shapes of symmetric standing waves. It has been shown that these modal imperfections have a significant influence on the onset of supersonic flutter of a cylindrical shell (Amabili and Pellicano, 2002) and also that the modal imperfections can significantly change the critical buckling load of a cylindrical shell conveying fluid, and this effect increases with the flow velocity (del Prado et al., 2009). Chaotic motions are observed in fluid-filled cylindrical shells with periodic compression and modal imperfections (Pellicano and Amabili, 2006). It is found that a finite dimple is created instead of a fully snap through when a spherical capsule is applied with external pressure and volume control (Knoche and Kierfeld, 2014).
A review of the influence of modal imperfections on the dynamics of shells conveying fluid can be found by Amabili and Païdoussis (Amabili and Païdoussis, 2003). Several studies exist on the response of a dry shell (i.e., with no fluid inside) with global geometric imperfections, and how the shell’s critical loading and its post-buckling behavior depend on these imperfections (Arbocz et al., 1971; Edlund, 2007; Hutchinson, 1965; 1967; Meyer-Piening et al., 2001; Shen and Q. S. Li, 2002; Shen and Noda, 2005). Discrepancies between theory and experiment could arise when these imperfections are neglected (Amabili and Païdoussis, 2003; Edlund, 2007).

Shell structures may also have localized imperfections. There are some recent studies on this problem for dry shells. For example, the elastic buckling due to external pressure of a long cylindrical shell with its thickness reduced in an angular section is studied by Xue and Fatt (Xue and Hoo Fatt, 2002). It was found that symmetric and anti-symmetric buckling modes can be observed depending on the magnitude and the width of the thin section. Vaziri and Estekanchi (Vaziri and Estekanchi, 2006) showed that internal pressure may stabilize or destabilize a cracked cylindrical shell, depending on the shell’s local buckling mode shapes and the crack’s orientation. Also it is shown that the thickness inhomogeneity of a colloidal capsule shell reduces the time delay before the onset of buckling by external osmotic pressure and guides the shape of the capsule’s post-buckling deformation (Datta et al., 2012). Paulose and Nelson (Paulose and Nelson, 2013) showed that a small circular soft region is capable of changing the critical buckling pressure and
post-buckling shape of a spherical shell with external pressure, which makes it possible to design the buckling shape of the shell by controlling the size and thickness of the soft spot. Nasto et al. (Nasto et al., 2013) found that a point indentation can create various shapes of axisymmetric deformations, depending on the local curvature and magnitude of the indentation.

Geometrical inhomogeneity can also be generated by a localized indentation applied on a perfect shell. Vaziri and Mahadevan (Vaziri and Mahadevan, 2008) studied the response of shells with positive or negative Gaussian curvatures to an indentation load and found that for a shell with positive curvature, a localized deformation is observed, and for a shell with a negative curvature, the deformation extends along the shell’s characteristic line. A numerical investigation by Nasto and Reis (Nasto and Reis, 2014) showed that even though large deformation can be observed for a shell under indentation, the maximum principal strain stays relatively small and is governed by the linear elastic theory. Vella et al. (Vella et al., 2012) showed that when an elastic shell is pressurized, its force-displacement relation has two linear regimes at small and large displacements compared with the shell's thickness, unlike unpressurized shells. Wrinkling instabilities can also be observed for highly pressurized spherical shells with an indentation (Vella et al., 2011).

Going back to the existing studies on flow-induced instabilities of shells with imperfection, we notice that they are limited to the shells with modal imperfections. For several applications, predicting the response of a shell with
a local imperfection to the internal flow loads is of great significance. As an example of a biological application, a cerebral aneurysm, before it ruptures, has the shape of a flexible shell with a local imperfection, which is in contact with internal flow forces (Raghavan et al., 2006). This local imperfection is in fact a local area with a smaller thickness compared with the rest of the structure. The existence of this local thin spot can influence the onset of the structural instability, when compared with a perfect shell, and therefore the time of the aneurysm rupture.

In the present work, we discuss the influence of local thin spots on the flow-induced response of shells of revolution with non-zero Gaussian curvatures.

3.2 Equations of motion and the method of solution

![Figure 3-1 Schematics of a shell of revolution with a thin spot (the shaded area).](image)

We consider a shell of revolution with a non-zero Gaussian curvature conveying fluid with a local thin spot as shown in Figure 3-1. In this figure, $a_1$
is the shell’s inlet radius, \( a_2 \) is the outlet radius, \( l \) is the length of the shell and\n\[ k_x(x) = 1/R_x(x) \quad \text{and} \quad k_\theta(x) = 1/R_\theta(x) \]
are the axial and circumferential principal curvatures, respectively. The profile of the meridian, \( R_x(x) \), is defined as \( R_x(x) = a_1 + (a_2 - a_1) \times x/l + a_1 \times \sin(x/l) \), where \( a_1 \) is a parameter defining the non-zero Gaussian curvature. The thin spot is located in a location defined by \( s_a < s < s_b \) and \( \theta_a < \theta < \theta_b \), the thickness of which is \( h \), while the thickness of the rest of the shell is \( h_0 \). For simplicity and because of the symmetry of the geometry of the shell of revolution, we assume that the thin spot always starts at \( \theta_a = 0 \), with no loss of generality. The doubly-curved shell’s equation (as discussed in (Soedel, 2015)) is used and the strain components in the shell’s arbitrary position can be found by Love’s hypothesis:

\[
\varepsilon_{xx} = \varepsilon_{xx,0} + z k_x, \quad \varepsilon_{\theta \theta} = \varepsilon_{\theta \theta,0} + z k_\theta, \quad \gamma_{x\theta} = \gamma_{x\theta,0} + z k_x \quad \gamma_{\theta x} = \gamma_{\theta x,0} + z k_\theta, \quad (3.1)
\]

where \( \varepsilon_{xx,0} \), \( \varepsilon_{\theta \theta,0} \) and \( \gamma_{x\theta,0} \) are the strain components in the mid-plane of the shell, and \( \varepsilon_{xx} \), \( \varepsilon_{\theta \theta} \) and \( \gamma_{x\theta} \) are the strain components in an arbitrary position. The mid-plane strain components can be calculated from the axial, circumferential and radial displacements, \( u \), \( v \) and \( w \):
where $A_x$ and $A_\theta$ are the Lamé parameters. The curvature components are subsequently calculated as:

\[
\begin{bmatrix}
  k_{xx} \\
  k \\
  k_y
\end{bmatrix} = 
\begin{bmatrix}
  \frac{1}{A_x} \frac{\partial A}{\partial x} & 0 & \frac{1}{R_x} \\
  \frac{1}{A_x R_x} \frac{\partial A}{\partial x} & 0 & \frac{1}{A_x} \frac{\partial A_x}{\partial x} \\
  \frac{1}{A_x R_x} \frac{\partial A}{\partial x} & \frac{1}{A_x} \frac{\partial}{\partial x} \left( \frac{1}{A_x} \frac{\partial A_x}{\partial x} \right) & 0
\end{bmatrix} \begin{bmatrix}
  u \\
  v \\
  w
\end{bmatrix},
\]

(3.2)

The strain components ($\sigma_{xx}$, $\sigma_{\theta\theta}$, $\sigma_{x\theta}$), force resultants ($N_{xx}$, $N_{\theta\theta}$, $N_{x\theta}$) and moment resultants ($M_{xx}$, $M_{\theta\theta}$, $M_{x\theta}$) can be found to be:
\[
\begin{bmatrix}
    xx \\
    x
\end{bmatrix} = E \begin{bmatrix}
    \frac{1}{(1 - n^2)} & \frac{1}{(1 - n^2)} & 0 \\
    \frac{1}{(1 - n^2)} & \frac{1}{(1 - n^2)} & 0 \\
    0 & 0 & \frac{1}{2(1 + 1/n^2)}
\end{bmatrix} \begin{bmatrix}
    xx \\
    x
\end{bmatrix}
+ \frac{h(x, y/2)}{h(x, y/2)} \int y \, dz,
\]

The shell's equations of motion are derived from Hamilton's principle,

\[
\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0,
\]

where \( q \) is the generalized coordinate and \( L \) is the system's Lagrangian, which can be calculated from inertia, force and moment resultants:

\[
L = \int_s \left( N_{xx} e_{xx} + N_{\theta y} e_{\theta y} + N_{x\theta} e_{x\theta} + M_{xx} k_{xx} + M_{\theta \theta} k_{\theta \theta} + M_{x \theta} k_{x \theta} \right) dx \, d\theta
- \int h(x, \theta)(\ddot{u}^2 + \dot{v}^2 + \dot{w}^2) \, dx \, d\theta.
\]

The equations of motion are discretized by the Galerkin technique using the following trial functions:

\[
u = \sum a_m \sin \left( \frac{m}{L} \right) \cos (n + ),
\]

\[
v = \sum a_m \sin \left( \frac{m}{L} \right) \sin (n + ),
\]

\[
w = \sum a_m \left[ \sinh \left( \frac{m}{L} \right) \sin \left( \frac{m}{L} \right) + m \left( \cosh \left( \frac{m}{L} \right) \cos \left( \frac{m}{L} \right) \right) \right] \cos (n + )
\]

where \( \sigma_m \) is the phase angle and \( \sigma_m \) are the eigenvalues of a clamped-clamped Euler beam. The final set of equations of motion has the form of \( M\ddot{q} + Kq = P \).
where \( \mathbf{M} \) and \( \mathbf{K} \) are the mass and stiffness matrices and the vector \( \mathbf{P} \) contains the contributions from the perturbed fluid pressure induced by the motion of the shell’s sidewall. The fluid is assumed to be inviscid and incompressible, and is modeled by the potential flow theory and satisfies the no-slip boundary condition:

\[
\frac{\partial W}{\partial t} + U(s) \frac{\partial W}{\partial s} = 0,
\]

(3.7)

where \( \varphi \) is the perturbed potential, \( S_1 \) denotes the shell’s sidewall, and \( U(s) \) is the mean flow velocity when there is no sidewall motion. The flow potential is solved numerically and finally the perturbed fluid pressure can be calculated as

\[
P = -\rho_f \left[ \frac{\partial \varphi}{\partial t} + U(s) \frac{\partial \varphi}{\partial s} \right]_{S_1}
\]

using Bernoulli’s principle. More details of this method can be found in (Chang and Modarres-Sadeghi, 2015).

### 3.3 Mode shapes of shells with thin spots versus perfect shells

We consider three shells of revolution with different shapes of meridian but the same length \( l \) and the same average of the inlet and outlet radii, \( (a_1 + a_2)/2 \), as shown in the first row of Figure 3-2. The first shell is a shell with a positive Gaussian curvature \( \alpha/a_1 = 10 \) and equal inlet and outlet radii (Figure 3-2a), the second one is another shell with a positive Gaussian curvature \( \alpha/a_1 = 10 \) and unequal inlet and outlet radii (Figure 3-2b), and the third one is a shell with a negative Gaussian curvature (}
\[ \alpha/a_i = -10 \] and unequal inlet and outlet radii (Figure 3-2c). We know that the buckling modes are always global (spread along the surface) for a shell with negative Gaussian curvature (Figure 3-2f) and local for a shell with positive Gaussian curvature. The local modes of shells with a positive curvature, however, sometimes look like a global mode (Vaziri and Mahadevan, 2008). For example, the shell with unequal inlet and outlet radii (Figure 3-2e) experiences a higher pressure in the area close to its inlet compared with any other area, since the mean flow velocity is inversely proportional to the cross-sectional area. This localized pressure results in a localized buckling mode shape with its maximum amplitude close to the inlet. In contrast, the shell with equal inlet and outlet (Figure 3-2d) has less variability in its cross-sectional area, and thus the perturbed flow pressure is distributed more uniformly in the axial direction, resulting in local buckling at two locations along the length of the shell, which looks like a global mode shape with a second axial-mode distribution. By introducing a thin spot somewhere on a shell with a nonzero Gaussian curvature, the structure will have a buckling preference, and as a result, the seemingly global modes in a shell with a positive Gaussian curvature will become local, but the global modes, in a shell with a negative Gaussian curvature, stay global as we discuss in what follows.
Figure 3-2 Shells with (a) a positive Gaussian curvature and equal inlet and outlet radii, (b) a positive Gaussian curvature and unequal inlet and outlet radii, (c) a negative Gaussian curvature, and their corresponding buckling modes (d-f) for perfect shells and (g-f) for shells with a thin spot.

The third row of Figure 3-2 shows the flow-induced buckling mode shapes of the shells in the presence of a thin spot located at \( s_d=0.4s, s_b=0.5s, \ \theta_b=\pi/4 \) and with a thickness ratio of \( \bar{h}/h_0=0.1 \). The location of the thin spot is indicated by the light color in the first row. In the case of a shell with a negative Gaussian curvature, the buckling mode shape remains global, but it becomes asymmetric as a result of the asymmetry introduced to the structure due to the thin spot (Figure 3-2i). The presence of a thin spot makes shells with positive Gaussian curvatures buckle with a clear localized mode shape around the thin spot (Figure 3-2g,h). Again in both cases the local mode
shapes are asymmetric due to the asymmetry introduced by the local thin spot. The seemingly global mode shape of the perfect shell of Figure 3-2(d) disappears when the thin spot is introduced.

3.4 Shells with Negative Gaussian Curvatures

For a shell with a negative Gaussian curvature, the buckling modes are always global even for an extremely thin spot, and independent from the location of the thin spot. By changing the location of the thin spot, the axial and circumferential mode numbers involved in the buckling mode change, but these modes stay global. Figure 3-3 shows the buckling mode shapes of a shell with a negative Gaussian curvature conveying fluid with uniform thickness, when a thin spot is either at $s_a=0.4s, s_b=0.5s$ (the middle column) or at $s_a=0.2s, s_b=0.3s$ (the right column). While the shapes of these modes are different, they are all global. The asymmetry introduced to the structure due to the thin spot breaks the circumferential symmetry of the mode shapes as seen in the second row of Figure 3-3, yet the modes stay global.
3.5 Shells with positive Gaussian curvatures

For the case of a shell with a positive Gaussian curvature, the initial buckling mode shape, before introducing a thin spot, is a local one. After introducing the thin spot, somewhere other than the peak amplitude of the buckled mode of the original perfect shell, a competition begins between these two points to be the peak amplitude of the resulting buckling mode. The thickness of the thin spot, its location and its size are the major parameters that could influence the resulting mode. In this section we discuss how each of these parameters influence the shell’s bucking mode.
The buckling mode shapes of a shell with positive Gaussian curvature and unequal inlet and outlet with a thin spot at $s_a=0.4s$, $s_b=0.5s$, $\theta_a=\pi/4$ with various thickness ratios: (a) the entirely perfect shell mode, (b) the combined mode, and (c) the entirely thin-spot dependent mode.

### 3.6 The thickness of the thin spot

For a thin spot, not much thinner than the rest of the shell, the buckling mode does not change compared with a perfect shell (i.e., with uniform thickness), which means that the thin spot has not influenced the instability yet. This is for the range of $h/h_0 > 0.6$ for the example of Figure 3-4. For a range of thinner spots ($0.4 < h/h_0 < 0.6$, for the example of Figure 3-4), a combined buckling mode is observed when a combination of the original local mode and a new local mode that is due to the thin spot contribute to the mode shape. For even thinner spots ($h/h_0 < 0.4$ for this example), the original local mode disappears completely and a new local mode, which depends entirely on the thin spot, takes over. Overall three bucking modes exist for a positive shell with a thin spot, independent from the thickness of the thin spot: (i) the
entirely perfect shell mode, (ii) the entirely thin-spot dependent mode, and (ii) the combined mode.

Once the transition from the original localized mode to the thin-spot dependent localized mode is complete, the shape of the shell's local mode does not change with reducing the thickness ratio further. This can be explained by noticing that with an extremely small thickness ratio, the bending and stretching stiffness of the thin spot become insignificant compared with the stiffness of the rest of the shell, and also to the added stiffness associated with the flow's perturbed pressure. Therefore, the shape of the localized buckling mode is decided not by the thickness of the thin spot, but by the properties of the neighboring areas of the thin spot, and the fluid loading. As an example, for all three shells of the first row of Figure 3-5, the thickness of the thin spot stays constant, and the global thickness is varied. This variation influences the shell’s buckling mode shapes in the circumferential direction. With smaller values of the global thickness, $h_0$, the wavelength of the shell's mode in the circumferential direction, $\ell_\theta$, decreases linearly from $\ell_\theta = 0.18$ mm for $h_0=0.0045$ mm to $\ell_\theta=0.11$ mm for $h_0=0.005$ mm. This behavior is very similar to what has been observed in wrinkling of pressurized thin shells, where the wavelength of the wrinkles, which are induced by a point indentation, is proportional to the shell’s thickness and it does not change with the external force (Vella et al., 2011). The same shells with the same location for the thin spot and with the same thickness ratio are shown in the second row of Figure 3-5, but this time the thickness ratio is
changed by changing the thickness of the thin spot. The buckling mode stays the same for all three cases.

Figure 3-5 Buckling mode shapes for a shell with a thin spot and (first row) a constant thickness for the thin spot, $\bar{h} = 0.001$, and various global thickness values: (a) $h_o = 0.01$ ($\bar{h} / h_o = 0.1$), (b) $h_o = 0.05$ ($\bar{h} / h_o = 0.02$), and (c) $h_o = 0.08$ ($\bar{h} / h_o = 0.125$), (second row) a constant global thickness, $h_o = 0.05$, and various thickness for the thin spot: (d) $\bar{h} = 0.01$ ($\bar{h} / h_o = 0.1$), (e) $\bar{h} = 0.002$ ($\bar{h} / h_o = 0.02$), and (f) $\bar{h} = 0.00125$ ($\bar{h} / h_o = 0.0125$).
3.7 The location of the thin spot

The location of the thin spot plays a significant role in transition from one local mode to another local mode. With a thin spot very close to the inlet \((s_a=0.2s, s_b=0.3s,\text{ green line in Figure 3-6})\), the stiffness of the shell in the neighborhood of the original asymmetric mode shape is weakened even if the thickness of the thin spot is just slightly smaller than the rest of the shell. Therefore, the value of the critical flux for the shell is reduced immediately. Since this location is very close to the location of the original peak amplitude in the mode shape of the shell with no thin spot, the thin spot and the original buckling mode work together toward making the system unstable. On the other hand, when the thin spot is far from the peak amplitude of the mode in the original perfect shell, then the thin spot has less influence on decreasing the onset of instability. For example the red line in Figure 3-4 The buckling mode shapes of a shell with positive Gaussian curvature and unequal inlet and outlet with a thin spot at \(s_a=0.4s, s_b=0.5s, \theta_b=\pi/4\) with various thickness ratios: (a) the entirely perfect shell mode, (b) the combined mode, and (c) the entirely thin-spot dependent mode. shows the reduced critical flux (\(\hat{U}\)) compared to its original value \((U_0)\) of a shell with a thin spot far from the inlet \((s_a=0.7s, s_b=0.8s,\text{ red line})\) and the shell keeps the same critical flux and the same buckling mode shape until \(\bar{h}/h_0 < 0.35\).
3.8 The size of the thin spot

The size of the thin spot does not influence the shells’ mode shapes, as long as the center of the thin spot is fixed. Figure 3-7 Buckling mode shapes of a shell with a positive Gaussian curvature and a constant global thickness, and with a thin spot with different circumferential spans: Blue: \( s_a=0.5s, s_b=0.6s \); Red: \( s_a=0.7s, s_b=0.8s \). The area with its amplitude larger than 30% of the maximum amplitude is contoured. shows the mode shapes for a shell with a positive curvature, with a thin spot placed at the same
location but with three different lengths for the thin spot in the circumferential direction. The overall shapes of the modes stay the same for all of these cases with the same location for the local maxima and minima, and only slight changes in the magnitude. This observation implies that the circumferential length scale of the modes is decided by the flow stiffness and the shell's global stiffness, and not by the length of the thin spot. This is of significance in analyzing the response of a shell with a thin spot, as the same set of mode shapes can be used for thin spots with different lengths.

![Figure 3-7 Buckling mode shapes of a shell with a positive Gaussian curvature and a constant global thickness, and with a thin spot with different circumferential spans: Blue: 0.5 - 0.5 . Red: 0.4 - 0.4 . Black: 0.3 - 0.3 . The area with its amplitude larger than 30% of the maximum amplitude is contoured.](image)
3.9 Shells with multiple thin spots

In this section, we discuss how multiple thin spots interact with each other on shells with non-zero Gaussian curvatures, as this is a common situation for shell structures in applications. We consider two thin spots with $\tilde{h}/h_0=0.2$ and $h_0=0.0025$, on different relative positions. The first one is always located at $s=0.2-0.3$, $\theta=0-0.2$, and the second one is placed at different locations.

For a shell with a positive Gaussian curvature, $\alpha/a_i = 10$ (Figure 3-8), two locations for the second thin spot are considered: The thin spot is first located at $s=0.7-0.8$, $\theta=0-0.2$ (the same angular location as spot 1) and then at $s=0.7-0.8$, $\theta=0.2-0.4$. In both cases, the deformations originated from each of the two thin spots are localized in their own neighborhood, and there is no coupling between the two local buckling modes, independent from their relative positions. This is similar to the local behavior of the buckling modes that we had observed for shells with positive Gaussian curvatures and a single thin spot in Section 3.5.
On the other hand, for a shell with a negative Gaussian curvature, the buckling mode shapes are global when there is only one thin spot as we discussed in Section 3.4. This implies that if we have more than one thin spot, the buckling modes due to each of these thin spots can spread all along the length of the shell thus interacting with the modes due to the other thin spots. We consider a shell with a negative Gaussian curvature, \( \rho / a_1 = -10 \), and equal inlet and outlet radii, with two thin spots at different angular locations. The first thin spot is located at \( s=0.2-0.3, \theta=0-0.2 \), and the second one at different locations relative to the first one (Figure 3-9). The influence of the thin spots extends along two characteristic lines, which make angles of \( \pm \tan^{-1}(\sqrt{R_x / R_y}) \) with respect to the \( x \)-axis. The positional difference in the angular direction subsequently rules how the two thin spots interact. When the difference in the angular locations between the two thin spots is zero or small (Figure 3-9a), the shell buckles in its second axial mode with its deformation parallel to the \( x \)-axis. When the difference in the angular locations increases and the two thin spots are located on each other’s characteristic lines (Figure 3-9b-c), the two extended deformations are coupled into one mode, cross the nodal line of zero Gaussian curvature at \( s=0.5 \).
The reduction in the critical flow velocities due to the existence of multiple thin spots is much smaller for shells with negative Gaussian curvatures than for shells with positive Gaussian curvatures. The critical flow velocity decreases by around 10% for shells with negative curvatures (numbers in Figure 3-9), and by around 30% for shells with positive curvatures (numbers in Figure 3-8). This is because for shells with positive Gaussian curvatures, the loading applied to one thin spot is solely resisted by the in-plane stress close to that thin spot, therefore such structures can be very imperfection-sensitive. For shells with negative Gaussian curvatures, however, the loading and the resulting deformation can travel to locations far from the thin spot, and therefore the critical loading is less sensitive to the imperfections, and is more influenced by the global properties of the system.
(the structure and the flow inside it). Furthermore, in the case of negative curvatures, the influence of one thin spot can be extended to another thin spot on the surface and form deformations that are distributed in large scales.

3.10 Conclusions

We have studied the flow-induced buckling of shells with nonzero Gaussian curvatures and thin spots. The equation of motion of doubly-curved shells with Love’s hypothesis is used to model a shell of revolution’s structure, and subsequently coupled with a potential flow model. The no-slip boundary condition is assumed for the potential flow. The fluid-structure interaction equation is discretized by the Galerkin method to study the critical flow velocities and the mode shapes of the shell’s buckling using instability analysis.

It is found that for a shell with a negative Gaussian curvature conveying fluid, the buckling modes are always global, both for a perfect shell (without any thin spot) and for a shell with a thin spot somewhere on its surface. For a shell with a positive Gaussian curvature and a thin spot conveying fluid, however, three distinct modes are obtained: (i) the original local mode shape of a perfect shell, for large thin spot ratios (defined as the ratio of the thickness of the thin spot to the thickness of the rest of the shell) (perfect mode), (ii) a mode that is a combination of the original local mode and a new local mode centered around the thin spot, for moderate thin spot ratios (combined mode), and (iii) a new local mode, centered around the thin
spot, for small thin spot ratios (thin spot mode). These mode shapes are
decided by the flow-induced stiffness, the shape and the location of the thin
spot, and the global thickness of the shell.

Furthermore, the sign of the shell’s Gaussian curvature decides on how
multiple thin spots interact with each other. For a shell with a negative
Gaussian curvature, the deformations due to each thin spot are spread along
the shell’s characteristic lines, resulting in coupled buckling mode shapes with
large deformations covering both thin spots. Thus the critical flow velocity for
the onset of buckling is not affected much by adding the second thin spot, the
main reduction in the critical flow velocity being due to the introduction of the
first thin spot. For a shell with positive curvature, however, the deformations
due to the thin spots are all localized in each thin spot’s neighborhood and
since the stress distributions are also localized around the thin spot, the
critical flow velocities are largely reduced.
CHAPTER 4

A REDUCED ORDER MODEL FOR WALL SHEAR STRESS IN ABDOMINAL AORTIC ANEURYSMS BY PROPER ORTHOGONAL DECOMPOSITION

In this chapter, a reduced-order model (ROM) is constructed to study the physiological flow and wall shear stress conditions for abdominal aortic aneurysms. The method of snapshot Proper Orthogonal Decomposition (POD) is utilized to construct the reduced-order bases using on a series of CFD results, which are subsequently improved using a QR-factorization technique to satisfy the various boundary conditions in physiological flow problems. This method can effectively construct a computationally efficient physiological model, which allows us to examine the fluid velocities and wall shear stress distributions under a range of different physiological flow parameters.

4.1 Introduction

Computational Fluid Dynamics (CFD) has become a useful tool in analyzing patient specific abdominal aortic aneurysm models to study the hemodynamic conditions and for planning surgical treatment (Suh et al., 2010; Tse et al., 2011). However, simulating physiological flows remains to be difficult for clinical usage, because of the computational difficulties and the limitations of the image-based models. In order to save computational cost, aneurysms are often treated as an isolated model and the computational
domain may include the aneurysms body and several segments of the
connected arteries, while in reality an aneurysm is part of the global and
complicated circulatory system and the far-field influence can be generated
both from upstream and downstream (Quarteroni et al., 2014).

Proper initial and boundary conditions for fluid and pressure need to
be provided for CFD simulations, however, the exact boundary conditions are
difficult to be measured right at the site of the computational domain in vivo
(Les et al., 2010). Often the analytical solution or the measurement from
another site of the patient are used (Olufsen et al., 2000; Quarteroni et al.,
2014). For example, three-dimensional ultrasound imaging is a time-saving
and cost-effective alternative to computed tomography (CT) imaging,
however, only the images with a limited field-of-view are captured (van
Disseldorp et al., 2016), therefore the full geometry of the aneurysm and its
connected arteries are difficult to be obtained, thus the uncertainties in the
assigned boundary conditions are increased. Aside from the difficulties in
computational fluid dynamics, the computational cost can even increased
enormously when fluid-structure interactions (FSI) and the aneurysm’s
material properties are considered, even with commercial software (Z. Li and
Kleinstreuer, 2007; Scotti et al., 2008). This makes it even less practical for
clinical usage.

This draws our interest to develop a type of reduced-order model
(ROM) for these kinds of problems in order to reduce the computational cost
and obtain reliable results. There is always a huge interest in the reduced
order modeling (ROM) of such ROMs have been extensively used for systems with high nonlinearity and geometrical non-uniformity to replace the full, nonlinear model with a low degree-of-freedom ROM, thus reducing the overall computational cost (Dowell and Hall, 2001). This method provides a possibility to investigate a wide range of different physiological flow boundary conditions without conducting CFD simulations repetitively. Therefore, the feasibility of using CFD results in practical usage is improved.

In the present chapter, we construct an ROM by proper orthogonal decomposition (POD) method to estimate the flow-induced wall shear stress (WSS) and pressure loading of a simplified abdominal aortic aneurysm (AAA) with various inflow angles. The variation of the inflow angle is of interest because the angulation of the angulation of the proximal neck has been known to be a dominating factor for the surgical outcome of endovascular abdominal aortic aneurysm (EAAA) repair (Sternbergh et al., 2002). Vessel asymmetry with large inflow angles is also known to be a significant indicator of an aneurysm’s potential to rupture (Doyle et al., 2009; Z. Li and Kleinstreuer, 2007; Scotti et al., 2005). The regular method to measure the angulation of aneurysm neck involves observer variations and often leads to over- or under-estimation (de Vries, 2012; van Keulen et al., 2010). Limited field-of-view of ultrasonic images can also increase the difficulty in estimating the angulation (van Disseldorp et al., 2016). Therefore, a numerical model which is robust enough to examine the physiological flow conditions over a range of neck angulation could be very beneficial. We also study the effectiveness of
the ROM model with varying frequencies in the hemodynamic waveforms, since it has been shown that different hemodynamic loadings in exercising condition may also influence the AAA growth (Les et al., 2010).

4.2 Reduced order modeling by proper orthogonal decomposition

The snapshot POD method is used here to construct the ROM. Consider the incompressible Navier-Stoked equation in the flow domain of an abdominal aneurysm model:

\[
\nabla \cdot \mathbf{u}(\mathbf{x}, t) = 0 \text{ for } \mathbf{x} \in \Omega_f
\]

\[
\rho_f \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \frac{1}{\text{Re}} \nabla^2 \mathbf{u} \text{ for } \mathbf{x} \in \Omega_f,
\]

\[
\mathbf{u}(\mathbf{x}, t) = \bar{\mathbf{u}}_\text{in}(\mathbf{x}, t) \text{ for } \mathbf{x} \in \Gamma_\text{in}
\]

\[
\nabla \bar{\mathbf{u}}(\mathbf{x}, t) = 0, \quad p(\mathbf{x}, t) = p_{\text{out}}(t) \text{ for } \mathbf{x} \in \Gamma_{\text{out}}
\]

where \( t \) is time, \( \mathbf{x} \) is the spatial coordinates, \( \mathbf{u} \) is the flow velocity, \( p \) is the flow pressure, \( \rho \) is the fluid density, \( \Omega_f \) is the flow domain and \( \Gamma_\text{in} \) is the inlet, \( \Gamma_\text{out} \) is the outlet, \( \text{Re} \) is the Reynolds number, \( \bar{\mathbf{u}}_\text{in} \) is the time-dependent inlet boundary condition and \( p_{\text{out}} \) is the time-dependent outlet boundary condition. The POD method starts by finding a series of deterministic functions, \( \bar{U}(\mathbf{x}, t) \), which best represent a given state variable, \( \bar{\mathbf{u}}(\mathbf{x}, t) \), and the approximate solution can be expressed by the summation of these deterministic functions:
\[ \tilde{u}(\bar{x},t) = \sum_{j=1}^{N} a(t) \cdot \tilde{U}_j(\bar{x}). \]  

(4.5)

where \( a(t) \) is the POD modal amplitudes and \( N \) is the number of POD modes used in the ROM. This is accomplished by finding the maximum of averaged normalized inner product of \( \tilde{u} \) and \( \tilde{U}_j \), that is:

\[
\max \frac{\langle (\tilde{u}, \tilde{U})^2 \rangle}{\langle (\tilde{U}, \tilde{U})^2 \rangle},
\]  

(4.6)

where \( \langle \cdot \rangle \) is the temporal average and \( \langle , \rangle \) denotes the usual \( L_2 \)-inner product in the related Hilbert space. In a two-dimensional case, the maximization problem leads to an eigenvalue problem:

\[
\int_{\Omega} R(\bar{x}, \bar{x}') \cdot \tilde{U}(\bar{x}') d\bar{x}' = \sigma U(\bar{x}),
\]  

(4.7)

where \( R(\bar{x}, \bar{x}') = \langle \tilde{u}(\cdot, \bar{x}) \otimes \tilde{u}(\cdot, \bar{x}') \rangle \) is the spatial correlation tensor and \( \sigma \) is the eigenvalues of \( R \). A CFD problem with complicated a geometry requires sizable degrees of freedom, which could make solving the eigenvalue problem computationally very costly. A so-called snapshot POD method was proposed to address this problem.

Assuming a collection of discrete snapshots are sampled by a fixed time interval, \( \tilde{u}_k(\bar{x}, t_k), k = 1 \cdots N \), and the collection of snapshots contains enough information to describe the behavior of the full-scale simulation, \( \tilde{u}(\bar{x}, t) \). By calculating the time-correlation coefficients between each snapshot and solving the discretized eigenvalue problem, the eigenvalues of the POD
modes, \( \bar{\phi}_j \), and the transformation matrix, \( T_j^k \), from the POD modes to the snapshots can be found (Rapún and Vega, 2010):

\[
\begin{align*}
\sum_{k=1}^{N} (u_j, u_k) \left( T_j^k \right) = \sum_{j=1}^{2} (u_j, u_j),
\end{align*}
\] (4.8)

and the original snapshots can be expressed in terms of the POD mode as:

\[
\bar{u}_i = \sum_{j=1}^{N} \sigma_j \alpha_j^i \bar{U}_j \text{ or } \bar{U}_j = \frac{1}{\sigma_j} \sum_{k=1}^{N} \alpha_j^k u_k,
\] (4.9)

and the reduced-order solution is found to be:

\[
\hat{u} = \sum_{i=1}^{n} a_i(t) \bar{U}_i,
\] (4.10)

where \( a_i = \langle U_i, u \rangle \) and \( n \ll N \) is the order of the reduced order solution. Using the snapshot method results in avoiding the requirement for calculating the eigenmodes from the CFD results, which is rather complicated in geometrically-complex problems. The time interval for sampling the snapshots should be carefully chosen so that significant flow is not neglected information between two snapshots with a given computational capacity. Also, in this method, the numerical simulations should run long enough for sufficient information to be gathered from the snapshots. Appropriate sampling frequency, location and number of snapshots depend on the system’s overall behavior, and should be chosen carefully to ensure a converged and physically meaningful result.

Because each snapshot is divergence free due to the incompressible flow condition, the linear combination of these snapshots will also be
divergence free, thus only the discretized Navier-Stokes equation in Equation 2.2 is required to be satisfied. Furthermore, using the fact that the POD modes are orthogonal and the expression in Equation 4.6, it can be shown that the magnitude of the eigenvalues of the unused POD modes can be used a priori as an estimator of the RMS error in the reconstructed velocities (Rapún and Vega, 2010):

\[
\sum_{j=1}^{N} |\tilde{u} - \bar{u}_j| = \sqrt{\frac{1}{N} \sum_{j=n+1}^{N} \sigma_j^2}.
\] (4.11)

This relation is critical in deciding on the accuracy of the resulting POD model and the ways the collections of snapshots are chosen from the simulation results. Because the eigenvalues of the POD modes are monotonically decreasing, the first \(n\)-POD modes will always be the optimal choice for the ROM model.

4.3 POD ROM model with multiple time dependent boundary conditions

Because we intend to utilize the POD method to construct a reduced order model, which is capable of predicting the behavior of a system in a large parameter space, it is to our interest to adapt multiple time-dependent boundary conditions of the physiological flow on the inlets and outlets of the computational flow domain. The method uses the QR decomposition algorithm (Gunzburger et al., 2007). Assuming a series of \(N\) snapshots is
generated from a simulation that contains $n^p$ time-dependent boundary conditions, which can be separated into spatial and temporal components:

$$G_i(\bar{x}, t) = \theta_i(\bar{x}) \cdot \hat{T}_i(t), \bar{x} \in \Gamma_i \text{ for } i = 1 \cdots n^p \quad (4.12)$$

where $G_i$ is the $i$-th time-dependent boundary condition, which exists on a portion of the boundary. $\theta_i(\bar{x})$ is the spatial part and $\hat{T}_i(t)$ is the temporal part of a boundary condition, respectively. Several different boundary conditions can exist in the same portion of the boundary. For example, a parabolic flow profile with an inflow angle can be separated into the axial flow component, $\theta_1(\bar{x})$, and the lateral flow component, $\theta_2(\bar{x})$. The flow velocity is subsequently approximated by the combination of a series of eigenmodes, chosen by the POD technique, to ensure that the dominating dynamic behavior is captured with a relatively small number of modes. The eigenmodes consist of particular ($\bar{U}_i^p$) and homogenous modes ($\bar{U}_i^h$) and the reconstructed flow velocity is written as:

$$\bar{U}(\bar{x}, t) = \sum_{i=1}^{n^p} a_i(t)\bar{U}_i^p(\bar{x}) + \sum_{i=n^p+1}^{n^p+n^h} a_i(t)\bar{U}_i^h(\bar{x}), \quad (4.13)$$

where $n^h$ is the number of modes for the homogenous modes and $n = n^p + n^h$ is the total degrees of freedom. The particular modes have non-zero values on the boundaries, and are used to satisfy the prescribed inflow condition. The number of particular modes equals to the number of discretized boundary conditions. Consequently, a limited number of particular modes are capable of satisfying multiple boundary conditions, which are continuous functions in
spatial coordinates. The homogenous modes have zero values on the boundaries, and are used to satisfy the reduced order model using Galerkin's technique to discretize the Navier-Stokes equation. The discretized Navier-Stokes equation has the following form:

\[
\frac{da_i}{dt} = \sum_{j=1}^{N} \sum_{k=1}^{N} a_ia_kB_{ijk} + \sum_{j=1}^{N} a_jC_{ij} + D_i \quad \text{for} \quad i = N^0 + N^w + 1 \cdots N, \tag{4.14}
\]

with

\[
B_{ijk} = -\langle \bar{U}_i, \bar{U}_j \cdot \nabla \bar{U}_k \rangle,
\]

\[
C_{ij} = \frac{1}{Re} \langle \bar{U}_i, \nabla^2 \bar{U}_j \rangle,
\]

\[
D_i = \langle \bar{U}_i, \nabla p_i \rangle, \tag{4.15}
\]

in which \(B_{ijk}\) is the convection force tensor, \(C_{ij}\) is the viscous force matrix, and \(D_i\) is the discretized pressure force. Because the flow is strongly driven by time-dependent inlet pressure, pressure POD modes are directly calculated from the fluctuation pressure value by subtracting the time-dependent outlet pressure from each snapshot,

\[
p_j = \frac{1}{N} \sum_{k=1}^{N} p_k. \tag{4.16}
\]

where \(p_k = p(\bar{x}, t) - p_{out}(t)\). The resulting pressure POD modes are used to model the fluctuating pressure distribution without additional pressure-velocity equation (Lassila et al., 2014).
4.4 A simplified abdominal aortic aneurysm model with various inflow angles

Figure 4-1 A simplified model for the abdominal aortic aneurysm

Figure 4-1 shows simplified aneurysm model that is considered here, where the diameter at the proximal neck is 16 mm, the diameter at the distal neck is 13.56 mm, and the diameter of the iliac arteries is 8 mm. The inflow angle, \( \alpha \), decides the ratio of axial to lateral inlet velocity components, namely, \( \alpha = \tan^{-1} \left( \frac{u_{\text{lateral}}}{u_{\text{axial}}} \right) \). The fluid is assumed to be laminar and Newtonian with a viscosity of \( 4 \times 10^{-3} \text{ Pa} \times \text{s} \) and density of \( 1050 \text{ kg/m}^3 \). Figure 4-2 shows the waveforms of the inlet flow velocity and outlet pressure boundary conditions with a maximum Reynolds number of 1700 and a maximum pressure of 120 mmHg. The simulation is conducted in OpenFoam using the PISO incompressible transient flow solver with a 24-core workstation. Besides its unlimited parallel capacity, OpenFoam is chosen because of its transparency and customizability in numerical scheme. The
exact governing equation can be projected onto the POD manifold without extra stabilizing source terms and ensure that the POD manifold is closest to the true dynamics, which is an ideal way to construct a POD ROM. Because the continuity equation is not included in the ROM, the time steps in the PISO solver are chosen to minimize the time step continuity error. Also, because the mesh quality of the original simulation decides the level of the geometrical details included in the ROM model, extra near-wall mesh layers are included in the computational mesh for the simulation to better reconstruct the WSS distributions. The total CPU time is around 200 hours with the 60 sec training simulation.

![Figure 4-2](image_url)  
(a) The inlet flow velocity and (b) the outlet pressure boundary conditions.
4.5 The local POD method

To construct a ROM that is capable of predicting the system behavior over a range of different parameters, we first obtain a set of snapshots from a flow simulation with multiple time-dependent parameters. We call these snapshots the training set. The training set should be simulated when the parameters of interest are changing to create the richest dynamical behavior for the snapshots and the resulting POD modes. In the present study, other than the time-dependent inlet flow velocities and outlet pressure boundary conditions, we have varied the angle of the incoming flow, \( \alpha \), varies from \(-30^\circ \sim +30^\circ\) in the original simulation, following the simple relation of \( \alpha = (30 - t)^3 / 30^3 \cdot 30^\circ \).

The local POD plus the Galerkin projection method is first introduced by (Rapún and Vega, 2010). Projecting the Navier-Stokes equation onto the POD manifold includes calculating the cubic convection term, and the computational effort increases rapidly with the number of POD modes. Therefore, it is computationally efficient to use local sets of POD modes in multiple intervals in the parameter space compared to using a whole set of POD modes covering the entire parameter space. It is subsequently necessary to select the interval of snapshots properly to ensure that every individual local-POD model is capable of reconstructing the original solution that from which the snapshots are sampled. The intervals of the local-POD modes are selected with the help of a priori error estimator introduced in Equation 4.9.
Because the time-dependent parameters in the training set include changing geometric profiles (the inflow angle), and it is difficult for geometrical information contained in the POD modes to be extrapolated, we separated the original simulation results into several intervals for the local-POD modes without skipping any snapshots.

![Graph showing the error in $L^2$-norm over the mode numbers of the local and complete POD ROM model.](image)

**Figure 4-3 The error in $L^2$-norm over the mode numbers of the local and complete POD ROM model.**

It is shown in Figure 4-3 that when attempting to construct a POD model using the snapshots from the entire training set, the RMS error in velocities reduces rapidly in the first 100 modes, but the speed of convergence decreases by further increasing the mode number, which gives the rapid
increasing computational cost for higher modes and resulting in an inefficient ROM. Instead, three local-POD models with $t \in [0,16], [16,44]$ and $[44,60]$ perform similar accuracy when 200 modes are used, and are far more effective than the original POD model. The exact range of time intervals can be decided in run-time by calculating the cross-correlations of the snapshots side-by-side the original simulation, based on the desired RMS error bound, size of the original problem, and the maximum mode number in the ROM model.

Figure 4-4(a) shows the first set of local-POD modes that are generated from snapshots within the first interval ($t \in [0,16]$). The first local-POD mode represent the dominating inflow with a positive inflow angle. It can be observed that except for the first and the second mode, all the other modes satisfy the homogenous boundary condition ($U = 0$) at the inlet, while at the outlet only the Neumann boundary condition ($\nabla \cdot U = 0$) is regulated and non-zero values are allowed. Figure 4-4(b) ($t \in [16,44]$) and Figure 4-4(c) ($t \in [44,60]$) show the second and the third sets of the local-POD modes, respectively. The dominating inflow with zero and negative inflow angles can be observed in their first modes, respectively.
Figure 4-4 The POD modes of the Local-POD models
In the present study, the flow behavior is reconstructed from the ROM for fixed system parameters (for example, fixed inflow angles and fixed frequencies of the pulsatile wave). Despite this, in the training CFD simulations, these parameters are varied in time. The reason behind this is twofold: (i) having continuously changing parameters in the simulation creates more significant transitional motion, which is shown to be valuable for constructing a Galerkin-POD model. (ii) because the parameters are evolved continuously, the previous behavior serves as an effective initial condition to minimize the required cycles to achieve converged solutions.

4.6 Flow velocity reconstruction

In this section, the previously constructed POD ROM is used to reconstruct the flow behavior with various inflow angles, and the reconstructed results are compared to the exact simulated results.
Figure 4-5 (a) Reconstructed and simulated flow velocities of the case of $\alpha = 30^\circ$, $T = 1$ at $t = 0.2$ and 0.3. (b) Reconstructed and simulated flow velocities of the case of $\alpha = 30^\circ$, $T = 1$ at $t = 0.4$ and 0.6.
Figure 4-5 shows the comparison between the reconstructed and simulated flow velocity distributions for $\alpha = 30^\circ$, $T=1.0$ and 200 POD modes at 4 times instance. It can be observed that the flow velocities are well reconstructed in all stages of the systolic cycle in terms of the amplitude and direction of the main flow. This also includes the large vortex that is generated near the aneurysm's sidewall, between $t=0.4-0.6$, which is captured in the reconstructed flow field as well.

In order to conduct a more qualitative comparison, we then compare the modal amplitudes of the POD modes between the reconstructed and the simulated results. The amplitudes of the POD modes in the simulated results are found as:

$$a(t)_j^{\text{sim}} = \langle \tilde{u}^{\text{sim}}(\tilde{x},t) \tilde{U}_j(\tilde{x}) \rangle.$$

where $\tilde{u}^{\text{sim}}(\tilde{x},t)$ is the simulated result, and $a(t)_j^{\text{sim}}$ is the amplitude of the POD modes in the simulated result. Figure 4-6 and Figure 4-7 show the comparison between the reconstructed modal amplitudes and the simulated results with $\alpha = 30^\circ$, $T=1.0$ and 200 POD modes. Agreement was found for both the dominating modes (modes 1-9 in Figure 4-6) and the higher modes (mode 151-159 in Figure 4-7), while in the higher modes the values of the modal amplitudes are significantly smaller than the values in the dominating modes.

Because the POD modes are sorted by their modal amplitude and kinetic energy in the original simulation, the ROM model constructed with the first 200 POD modes capture most of the kinetic energy and the distribution of the kinetic energy is adequately reconstructed subsequently. Figure 4-8a
shows the comparison between the reconstructed and simulated total kinetic energy. At the systolic peak, the kinetic energy is reconstructed well with less than 3 percent of relative error \( \text{defined as } \int_\Omega \left[ \frac{(\tilde{u}^{\text{sim}})^2 - (\tilde{u}^{\text{rec}})^2}{(\tilde{u}^{\text{sim}})^2} \right] dV, \)

where \( \tilde{u}^{\text{rec}} \) is the reconstructed flow velocity distribution). The relative error increases in the diastolic phase (up to 22 percent), but since the flow velocities is much smaller in this phase than in the systolic phase, the majority of the kinetic energy is reconstructed successfully in a systolic cycle. Figure 4-8b shows the reconstructed total kinetic energy using 50, 100 and 200 modes. It can be observed that in the present case the reconstructed kinetic energy converged with relatively low number of modes (~100 modes), while using insufficient number of POD modes the mean flow and the kinetic energy is overestimated.

Figure 2-9 shows the case with a smaller inflow angle \( \alpha = 6^\circ \). The second local-POD model is used for reconstruction. In this case, the vortices are generated closer to the anterior distal side, and are accurately reconstructed by the ROM similar to the previous case. For all the simulated inflow angles \( \alpha = -30 \sim 30^\circ \), the reconstructed flow distributions and kinetic energy match well with the simulated values.
Figure 4-6 Reconstructed and simulated modal amplitudes for $\alpha=30^\circ$, $T=1$ (dominating modes.).

Figure 4-7 Reconstructed and simulated modal amplitudes for $\alpha=30^\circ$, $T=1$ (higher modes).
Figure 4-8 (a) Reconstructed and simulated total kinetic energy and (b) reconstructed total kinetic energy with various mode numbers in the ROM model of the case of $\alpha = 30^\circ$, $T = 1$. 
Figure 4-9 Reconstructed and simulated flow velocities for $\alpha=6^\circ$, $T=1$. 
4.7 Reconstruction of the wall shear stress

In this section, we focus using the ROMS for calculating the WSS distribution. Because the distribution of WSS is closely related to the near-wall flow pattern, the first $N$ POD modes, which are sorted by the flow's kinetic energy, are not guaranteed to contain the most dominant information for the WSS reconstruction. Therefore, reconstructing the distribution of WSS is more challenging than reconstructing the flow velocity distribution. To overcome this problem, we propose a remedy by replacing some POD modes (the last 10 modes for the 50-mode model, and the last 20 modes for the 100 and 200-mode model in the present study) with some other originally unused higher POD modes, which have the most significant WSS magnitude, in the ROM model. Figure 4-10 shows the reconstructed and simulated WSS distributions for $\alpha=30^\circ$, $T=1.0$. While the largest magnitude of WSS happens around the aneurysm's distal neck and iliac bifurcation right at the systolic peak, the large-angle inflow induced a large vortex near the sidewall ($t=0.3\sim0.4$) and generated large WSS on the sidewall.

For a smaller inflow angle ($\alpha=6^\circ$, $T=1.0$, Figure 4-11), a pair of small vortices is generated close to the aneurysm's distal neck, instead of one one-sided vortex. In this case, the WSS distribution on the aneurysm's sidewall is much smaller in magnitude and the area with high WSS is much closer to the
aneurysm’s distal neck. Higher flow velocities and WSS are observed in the iliac branches. ROM reconstructs all the behavior successfully.

To quantify the evolution of WSS distribution over inflow angles, Figure 4-12 shows the reconstructed and simulated time evolution of the WSS on the proximal and distal neck (green) and on the sidewall (blue) with various inflow angles. The WSS distribution on both the proximal and distal neck area is mainly generated by the main inflow, and is reconstructed accurately by the ROM model. The overall time evolution of WSS applied on the aneurysm’s sidewall is also well reconstructed, although that the maximum magnitude of the WSS is slightly overestimated.
Figure 4-10 The reconstructed and simulated WSS for $\alpha = 30^\circ$ and $T=1$. 
Figure 4-11: Reconstructed and simulated WSS for $\alpha = 6$ and $T=1$. 
Figure 4-12 The reconstructed and the simulated total WSS on the aneurysm’s sidewall and neck area for $\alpha = (a) \ 30^\circ \ (b) \ 6^\circ \ (c) \ 0^\circ \text{ and } T=1$.

Figure 4-13 shows the reconstructed WSS results for $\alpha = 30^\circ \text{ and } T=1.0$, using different numbers of POD modes in the ROM model. Again, relatively
small number of POD modes is sufficient for a successful WSS estimation (~100 modes). Using insufficient number of modes has different effects on the WSS estimations at different locations of the aneurysm model. With insufficient modes, the WSS magnitudes on the aneurysm’s proximal and distal neck area are overestimated, similar to the behavior of the kinetic energy estimations (Figure 4-8), since both characteristics are closely related to the main flow pattern. On the other hand, the WSS magnitudes on the aneurysm’s sidewalls are underestimated with insufficient number of modes. This phenomenon confirms the fact that the smaller flow structures and the associated WSS distribution require higher POD modes in the reconstruction.

Figure 4-13 (a) Reconstructed and simulated total WSS on the neck area and (b) the reconstructed total WSS on the aneurysm’s sidewall with various number of mode in the ROM model for α=30° and T=1.

The present ROM model approximates the distribution of WSS efficiently over a range of inflow angles. The comparison for the average
value of the reconstructed and simulated WSS values on the aneurysm’s sidewall over a cardiac cycle is shown in Figure 4-14. Overall, the magnitude of the wall shear stress on the aneurysm’s proximal and distal neck decrease and the magnitude of the wall shear stress on the aneurysm’s sidewall increase with the inflow angle.

![Graph showing mean WSS vs. inflow angle](image)

**Figure 4-14**  
Reconstructed and simulated mean WSS on the aneurysm’s sidewall over different inflow angle.

Figure 4-15 shows the comparison between the reconstructed and simulated WSS distributions with three different frequencies of the hemodynamics waveform. Considering the fact that the frequency of the cardiovascular waveform used in the original training set was set to 1 Hz, the results of Figure 4-14 show that it is possible for a ROM to predict the behavior of a system with parameters that are totally different from those in the original simulation. This fact greatly increases the robustness of the ROM.
Using the ROM, the areas with low and high WSS can be easily recognized and the magnitude of peak WSS can be efficiently estimated with very little computational time. A ROM with 100 POD modes usually required about 5 min for a 10 sec simulation, and is negligible when compared to the cost of the CFD simulation (~30 CPU hours for the same 10 sec simulation).
Figure 4-15: Reconstructed and simulated total WSS on the aneurysm’s sidewall and neck area for $1/T = (a) 1.0$ (b) 1.3 (c) 1.5 and $\alpha = 30^\circ$. 
4.8 Conclusions

A reduced-order model (ROM) is constructed to study the physiological flow in abdominal aortic aneurysms. The method of snapshot Proper Orthogonal Decomposition (POD) is used to construct the reduced-order bases based on a highly robust training CFD simulation, thus a rich dynamic behavior can be captured in the POD modes. Reconstructed distribution of the flow velocities and the wall shear stress are calculated based on several different physiological flow parameters and are in agreement with the exact simulated solutions, while maintaining the ROM’s computational efficiency. The main flow pattern and the kinetic energy can be reconstructed from the ROM models very efficiently, while the reconstruction of the wall shear stress distributions requires higher POD modes. The ROM efficiently predicts the location and magnitude for maximum WSS to appear on the aneurysm’s sidewall with various inflow angles and frequencies.

The ability for a POD ROM to predict the flow and WSS distributions with various parameters is promising for clinical applications. A robust ROM, trained by well-designed CFD simulation, can be used to investigate the flow conditions in patient-specific image-based aneurysm models with various uncertainties in flow conditions. Even if high-resolution meshes are used for patient-specific aneurysm CFD simulation, the computational cost for ROM will remain very low because the spatial information of the mesh is removed from the POD process. This POD-ROM can also be useful for reconstructing
and analyzing patient specific flow measurements. This greatly enhances the ability for computational fluid dynamics for personal treatment and clinical usage.
CHAPTER 5

GROWTH AND REMODELING OF ANEURYSMS WITH NON-UNIFORM THICKNESS DISTRIBUTION

5.1 Introduction

In this chapter, we demonstrate how the reduced order model of the hemodynamics load and the shell model with non-uniform thickness can be utilized to model the growth and remodeling (G&R) process of non-uniform AAAs. Since the discoveries of unruptured aneurysms have become more frequent, it is critical to find a reliable model to predict the aneurysm’s growth and rupture to determine accurate criteria for clinical decision-making (Ambrosi et al., 2011). AAAs are often characterized by a thinned media and a reduced elastin. It was reported that the volume fraction of elastin can be dramatically reduced from 22.7% ± 5.7% to 2.4% ± 2.2% in AAAs when compared with nonaneurysmal aortas (He and Roach, 1994). While normal magnitude of WSS supports endothelial cell survival within healthy aortas (Paszkowiak and Dardik, 2003), abnormally low and high magnitude WSS lead to pathological changes of arterial wall and results in the degradation of aneurysm’s tissue (Kadasi et al., 2013a).

In the breakthrough work by Boussel et al. (2008), the MRI data acquired from two different times were compared and a significant correlation was found between the regions with low time-average WSS, which is predicted by computational fluid dynamics (CFD), and the regions with
large radial growth of cerebral aneurysms (Boussel et al., 2008). This is of great significance for using medical images and CFD to predict future aneurysm growth. On the other hand, abnormally high magnitude of WSS and high spatial gradients of WSS (WSSG) have been related to the initiation and development of cerebral aneurysms from an arterial bifurcation (Hoi et al., 2004).

Based on these understandings, several numerical models have been proposed to model the G&R process of aneurysms in order to describe the changes in the aneurysmal wall’s mass and properties and the resulting enlargement of cerebral or abdominal aneurysms. In traditional G&R models, the stress-mediated collagen turnover is assumed to be mainly responsible for the aneurysm’s enlargement (Ambrosi et al., 2011; Volokh and Vorp, 2008). However, in more recent studies, hemodynamic environment and the resulting elastin degradation have been included in the G&R models because of the recent advanced understanding in molecular and cell biology.

Zeinali-Davarani et al. (2011a) assumed that the elastin degradation has the form of multiple circular spatial functions on an aorta segment and results in aneurysms shapes with different stress distributions and growth rates. Most interestingly, the simulation result suggests that a degraded region located on the convex side (with negative Gaussian curvature) of the aneurysmal wall leads to larger enlargement of the aneurysm than a degraded region located on the concave side (with positive Gaussian curvature) of the
aneurysmal wall. This result indicates the potential clinical application with computational G&R model with patient-specific geometries.

5.2 Wall Shear Stress (WSS) and elastin degradation

Despite the fact that the quantitative relation between WSS and the rate of elastin degradation is still unknown, it has been found that a baseline magnitude of WSS is necessary to maintain the mass of the elastin, and the relation between the rate of degradation and WSS is often assumed to have a simplified relation of (Sheidaei et al., 2011; Watton et al., 2009):

$$F_D(\tau_w) = \begin{cases} 0, & \tau_w \geq \tau_{\text{crit}} \\ \left( \frac{\tau_{\text{crit}} - \tau_w}{\tau_{\text{crit}} - \tau_X} \right)^2, & 1 < \tau_w < \tau_{\text{crit}} \\ 1, & \tau_w \leq \tau_X \end{cases}$$

(5.1)

where $F_D$ is the degradation function, $\tau_w$ is the minimum magnitude of WSS over a cardiac cycle, $\tau_{\text{crit}}$ is the minimum WSS to maintain the arterial structure and $\tau_X$ is the amount of WSS to allow maximum elastin degradation.

Following Watton et al. (2009), here we pick $\tau_{\text{crit}} = 2$ Pa and $\tau_X = 0.5$ Pa.
To demonstrate the influence of asymmetric hemodynamic load on the G&R process, we consider a case with a maximum Re of 3400 (defined based on the diameter of the inlet) and a large inflow angle of 30°. The WSS estimation is based on the reduced order model introduced in Chapter 2. Figure 5-1(a) shows the minimum magnitude of WSS over a cardiac cycle and Figure 5-1(b) shows the estimated degradation function based on Equation 5.1. It can be observed that large inflow angle generates very asymmetric distribution of WSS. One low WSS region is located at the middle section of the aneurysm, and another is located closed to the aneurysm’s proximal neck. As
discussed by Figueroa et al. (2009), the G&R process is likely to be originated from these localized regions with abnormal magnitudes of WSS and creates localized inhomogeneity in the aneurysmal wall.

5.3 A shell-based aneurysmal wall model with non-uniform thickness

One central issue about G&R modeling of aneurysms is to model the non-uniform and localized mechanical properties of the aneurysmal wall (Figueroa et al., 2009; Zeinali-Davarani et al., 2011a). It has been found by dissection that AAAs can have a very non-uniform distribution of thickness, modulus, and failure tension, especially in the neighborhoods of the rupture site (Raghavan et al., 2006). It has been reported that the measured thickness of AAA’s sidewall can be reduced by 75% within several millimeters while the difference between the maximum and the minimum thickness can be more than ten times (Raghavan et al., 2011). Therefore, the G&R simulations aim to model the non-uniform and localized spatial distributions of thickness degradation.

Despite these facts, constant thickness or a simple thickness-radius relation is often assumed in traditional FEM-based simulations to study an aneurysm’s in-plane stress and deformations under hemodynamic loads. These simplified models often result in overly gradual, or sometimes totally erroneous stress distributions when compared to numerical models with non-uniform thickness distribution, which, alas, requires measurements from a harvested AAA (Raghavan et al., 2006) or scanning images with ultra-high
Because the quantitative relation between WSS and elastin degradation is still under speculation, the locations for the weaken regions of aneurysmal wall can be anticipated, but the exact magnitude of elastin degradation cannot. Also for in-vivo measurements, quantitative details of the non-uniform thickness distributions are still difficult to be obtained through non-invasive medical images. For example, for cerebral aneurysms, intraoperative microscopy images are used to identify the “superthin” translucent areas, intermediate areas and thick calcified areas (Kadasi et al., 2013b). However, no quantitative detail of aneurysmal wall thickness is provided in this approach.

Therefore, a numerical tool that is capable of modeling different magnitudes of localized thickness inhomogeneity of the AAA wall efficiently will be greatly beneficial for studying the stress, development and rupture of aneurysms. This model is presented here.

Under the thin shell theory with Love’s hypothesis, the equation of doubly curved shells (Equation 2.2-5) can be rearranged and the elastic energy of thin shells can be separated into stretching and bending parts:


\[ E_{\text{Stretching}} + E_{\text{Bending}} \]

\[ = \iint (N_{xx} \varepsilon_{xx} + N_{\theta \theta} \varepsilon_{\theta \theta} + N_{x \theta} \gamma_{x \theta}) dxd\theta + \iint (M_{xx} k_{xx} + M_{\theta \theta} k_{\theta \theta} + M_{x \theta} k_{x \theta}) dxd\theta \]

\[ + \iint \frac{Eh(\tilde{r})}{1 - \nu^2} \left[ \left( \varepsilon_{x,0} + v \cdot \varepsilon_{\theta,0} \right) \cdot \varepsilon_{x,0} + \left( \varepsilon_{x,0} + \varepsilon_{\theta,0} \right) \cdot \varepsilon_{\theta,0} + \frac{\varepsilon_{x,0}}{2(1 - \nu)} \right] dxd\theta + \]

\[ + \iint \frac{Eh^3(\tilde{r})}{12(1 - \nu^2)} \left[ \left( k_{x,0} + v \cdot k_{\theta,0} \right) \cdot k_{x,0} + \left( v \cdot k_{x,0} + k_{\theta,0} \right) \cdot k_{\theta,0} + \frac{k_{x,0}}{2(1 - \nu)} \right] dxd\theta \]

\[ = \iint \left( \frac{Eh(\tilde{r})}{1 - \nu^2} \Gamma_s \right) dxd\theta + \iint \left( \frac{Eh^3(\tilde{r})}{12(1 - \nu^2)} \Gamma_b \right) dxd\theta \]

\( (5.2) \)

where \( s \) contains the terms for stretching and \( b \) contains the terms for bending. The parameters used in this equation have been defined in Section 3.2 of the Thesis. For incompressible elastic material and a constant \( \nu \), \( s \) and \( b \) are functions of the shell’s geometrical profiles (mid-plane strain and curvature's components) and are independent from the shell’s mechanical properties (thickness and stiffness). We can therefore separate the elastic energy stored in the weakened area with reduced thickness from the area with uniform thickness:

\[ \iint \left( \frac{Eh(\tilde{r})}{1 - \nu^2} \Gamma_s \right) dxd\theta + \iint \left( \frac{Eh^3(\tilde{r})}{12(1 - \nu^2)} \Gamma_b \right) dxd\theta \]

\[ = \frac{E}{1 - \nu^2} h_0 \iiint \Gamma_s dxd\theta + \frac{E}{1 - \nu^2} h_0 \iiint (\varepsilon \cdot \eta(\tilde{r}) - 1) \Gamma_s dxd\theta \]

\[ + \frac{E}{12(1 - \nu^2)} h_0^3 \iiint \Gamma_b dxd\theta + \frac{E}{12(1 - \nu^2)} h_0^3 \iiint (\varepsilon^3 \cdot \eta(\tilde{r})^3 - 1) \Gamma_b dxd\theta , \]

\( (5.3) \)

where the thickness distribution in the thin area is defined as \( h(\tilde{r}) = h_0 \cdot \varepsilon \cdot \eta(\tilde{r}) \), \( \varepsilon \) is a parameter to define the magnitude of the thickness reduction; \( \int_\omega \) denotes the integral over the entire shell; \( \int_\omega \) denotes the integral over the weak area; \( \tilde{r} \) is the spatial vector pointed to the weak area,
and $h_0$ is the uniform thickness of the rest of the aneurysm. Since only the geometrical profiles are integrated in Equation 5.3, one will be able to investigate different magnitudes and mechanical properties of the local imperfection in the thickness without calculating the surface integrals repeatedly.

Figure 5-2: Schematic of a simplified AAA model with a circular thin area.

To model the localized elastin degradation in AAAs, in the present numerical model, the degraded thin area is modeled as a small circular spatial function with its thickness defined as

$$h(\bar{r}) = h_0 \cdot e^{-\left(\frac{d\bar{r}}{\varepsilon}\right)^2},$$

(5.4)
where \( \vec{r}_0 \) is the spatial vector pointed to the center of the weak area and \( a \) is the geometrical parameter that decides on the radius of the thin area (Figure 5-2). The AAA shape of Figure 5-2 has negative Gaussian curvatures on the area of distal side and close to the proximal neck, and positive Gaussian curvatures in the middle where the radius has its maximum. The aneurysm’s geometrical profile is modeled by a 4th-order spline curve to ensure the curvature and its derivatives are always continuous.

The shell-based model is chosen instead of a membrane-based model because AAAs usually have larger thickness to radius ratios, and therefore the bending stress, which is proportional to the cubic power of the thickness, is significantly higher than the bending stress in cerebral aneurysms. This also leads to a need for a more accurate estimation of curvature and an increased spatial and temporal resolution for patient-specific scanning for the AAA wall’s in-plane stress.

5.4 Deformation of aneurysmal wall under hemodynamic loads

It has been shown that the magnitude of pressure-induced in-plane stress is closely related to the curvature of the aneurysm’s surface (Lu et al., 2013). With the same thickness and hemodynamic loads, an aneurysmal wall with a positive Gaussian curvature has its two in-plane stress components accumulated to resist the loading in the normal direction, while an aneurysmal wall with a negative Gaussian curvature (a saddle region) has its
two in-plane stress components compete with each other. Therefore, high magnitudes of tension can often be observed in the saddle regions.

![Diagrams showing pressure and displacement](image)

**Figure 5-3:** The reconstructed (a) pressure and (b) wall displacement of the aneurysm for a case with constant wall thickness under peak systolic pressure.

To find the maximum deformation of the aneurysm under hemodynamic loads, we apply the reconstructed maximum systolic pressure from the reduced order model discussed in Chapter 4, to the aneurysm’s structural model. Since smaller magnitudes of displacement and stress are usually reported for the healthy abdominal aorta and iliac artery compared to the values reported in the aneurysm's main body (Li and Kleinstreuer, 2007; Scotti et al., 2005), the aneurysm is assumed to be fixed at the proximal and
distal neck and only the deformation of the aneurysm’s main body is considered.

First, we consider an aneurysm model with uniform thickness (Figure 5-3). The aneurysmal wall is assumed to have an original uniform thickness of 2 mm. At peak systolic, the pressure distribution is close to uniform. The pressure difference between the inlet and outlet of the aneurysm is less than 5 mmHg (Figure 5-3(a)). For an aneurysm model with uniform thickness, the pressure loading creates an axis-symmetric deformation (Figure 5-3(b)). The maximum amplitude is observed near the proximal and distal necks of the aneurysm, where the aneurysmal wall has zero or negative Gaussian curvatures. Despite having the smallest thickness to radius value, the middle section of the aneurysm has the minimum displacement, because the positive value of Gaussian curvature there allows the pressure loading to be resisted by the in-plane stress in both the axial and circumferential directions.
Figure 5-4 (a) A weak area close to the distal neck of the aneurysm, and the deformation of the aneurysmal under peak systolic pressure for (b) 30%, (c) 50%, and (d) 70% reduction in thickness.

Now, we investigate the influence of a thin spot on the deformation of the aneurysm under pressure loading. As discussed in Chapter 4, the influence of a thin spot largely depends on the sign of the Gaussian curvature of where the weak spot is located. For a weak spot close to the aneurysm’s distal neck (Figure 5-4(a)), a small amount of imperfection leads to large-magnitude deformation. With a maximum 30% reduction in the thickness (Figure
5-4(b)), the maximum amplitude of the aneurysmal wall’s deformation is nearly doubled. As the thickness is further reduced, the deformation originated from the weak area also extends to its neighboring regions and creates large-scale deformation compared with the original axis-symmetric deformation (Figure 5-4(d)).

![Figure 5-5](image)

**Figure 5-5** (a) a weak area in the middle of the aneurysm, and the deformation of the aneurysmal wall under peak systolic pressure for (b) 30%, (c) 50%, and (d) 70% reduction in thickness.
For a weak spot located at the aneurysm’s midpoint (Figure 5-5(a)) where the positive Gaussian curvature has its maximum, the influence of the thin area is insignificant when compared to the previous case. With a 50% reduction in the thickness (Figure 5-5(c)), the amplitude of deformation observed at the location of the weak area is merely equal to the maximum amplitude of the original axis-symmetric deformation with constant thickness. The deformation originated from the weak area is also localized and does not extend to its neighboring regions.

5.5 Degradation of the aneurysm wall

In this section, the hemodynamic loading is applied to the AAA model with the thickness degradation estimated by the degradation function of Equation 5.4 and the reconstructed WSS shown in Figure 5.1. We consider the spatial distribution of the regions of low WSS area in Figure 5-1 and for computational simplicity, we assume the spatial distributions of the degraded regions to be two circular areas (Figure 5-6a), similar to the spatial functions assumed by Zeinali-Davarani et al.(2011b). The degradation of the AAA wall’s thickness is related to the survival rate function as (Sheidaei et al., 2011):

\[
h^j = h^b \cdot \exp\left(-\int_0^{t_j} 0.02 \cdot F_D(t) dt\right),
\]  

(5.5)

where \(h^j\) is the thickness of the weak area at time \(t = t_j\). Using this function, the thickness of the thin area is assumed to be reduced by 18% after 100 days
and 48% after 300 days of elastin degradation. Figure 5-6 shows the aneurysmal wall's hemostatic state after 100, 200 and 300 days of growth and remodeling.

![Figure 5-6](image)

**Figure 5-6** (a) The locations of the two areas with elastin degradation and the deformed geometry after (b) 100, (c) 200, and (d) 300 days of growth and remodeling.

Based on the results of Figure 5-6, it is clear that the saddle regions are most sensitive to the reduction of the wall thickness. Furthermore, the saddle regions are likely to be influenced by the thickness reduction of their neighboring areas (points A and B in Figure 5-6d) even when the thickness in that location itself is intact. This phenomenon is similar to that found in by Zeinali-Davarani et al. (2011), where larger aneurysm enlargement is
observed in the convex side of aneurysmal wall with a negative Gaussian curvature.

Therefore, it is most critical to identify the potential thickness reduction in the saddle regions and the neighboring area when constructing 3D models from medical images. Regions with abnormally low WSS and negative Gaussian curvature should be carefully monitored for potential aneurysm growth or rupture. Considering the existing uncertainties in measuring the mechanical and geometrical properties either from image-based models or computational G&R models, mentioned in Section 5.1, the present approach allows us to easily find the dependency of the bending and stretching energy on different magnitudes of thickness reduction of the aneurysmal wall. Therefore, the model can be very beneficial for clinical assessment when a thin translucent area is recognized but quantitative detail of wall thickness is unknown. It can also be applied to computational G&R models with different assumptions in WSS-dependency of elastic degradation.

Finally, because the present model is focused on the short-term elastic degradation to identify the potential region for aneurysm enlargement and rupture, the long-term stress-mediated turnover of collagen and smooth muscle is neglected. Nevertheless, the same approach could be generalized to the shell models with multiple layers to include the mechanical properties of collagen fibers and muscles (Amabili et al., 2012).
CONCLUSIONS AND FUTURE WORK

In this thesis, we derive a reduced order fluid-structure interaction model of thin shells and use it to study the hemodynamic loads and pressure-induced wall stress in aneurysms with non-uniform thickness. Since it is well known that a shell's stress and deformation and closely related to the sign and magnitude of a shell's surface curvature, in Chapter 2, a new numerical method to calculate the flow-induced instabilities of shells of revolution with non-zero Gaussian curvatures is discussed. The numerical method contains two parts: the first part calculates the coupled eigenmodes of shells of revolution with the Galerkin method based on the exact doubly-curved thin shell equation, and the second part calculates the contribution of flow forces associated with the shell's different modes using the boundary elements method (BEM). It is found that shells with negative Gaussian curvatures conveying fluid change their buckling mode shapes frequently as the magnitude of the negative curvature increases, while their buckling mode shapes are similar to those observed in uniform and conical shells. On the other hand, shells with positive Gaussian curvatures conveying fluid buckle with their deformations localized close to the area with higher local flow velocities.

In Chapter 3, the numerical model discussed in Chapter 3 is further extended to study the flow-induced buckling of shells with nonzero Gaussian curvatures and thin spots to investigate a shell structure's sensitivity to
thickness imperfection with non-zero Gaussian curvature. It is found that for a shell with a negative Gaussian curvature conveying fluid, the buckling modes are always global, both for a perfect shell (without any thin spot) and for a shell with a thin spot somewhere on its surface. For a shell with a positive Gaussian curvature and a thin spot conveying fluid, however, three distinct modes are obtained: (i) the original local mode shape of a perfect shell, for large thin spot ratios (defined as the ratio of the thickness of the thin spot to the thickness of the rest of the shell) (perfect mode), (ii) a mode that is a combination of the original local mode and a new local mode centered around the thin spot, for moderate thin spot ratios (combined mode), and (iii) a new local mode, centered around the thin spot, for small thin spot ratios (thin spot mode). The flow-induced stiffness, the shape and the location of the thin spot, and the global thickness of the shell decide on these mode shapes.

To utilize this newly developed shell model to study the deformations of aneurysmal wall under flow loading, in chapter 4, we construct a reduced-order model (ROM) to study the hemodynamic loads in aneurysms, with a focus on a simplified geometry of fusiform aneurysm. The method of local Proper Orthogonal Decomposition (local-POD) is used to construct the reduced order bases based on a highly robust training CFD simulation. The main flow pattern and the wall shear stress (WSS) distributions can be reconstructed from the ROM efficiently. Using the ROM, the areas with low and high WSS can be easily recognized and the magnitude of peak WSS can be
efficiently estimated with very little computational time when compared to the duration of the original simulation.

In Chapter 5, we demonstrate how the ROM of the hemodynamics loads and the shell model with non-uniform thickness can contribute to the growth and remodeling (G&R) process of non-uniform AAAs. It is found that the saddle regions, where the pressure-induced stress is most concentrated, are also the most sensitive to the reduction of the wall degradation. Furthermore, the saddle regions are likely to be influenced by the thickness reduction of their neighboring areas even when the thickness in that location itself is intact. Therefore, it is most critical to identify the potential thickness reduction in the saddle regions and the neighboring area when constructing 3D models from medical images.

This thesis leads to several future work directions. One future work that emerges naturally is applying the present numerical models to image-based patient-specific models of abdominal or cerebral aneurysms. For the POD-ROM physiological flow model, the present approach can be applied to geometries with patient-specific and asymmetric models easily, because the geometrical information is irrelevant in the POD technique. On the other hand, the shell models with non-uniform thickness and asymmetric geometries require further study.

The FSI coupling scheme can also be further improved. Currently, only the one-way coupling is utilized to study the aneurysmal wall’s pressure-induced stress. A two-way coupling scheme can be used for a more accurate
estimation of flow velocity, pressure and WSS. Similar to the application of POD-ROM in aeroelasticity, one can therefore excite the flow field with the aneurysmal wall's structural modes and couple the structural modes with their corresponding POD flow modes.

The present numerical formulation of shells with non-uniform thickness can also be applied to various material descriptions that the elastic energy can be written in a conservative form in strain and stress. Therefore, it is possible to investigate the aneurysmal walls with hyperelastic or viscoelastic properties and thickness inhomogeneity. However, further research is required to investigate the robustness and efficiency of the numerical models in those scenarios.

Finally, in this research, it can be clearly found that the distributions of the in-plane stress of the aneurysmal wall and the aneurysmal wall’s sensitivity to thickness degradation are closely related to the wall’s distribution of Gaussian curvatures. However, accurate estimation of the curvature required high-resolution medical images. A more robust numerical model of the aneurysmal wall that is able to handle the uncertainties in curvatures from the image-based models will be highly desired.
APPENDIX

EQUATION OF MOTION OF SHELLS OF REVOLUTION WITH NON-ZERO GAUSSIAN CURVATURE AND THIN SPOTS

For a shell of revolution with non-uniform cross-sectional area, both principal curvatures are functions of the $x$-axis. If the shell is thin and zero strain values are assumed for small rigid body motions, Love's hypothesis can be applied (Soedel, 2004):

$$
\begin{align*}
\epsilon_{xx} &= \epsilon_{x,0} + zk_x, \\
\epsilon_{\rho \rho} &= \epsilon_{\rho,0} + zk, \\
\epsilon_{x \rho} &= \epsilon_{x,\rho} + zk_x,
\end{align*}
$$

(A.1)

where $\epsilon_{x,0}$ and $\epsilon_{\rho,0}$ are the strain components in the mid-plane of the shell, and $\epsilon_{xx}$ and $\epsilon_{x \rho}$ are the strain components in an arbitrary position, $k_x = k_x(x) = 1/R_x(x)$ is the principal curvature of the meridian, and $k = k(x) = 1/R(x)$ is the principal curvature of the parallel circles. The mid-plane strain-displacement relations of shells of revolution are (Soedel, 2004):

$$
\begin{align*}
\epsilon_{x,0} &= \frac{1}{A_x} u + \frac{1}{A_x A_x} \frac{A_x}{A_x} v + \frac{w}{R_x}, \\
\epsilon_{\rho,0} &= \frac{1}{A} v + \frac{1}{A_x A_x} \frac{A_x}{A_x} u + \frac{w}{R}, \\
\epsilon_{x,\rho} &= \frac{1}{A_x} v + \frac{1}{A_x} u - \frac{1}{A_x A_x} \frac{A_x}{A_x} v,
\end{align*}
$$

(A.2)

where $u = u(s, \ )$, $v = v(s, \ )$ and $w = w(s, \ )$ are the axial, circumferential and radial displacements, respectively. The Lamé parameters, $A_x$ and $A$, can be found from the principal curvature (Soedel, 2004):
\[ A_x = \sqrt{1 + \left( \frac{dR_x}{dx} \right)^2}, \quad A = R. \quad (A.3) \]

The curvature components in each direction are:

\[
k_{xx} = \frac{w}{R_x R} + \frac{1}{A_x A R_x} \frac{\partial A}{\partial x} u + \frac{1}{A_x A R_x} \frac{\partial A}{\partial x} v + \frac{1}{A_x R_x} \frac{\partial u}{\partial x} + \frac{1}{A_x R_x} \frac{\partial v}{\partial x},
\]

\[
k_x = \frac{w}{R_x R} + \frac{1}{A_x A R_x} \frac{\partial A}{\partial x} u + \frac{1}{A_x A R_x} \frac{\partial A}{\partial x} v + \frac{1}{A_x R_x} \frac{\partial u}{\partial x} + \frac{1}{A_x R_x} \frac{\partial v}{\partial x},
\]

\[
k_{xq} = \frac{1}{A} \frac{\partial}{\partial} \left( \frac{1}{R_x} \frac{1}{R} \right) + \frac{1}{A_x} \frac{\partial}{\partial x} \left( \frac{1}{R} \frac{1}{R_x} \right) + \frac{2}{A_x A^2} \frac{\partial A}{\partial x} \frac{\partial w}{\partial x}
\]

\[
+ \frac{2}{A_x} \frac{\partial}{\partial} \frac{\partial A}{\partial x} \frac{\partial w}{\partial x},
\]

The stress components can be calculated by:

\[
s_{xx} = \frac{E}{(1 - v^2)} \left( \sigma_{xx} + \nu \sigma_{xy} \right)
\]

\[
= \frac{E}{(1 - v^2)} \left( \sigma_{xx} + \nu \sigma_{xy} \right)
\]

\[
s_{xq} = \frac{E}{2(1 - v)} \sigma_{xq},
\]

\[
(A.4)
\]

The system’s Lagrangian, \( L = U - T \), where \( U \) is the potential energy and \( T \) is the kinetic energy, is found to be (Soedel, 2004):

\[
L = \iiint (N_x \varepsilon_x + N_\theta \varepsilon_\theta + N_{\phi} \varepsilon_{\phi} + M_x k_x + M_\theta k_\theta + M_{\phi} k_{\phi} + M_x k_x) dx d\theta
\]

\[
- \iiint h(u^2 + v^2 + w^2) dx d\theta,
\]

\[
(A.6)
\]

where \( N_x, N \) and \( N_\phi \) are the resultant forces; \( M_x, M \) and \( M_\phi \) are the resultant moments, and \( h = h(x, \theta) \) is the non-uniform distribution of the shell’s thickness. The resultant forces and moments in different directions are
obtained by integrating the strain tensors over the thickness of the shell (Soedel, 2004):

\[
N_{xx} = \int_{-h(x,q)/2}^{h(x,q)/2} h(x,q) N_{xx} dz, \quad N = \int_{-h(x,q)/2}^{h(x,q)/2} h(x,q) N_x dz, \quad N_x = \int_{-h(x,q)/2}^{h(x,q)/2} h(x,q) N_{x} dz, \\
M_{xx} = \int_{-h(x,q)/2}^{h(x,q)/2} h(x,q) M_{xx} z dz, \quad M = \int_{-h(x,q)/2}^{h(x,q)/2} h(x,q) M_x z dz, \quad M_x = \int_{-h(x,q)/2}^{h(x,q)/2} h(x,q) M_{x} z dz.
\]

The elastic energy can be categorized into the stretching and bending energy:

\[
E_S + E_B = \int \left( N_{xx} + N_{xx} x + N_x x \right) dx \, dq + \int \left( M_{xx} + M_x x \right) dx \, dq \\
= \int \frac{E(h)}{2} \left[ \left( x_{,0} + x_{,0} \right) \cdot x_{,0} + \left( x_{,0} + x_{,0} \right) \cdot x_{,0} + \frac{x_{,0} x_{,0}}{2(1-n^2)} \right] dx \, dq + \int \frac{E h^3}{12(1-n^2)} \left[ \left( k_{x,0} + k_{x,0} \right) \cdot k_{x,0} + \left( k_{x,0} + k_{x,0} \right) \cdot k_{x,0} + \frac{k_{x,0} k_{x,0}}{2(1-n^2)} \right] dx \, dq
\]

Where \( E_S \) and \( E_B \) are the stretching and bending energy of the shell, respectively. After applying Hamilton’s principle to the system’s Lagrangian, \( L \), in equation A.6:

\[
\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt} \frac{\partial L}{\partial q} = 0,
\]

the final equation of motion has a form of:

\[
M \ddot{q} = (K_S + K_B) q = (h \kappa_S + h^3 \kappa_B) q,
\]

where \( K_S \) is the shell’s stretching stiffness, \( K_B \) is the shell’s bending stiffness, \( \kappa_S \) and \( \kappa_B \) are terms that depend on the shell’s geometrical profile other than the thickness and \( q \) is the generalized coordinate in the axial, circumferential or radial direction.

Considering a thin spot located at a location defined by \( x_a < x < x_b \) and \( \theta_a < \theta < \theta_b \),

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and its thickness is \( h \), while the thickness of the rest of the shell is \( h_0 \),
equation A.10 can be further written as the summation of the stiffness over
the thin area, \( _1 \), and the rest of shell, \( _2 \):

\[
M_h \frac{\partial^2 q}{\partial t^2} = \left( h \iint_{_1} s \, dx \, d \beta + h^3 \iint_{_2} \beta \, dx \, d \beta \right) q
\]

\[
= \left[ h_0 \iint_{_1} s \, dx \, d \beta + \bar{h} \iint_{_2} s \, dx \, d \beta + h_0^3 \iint_{_2} \beta \, dx \, d \beta \right] q.
\]

The final form of the equations is presented as:

\[
\rho h \frac{\partial^2 \nu}{\partial t^2} = -\frac{1}{R^2} \left( \frac{1}{A} (V - 1) \left( \frac{\partial (1/R_\nu)}{\partial \nu} \frac{\partial u}{\partial \theta} + \frac{1}{R_\nu} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{A} \frac{\partial^3 u}{\partial \xi^3} + \frac{\partial A}{\partial \xi} \frac{\partial^2 w}{\partial \xi^2} + \frac{\partial A}{\partial \xi} \frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \xi^2} \right) \right) + \frac{1}{R^2} \left( \frac{1}{B} \frac{\partial^2 u}{\partial \beta^2} - \frac{1}{B} \frac{\partial A}{\partial \beta} \frac{\partial^2 w}{\partial \beta^2} + \frac{1}{A} \frac{\partial^3 w}{\partial \beta^3} + \frac{\partial A}{\partial \beta} \frac{\partial^2 w}{\partial \alpha^2} + \frac{\partial A}{\partial \beta} \frac{\partial^2 w}{\partial \alpha^2} \right)
\]

\[
+ \left( \frac{1}{R_\nu} \frac{\partial u}{\partial \beta} + \frac{1}{R_\nu} \frac{\partial^2 w}{\partial \beta^2} + \frac{1}{A} \frac{\partial^3 w}{\partial \beta^3} + \frac{\partial A}{\partial \beta} \frac{\partial^2 w}{\partial \beta^2} + \frac{\partial A}{\partial \beta} \frac{\partial^2 w}{\partial \beta^2} \right) \frac{\partial A}{\partial \beta} \frac{\partial^2 w}{\partial \beta^2} + \frac{\partial A}{\partial \beta} \frac{\partial^2 w}{\partial \beta^2} \right)
\]

\[
= \frac{1}{h_0} \left( \frac{1}{k} \int \int s \, dx \, d \beta + h_0^3 \int \int \beta \, dx \, d \beta \right) q.
\]
\[
\frac{\partial^2 u}{\partial t^2} = \frac{1}{AB} (K \frac{\partial B}{\partial x} \frac{1}{A} \frac{\partial u}{\partial x} + \frac{1}{R_x} w + V(1 \frac{\partial v}{\partial x} + \frac{1}{R_0} \frac{\partial B}{\partial x} v + \frac{\partial A}{\partial x} u) - \frac{\partial A}{\partial B} v) - \frac{\partial B}{\partial B} \frac{1}{\partial t} \frac{\partial v}{\partial x} + \frac{1}{R_x} w + V(1 \frac{\partial u}{\partial x} + \frac{1}{R_0} \frac{\partial B}{\partial x} v + \frac{\partial A}{\partial x} u) + (K \frac{\partial B}{\partial x} + \frac{\partial A}{\partial x}) w
\]
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