2008

Asymptotic Behavior of the Magnetization Near Critical and Tricritical Points via Ginzburg–Landau Polynomials

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Ginzburg-Landau Polynomials and the Asymptotic Behavior of the Magnetization Near Critical and Tricritical Points

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May 9, 2008

Abstract

The purpose of this paper is to prove unexpected connections among the asymptotic behavior of the magnetization, the structure of the phase transitions, and a class of polynomials that we call the Ginzburg-Landau polynomials. The model under study is a mean-field version of an important lattice-spin model due to Blume and Capel. It is defined by a probability distribution that depends on the parameters β and K, which represent, respectively, the inverse temperature and the interaction strength. Our main focus is on the asymptotic behavior of the magnetization $m(\beta_n, K_n)$ for appropriate sequences (β_n, K_n) that converge to a second-order point or to the tricritical point of the model and that lie inside various subsets of the phase-coexistence region. The main result states that as (β_n, K_n) converges to one of these points (β, K), $m(\beta_n, K_n) \sim \bar{x}|\beta - \beta_n|^\gamma \to 0$. In this formula γ is a positive constant, and $\bar{x}$ is the unique positive, global minimum point of a certain polynomial g. We call g the Ginzburg-Landau polynomial because of its close connection with the Ginzburg-Landau phenomenology of critical phenomena. This polynomial arises as a limit of appropriately scaled free-energy functionals, the global minimum points of which define the phase-transition structure of the model. In the asymptotic formula $m(\beta_n, K_n) \sim \bar{x}|\beta - \beta_n|^\gamma$, both γ and $\bar{x}$ depend on the sequence (β_n, K_n). Six examples of such sequences are considered, each leading to a different asymptotic behavior of $m(\beta_n, K_n)$. Our approach to studying the asymptotic behavior of the magnetization has three advantages. First, for each sequence (β_n, K_n) under study, the structure of the global minimum points of the associated Ginzburg-Landau polynomial mirrors the structure of the global minimum points of the free-energy functional in the region through which (β_n, K_n) passes and thus reflects the phase-transition structure of the model in that region. In this way the properties of the Ginzburg-Landau polynomials make rigorous the predictions of the Ginzburg-Landau phenomenology of critical phenomena. Second, we use these properties to discover new features of the first-order curve in a neighborhood of the tricritical point. Third, the predictions of the heuristic scaling theory of the tricritical point are made rigorous by the asymptotic formula $m(\beta_n, K_n) \sim \bar{x}|\beta - \beta_n|^\gamma$, which is the main result in the paper.

American Mathematical Society 2000 Subject Classifications. Primary 82B20

Key words and phrases: Ginzburg-Landau phenomenology, second-order phase transition, first-order phase transition, tricritical point, scaling theory, Blume-Capel model

1 Introduction

In this paper we prove unexpected connections among the asymptotic behavior of the magnetization, the structure of the phase transitions, and a class of polynomials that we call the Ginzburg-Landau polynomials. The investigation is carried out for a mean-field version of an important lattice-spin model due to Blume and Capel, to which we refer as the B-C model [2, 6, 7, 8]. This mean-field model is equivalent to the B-C model on the complete graph on n vertices. It is certainly one of the simplest models that exhibit the following intricate phase-transition structure: a curve of second-order points; a curve of first-order points; and a tricritical point, which separates the two curves. A generalization of the B-C model is studied in [3].

The main result in the present paper is Theorem 4.2, a general theorem that gives the asymptotic behavior of the magnetization in the mean-field B-C model for suitable sequences. With only changes in notation, the theorem also applies to other mean-field models including the Curie-Weiss model [12] and the Curie-Weiss-Potts model [16].

The mean-field B-C model is defined by a canonical ensemble that we denote by $P_{N,\beta,K}$; N equals the number of spins, β is the inverse temperature, and K is the interaction strength. $P_{N,\beta,K}$ is defined in terms of the Hamiltonian

$$H_{N,K}(\omega) = \sum_{j=1}^{N} \omega_j^2 - \frac{K}{N}\left(\sum_{j=1}^{N} \omega_j\right)^2,$$

in which ω_j represents the spin at site $j \in \{1, 2, \ldots, N\}$ and takes values in $\Lambda = \{1, 0, -1\}$. The configuration space for the model is the set Λ^N containing all sequences $\omega = (\omega_1, \omega_2, \ldots, \omega_N)$ with each $\omega_j \in \Lambda$.

Before introducing the results in this paper, we summarize the phase-transition structure of the model. For $\beta > 0$ and $K > 0$ we denote by $\mathcal{M}_{\beta,K}$ the set of equilibrium values of the magnetization. $\mathcal{M}_{\beta,K}$ coincides with the set of zeroes of the rate function in a large deviation principle for the spin per site [Thm. 2.1(a)] and with the set of global minimum points of the free-energy functional $G_{\beta,K}$, which is related to the rate function via a Legendre-Fenchel transform [see (2.6)]. It is known from heuristic arguments and is proved in [15] that there exists a critical inverse temperature $\beta_c = \log 4$ and that for $0 < \beta \leq \beta_c$ there exists a quantity $K(\beta)$ and for $\beta > \beta_c$ there exists a quantity $K_1(\beta)$ having the following properties:

1. For $0 < \beta \leq \beta_c$ and $0 < K \leq K(\beta)$, $\mathcal{M}_{\beta,K}$ consists of the unique pure phase 0.

2. For $0 < \beta \leq \beta_c$ and $K > K(\beta)$, $\mathcal{M}_{\beta,K}$ consists of two symmetric, nonzero values of the magnetization $\pm m(\beta, K)$.

3. For $0 < \beta \leq \beta_c$, $\mathcal{M}_{\beta,K}$ undergoes a continuous bifurcation at $K = K(\beta)$, changing continuously from $\{0\}$ for $K \leq K(\beta)$ to $\{\pm m(\beta, K)\}$ for $K > K(\beta)$. This continuous bifurcation corresponds to a second-order phase transition.

4. For $\beta > \beta_c$ and $0 < K < K_1(\beta)$, $\mathcal{M}_{\beta,K}$ consists of the unique pure phase 0.

5. For $\beta > \beta_c$ and $K = K_1(\beta)$, $\mathcal{M}_{\beta,K}$ consists of 0 and two symmetric, nonzero values of the magnetization $\pm m(\beta, K_1(\beta))$.

6. For $\beta > \beta_c$ and $K > K_1(\beta)$, $\mathcal{M}_{\beta,K}$ consists of two symmetric, nonzero values of the magnetization $\pm m(\beta, K)$.

7. For $\beta > \beta_c$, $\mathcal{M}_{\beta,K}$ undergoes a discontinuous bifurcation at $K = K_1(\beta)$, changing discontinuously from $\{0\}$ for $K < K(\beta)$ to $\{0, \pm m(\beta, K)\}$ for $K = K_1(\beta)$ to $\{\pm m(\beta, K)\}$ for $K > K_1(\beta)$. This discontinuous bifurcation corresponds to a first-order phase transition.

Because of items 3 and 7, we refer to the curve $\{(\beta, K(\beta)), 0 < \beta < \beta_c\}$ as the second-order curve and to the curve $\{(\beta, K_1(\beta)), \beta > \beta_c\}$ as the first-order curve. Points on the second-order curve are called second-order points, and points on the first-order curve first-order points. The point $(\beta_c, K(\beta_c)) = (\log 4, 3/2 \log 4)$ separates the second-order curve from the first-order curve and is called the tricritical point. The phase-coexistence region consists of all points in the positive β-K quadrant for which $\mathcal{M}_{\beta,K}$ consists of more than one value. Thus this region consists of all (β, K) above the second-order curve, above the tricritical point, on the first-order curve, and above the first-order curve; i.e., all (β, K) satisfying $0 < \beta \leq \beta_c$ and $K > K(\beta)$ and satisfying $\beta > \beta_c$ and $K \geq K_1(\beta)$. The sets that describe the phase-transition structure of the model are shown in Figure 1.

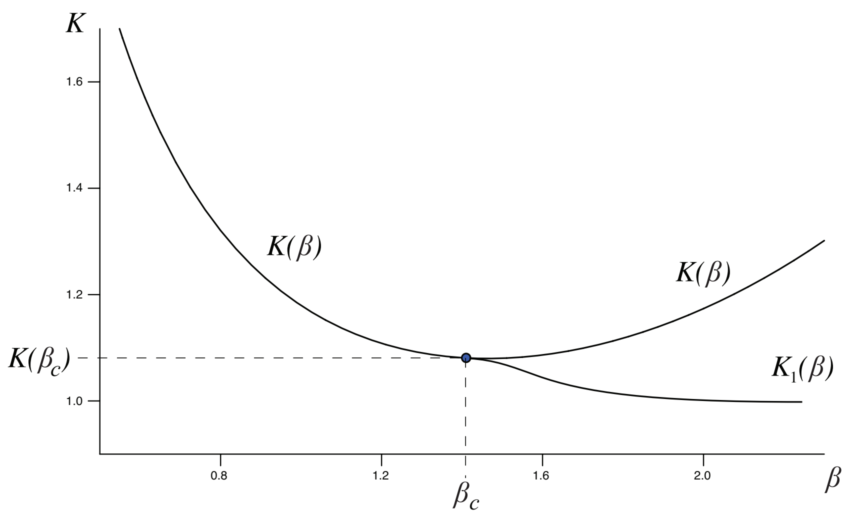

Figure 1: The sets that describe the phase-transition structure of the BEG model: the second-order curve $\{(\beta, K(\beta)), 0 < \beta < \beta_c\}$, the first-order curve $\{(\beta, K_1(\beta)), \beta > \beta_c\}$, and the tricritical point $(\beta_c, K(\beta_c))$. The phase-coexistence region consists of all (β, K) above the second-order curve, above the tricritical point, on the first-order curve, and above the first-order curve. The extension of the second-order curve to $\beta > \beta_c$ is called the spinodal curve.

We now turn to the main focus of this paper, which is the asymptotic behavior of the magnetization $m(\beta_n, K_n)$ for appropriate sequences (β_n, K_n) that converge either to a second-order point or to the tricritical point from various subsets of the phase-coexistence region. In the case of second-order points we consider two such sequences in Theorems 3.1 and 3.2, and in the case of the tricritical point we consider four such sequences in Theorems 5.1–5.4. Denoting the second-order point or the tricritical point by (β, K), in each case we prove as a consequence of the general result in Theorem 4.2 that $m(\beta_n, K_n) \to 0$ according to the asymptotic formula

$$m(\beta_n, K_n) \sim \bar{x}|\beta - \beta_n|^{\gamma}; \text{ i.e. } \lim_{n\to\infty} |\beta - \beta_n|^{-\gamma} m(\beta_n, K_n) = \bar{x}. \tag{1.1}$$

In this formula γ is a positive constant, and $\bar{x}$ is the unique positive, global minimum point of a certain polynomial g. We call g the Ginzburg-Landau polynomial because of its close connection with the Ginzburg-Landau phenomenology of critical phenomena [17]. Both γ and $\bar{x}$ depend on the sequence (β_n, K_n). The exponent γ and the polynomial g arise via the limit of suitably scaled free-energy functionals; specifically, for appropriate choices of $u \in \mathbb{R}$ and $\gamma > 0$ and uniformly for x in compact subsets of $\mathbb{R}$

$$\lim_{n\to\infty} n^{1-u} G_{\beta_n, K_n}(x/n^{\gamma}) = g(x). \tag{1.2}$$

The Ginzburg-Landau polynomials g play several roles in this paper. First, the structure of the set of global minimum points of each g mirrors the structure of the set of global minimum points of the free-energy functional in the subset of the phase-coexistence region through which the corresponding sequence (β_n, K_n) passes. Details of this mirroring are given in the discussions leading up to Theorem 3.1 and Theorem 5.2; of all the sequences that we consider, the sequence considered in the latter theorem shows the most varied behavior. Since the global minimum points of the free-energy functional determine the phase-transition structure of the model, one can also investigate the phase-transition structure using properties of the Ginzburg-Landau polynomials, which are polynomials of degree 4 or 6 and thus have a much simpler form than the free-energy functional. In this way, properties of the Ginzburg-Landau polynomials make rigorous the predictions of the Ginzburg-Landau phenomenology of critical phenomena, which replaces appropriate thermodynamic quantities by the first few terms of their Taylor expansions in an ad hoc manner. An example of such an application of the Ginzburg-Landau polynomials is given in section 6, where we use the polynomials to discover new features of the first-order curve. The Ginzburg-Landau phenomenology is used in section 2 to motivate the phase-transition structure of the model.

The Ginzburg-Landau polynomials are also intimately related to probabilistic limit theorems for the total spin S_N with respect to the canonical ensemble $P_{N,\beta,K}$. These limit theorems are studied in the sequel to the present paper when $N = n$ [14]; i.e., when the system size N coincides with the index n parametrizing the sequence (β_n, K_n). For each sequence (β_n, K_n)

for which the asymptotic behavior of $m(\beta_n, K_n)$ is studied, there exists $\gamma_0 \in (0, 1/4]$ such that up to a multiplicative constant, the exponential of the associated Ginzburg-Landau polynomial is the limiting density in a scaling limit for $S_n/n^{1-\gamma_0}$. This limit is supplemented by other scaling limits for $\gamma \neq \gamma_0$. In addition, for all $\gamma \in (0, \gamma_0)$, up to an additive constant the Ginzburg-Landau polynomial is the rate function in a moderate deviation principle (MDP) for $S_n/n^{1-\gamma}$., After deriving the MDPs in [14], we apply them to study refined asymptotics of the spin. This fascinating feature is discussed at the end of the introduction.

The connection with probabilistic limit theorems reveals a close relationship between the present study and our previous paper [11]. In that paper we reveal the intricate structure of the phase transitions in the BEG model by proving a total of 18 scaling limits and 18 MDPs for $S_n/n^{1-\gamma}$. These results are obtained for appropriate sequences (β_n, K_n) converging either to points in the single-phase region located under the second-order curve (1 scaling limit and 1 MDP), to second-order points (4 scaling limits and 4 MDPs), and to the tricritical point (13 scaling limits and 13 MDPs). Our goal in that paper was to obtain the maximal number of probabilistic limit theorems. In order to achieve that goal, we chose sequences (β_n, K_n) for which certain terms in a Taylor expansion have appropriate large-n behavior. However, the physical significance of those sequences is not obvious.

By contrast, in the present paper we focus, not on probabilistic limit theorems as in [11, 14], but on the asymptotic behavior of the magnetization using physically significant sequences (β_n, K_n). These sequences converge either to second-order points or to the tricritical point from specific subsets of the phase-coexistence region. Doing so allows us to use properties of the Ginzburg-Landau polynomials in order to study the phase transitions in these various subsets.

This paper puts on a rigorous footing the idea, first introduced by Ginzburg and Landau, that low-order polynomial approximations to the free energy functional give correct asymptotic results near continuous phase transitions for mean-field models [17]. The use of sequences (β_n, K_n) that approach second-order points or the tricritical point permits us to establish the validity of truncating the expansion of the free-energy functional at an appropriate low order. The higher order terms are driven to zero by a power of n and are shown to be asymptotically irrelevant. While the renormalization group methodology also demonstrates the irrelevance of higher order terms in the expansion of the free-energy functional, it does so via a different route that depends on heuristics. By contrast, our approach is rigorous and shows in (1.2) how to obtain the Ginzburg-Landau polynomial as a limit of suitably scaled free-energy functionals. No heuristic approximations appear.

Let (β_n, K_n) be any particular sequence converging to a second-order point or the tricritical point from the phase-coexistence region and denote the limiting point by (β, K). It is not difficult to obtain an asymptotic formula expressing the rate at which $m(\beta_n, K_n)$ converges to 0. Since $m(\beta_n, K_n)$ is the unique positive minimum point of G_{β_n, K_n}, it solves the equation

$G'_{\beta_n,K_n}(m(\beta_n, K_n)) = 0$. As we illustrate in appendix B in two examples, expanding G'_{β_n,K_n} in a Taylor series of appropriate order, one obtains the formula $m(\beta_n, K_n) \sim \bar{x}|\beta - \beta_n|^\gamma$ for some $\bar{x} > 0$ and $\gamma > 0$. However, this method gives only the functional form for $\bar{x}$, not associating it with the model via the Ginzburg-Landau polynomial. This is in contrast to our general result in Theorem 4.2. That result identifies $\bar{x}$ as the unique positive, global minimum point of the Ginzburg-Landau polynomial, using the uniform convergence in (1.2). The proof of that result completely avoids Taylor expansions, making use of the fact that under appropriate conditions, the positive global minimum points of n-dependent minimization problems converge to the positive global minimum point of a limiting minimization problem when such a minimum point is unique.

Our work is also closely related to the scaling theory for critical and tricritical points. By choosing sequences that approach second-order points or the tricritical point from various directions and at various rates, we are able to verify a number of predictions of scaling theory. The sequences that approach the tricritical point reveal the subtle geometry of the crossover between critical and tricritical behavior described in Riedel's tricritical scaling theory [22]. In section 7 we will see that a proper application of scaling theory near the tricritical point requires that the scaling parameters be defined in a curvilinear coordinate system, an idea proposed in [22] but, to our knowledge, not previously explored.

Some of the results proved here are contained in the work of Capel and collaborators [6, 7, 8, 9, 19, 20, 21]. These papers introduce the mean-field B-C model and provide a general framework for studying mean-field models and obtaining the thermodynamic properties of these systems. The work of Capel and his collaborators does not encompass the contributions of the present paper, which include a new and rigorous methodology involving Ginzburg-Landau polynomials for treating mean-field models. We believe that this methodology will be valuable in future mathematical investigations of related systems in statistical mechanics.

In order to highlight our new results, we summarize them for the six sequences considered in Theorems 3.1–3.2 and in Theorems 5.1–5.4. Two of the six cases involve the spinodal curve, which is the extension of the second-order curve $\{(\beta, K(\beta)), 0 < \beta < \beta_c\}$ to values $\beta > \beta_c$. In cases 1, 2, and 6 the limiting Ginzburg-Landau polynomial has degree 4 while in cases 3, 4, and 5 the limiting Ginzburg-Landau polynomial has degree 6. In each case the quantity $\bar{x}$ equals the unique positive, global minimum point of the associated Ginzburg-Landau polynomial. In the following six items the asymptotic behavior of $m(\beta_n, K_n) \to 0$ is expressed as an appropriate function of $\beta - \beta_n$.

1. In case 1 the sequence (β_n, K_n) converges to a second-order point $(\beta, K(\beta))$ along a ray that lies in the phase-coexistence region. This ray is above the tangent line to the second-order curve at that point. Given $0 < \beta < \beta_c$, $\alpha > 0$, $b \in \{1, 0, -1\}$, and $k \in \mathbb{R}$ satisfying $K'(\beta)b - k < 0$, the sequence is defined by $\beta_n = \beta + b/n^\alpha$ and

$K_n = K(\beta) + k/n^\alpha$. As described in Theorem 3.1, $m(\beta_n, K_n) \sim \bar{x}/n^{\alpha/2}$. If $b \neq 0$, then this becomes $m(\beta_n, K_n) \sim \bar{x}|\beta - \beta_n|^{1/2}$.

2. In case 2 the sequence (β_n, K_n) converges to a second-order point $(\beta, K(\beta))$ along a curve that lies in the phase-coexistence region. It coincides with the second-order curve to order $p-1$ in powers of $\beta_n - \beta$, where $p \geq 2$ is an integer. Hence the two curves have the same tangent at $(\beta, K(\beta))$. Parametrized by $\alpha > 0$, $b \in \{1, -1\}$, an integer $p \geq 2$, and a real number ℓ satisfying $(K^{(p)}(\beta) - \ell)b^p < 0$, the sequence is defined by
$$\beta_n = \beta + b/n^\alpha \ \text{ and } \ K_n = K(\beta) + \sum_{j=1}^{p-1} K^{(j)}(\beta) b^j/(j! n^{j\alpha}) + \ell b^p/(p! n^{p\alpha}). \tag{1.3}$$
As described in Theorem 3.2, $m(\beta_n, K_n) \sim \bar{x}/n^{p\alpha/2} = \bar{x}|\beta - \beta_n|^{p/2}$.

3. In case 3 the sequence (β_n, K_n) converges to the tricritical point $(\beta_c, K(\beta_c))$ along a ray that lies in the phase-coexistence region. The ray is above the tangent line to the phase-transition curve at the tricritical point. The sequence is defined as in case 1 with β replaced by β_c. As described in Theorem 5.1, $m(\beta_n, K_n) \sim \bar{x}/n^{\alpha/4}$. If $b \neq 0$, then this becomes $m(\beta_n, K_n) \sim \bar{x}|\beta_c - \beta_n|^{1/4}$.

4. In case 4 the sequence (β_n, K_n) converges to the tricritical point $(\beta_c, K(\beta_c))$ along a curve that lies in the phase-coexistence region and is tangent to the spinodal curve at the tricritical point. Given Conjectures 1–3 in section 6, in a neighborhood of the tricritical point this curve either lies above the first-order curve or coincides with that curve to order 2 in powers of $\beta - \beta_c$. The sequence is defined in (1.4) in terms of a curvature parameter ℓ and another parameter $\tilde{\ell}$, and four different cases are listed in items 4a–4d after the definition (1.4). As described in Theorem 5.2, in all four cases $m(\beta_n, K_n) \sim \bar{x}/n^{\alpha/2} = \bar{x}(\beta_n - \beta_c)^{1/2}$.

5. In case 5 the sequence (β_n, K_n) converges to the tricritical point $(\beta_c, K(\beta_c))$ along a curve that lies in the phase-coexistence region and coincides with the second-order curve to order 2 in powers of $\beta - \beta_c$. Hence the two curves have the same tangent at the tricritical point. The sequence is defined as in (1.4) with $\beta_n = \beta_c + 1/n^\alpha$ replaced by $\beta_n = \beta_c - 1/n^\alpha$ and with $\tilde{\ell} = 0$. As described in Theorem 5.3, for $\ell > K''(\beta_c)$, $m(\beta_n, K_n) \sim \bar{x}/n^{\alpha/2} = \bar{x}(\beta_c - \beta_n)^{1/2}$.

6. In case 6 the sequence (β_n, K_n) converges to the tricritical point $(\beta_c, K(\beta_c))$ along a curve that lies in the phase-coexistence region and coincides with the second-order curve to order $p-1$ in powers of $\beta - \beta_c$, where $p \geq 3$ is an integer. Hence the two curves

have the same tangent at the tricritical point. The sequence is defined as in (1.3) with $b = -1$ and β replaced by β_c. As described in Theorem 5.4, for appropriate choices of $\ell \neq K^{(p)}(\beta_c)$, $m(\beta_n, K_n) \sim \bar{x}/n^{(p-1)\alpha/2} = \bar{x}(\beta_c - \beta_n)^{(p-1)/2}$.

Possible paths followed by the sequences in items 1–6 are shown in Figure 2. Two different paths are shown for each of the sequences in items 1, 2, and 3, four different paths for the sequences in item 4, and one path for each of the sequences in items 5 and 6. We believe that modulo uninteresting scale changes, these are all the sequences of the form $\beta_n = \beta + b/n^\alpha$ and K_n equal to $K(\beta)$ plus a polynomial in $1/n^\alpha$, where $(\beta, K(\beta))$ is either a second-order point or the tricritical point and for which $m(\beta_n, K_n) > 0$.

It is interesting to compare the sequences in items 1 and 3 and the sequences in items 2 and 6. Although in both cases the sequences are defined similarly, the asymptotic formulas for $m(\beta_n, K_n)$ involve different powers of n. From the viewpoint of the scaling theory for critical phenomena, the discrepancies arise because the sequences in items 1 and 2 converge to a second-order point while those in items 3 and 6 converge to the tricritical point; this is discussed in section 7.

Table 1 summarizes the asymptotic behavior of $m(\beta_n, K_n)$ for the sequences depicted in Figure 2 and indicates the theorem where the asymptotic behavior is proved.

sequence	**converges to**	**theorem**	**asymptotic behavior of $m(\beta_n, K_n)$**
1	second-order point	Thm. 3.1	$m(\beta_n, K_n) \sim \bar{x}\lvert\beta - \beta_n\rvert^{1/2}$
2	second-order point	Thm. 3.2	$m(\beta_n, K_n) \sim \bar{x}\lvert\beta - \beta_n\rvert^{p/2}$
3	tricritical point	Thm. 5.1	$m(\beta_n, K_n) \sim \bar{x}\lvert\beta_c - \beta_n\rvert^{1/4}$
4a–4d	tricritical point	Thm. 5.2	$m(\beta_n, K_n) \sim \bar{x}(\beta_n - \beta_c)^{1/2}$
5	tricritical point	Thm. 5.3	$m(\beta_n, K_n) \sim \bar{x}(\beta_c - \beta_n)^{1/2}$
6	tricritical point	Thm. 5.4	$m(\beta_n, K_n) \sim \bar{x}(\beta_c - \beta_n)^{(p-1)/2}$

Table 1: Asymptotic behavior of $m(\beta_n, K_n) \to 0$ for the sequences in Figure 2. For sequences 1 and 3 the asymptotic formula is valid provided $b \neq 0$ in the definition of the sequence.

The sequences mentioned in item 4 and labeled 4a–4d in Figure 2 are particularly interesting. Parameterized by $\alpha > 0$, a curvature parameter $\ell \in \mathbb{R}$, and another parameter $\tilde{\ell} \in \mathbb{R}$, these sequences are defined by

$$\beta_n = \beta_c + 1/n^\alpha \text{ and } K_n = K(\beta_c) + K'(\beta_c)/n^\alpha + \ell/(2n^{2\alpha}) + \tilde{\ell}/(6n^{3\alpha}). \tag{1.4}$$

For appropriate choices of ℓ and $\tilde{\ell}$, these sequences converge to the tricritical point while passing through the following interesting subsets of the phase-coexistence region.

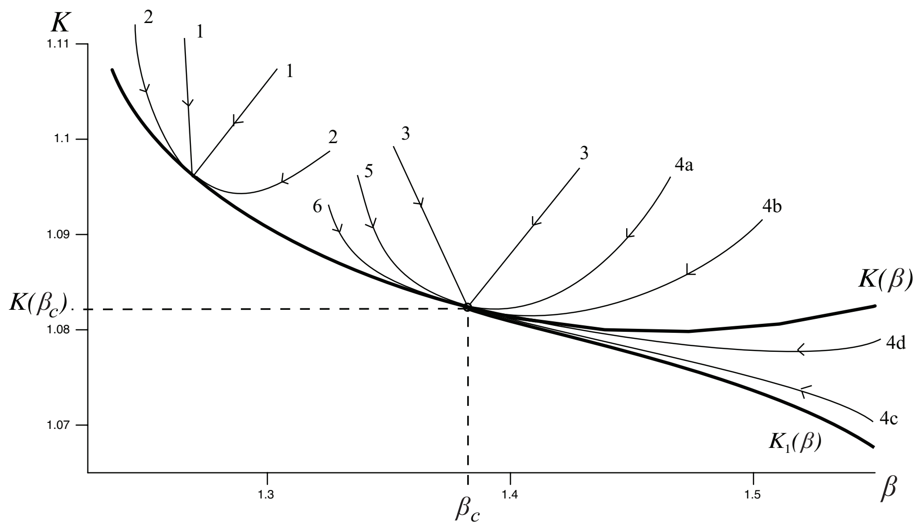

Figure 2: Possible paths for sequences converging to a second-order point and to the tricritical point. The curves labeled 1, 2, 3, 4a–4d, 5, and 6 are discussed in the respective items 1, 2, 3, 4, 5, and 6. The sequences on the curves labeled 4a–4d are defined in (1.4) and are discussed in more detail in the respective items 4a–4d appearing after (1.4).

4a. For $\ell > K''(\beta_c)$, (β_n, K_n) passes through the phase-coexistence region located above the spinodal curve.

4b. For $\ell = K''(\beta_c)$, (β_n, K_n) converges to the tricritical point along a curve that coincides with the spinodal curve to order 2 in powers of $\beta - \beta_c$.

4c. For $\ell = \ell_c = K''(\beta_c) - 5/(4\beta_c)$, (β_n, K_n) converges to the tricritical point along a curve that coincides, to order 2 in powers of $\beta - \beta_c$, with what we conjecture is the first-order curve.

4d. For ℓ in the open interval $(\ell_c, K''(\beta_c))$, (β_n, K_n) converges to the tricritical point along a curve that passes between what we conjecture is the first-order curve and the spinodal curve in a neighborhood of the tricritical point.

In Figure 2 we do not show the curve along which (β_n, K_n) converges to the tricritical point when $\ell < \ell_c$ and $\tilde{\ell} \in \mathbb{R}$. This curve is conjectured to lie in the single-phase region under the first-order curve.

A number of examples are given in the paper of how the structure of the set of global minimum points of the associated Ginzburg-Landau polynomials g mirrors the phase-transition structure of the subsets through which (β_n, K_n) passes. For example, for the sequence defined in (1.4), we have the following picture. In all cases g depends only on ℓ, not on $\tilde{\ell}$.

1. For $\ell > \ell_c$ and any $\tilde{\ell} \in \mathbb{R}$, the global minimum points of g are a symmetric, nonzero pair $\pm\bar{x}(\ell)$, mirroring the fact that for (β, K) above the first-order curve the equilibrium values of the magnetization are the symmetric, nonzero pair $\pm m(\beta, K)$.

2. For $\ell = \ell_c$ and all sufficiently large $\tilde{\ell}$, the global minimum points of g are 0 and a symmetric, nonzero pair $\pm\bar{x}(\ell_c)$, mirroring the fact that for $(\beta, K) = (\beta, K_1(\beta))$ on the first-order curve the equilibrium values of the magnetization are 0 and the symmetric, nonzero pair $\pm m(\beta, K_1(\beta))$.

3. For $\ell < \ell_c$ and any $\tilde{\ell} \in \mathbb{R}$, g has a unique global minimum point at 0, mirroring the fact that for (β, K) under the first-order curve there is a unique pure phase at 0.

The values of the magnetization and the positive, global minimum points $\bar{x}(\ell)$ are connected through the main result of the paper, which is that as (β_n, K_n) converges to the tricritical point $m(\beta_n, K_n) \sim \bar{x}(\ell)/n^{\alpha/4} = \bar{x}(\ell)(\beta_c - \beta_n)^{1/4}$.

In section 6 we reverse this procedure, using properties of the appropriate Ginzburg-Landau polynomials not to mirror, but to predict features of the phase-transition structure. There we argue that at β_c the first-order curve defined by $(\beta, K_1(\beta))$ and the spinodal curve have the same right-hand tangent, that $K_1''(\beta_c) = \ell_c < K''(\beta_c)$, and that $K_1'''(\beta_c) > 0$ (see Conjectures 1, 2, and 3). These conjectures are used to verify the asymptotic behavior of $m(\beta_n, K_n) \to 0$ given in part (c) of Theorem 5.2 when ℓ satisfies $\ell_c \leq \ell < K''(\beta_c)$.

We end the introduction by previewing our results on the refined asymptotics of S_N when the system size N coincides with the index n parametrizing the sequence (β_n, K_n). These asymptotics are the main focus of the sequel to the present paper [14]. These refined asymptotics reveal a fascinating relationship between the asymptotic formulas for $m(\beta_n, K_n)$ obtained here and the finite-size expectation $E_{n,\beta_n,K_n}\{|S_n/n|\}$, where E_{n,β_n,K_n} denotes expectation with respect to P_{n,β_n,K_n}. In order to illustrate this relationship, we focus on the sequence (β_n, K_n) in Theorem 3.1 that converges to a second-order point $(\beta, K(\beta))$. A general result covering the other five sequences considered in the present paper is given in [14].

According to part (c) of Theorem 3.1, for any $\alpha > 0$, $m(\beta_n, K_n)$ has the asymptotic behavior $m(\beta_n, K_n) \sim \bar{x}/n^{\alpha/2}$, where $\bar{x}$ is the positive global minimum point of the associated Ginzburg-Landau polynomial. When $\alpha \in (0, 1/2)$, we prove in [14] that $m(\beta_n, K_n)$ is asymptotic to the expectation of $|S_n/n|$; i.e.,

$$E_{n,\beta_n,K_n}\{|S_n/n|\} \sim m(\beta_n, K_n) \sim \bar{x}/n^{\alpha/2}.$$

A more refined statement is that when $\alpha \in (0, 1/2)$, the probability distribution $P_{n,\beta_n,K_n}\{S_n/n \in dx\}$ is sharply peaked at $\pm m(\beta_n, K_n)$ as $n \to \infty$. We prove this by considering the asymptotic behavior of the expectation

$$E_{n,\beta_n,K_n}\{|\,|S_n/n| - m(\beta_n, K_n)|\},$$

showing that the fluctuations of $|S_n/n|$ around $m(\beta_n, K_n)$ as measured by this expectation are asymptotic to $\bar{z}/n^{1/2-\alpha/2}$, where $\bar{z} > 0$ is given explicitly. Since $\alpha \in (0, 1/2)$, the rate $\bar{z}/n^{1/2-\alpha/2}$ at which this expectation converges to 0 is much faster than the rate $\bar{x}/n^{\alpha/2}$ at which $m(\beta_n, K_n)$ converges to 0. In this case $(\beta_n, K_n) \to (\beta, K(\beta))$ slowly, and the system is in the phase-coexistence regime, where it is effectively infinite. Interestingly, the range $\alpha \in (0, 1/2)$, for which $m(\beta_n, K_n)$ and $E_{n,\beta_n,K_n}\{|S_n/n|\}$ have the same asymptotic behavior, is precisely the range of α for which we have a moderate deviation principle for $S_n/n^{1-\gamma}$ for appropriate $\gamma \in (0, 1/4)$. The moderate deviation principle plays a key role in the proofs of the asymptotic behaviors of the two expectations mentioned in this paragraph.

On the other hand, when $\alpha > 1/2$, $m(\beta_n, K_n)$ is not related to the finite-size expectation $E_{n,\beta_n,K_n}\{|S_n/n|\}$. In the $\alpha > 1/2$ regime, $E_{n,\beta_n,K_n}\{|S_n/n|\}$ is asymptotic to $\bar{y}/n^{1/4}$, where $\bar{y} > 0$ is given explicitly. In this case the fluctuations of $|S_n/n|$ as measured by this expectation are much larger than $m(\beta_n, K_n)$, which converges to 0 at the much faster rate $\bar{x}/n^{\alpha/2}$. When $\alpha > 1/2$, $(\beta_n, K_n) \to (\beta, K(\beta))$ quickly, and the system is in the critical regime. The theory of finite-size scaling predicts that when $\alpha > 1/2$, critical singularities are controlled by the size of the system rather than by the distance in parameter space from the phase transition [1, 5, 10, 24].

The contents of the present paper are as follows. In section 2 we use the Ginzburg-Landau phenomenology of critical phenomena to motivate the phase-transition structure of the model. We then present two theorems proved in [15] justifying the predictions of this phenomenology. In section 3 we illustrate the use of our main result on the asymptotic behavior of the magnetization by applying it to two particular sequence (β_n, K_n) converging to second-order points. In section 4 we prove our main result (1.1) on the asymptotic behavior of $m(\beta_n, K_n) \to 0$ [Thm. 4.2]. In section 5 that result is applied to four different sequences (β_n, K_n) converging to the tricritical point from different subsets of the phase-coexistence region. Section 6 is devoted to using the properties of appropriate Ginzburg-Landau polynomials to discover new properties of the first-order curve. In section 7 we relate the results obtained earlier in this paper to the scaling theory of critical phenomena.

The paper also has two appendices. In appendix A we collect a number of results on polynomials of degree 6 needed earlier in the paper. In appendix B we illustrate another technique for determining the asymptotic behavior of $m(\beta_n, K_n) \to 0$ for two of the sequences considered earlier in the paper. This technique is based on the fact that $m(\beta_n, K_n)$ is a positive global minimum point of the associated free-energy functional and thus a positive zero of the derivative of this function. Despite the naturalness of this characterization of $m(\beta_n, K_n)$, the proofs of the asymptotic behavior of $m(\beta_n, K_n)$ given in this appendix are much more complicated and the respective theorems give less information than the proofs in the main part of the paper that are based on properties of the Ginzburg-Landau polynomials. This emphasizes once again the elegance of the approach in Theorem 4.2, where we use properties of these polynomials to deduce the asymptotic behavior of $m(\beta_n, K_n)$.

Acknowledgments. The research of Richard S. Ellis is supported in part by a grant from the National Science Foundation (NSF-DMS-0604071).

2 Phase-Transition Structure of the BEG Model

After defining the mean-field B-C model, we introduce a function $G_{\beta,K}$, called the free-energy functional. The global minimum points of this function define the equilibrium values of the magnetization, and the minimum value of this function over $\mathbb{R}$ gives the canonical free energy. We then apply the Ginzburg-Landau phenomenology to $G_{\beta,K}$ in order to motivate the phase-transition structure of the model. The predictions of the Ginzburg-Landau phenomenology are shown to be correct in Theorems 2.2 and 2.3.

The mean-field B-C model is a lattice-spin model defined on the complete graph on N vertices $1, 2, \ldots, N$. The spin at site $j \in \{1, 2, \ldots, N\}$ is denoted by ω_j, a quantity taking values in $\Lambda = \{1, 0, -1\}$. The configuration space for the model is the set Λ^N containing all sequences $\omega = (\omega_1, \omega_2, \ldots, \omega_N)$ with each $\omega_j \in \Lambda$. In terms of a positive parameter K representing the interaction strength, the Hamiltonian is defined by

$$H_{N,K}(\omega) = \sum_{j=1}^{N} \omega_j^2 - \frac{K}{N}\left(\sum_{j=1}^{N} \omega_j\right)^2$$

for each $\omega \in \Lambda^N$. Let P_N be the product measure on Λ^N with identical one-dimensional marginals $\rho = \frac{1}{3}(\delta_{-1} + \delta_0 + \delta_1)$. Thus P_N assigns the probability 3^{-N} to each $\omega \in \Lambda^N$. For $N \in \mathbb{N}$, inverse temperature $\beta > 0$, and $K > 0$, the canonical ensemble for the BEG model is the sequence of probability measures that assign to each subset B of Λ^N the probability

$$\begin{aligned} P_{N,\beta,K}(B) &= \frac{1}{Z_N(\beta,K)} \cdot \int_B \exp[-\beta H_{N,K}]\, dP_N \\ &= \frac{1}{Z_N(\beta,K)} \cdot \sum_{\omega \in B} \exp[-\beta H_{N,K}(\omega)] \cdot 3^{-N}. \end{aligned} \tag{2.1}$$

In this formula $Z_N(\beta,K)$ is the partition function equal to

$$\int_{\Lambda^N} \exp[-\beta H_{N,K}]\, dP_N = \sum_{\omega \in \Lambda^N} \exp[-\beta H_{N,K}(\omega)] \cdot 3^{-N}.$$

The analysis of the canonical ensemble $P_{N,\beta,K}$ is facilitated by expressing it in the form of a Curie-Weiss-type model. This is done by absorbing the noninteracting component of the Hamiltonian into the product measure P_N, obtaining

$$P_{N,\beta,K}(d\omega) = \frac{1}{\tilde{Z}_N(\beta,K)} \cdot \exp\left[N\beta K \left(\frac{S_N(\omega)}{N}\right)^2\right] P_{N,\beta}(d\omega). \tag{2.2}$$

In this formula $S_N(\omega)$ equals the total spin $\sum_{j=1}^N \omega_j$, $P_{N,\beta}$ is the product measure on Λ^N with identical one-dimensional marginals

$$\rho_\beta(d\omega_j) = \frac{1}{Z(\beta)} \cdot \exp(-\beta\omega_j^2)\, \rho(d\omega_j), \tag{2.3}$$

$Z(\beta)$ is the normalization equal to $\int_\Lambda \exp(-\beta\omega_j^2)\rho(d\omega_j) = (1+2e^{-\beta})/3$, and $\tilde{Z}_N(\beta,K)$ is the normalization equal to $[Z(\beta)]^N/Z_N(\beta,K)$.

When rewritten as in (2.2), $P_{N,\beta,K}$ is reminiscent of the canonical ensemble for the Curie-Weiss model [12, §IV.4]. However, $P_{N,\beta,K}$ is much more complicated because of the β-dependent product measure $P_{N,\beta}$ and the presence of the parameter K. As we will show in this section, the canonical ensemble $P_{N,\beta,K}$ for the mean-field B-C model gives rise to a second-order phase transition, a first-order phase transition, and a tricritical point, which separates the two phase transitions and is one of the main focuses of the present paper.

The starting point of the analysis of the phase-transition structure of the mean-field B-C model is the large deviation principle (LDP) satisfied by the spin per site S_N/N with respect to $P_{N,\beta,K}$. In order to state the form of the rate function, we introduce the cumulant generating function c_β of the measure ρ_β defined in (2.3); for $t \in \mathbb{R}$ this function is defined by

$$\begin{aligned} c_\beta(t) &= \log \int_\Lambda \exp(t\omega_1)\, \rho_\beta(d\omega_1) \\ &= \log\left(\frac{1+e^{-\beta}(e^t+e^{-t})}{1+2e^{-\beta}}\right). \end{aligned} \tag{2.4}$$

We also introduce the Legendre-Fenchel transform of c_β, which is defined for $x \in [-1,1]$ by

$$J_\beta(x) = \sup_{t \in \mathbb{R}} \{tx - c_\beta(t)\};$$

The smallest interval containing the range of S_N/N is $[-1,1]$, and $J_\beta(x)$ is finite for x in that interval. J_β is the rate function in Cramér's theorem, which is the LDP for S_n/n with respect to the product measures $P_{N,\beta}$ [12, Thm. II.4.1] and is one of the components of the proof of the LDP for S_N/N with respect to $P_{N,\beta,K}$. This LDP and a related limit are stated in parts (a) and (b) of the next theorem. The theorem is proved like Theorem 3.1 in [4] and Theorem 2.4 in [13].

Theorem 2.1. *For all $\beta > 0$ and $K > 0$ the following conclusions hold.*

(a) *With respect to the canonical ensemble $P_{N,\beta,K}$, S_N/N satisfies the LDP on $[-1,1]$ with rate function*

$$I_{\beta,K}(x) = J_\beta(x) - \beta K x^2 - \inf_{y \in [-1,1]} \{J_\beta(y) - \beta K y^2\}.$$

In particular, for any closed set F in $[-1,1]$ we have the large deviation upper bound

$$\limsup_{N \to \infty} \frac{1}{N} \log P_{N,\beta,K}\{S_N/N \in F\} \leq -I_{\beta,K}(F) = -\inf_{x \in F} I_{\beta,K}(x).$$

(b) *We define the canonical free energy*

$$\varphi(\beta, K) = -\lim_{N \to \infty} \frac{1}{N} \log Z_N(\beta, K),$$

where $Z_N(\beta,K)$ is the partition function defined in (2.1). *Then $\varphi(\beta,K) = \inf_{y \in \mathbb{R}}\{J_\beta(y) - \beta K y^2\}$.*

By definition, the infimum of $I_{\beta,K}$ over $[-1,1]$ equals 0. We define the set of equilibrium macrostates by

$$\mathcal{M}_{\beta,K} = \{x \in [-1,1] : I_{\beta,K}(x) = 0\}.$$

In order to justify this definition, let F be any closed subset of $[-1,1]$ that is disjoint from the closed set $\mathcal{M}_{\beta,K}$. Then $I_{\beta,K}(F) > 0$, and the large deviation upper bound implies that for all sufficiently large n

$$P_{N,\beta,K}\{S_N/N \in F\} \leq \exp[-N I_{\beta,K}(F)/2] \to 0. \tag{2.5}$$

It follows that those $x \in [-1,1]$ satisfying $I_{\beta,K}(x) > 0$ have an exponentially small probability of being observed in the canonical ensemble. Each $x \in \mathcal{M}_{\beta,K}$ is a global minimum point of

$I_{\beta,K}$ and represents an equilibrium value of the magnetization. In [15], where Theorems 2.2 and 2.3 are proved, the set $\mathcal{M}_{\beta,K}$ was denoted by $\tilde{\mathcal{E}}_{\beta,K}$.

For $x \in \mathbb{R}$ we define

$$G_{\beta,K}(x) = \beta K x^2 - c_\beta(2\beta K x). \tag{2.6}$$

The calculation of the zeroes of $I_{\beta,K}$ — equivalently, the global minimum points of $J_{\beta,K}(x) - \beta K x^2$ — is greatly facilitated by the following observations made in Proposition 3.4 in [15]:

1. The global minimum points of $J_{\beta,K}(x) - \beta K x^2$ coincide with the global minimum points of $G_{\beta,K}$, which are much easier to calculate.

2. The minimum values $\min_{x\in\mathbb{R}}\{J_{\beta,K}(x) - \beta K x^2\}$ and $\min_{x\in\mathbb{R}} G_{\beta,K}(x)$ coincide and both equal the canonical free energy $\varphi(\beta, K)$ defined in part (b) of Theorem 2.1.

Item 1 gives the alternate characterization

$$\mathcal{M}_{\beta,K} = \{x \in [-1, 1] : x \text{ is a global minimum point of } G_{\beta,K}(x)\}. \tag{2.7}$$

Because of item 2 we call $G_{\beta,K}$ the free-energy functional of the mean-field B-C model. In the context of Curie-Weiss-type models, the form of $G_{\beta,K}$ is explained on page 2247 of [15].

We next apply the Ginzburg-Landau phenomenology to $G_{\beta,K}$ in order to reveal the phase-transition structure of the model. We then state two theorems showing that the predictions of the Ginzburg-Landau phenomenology are correct. As explained in [17], the starting point is to represent $G_{\beta,K}$ in the positive quadrant of the β–K plane by a polynomial consisting of the first few terms in its Taylor expansion. The art in applying this phenomenology is to have a feel for how many terms should be kept. The global minimum points of this polynomial, which are easily determined, should indicate the structure of the global minimum points of $G_{\beta,K}$ and thus the phase-transition structure of the model. The sets that describe the phase-transition structure of the model are shown in Figure 1 in the introduction.

One additional aspect of the Ginzburg-Landau phenomenology is to correctly capture the symmmetries of the order parameter. In the case of mean-field B-C model the order parameter is the scalar magnetization $m(\beta, K)$, and the only symmetry is sign change, which rules out odd powers in the approximations (2.9) and (2.10). In more complicated models it becomes an important challenge to construct the correct Ginzburg-Landau approximation to the free-energy functional that captures all the symmetries.

In the mean-field B-C model $G_{\beta,K}$ is an even function vanishing at 0. Define $K(\beta) = (e^\beta + 2)/4\beta$. For $\beta > 0$ and $K > 0$ the first three terms in the Taylor expansion of $G_{\beta,K}(x)$ at 0 are

$$\frac{G^{(2)}_{\beta,K}(0)}{2!}x^2 + \frac{G^{(4)}_{\beta,K}(0)}{4!}x^4 + \frac{G^{(6)}_{\beta,K}(0)}{6!}x^6, \tag{2.8}$$

where $G^{(2)}_{\beta,K}(0) = 2\beta K(K(\beta) - K)/K(\beta)$ and $G^{(4)}_{\beta,K}(0) = 2(2\beta K)^4(4 - e^\beta)/(e^\beta + 2)^2$. We will comment on the sixth-order term later. The coefficient of x^2 in (2.8) is positive, zero, or negative according to whether $K < K(\beta)$, $K = K(\beta)$, or $K > K(\beta)$. Also, the coefficient of x^4 in (2.8) is positive, zero, or negative according to whether $\beta < \log 4$, $\beta = \log 4$, or $\beta > \log 4$. The value $\log 4$ defines the critical inverse temperature β_c. The coefficients of x^2 and x^4 both vanish when $(\beta, K) = (\beta_c, K(\beta_c)) = (\log 4, 3/2 \log 4)$, which is the tricritical point.

For $0 < \beta < \beta_c$, the coefficient of x^4 in (2.8) is positive. We represent $G_{\beta,K}$ by the first two terms in its Taylor expansion, obtaining for x near 0

$$G_{\beta,K}(x) \approx \frac{\beta K[K(\beta) - K]}{K(\beta)} x^2 + \frac{4(\beta K)^4(4 - e^\beta)}{3(e^\beta + 2)^2} x^4.$$

In order to simplify the calculation, we replace K by $K(\beta)$ both in the term multiplying $K(\beta) - K$ in the coefficient of x^2 and in the coefficient of x^4. After the substitution $\beta K(\beta) = (e^\beta + 2)/4$, the coefficient of x^2 becomes $\beta(K(\beta) - K)$, and the coefficient of x^4 becomes $c_4(\beta) = (e^\beta + 2)^2(4 - e^\beta)/3 \cdot 4^3$. Thus for $0 < \beta < \beta_c$, K near $K(\beta)$, and x near 0 we have the ad hoc approximation

$$G_{\beta,K}(x) \approx \tilde{G}_{\beta,K}(x) = \beta(K(\beta) - K)x^2 + c_4(\beta)x^4. \tag{2.9}$$

In a different guise a polynomial having a similar form arises in (3.4) in the derivation of the asymptotic behavior of $m(\beta_n, K_n)$ for appropriate sequences (β_n, K_n) converging to a point $(\beta, K(\beta))$ for $0 < \beta < \beta_c$.

We now describe the structure of the set of global minimum points of $\tilde{G}_{\beta,K}$ for fixed $0 < \beta < \beta_c$ and variable K. For $0 < K \leq K(\beta)$ the coefficient of x^2 in $\tilde{G}_{\beta,K}$ is nonnegative, and $\tilde{G}_{\beta,K}$ has a unique global minimum point at 0. As K increases through the value $K(\beta)$, 0 becomes a local maximum point of $\tilde{G}_{\beta,K}$, and $\tilde{G}_{\beta,K}$ picks up two symmetric global minimum points, which we label $\pm\bar{x}(\beta, K)$. The quantity $\bar{x}(\beta, K)$ is a positive, increasing, continuous function for $K > K(\beta)$, and as $K \to (K(\beta))^+$, $\bar{x}(\beta, K) \to 0^+$. Thus the set of global minimum points of $\tilde{G}_{\beta,K}$ exhibits a continuous bifurcation at $K = K(\beta)$, changing continuously from $\{0\}$ for $0 < K \leq K(\beta)$ to $\{\pm\bar{x}(\beta, K)\}$ for $K > K(\beta)$. This continuous bifurcation corresponds to a second-order phase transition. As we will see in Theorem 2.2, for $0 < \beta < \beta_c$ the behavior of the set of global minimum points of $\tilde{G}_{\beta,K}$ has the same qualitative behavior as the behavior of the set $\mathcal{M}_{\beta,K}$ of the set of global minimum points of the free-energy functional $G_{\beta,K}$; namely, a continuous bifurcation corresponding to a second-order phase transition as K increases through the value $K(\beta)$. For $0 < \beta < \beta_c$, $K(\beta)$ describes the second-order curve.

The analysis for $\beta > \beta_c$ is much more complicated because of the much more intricate phase transition structure in the neighborhood of the tricritical point. For $\beta > \beta_c$ and (β, K) in a neighborhood of the tricritical point we represent $G_{\beta,K}$ by the first three terms in its Taylor expansion, obtaining

$$G_{\beta,K}(x) \approx \frac{\beta K(K(\beta)-K)}{K(\beta)}x^2 + \frac{4(\beta K)^4(4-e^\beta)}{3(e^\beta+2)^2}x^4 + \frac{G^{(6)}_{\beta,K}(0)}{6!}x^6.$$

In order to simplify the calculation, we replace the sixth-order coefficient $G^{(6)}_{\beta,K}(0)$ by the value of this derivative at the tricritical point $(\beta_c, K(\beta_c)) = (\log 4, 3/2\beta_c)$. This value is $G^{(6)}_{\beta_c,K(\beta_c)}(0) = 2 \cdot 3^4$. We also replace β and K by β_c and $K(\beta_c)$ both in the terms multiplying $K(\beta) - K$ in the coefficient of x^2 and in the term multiplying $4 - e^\beta$ in the coefficient of x^4. With these replacements the coefficient of x^2 becomes $\beta_c(K(\beta) - K)$, and the coefficient of x^4 becomes $3(4 - e^\beta)/16$. Thus for $\beta > \beta_c$, (β, K) near the tricritical point, and x near 0 we have the ad hoc approximation

$$G_{\beta,K}(x) \approx \tilde{G}_{\beta,K}(x) = \beta_c(K(\beta)-K)x^2 + c_4(4-e^\beta)x^4 + c_6x^6, \tag{2.10}$$

where $c_4 = 3/16$ and $c_6 = 2 \cdot 3^4/6! = 9/40$. In a different guise a polynomial having a similar form arises in (5.4) in the derivation of the asymptotic behavior of $m(\beta_n, K_n)$ for appropriate sequences (β_n, K_n) converging to the tricritical point.

$\tilde{G}_{\beta,K}$ is a polynomial of degree 6, the set of global minimum points of which can be analyzed using Theorem A.1. The details of this analysis are omitted. The main point is that for $\beta > \beta_c$ the set of global minimum points of $\tilde{G}_{\beta,K}$ exhibits a discontinuous bifurcation at a certain value $K = \tilde{K}_1(\beta)$, changing discontinuously from $\{0\}$ for $K < \tilde{K}_1(\beta)$ to $\{0, \pm\bar{x}(\beta, \tilde{K}_1(\beta))\}$ for $K = \tilde{K}_1(\beta)$ to $\{\pm\bar{x}(\beta, K)\}$ for $K > \tilde{K}_1(\beta)$. In these formulas $\bar{x}(\beta, K)$ is a positive quantity defined for $\beta > \beta_c$ and $K \geq \tilde{K}_1(\beta)$. This discontinuous bifurcation corresponds to a first-order phase transition. As we will see in Theorem 2.3, for $\beta > \beta_c$ the behavior of the set of global minimum points of $\tilde{G}_{\beta,K}$ has the same qualitative behavior as the behavior of the set $\mathcal{M}_{\beta,K}$ of global minimum points of the free-energy functional $G_{\beta,K}$; namely, a discontinuous bifurcation corresponding to a first-order phase transition as K increases through a certain value $K_1(\beta)$.

The next two theorems give the structure of $\mathcal{M}_{\beta,K}$ first for $0 < \beta < \beta_c = \log 4$ and then for $\beta > \beta_c$. These theorems make rigorous the discussion based on the structure of the set of global minimum points of the approximating polynomials $\tilde{G}_{\beta,K}$ defined in (2.9) and (2.10). The first theorem, proved in Theorem 3.6 in [15], describes the continuous bifurcation in $\mathcal{M}_{\beta,K}$ for $0 < \beta < \beta_c$. This bifurcation corresponds to a second-order phase transition. The quantity $K(\beta)$ is denoted by $K_c^{(2)}(\beta)$ in [15] and by $K_c(\beta)$ in [11].

Theorem 2.2. *For $0 < \beta \leq \beta_c$, we define*

$$K(\beta) = 1/[2\beta c''_\beta(0)] = (e^\beta + 2)/(4\beta). \tag{2.11}$$

For these values of β, $\mathcal{M}_{\beta,K}$ has the following structure.

(a) *For* $0 < K \leq K(\beta)$, $\mathcal{M}_{\beta,K} = \{0\}$.

(b) *For* $K > K(\beta)$, *there exists* $m(\beta, K) > 0$ *such that* $\mathcal{M}_{\beta,K} = \{\pm m(\beta, K)\}$.

(c) $m(\beta, K)$ *is a positive, increasing, continuous function for* $K > K(\beta)$, *and as* $K \to (K(\beta))^+$, $m(\beta, K) \to 0^+$. *Therefore,* $\mathcal{M}_{\beta,K}$ *exhibits a continuous bifurcation at* $K(\beta)$.

The next theorem, proved in Theorem 3.8 in [15], describes the discontinuous bifurcation in $\mathcal{M}_{\beta,K}$ for $\beta > \beta_c$. This bifurcation corresponds to a first-order phase transition. As shown in that theorem, for all $\beta > \beta_c$, $K_1(\beta) < K(\beta)$. The quantity $K_1(\beta)$ is denoted by $K_c^{(1)}(\beta)$ in [15] and by $K_c(\beta)$ in [11].

Theorem 2.3. *For* $\beta > \beta_c$, $\mathcal{M}_{\beta,K}$ *has the following structure in terms of the quantity* $K_1(\beta)$, *denoted by* $K_c^{(1)}(\beta)$ *in* [15] *and defined implicitly for* $\beta > \beta_c$ *on page* 2231 *of* [15].

(a) *For* $0 < K < K_1(\beta)$, $\mathcal{M}_{\beta,K} = \{0\}$.

(b) *For* $K = K_1(\beta)$ *there exists* $m(\beta, K_1(\beta)) > 0$ *such that* $\mathcal{M}_{\beta,K_1(\beta)} = \{0, \pm m(\beta, K_1(\beta))\}$.

(c) *For* $K > K_1(\beta)$ *there exists* $m(\beta, K) > 0$ *such that* $\mathcal{M}_{\beta,K} = \{\pm m(\beta, K)\}$.

(d) $m(\beta, K)$ *is a positive, increasing, continuous function for* $K \geq K_1(\beta)$, *and as* $K \to K_1(\beta)^+$, $m(\beta, K) \to m(\beta, K_1(\beta)) > 0$. *Therefore,* $\mathcal{M}_{\beta,K}$ *exhibits a discontinuous bifurcation at* $K_1(\beta)$.

We recall from (2.7) that $\mathcal{M}_{\beta,K}$ can be characterized as the set of global minimum points of $G_{\beta,K}$. In Figure 3 we exhibit the graphs of $G_{\beta,K}$ for $0 < \beta \leq \beta_c$ and increasing values of $K > 0$. These graphs are based on the detailed analysis of the global minimum points of a related function in section 3.2 of [15]. The graph for $0 < K \leq K(\beta)$ is shown in (a), and the graph for $K > K(\beta)$ is shown in (b).

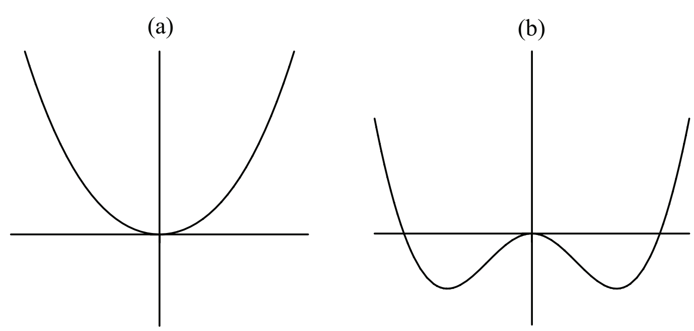

Figure 3: Graphs of $G_{\beta,K}$ for $0 < \beta \leq \beta_c$. (a) $0 < K \leq K(\beta)$, (b) $K > K(\beta)$.

In Figures 4, 5, and 6, we exhibit the graphs of $G_{\beta,K}$ for $\beta > \beta_c$ and increasing values of $K > 0$. These graphs are based on the detailed analysis of the global minimum points of a related function in section 3.3 of [15]. The graphs for $0 < K < K_1(\beta)$ are shown in Figure 4, the graph for $K = K_1(\beta)$ in Figure 5, and the graphs for $K > K_1(\beta)$ in Figure 6. These three figures correspond, respectively, to parts (a), (b), and (c) of Theorem 2.3.

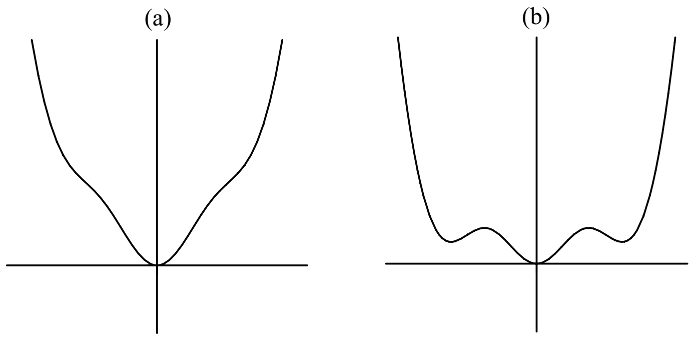

Figure 4: Graphs of $G_{\beta,K}$ for $\beta > \beta_c$. (a) $0 < K < \kappa(\beta)$, (b) $\kappa(\beta) < K < K_1(\beta)$. For $\beta > \beta_c$ the set of minimum points of $G_{\beta,K}$ undergoes the bifurcation shown in graphs (a) and (b) as K increases through the value $\kappa(\beta)$.

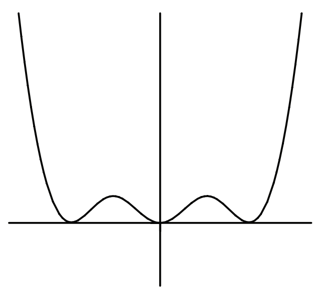

Figure 5: Graph of $G_{\beta,K}$ for $\beta > \beta_c$ and $K = K_1(\beta)$.

In the next section we determine the asymptotic behavior of $m(\beta_n, K_n)$ for appropriate sequences (β_n, K_n) converging from the phase-coexistence region to a second-order point. In determining the asymptotic behavior, we will see how to make rigorous the Ginzburg-Landau phenomenology by replacing, with a well defined limit, the approximation of the free-energy functional $G_{\beta,K}$ by the fourth-degree polynomial $\tilde{G}_{\beta,K}$ in (2.9).

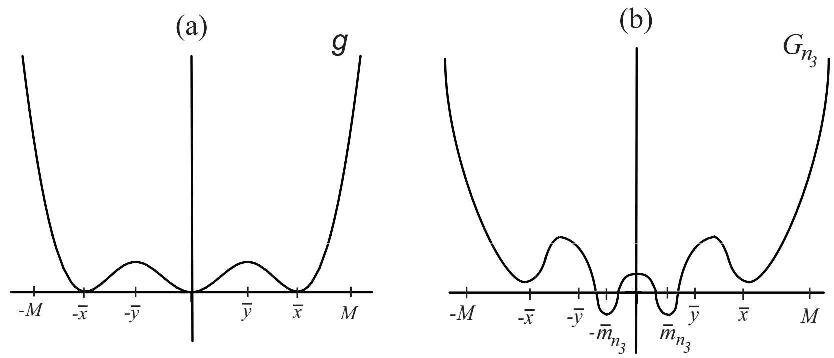

Figure 6: Graphs of $G_{\beta,K}$ for $\beta > \beta_c$. (a) $K_1(\beta) < K < K(\beta)$, (b) $K \geq K(\beta)$.

3 Asymptotic Behavior of $m(\beta_n, K_n)$ Near a Second-Order Point

In this section we derive the asymptotic behavior of the magnetization $m(\beta_n, K_n)$ for two sequences (β_n, K_n). For $0 < \beta < \beta_c$ each of these sequences converges to a second-order point $(\beta, K(\beta))$ from the phase-coexistence region located above the second-order curve. This section is a warm-up for section 5, in which we analyze the much more complicated asymptotic behavior of $m(\beta_n, K_n)$ in the neighborhood of the tricritical point.

By definition, when (β_n, K_n) lies in the phase-coexistence region, $m(\beta_n, K_n)$ is the unique positive, global minimum point of the free-energy functional G_{β_n,K_n}. For each of the sequences considered in this section the asymptotic behavior of $m(\beta_n, K_n)$ is expressed in terms of the unique positive, global minimum point $\bar{x}$ of the limit of a suitable scaled version of G_{β_n,K_n}.

This limit is a fourth degree polynomial called the Ginzburg-Landau polynomial. As we will see, properties of this polynomial reflect the phase-transition structure of the mean-field B-C model, thus making rigorous the predictions of the Ginzburg-Landau phenomenology of critical phenomena mentioned in section 2.

The two sequences (β_n, K_n) to be considered in this section are defined in terms of a positive parameter α that regulates the speed of approach of (β_n, K_n) to a second-order point. The asymptotic behavior of $m(\beta_n, K_n)$ for each of the two sequences is given in Theorems 3.1 and 3.2. This behavior is derived from the general result in Theorem 4.2.

For $0 < \beta < \beta_c$ let (β_n, K_n) be an arbitrary positive sequence converging to a second-order point $(\beta, K(\beta))$ and let $\gamma > 0$ be given. In the preceding section we motivated the phase-transition structure for $0 < \beta < \beta_c$ by approximating $G_{\beta,K}(x)$ in (2.9) by a polynomial of degree 4 derived from the first two terms of its Taylor expansion. The starting point in determining the asymptotic behavior of $m(\beta_n, K_n)$ is to replace this two-term Taylor expansion for $G_{\beta,K}(x)$ by the two-term Taylor expansion for $nG_{\beta_n,K_n}(x/n^\gamma)$ with an error term. According to Taylor's Theorem, for all $n \in \mathbb{N}$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$ there exists $\xi_n(x/n^\gamma) \in [-x/n^\gamma, x/n^\gamma]$ such that

$$\begin{aligned} &nG_{\beta_n,K_n}(x/n^\gamma) \\ &\quad = \frac{1}{n^{2\gamma-1}}\frac{G^{(2)}_{\beta_n,K_n}(0)}{2!}x^2 + \frac{1}{n^{4\gamma-1}}\frac{G^{(4)}_{\beta_n,K_n}(0)}{4!}x^4 + \frac{1}{n^{5\gamma-1}}\frac{G^{(5)}_{\beta_n,K_n}(\xi_n(x/n^\gamma))}{5!}x^5. \end{aligned} \tag{3.1}$$

In deriving this formula, we use the fact that $G_{\beta_n,K_n}(0) = 0$ and that since G_{β_n,K_n} is an even function, $G^{(1)}_{\beta_n,K_n}(0) = 0 = G^{(3)}_{\beta_n,K_n}(0)$. Because the sequence (β_n, K_n) is positive and bounded, there exists $a \in (0,\infty)$ such that $0 < \beta_n \le a$ and $0 < K_n \le a$ for all n. As a continuous function of (β, K, y) on the compact set $[0,a] \times [0,a] \times [-R, R]$, $G^{(5)}_{\beta,K}(y)$ is uniformly bounded. It follows that the quantity $G^{(5)}_{\beta_n,K_n}(\xi_n(x/n^\gamma))$ appearing in the error term in the Taylor expansion is uniformly bounded for $n \in \mathbb{N}$ and $x \in (-Rn^\gamma, Rn^\gamma)$. We summarize this expansion by writing

$$nG_{\beta_n,K_n}(x/n^\gamma) = \frac{1}{n^{2\gamma-1}}\frac{G^{(2)}_{\beta_n,K_n}(0)}{2!}x^2 + \frac{1}{n^{4\gamma-1}}\frac{G^{(4)}_{\beta_n,K_n}(0)}{4!}x^4 + \mathrm{O}\left(\frac{1}{n^{5\gamma-1}}\right)x^5, \tag{3.2}$$

where the big-oh term is uniform for $x \in (-Rn^\gamma, Rn^\gamma)$.

In terms of the quantity $K(\beta) = (e^\beta + 2)/(4\beta)$, the coefficients $G^{(2)}_{\beta_n,K_n}(0)$ and $G^{(4)}_{\beta_n,K_n}(0)$ in the Taylor expansion are given by

$$G^{(2)}_{\beta_n,K_n}(0) = \frac{2\beta_n K_n(K(\beta_n) - K_n)}{K(\beta_n)} = 2\beta(K(\beta_n) - K_n) \cdot \frac{\beta_n K_n}{\beta K(\beta_n)}$$

and

$$G^{(4)}_{\beta_n,K_n}(0) = \frac{2(2\beta_n K_n)^4(4 - e^{\beta_n})}{(e^{\beta_n} + 2)^2}.$$

In order to ease the notation, we let ε_n denote a sequence that converges to 0 and that represents the various error terms arising in the following calculation; we use the same notation ε_n to represent different error terms. Since (β_n, K_n) converges to $(\beta, K(\beta))$ and the function $K(\cdot)$ is continuous, we have $\beta_n K_n / K(\beta_n) \to \beta$. Thus

$$G^{(2)}_{\beta_n,K_n}(0)/2! = \beta(K(\beta_n) - K_n)(1 + \varepsilon_n).$$

Define $c_4(\beta) = (e^\beta + 2)^2(4 - e^\beta)/(8 \cdot 4!)$. Since $2\beta_n K_n \to 2\beta K(\beta) = (e^\beta + 2)/2$, we also have

$$G^{(4)}_{\beta_n,K_n}(0)/4! = (e^\beta + 2)^2(4 - e^\beta)(1 + \varepsilon_n)/(8 \cdot 4!) = c_4(\beta)(1 + \varepsilon_n); \tag{3.3}$$

$c_4(\beta) > 0$ since $4 - e^\beta = e^{\beta_c} - e^\beta > 0$. Thus for all $n \in \mathbb{N}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$\begin{aligned} nG_{\beta_n,K_n}(x/n^\gamma) &= \frac{1}{n^{2\gamma-1}}\beta(K(\beta_n) - K_n)(1 + \varepsilon_n)x^2 \\ &\quad + \frac{1}{n^{4\gamma-1}}c_4(\beta)(1 + \varepsilon_n)x^4 + \mathrm{O}\left(\frac{1}{n^{5\gamma-1}}\right)x^5, \end{aligned} \tag{3.4}$$

where the big-oh term is uniform for $x \in (-Rn^\gamma, Rn^\gamma)$.

For the moment, on the right side of the last display let us replace (β_n, K_n) by (β, K), set $n = 1$, and drop the big-oh term. Doing so, we obtain the polynomial $\tilde{G}_{\beta,K}$ that approximates the free energy functional $G_{\beta,K}$ in (2.9) for $0 < \beta < \beta_c$, K near $K(\beta)$, and x near 0. Arising via the Ginzburg-Landau phenomenology, this polynomial is used in section 2 to motivate the continuous bifurcation in the set of equilibrium values of the magnetization that is described rigorously in Theorem 2.2. As we will soon see, by a suitable choice of (β_n, K_n) and other parameters the right side of the last display converges to a Ginzburg-Landau polynomial in terms of which the asymptotic behavior of $m(\beta_n, K_n)$ is described.

We return to (3.4), in which the term $(K(\beta_n) - K_n)$ converges to 0 as $n \to \infty$. The two different asymptotic behaviors of $m(\beta_n, K_n)$ to be considered in this section each depends on the choice of the sequence (β_n, K_n) converging to the second-order point $(\beta, K(\beta))$. Each choice controls, in a different way, the rate at which $(K(\beta_n) - K_n) \to 0$. We analyze two different cases, each giving rise to a Ginzburg-Landau polynomial having a unique positive, global minimum points at $\bar{x}$ for some $\bar{x} > 0$. This quantity enters the respective asymptotic formula for $m(\beta_n, K_n) \to 0$.

Fix $0 < \beta < \beta_c$. For the first choice of sequence we take $\alpha > 0$, $b \in \{1, 0, -1\}$, and $k \in \mathbb{R}$ and define

$$\beta_n = \beta + b/n^\alpha \text{ and } K_n = K(\beta) + k/n^\alpha. \tag{3.5}$$

If $b \neq 0$, then (β_n, K_n) converges to $(\beta, K(\beta))$ along a ray with slope k/b (see path 1 in Figure 2). We assume that $K'(\beta)b - k \neq 0$. Since

$$K(\beta_n) = K(\beta + b/n^\alpha) = K(\beta) + K'(\beta)b/n^\alpha + \mathrm{O}(1/n^{2\alpha}),$$

we have

$$K(\beta_n) - K_n = (K'(\beta)b - k)/n^\alpha + \mathrm{O}(1/n^{2\alpha}). \tag{3.6}$$

It follows from (3.4) that for all $n \in \mathbb{N}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$\begin{aligned} &nG_{\beta_n,K_n}(x/n^\gamma) \\ &\quad = \frac{1}{n^{2\gamma+\alpha-1}}\beta(K'(\beta)b - k)(1+\varepsilon_n)x^2 \\ &\qquad + \frac{1}{n^{4\gamma-1}}c_4(\beta)(1+\varepsilon_n)x^4 + \mathrm{O}\left(\frac{1}{n^{2\gamma+2\alpha-1}}\right)x^2 + \mathrm{O}\left(\frac{1}{n^{5\gamma-1}}\right)x^5. \end{aligned} \tag{3.7}$$

Because $K'(\beta)b - k \neq 0$ and $c_4(\beta) > 0$, the coefficients of x^2 and of x^4 are both nonzero.

The case where $K'(\beta)b - k = 0$ must be handled differently. If this equality holds, then the asymptotic expression (3.6) for $(K(\beta_n) - K_n)$ is indeterminate. In order to calculate the correct asymptotic expression for $(K(\beta_n) - K_n)$ when $K'(\beta)b - k = 0$, one must consider the next term in the Taylor expansion of $K(\beta + b/n^\alpha)$, obtaining (3.14) with $p = 2$ and $\ell = 0$. We carry out the asymptotic analysis for this case in Theorem 3.2.

We return to the sequence (β_n, K_n) in (3.5) when $K'(\beta_c)b - k \neq 0$. In order to obtain the limit of $nG_{\beta_n,K_n}(x/n^\gamma)$, we impose the condition that the powers of n appearing in the first two terms in (3.7) equal 0; i.e., $2\gamma + \alpha - 1 = 0 = 4\gamma - 1$, which is equivalent to $\gamma = 1/4$ and $\alpha = 1 - 2\gamma = 1/2$. With this choice of γ and α, the powers of n appearing in the last two terms in (3.7) are positive, and so for all $x \in \mathbb{R}$ both terms converge to 0 as $n \to \infty$. It follows that for $\gamma = 1/4$ and $\alpha = 1/2$, as $n \to \infty$ we have for all $x \in \mathbb{R}$

$$nG_{\beta_n,K_n}(x/n^\gamma) \to g(x) = \beta(K'(\beta)b - k)x^2 + c_4(\beta)x^4.$$

We call g the Ginzburg-Landau polynomial. Since the big-oh terms in (3.7) are uniform for $x \in (-Rn^\gamma, Rn^\gamma)$, the convergence of $nG_{\beta_n,K_n}(x/n^\gamma)$ to $g(x)$ is uniform for x in compact subsets of $\mathbb{R}$.

By a simple trick, we are able to derive a similar limit that is valid for all $\alpha > 0$. Let u be a real number that will be chosen momentarily. Multiplying the numerator and denominator of the right side of (3.7) by n^u, we obtain $nG_{\beta_n,K_n}(x/n^\gamma) = n^u G_n(x)$, where for all $n \in \mathbb{N}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$\begin{aligned} &G_n(x) \\ &\quad = \frac{1}{n^{2\gamma+\alpha-1+u}} \beta(K'(\beta)b - k)(1+\varepsilon_n)x^2 \\ &\quad\quad + \frac{1}{n^{4\gamma-1+u}} c_4(\beta)(1+\varepsilon_n)x^4 + \mathrm{O}\!\left(\frac{1}{n^{2\gamma+2\alpha-1+u}}\right)x^2 + \mathrm{O}\!\left(\frac{1}{n^{5\gamma-1+u}}\right)x^5. \end{aligned} \tag{3.8}$$

In this formula $\varepsilon_n \to 0$ and the big-oh terms are uniform for $x \in (-Rn^\gamma, Rn^\gamma)$. Again we impose the condition that the powers of n appearing in the first two terms in (3.8) equal 0; i.e., $2\gamma + \alpha - 1 + u = 0 = 4\gamma - 1 + u$. These two equalities are equivalent to $\gamma = \alpha/2$ and $u = 1 - 4\gamma = 1 - 2\alpha$. With this choice of γ and u, the powers of n appearing in the last two terms in (3.8) are positive, and so for all $x \in \mathbb{R}$ both terms converge to 0 as $n \to \infty$. It follows that as $n \to \infty$, we have for all $x \in \mathbb{R}$

$$G_n(x) = n^{1-u} G_{\beta_n,K_n}(x/n^\gamma) \to g(x) = \beta(K'(\beta)b - k)x^2 + c_4(\beta)x^4. \tag{3.9}$$

Again, since the big-oh terms in (3.8) are uniform for $x \in (-Rn^\gamma, Rn^\gamma)$, the convergence of $G_n(x)$ to $g(x)$ is uniform for x in compact subsets of $\mathbb{R}$.

As we will see in Theorem 4.2, by proving the convergence in (3.9) for $u = 1 - 2\alpha \in (-\infty, 1)$, we obtain the asymptotic behavior of $m(\beta_n, K_n)$ for any $\alpha > 0$. If we worked only with $u = 0$, then we would obtain the asymptotic behavior of $m(\beta_n, K_n)$ only for $\alpha = 1/2$.

The Ginzburg-Landau polynomial g has two different behaviors depending on the sign of $\beta(K'(\beta)b - k)$, which is the coefficient of x^2. We first consider the case where $\beta(K'(\beta)b - k) > 0$. This corresponds to (β_n, K_n) converging to $(\beta, K(\beta))$ along a ray lying beneath the tangent line to $(\beta, K(\beta))$. Since $K(\beta)$ is a convex function for $0 < \beta < \beta_c$ [Lem. 6.1(c)], in this situation (β_n, K_n) converges to $(\beta, K(\beta))$ from the single-phase region located beneath the second-order curve. In this case, for all n the free energy functional G_{β_n,K_n} has a unique global minimum point 0 [Thm. 2.2(a)], a property reflected in the fact that the Ginzburg-Landau polynomial

$$g(x) = \beta(K'(\beta)b - k)x^2 + c_4(\beta)x^4$$

also has a unique global minimum point 0.

We now consider the case where $\beta(K'(\beta)b - k) < 0$, which corresponds to (β_n, K_n) converging to $(\beta, K(\beta))$ along a ray lying above the tangent line to $(\beta, K(\beta))$. If $b = 1$, then the slope of the ray satisfies $k/b > K'(\beta)$, which corresponds to $(\beta_n, K_n) \to (\beta, K(\beta))$ from the northeast; if $b = 0$, then (β_n, K_n) converges to $(\beta, K(\beta))$ from the north; and if $b = -1$, then the slope of the ray satisfies $k/b < K'(\beta)$, which corresponds to (β_n, K_n) converging to

$(\beta, K(\beta))$ from the northwest. In each of these situations (β_n, K_n) lies in the phase-coexistence region for all sufficiently large n. For such values of n the global minimum points of the free energy functional G_{β_n,K_n} are $\pm m(\beta_n, K_n)$ [Thm. 2.2(b)], a property reflected in the fact that the global minimum points of the Ginzburg-Landau polynomial g are also a pair of symmetric nonzero points; these points are $\pm\bar{x}$ for $\bar{x} > 0$ defined in (3.10).

In the next theorem we derive the asymptotic behavior of $m(\beta_n, K_n)$ when the sequence (β_n, K_n) defined in (3.5) converges to a second-order point $(\beta, K(\beta))$ from the phase-coexistence region located above the second-order curve; equivalently, when the coefficient $\beta(K'(\beta)b - k)$ of x^2 in the Ginzburg-Landau polynomial g is negative. According to Theorem 4.1, in this case $m(\beta_n, K_n) \to 0$. Theorem 4.2 shows that as a consequence of the uniform convergence of G_n to g and other hypotheses, the sequence $n^{\alpha/2} m(\beta_n, K_n)$ of positive global minimum points of G_n converges to the positive global minimum point $\bar{x}$ of g. This yields the asymptotic formula $m(\beta_n, K_n) \sim \bar{x}/n^{\alpha/2} \to 0$ given in part (c) of the next theorem.

Theorem 3.1. *For $\beta \in (0, \beta_c)$, $\alpha > 0$, $b \in \{1, 0, -1\}$, and a real number $k \neq K'(\beta)b$, define*

$$\beta_n = \beta + b/n^\alpha \text{ and } K_n = K(\beta) + k/n^\alpha$$

as well as $c_4(\beta) = (e^\beta + 2)^2(4 - e^\beta)/(8 \cdot 4!)$. Then (β_n, K_n) converges to the second-order point $(\beta, K(\beta))$. The following conclusions hold.

(a) *For any $\alpha > 0$, $u = 1 - 2\alpha$, and $\gamma = \alpha/2$*

$$G_n(x) = n^{1-u} G_{\beta_n,K_n}(x/n^\gamma) \to g(x) = \beta(K'(\beta)b - k)x^2 + c_4(\beta)x^4$$

uniformly for x in compact subsets of $\mathbb{R}$.

(b) *The Ginzburg-Landau polynomial g has nonzero global minimum points if and only if $K'(\beta)b - k < 0$. If this inequality holds, then the global minimum points of g are $\pm\bar{x}$, where*

$$\bar{x} = (\beta(k - K'(\beta)b)/[2c_4(\beta)])^{1/2} \tag{3.10}$$

(c) *Assume that $K'(\beta)b - k < 0$. Then for any $\alpha > 0$, $m(\beta_n, K_n) \to 0$ and has the asymptotic behavior*

$$m(\beta_n, K_n) \sim \bar{x}/n^{\alpha/2}; \text{ i.e., } \lim_{n\to\infty} n^{\alpha/2} m(\beta_n, K_n) = \bar{x}.$$

If $b \neq 0$, then this becomes $m(\beta_n, K_n) \sim \bar{x}|\beta - \beta_n|^{1/2}$.

Proof. Part (a) follows from the discussion leading up to the statement of the theorem. The first assertion in part (b) is elementary. If $K'(\beta)b - k < 0$, then the equation $g'(x) = 2\beta(K'(\beta)b -$

$k)x + 4c_4(\beta)x^3 = 0$ has solutions at $\pm\bar{x}$ and at 0, where $\bar{x}$ is defined in (3.10). One easily checks that $\pm\bar{x}$ are global minimum points and 0 a local maximum point.

We now verify the asymptotic behavior of $m(\beta_n, K_n)$ in part (c). The convergence $m(\beta_n, K_n) \to 0$ is proved in Theorem 4.1. The asymptotic behavior $m(\beta_n, K_n) \sim \bar{x}/n^{\alpha/2}$ is a consequence of Theorem 4.2, a general result covering the present case, the sequence considered in Theorem 3.2, and the sequences (β_n, K_n) converging to the tricritical point that are considered in Theorems 5.1–5.4. We now verify the four hypotheses of that theorem for the sequence (β_n, K_n) in Theorem 3.1, which converges to the second-order point $(\beta, K(\beta))$. Hypothesis (i) is valid since $K'(\beta)b - k < 0$ is equivalent to $K_n > K(\beta_n)$ for all sufficiently n [see (3.6)]. Hence the inequality $K'(\beta)b - k < 0$ guarantees that for all sufficiently large n, (β_n, K_n) lies in the phase-coexistence region above the second-order curve. Hypothesis (ii) is valid since $n^\alpha(\beta_n - \beta) = b$, $n^\alpha(K_n - K(\beta)) = k$, and either b or k is nonzero. Hypothesis (iii) involves the Ginzburg-Landau polynomial g in part (a), which is an even polynomial of degree 4 satisfying $g(x) \to \infty$ as $|x| \to \infty$. Hypothesis (iii)(a) states that there exist $\alpha_0 > 0$ and $\theta > 0$ such that for any $\alpha > 0$, if $u = 1 - \alpha/\alpha_0$ and $\gamma = \theta\alpha$, then

$$\lim_{n\to\infty} n^{1-u} G_{\beta_n,K_n}(x/n^\gamma) = g(x) = \beta(K'(\beta)b - k)x^2 + c_4(\beta)x^4$$

uniformly for x in compact subsets of $\mathbb{R}$. As verified by the calculation leading up to the limit (3.9) and as stated in part (a) of Theorem 3.1, hypothesis (iii)(a) is valid with $\alpha_0 = 1/2$ and $\theta = 1/2$. Hypothesis (iii)(b) is satisfied since g has a unique positive, global minimum point $\bar{x}$, Hypothesis (iv), the most technical of the four, is verified in the paragraph after the next one. Since $\theta = 1/2$, the general result in Theorem 4.2 takes the form $m(\beta_n, K_n) \sim \bar{x}/n^{\theta\alpha} = \bar{x}/n^{\alpha/2}$. This is the conclusion of part (c) of Theorem 3.1.

We next consider hypothesis (iv) in Theorem 4.2, which states that there exists a polynomial $H(x)$ satisfying $H(x) \to \infty$ as $|x| \to \infty$ together with the following property: for any $\alpha > 0$ there exists $R > 0$ such that, if $u = 1 - \alpha/\alpha_0 = 1 - 2\alpha$ and $\gamma = \theta\alpha = 2\alpha$, then for all sufficiently large $n \in \mathbb{N}$ and for all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$, $n^{1-u} G_{\beta_n,K_n}(x/n^\gamma) \geq H(x)$. This hypothesis is used in the proof of the theorem to verify that the sequence $n^\gamma m(\beta_n, K_n)$ is bounded, a key component in the derivation of the asymptotic behavior of $m(\beta_n, K_n)$.

In order to verify hypothesis (iv) in Theorem 4.2, we fix $\alpha > 0$ and substitute $u = 1 - 2\alpha$ and $\gamma = \alpha/2$ in the expansion (3.8) for $G_n(x) = n^{1-u} G_{\beta_n,K_n}(x/n^\gamma)$. It follows that for any $\alpha > 0$, any $R > 0$, all $n \in \mathbb{N}$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$\begin{aligned} n^{1-u} G_{\beta_n,K_n}(x/n^\gamma) &= \beta(K'(\beta)b - k)(1 + \varepsilon_n)x^2 \\ &\quad + c_4(\beta)(1 + \varepsilon_n)x^4 + \mathrm{O}(1/n^\alpha)x^2 + \mathrm{O}(1/n^\gamma)x^5 \\ &= [\beta(K'(\beta)b - k)(1 + \varepsilon_n) + \mathrm{O}(1/n^\alpha)]\, x^2 \\ &\quad + [c_4(\beta)(1 + \varepsilon_n) + \mathrm{O}(x/n^\gamma)]\, x^4. \end{aligned}$$

Since the big-oh terms are uniform for $x \in (-Rn^\gamma, Rn^\gamma)$, we conclude that for any $\alpha > 0$ there exists $R > 0$ such that for all sufficiently large $n \in \mathbb{N}$ and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$n^{1-u} G_{\beta_n, K_n}(x/n^\gamma) \geq H(x) = -2\beta |K'(\beta) b - k| x^2 + \tfrac{1}{2} c_4(\beta) x^4. \tag{3.11}$$

Since $H(x) \to \infty$ as $|x| \to \infty$, hypothesis (iv) in Theorem 4.2 is satisfied. This completes the verification that the four hypotheses of Theorem 4.2 are valid in the context of Theorem 3.1 and so completes the proof of the latter theorem. ■

We now consider the second choice of the sequence (β_n, K_n) converging to a second-order point. This sequence gives a different asymptotic behavior of $m(\beta_n, K_n) \to 0$ from the sequence considered in Theorem 3.1. Let $(\beta_0, K(\beta_0))$ be a second-order point corresponding to $0 < \beta_0 < \beta_c$. Given $\alpha > 0$, $b \in \{1, -1\}$, an integer $p \geq 2$, and $\ell \in \mathbb{R}$ we define

$$\beta_n = \beta_0 + b/n^\alpha \ \text{ and } \ K_n = K(\beta_0) + \sum_{j=1}^{p-1} K^{(j)}(\beta_0) b^j/(j! n^{j\alpha}) + \ell b^p/(p! n^{p\alpha}). \tag{3.12}$$

In order to simplify the analysis, we assume that $\ell \neq K^{(p)}(\beta_0)$. The choice $\ell = K^{(p)}(\beta_0)$ will be discussed after Theorem 3.2.

Since $\beta_n - \beta_0 = b/n^\alpha$, we can write

$$K_n = K(\beta_0) + \sum_{j=1}^{p-1} K^{(j)}(\beta_0)(\beta_n - \beta_0)^j/j! + \ell(\beta_n - \beta_0)^p/p!.$$

Thus (β_n, K_n) converges to $(\beta_0, K(\beta_0))$ along the curve $(\beta, \tilde{K}(\beta))$, where for $0 < \beta < \beta_c$

$$\tilde{K}(\beta) = K(\beta_0) + \sum_{j=1}^{p-1} K^{(j)}(\beta_0)(\beta - \beta_0)^j/j! + \ell(\beta - \beta_0)^p/p!. \tag{3.13}$$

This curve coincides with the second-order curve to order $p-1$ in powers of $\beta - \beta_0$ in a neighborhood of $(\beta_0, K(\beta_0))$ (see path 2 in Figure 2). Thus the two curves have the same tangent at $(\beta_0, K(\beta_0))$.

The relationship of the sequence (β_n, K_n) to the second-order curve depends on the sign of b. We first assume that $b = 1$. For all sufficiently large n, $\ell > K^{(p)}(\beta_c)$ corresponds to (β_n, K_n) lying in the phase-coexistence region located above the second-order curve and thus to the free energy functional G_{β_n, K_n} having its global minimum points at $\pm m(\beta_n, K_n) \neq 0$. On the other hand, for all sufficiently large n, $\ell < K^{(p)}(\beta_c)$ corresponds to (β_n, K_n) lying in the single-phase region located under the second-order curve and thus to G_{β_n,K_n} having a unique global minimum point at 0.

When $b = -1$, we must take into account the parity of p. If p is even, then the situation is as in the last paragraph. If p is odd, then the situation is reversed. As we will see, in all cases the structure of the set of global minimum points of the associated Ginzburg-Landau polynomial mirrors the structure of the set of global minimum points of G_{β_n,K_n}.

In order to calculate the asymptotic behavior of $m(\beta_n, K_n)$ when (β_n, K_n) is the sequence in (3.12), we follow the pattern of proof of Theorem 3.1. Let $\alpha > 0$ be given. The first step is to calculate the appropriate expansion of $nG_{\beta_n,K_n}(x/n^\gamma)$ in (3.4). In order to ease the notation, we indicate the second-order point by $(\beta, K(\beta))$ instead of by $(\beta_0, K(\beta_0))$. Since

$$K(\beta_n) = K(\beta + b/n^\alpha) = K(\beta) + \sum_{j=1}^{p} K^{(j)}(\beta)b^j/(j!n^{j\alpha}) + \mathrm{O}(1/n^{(p+1)\alpha}),$$

we have

$$K(\beta_n) - K_n = (K^{(p)}(\beta) - \ell)b^p/(p!n^{p\alpha}) + \mathrm{O}(1/n^{(p+1)\alpha}). \tag{3.14}$$

Substituting this expression into (3.4), we see that for all $n \in \mathbb{N}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$\begin{aligned} &nG_{\beta_n,K_n}(x/n^\gamma) \\ &\quad = \frac{1}{n^{2\gamma+p\alpha-1}}\frac{1}{p!}\beta(K^{(p)}(\beta) - \ell)b^p(1+\varepsilon_n)x^2 \\ &\qquad + \frac{1}{n^{4\gamma-1}}c_4(\beta)(1+\varepsilon_n)x^4 + \mathrm{O}\left(\frac{1}{n^{2\gamma+(p+1)\alpha-1}}\right)x^2 + \mathrm{O}\left(\frac{1}{n^{5\gamma-1}}\right)x^5. \end{aligned} \tag{3.15}$$

In this formula $c_4(\beta) = (e^\beta+2)^2(4-e^\beta)/(8\cdot 4!)$, $\varepsilon_n \to 0$, and the big-oh terms are uniform for $x \in (-Rn^\gamma, Rn^\gamma)$. Given $u \in \mathbb{R}$, we multiply the numerator and denominator of the right side of the last display by n^u, obtaining $nG_{\beta_n,K_n}(x/n^\gamma) = n^u G_n(x)$, where for any $R > 0$ and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$\begin{aligned} &G_n(x) \\ &\quad = \frac{1}{n^{2\gamma+p\alpha-1+u}}\frac{1}{p!}\beta(K^{(p)}(\beta) - \ell)b^p(1+\varepsilon_n)x^2 \\ &\qquad + \frac{1}{n^{4\gamma-1+u}}c_4(\beta)(1+\varepsilon_n)x^4 + \mathrm{O}\left(\frac{1}{n^{2\gamma+(p+1)\alpha-1+u}}\right)x^2 + \mathrm{O}\left(\frac{1}{n^{5\gamma-1+u}}\right)x^5. \end{aligned} \tag{3.16}$$

Since $\ell \neq K^{(p)}(\beta)$ and $c_4(\beta) > 0$, the coefficients of x^2 and x^4 are both nonzero.

The next step in calculating the asymptotic behavior of $m(\beta_n, K_n)$ is to obtain the limit of G_n. In order to carry this out, we impose the condition that the two powers of n appearing in the first two terms in (3.16) equal 0; i.e., $2\gamma + p\alpha - 1 + u = 0 = 4\gamma - 1 + u$. These two equalities are equivalent to $\gamma = p\alpha/2$ and $u = 1 - 4\gamma = 1 - 2p\alpha$. With this choice of γ and u, the powers of n appearing in the last two terms in (3.16) are positive, and so for all $x \in \mathbb{R}$ both terms converge to 0 as $n \to \infty$. It follows that as $n \to \infty$, we have for all $x \in \mathbb{R}$

$$G_n(x) = n^{1-u} G_{\beta_n, K_n}(x/n^\gamma) \to g(x) = \tfrac{1}{p!}\beta(K^{(p)}(\beta) - \ell) b^p x^2 + c_4(\beta) x^4.$$

Since the big-oh terms in (3.16) are uniform for $x \in (-Rn^\gamma, Rn^\gamma)$, the convergence of $G_n(x)$ to $g(x)$ is uniform for x in compact subsets of $\mathbb{R}$.

The structure of the set of global minimum points of the Ginzburg-Landau polynomial g mirrors precisely the structure of the set of global minimum points of G_{β_n, K_n} in the region through which (β_n, K_n) passes. The structure of the set of global minimum points of G_{β_n, K_n} is noted in the two paragraphs after (3.13). We first assume that $b = 1$. The choice $\ell > K^{(p)}(\beta)$ yields a polynomial g for which the global minimum points are a symmetric nonzero pair $\pm\bar{x}$, where $\bar{x}$ is defined in (3.17). On the other hand, the choice $\ell < K^{(p)}(\beta)$ yields a polynomial g having a unique global minimum point at 0. When $b = -1$, we must take into account the parity of p. If p is even, then the situation is the same as for $b = 1$. If p is odd, then the situation is reversed. The choice $\ell < K^{(p)}(\beta)$ yields a polynomial g for which the global minimum points are the symmetric nonzero pair $\pm\bar{x}$, while the choice $\ell > K^{(p)}(\beta)$ yields a polynomial g having a unique global minimum point at 0.

We are now ready to state the asymptotic behavior of $m(\beta_n, K_n)$ for the sequence (β_n, K_n) defined in (3.12) and for the choices of ℓ for which the Ginzburg-Landau polynomial has a unique positive, global minimum point $\bar{x}$. The relationships between ℓ and $K^{(p)}(\beta)$ in parts (b)(i) and (b)(ii) guarantee that for all sufficiently large n, (β_n, K_n) lies in the phase-coexistence region above the second-order curve.

Theorem 3.2. *For $\beta \in (0, \beta_c)$, $\alpha > 0$, $b \in \{1, -1\}$, an integer $p \geq 2$, and a real number $\ell \neq K^{(p)}(\beta)$, define*

$$\beta_n = \beta + b/n^\alpha \;\text{ and }\; K_n = K(\beta) + \sum_{j=1}^{p-1} K^{(j)}(\beta) b^j/(j! n^{j\alpha}) + \ell b^p/(p! n^{p\alpha})$$

as well as $c_4(\beta) = (e^\beta + 2)^2(4 - e^\beta)/(8 \cdot 4!)$. Then (β_n, K_n) converges to the second-order point $(\beta, K(\beta))$. The following conclusions hold.

(a) *For any $\alpha > 0$, $u = 1 - 2p\alpha$, and $\gamma = p\alpha/2$*

$$G_n(x) = n^{1-u} G_{\beta_n, K_n}(x/n^\gamma) \to g(x) = \tfrac{1}{p!}\beta(K^{(p)}(\beta) - \ell) b^p x^2 + c_4(\beta) x^4$$

uniformly for x in compact subsets of $\mathbb{R}$.

(b) *The Ginzburg-Landau polynomial g has nonzero global minimum points if and only if $(K^{(p)}(\beta) - \ell)b^p < 0$. If this inequality holds, then the global minimum points of g are $\pm\bar{x}$, where*

$$\bar{x} = \left(\beta(\ell - K^{(p)}(\beta))b^p/[2c_4(\beta)p!]\right)^{1/2}. \tag{3.17}$$

(c) *Assume that $(K^{(p)}(\beta) - \ell)b^p < 0$. Then for any $\alpha > 0$, $m(\beta_n, K_n) \to 0$ and has the asymptotic behavior*

$$m(\beta_n, K_n) \sim \bar{x}/n^{p\alpha/2} = \bar{x}|\beta - \beta_n|^{p/2}; \text{ i.e., } \lim_{n\to\infty} n^{p\alpha/2} m(\beta_n, K_n) = \bar{x}.$$

Proof. Part (a) follows from the discussion leading up to the statement of the theorem. The assertions in part (b) are elementary. If $(K^{(p)}(\beta) - \ell)b^p < 0$, then the equation $g'(x) = 2\beta(K^{(p)}(\beta) - \ell)b^p x/p! + 4c_4(\beta)x^3 = 0$ has the three real solutions 0 and $\pm\bar{x}$, where $\bar{x}$ is defined in (3.17). One easily checks that $\pm\bar{x}$ are global minimum points and 0 a local maximum point.

We now verify the asymptotic behavior of $m(\beta_n, K_n)$ in part (c). According to Theorem 4.1, $m(\beta_n, K_n) \to 0$. The validity of hypotheses (i) and (ii) of Theorem 4.2 follows from the definition of the sequence (β_n, K_n) and the inequality $(K^{(p)}(\beta) - \ell)b^p < 0$, which by (3.14) is equivalent to $K_n > K(\beta_n)$ for all sufficiently large n. Thus if $(K^{(p)}(\beta) - \ell)b^p < 0$, then for all sufficiently large n, (β_n, K_n) lies in the phase-coexistence region above the second-order curve. Hypothesis (iii) of Theorem 4.2 is parts (a) and (b) of the present theorem. We now verify hypothesis (iv) of Theorem 4.2. Using (3.16) with $u = 1 - 2\alpha$ and $\gamma = p\alpha/2$, one easily proves that for any $\alpha > 0$ there exists $R > 0$ such that for all sufficiently large $n \in \mathbb{N}$ and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$G_n(x) = n^{1-u} G_{\beta_n, K_n}(x/n^\gamma) \geq H(x) = -\tfrac{2}{p!}\beta|K^{(p)}(\beta) - \ell|x^2 + \tfrac{1}{2}c_4(\beta)x^4.$$

Since $H(x) \to \infty$ as $|x| \to \infty$, hypothesis (iv) of Theorem 4.2 is satisfied. This completes the verification of the four hypotheses of Theorem 4.2. We now apply the theorem to conclude that for any $\alpha > 0$, $m(\beta_n, K_n) \sim \bar{x}/n^\gamma = \bar{x}/n^{p\alpha/2}$. Part (c) of the present theorem is proved. ■

In order to derive the asymptotic behavior of $m(\beta_n, K_n) \to 0$ for the sequence (β_n, K_n) in the last theorem, we choose $\ell \neq K^{(p)}(\beta)$. The choice $\ell = K^{(p)}(\beta)$ corresponds to the sequence (β_n, K_n) lying on a curve that coincides with the second-order curve to order p in powers of $(\beta - \beta_c)$. In order to analyze this case, we must know the sign of $K^{(p+1)}(\beta)$. Because we are unable to determine this sign analytically for arbitrary $\beta \in (0, \beta_c)$, the discussion of this case is omitted.

We have seen several examples in which the structure of the set of the global minimum points of the Ginzburg-Landau polynomial mirrors the structure of the set of global minimum points of the free energy functional, and thus the phase-transition structure, in the region through which the associated sequence (β_n, K_n) passes. We now reverse this procedure by using the structure of the global minimum points of the Ginzburg-Landau polynomial g in the last theorem not to mirror, but to explore the phase-transition structure in the region through which the sequence (β_n, K_n) passes. If $p = 2$ and $\ell = 0$, then the Ginzburg-Landau polynomial takes the form

$$g(x) = \tfrac{1}{2}\beta K''(\beta) b^2 x^2 + c_4(\beta) x^4.$$

According to part (c) of Lemma 6.1, $K''(\beta) > 0$. With this choice of parameters, g has a unique global minimum point at 0, and the sequence (β_n, K_n) converges to $(\beta, K(\beta))$ along the tangent line to $(\beta, K(\beta))$. This suggests that for each $0 < \beta < \beta_c$ the points on the tangent line sufficiently close to $(\beta, K(\beta))$ lie in the single-phase region located beneath the second-order curve. In turn, this suggests that for each $0 < \beta < \beta_c$ the second-order curve lies above the tangent line at $(\beta, K(\beta))$ except at the point of tangency. This feature of the second-order curve is equivalent to the strict convexity of the function $K(\beta)$ that defines this curve, a property that can be verified directly [Lem. 6.1(c)].

This completes our analysis of the asymptotic behavior of $m(\beta_n, K_n) \to 0$ for the sequences considered in Theorems 3.1 and 3.2. In each case the asymptotic behavior of $m(\beta_n, K_n)$ is expressed in terms of the unique positive, global minimum point of the Ginzburg-Landau polynomial appearing in the statement of the theorem. This is a consequence of the general asymptotic result given in Theorem 4.2, which we derive in the next section.

4 Asymptotic Behavior of $m(\beta_n, K_n)$ in Terms of Ginzburg-Landau Polynomials

Theorem 4.2 is the main result in this paper. It gives the asymptotic behavior of $m(\beta_n, K_n)$ for appropriate sequences (β_n, K_n) lying in the phase-coexistence region and converging either to a second-order point or to the tricritical point. The asymptotic behavior is expressed in terms of the unique positive, global minimum point of the associated Ginzburg-Landau polynomial. We already illustrated the use of this theorem in the previous section, where we considered (β_n, K_n) converging to a second-order point along a ray [Thm. 3.1] and along a curve [Thm. 3.2]. The theorem will be applied again in the next section, where we study the much more complicated asymptotic behavior of $m(\beta_n, K_n)$ in the neighborhood of the tricritical point.

The phase-coexistence region is defined to be all (β, K) satisfying $0 < \beta \leq \beta_c$ and $K > K(\beta)$ and all (β, K) satisfying $\beta > \beta_c$ and $K \geq K_1(\beta)$. Thus for $0 < \beta \leq \beta_c$, the phase-coexistence region consists of the region located above the second-order curve and above the tricritical point. For $\beta > \beta_c$, the phase-coexistence region consists of the first-order curve $(\beta, K_1(\beta))$ and the region located above that curve. For all (β, K) in the phase-coexistence region there exists $m(\beta, K) > 0$ such that $\{\pm m(\beta, K)\} \subset \mathcal{M}_{\beta,K}$. This is an equality for all (β, K) in the phase-coexistence region except for $\beta > \beta_c$ and $K = K_1(\beta)$, in which case $\mathcal{M}_{\beta,K} = \{0, \pm m(\beta, K)\}$.

The first theorem in this section shows that for any sequence (β_n, K_n) converging either to a second-order point or to the tricritical point, $m(\beta_n, K_n) \to 0$. For appropriate sequences (β_n, K_n) lying in the phase-coexistence region and converging either to a second-order point or to the tricritical point, the exact asymptotic behavior of $m(\beta_n, K_n) \to 0$ is expressed in Theorem 4.2. The next theorem is an essential component in the proof of the asymptotic behavior of $m(\beta_n, K_n)$ given in Theorem 4.2.

We are able to prove an extension of Theorem 4.1 valid for first-order points. Given $\beta > \beta_c$, let $(\beta, K_1(\beta))$ be a point on the first-order curve. If (β_n, K_n) is a positive sequence converging to $(\beta, K_1(\beta))$ from the phase-coexistence region located above the first-order curve, then $\lim_{n\to\infty} m(\beta_n, K_n) = m(\beta, K_1(\beta)) > 0$. Because this extension of the theorem is not used in the paper, the proof is omitted.

Theorem 4.1. *Let (β_n, K_n) be an arbitrary positive sequence converging either to a second-order point $(\beta, K(\beta)), 0 < \beta < \beta_c$, or to the tricritical point $(\beta, K(\beta)) = (\beta_c, K(\beta_c))$. Then* $\lim_{n\to\infty} m(\beta_n, K_n) = 0$.

Proof. Since G_{β_n,K_n} is a real analytic function, $G_{\beta_n,K_n}(m(\beta_n, K_n)) \leq 0$, and $G_{\beta_n,K_n}(x) \to \infty$ as $|x| \to \infty$, G_{β_n,K_n} has a largest positive zero, which we denote by x_n. We have the inequality $0 < m(\beta_n, K_n) < x_n$. For any $t \in \mathbb{R}$, $c_\beta(t) \leq \log(4e^{|t|}) = \log 4 + |t|$. Because the sequence (β_n, K_n) is bounded and remains a positive distance from the origin and the coordinate axes, there exist numbers $0 < b_1 < b_2 < \infty$ such that $b_1 \leq \beta_n \leq b_2$ and $b_1 \leq K_n \leq b_2$ for all $n \in \mathbb{N}$. Hence

$$\begin{aligned} G_{\beta_n,K_n}(x) &= \beta_n K_n x^2 - c_{\beta_n}(2\beta_n K_n x) \\ &\geq \beta_n K_n x^2 - 2\beta_n K_n |x| - \log 4 \;\geq\; b_1^2(|x| - 1)^2 - b_2^2 - \log 4. \end{aligned}$$

Therefore, if x^* denotes the positive zero of the quadratic $b_1^2(|x| - 1)^2 - b_2^2 - \log 4$, then

$$0 < \sup_{n\in\mathbb{N}} m(\beta_n, K_n) \leq \sup_{n\in\mathbb{N}} x_n \leq x^*.$$

It follows that $m(\beta_n, K_n)$ is a bounded sequence. Thus given any subsequence $m(\beta_{n_1}, K_{n_1})$, there exists a further subsequence $m(\beta_{n_2}, K_{n_2})$ and $\tilde{x} \in \mathbb{R}$ such that $m(\beta_{n_2}, K_{n_2}) \to \tilde{x}$ as $n_2 \to \infty$. We complete the proof by showing that independently of the subsequence chosen, $\tilde{x} = 0$. To prove this, we use the fact that

$$G_{\beta_{n_2},K_{n_2}}(m(\beta_{n_2}, K_{n_2})) = \inf_{y\in\mathbb{R}} G_{\beta_{n_2},K_{n_2}}(y).$$

Hence for any $y \in \mathbb{R}$, $G_{\beta_{n_2},K_{n_2}}(m(\beta_{n_2},K_{n_2})) \leq G_{\beta_{n_2},K_{n_2}}(y)$. Since $G_{\beta_{n_2},K_{n_2}}(x) \to G_{\beta,K(\beta)}(x)$ uniformly for x in compact subsets of $\mathbb{R}$, it follows that for all $y \in \mathbb{R}$

$$G_{\beta,K(\beta)}(\tilde{x}) = \lim_{n_2\to\infty} G_{\beta_{n_2},K_{n_2}}(m(\beta_{n_2},K_{n_2})) \leq \lim_{n_2\to\infty} G_{\beta_{n_2},K_{n_2}}(y) = G_{\beta,K(\beta)}(y).$$

Therefore $\tilde{x}$ is a minimum point of $G_{\beta,K(\beta)}$. Because $(\beta, K(\beta))$ is either a second-order point or the tricritical point, $\tilde{x}$ must coincide with the unique positive, global minimum point of $G_{\beta,K(\beta)}$ at 0 [Thm. 2.2(a), Thm. 2.3(a)]. We have proved that any subsequence $m(\beta_{n_1}, K_{n_1})$ of $m(\beta_n, K_n)$ has a further subsequence $m(\beta_{n_2}, K_{n_2})$ such that $m(\beta_{n_2}, K_{n_2}) \to 0$ as $n_2 \to \infty$. The conclusion is that $\lim_{n\to\infty} m(\beta_n, K_n) = 0$, as claimed. ■

In sections 3 and 5 we consider six different sequences (β_n, K_n) converging either to a second-order point or to the tricritical point. The fact that each of these sequences lies in the phase-coexistence region for all sufficiently large n is the first hypothesis of Theorem 4.2; this property implies that $m(\beta_n, K_n) > 0$ for all sufficiently large n and $m(\beta_n, K_n) \to 0$ [Thm. 4.1]. Under three additional hypotheses Theorem 4.2 describes the exact asymptotic behavior of $m(\beta_n, K_n) \to 0$. Examples of sequences for which the hypotheses of the theorem are valid are given in Theorems 3.1 and 3.2 for sequences converging to a second-order point and in Theorems 5.1–5.4 for sequences converging to the tricritical point.

Theorem 4.2. *Let (β_n, K_n) be a positive sequence that converges either to a second-order point $(\beta, K(\beta))$, $0 < \beta < \beta_c$, or to the tricritical point $(\beta, K(\beta)) = (\beta_c, K(\beta_c))$. We assume that (β_n, K_n) satisfies the following four hypotheses*:

(i) *(β_n, K_n) lies in the phase-coexistence region for all sufficiently large n.*

(ii) *The sequence (β_n, K_n) is parametrized by $\alpha > 0$. This parameter regulates the speed of approach of (β_n, K_n) to the second-order point or the tricritical point in the following sense:*

$$b = \lim_{n\to\infty} n^\alpha(\beta_n - \beta) \ \text{ and } \ k = \lim_{n\to\infty} n^\alpha(K_n - K(\beta))$$

both exist, and b and k are not both zero; if $b \neq 0$, then b equals 1 or -1.

(iii) *There exists an even polynomial g of degree 4 or 6 satisfying $g(x) \to \infty$ as $|x| \to \infty$ together with the following two properties*; *g is called the Ginzburg-Landau polynomial.*

(a) *$\exists \alpha_0 > 0$ and $\exists \theta > 0$ such that $\forall \alpha > 0$, if $u = 1 - \alpha/\alpha_0$ and $\gamma = \theta\alpha$, then*

$$\lim_{n\to\infty} n^{1-u} G_{\beta_n,K_n}(x/n^\gamma) = g(x)$$

uniformly for x in compact subsets of $\mathbb{R}$.

(b) *g has a unique positive, global minimum point $\bar{x}$; thus the set of global minimum points of g equals $\{\pm\bar{x}\}$ or $\{0, \pm\bar{x}\}$.*

(iv) *There exists a polynomial H satisfying $H(x) \to \infty$ as $|x| \to \infty$ together with the following property*: $\forall \alpha > 0$ $\exists R > 0$ *such that, if $u = 1 - \alpha/\alpha_0$ and $\gamma = \theta\alpha$, then $\forall n \in \mathbb{N}$ sufficiently large and $\forall x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$, $n^{1-u} G_{\beta_n,K_n}(x/n^\gamma) \geq H(x)$.*

Under hypotheses (i)–(iv)*, for any $\alpha > 0$*

$$m(\beta_n, K_n) \sim \bar{x}/n^{\theta\alpha}; \ \textit{ i.e., } \lim_{n\to\infty} n^{\theta\alpha} m(\beta_n, K_n) = \bar{x}.$$

If $b \neq 0$, then this becomes $m(\beta_n, K_n) \sim \bar{x}|\beta - \beta_n|^\theta$.

Proof. Since $G_{\beta_n,K_n}(0) = 0$ and G_{β_n,K_n} is even, by hypotheses (iii) g is an even polynomial of degree 4 or 6 satisfying $g(0) = 0$. Hence the global minimum points of g are either $\pm\bar{x}$ for some $\bar{x} > 0$ or 0 and $\pm\bar{x}$ for some $\bar{x} > 0$. The proof of the asymptotic relationship $m(\beta_n, K_n) \sim \bar{x}/n^{\theta\alpha}$ is much easier in the case where the global minimum points of g are $\pm\bar{x}$ for some $\bar{x} > 0$. After a number of preliminary steps, we will prove the theorem for such polynomials g. We will then turn to the case where the global minimum points of g are 0 and $\pm\bar{x}$ for some $\bar{x} > 0$.

As in hypothesis (iii)(a), let $\alpha > 0$ be given and define $u = 1 - \alpha/\alpha_0$ and $\gamma = \theta\alpha$. In order to ease the notation, we write $\bar{m}_n = n^\gamma m(\beta_n, K_n)$ and $G_n(x) = n^{1-u} G_{\beta_n,K_n}(x/n^\gamma)$. For all sufficiently large n, since (β_n, K_n) lies in the phase-coexistence region, we have $m(\beta_n, K_n) > 0$ and

$$G_{\beta_n,K_n}(m(\beta_n, K_n)) = \inf_{y\in\mathbb{R}} G_{\beta_n,K_n}(y).$$

It follows that for all sufficiently large n

$$\begin{aligned} G_n(\bar{m}_n) &= n^{1-u} G_{\beta_n,K_n}(m(\beta_n, K_n)) \\ &= \inf_{y\in\mathbb{R}} [n^{1-u} G_{\beta_n,K_n}(y)] = \inf_{y\in\mathbb{R}} G_n(y); \end{aligned} \tag{4.1}$$

i.e., G_n attains its minimum over $\mathbb{R}$ at $\bar{m}_n > 0$. This fact will be used several times in the proof.

We first prove that the sequence $\{\bar{m}_n, n \in \mathbb{N}\}$ is bounded. If the sequence $\bar{m}_n$ is not bounded, then there exists a subsequence $\bar{m}_{n_1}$ of $\bar{m}_n$ such that $\bar{m}_{n_1} \to \infty$ as $n_1 \to \infty$. Let R be the quantity in hypothesis (iv). Since $m(\beta_{n_1}, K_{n_1}) > 0$ and $m(\beta_{n_1}, K_{n_1}) \to 0$ [Thm. 4.1], we have $0 < \bar{m}_{n_1}/n^\gamma = m(\beta_{n_1}, K_{n_1}) < R$ for all sufficiently large n_1, and so by hypothesis (iv)

$$G_{n_1}(\bar{m}_{n_1}) \geq H(\bar{m}_{n_1}) \to \infty \ \text{ as } n_1 \to \infty.$$

However, this contradicts the inequality

$$G_n(\bar{m}_n) = \inf_{y \in \mathbb{R}} G_n(y) \leq G_n(0) = 0,$$

which is valid for all n. This contradiction proves that the sequence $\bar{m}_n$ is bounded.

We now prove that $m(\beta_n, K_n) \sim \bar{x}/n^\gamma = \bar{x}/n^{\theta\alpha}$ in the case where the global minimum points of g are $\pm\bar{x}$ for some $\bar{x} > 0$. Let $\bar{m}_{n_1}$ be any subsequence of $\bar{m}_n$. Since the sequence $\bar{m}_{n_1}$ is bounded, there exists a further subsequence $\bar{m}_{n_2}$ and $\hat{x} \geq 0$ such that $\bar{m}_{n_2} \to \hat{x}$ as $n_2 \to \infty$. According to (4.1), for any $y \in \mathbb{R}$, $G_{n_2}(\bar{m}_{n_2}) \leq G_{n_2}(y)$. Since $G_n(x) \to g(x)$ uniformly for x in compact subsets of $\mathbb{R}$, it follows that

$$g(\hat{x}) = \lim_{n_2 \to \infty} G_{n_2}(\bar{m}_{n_2}) \leq \lim_{n_2 \to \infty} G_{n_2}(y) = g(y).$$

Hence $\hat{x}$ is a nonnegative global minimum point of g. Because g has a unique nonnegative, global minimum point $\bar{x}$, which is positive, $\hat{x}$ coincides with $\bar{x}$. We have proved that any subsequence $\bar{m}_{n_1}$ of $\bar{m}_n$ has a further subsequence $\bar{m}_{n_2}$ such that $\bar{m}_{n_2} \to \bar{x}$ as $n_2 \to \infty$. The conclusion is that $\lim_{n\to\infty} \bar{m}_n = \bar{x}$, which implies that $m(\beta_n, K_n) \sim \bar{x}/n^\gamma = \bar{x}/n^{\theta\alpha}$.

We now prove that $m(\beta_n, K_n) \sim \bar{x}/n^\gamma = \bar{x}/n^{\theta\alpha}$ in the case where the global minimum points of g are 0 and $\pm\bar{x}$ for some $\bar{x} > 0$. In this case g is a polynomial of degree 6. There are two subcases to consider: (1) there exists an infinite subsequence n_1 in $\mathbb{N}$ such that the global minimum points of G_{n_1} are $\pm\bar{m}_{n_1}$; (2) there exists an infinite subsequence n_4 in $\mathbb{N}$ such that the global minimum points of G_{n_4} are 0 and $\pm\bar{m}_{n_4}$. Examples of sequences for which both subcases hold are given in the second paragraph before Theorem 5.2.

In subcase 1 we will prove that any subsequence n_2 of n_1 has a further subsequence n_3 for which $\bar{m}_{n_3} \to \bar{x}$. This implies that $\bar{m}_{n_1} \to \bar{x}$. In subcase 2 a similar proof shows that any subsequence n_5 of n_4 has a further subsequence n_6 for which $\bar{m}_{n_6} \to \bar{x}$. This implies that $\bar{m}_{n_4} \to \bar{x}$. Now let n_7 be an arbitrary subsequence in $\mathbb{N}$. Then n_7 contains either infinitely many elements of the subsequence n_1 or infinitely many elements of the subsequence n_4. In either case n_7 contains a further subsequence n_8 for which $\bar{m}_{n_8} \to \bar{x}$. The conclusion is that $\bar{m}_n \to \bar{x}$, which yields the desired conclusion, namely, $m(\beta_n, K_n) \sim \bar{x}/n^\gamma = \bar{x}/n^{\theta\alpha}$.

We focus on subcase 1; subcase 2 is handled similarly. In order to understand the subtlety of the proof, we return to the argument just given in the case where the global minimum points of g are $\pm\bar{x}$ for some $\bar{x} > 0$. Let n_1 be the subsequence in subcase 1 and let n_2 be any further subsequence. Since the sequence $\bar{m}_{n_2}$ is bounded, the same argument shows that there exists a further subsequence n_3 such that $\bar{m}_{n_3} \to \hat{x}$ as $n_2 \to \infty$, where $\hat{x}$ is a nonnegative global minimum point of g. When the global minimum points of g are $\pm\bar{x}$ for some $\bar{x} > 0$, we are able to conclude in fact that $\hat{x}$ equals $\bar{x}$. However, in the present case where the global minimum points of g are 0 and $\pm\bar{x}$ for some $\bar{x} > 0$, it might turn out that $\hat{x}$ equals the global minimum point of g at 0. In this situation we would conclude that $\bar{m}_{n_3} \to 0$, which is not the asymptotic relationship that we want.

As this discussion shows, in subcase 1 it suffices to prove that there exists no subsequence n_3 of n_2 for which $\bar{m}_{n_3} \to 0$ as $n_3 \to \infty$. Under the assumption that there exists such a subsequence, we will reach a contradiction of the fact that $\bar{m}_{n_3}$ is the largest critical point of G_{n_3}. This fact is not directly stated in [15], but it is a straightforward consequence of Lemmas 3.9 and 3.10(a) and Theorem 3.5 in that paper. From these three results it follows that when (β, K) lies in the two-phase region, the positive global minimum point of the function $F_{\beta,K}(z) = G_{\beta,K}(z/2\beta K)$ is also its largest critical point. From the definition of G_n in terms of G_{β_n,K_n}, it then follows that $\bar{m}_n$ is the largest critical point of G_n for all n.

Since the global minimum points of g are 0 and $\pm\bar{x}$ for some $\bar{x} > 0$, there exists $\bar{y} \in (0, \bar{x})$ such that g attains its maximum on the interval $[0, \bar{x}]$ at $\bar{y}$ and attains its maximum on the interval $[-\bar{x}, 0]$ at $-\bar{y}$. In addition, $g(\pm\bar{y}) > 0 = g(0) = g(\pm\bar{x})$. The graph of g is shown in graph (a) in Figure 7. The graph of G_{n_3} under the assumption that $\bar{m}_{n_3} \to 0$ is shown in graph (b) in Figure 7. Referring to these graphs should help the reader follow the proof.

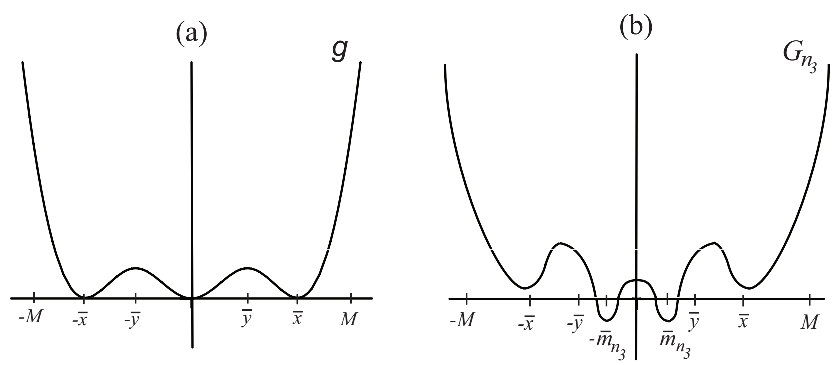

Figure 7: Proof of Theorem 4.2 in subcase 1. (a) Graph of Ginzburg-Landau polynomial g having three global minimum points, (b) graph of G_{n_3} showing $\bar{m}_{n_3} \to 0$.

By hypothesis (iii)(a), as $n_3 \to \infty$, $G_{n_3}(z) \to g(z)$ uniformly on compact subsets of $\mathbb{R}$. Thus for all sufficiently large n_3 and each choice of sign

$$G_{n_3}(\pm\bar{y}) \geq 2g(\bar{y})/3 > 0, \; G_{n_3}(\pm\bar{x}) \leq g(\bar{y})/3. \tag{4.2}$$

By definition of subcase 1 the global minimum points of G_{n_3} are $\pm\bar{m}_{n_3}$, and by assumption $\bar{m}_{n_3} \to 0$ as $n_3 \to \infty$. For all sufficiently large n_3, the inequality $G_{n_3}(\bar{m}_{n_3}) < G_{n_3}(0) = 0$

and the two inequalities in (4.2) imply that there exists $\bar{y}_{n_3} \in (\bar{m}_{n_3}, \bar{x})$ such that G_{n_3} attains its maximum on the interval $[\bar{m}_{n_3}, \bar{x}]$ at $\bar{y}_{n_3}$ and attains its maximum on the interval $[-\bar{x}, -\bar{m}_{n_3}]$ at $-\bar{y}_{n_3}$. Therefore, $\bar{y}_{n_3}$ is a critical point of G_{n_3} greater than $\bar{m}_{n_3}$. This contradicts the fact that $\bar{m}_{n_3}$ is the largest critical point of G_{n_3}. The proof of subcase 1 is complete.

Subcase 2 is handled similarly. Let n_4 be the subsequence in subcase 2 and let n_5 be any further subsequence. If there exists a subsequence n_6 of n_5 for which $\bar{m}_{n_6} \to 0$ as $n_6 \to \infty$, then for all sufficiently large n_6, there exists $\bar{y}_{n_6} \in (\bar{m}_{n_6}, \bar{x})$ such that G_{n_6} attains its maximum on the interval $[\bar{m}_{n_6}, \bar{x}]$ at $\bar{y}_{n_6}$ and attains its maximum on the interval $[-\bar{x}, -\bar{m}_{n_6}]$ at $-\bar{y}_{n_6}$. Again this contradicts the fact that $\bar{m}_{n_6}$ is the largest critical point of G_{n_6}. This completes the proof of the theorem. ■

In the next section we use Theorem 4.2 to derive the asymptotic behavior of $m(\beta_n, K_n)$ for appropriate sequences (β_n, K_n) converging to the tricritical point $(\beta_c, K(\beta_c))$ from various subsets of the phase-coexistence region. A number of new phenomena arise in this case that are not observed in the cases studied in section 3.

5 Asymptotic Behavior of $m(\beta_n, K_n)$ Near the Tricritical Point

In this section we derive the asymptotic behavior of the magnetization $m(\beta_n, K_n)$ for appropriate sequences (β_n, K_n) converging to the tricritical point $(\beta_c, K(\beta_c))$ from various subsets of the phase-coexistence region. The situation is much more complicated than in section 3, in which we studied the asymptotic behavior of $m(\beta_n, K_n)$ for two different sequences (β_n, K_n) converging to points $(\beta, K(\beta))$ on the second-order curve from the phase-coexistence region located above that curve. For each of these sequences there is a different asymptotic behavior.

By contrast, in the present section there are four distinct asymptotic behaviors of $m(\beta_n, K_n)$ corresponding to four different choices of the sequences (β_n, K_n). These are treated in Theorems 5.1–5.4. In the first three cases the limiting Ginzburg-Landau polynomial has degree 6, and in the fourth case it has degree 4. The most interesting example is treated in Theorem 5.2. The sequence (β_n, K_n) in that theorem converges to the tricritical point for $\beta_n > \beta_c$ along a curve that is tangent to the spinodal curve at the tricritical point and depends on a curvature parameter. For those sequences that lie in the phase-coexistence region, Theorem 5.2 shows that $m(\beta_n, K_n) \sim \bar{x}(\beta_n - \beta_c)^{1/2}$.

As in section 3, properties of the Ginzburg-Landau polynomials in these four cases reflect the phase-transition structure of the mean-field B-C model in the region through which the associated sequence (β_n, K_n) passes. This again makes rigorous the predictions of the Ginzburg-

Landau phenomenology of critical phenomena mentioned in section 2.

Let (β_n, K_n) be an arbitrary positive sequence converging to the tricritical point $(\beta_c, K(\beta_c)) = (\log 4, 3/2 \log 4)$ and let $\gamma > 0$ be given. In section 2 we motivated the phase-transition structure for $\beta > \beta_c$ by approximating $G_{\beta,K}(x)$ in (2.10) by a polynomial of degree 6 derived from the first three terms in its Taylor expansion. The starting point in determining the asymptotic behavior of $m(\beta_n, K_n)$ is to replace this three-term Taylor expansion for $G_{\beta,K}(x)$ by the three-term Taylor expansion for $nG_{\beta_n,K_n}(x/n^\gamma)$ with an error term. According to Taylor's Theorem, for any $R > 0$ and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$ there exists $\xi_n(x/n^\gamma) \in [-x/n^\gamma, x/n^\gamma]$ such that

$$\begin{aligned} & nG_{\beta_n,K_n}(x/n^\gamma) \\ & \quad = \frac{1}{n^{2\gamma-1}} \frac{G^{(2)}_{\beta_n,K_n}(0)}{2!} x^2 + \frac{1}{n^{4\gamma-1}} \frac{G^{(4)}_{\beta_n,K_n}(0)}{4!} x^4 \\ & \quad \quad + \frac{1}{n^{6\gamma-1}} \frac{G^{(6)}_{\beta_n,K_n}(0)}{6!} x^6 + \frac{1}{n^{7\gamma-1}} \frac{G^{(7)}_{\beta_n,K_n}(\xi_n(x/n^\gamma))}{7!} x^7. \end{aligned} \tag{5.1}$$

In deriving this formula, we use the fact that $G_{\beta_n,K_n}(0) = 0$ and that since G_{β_n,K_n} is an even function, $G^{(1)}_{\beta_n,K_n}(0) = 0 = G^{(3)}_{\beta_n,K_n}(0) = G^{(5)}_{\beta_n,K_n}(0)$. Because the sequence (β_n, K_n) is positive and bounded, there exists $a \in (0, \infty)$ such that $0 < \beta_n \leq a$ and $0 < K_n \leq a$ for all n. As a continuous function of (β, K, y) on the compact set $[0, a] \times [0, a] \times [-R, R]$, $G^{(7)}_{\beta,K}(y)$ is uniformly bounded. It follows that the quantity $G^{(7)}_{\beta_n,K_n}(\xi_n(x/n^\gamma))$ appearing in the error term in the Taylor expansion is uniformly bounded for $x \in (-Rn^\gamma, Rn^\gamma)$. We summarize this expansion by writing

$$\begin{aligned} nG_{\beta_n,K_n}(x/n^\gamma) &= \frac{1}{n^{2\gamma-1}} \frac{G^{(2)}_{\beta_n,K_n}(0)}{2!} x^2 + \frac{1}{n^{4\gamma-1}} \frac{G^{(4)}_{\beta_n,K_n}(0)}{4!} x^4 \\ & \quad + \frac{1}{n^{6\gamma-1}} \frac{G^{(6)}_{\beta_n,K_n}(0)}{6!} x^6 + \mathrm{O}\left(\frac{1}{n^{7\gamma-1}}\right) x^7, \end{aligned} \tag{5.2}$$

where the big-oh term is uniform for $x \in (-Rn^\gamma, Rn^\gamma)$.

In terms of the quantity $K(\beta) = (e^\beta + 2)/4\beta$, the coefficients $G^{(2)}_{\beta_n,K_n}(0)$ and $G^{(4)}_{\beta_n,K_n}(0)$ in the Taylor expansion are given by

$$G^{(2)}_{\beta_n,K_n}(0) = \frac{2\beta_n K_n (K(\beta_n) - K_n)}{K(\beta_n)} = 2\beta(K(\beta_n) - K_n) \cdot \frac{\beta_n K_n}{\beta K(\beta_n)}$$

and

$$G^{(4)}_{\beta_n,K_n}(0) = \frac{2(2\beta_n K_n)^4 (4 - e^{\beta_n})}{(e^{\beta_n} + 2)^2}.$$

In order to ease the notation, we let ε_n denote a sequence that converges to 0 and that represents the various error terms arising in the following calculation; we use the same notation ε_n to represent different error terms. Since (β_n, K_n) converges to $(\beta_c, K(\beta_c))$ and the function $K(\cdot)$ is continuous, we have $\beta_n K_n / K(\beta_n) \to \beta_c$. Thus

$$G^{(2)}_{\beta_n,K_n}(0)/2! = \beta_c(K(\beta_n) - K_n)(1 + \varepsilon_n).$$

Let $c_4 = 3/16$. Since $2\beta_n K_n \to 2\beta_c K(\beta_c) = (e^{\beta_c} + 2)/2 = 3$ and $e^{\beta_n} + 2 \to e^{\beta_c} + 2 = 6$, we also have

$$G^{(4)}_{\beta_n,K_n}(0)/4! = 2 \cdot 3^4(4 - e^{\beta_n})(1 + \varepsilon_n)/6^2 \cdot 4! = c_4(4 - e^{\beta_n})(1 + \varepsilon_n). \tag{5.3}$$

In section 3 we have $\beta_n \to \beta \in (0, \beta_c)$ and thus $4 - e^{\beta_n} = (4 - e^{\beta})(1 + \varepsilon_n) > 0$. In the present section, however, $4 - e^{\beta_n} \to 0$ as $\beta_n \to \beta_c$, and so we must keep this term in the last display. Finally, let $c_6 = 9/40$. Since $G^{(6)}_{\beta_n,K_n}(0) \to G^{(6)}_{\beta_c,K(\beta_c)}(0) = 2 \cdot 3^4$, we have

$$G^{(6)}_{\beta_n,K_n}(0)/6! = 2 \cdot 3^4(1 + \varepsilon_n)/6! = c_6(1 + \varepsilon_n).$$

Substituting these expressions into the Taylor expansion (5.2), we obtain for all $n \in \mathbb{N}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$\begin{aligned} &nG_{\beta_n,K_n}(x/n^\gamma) \\ &\quad = \frac{1}{n^{2\gamma-1}}\beta_c(K(\beta_n) - K_n)(1 + \varepsilon_n)x^2 + \frac{1}{n^{4\gamma-1}}c_4(4 - e^{\beta_n})(1 + \varepsilon_n)x^4 \\ &\quad + \frac{1}{n^{6\gamma-1}}c_6(1 + \varepsilon_n)x^6 + \mathrm{O}\left(\frac{1}{n^{7\gamma-1}}\right)x^7, \end{aligned} \tag{5.4}$$

where $c_4 = 3/16$ and $c_6 = 9/40$.

For the moment, in the polynomial on the right side of the last display, let us replace (β_n, K_n) by (β, K) and set $n = 1$. Doing so, we obtain the polynomial $\tilde{G}_{\beta,K}$ that approximates the free energy functional $G_{\beta,K}$ in (2.10) for $\beta > \beta_c$, (β, K) near the tricritical point, and x near 0. Arising via the Ginzburg-Landau phenomenology, this polynomial is used in section 2 to motivate the discontinuous bifurcation in the set of equilibrium values of the magnetization that is described rigorously in Theorem 2.3. As we will soon see, by suitable choices of (β_n, K_n) and other parameters the polynomial on the right side of the last display converges to a Ginzburg-Landau polynomial in terms of which the convergence of $m(\beta_n, K_n)$ to 0 is described.

We return to (5.4), in which the terms $K(\beta_n) - K_n$ and $4 - e^{\beta_n}$ both converge to 0 as $n \to \infty$. This formula is the seed from which will blossom the various asymptotic behaviors of $m(\beta_n, K_n)$, each depending on the choice of the sequence (β_n, K_n) converging to the tricritical point. Each choice controls, in a different way, the rate at which $K(\beta_n) - K_n \to 0$ and $e^{\beta_n} - 4 \to 0$. We analyze four separate cases, each giving rise to a Ginzburg-Landau polynomial having a unique positive, global minimum points at $\bar{x}$ for some $\bar{x} > 0$. This quantity enters the respective asymptotic formula for $m(\beta_n, K_n) \to 0$.

For the first choice we take $\alpha > 0$, $b \in \{1, 0, -1\}$, and $k \in \mathbb{R}$ and define

$$\beta_n = \beta_c + b/n^\alpha \text{ and } K_n = K(\beta_c) + k/n^\alpha. \tag{5.5}$$

If $b \neq 0$, then (β_n, K_n) converges to the tricritical point along a ray with slope k/b (see path 3 in Figure 2). We assume that $K'(\beta_c)b - k \neq 0$. Since

$$K(\beta_n) = K(\beta_c + b/n^\alpha) = K(\beta_c) + K'(\beta_c)b/n^\alpha + \mathrm{O}(1/n^{2\alpha}),$$

we have

$$K(\beta_n) - K_n = (K'(\beta_c)b - k)/n^\alpha + \mathrm{O}(1/n^{2\alpha}) \tag{5.6}$$

and

$$4 - e^{\beta_n} = e^{\beta_c}(1 - e^{b/n^\alpha}) = -4b/n^\alpha + \mathrm{O}(1/n^{2\alpha}). \tag{5.7}$$

The case where $K'(\beta_c)b - k = 0$ must be handled differently. If this equality holds, then the expression for $K(\beta_n) - K_n$ given here is indeterminate. In order to calculate the correct behavior of $K(\beta_n) - K_n$ when $K'(\beta_c)b - k = 0$, one must consider the next term in the Taylor expansion of $K(\beta_c + b/n^\alpha)$, obtaining (5.13) with $\ell = 0 = \tilde{\ell}$. We carry out the analysis for this case after Theorem 5.1.

We return to the sequence (β_n, K_n) in (5.5) when $K'(\beta_c)b - k \neq 0$. Substituting (5.6) and (5.7) into (5.4), we obtain for all $n \in \mathbb{N}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$\begin{aligned} &nG_{\beta_n,K_n}(x/n^\gamma) \\ &\quad = \frac{1}{n^{2\gamma+\alpha-1}}\beta_c(K'(\beta_c)b - k)(1+\varepsilon_n)x^2 - \frac{1}{n^{4\gamma+\alpha-1}}4c_4 b(1+\varepsilon_n)x^4 \\ &\qquad + \frac{1}{n^{6\gamma-1}}c_6(1+\varepsilon_n)x^6 + \mathrm{O}\left(\frac{1}{n^{2\gamma+2\alpha-1}}\right)x^2 \\ &\qquad + \mathrm{O}\left(\frac{1}{n^{4\gamma+2\alpha-1}}\right)x^4 + \mathrm{O}\left(\frac{1}{n^{7\gamma-1}}\right)x^7. \end{aligned} \tag{5.8}$$

Given $u \in \mathbb{R}$, we multiply the numerator and denominator of the right side of the last display by n^u, obtaining $nG_{\beta_n,K_n}(x/n^\gamma) = n^u G_n(x)$, where for any $R > 0$ and all $x \in \mathbb{R}$ satisfying

$|x/n^\gamma| < R$

$$\begin{aligned} G_n(x) & \quad (5.9) \\ &= \frac{1}{n^{2\gamma+\alpha-1+u}} \beta_c (K'(\beta_c)b - k)(1+\varepsilon_n)x^2 - \frac{1}{n^{4\gamma+\alpha-1+u}} 4c_4 b(1+\varepsilon_n)x^4 \\ &\quad + \frac{1}{n^{6\gamma-1+u}} c_6(1+\varepsilon_n)x^6 + \mathrm{O}\!\left(\frac{1}{n^{2\gamma+2\alpha-1+u}}\right)x^2 \\ &\quad + \mathrm{O}\!\left(\frac{1}{n^{4\gamma+2\alpha-1+u}}\right)x^4 + \mathrm{O}\!\left(\frac{1}{n^{7\gamma-1+u}}\right)x^7. \end{aligned}$$

In this formula $\varepsilon_n \to 0$ and the big-oh terms are uniform for $x \in (-Rn^\gamma, Rn^\gamma)$. Since $K'(\beta_c)b - k \neq 0$ and $c_6 > 0$, the coefficients of x^2, x^4, and x^6 in the first three terms are all nonzero.

In order to obtain the limit of G_n, we impose the condition that the powers of n appearing in the first and third terms in the last display equal 0; i.e., $2\gamma + \alpha - 1 + u = 0 = 6\gamma - 1 + u$. These two equalities are equivalent to $\gamma = \alpha/4$ and $u = 1 - 6\gamma = 1 - 3\alpha/2$. With this choice of γ and u, the powers of n in the second term and the last three terms in (5.9) are positive, and so for all $x \in \mathbb{R}$ these four terms converge to 0 as $n \to \infty$. It follows that as $n \to \infty$, we have for all $x \in \mathbb{R}$

$$G_n(x) = n^{1-u} G_{\beta_n, K_n}(x/n^\gamma) \to g(x) = \beta_c(K'(\beta)b - k)x^2 + c_6 x^6.$$

Since the big-oh terms in (5.9) are uniform for $x \in (-Rn^\gamma, Rn^\gamma)$, the convergence of $G_n(x)$ to $g(x)$ is uniform for x in compact subsets of $\mathbb{R}$.

In the next theorem we derive the asymptotic behavior of $m(\beta_n, K_n)$ for the sequence (β_n, K_n) defined in (5.5) with $K'(\beta_c)b - k < 0$. This inequality is equivalent to (β_n, K_n) lying in the phase-coexistence region for all sufficiently large n. Derived from the general asymptotic result in Theorem 4.2, part (c) of the next theorem expresses the asymptotic behavior of $m(\beta_n, K_n) \to 0$ in terms of the unique positive, global minimum point $\bar{x}$ of the associated Ginzburg-Landau polynomial g.

Theorem 5.1. *For $\alpha > 0$, $b \in \{1, 0, -1\}$, and a real number $k \neq K'(\beta_c)b$, define*

$$\beta_n = \beta_c + b/n^\alpha \;\; \textit{and} \;\; K_n = K(\beta_c) + k/n^\alpha$$

as well as $c_6 = 9/40$. Then (β_n, K_n) converges to the tricritical point $(\beta_c, K(\beta_c))$. The following conclusions hold.

(a) *For any $\alpha > 0$, $u = 1 - 3\alpha/2$, and $\gamma = \alpha/4$*

$$G_n(x) = n^{1-u} G_{\beta_n, K_n}(x/n^\gamma) \to g(x) = \beta_c(K'(\beta_c)b - k)x^2 + c_6 x^6$$

uniformly for x in compact subsets of $\mathbb{R}$.

(b) *The Ginzburg-Landau polynomial g has nonzero global minimum points if and only if $K'(\beta_c)b - k < 0$. If this inequality holds, then the global minimum points of g are $\pm\bar{x}$, where*

$$\bar{x} = (\beta_c(k - K'(\beta_c)b)/[3c_6])^{1/4} \tag{5.10}$$

(c) *Assume that $K'(\beta_c)b - k < 0$. Then for any $\alpha > 0$, $m(\beta_n, K_n) \to 0$ and has the asymptotic behavior*

$$m(\beta_n, K_n) \sim \bar{x}/n^{\alpha/4}; \;\; \textit{i.e.,} \;\; \lim_{n\to\infty} n^{\alpha/4} m(\beta_n, K_n) = \bar{x}.$$

When $b \neq 0$, this becomes $m(\beta_n, K_n) \sim \bar{x}|\beta_c - \beta_n|^{1/4}$.

Proof. Part (a) follows from the discussion leading up to the statement of the theorem. The first assertion in part (b) is elementary. If $K'(\beta_c)b - k < 0$, then the equation $g'(x) = 6c_6x^5 + 2\beta_c(K'(\beta_c)b - k)x = 0$ has solutions at $\pm\bar{x}$ and at 0, where $\bar{x}$ is defined in (5.10). One easily checks that $\pm\bar{x}$ are global minimum points and 0 a local maximum point.

We now verify the asymptotic behavior of $m(\beta_n, K_n)$ in part (c). According to Theorem 4.1, $m(\beta_n, K_n) \to 0$. The validity of hypotheses (i) and (ii) of Theorem 4.2 follows from the definition of the sequences (β_n, K_n) and the inequality $K'(\beta_c)b - k < 0$, which by (5.6) is equivalent to $K_n > K(\beta_n)$ for all sufficiently large n. Thus if $K'(\beta_c)b - k < 0$, then for all sufficiently large n, (β_n, K_n) lies in the phase-coexistence region; (β_n, K_n) is above the spinodal curve if $b = 1$, above the second-order curve if $b = -1$, and above the tricritical point if $b = 0$. Hypothesis (iii) of Theorem 4.2 is parts (a) and (b) of the present theorem. We now verify hypothesis (iv) of Theorem 4.2. Using (5.9) with $\gamma = \alpha/4$ and $u = 1 - 3\alpha/2$, one easily proves that for any $\alpha > 0$ there exists $R > 0$ such that for all sufficiently large $n \in \mathbb{N}$ and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$G_n(x) = n^{1-u} G_{\beta_n, K_n}(x/n^\gamma) \geq H(x) = -2\beta_c|K'(\beta_c)b - k|x^2 - 8c_4x^4 + \tfrac{1}{2}c_6x^6.$$

Since $H(x) \to \infty$ as $|x| \to \infty$, hypothesis (iv) of Theorem 4.2 is satisfied. The new element is that the term $-8c_4|b|x^4$ in $H(x)$ must be included in order to bound below the two x^4-terms in (5.8) for all n. This completes the verification of the four hypotheses of Theorem 4.2. We now apply the theorem to conclude that for any $\alpha > 0$, $m(\beta_n, K_n) \sim \bar{x}/n^\gamma = \bar{x}/n^{\alpha/4}$. Part (c) of the present theorem is proved. ■

We now consider the second choice of sequence (β_n, K_n) converging to the tricritical point, which gives a different asymptotic behavior of $m(\beta_n, K_n) \to 0$. Given $\alpha > 0$, $\ell \in \mathbb{R}$, and $\tilde{\ell} \in \mathbb{R}$ we define

$$\beta_n = \beta_c + 1/n^\alpha \;\text{ and }\; K_n = K(\beta_c) + K'(\beta_c)/n^\alpha + \ell/(2n^{2\alpha}) + \tilde{\ell}/(6n^{3\alpha}). \tag{5.11}$$

The term involving $\tilde{\ell}$ is needed only in the case where $\ell = \ell_c$ in order to assure that (β_n, K_n) lies in the phase-coexistence region for all sufficiently large n; see item (ii) after (5.12). In all other cases the inclusion of the term involving $\tilde{\ell}$ adds no new features, and we can take $\tilde{\ell} = 0$.

The choice $\ell = \tilde{\ell} = 0$ in (5.11) reduces to the sequence (β_n, K_n) in (5.5) when $K'(\beta_c)b - k = 0$ in that formula. However, as pointed out in the paragraph after (5.5), the analysis given there is valid only when $K'(\beta_c)b - k \neq 0$.

We now consider the behavior of (β_n, K_n) for various choices of ℓ and $\tilde{\ell}$. Since $\beta_n - \beta_c = 1/n^\alpha$, we can write

$$K_n = K(\beta_c) + K'(\beta_c)(\beta_n - \beta_c) + \ell(\beta_n - \beta_c)^2/2 + \tilde{\ell}(\beta_n - \beta_c)^3/6.$$

Thus (β_n, K_n) converges to the tricritical point along the curve $(\beta, \tilde{K}(\beta))$, where for $\beta > \beta_c$

$$\tilde{K}(\beta) = K(\beta_c) + K'(\beta_c)(\beta - \beta_c) + \ell(\beta - \beta_c)^2/2 + \tilde{\ell}(\beta - \beta_c)^3/6.$$

This curve is tangent to the spinodal curve at the tricritical point and satisfies $\tilde{K}''(\beta_c) = \ell$. Thus $\ell > K''(\beta_c)$ and any $\tilde{\ell} \in \mathbb{R}$ correspond to the sequence (β_n, K_n) converging to the tricritical point from the phase-coexistence region located above the spinodal curve (see path 4a in Figure 2). The value $\ell = K''(\beta_c)$ corresponds to the sequence (β_n, K_n) converging to $(\beta_c, K(\beta_c))$ along a curve that coincides with the spinodal curve to order 2 in powers of $\beta - \beta_c$ in a neighborhood of the tricritical point. When $\ell = K''(\beta_c)$ and $\tilde{\ell} > K'''(\beta_c)$, (β_n, K_n) lies in the phase-coexistence region above the spinodal curve for all sufficiently large n (see path 4b in Figure 2). Since $K'''(\beta_c) < 0$ [Lem. 6.1(d)], we can take $\tilde{\ell} = 0$.

The situation for $\ell < K''(\beta_c)$ is considerably more complicated. The discussion is based on three conjectured properties of the function $K_1(\beta)$, which for $\beta > \beta_c$ defines the first-order curve. Since $\lim_{\beta \to \beta_c^+} K_1(\beta) = K(\beta_c)$, by continuity we extend the definition of $K_1(\beta)$ to $\beta = \beta_c$ by defining $K_1(\beta_c) = K(\beta_c)$. In the next paragraph, we assume that the first three right-hand derivatives of $K_1(\beta)$ exist at β_c and denote them by $K_1'(\beta_c)$, $K_1''(\beta_c)$, and $K_1'''(\beta_c)$.

We define

$$\ell_c = -1/(4\beta_c) - 2/\beta_c^2 + 3/\beta_c^3. \tag{5.12}$$

In section 6 we use properties of the appropriate Ginzburg-Landau polynomials plus numerical evidence to support Conjectures 1, 2, and 3, which state the following: (1) $K_1'(\beta_c) = K'(\beta_c)$ — i.e., at β_c the first-order curve and the spinodal curve have the same right-hand tangent; (2) $K_1''(\beta_c) = \ell_c < 0 < K''(\beta_c)$; (3) $K_1'''(\beta_c) > 0$. If these conjectures are true, then we have the following picture:

1. Assume that ℓ satisfies $K''(\beta_c) > \ell > \ell_c$ and take any $\tilde{\ell} \in \mathbb{R}$. Then by Conjectures 1 and 2, (β_n, K_n) converges to the tricritical point from the phase-coexistence region along a curve that passes above the first-order curve and below the spinodal curve in a neighborhood of the tricritical point (see path 4d in Figure 2).

2. Assume that $\ell = \ell_c$ and take any $\tilde{\ell} > K_1'''(\beta_c)$. Then by Conjectures 1 and 2 (β_n, K_n) converges to the tricritical point along a curve that coincides with the first-order curve to order 2 in powers of $\beta - \beta_c$ in a neighborhood of the tricritical point (see path 4c in Figure 2). By Conjecture 3, for all sufficiently large n, (β_n, K_n) lies in the phase-coexistence region above the first-order curve and below the spinodal curve. If we did not include the term involving $\tilde{\ell}$ in (5.11), then by Conjecture 3 the sequence (β_n, K_n) would lie in the single-phase region below the first-order curve for all sufficiently large n.

3. Assume that $\ell < \ell_c$ and take any $\tilde{\ell} \in \mathbb{R}$. Then by Conjectures 1 and 2 (β_n, K_n) converges to the tricritical point from the single-phase region located under the first-order curve.

The structure of the set of global minimum points of the associated Ginzburg-Landau polynomials g mirrors the phase-transition structure of the region through which the corresponding sequence (β_n, K_n) passes. As we will see in items 1–3 before the statement of Theorem 5.2, the following three cases arise: (1) for $\ell > \ell_c$, the global minimum points of g are $\pm\bar{x}(\ell)$, where $\bar{x}(\ell) > 0$ is defined in (5.19); (2) for $\ell = \ell_c$, the global minimum points of g are 0 and $\pm\bar{x}(\ell_c) = \sqrt{5/3}$; (3) for $\ell < \ell_c$, g has a unique global minimum point at 0. These three cases mirror the following features of G_{β_n,K_n}: (1) for (β_n, K_n) above the first-order curve, the global minimum points of G_{β_n,K_n} are the symmetric nonzero pair $\pm m(\beta_n, K_n)$; (2) for (β_n, K_n) on the first-order curve, the global minimum points of G_{β_n,K_n} are 0 and the symmetric nonzero pair $\pm m(\beta_n, K_n)$; (3) for (β_n, K_n) below the first-order curve, G_{β_n,K_n} has a unique global minimum point at 0. In addition, as pointed out in item 4 before the statement of Theorem 5.2, the set of global minimum points of g undergoes a discontinuous bifurcation at $\ell = \ell_c$. This mirrors the discontinuous bifurcation that occurs in the set of global minimum points of $G_{\beta,K}$ at $K = K_1(\beta)$ for $\beta > \beta_c$ [Thm. 2.3(d)].

In order to verify these properties of the Ginzburg-Landau polynomials, we must calculate the relevant expansion of $nG_{\beta_n,K_n}(x/n^\gamma)$ in (5.4). We first consider the case where $\ell \neq K''(\beta_c)$; the case where $\ell = K''(\beta_c)$ will be discussed later. Since

$$\begin{aligned} K(\beta_n) &= K(\beta_c + 1/n^\alpha) \\ &= K(\beta_c) + K'(\beta_c)/n^\alpha + K''(\beta_c)/(2n^{2\alpha}) + K'''(\beta_c)/(6n^{3\alpha}) + \mathrm{O}(1/n^{4\alpha}), \end{aligned}$$

we have

$$K(\beta_n) - K_n = (K''(\beta_c) - \ell)/(2n^{2\alpha}) + (K'''(\beta_c) - \tilde{\ell})/(6n^{3\alpha}) + \mathrm{O}(1/n^{4\alpha}) \qquad (5.13)$$

and

$$4 - e^{\beta_n} = 4(1 - e^{1/n^\alpha}) = -4/n^\alpha + \mathrm{O}(1/n^{2\alpha}).$$

Substituting the last two formulas into (5.4), we see that for all $n \in \mathbb{N}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$\begin{aligned} & nG_{\beta_n,K_n}(x/n^\gamma) \\ & = \frac{1}{n^{2\gamma+2\alpha-1}} \frac{1}{2}\beta_c(K''(\beta_c)-\ell)(1+\varepsilon_n)x^2 - \frac{1}{n^{4\gamma+\alpha-1}} 4c_4(1+\varepsilon_n)x^4 \\ & \quad + \frac{1}{n^{6\gamma-1}} c_6(1+\varepsilon_n)x^6 + \mathrm{O}\bigg(\frac{1}{n^{2\gamma+3\alpha-1}}\bigg)x^2 + \mathrm{O}\bigg(\frac{1}{n^{2\gamma+4\alpha-1}}\bigg)x^2 \\ & \quad + \mathrm{O}\bigg(\frac{1}{n^{4\gamma+2\alpha-1}}\bigg)x^4 + \mathrm{O}\bigg(\frac{1}{n^{7\gamma-1}}\bigg)x^7, \end{aligned} \tag{5.14}$$

where $c_4 = 3/16$ and $c_6 = 9/40$. Given $u \in \mathbb{R}$, we multiply the numerator and denominator of the right side of the last display by n^u, obtaining $nG_{\beta_n,K_n}(x/n^\gamma) = n^u G_n(x)$, where for all $n \in \mathbb{N}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$\begin{aligned} & G_n(x) \\ & = \frac{1}{n^{2\gamma+2\alpha-1+u}} \frac{1}{2}\beta_c(K''(\beta_c)-\ell)(1+\varepsilon_n)x^2 - \frac{1}{n^{4\gamma+\alpha-1+u}} 4c_4(1+\varepsilon_n)x^4 \\ & \quad + \frac{1}{n^{6\gamma-1+u}} c_6(1+\varepsilon_n)x^6 + \mathrm{O}\bigg(\frac{1}{n^{2\gamma+3\alpha-1+u}}\bigg)x^2 + \mathrm{O}\bigg(\frac{1}{n^{2\gamma+4\alpha-1+u}}\bigg)x^2 \\ & \quad + \mathrm{O}\bigg(\frac{1}{n^{4\gamma+2\alpha-1+u}}\bigg)x^4 + \mathrm{O}\bigg(\frac{1}{n^{7\gamma-1+u}}\bigg)x^7. \end{aligned} \tag{5.15}$$

In this formula $\varepsilon_n \to 0$ and the big-oh terms are uniform for $x \in (-Rn^\gamma, Rn^\gamma)$. Since $\ell \neq K''(\beta_c)$, $c_4 > 0$, and $c_6 > 0$, the coefficients of x^2, x^4, and x^6 in the first three terms are all nonzero.

In order to obtain the limit of G_n, we impose the condition that the powers of n appearing in the first three terms in the last display equal 0; i.e., $2\gamma + 2\alpha - 1 + u = 0 = 4\gamma + \alpha - 1 + u = 6\gamma - 1 + u$. These three equalities are equivalent to $\gamma = \alpha/2$ and $u = 1 - 6\gamma = 1 - 3\alpha$. With this choice of γ and u, the powers of n in the last four terms in (5.15) are positive, and so for all $x \in \mathbb{R}$ these four terms converge to 0 as $n \to \infty$. It follows that as $n \to \infty$, we have for all $x \in \mathbb{R}$

$$G_n(x) = n^{1-u} G_{\beta_n,K_n}(x/n^\gamma) \to g_\ell(x) = \tfrac{1}{2}\beta_c(K''(\beta_c)-\ell)x^2 - 4c_4x^4 + c_6x^6. \tag{5.16}$$

Because the big-oh terms in (5.15) are uniform for $x \in (-Rn^\gamma, Rn^\gamma)$, the convergence of $G_n(x)$ to $g(x)$ is uniform for x in compact subsets of $\mathbb{R}$. Since $\ell \neq K''(\beta_c)$, the three coefficients in g_ℓ are all nonzero. We write the Ginzburg-Landau polynomial as g_ℓ in order to emphasize the dependence on the parameter ℓ; g_ℓ does not depend on the choice of $\tilde{\ell}$ in (5.11).

We now briefly consider the case where $\ell = K''(\beta_c)$. In this situation we have the right side of (5.13) with $\ell = K''(\beta_c)$. We omit the rest of the calculation showing that when $\ell = K''(\beta_c)$, the limit of $G_n(x) = n^{1-u} G_{\beta_n, K_n}(x/n^\gamma)$ equals the same Ginzburg-Landau polynomial g_ℓ.

The Ginzburg-Landau polynomial g_ℓ has the form $a_2x^2 - a_4x^4 + a_6x^6$, where

$$a_2 = \beta_c(K''(\beta_c) - \ell)/2, a_4 = 4c_4 = 3/4 > 0, \text{ and } a_6 = c_6 = 9/40 > 0; \tag{5.17}$$

depending on the value of ℓ, a_2 can be positive, 0, or negative. We are interested in the structure of the set of global minimum points of g_ℓ for variable ℓ. In order to analyze this structure, we define the critical value $a_c = a_4^2/4a_6$. According to Theorem A.1, if $a_2 < a_c$, then the global minimum points of $a_2x^2 - a_4x^4 + a_6x^6$ are $\pm\bar{x}(a_2)$, where $\bar{x}(a_2) > 0$ is defined in (A.1); if $a_2 = a_c$, then the global minimum points of this polynomial are 0 and $\pm(2a_2/a_4)^{1/2}$; and if $a_2 > a_c$, then the unique global minimum point of this polynomial is 0. Substituting the values of a_2, a_4, and a_6, we see that $a_c = 5/8$. Defining ℓ_c to be the value of ℓ for which $a_2 = a_c$, we find that

$$\ell_c = K''(\beta_c) - 5/(4\beta_c) = -1/(4\beta_c) - 2/\beta_c^2 + 3/\beta_c^3 = -0.094979. \tag{5.18}$$

The second equality follows from the formula for $K''(\beta_c)$ given in part (d) of Lemma 6.1. The structure of the set of global minimum points of the polynomial $a_2x^2 - a_4x^4 + a_6x^6$ translates into the following structure of the set of global minimum points of the Ginzburg-Landau polynomial

$$g_\ell(x) = \tfrac{1}{2}\beta_c(K''(\beta_c) - \ell)x^2 - 4c_4x^4 + c_6x^6.$$

The formula for $\bar{x}(\ell)$ in (5.19) is obtained by substituting a_2, a_4, and a_6 from (5.17) into the definition (A.1) of $\bar{x}(a_2)$.

1. For $\ell > \ell_c$, which corresponds to $a_2 < a_c$, the global minimum points of g_ℓ are $\pm\bar{x}(\ell)$, where
$$\bar{x}(\ell) = \tfrac{\sqrt{10}}{3}\left(1 + \left(1 - \tfrac{3\beta_c}{5}(K''(\beta_c) - \ell)\right)^{1/2}\right)^{1/2}. \tag{5.19}$$
The polynomial g_ℓ has the same shape as $G_{\beta,K}$ in Figure 6 in section 2.

2. For $\ell = \ell_c$, which corresponds to $a_2 = a_c$, the global minimum points of g_ℓ are 0 and $\pm\bar{x}(\ell_c)$, where $\bar{x}(\ell_c) = \sqrt{5/3}$. The polynomial g_ℓ has the same shape as $G_{\beta,K}$ in Figure 5 in section 2.

3. For $\ell < \ell_c$, which corresponds to $a_2 > a_c$, g_ℓ has a unique global minimum point at 0. The polynomial g_ℓ has the same shape as $G_{\beta,K}$ in Figure 4 in section 2.

4. The set of global minimum points of g_ℓ undergoes a discontinuous bifurcation at $\ell = \ell_c$, changing discontinuously from $\{0\}$ for $\ell < \ell_c$ to $\{0, \pm\bar{x}(\ell_c)\}$ for $\ell = \ell_c$ to $\{\pm\bar{x}(\ell)\}$ for $\ell > \ell_c$. This mirrors an analogous property of $G_{\beta,K}$ given in part (d) of Theorem 2.3.

Assuming Conjectures 1–3 in section 6, we contrast properties of the sequence (β_n, K_n) for $\ell = \ell_c$ and $\tilde{\ell} > K_1'''(\beta_c)$ with properties of another sequence having the same Ginzburg-Landau polynomial but a different structure of the set of global minimum points of G_{β_n,K_n}. For the first sequence, since (β_n, K_n) lies in the phase-coexistence region above the first-order curve for all sufficiently large n, the global minimum points of G_{β_n,K_n} are the symmetric nonzero pair $\pm m(\beta_n, K_n)$, and as we have just pointed out, the global minimum points of g_{ℓ_c} are 0 and $\pm\bar{x}(\ell_c)$. This is to be contrasted with the sequence $\beta_n = \beta_c + 1/n^\alpha$, $K_n = K_1(\beta_n)$. Since this sequence lies on the first-order curve for all n, the global minimum points of G_{β_n,K_n} are 0 and $\pm m(\beta_n, K_n)$. If one replaces $K_1(\beta_n)$ by the first three terms in its Taylor expansion plus an error term and uses Conjectures 1 and 2 in section 6, then one sees that the associated Ginzburg-Landau polynomial coincides with g_{ℓ_c}.

For the sequence (β_n, K_n) defined in (5.11) the asymptotic behavior of $m(\beta_n, K_n)$ is given in the next theorem. As we verify in the discussion after (5.11), (β_n, K_n) lies in the phase-coexistence region for all sufficiently large n under the following conditions on the parameters ℓ and $\tilde{\ell}$; in the last two cases these conditions are supplemented by the appropriate conjectures in section 6:

(i) $\ell > K''(\beta_c)$ and $\tilde{\ell} \in \mathbb{R}$,

(ii) $\ell = K''(\beta_c)$ and $\tilde{\ell} > K'''(\beta_c)$,

(iii) $K''(\beta_c) > \ell > \ell_c$, $\tilde{\ell} \in \mathbb{R}$, and Conjectures 1 and 2,

(iv) $\ell = \ell_c$, $\tilde{\ell} > K_1'''(\beta_c)$, and Conjectures 1–3.

Theorem 5.2. *For $\alpha > 0$, $\ell \in \mathbb{R}$, and $\tilde{\ell} \in \mathbb{R}$, define*

$$\beta_n = \beta_c + 1/n^\alpha \ \text{ and } \ K_n = K(\beta_c) + K'(\beta_c)/n^\alpha + \ell/(2n^{2\alpha}) + \tilde{\ell}/(6n^{3\alpha})$$

as well as $c_4 = 3/16$ and $c_6 = 9/40$. Then (β_n, K_n) converges to the tricritical point $(\beta_c, K(\beta_c))$. The following conclusions hold.

(a) *For any $\alpha > 0$, $u = 1 - 3\alpha$, and $\gamma = \alpha/2$*

$$G_n(x) = n^{1-u} G_{\beta_n,K_n}(x/n^\gamma) \to g_\ell(x) = \tfrac{1}{2}\beta_c(K''(\beta_c) - \ell)x^2 - 4c_4x^4 + c_6x^6$$

uniformly for x in compact subsets of $\mathbb{R}$.

(b) *The Ginzburg-Landau polynomial g_ℓ has nonzero global minimum points if and only if $\ell \geq \ell_c = (-\beta_c^2 - 8\beta_c + 12)/4\beta_c^3$.*

(i) *Assume that* $\ell > \ell_c$. *Then the global minimum points of* g_ℓ *are* $\pm\bar{x}(\ell)$, *where* $\bar{x}(\ell)$ *is defined in* (5.19).

(ii) *Assume that* $\ell = \ell_c$. *Then the global minimum points of the Ginzburg-Landau polynomial* $g_\ell(x)$ *are* 0 *and* $\pm\bar{x}(\ell_c)$, *where* $\bar{x}(\ell_c) = (5/3)^{1/2}$.

(c) *In each of the cases* (i)–(iv) *appearing before the statement of the theorem and for any* $\alpha > 0$, $m(\beta_n, K_n) \to 0$ *and has the asymptotic behavior*

$$m(\beta_n, K_n) \sim \bar{x}(\ell)/n^{\alpha/2} = \bar{x}(\ell)(\beta_n - \beta_c)^{1/2}; \;\; \textit{i.e.,} \;\; \lim_{n\to\infty} n^{\alpha/2} m(\beta_n, K_n) = \bar{x}(\ell).$$

Proof. Part (a) follows from the discussion leading up to the statement of the theorem. Parts (b), (b)(i), and (b)(ii) are consequences of Theorem A.1.

We now verify the asymptotic behavior of $m(\beta_n, K_n)$ in part (c). According to Theorem 4.1, $m(\beta_n, K_n) \to 0$. The validity of hypotheses (i) and (ii) of Theorem 4.2 follows from the definition of the sequence (β_n, K_n) and the discussion leading up to the statement of the present theorem. Hypothesis (iii) of Theorem 4.2 is parts (a) and (b) of the present theorem. We now verify hypothesis (iv), first in the case where $\ell \neq K''(\beta_c)$. Using (5.15) with $\gamma = \alpha/2$ and $u = 1 - 3\alpha$, one easily proves that for any $\alpha > 0$ there exists $R > 0$ such that for all sufficiently large $n \in \mathbb{N}$ and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$G_n(x) = n^{1-u} G_{\beta_n, K_n}(x/n^\gamma) \geq H(x) = -\beta_c |K''(\beta_c) - \ell| x^2 - 8c_4 x^4 + \tfrac{1}{2} c_6 x^6.$$

Since $H(x) \to \infty$ as $|x| \to \infty$, hypothesis (iv) of Theorem 4.2 is satisfied when $\ell \neq K''(\beta_c)$. When $\ell = K''(\beta_c)$, we obtain the lower bound in the last display by replacing H there by

$$H(x) = -\tfrac{1}{3}\beta_c |K'''(\beta_c) - \tilde{\ell}| x^2 - 8c_4 x^4 + \tfrac{1}{2} c_6 x^6,$$

where $K'''(\beta_c) = -1.024398 < 0$ [Lem. 6.1(d)]. This follows from the expansion replacing (5.15) when $\ell = K''(\beta_c)$; the proof is omitted. The verification of the four hypotheses of Theorem 4.2 is complete. We now apply the theorem to conclude that for any $\alpha > 0$, $m(\beta_n, K_n) \sim \bar{x}/n^\gamma = \bar{x}/n^{\alpha/2}$. Part (c) of the present theorem is proved. ■

We now consider the third choice of sequence (β_n, K_n) converging to the tricritical point, which gives yet a different asymptotic behavior of $m(\beta_n, K_n) \to 0$. Given $\alpha > 0$, an integer $p \geq 2$, and $\ell \in \mathbb{R}$, we define

$$\beta_n = \beta_c - 1/n^\alpha \;\text{ and }\; K_n = K(\beta_c) + \sum_{j=1}^{p-1} K^{(j)}(\beta_c)(-1)^j/(j! n^{j\alpha}) + \ell(-1)^p/(p! n^{p\alpha}). \quad (5.20)$$

In order to simplify the analysis here, we assume that $\ell \neq K^{(p)}(\beta_c)$. The choice $\ell = K^{(p)}(\beta_c)$ will be discussed in the third paragraph before Theorem 5.3 and after Theorem 5.4.

The sequence (β_n, K_n) defined in (5.20) converges to the tricritical point $(\beta_c, K(\beta_c))$. Since $\beta_n - \beta_c = -1/n^\alpha$, the convergence takes place along the curve

$$\tilde{K}_p(\beta) = K(\beta_c) + \sum_{j=1}^{p-1} K^{(j)}(\beta_c)(\beta - \beta_c)^j/j! + \ell(\beta - \beta_c)^p/p!.$$

This curve coincides with the second-order curve to order $p-1$ in powers of $\beta - \beta_c$ in a neighborhood of the tricritical point and satisfies $\tilde{K}_p^{(p)}(\beta_c) = \ell$ (see paths 5 and 6 in Figure 2). The relationship of the sequence (β_n, K_n) to the second-order curve depends on the parity of p. We first assume that p is even. For all sufficiently large n, $\ell > K^{(p)}(\beta_c)$ corresponds to (β_n, K_n) lying in the phase-coexistence region located above the second-order curve for $\beta < \beta_c$ and thus to the free energy functional G_{β_n,K_n} having its global minimum points at $\pm m(\beta_n, K_n) \neq 0$. On the other hand, for all sufficiently large n, $\ell < K^{(p)}(\beta_c)$ corresponds to (β_n, K_n) lying in the single-phase region under the second-order curve and thus to G_{β_n,K_n} having a unique global minimum point at 0. If p is odd, then the situation is reversed. As one can check, in all cases the structure of the set of global minimum points of the Ginzburg-Landau polynomial mirrors the structure of the set of global minimum points of G_{β_n,K_n}.

We now determine the relevant expansion of $nG_{\beta_n,K_n}(x/n^\gamma)$ in (5.4) where (β_n, K_n) is the sequence in (5.20). Since

$$K(\beta_n) = K(\beta_c + b/n^\alpha) = K(\beta_c) + \sum_{j=1}^{p} K^{(j)}(\beta_c)(-1)^j/(j!n^{j\alpha}) + \mathrm{O}(1/n^{(p+1)\alpha}),$$

we have

$$K(\beta_n) - K_n = (K^{(p)}(\beta_c) - \ell)(-1)^p/(p!n^{p\alpha}) + \mathrm{O}(1/n^{(p+1)\alpha}) \tag{5.21}$$

and

$$4 - e^{\beta_n} = 4(1 - e^{-1/n^\alpha}) = 4/n^\alpha + \mathrm{O}(1/n^{2\alpha}).$$

Substituting the last two expressions into (5.4), we see that for all $n \in \mathbb{N}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$\begin{aligned} &nG_{\beta_n,K_n}(x/n^\gamma) \\ &\quad = \frac{1}{n^{2\gamma+p\alpha-1}}\frac{1}{p!}\beta_c(K^{(p)}(\beta_c) - \ell)(-1)^p(1+\varepsilon_n)x^2 \\ &\qquad + \frac{1}{n^{4\gamma+\alpha-1}}4c_4(1+\varepsilon_n)x^4 + \frac{1}{n^{6\gamma-1}}c_6(1+\varepsilon_n)x^6 \\ &\qquad + \mathrm{O}\!\left(\frac{1}{n^{2\gamma+(p+1)\alpha-1}}\right)x^2 + \mathrm{O}\!\left(\frac{1}{n^{4\gamma+2\alpha-1}}\right)x^4 + \mathrm{O}\!\left(\frac{1}{n^{7\gamma-1}}\right)x^7, \end{aligned} \tag{5.22}$$

where $c_4 = 3/16$ and $c_6 = 9/40$. Given $u \in \mathbb{R}$, we multiply the numerator and denominator of the right side of the last display by n^u, obtaining $nG_{\beta_n,K_n}(x/n^\gamma) = n^u G_n(x)$, where for all $n \in \mathbb{N}$, any $\gamma > 0$, any $R > 0$, and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$\begin{aligned} G_n(x) & \qquad (5.23) \\ &= \frac{1}{n^{2\gamma+p\alpha-1+u}} \frac{1}{p!} \beta_c (K^{(p)}(\beta_c) - \ell)(-1)^p (1+\varepsilon_n) x^2 \\ &\quad + \frac{1}{n^{4\gamma+\alpha-1+u}} 4c_4 (1+\varepsilon_n) x^4 + \frac{1}{n^{6\gamma-1+u}} c_6 (1+\varepsilon_n) x^6 \\ &\quad + \mathrm{O}\left(\frac{1}{n^{2\gamma+(p+1)\alpha-1+u}}\right) x^2 + \mathrm{O}\left(\frac{1}{n^{4\gamma+2\alpha-1+u}}\right) x^4 + \mathrm{O}\left(\frac{1}{n^{7\gamma-1+u}}\right) x^7. \end{aligned}$$

In this formula $\varepsilon_n \to 0$ and the big-oh terms are uniform for $x \in (-Rn^\gamma, Rn^\gamma)$. Since $\ell \neq K^{(p)}(\beta_c)$, $b < 0$, $c_4 > 0$, and $c_6 > 0$, the coefficients of x^2, x^4, and x^6 in the first three terms are all nonzero.

We first consider $p = 2$, which gives rise to a different asymptotic behavior of $m(\beta_n, K_n) \to 0$ from $p \geq 3$. We continue to assume that $\ell \neq K''(\beta_c)$. In order to obtain the limit of G_n, we fix $\alpha > 0$ and impose the condition that the three powers of n appearing in the first three terms in (5.23) equal 0; i.e., $2\gamma + 2\alpha - 1 + u = 0 = 4\gamma + \alpha - 1 + u = 6\gamma - 1 + u$. These three equalities are equivalent to $\gamma = \alpha/2$ and $u = 1 - 6\gamma = 1 - 3\alpha$. With this choice of γ and u, the powers of n in the last three terms in (5.23) are positive, and so for all $x \in \mathbb{R}$ these three terms converge to 0 as $n \to \infty$. It follows that as $n \to \infty$, we have for all $x \in \mathbb{R}$

$$G_n(x) = n^{1-u} G_{\beta_n,K_n}(x/n^\gamma) \to g(x) = \tfrac{1}{2}\beta_c(K''(\beta_c) - \ell)x^2 + 4c_4 x^4 + c_6 x^6. \qquad (5.24)$$

Since the big-oh terms in (5.23) are uniform for $x \in (-Rn^\gamma, Rn^\gamma)$, the convergence of G_n to g is uniform for x in compact subsets of $\mathbb{R}$.

We now briefly consider the case where $\ell = K''(\beta_c)$. In this situation the right side of (5.21) must be replaced by $-K'''(\beta_c)/(6n^{3\alpha}) + \mathrm{O}(1/n^{4\alpha})$. We omit the rest of the calculation showing that when $\ell = K''(\beta_c)$, the limit of $G_n(x) = n^{1-u} G_{\beta_n,K_n}(x/n^\gamma)$ equals the same Ginzburg-Landau polynomial g in (5.24). When $\ell = K''(\beta_c)$, the coefficient of x^2 equals 0, and thus g has a unique global minimum point at 0, an uninteresting case from the viewpoint of the asymptotic behavior of $m(\beta_n, K_n)$.

A major difference between the present situation and that considered in Theorem 5.2 arises in the condition guaranteeing that the Ginzburg-Landau polynomial g in (5.24) has nonzero global minimum points. It follows from Theorem A.2 that since the coefficient of x^4 is positive, g has nonzero global minimum points if and only if the coefficient of x^2 is negative; i.e., if and only if $\ell > K''(\beta_c)$. If $\ell \leq K''(\beta_c)$, then g has a unique global minimum point at 0, again an uninteresting case from the viewpoint of the asymptotic behavior of $m(\beta_n, K_n)$.

For the sequence (β_n, K_n) defined in (5.20) with $p = 2$ and $\ell > K''(\beta_c)$, the next theorem describes the asymptotic behavior of $m(\beta_n, K_n)$. The inequality $\ell > K''(\beta_c)$ in parts (b) and (c) guarantees that for all sufficiently large n, (β_n, K_n) lies in the phase-coexistence region above the second-order curve.

Theorem 5.3. *For $\alpha > 0$ and $\ell \in \mathbb{R}$ define*

$$\beta_n = \beta_c - 1/n^\alpha \ \textit{ and } \ K_n = K(\beta_c) + K'(\beta_c)/n^\alpha + \ell/2n^{2\alpha}$$

as well as $c_4 = 3/16$ and $c_6 = 9/40$. Then the sequence (β_n, K_n) converges to the tricritical point $(\beta_c, K(\beta_c))$. The following conclusions hold.

(a) *For any $\alpha > 0$, $u = 1 - 3\alpha$. and $\gamma = \alpha/2$*

$$G_n(x) = n^{1-u} G_{\beta_n, K_n}(x/n^\gamma) \to g(x) = \tfrac{1}{2}\beta_c(K''(\beta_c) - \ell)x^2 + 4c_4 x^4 + c_6 x^6$$

uniformly for x in compact subsets of $\mathbb{R}$.

(b) *The Ginzburg-Landau polynomial g has nonzero global minimum points if and only if $\ell > K''(\beta_c)$. If this inequality holds, then the global minimum points of g are $\pm\bar{x}$, where*

$$\bar{x} = \tfrac{\sqrt{10}}{3}\left(-1 + \left(1 + \tfrac{3\beta_c}{5}(\ell - K''(\beta_c))\right)^{1/2}\right)^{1/2}. \tag{5.25}$$

(c) *Assume that $\ell > K''(\beta_c)$. Then for any $\alpha > 0$, $m(\beta_n, K_n) \to 0$ and has the asymptotic behavior*

$$m(\beta_n, K_n) \sim \bar{x}/n^{\alpha/2} = \bar{x}(\beta_c - \beta_n)^{1/2}; \ \textit{ i.e., } \lim_{n\to\infty} n^{\alpha/2} m(\beta_n, K_n) = \bar{x}.$$

Proof. Part (a) follows from the discussion leading up to the statement of the theorem. The first assertion in part (b) is a consequence of Theorem A.2. The formula for $\bar{x}$ comes from the formula for $\bar{x}(a_2)$ in (A.2) by substituting $a_2 = \frac{1}{2}\beta_c(K''(\beta_c) - \ell)$, $a_4 = 4c_4$, and $a_6 = c_6$.

We now verify the asymptotic behavior of $m(\beta_n, K_n)$ in part (c). According to Theorem 4.1, $m(\beta_n, K_n) \to 0$. The validity of hypotheses (i) and (ii) of Theorem 4.2 follows from the definition of the sequences (β_n, K_n) and the inequality $\ell > K''(\beta_c)$, which by (5.21) is equivalent to $K_n > K(\beta_n)$ for all sufficiently large n. Thus, if $\ell > K''(\beta_c)$, then (β_n, K_n) lies in the phase-coexistence region for all sufficiently large n. Hypothesis (iii) of Theorem 4.2 is parts (a) and (b) of the present theorem. We now verify hypothesis (iv) of Theorem 4.2. Using (5.23) with $\gamma = \alpha/2$ and $u = 1 - 3\alpha$, one easily proves that for any $\alpha > 0$ there exists $R > 0$ such that for all sufficiently large $n \in \mathbb{N}$ and all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$G_n(x) = n^{1-u} G_{\beta_n, K_n}(x/n^\gamma) \geq H(x) = -\beta_c|K''(\beta_c) - \ell|x^2 + \tfrac{1}{2}c_6 x^6.$$

Since $H(x) \to \infty$ as $|x| \to \infty$, hypothesis (iv) in Theorem 4.2 is satisfied. This completes the verification of the four hypotheses of Theorem 4.2. We now apply the theorem to conclude that for any $\alpha > 0$, $m(\beta_n, K_n) \sim \bar{x}/n^\gamma = \bar{x}/n^{\alpha/2}$. Part (c) of the present theorem is proved. ■

This theorem completes the analysis for $p = 2$. We now continue with the analysis for the sequences (β_n, K_n) defined in (5.20) for $p \geq 3$, $\alpha > 0$, and $\ell \neq K^{(p)}(\beta_c)$. As we saw in the discussion leading up to Theorem 5.3, for all $n \in \mathbb{N}$, any $\gamma > 0$, any $R > 0$, all $x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$, we have $nG_{\beta_n,K_n}(x/n^\gamma) = n^u G_n(x)$, where G_n is given by the expansion (5.23):

$$\begin{aligned} G_n(x) & \qquad\qquad (5.26) \\ &= \frac{1}{n^{2\gamma+p\alpha-1+u}} \tfrac{1}{p!}\beta_c(K^{(p)}(\beta_c) - \ell)(-1)^p(1+\varepsilon_n)x^2 \\ &\quad + \frac{1}{n^{4\gamma+\alpha-1+u}} 4c_4(1+\varepsilon_n)x^4 + \frac{1}{n^{6\gamma-1+u}} c_6(1+\varepsilon_n)x^6 \\ &\quad + \mathrm{O}\!\left(\frac{1}{n^{2\gamma+(p+1)\alpha-1+u}}\right)x^2 + \mathrm{O}\!\left(\frac{1}{n^{4\gamma+2\alpha-1+u}}\right)x^4 + \mathrm{O}\!\left(\frac{1}{n^{7\gamma-1+u}}\right)x^7. \end{aligned}$$

In this formula $\varepsilon_n \to 0$ and the big-oh terms are uniform for $x \in (-Rn^\gamma, Rn^\gamma)$.

The analysis for $p \geq 3$ is considerably more complicated than in the case $p = 2$ just considered. In order to obtain the limit of G_n, it is useful to denote by $f(\gamma, \alpha, u)$, $g(\gamma, \alpha, u)$, and $h(\gamma, \alpha, u)$ the respective exponents of n in the coefficients of x^2, x^4, and x^6 in the first three terms in the last display. Thus, $f(\gamma, \alpha, u) = 2\gamma + p\alpha - 1 + u$, $g(\gamma, \alpha, u) = 4\gamma + \alpha - 1 + u$, and $h(\gamma, \alpha, u) = 6\gamma - 1 + u$. In order to obtain a limiting Ginzburg-Landau polynomial g having nonzero global minimum points, g must have either three terms or two terms. The polynomial g has three terms if and only if there exist γ, α, and u for which the three equalities $f(\gamma, \alpha, u) = 0 = g(\gamma, \alpha, u) = h(\gamma, \alpha, u)$ are compatible. However, a short calculation shows that the three equalities are incompatible. The polynomial g has two terms if and only if there exist γ, α, and u for which at least one of the following sets of two equalities and one inequality are compatible:

1. $f(\gamma, \alpha, u) = 0 = g(\gamma, \alpha, u) < h(\gamma, \alpha, u)$,
2. $g(\gamma, \alpha, u) = 0 = h(\gamma, \alpha, u) < f(\gamma, \alpha, u)$,
3. $f(\gamma, \alpha, u) = 0 = h(\gamma, \alpha, u) < g(\gamma, \alpha, u)$.

Another short calculation shows the two equalities and the one inequality in item 3 are incompatible. By contrast, the two equalities and the one inequality in item 2 are compatible. In fact, $g(\gamma, \alpha, u) = 0 = h(\gamma, \alpha, u)$ when $\gamma = \alpha/2$ and $u = 1 - 6\gamma = 1 - 3\alpha$, and $g(\gamma, \alpha, u) < f(\gamma, \alpha, u)$ when $\gamma < (p-1)\alpha/2$; the latter inequality is compatible with $\gamma = \alpha/2$ since $p \geq 3$.

With this choice of γ and u, the powers of n in the first term and in the last three terms in (5.26) are positive, and so for all $x \in \mathbb{R}$ these four terms converge to 0 as $n \to \infty$. It follows that as $n \to \infty$, we have for all $x \in \mathbb{R}$

$$G_n(x) = n^{1-u} G_{\beta_n, K_n}(x/n^\gamma) \to g(x) = 4c_4 x^4 + c_6 x^6.$$

The polynomial g has a unique global minimum point at 0, an uninteresting case from the viewpoint of the asymptotic behavior of $m(\beta_n, K_n)$.

The final case to consider is the compatibility of the two equalities and the one inequality in item 1. In fact, $f(\gamma, \alpha, u) = 0 = g(\gamma, \alpha, u)$ when $\gamma = (p-1)\alpha/2$ and $u = 1 - p\alpha - 2\gamma = 1 - (2p-1)\alpha$. On the other hand, $g(\gamma, \alpha, u) < h(\gamma, \alpha, u)$ when $\gamma > \alpha/2$, which is compatible with $\gamma = (p-1)\alpha/2$ since $p \geq 3$. With this choice of γ and u, the powers of n in the last four terms in (5.26) are positive, and so for all $x \in \mathbb{R}$ these four terms converge to 0 as $n \to \infty$. It follows that as $n \to \infty$, we have for all $x \in \mathbb{R}$

$$G_n(x) = n^{1-u} G_{\beta_n, K_n}(x/n^\gamma) \to g(x) = \tfrac{1}{p!}\beta_c (K^{(p)}(\beta_c) - \ell)(-1)^p x^2 + 4c_4 x^4. \tag{5.27}$$

Since the big-oh terms in (5.26) are uniform for $x \in (-Rn^\gamma, Rn^\gamma)$, the convergence of $G_n(x)$ to $g(x)$ is uniform for x in compact subsets of $\mathbb{R}$. The Ginzburg-Landau polynomial g has nonzero global minimum points if and only if $(K^{(p)}(\beta_c) - \ell)(-1)^p < 0$.

For the sequence (β_n, K_n) defined in (5.20) with $p \geq 3$ a positive integer and $(K^{(p)}(\beta_c) - \ell)(-1)^p < 0$, the asymptotic behavior of $m(\beta_n, K_n)$ is given in the next theorem. This inequality involving ℓ and $K^{(p)}(\beta_c)$ guarantees that for all sufficiently large n, (β_n, K_n) lies in the phase-coexistence region above the second-order curve.

Theorem 5.4. *For p a positive integer satisfying $p \geq 3$, $\alpha > 0$, and a real number $\ell \neq K^{(p)}(\beta_c)$, define*

$$\beta_n = \beta_c - 1/n^\alpha \;\; \text{and} \;\; K_n = K(\beta_c) + \sum_{j=1}^{p-1} K^{(j)}(\beta_c)(-1)^j/(j! n^{j\alpha}) + \ell(-1)^p/(p! n^{p\alpha})$$

as well as $c_4 = 3/16$. Then (β_n, K_n) converges to the tricritical point $(\beta_c, K(\beta_c))$. The following conclusions hold.

(a) *For any $\alpha > 0$, $u = 1 - (2p-1)\alpha$, and $\gamma = (p-1)\alpha/2$*

$$G_n(x) = n^{1-u} G_{\beta_n, K_n}(x/n^\gamma) \to g(x) = \tfrac{1}{p!}\beta_c (K^{(p)}(\beta_c) - \ell)(-1)^p x^2 + 4c_4 x^4$$

uniformly for x in compact subsets of $\mathbb{R}$.

(b) *The Ginzburg-Landau polynomial has nonzero global minimum points if and only if* $(K^{(p)}(\beta_c) - \ell)(-1)^p < 0$. *If this inequality holds, then the global minimum points of* g *are* $\pm\bar{x}$, *where*

$$\bar{x} = \big(\beta_c\big(\ell - K^{(p)}(\beta_c)(-1)^p\big)/[8c_4 p!]\big)^{1/2}. \tag{5.28}$$

(c) *Assume that* $(K^{(p)}(\beta_c) - \ell)(-1)^p < 0$. *Then for any* $\alpha > 0$, $m(\beta_n, K_n) \to 0$ *and has the asymptotic behavior*

$$m(\beta_n, K_n) \sim \bar{x}/n^{(p-1)\alpha/2} = \bar{x}(\beta_n - \beta_c)^{(p-1)/2}; \;\; \textit{i.e.,} \; \lim_{n\to\infty} n^{(p-1)\alpha/2} m(\beta_n, K_n) = \bar{x}.$$

Proof. Part (a) follows from the discussion leading up to the statement of the theorem. The first assertion in part (b) is elementary. If $(K^{(p)}(\beta_c) - \ell)(-1)^p < 0$, then $g'(x) = 2\beta_c(K^{(p)}(\beta_c) - \ell)(-1)^p x/p! + 16c_4 x^3 = 0$ has solutions $\pm\bar{x}$ and 0, where $\bar{x}$ is defined in (5.28). One easily checks that $\pm\bar{x}$ are global minimum points and 0 a local maximum point.

We now verify the asymptotic behavior of $m(\beta_n, K_n)$ in part (c). According to Theorem 4.1, $m(\beta_n, K_n) \to 0$. The validity of hypotheses (i) and (ii) of Theorem 4.2 follows from the definition of the sequences (β_n, K_n) and the inequality $(K^{(p)}(\beta_c) - \ell)(-1)^p < 0$, which by (5.21) is equivalent to $K_n > K(\beta_n)$ for all sufficiently large n. Thus if $(K^{(p)}(\beta_c) - \ell)(-1)^p < 0$, then for all sufficiently large n, (β_n, K_n) lies in the phase-coexistence region above the second-order curve. Hypothesis (iii) of Theorem 4.2 is parts (a) and (b) of the present theorem. The verification of hypothesis (iv) of Theorem 4.2 is more subtle than in the previous theorems. The limiting Ginzburg-Landau polynomial is of degree 4, not of degree 6, because for all $x \in \mathbb{R}$ the x^6-term in (5.26) converges to 0 as $n \to \infty$. It is much more efficient to recalculate this limit by using the formula (3.2) based on the two-term Taylor expansion for $nG_{\beta_n,K_n}(x/n^\gamma)$, rather than the formula (5.1) based on the three-term Taylor expansion for $nG_{\beta_n,K_n}(x/n^\gamma)$. Inserting the expressions for $K(\beta_n) - K_n$ and for $4 - e^{\beta_n}$, we obtain in place of (5.26) the expansion

$$\begin{aligned} &G_n(x) \\ &\quad = \frac{1}{n^{2\gamma+p\alpha-1+u}} \frac{1}{p!} \beta_c(K^{(p)}(\beta_c) - \ell)(-1)^p(1+\varepsilon_n)x^2 \\ &\qquad + \frac{1}{n^{4\gamma+\alpha-1+u}} 4c_4(1+\varepsilon_n)x^4 + \mathrm{O}\bigg(\frac{1}{n^{2\gamma+(p+1)\alpha-1+u}}\bigg)x^2 \\ &\qquad + \mathrm{O}\bigg(\frac{1}{n^{4\gamma+2\alpha-1+u}}\bigg)x^4 + \mathrm{O}\bigg(\frac{1}{n^{5\gamma-1+u}}\bigg)x^5. \end{aligned}$$

Given $\alpha > 0$ and choosing $\gamma = (p-1)\alpha/2$ and $u = 1 - (2p-1)\alpha$, for all $x \in \mathbb{R}$ we obtain the same $n \to \infty$ limit as in (5.27). Using the last display with these values of γ and u, one easily proves that for any $\alpha > 0$ there exists $R > 0$ such that for all sufficiently large $n \in \mathbb{N}$ and all

$x \in \mathbb{R}$ satisfying $|x/n^\gamma| < R$

$$G_n(x) = n^{1-u} G_{\beta_n, K_n}(x/n^\gamma) \geq H(x) = -2\tfrac{1}{p!}\beta_c |K^{(p)}(\beta_c) - \ell| x^2 + 2c_4 x^4.$$

Since $H(x) \to \infty$ as $|x| \to \infty$, hypothesis (iv) of Theorem 4.2 is satisfied. This completes the verification of the four hypotheses of that theorem. We now apply the theorem to conclude that for any $\alpha > 0$, $m(\beta_n, K_n) \sim \bar{x}/n^\gamma = \bar{x}/n^{(p-1)\alpha/2}$. Part (c) of the present theorem is proved. ■

In order to derive the asymptotic behavior of $m(\beta_n, K_n) \to 0$ for the sequence (β_n, K_n) in the last theorem, we choose $\ell \neq K^{(p)}(\beta_c)$. The choice $\ell = K^{(p)}(\beta_c)$ corresponds to the sequence (β_n, K_n) lying on a curve that coincides with the second-order curve to order p in powers of $\beta - \beta_c$. In order to analyze this case, we must know the sign of $K^{(p+1)}(\beta_c)$. Because we are unable to determine this sign analytically, the discussion of this case is omitted.

This completes the analysis of the asymptotic behavior of $m(\beta_n, K_n) \to 0$ for the four sequences considered in Theorems 5.1–5.4. In the next section we combine several results derived in the present section with other calculations to conjecture a number of properties of the first-order curve.

6 Properties of the First-Order Curve

The starting point of the present section is our analysis in section 5 of the asymptotics of the magnetization $m(\beta_n, K_n) \to 0$ for sequences (β_n, K_n) converging to the tricritical point $(\beta_c, K(\beta_c))$. As in section 3, the structure of the sets of global minimum points of the associated Ginzburg-Landau polynomials mirrors features of the phase transitions of the model. In the present section we operate differently. We shall use the structure of the sets of global minimum points of the Ginzburg-Landau polynomials not to mirror, but to determine features of the phase transitions in the subsets of the phase-coexistence region through which (β_n, K_n) passes. These features focus on properties of the first-order curve, properties that are notoriously difficult to derive rigorously. Although the insights into the properties of this curve given by the Ginzburg-Landau polynomials are not rigorous, we back them up with convincing numerical evidence.

The properties of the first-order curve to be presented in this section are stated in the form of three conjectures involving the first three right-hand derivatives of $K_1(\beta)$ at β_c. These three conjectures are used in the proof of part (c) of Theorem 5.2 to verify the asymptotic behavior of $m(\beta_n, K_n) \to 0$ given there.

Before using properties of the Ginzburg-Landau polynomials to determine properties of the first-order curve, in the next theorem we record properties of the function $K(\beta) = (e^\beta + 2)/4\beta$. For $0 < \beta < \beta_c$, $K(\beta)$ defines the second-order curve, while for $\beta \geq \beta_c$, $K(\beta)$ defines the spinodal curve. Parts (e) and (f) express relationships between $K(\beta)$ and $K_1(\beta)$, which for $\beta > \beta_c$ defines the first-order curve. $K_1(\beta)$ has the properties given in Theorem 2.3.

Lemma 6.1. *The function $K(\beta) = (e^\beta + 2)/4\beta$ has the following properties.*

(a) *For $\beta > 0$, $K'(\beta) = ((\beta - 1)e^\beta - 2)/(4\beta^2)$. There exists $\beta_1 > \beta_c = \log 4$ such that $K'(\beta) < 0$ for $0 < \beta < \beta_1$.*

(b) $4K(\beta) - e^\beta = -4\beta K'(\beta)$.

(c) *For $\beta > 0$, $K(\beta) > 0$ and $K''(\beta) = ((\beta^2 - 2\beta + 2)e^\beta + 4)/(4\beta^3) > 0$. Thus $K(\beta)$ is a positive, convex function of $\beta > 0$.*

(d) $K'(\beta_c) = (2\beta_c - 3)/(2\beta_c^2) = -0.059166$, $K''(\beta_c) = (\beta_c^2 - 2\beta_c + 3)/\beta_c^3 = 0.806706$, *and* $K'''(\beta_c) = (\beta_c^3 - 3\beta_c^2 + 6\beta_c - 9)/\beta_c^4 = -1.024398$.

(e) *For $\beta > \beta_c$, $K(\beta) > K_1(\beta)$.*

(f) $\lim_{\beta \to \beta_c^+} K_1(\beta) = K(\beta_c)$.

Proof. (a) For $\beta > 0$ we calculate $K'(\beta) = ((\beta - 1)e^\beta - 2)/(4\beta^2)$. For $0 < \beta < \beta_c$ the numerator satisfies

$$(\beta - 1)e^\beta - 2 < (\beta_c - 1)e^{\beta_c} - 2 = 4(\log 4 - 1) - 2 = -0.454823 < 0.$$

By continuity it follows that there exists $\beta_1 > \beta_c = \log 4$ such that $K'(\beta) < 0$ for $0 < \beta < \beta_1$.

(b) This follows from the formula for $K'(\beta)$ given in part (a).

(c) Clearly $K(\beta) > 0$ for $\beta > 0$. For $\beta > 0$, we calculate

$$K''(\beta) = ((\beta^2 - 2\beta + 2)e^\beta + 4)/(4\beta^3).$$

This is positive for all $\beta > 0$ since $\beta^2 - 2\beta + 2 > 0$. It follows that $K(\beta)$ is a positive, convex function of $\beta > 0$.

(d) Since $e^{\beta_c} = 4$, the formulas for $K'(\beta_c)$, $K''(\beta_c)$, and $K'''(\beta_c)$ follow from parts (a) and (c).

(e) This is proved in Theorem 3.8 in [15].

(f) We omit the proof, which is based on a number of technical results in sections 3.1 and 3.3 of [15]. ■

The first-order curve is defined by the function $K_1(\beta)$ for $\beta > \beta_c$. We use part (f) of the last lemma to extend the definition of this function to $\beta = \beta_c$ by defining $K_1(\beta_c) = K(\beta_c)$. We next state three properties of the first-order curve in the form of three conjectures. They involve the first three right-hand derivatives of $K_1(\beta)$ at β_c. In doing so, we assume that these derivatives exist and denote them by $K_1'(\beta_c)$, $K_1''(\beta_c)$, and $K_1'''(\beta_c)$. In combination with the definition $K_1(\beta_c) = K(\beta_c)$, the first two conjectures are consistent with the fact that $K_1(\beta) < K(\beta)$ for all $\beta > \beta_c$ [Lem. 6.1(e)]. Our main goal in this section is to use properties of the appropriate Ginzburg-Landau polynomials plus numerical evidence to support these conjectures.

Conjecture 1. The right-hand derivative $K_1'(\beta_c)$ exists, and $K_1'(\beta_c) = K'(\beta_c)$; i.e., at β_c the first-order curve and the spinodal curve have the same right-hand tangent. Numerically

$$K_1'(\beta_c) = K'(\beta_c) = 1/\beta_c - 3/(2\beta_c^2) = -0.059166.$$

Conjecture 2. The right-hand derivative $K_1''(\beta_c)$ exists, and $K_1''(\beta_c) < 0 < K''(\beta_c)$; numerically,

$$\begin{aligned} K_1''(\beta_c) &= K''(\beta_c) - 5/(4\beta_c) = -1/(4\beta_c) - 2/\beta_c^2 + 3/\beta_c^3 = -0.094979 \\ &< K''(\beta_c) = 1/\beta_c - 2/\beta_c^2 + 3/\beta_c^3 = 0.806706. \end{aligned}$$

Conjecture 3. The right-hand derivative $K_1'''(\beta_c)$ exists, and $K_1'''(\beta_c) > 0$; numerically,

$$K_1'''(\beta_c) = 219/(224\beta_c) + 3/(4\beta_c^2) + 6/\beta_c^3 - 9/\beta_c^4 = 0.910784.$$

Support for Conjecture 1 comes from the form of the Ginzburg-Landau polynomial when the sequence (β_n, K_n) in Theorem 5.1 satisfies $\beta_n > \beta_c$ and converges to the tricritical point $(\beta_c, K(\beta_c))$ along a ray lying under the tangent line to the spinodal curve $\{(\beta, K(\beta)), \beta > \beta_c\}$ at the tricritical point. Given $\alpha > 0$ and $k \in \mathbb{R}$, the sequence has the form

$$\beta_n = \beta_c + 1/n^\alpha \text{ and } K_n = K(\beta_c) + k/n^\alpha.$$

The requirement that the sequence converges along a ray lying under the tangent line means that $k < K'(\beta_c)$. In this case the Ginzburg-Landau polynomial is given by $g(x) = c_6 x^6 + \beta_c(K'(\beta_c) - k)x^2$, where $c_6 = 9/40$. This has the same form as the Ginzburg-Landau polynomial in part (b) of Theorem 5.1 except that there $K'(\beta_c) - k < 0$ and here $K'(\beta_c) - k > 0$. With this choice of sign the Ginzburg-Landau polynomial has a unique global minimum point at 0, implying that the sequence (β_n, K_n) approaches the tricritical point from the single-phase region. However, as Theorem 2.3 indicates, when $\beta > \beta_c$, the single-phase region is located under the first-order curve $(\beta, K_1(\beta))$. It follows that at the tricritical point the right-hand tangent line to the first-order curve coincides with or lies above the right-hand tangent line to the spinodal curve; i.e., $K_1'(\beta_c) \geq K'(\beta_c)$. On the other hand, parts (e) and (f) of Lemma 6.1 imply that for any $\beta > \beta_c$

$$\frac{K_1(\beta) - K_1(\beta_c)}{\beta - \beta_c} < \frac{K(\beta) - K(\beta_c)}{\beta - \beta_c}$$

and thus that $K_1'(\beta_c) \leq K'(\beta_c)$. We conclude that $K_1'(\beta_c) = K'(\beta_c)$, which is the content of Conjecture 1.

By a more detailed argument that involves the uniform convergence of the scaled free-energy functionals to the Ginzburg-Landau polynomial, we can prove Conjecture 1 under the assumption that the right-hand derivative $K_1'(\beta_c)$ exists. Again, because of parts (e) and (f) of Lemma 6.1 it suffices to prove that $K_1'(\beta_c) \geq K'(\beta_c)$. We carry this out by showing that $K_1'(\beta_c) < K'(\beta_c)$ leads to a contradiction. If this strict inequality holds, then we consider the same sequence $(\beta_n, K_n) \to (\beta_c, K(\beta_c))$ in Theorem 5.1 that we considered in the preceding paragraph, choosing k to satisfy $K_1'(\beta_c) < k < K'(\beta_c)$. According to Theorem 5.1, for any $\alpha > 0$, $\gamma = \alpha/4$, and $u = 1 - 3\alpha/2$, $G_n(x) = n^{1-u} G_{\beta_n, K_n}(x/n^\gamma)$ converges uniformly to $g(x)$ uniformly for x in compact subsets of $\mathbb{R}$. Since $k > K_1'(\beta_c)$, for all sufficiently large n, (β_n, K_n) lies in the phase-coexistence region located above the first-order curve. Hence the global minimum points of G_{β_n, K_n} are the symmetric pair $\pm m(\beta_n, K_n)$ [Thm. 2.3(c)]. It follows that the global minimum points of G_n are the symmetric pair $\pm \bar{m}_n = \pm n^\gamma m(\beta_n, K_n)$ [see (4.1)], which converge to $\pm \bar{x}$ as $n \to \infty$ [Thm. 5.1(b)].

In order to complete the proof, we appeal to several standard results in the theory of analytic functions. Since $k < K'(\beta_c)$, as a function of $z \in \mathbb{C}$ the derivative $g'(z) = 6c_6 z^5 + 2\beta_c(K'(\beta_c) - k)z$ of the Ginzburg-Landau polynomial has 1 real zero at $z = 0$ and 4 nonreal zeroes at the 4 fourth roots of $\beta_c(k - K'(\beta_c))/3c_6$. There exists an open set V in the complex plane having the following properties: the boundary of V is a smooth, simple, closed curve; V contains the set $\{z \in \mathbb{C} : \Re(z) \in [-2\bar{x}, 2\bar{x}], \Im(z) = 0\}$, in which the real zero of g' at $z = 0$ lies, but V does not contain the 4 nonreal zeroes of g'; G_n and g are analytic on V; as $n \to \infty$, $G_n(z) \to g(z)$ uniformly for $z \in V$. It follows that as $n \to \infty$, $G_n'(z) \to g'(z)$ uniformly for z in any closed disk contained in V [18, Thm. 3.1.8(i)]. Furthermore, by a corollary of Rouché's Theorem [18, p. 389], for all sufficiently large n, G_n' has the same number of zeroes in V as g', namely 1. However, this contradicts the fact that for all sufficiently large n, G_n' has two zeroes in V at $\pm \bar{m}_n$. The contradiction shows that the inequality $K_1'(\beta_c) < K'(\beta_c)$ is not valid and thus that $K_1'(\beta_c) \geq K'(\beta_c)$, which is what we want to show. This completes the proof of Conjecture 1.

Support for Conjecture 2 comes from Theorem 5.2. Given $\alpha > 0$, $\ell \in \mathbb{R}$, and $\tilde{\ell} \in \mathbb{R}$, in that theorem we consider the sequence

$$\beta_n = \beta_c + 1/n^\alpha \ \text{ and } \ K_n = K(\beta_c) + K'(\beta_c)/n^\alpha + \ell/(2n^{2\alpha}) + \tilde{\ell}/(6n^{3\alpha}).$$

This sequence converges to the tricritical point along the curve $(\beta, \tilde{K}(\beta))$, where

$$\tilde{K}(\beta) = K(\beta_c) + K'(\beta_c)(\beta - \beta_c) + \ell(\beta - \beta_c)^2/2 + \tilde{\ell}(\beta - \beta_c)^3/6.$$

For $\beta > \beta_c$, for points $(\beta, K_1(\beta))$ on the first-order curve the free-energy functional $G_{\beta, K_1(\beta)}$ has three global minimum points at 0 and $\pm m(\beta, K_1(\beta))$ [Thm. 2.3(b)]. In addition, as we determine in part (c) of Theorem 5.2, when $\ell = \ell_c = (-\beta_c^2 - 8\beta_c + 12)/4\beta_c^3$ the limiting Ginzburg-Landau polynomial g_ℓ has three global minimum points at 0 and $\pm(5/3)^{1/2}$. This analogous structure of global minimum points both for $G_{\beta,K}$ when $K = K_1(\beta)$ and for g_ℓ when $\ell = \ell_c$ suggests the following conclusion: when $\ell = \ell_c$, the curve $(\beta, \tilde{K}(\beta))$ along which the given sequence (β_n, K_n) converges to the tricritical point coincides with the first-order curve $(\beta, K_1(\beta))$ to order 2 in powers of $\beta - \beta_c$. If this is true, then it follows that when $\ell = \ell_c$,

$$K_1''(\beta_c) = \tilde{K}''(\beta_c) = \ell_c.$$

Since $\ell_c = K''(\beta_c) - 5/4\beta_c$ [see (5.18)], Conjecture 2 follows.

We complete this section by citing numerical evidence that supports all three conjectures. Let c_β be the function defined in (2.4). For $\beta > \beta_c$ the function $K_1(\beta)$ defining the first-order curve is determined by the property that the global minimum points of $G_{\beta,K_1(\beta)}(x) = \beta K_1(\beta)x^2 - c_\beta(2\beta K_1(\beta)x)$ are 0 and $\pm m$ for $m = m(\beta, K_1(\beta)) > 0$ [Thm. 2.3(b)]. This holds if and only if

$$G_{\beta,K_1(\beta)}(m) = 0 \ \text{ and } \ G'_{\beta,K_1(\beta)}(m) = 0. \tag{6.1}$$

Approximating $c_\beta(2\beta K_1(\beta)x)$ by its Taylor expansion to order 6, we solve the equations in the last display, obtaining an approximation $\bar{K}_1(\beta)$ to $K_1(\beta)$ having the following form:

$$\bar{K}_1(\beta) = \frac{2(2+e^\beta)(64 - 26e^\beta + e^{2\beta})}{3\beta(144 - 56e^\beta + e^{2\beta})}. \tag{6.2}$$

When $\beta = \beta_c = \log 4$, we have $\bar{K}_1(\beta_c) = 3/2\beta_c = K(\beta_c)$. This is consistent with the definition of $K_1(\beta_c) = K(\beta_c)$ given before Conjecture 1. The formula for $\bar{K}_1(\beta)$ is also consistent with the values of $K_1'(\beta_c)$ and $K_1''(\beta_c)$ given in Conjectures 1 and 2. In fact, we calculate

$$\bar{K}_1'(\beta_c) = 1/\beta_c - 3/(2\beta_c)^2 = K'(\beta_c) \ \text{ and } \ \bar{K}_1''(\beta_c) = -1/(4\beta_c) - 2/\beta_c^2 + 3/\beta_c^3.$$

However, the evidence cited in the next paragraph suggests that $\bar{K}_1'''(\beta_c)$ is not the correct value of $K_1'''(\beta_c)$.

In order to calculate numerically the value of $K_1'''(\beta_c)$, we approximate $c_\beta(2\beta K_1(\beta)x)$ by its Taylor expansion to order 8 and solve equations (6.1), obtaining an approximation $\hat{K}_1(\beta)$ to $K_1(\beta)$ that is too complicated to display here. $\hat{K}_1'''(\beta_c)$ is the value given in Conjecture 3. Like $\bar{K}_1$ in (6.2), $\hat{K}_1(\beta_c) = K(\beta_c)$, $\hat{K}_1'(\beta_c) = K'(\beta_c)$, and $\hat{K}_1''(\beta_c) = K''(\beta_c)$; however, $\hat{K}_1'''(\beta_c) = 0.910784 < 4.53362 = \bar{K}_1'''(\beta_c)$. If we approximate $c_\beta(2\beta K_1(\beta)x)$ by its Taylor expansion to order 10 and solve equations (6.1), then the resulting approximation to $K_1(\beta)$ has the same value at β_c and the same first three right-hand derivatives at β_c as $\hat{K}_1(\beta)$. It is reasonable to assume that the same properties hold for any approximation to $K_1(\beta)$ that arises by replacing $c_\beta(2\beta K_1(\beta)x)$ by its Taylor expansion to order 12 or higher.

This completes our discussion of properties of the first-order curve that are consistent with properties of the Ginzburg-Landau polynomials plus numerical evidence. In the next section we relate the results obtained in this paper to the scaling theory of critical phenomena.

7 Relationship with Scaling Theory of Critical Phenomena

The results on the asymptotic behavior of $m(\beta_n, K_n)$ obtained in sections 3 and 5 are related to scaling theory for critical and tricritical points [22, 25]. In this section we review scaling theory and show that its predictions for the magnetization are consistent with the results obtained in those sections.

Scaling theory is based on the idea that the singular parts of thermodynamic functions near continuous phase transitions are homogeneous functions of the distance to the phase transition. If there is a single parameter controlling the approach to the phase transition, then the content of scaling theory for a single thermodynamic quantity is simply that its singularities are power laws. If there is more than one parameter, as is the case here, then scaling theory has a richer content, especially near the tricritical point where the type of phase transition changes in a small neighborhood.

We are interested in the magnetization m as a function of (β_n, K_n), a sequence converging either to a second-order point $(\beta, K(\beta))$ with $0 < \beta < \beta_c$ or to the tricritical point $(\beta, K(\beta)) = (\beta_c, K(\beta_c))$. In either case, the relevant parameter-space is two dimensional. Given any phase-transition point $(\beta, K(\beta))$ with $0 < \beta \leq \beta_c$, the natural coordinate system for scaling theory is a curvilinear system (μ_1, μ_2) measuring the signed distances from the phase transition point; μ_1 is the signed distance from the curve of phase transitions and μ_2 the signed distance from the chosen point along the curve of phase transitions. Since we are concerned with the phase-coexistence region, in all our considerations $\mu_1 \geq 0$; however, μ_2 may take either sign. At the tricritical point, $\mu_2 > 0$ and $\mu_1 = 0$ correspond to the first-order line to the right of the tricritical point while $\mu_2 < 0$ and $\mu_1 = 0$ correspond to the second order line to the left of the tricritical point. At a second-order point, for sufficiently small $|\mu_2|$, $(0, \mu_2)$ is also a second-order point. Figure 4 shows this coordinate system for the special case of the tricritical point.

Scaling theory for the magnetization in a two-dimensional parameter space takes the general form

$$m(\mu_1 \tau, \mu_2 \tau^a) = \tau^b m(\mu_1, \mu_2), \tag{7.1}$$

where τ is an arbitrary scale factor and a and b are exponents to be determined [22]. The exponents a and b are chosen so that the theory is consistent with known exponents for the particular type of phase transition. In our case, a and b depend on whether the phase transition point is a second-order point or the tricritical point.

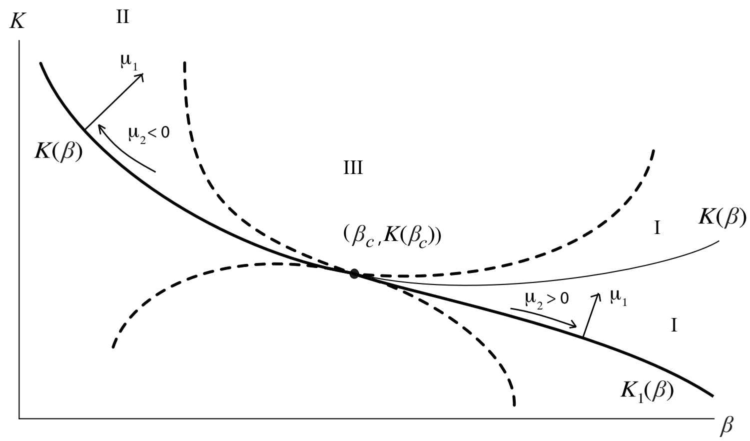

Figure 8: Curvilinear coordinate system for scaling theory showing the coordinates μ_1 and μ_2; μ_1 is the signed distance from the phase transition line and μ_2 the signed distance from the tricritical point along the phase transition line. Regions I, II, and III are dominated, respectively, by the first-order, second-order, and tricritical phase transition. A similar coordinate system can be defined for any point along the second-order curve.

We first consider the simpler case of a second-order point. Then the neighboring points along the phase-transition curve are also second-order points, and there is no singular dependence on μ_2, implying that $a = 0$. The singular behavior of the magnetization is controlled by $\tilde{\beta}$, the mean-field magnetization exponent for second-order transitions, which has the value $\tilde{\beta} = 1/2$ [25]. Choosing $b = \tilde{\beta} = 1/2$, we obtain from (7.1)

$$m(\mu_1\tau, \mu_2) = \tau^{\tilde{\beta}} m(\mu_1, \mu_2) = \tau^{1/2} m(\mu_1, \mu_2). \tag{7.2}$$

Setting $\tau = 1/\mu_1$ yields

$$m(\mu_1, \mu_2) = \mu_1^{\tilde{\beta}} m(1, \mu_2) = \mu_1^{1/2} f(\mu_2); \tag{7.3}$$

$f(\mu_2)$ is a smooth function of μ_2 that depends on the chosen point $(\beta, K(\beta))$, and the critical amplitude $f(0)$ is presumed to be positive. Equation (7.3) reflects the standard power-law behavior of the magnetization near a critical point.

We now show that (7.3) is consistent with Theorems 3.1 and 3.2. These theorems give the exact asymptotic behavior of $m(\beta_n, K_n)$ for sequences (β_n, K_n) converging to a second-order point. For ease of exposition, we refer to the definitions of the sequences according to the labeling in Figure 2 in the introduction, calling them sequences of type 1 and type 2, respectively.

We first consider the sequence of type 1, which converges to a second-order point $(\beta_0, K(\beta_0))$ along a ray that is above the tangent line to the second-order curve at that point. Defined in (3.5), this sequence takes the form

$$\beta_n = \beta_0 + b/n^\alpha \ \text{ and } \ K_n = K(\beta_0) + k/n^\alpha, \tag{7.4}$$

where $b \in \{1, 0, -1\}$ and $K'(\beta_0)b - k < 0$. To leading order, the coordinate μ_1 is given by the distance to the tangent to the second-order curve at $(\beta_0, K(\beta_0))$; i.e.,

$$\mu_1 \approx (K - K(\beta_0)) - K'(\beta_0)(\beta - \beta_0). \tag{7.5}$$

Hence we obtain

$$\mu_1 \approx (k - K'(\beta_0)b)/n^\alpha. \tag{7.6}$$

The distance μ_2 is also of order $1/n^\alpha$. However, f is a smooth function of μ_2 that converges to $f(0) > 0$ as $\mu_2 \to 0$. Hence we need only know that $\mu_2 \to 0$ in order to obtain the leading-order behavior of m from (7.3) and (7.6), namely, $m \approx (k - K'(\beta_0)b)^{1/2}/n^{\alpha/2}$. This asymptotic formula is consistent with the exact asymptotic behavior of $m(\beta_n, K_n)$ given in Theorem 3.1, correctly predicting both the exponent of n and the dependence on k and b in the prefactor $\bar{x}$ as given in (3.10) with $\beta = \beta_0$.

We next consider the sequence of type 2, which converges to a second-order point $(\beta_0, K(\beta_0))$ along a curve lying in the phase-coexistence region and having the same tangent as the second-order curve at that point. Defined in (3.12), this sequence takes the form

$$\beta_n = \beta_0 + b/n^\alpha \ \text{ and } \ K_n = K(\beta_0) + \sum_{j=1}^{p-1} K^{(j)}(\beta_0)b^j/(j!n^{j\alpha}) + \ell b^p/(p!n^{p\alpha}), \tag{7.7}$$

where $b \in \{1, -1\}$, $p \geq 2$, and $(K^{(p)}(\beta_0) - \ell)b^p < 0$. In this case it is crucial to recall that the scaling variables comprise a curvilinear coordinate system. In particular, the coordinate μ_1 measures the distance from the second-order curve, not the distance from the tangent to this curve at $(\beta_0, K(\beta_0))$; as a result (7.5) is not sufficient to determine the asymptotic behavior of m. The sequence of type 2 converges to the second-order point along a curve that agrees with the second-order curve to order $p - 1$ in powers of $\beta - \beta_0$. Hence to leading order μ_1 is proportional to the difference between the last term in the definition of K_n and the term of order p in the Taylor expansion of $K(\beta_n)$, namely, $\mu_1 \approx |\ell - K^{(p)}(\beta_0)|/(p!n^{p\alpha})$. Substituting this expression into (7.3) yields

$$m \approx (|\ell - K^{(p)}(\beta_0)|/p!n^{p\alpha})^{\tilde{\beta}} = (|\ell - K^{(p)}(\beta_0)|/p!)^{1/2}/n^{p\alpha/2}.$$

Again, this asymptotic formula is consistent with the exact asymptotic behavior of $m(\beta_n, K_n)$ given in Theorem 3.2 and correctly captures the square-root dependence of the prefactor $\bar{x}$ on $|\ell - K^{(p)}(\beta_0)|$ given in (3.17) with $\beta = \beta_0$.

If there is more than one type of phase transition in a neighborhood of a phase-transition point, as is the case near a tricritical point, then scaling theory becomes more complicated [22]. In the case of the tricritical point, the theory involves crossovers between the nearby first-order, second-order, and tricritical phase transitions. Figure 4 shows the region near the tricritical point.

The three regions I, II, and III separated by dotted lines are controlled by the first-order, the second-order, and the tricritical phase transitions, respectively. The mean-field tricritical crossover exponent φ_t determines the boundaries of the regions. In regions I and II we have $|\mu_1| \ll |\mu_2|^{1/\varphi_t}$ while in region III $|\mu_1| \gg |\mu_2|^{1/\varphi_t}$. In region II the magnetization m is controlled by $\tilde{\beta}$, the mean-field magnetization exponent for second-order transitions. In region III the magnetization m is controlled by $\tilde{\beta}_t$, the mean-field magnetization exponent for tricritical transitions, while in region I the magnetization m approaches a constant value as the first-order line is approached. These insights are incorporated in the scaling hypothesis

$$m(\mu_1\tau, \mu_2\tau^{\varphi_t}) = \tau^{\tilde{\beta}_t} m(\mu_1, \mu_2), \tag{7.8}$$

where τ is an arbitrary scale factor [22]. This corresponds to (7.1) with $a = \varphi_t$ and $b = \tilde{\beta}_t$. Setting $\tau = |\mu_2|^{-1/\varphi_t}$ yields the alternate form

$$m(\mu_1, \mu_2) = |\mu_2|^{\tilde{\beta}_t/\varphi_t} m(\mu_1/|\mu_2|^{1/\varphi_t}, 1) = |\mu_2|^{\tilde{\beta}_t/\varphi_t} f_{\pm}(\mu_1/|\mu_2|^{1/\varphi_t}), \tag{7.9}$$

where f_+ is used on the first-order side of the tricritical point ($\mu_2 > 0$) and f_- is used on the second-order side of the tricritical point ($\mu_2 < 0$). The values of the three relevant mean-field exponents are $\varphi_t = 1/2$, $\tilde{\beta} = 1/2$, and $\tilde{\beta}_t = 1/4$ [23].

We now consider the form taken by the right side of (7.9) in each of the three regions. In region III the arguments of f_+ and of f_- are large. Hence in order to recover the tricritical power-law behavior of m we require that $f_+(x) \approx x^{\tilde{\beta}_t}$ and $f_-(x) \approx x^{\tilde{\beta}_t}$ as $x \to \infty$, yielding

$$m(\mu_1, \mu_2) \approx \mu_1^{\tilde{\beta}_t} = \mu_1^{1/4} \text{ [region III]}. \tag{7.10}$$

In region II with fixed μ_2 we expect that the scaling is the one given in (7.2) for the second-order curve; i.e.,

$$m(\mu_1\tau, \mu_2) = \tau^{\tilde{\beta}} m(\mu_1, \mu_2) = \tau^{1/2} m(\mu_1, \mu_2). \tag{7.11}$$

The requirement that the two scaling assumptions (7.9) and (7.11) are consistent yields an interesting result for the behavior of m in region II for small $|\mu_2|$. The asymptotic behavior of

$f_-(x)$ as $x \to 0^+$ must be of the form $x^{\tilde{\beta}}$ in order that second-order scaling is recovered. Thus in region II we find

$$m(\mu_1, \mu_2) \approx \mu_1^{\tilde{\beta}} |\mu_2|^{(\tilde{\beta}_t - \tilde{\beta})/\varphi_t} = \mu_1^{1/2} |\mu_2|^{-1/2} \ \text{[region II]}. \tag{7.12}$$

Near the first-order curve in region I, for small positive μ_2 a similar result can be obtained except that $m(\mu_1, \mu_2)$ must converge to a constant as $\mu_1 \to 0^+$. For (7.9) to be consistent with first-order behavior, $f_+(x)$ must also converge to a constant as $x \to 0^+$. Hence along the first-order curve, which is defined by $\mu_1 = 0$ and $\mu_2 > 0$, we have

$$m(0, \mu_2) \approx \mu_2^{\tilde{\beta}_t/\varphi_t} = \mu_2^{1/2} \ \text{[region I]}. \tag{7.13}$$

We now show that these results in tricritical scaling theory are consistent with Theorems 5.1–5.4.

We first consider the sequence of type 3, which converges to the tricritical point along a ray that is above the tangent line to the phase-transition curve at the tricritical point. This sequence is defined as in (7.4) with β_0 replaced by β_c, $b \in \{1, 0, -1\}$, and $K'(\beta_c)b - k < 0$. For this sequence (7.6) holds with β_0 replaced by β_c. Thus $\mu_1 \approx (k - K'(\beta_c)b)/n^\alpha$, and μ_2 is of order $1/n^\alpha$. Since this sequence lies in region III, the asymptotic formula (7.10) predicts $m \approx \mu_1^{\tilde{\beta}_t} \approx (k - K'(\beta_c)b)^{1/4}/n^{\alpha/4}$. This asymptotic formula is consistent with the exact asymptotic behavior of $m(\beta_n, K_n)$ given in Theorem 5.1 and correctly predicts the 1/4-power dependence on $(k - K'(\beta_c)b)$ given in (5.10).

The sequence of type 4 is defined in (5.11) in terms of real parameters ℓ and $\tilde{\ell}$. For $\ell > \ell_c = K''(\beta_c) - 5/(4\beta_c)$ and appropriate choices of $\tilde{\ell}$, the sequences of type 4a, 4b, and 4d converge to the tricritical point in the crossover region between regions I and III in a neighborhood of the first-order curve. For these sequences $\mu_2 \approx 1/n^\alpha$ and $\mu_1 \approx (\ell - \ell_c)/n^{2\alpha}$. Hence the scaling expression for the magnetization in (7.9) becomes

$$m \approx n^{-\alpha \tilde{\beta}_t/\varphi_t} f_+(\ell - \ell_c) = f_+(\ell - \ell_c)/n^{\alpha/2}.$$

We note that n does not appear in the argument of f_+ since $1/\varphi_t = 2$ and the powers of n cancel. This asymptotic formula is consistent with the exact asymptotic behavior of $m(\beta_n, K_n)$ given in Theorem 5.2.

The sequence of type 4c is defined in (5.11) with $\ell = \ell_c$ and $\tilde{\ell} > K_1'''(\beta_c)$. We conjecture that this sequence converges to the tricritical point along a curve that coincides with the first-order curve to order 2 in powers of $\beta - \beta_c$ and lies in the phase-coexistence region for all sufficiently large n. Thus when $\ell = \ell_c$, $\mu_1 \approx 0$ and (7.13) holds. Since $\mu_2 \approx 1/n^\alpha$, we have $m \approx \mu_2^{1/2} \approx 1/n^{\alpha/2}$. This result is consistent with part (c) of Theorem 5.2.

The sequences of type 5 and type 6 approach the tricritical point along a curve that coincides with the second-order curve to order $p-1$ in powers of $\beta - \beta_c$. These sequences are defined in terms of a parameter ℓ as in (7.7) with β_0 replaced by β_c and $b = -1$; the sequence of type 5 corresponds to the choice $p = 2$ while the sequence of type 6 corresponds to $p \geq 3$. Since $1/\varphi_t = 2$, the dotted line separating regions II and III deviates quadratically from the second-order curve. Thus the sequence of type 5, defined for $\ell > K''(\beta_c)$, lies in the crossover range between region II and region III. The sequence of type 6 lies within region II since it approaches the second-order curve faster than quadratically. For a sequence of type 5 we have $\mu_1 \approx (\ell - K''(\beta_c))/n^{2\alpha}$ and $\mu_2 \approx 1/n^\alpha$. From the general expression (7.9) we obtain

$$m \approx n^{-\alpha\tilde{\beta}_t/\varphi_t} f_-(\ell - K''(\beta_c)) = n^{-\alpha/2} f_-(\ell - K''(\beta_c)).$$

Since $f_-(x) \approx x^{\tilde{\beta}}$, for small x we find that $m \approx (\ell - K''(\beta_c))^{1/2}/n^{\alpha/2}$. This asymptotic formula is consistent with the exact asymptotic behavior of $m(\beta_n, K_n)$ given in Theorem 5.3. It captures the correct dependence of the prefactor $\bar{x}$ on $\ell - K''(\beta_c)$ for small $\ell - K''(\beta_c)$ that follows from (5.25).

The sequence of type 6 is defined as in (7.7) with β_0 replaced by β_c, $p \geq 3$, $b = -1$, and $(K^{(p)}(\beta_c) - \ell)(-1)^p < 0$. Because this sequence converges to the tricritical point in region II, the scaling expression (7.12) is valid. In this case $\mu_1 \approx |\ell - K^{(p)}(\beta_c)|/n^{p\alpha}$ and $\mu_2 \approx 1/n^\alpha$. Substituting these values into (7.12) yields

$$m \approx (|\ell - K^{(p)}(\beta_c)|/p!)^{1/2}/n^{(p-1)\alpha/2}.$$

Once again this asymptotic formula is consistent with the exact asymptotic behavior of $m(\beta_n, K_n)$ given in Theorem 5.4. We note that scaling theory predicts the correct square-root dependence of the prefactor $\bar{x}$ on $|\ell - K^{(p)}(\beta_c)|$ given in (5.28).

This completes the discussion of the relationship between the results obtained in sections 3 with scaling theory for critical and tricritical points [25]. We have shown that scaling theory, together with the known mean-field exponents, predicts many of the exact results for $m(\beta_n, K_n)$, capturing both the correct power laws and, in some cases, the dependence on the parameters defining the sequences.

In the next two appendices we address a number of issues related to several results contained in the main body of the paper.

Appendix

A Properties of Polynomials of Degree 6

The main purpose of this section is to analyze the structure of the set of global minimum points of polynomials of degree 6 having the form $g(x) = a_2x^2 - a_4x^4 + a_6x^6$, where $a_2 \in \mathbb{R}$, $a_4 > 0$, and $a_6 > 0$. We use this information in section 2 in order to motivate the phase-transition structure of the mean-field B-C model via the Ginzburg-Landau phenomenology. We also use it in the discussion leading up to Theorem 5.2 when we show how the predictions of the Ginzburg-Landau phenomenology can be made rigorous via properties of the Ginzburg-Landau polynomials. Specifically, items 1–4 preceding Theorem 5.2 exhibit the structure of the set of global minimum points of the associated Ginzburg-Landau polynomial, which precisely mirror the phase-transition structure of the model. We end the section by briefly considering the polynomial $h(x) = a_2x^2 + a_4x^4 + a_6x^6$, where $a_2 \in \mathbb{R}$, $a_4 > 0$, $a_6 > 0$.

For variable $a_2 \in \mathbb{R}$ and fixed $a_4 > 0$ and $a_6 > 0$, the analysis of the set of global minimum points of $g(x) = a_2x^2 - a_4x^4 + a_6x^6$ is given in Theorem A.1 in terms of the quantity $a_c = a_4^2/4a_6$. Here are the main features. If $a_2 > a_c$, then g has a unique global minimum point at 0; if $a_2 = a_c$, then the global minimum points of g are 0 and nonzero numbers $\pm\bar{x}(a_c)$; and if $a_2 < a_c$, then the global minimum points of g are nonzero numbers $\pm\bar{x}(a_2)$, where the positive number $\bar{x}(a_2)$ converges to the positive number $\bar{x}(a_c)$ as $a_2 \to (a_c)^+$. Thus the set of global minimum points of g undergoes a discontinuous bifurcation at a_c, changing discontinuously from $\{0\}$ for $a_2 < a_c$ to $\{0, \pm\bar{x}(a_c)\}$ for $a_2 = a_c$ to $\{\pm\bar{x}(a_2)\}$ for $a_2 > a_c$. The discontinuous bifurcation in the set of global minimum points of g is reminiscent of a first-order phase transition, and the value a_c corresponds to a point on the first-order curve $K_1(\beta)$. In fact, in section 6 we use the analogous behavior of the set of global minimum points of the appropriate Ginzburg-Landau polynomials to deduce properties of the first-order curve.

In order to highlight the discontinuous bifurcation in the set of global minimum points of g at a_c, we give a quick proof of the fact that g has three global minimum points if and only if $a_2 = a_c = a_4^2/4a_6$. Since $g(x) \to \infty$ as $|x| \to \infty$, the set of global minimum points of g is nonempty. By symmetry, g has 3 global minimum points at 0 and at $\pm\bar{x}$ for some $\bar{x} > 0$ if and only if $g(\bar{x}) = 0 = g(0)$ and $g'(\bar{x}) = 0$. Since $\bar{x} > 0$, these equations reduce to $a_2 - a_4\bar{x}^2 + a_6\bar{x}^4 = 0$ and $2a_2 - 4a_4\bar{x}^2 + 6a_6\bar{x}^4 = 0$. Eliminating a_6 from both equations yields $\bar{x}^2 = 2a_2/a_4$ or $\bar{x} = \pm(2a_2/a_4)^{1/2}$. Substituting this value back into $g(\bar{x}) = 0$ gives $4a_2a_6 = a_4^2$ or $a_2 = a_c$. We conclude that g has three global minimum points if and only if $a_2 = a_c$, and then the global minimum points are 0 and $\pm(2a_2/a_4)^{1/2}$. This fact is recorded in part (b) of the next theorem.

In parts (a), (b), and (c) of the next theorem we give information about the global minimum points of g for variable $a_2 \in \mathbb{R}$. Part (d) highlights properties of the positive global minimum point of g for $a_2 \le a_c$. These properties underlie the discontinuous bifurcation in the set of global minimum points of g at a_c. For $a_2 > a_c$ [part (a)], g has a similar shape as $G_{\beta,K}$ in

Figure 4 in section 2; for $a_2 = a_c$ [part (b)], g has a similar shape as $G_{\beta,K}$ in Figure 5 in section 2; and for $a_2 < a_c$ [part (c)], g has a similar shape as $G_{\beta,K}$ in Figure 6 in section 2. The elementary proof of the next theorem is omitted.

Theorem A.1. *For variable $a_2 \in \mathbb{R}$ and fixed $a_6 > 0$ and $a_4 > 0$, define $g(x) = a_2x^2 - a_4x^4 + a_6x^6$ and $a_c = a_4^2/4a_6$. If $0 \leq a_2 \leq a_4^2/3a_6$, then also define the positive number*

$$\bar{x}(a_2) = \tfrac{1}{\sqrt{3a_6}} \left(a_4 + (a_4^2 - 3a_2a_6)^{1/2}\right)^{1/2}. \tag{A.1}$$

The structure of the set of global minimum points of g is as follows.

(a) *If $a_2 > a_c$, then g has a unique global minimum point at 0.*

(b) *If $a_2 = a_c$, then the global minimum points of g are 0 and $\pm\bar{x}(a_c) = \pm(2a_2/a_4)^{1/2}$.*

(c) *If $a_2 < a_c$, then the global minimum points of g are $\pm\bar{x}(a_2)$.*

(d) *$\bar{x}(a_2)$ is a positive, decreasing, continuous function for $a_2 < a_c$, and as $a_2 \to (a_c)^-$, $\bar{x}(a_2) \to \bar{x}(a_c)$, the unique positive, global minimum point in part* (b).

We end this appendix by making several observations about the polynomial $h(x) = a_2x^2 + a_4x^4 + a_6x^6$, where $a_2 \in \mathbb{R}$, $a_4 > 0$, and $a_6 > 0$. Such polynomials arise in Theorem 5.3. The elementary proof of the next theorem is omitted.

Theorem A.2. *For variable $a_2 \in \mathbb{R}$ and fixed $a_4 > 0$ and $a_6 > 0$, define $h(x) = a_2x^2 + a_4x^4 + a_6x^6$. The following conclusions hold.*

(a) *If $a_2 \geq 0$, then h has a unique global minimum point at 0.*

(b) *If $a_2 < 0$, then the global minimum points of h are $\pm\bar{x}(a_2)$, where*

$$\bar{x}(a_2) = \tfrac{1}{\sqrt{3a_6}} \left(-a_4 + (a_4^2 - 3a_2a_6)^{1/2}\right)^{1/2}. \tag{A.2}$$

In the next section we give a second calculation of the asymptotics of $m(\beta_n, K_n) \to 0$ in the two cases treated in Theorems 3.1 and 5.1.

B Exact Asymptotics by Another Method

In this appendix we give another proof of the asymptotic behavior of $m(\beta_n, K_n)$ in two separate cases, using an argument based directly on the fact that $m(\beta_n, K_n)$ is a positive zero of G'_{β_n,K_n}. These two cases correspond to Theorem 3.1, which deals with sequence (β_n, K_n) converging to a second-order point, and to Theorem 5.1, which deals with sequence (β_n, K_n) having a similar form as the sequences in Theorem 3.1 but converging to the tricritical point. Despite the naturalness of the characterization of $m(\beta_n, K_n)$ as a positive zero of G'_{β_n,K_n}, the proofs of the asymptotic behaviors of $m(\beta_n, K_n)$ given in this appendix are much more complicated and the respective theorems give less information than Theorems 3.1 and 5.2, which are based on properties of the Ginzburg-Landau polynomials. This emphasizes once again the elegance of the approach of using properties of these polynomials to deduce the asymptotic behavior of $m(\beta_n, K_n)$. As in Theorems 3.1 and 5.2, the proofs of the asymptotic behaviors of $m(\beta_n, K_n)$ given in this appendix rely on the fact that $m(\beta_n, K_n) \to 0$, which is proved in Theorem 4.1.

We first consider the sequence (β_n, K_n) in Theorem 3.1, which converge to a point on the second-order curve. The next theorem expresses the asymptotic behavior of $m(\beta_n, K_n)$ in terms of an explicitly given quantity $f(b, k)$, which can be seen to agree with the explicit formula for $\bar{x}$ given in (3.10) in Theorem 3.1. However, in contrast to Theorem 3.1 the next theorem does not identify $f(b, k)$ as the unique, positive, global minimum point of the Ginzburg-Landau polynomial. In this sense, the next theorem gives less information than Theorem 3.1.

Theorem B.1. *For $\beta \in (0, \beta_c)$, $\alpha > 0$, $b \in \{1, 0, -1\}$, and $k \in \mathbb{R}$ satisfying $K'(\beta)b - k < 0$, define*

$$\beta_n = \beta + b/n^\alpha \ \text{ and } \ K_n = K(\beta) + k/n^\alpha$$

as well as

$$f(b, k) = \left(\frac{96\beta(k - bK'(\beta)|)}{(e^\beta + 2)^2\,(4 - e^\beta)}\right)^{1/2}.$$

The sequence (β_n, K_n) converges to the second-order point $(\beta, K(\beta))$. The following conclusions hold: $m(\beta_n, K_n) \to 0$ and has the asymptotic behavior

$$m(\beta_n, K_n) \sim f(b, k)/n^{\alpha/2}; \ \text{ i.e., } \ \lim_{n\to\infty} n^{\alpha/2} m(\beta_n, K_n) = f(b, k).$$

Proof. In order to simplify the notation, we write m_n in place of $m(\beta_n, K_n)$. For all sufficiently large n, (β_n, K_n) lies above the curve $(\beta, K(\beta))$ for $0 < \beta < \beta_c$. Hence for all sufficiently large n, m_n is a positive global minimum point of G_{β_n, K_n} and thus a positive zero of G'_{β_n, K_n}. In fact, m_n is the unique, positive, global minimum point of G and the unique positive zero of G'_{β_n, K_n}, but this fact is not needed in the proof. Since

$$\begin{aligned} 0 &= G'_{\beta_n, K_n}(m_n) \\ &= 2\beta_n K_n m_n - (2\beta_n K_n) c'\beta_n(2\beta_n K_n m_n) \\ &= 2\beta_n K_n m_n - 2\beta_n K_n \cdot \frac{e^{-\beta_n}\big(e^{2\beta_n K_n m_n} - e^{-2\beta_n K_n m_n}\big)}{1 + e^{-\beta_n}\big(e^{2\beta_n K_n m_n} + e^{-2\beta_n K_n m_n}\big)}, \end{aligned}$$

it follows that for all sufficiently large n, m_n is the unique positive solution of

$$m_n = \frac{e^{2\beta_n K_n m_n} - e^{-2\beta_n K_n m_n}}{e^{\beta_n} + e^{2\beta_n K_n m_n} + e^{-2\beta_n K_n m_n}}. \tag{B.1}$$

We now use Taylor's Theorem to replace the exponentials in the numerator by a Taylor expansion to order 3 and the exponentials in the denominator by a Taylor expansion to order 2. Since $0 < \beta < \beta_c$ and $(\beta_n, K_n) \to (\beta, K(\beta))$, $m_n \to 0$ by Theorem 4.1. Hence there exist error terms $\varepsilon_n = \mathrm{O}(m_n^2) \to 0$ and $\delta_n = \mathrm{O}(m_n^2) \to 0$ such that

$$m_n = \frac{4\beta_n K_n m_n + \frac{8}{3}\beta_n^3 K_n^3 m_n^3 (1+\varepsilon_n)}{e^{\beta_n} + 2 + 4\beta_n^2 K_n^2 m_n^2 (1+\delta_n)}.$$

Because $m_n > 0$, this equation can be rewritten in the form

$$m_n^2 (2\beta_n K_n)^2 (B_n + \bar{\varepsilon}_n) + C_n = 0, \tag{B.2}$$

where $\bar{\varepsilon}_n = \mathrm{O}(m_n^2) \to 0$ and

$$B_n = 1 - 2\beta_n K_n/3, \; C_n = 2 + e^{\beta_n} - 4\beta_n K_n.$$

Thus

$$m_n = \left(\frac{-C_n}{(2\beta_n K_n)^2 (B_n + \bar{\varepsilon}_n)} \right)^{1/2}. \tag{B.3}$$

We work out the asymptotic behavior of the terms in this equation. We have $\beta_n K_n \to \beta K(\beta) = (e^\beta + 2)/4$ and thus

$$(2\beta_n K_n)^2 (B_n + \bar{\varepsilon}_n) \to \frac{(e^\beta + 2)^2}{4} \left(1 - \frac{(e^\beta + 2)}{6} \right) = \frac{1}{24} (e^\beta + 2)^2 (4 - e^\beta). \tag{B.4}$$

Since $4\beta K(\beta) = e^\beta + 2$, $4K(\beta) - e^\beta = -4\beta K'(\beta)$ [Lem. 6.1(b)], and $e^{b/n^\alpha} \sim 1 + b/n^\alpha$, the asymptotic behavior of C_n is given by

$$\begin{aligned} C_n &= 2 + e^{\beta_n} - 4\beta_n K_n \\ &= 2 + e^\beta e^{b/n^\alpha} - 4(\beta + b/n^\alpha)(K(\beta) + k/n^\alpha) \\ &\sim -4\beta(k - bK'(\beta))/n^\alpha. \end{aligned} \tag{B.5}$$

Substituting (B.4) and (B.5) into (B.2) yields the desired asymptotic behavior of m_n:

$$m_n \sim \left(\frac{96\beta(k - bK'(\beta))}{(e^\beta + 2)^2 \, (4 - e^\beta)} \right)^{1/2} \cdot \frac{1}{n^{\alpha/2}}$$

The proof of the theorem is complete. ■

When $\beta = \beta_c$, the sequence (β_n, K_n) considered in Theorem B.1 converges to the tricritical point, and the asymptotic behavior of $m(\beta_n, K_n)$ given in Theorem B.1 fails because in (B.4)

$4 - e^{\beta_c} = 0$. As we will see in the proof of the next theorem, in this case the asymptotic behavior of $m(\beta_n, K_n) \to 0$ is determined by taking higher order terms in the Taylor expansions of the exponentials in (B.1).

The next theorem expresses the asymptotic behavior of $m(\beta_n, K_n)$ in terms of an explicitly given quantity $g(b, k)$, which can be seen to agree with the explicit formula for $\bar{x}$ given in (5.10). However, in contrast to Theorem 5.1 the next theorem does not identify $g(b, k)$ as the unique, positive, global minimum point of the Ginzburg-Landau polynomial. In this sense, the next theorem gives less information than Theorem 3.1.

Theorem B.2. *For $\alpha > 0$, $b \in \{1, 0, -1\}$, and $k \in \mathbb{R}$ satisfying $bK'(\beta_c) - k < 0$, define*

$$\beta_n = \beta_c + b/n^\alpha \ \text{ and } \ K_n = K(\beta_c) + k/n^\theta$$

as well as

$$g(b, k) = (40\beta_c(k - K'(\beta_c)b)/27)^{1/4}.$$

Then (β_n, K_n) converges to the tricritical point $(\beta_c, K(\beta_c))$. The following conclusions hold: $m(\beta_n, K_n) \to 0$ and has the the following asymptotic behavior:

$$m(\beta_n, K_n) \sim g(k, b)/n^{\alpha/4}.$$

Proof. For all sufficiently large n, (β_n, K_n) lies above the curve $(\beta, K(\beta))$ for $\beta > \beta_c$. Hence for all sufficiently large n, m_n is a positive global minimum point of G_{β_n, K_n} and a positive zero of G'_{β_n, K_n}. As in the proof of the preceding theorem, m_n is the unique, positive, global minimum point of G and the unique positive zero of G'_{β_n, K_n}, but this fact is not needed in the present proof. Since

$$\begin{aligned} 0 &= G'_{\beta_n, K_n}(m_n) \\ &= 2\beta_n K_n m_n - (2\beta_n K_n) c'_{\beta_n}(2\beta_n K_n m_n) \\ &= 2\beta_n K_n m_n - 2\beta_n K_n \cdot \frac{e^{-\beta_n}\left(e^{2\beta_n K_n m_n} - e^{-2\beta_n K_n m_n}\right)}{1 + e^{-\beta_n}\left(e^{2\beta_n K_n m_n} + e^{-2\beta_n K_n m_n}\right)}, \end{aligned}$$

it follows that for all sufficiently large n, m_n is the unique positive solution of

$$m_n = \frac{e^{2\beta_n K_n m_n} - e^{-2\beta_n K_n m_n}}{e^{\beta_n} + e^{2\beta_n K_n m_n} + e^{-2\beta_n K_n m_n}}. \tag{B.6}$$

We now use Taylor's Theorem to replace the exponentials in the numerator by a Taylor expansion to order 5 and the exponentials in the denominator by a Taylor expansion to order

4. Since $(\beta_n, K_n) \to (\beta_c, K(\beta_c))$, $m_n \to 0$ by Theorem 4.1. Hence there exist error terms $\varepsilon_n = \mathrm{O}(m_n^2) \to 0$ and $\delta_n = \mathrm{O}(m_n^2) \to 0$ such that

$$m_n = \frac{2\left(2\beta_n K_n z_n + \frac{(2\beta_n K_n)^3}{3!} z_n^3 + \frac{(2\beta_n K_n)^5}{5!} z_n^5 (1+\varepsilon_n)\right)}{e^{\beta_n} + 2\left(1 + \frac{(2\beta_n K_n)^2}{2} z_n^2 + \frac{(2\beta_n K_n)^4}{4!} z_n^4 (1+\delta_n)\right)}.$$

Because $m_n > 0$, this equation can be rewritten in the form

$$m_n^4 \frac{(2\beta_n K_n)^4}{12}(A_n + \bar{\varepsilon}_n) + m_n^2 (2\beta_n K_n)^2 B_n + C_n = 0, \tag{B.7}$$

where $\bar{\varepsilon}_n = \mathrm{O}(m_n^2) \to 0$,

$$A_n = 1 - 2\beta_n K_n/5, \; B_n = 1 - 2\beta_n K_n/3, \; C_n = 2 + e^{\beta_n} - 4\beta_n K_n.$$

Without the term involving m_n^4, the formula in (B.7) reduces to the quadratic (B.2) without the error term $\bar{\varepsilon}_n$; this quadratic was used to determine the asymptotic behavior of m_n in Theorem B.1. It follows from (B.7) that

$$\begin{aligned} z_n^2 &= \frac{1}{\frac{(2\beta_n K_n)^4}{6}(A_n + \bar{\varepsilon}_n)} \times \\ &\qquad \left(-(2\beta_n K_n)^2 B_n \pm \left((2\beta_n K_n)^4 B_n^2 - \frac{1}{3}(2\beta_n K_n)^4 (A_n + \bar{\varepsilon}_n) C_n\right)^{1/2}\right) \\ &= \frac{6}{(2\beta_n K_n)^2 (A_n + \bar{\varepsilon}_n)} \left(-B_n \pm \left(B_n^2 - \frac{1}{3}(A_n + \bar{\varepsilon}_n) C_n\right)^{1/2}\right). \end{aligned} \tag{B.8}$$

We now work out the asymptotic behavior of the terms in this equation. We have $\beta_n K_n \to \beta_c K(\beta_c) = 3/2$ and thus

$$A_n \to 2/5,$$

$$\begin{aligned} B_n &= 1 - 2(\beta_c + b/n^\alpha)(K(\beta_c) + k/n^\theta)/3 \\ &\sim -2bK(\beta_c)/3n^\alpha - 2\beta_c k/3n^\alpha, \end{aligned}$$

and as in (B.5),

$$C_n \sim -4\beta(k - bK'(\beta_c))/n^\alpha.$$

Since $A_n \to 2/5$, $\bar{\varepsilon}_n \to 0$, and $B_n^2/C_n \to 0$, it follows from (B.8) that

$$m_n^2 \sim \frac{6}{(2\beta_n K_n)^2 A_n}\left(-\frac{1}{3} A_n C_n\right)^{1/2}. \tag{B.9}$$

Combining this with the preceding display yields the desired asymptotic behavior of m_n:

$$m_n \sim \left(\frac{40\beta_c(k - K'(\beta_c)b)}{27}\right)^{1/4} \cdot \frac{1}{n^{\alpha/4}}$$

The proof of the theorem is complete. ■

The proofs of the last two theorems make clear that deriving the asymptotic behavior of $m(\beta_n, K_n) \to 0$ in this way requires a detailed asymptotic analysis that is highly dependent on the specific situation. In the case of Theorem B.1 the starting point is (B.3), while in the case of Theorem B.1 the starting point is (B.8). The latter formula can also be used to deduce the asymptotic behavior of $m(\beta_n, K_n) \to 0$ for the sequences appearing in Theorems 5.2–5.4. The asymptotic expressions for B_n and C_n in (B.8) are different for each of these sequences, and in each case one must redo the tedious calculation leading from (B.8) to the answer. By contrast, the proofs of the asymptotic behavior of $m(\beta_n, K_n)$ given in Theorems 5.1–5.4 all rely on the general result given in Theorem 4.2. In each of these four cases the asymptotic behavior depends on the unique, positive, global minimum point $\bar{x}$ of the associated Ginzburg-Landau polynomial. One uses the explicit formula for the sequences only in calculating the form of $\bar{x}$, a much simpler task than the detailed asymptotics that are required when applying the method in this appendix.

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