THREE ESSAYS ON THE THEORY OF ENVIRONMENTAL REGULATION: HYBRID PRICE AND QUANTITY POLICIES AND REGULATION IN THE PRESENCE OF CO-POLLUTANTS

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THREE ESSAYS ON THE THEORY OF ENVIRONMENTAL REGULATION: HYBRID PRICE AND QUANTITY POLICIES AND REGULATION IN THE PRESENCE OF CO-POLLUTANTS

A Dissertation Presented

by

INSUNG SON

Submitted to the Graduate School of the University of Massachusetts Amherst in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

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Resource Economics
THREE ESSAYS ON THE THEORY OF
ENVIRONMENTAL REGULATION: HYBRID PRICE
AND QUANTITY POLICIES AND REGULATION IN
THE PRESENCE OF CO-POLLUTANTS

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INSUNG SON

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This dissertation contains three original essays in the economic theory of environmental regulation. The main motivations for this work are two problems: the design of greenhouse gas (GHG) policies when emissions of these gases interact with so-called co-pollutants and the design of hybrid price and quantity policies to deal with the uncertainty in the benefits and costs of controlling GHG emissions.

Concerns about how best to control GHGs have generated intense interest in the co-benefits and adverse side-effects of climate policies. Efforts to reduce CO\textsubscript{2} emissions can reduce emissions of flow pollutants that are emitted along with CO\textsubscript{2}, which provides a co-benefit of climate policy. However, it is not always the case that efforts to reduce CO\textsubscript{2} emissions have positive co-benefits.

The challenge of climate change has also intensified research in policy design under uncertainty about the benefits and costs of controlling GHG emissions. Literature on
this problem suggest that a carbon tax is more efficient than carbon trading. However, given that many existing GHG control policies feature tradable permit markets, there have been a lot of interest and innovation in hybrid schemes. The most popular form of these hybrids involves tradable emissions permits with price controls.

While there is a significant literature on designing hybrid price and quantity environmental regulations under uncertainty, and another literature on regulating multiple interacting pollutants, no one has addressed the design of an emission markets with price controls for a pollutant that interacts with a co-pollutant in emission control. In Chapter 2, we investigate the optimal regulation of a pollutant given its abatement interaction with another pollutant under asymmetric information about firms’ abatement costs. The co-pollutant is regulated, but perhaps not efficiently. Our focus is on optimal instrument choice in this setting, and we derive rules for determining whether a pollutant should be regulated with an emissions tax, tradable permits, or an emissions market with price controls. The policy choices depend on the relative slopes of the damage functions for both pollutants and the aggregate marginal abatement cost function, including whether the pollutants are complements or substitutes in abatement and whether the co-pollutant is controlled with a tax or tradable permits.

In Chapter 3, we extend the model in Chapter 2 by allowing a pollutant to interact with a co-pollutant in both abatement and damage. In this situation, we examine the expected performance of optimal price-based regulations for the primary pollutant. We find that, given exogenous but possibly inefficient regulation of a co-pollutant, an optimal permit market, an optimal tax, and an optimal permit market with price controls all produce the same expected emissions for the primary pollutant, which deviates from its ex ante optimal emissions if the co-pollutant is regulated inefficiently. This deviation depends on 1) the interactions of the two pollutants in abatement costs and damages, 2) the deviation of the expected emissions of the co-pollutant from its
ex ante optimal emissions, and 3) whether it is regulated with a fixed number of tradable permits or an emissions tax.

Another important concern about permit trading has been how much regulations induce investments in abatement capital or technology. As concern about cost containment has increased, the effects of cost-containment policies on abatement investments have gained attention among researchers. In Chapter 4 we examine the effects of a hybrid policy on investment in abatement capital. We construct a dynamic stochastic model to study the decision to invest in irreversible abatement capital under an emissions market with price controls. We consider investment decisions in an emissions market with price controls, and compare these to the decisions in a market without price controls. We found that a price floor tends to increase the opportunity of investment while a price ceiling always reduces the opportunity of investment by imposing an upper bound of investment intervals. Under a hybrid regulation there exists an upper bound of abatement capital stock such that no additional investment occurs. No such upper bound exists for a pure permit trading. On the other hand, there may exist investment opportunities for low marginal abatement costs under a hybrid policy that are not available under a pure permit trading. However, when investments are required under both regulations, increases in capital stock under a hybrid regulation are likely to be less than under pure permit trading.
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CHAPTER 1
INTRODUCTION

This dissertation contains three original essays in the economic theory of environmental regulation. The main motivations for this work are two problems that have received special attention in the literature on formulating policies to control greenhouse gases that are responsible for global climate change. These problems are the design of greenhouse gas policies when emissions of these gases interact with so-called co-pollutants in production processes, and the design of hybrid price and quantity policies to deal with the immense uncertainty in the benefits and costs of controlling greenhouse gas emissions.

Concerns about how best to control greenhouse gases have generated intense interest in the co-benefits and adverse side-effects of climate policies. Perhaps the most well studied co-benefits of climate policy are the effects on flow pollutants like sulfur dioxide (SO$_2$), nitrous dioxide (NO$_2$), and fine and coarse particulate matter (PM2.5 and PM10) that are emitted along with CO$_2$ in combustion processes. Efforts to reduce CO$_2$ emissions can reduce emissions of these pollutants providing a co-benefit of climate policy. The Intergovernmental Panel on Climate Change (IPCC) has reviewed many empirical studies of these co-benefits in Chapter 6 of IPCC (2014), and they have concluded that the benefits of reductions in emissions of CO$_2$ co-pollutants can be substantial. Burtraw et al. (2003) found that a tax of $25 per ton of carbon emissions would cause further reductions in NO$_X$ emissions and they evaluate health co-benefits at about $8 per ton of carbon reduction (1997 dollars). From a survey of previous research, Nemet et al. (2010) find that
air-quality co-benefits of climate change mitigation has a mean value of $48 per ton of carbon reduction (2008 dollars). Groosman et al. (2011) calculate the effects of U.S. climate policy on local air pollutants and they assess the health co-benefits at between $103 billion and $1.2 trillion (2006 dollars). Parry et al. (2015) calculate an average co-benefit of $57.5 per ton of CO$_2$ (2010 dollars) among the top 20 CO$_2$ emitting countries.\(^1\) It is not always the case that efforts to reduce CO$_2$ emissions have positive co-benefits. For example, Ren et al. (2011) show that when the use of biofuels increases as part of an effort to reduce CO$_2$ emissions, fertilizer runoff can also increase. Depending on the level of nitrogen runoff regulation, the effects on social welfare can be negative.

The challenge of climate change has also intensified research in policy design under uncertainty about the benefits and costs of controlling greenhouse gas emissions. The seminal work in the area of policy instrument choice under uncertainty is Weitzman (1974) whose work demonstrates that, under certain conditions, a pure emissions tax is more efficient than competitively traded emissions permits when the slope of the aggregate marginal damage function is less than the absolute value of the slope of the aggregate marginal abatement cost function. Tradable permits are preferred to taxes if that slope-relationship is reversed. This work is still relevant today, because the marginal damage associated with carbon emissions is almost perfectly flat over a relatively short compliance period (e.g., Pizer (2002)). Hence, uncertainty in the costs and benefits of controlling greenhouse gases suggests that a carbon tax is more efficient than carbon trading.

However, given that many existing greenhouse gas control policies feature tradable permit markets, there have been a lot of interest and innovation in hybrid schemes.\(^1\) They focus on domestic co-benefits like mortality risks, road congestion, accident risk and road damage while they exclude the global benefits from reduced CO$_2$ emissions. The cited figures do not take account of the revenue-recycling effects from a carbon tax and tax-interaction effects with pre-existing fuel taxes.

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\(^1\)They focus on domestic co-benefits like mortality risks, road congestion, accident risk and road damage while they exclude the global benefits from reduced CO$_2$ emissions. The cited figures do not take account of the revenue-recycling effects from a carbon tax and tax-interaction effects with pre-existing fuel taxes.
The most popular form of these hybrids, in the literature and in actual practice, involves tradable emissions permits with price controls. The conceptual foundation for these policies originated with Roberts and Spence (1976), who demonstrate that since an emissions tax and a simple permit market are special cases of such a hybrid policy, emissions markets with price controls cannot be less efficient and will often be more efficient than either of the pure instruments. The performance of alternative hybrid policies has been examined theoretically (e.g., Grüßl and Taschini (2011)), with simulations (Burtraw et al. (2010); Fell and Morgenstern (2010); Fell et al. (2012)), and with laboratory experiments (Stranlund et al. (2014)). Recent theoretical work has also examined technology choices in emissions markets with price controls (Weber and Neuhoff (2010)) and the enforcement of these policy schemes (Stranlund and Moffitt (2014)).

In brief the essays in this dissertation are the following:

 Prices versus quantities versus hybrids in the presence of co-pollutants

While there is a significant literature on the economic theory of designing hybrid price and quantity environmental regulations under uncertainty, and another literature on regulating multiple interacting pollutants, no one has addressed the design of an emission markets with price controls ala Roberts and Spence (1976) for a pollutant that interacts with a co-pollutant in emission control. In this essay we investigate the optimal regulation of a pollutant given its abatement interaction with another pollutant under asymmetric information about firms' abatement costs. The co-pollutant is regulated, but perhaps not efficiently. Our focus is on optimal instrument choice in this setting, and we derive rules for determining whether a pollutant should be regulated with an emissions tax, tradable permits, or an emissions market with price controls. The policy choices depend on the relative slopes of the damage functions for both pollutants and the aggregate marginal
abatement cost function, including whether the pollutants are complements or substitutes in abatement and whether the co-pollutant is controlled with a tax or tradable permits.

**Second best regulation in the presence of co-pollutants**

The second essay builds on the first essay by examining the expected performance of optimal price-based regulations for a pollutant when it interacts with a co-pollutant in both abatement cost and damage under asymmetric information about abatement costs. By allowing its co-pollutant to be regulated exogenously but possibly inefficiently, we take a second-best approach to regulating the primary pollutant. We find that, given the regulation of a co-pollutant, an optimal permit market, an optimal tax, and an optimal permit market with price controls all produce the same expected emissions for the primary pollutant, which deviates from its ex ante optimal emissions if the co-pollutant is regulated inefficiently. This deviation depends on 1) the interactions of the two pollutants in abatement and damages, 2) the deviation of the expected emissions of the co-pollutant from its ex ante optimal emissions, and 3) the form of the regulation for the co-pollutant, that is, whether it is regulated with a fixed number of tradable permits or an emissions tax.

**Irreversible investments in emissions control under a hybrid price and quantity regulation**

Despite its theoretical efficiency, implementing emissions permit market has generated some practical concerns. Containing costs has been one of those concerns. Another important concern about permit trading has been how much regulations induce investments in abatement capital or technology. As concern about cost containment has increased, the effects of cost-containment policies on abatement investments have gained attention among researchers (Phaneuf and Requate (2002);
The third essay in this dissertation contributes to this literature by studying the effects of a hybrid policy on investment in abatement capital. Our approach to the problem is to construct a dynamic stochastic model based on the real option approach to study the decision to invest in irreversible abatement capital under an emissions market with price controls. Our model is an extension of Zhao (2003), who considered the differences in investment under a pure emissions market and an emissions tax. In contrast, we consider investment decisions in an emissions market with price controls, and compare these to the decisions in a market without price controls. We found that a price floor tends to increase the opportunity of investment while a price ceiling always reduces the opportunity of investment by imposing an upper bound of investment intervals. Under a hybrid regulation there exists an upper bound of abatement capital stock such that no additional investment occurs. No such upper bound exists for a pure permit trading. On the other hand, there may exist investment opportunities for low marginal abatement costs under a hybrid policy that are not available under a pure permit trading. However, when investments are required under both regulations, increases in capital stock under a hybrid regulation are likely to be less than under pure permit trading.
CHAPTER 2

PRICES VERSUS QUANTITIES VERSUS HYBRIDS IN THE PRESENCE OF CO-POLLUTANTS

2.1 Introduction

Concerns about how best to control greenhouse gases have generated intense interest in the co-benefits and adverse side-effects of climate policies. Perhaps the most well studied co-benefits of climate policy are the effects on flow pollutants like NO\textsubscript{X}, SO\textsubscript{2} and PM that are emitted along with CO\textsubscript{2} in combustion processes. Efforts to reduce CO\textsubscript{2} emissions can reduce emissions of these pollutants, thereby providing a co-benefit of climate policy. The Intergovernmental Panel on Climate Change (IPCC) has reviewed many empirical studies of these co-benefits in Chapter 6 of IPCC (2014), and they have concluded that the benefits of reductions in emissions of CO\textsubscript{2} co-pollutants can be substantial.\(^1\) On the other hand, climate policy can also have adverse consequences, some of which come from increases in related pollutants. For example, Ren et al. (2011) suggest that increased use of biofuels as part of a policy to reduce CO\textsubscript{2} emissions can result in greater water pollution from agricultural runoff.\(^2\)

\(^1\)Nemet et al. (2010) surveyed empirical studies of air pollutant co-benefits of climate change mitigation and found a mean value of $49 (2008 dollars) per ton of CO\textsubscript{2} reduction. Similarly, Parry et al. (2015) calculated the average co-benefits for the top 20 CO\textsubscript{2} emitting countries to be about $57.5 for 2010 (in 2010 dollars). These values are about the same magnitude as estimates for the climate-related benefit per ton of CO\textsubscript{2} reduction developed by the US Interagency Working Group on the Social Cost of Carbon. Using a 3\% discount rate, the Interagency Working Group proposes a schedule for the social cost of carbon dioxide to be used in regulatory impact analysis that starts at $31 (2007 dollars) per ton CO\textsubscript{2} in 2010 and rises to $69 per ton in 2050 (IAWG (2013)).

\(^2\)The IPCC considers many more ancillary consequences of climate policy besides those generated by co-pollutants. These include the effects of climate policy on other social goals like food security,
The presence of co-benefits or adverse side-effects presents challenges for efficient pollution regulation. The efficient regulation of one pollutant must account for how its control affects the abatement of its co-pollutants, and how the abatement interactions translate into changes in the damages associated with its co-pollutants. In addition, accounting for existing regulations of co-pollutants is critical for determining the net co-benefits or adverse consequences of pollution control. Of course, full efficiency would require that the regulations of multiple interacting pollutants be determined jointly to maximize the net social benefits of a complex environmental regulatory system, but this may not be realistic. Instead environmental regulations tend to focus on single pollutants, not joint regulation of multiple pollutants, and these single-pollutant regulations may be inefficient for a host of reasons. At best, regulation of a particular pollutant may strive for efficiency, given the not-necessarily-efficient regulation of its co-pollutants.

That is the situation we address in this paper. In particular, we investigate the second-best optimal regulation of a pollutant given its abatement-interaction with another controlled pollutant under asymmetric information about firms’ abatement costs. The co-pollutant is regulated, but perhaps not efficiently. Like most of the related literature the pollutants in our analysis interact in terms of abatement: consequently, we are concerned with pollutants that are either complements or substitutes in abatement. Our focus is on optimal instrument choice in this setting, preservation of biodiversity, energy access, and sustainable development. In this paper we focus on regulation in the presence of co-pollutants.

3The theory of second-best Lipsey and Lancaster (1956) suggests in the environmental policy context that inefficient control of a related activity may lead to optimal regulation of a pollutant that deviates from first-best control. For example, Ren et al. (2011) present a general equilibrium model of interacting pollutants and show that second-best optimal tax on a pollutant may deviate from the first-best tax. The design of environmental policy in the presence of other taxes that cause distortions in the economy is another important example of second-best environmental regulation (e.g., Goulder et al. (2010)).
and we derive rules for determining whether a pollutant should be regulated with an emissions tax, an emissions market, or a hybrid market with price controls.

Our work contributes to two literatures, instrument choice under uncertainty and the regulation of multiple pollutants. Like the research interest in regulating multiple interacting pollutants, the challenge of climate change has also intensified research in policy design under the immense uncertainty about the benefits and costs of controlling greenhouse gas emissions. The seminal work of Weitzman (1974) is still relevant, because the marginal damage associated with carbon emissions is almost perfectly flat over a relatively short compliance period (Pizer (2002)). Hence, uncertainty in the costs and benefits of controlling greenhouse gases suggest that a carbon tax is more efficient than carbon trading. However, the preference in some circles for emissions markets over emissions taxes has generated much interest and innovation in hybrid schemes. The most popular form of these hybrids, in the literature and in actual practice, involve tradable emissions permits with price controls. This is the form of hybrid policy that we model. The conceptual foundation for these policies originated with Roberts and Spence (1976), who demonstrated how an emissions market with price controls can outperform a pure emissions tax or a pure permit market. The performance of alternative hybrid policies has been examined theoretically (Grüll and Taschini (2011)), with simulations (Burtraw et al. (2010); Fell and Morgenstern (2010); Fell et al. (2012)), and with laboratory experiments (Stranlund et al. (2014)). Recent theoretical work has also examined technology choices in emissions markets with price controls (Weber and Neuhoff (2010)) and the enforcement of these policy schemes (Stranlund and Moffitt (2014)). However, no one has yet examined the design of emissions market with price controls in the presence of co-pollutants.

While there is a substantial empirical literature on the co-benefits and adverse side-effects of pollutant interactions in climate change policy, the theoretical literature
on regulating multiple interacting pollutants is much smaller, and much of it focuses on integrating markets for co-pollutants. For example, Montero (2001) examines the welfare effects of integrating the policies for two pollutants under uncertainty about abatement costs and imperfect enforcement. Woodward (2011) asks whether firms that undertake a single abatement activity that reduces two kinds of emissions should be able to sell emissions reduction credits for both pollutants. Under complete information, Caplan and Silva (2005) demonstrate that a global market for carbon with transfers across countries can be linked with markets to control more localized pollutants to produce an efficient outcome. In contrast, Caplan (2006) shows that an efficient outcome cannot be achieved with taxes. While most models in this literature are static, the climate change setting has led several authors to examine dynamically efficient paths for the multiple greenhouse gases that contribute to climate change (e.g., Kuosmanen and Laukkanen (2011); Moslener and Requate (2007)). None of these articles consider alternative policy choices under uncertainty in the presence of co-pollutants.

The only published work that we are aware of that examines instrument choice in the presence of multiple interacting pollutants under uncertainty is Ambec and Coria (2013). They focus on the choice between taxes and emissions markets for two interacting pollutants under the assumption that the policies for the two pollutants are jointly optimal. They also consider a hybrid policy due to Weitzman (1978) that can designed to achieve the ex-post efficient outcome (i.e., the efficient outcome after uncertainty has been resolved), and hence, can never be dominated by combinations of taxes and markets that are only optimal ex ante.

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4 Evans (2007) conducts a similar analysis in an unpublished dissertation. In contrast to most of the literature on multiple interacting pollutants, which typically assume one regulator in charge of multiple pollutants or an exogenous regulation of a co-pollutant (as in our case), Ambec and Coria (2015) and Burtraw et al. (2012) examine alternative policies for co-pollutants when they are controlled by different regulators. None of these works include an emissions market with price controls as an alternative policy.
While our work complements Ambec’s and Coria’s, we believe that our approach confronts a more realistic policy environment for two reasons. First, rather than consider the jointly optimal design of policies for interacting pollutants, we investigate the second-best regulation of a single pollutant, given the regulation of its co-pollutant which may not be efficient. Our motivation for taking this approach comes from IPCC (2014, Chapter 3) which treats the problem of co-benefits or adverse side-effects of greenhouse gas regulation as one of choosing the appropriate climate policy, given existing policies for related activities. Second, the hybrid policy of Weitzman (1978) that Ambec and Coria (2013) employ is not used in actual practice. In contrast, the hybrid model of Roberts and Spence (1976) that we use, one that consists of a market with price controls, is very much in line with current policy proposals. See Hood (2010) and Newell et al. (2013) for several examples of recent proposed and implemented greenhouse gas markets with some form of price control. In fact, we are the first to consider the impact of a regulated co-pollutant on the design of an emissions market with price controls.\footnote{There are other differences between our work and Ambec’s and Coria’s. In particular, they extend their base model to examine pollutant interactions in both damages and abatement costs, as well as uncertainty about whether two pollutants are substitutes or complements in abatement. We do not extend our work to these cases, but they may be important extensions for the future.}

Our main contribution is a set of rules for the choice of regulation of a pollutant with a tax, an emissions market, or a market with price controls, given the regulation of its co-pollutant with either a tax or tradable permits. We find that the policy choice for a pollutant is unaffected by its interaction with a co-pollutant that is controlled with a fixed number of tradable permits, because changes in emissions of the primary pollutant do not affect emissions of the co-pollutant. However, when a co-pollutant is controlled with a tax, the instrument choice for the primary pollutant must account for its effect on co-pollutant emissions. In particular, how potential variation in emissions of the primary pollutant affects expected damage of
the co-pollutant becomes an important determinant of the optimal policy choice. If damage from the co-pollutant is strictly convex and the two pollutants are complements, reduced variance of emissions of the primary pollutant decreases expected co-pollutant damage by decreasing the variance of co-pollutant emissions. On the other hand, expected co-pollutant damage increases with a decrease in the variance of the emissions of the primary pollutant if the two pollutants are substitutes, because the variance of co-pollutant emissions increases. Consequently, since the variance of emissions of the primary pollutant is highest under the tax, lower but not zero under a market with price controls, and zero for a pure trading program, complementarity between the two pollutants tends to favor an emissions market (perhaps with price controls), while substitutability tends to favor fixed prices (perhaps as part of a hybrid policy).

Since the rules for determining the optimal policy for a pollutant depend on how its co-pollutant is regulated, many examples exist in which the optimal policy for the primary pollutant changes as the form of regulation of the co-pollutant is changed. For just one example, recall the conventional wisdom that the optimal instrument for carbon emissions is a tax because the marginal damage function is essentially flat in a compliance period. This remains true in the multiple pollutant case as long as the co-pollutant is regulated with tradable permits. However, a constant marginal damage is neither necessary or sufficient for a tax to be optimal when accounting for a co-pollutant that is controlled with a tax. A tax may be the optimal choice if the marginal damage function for the primary pollutant is upward sloping, provided that the pollutants are substitutes in abatement. Moreover, regulation when the primary pollutant has a constant marginal damage must involve a market if the co-pollutant marginal damage is upward sloping and the pollutants are complements. Many such policy reversals are possible, so the intuition about instrument choice that
environmental economists have developed over many years must be modified when policies account for co-pollutants.

The remainder of this chapter proceeds as follows. In the next section we lay out the fundamentals of our model and characterize an optimal tax, an optimal emissions market, and an optimal market with price controls for the primary pollutant, given that the co-pollutant is regulated with tradable permits or an emissions tax. Section 3 contains the main results of the paper, which are the rules for the policy choice of the primary pollutant. Since the policy choice rules are very simple when the co-pollutant is regulated with tradable permits, much of the discussion of this section focuses on the policy choice when the co-pollutant is taxed. Motivated as we are by greenhouse gas control, we use section 4 for a largely graphical analysis of the choice between a tax and a hybrid policy for the primary pollutant when it produces constant marginal damage. We conclude in section 5.

2.2 Optimal policies in the presence of a regulated co-pollutant

The analysis throughout considers regulation of a fixed number of $n$ heterogeneous, risk-neutral firms, each of which emits two pollutants. Both pollutants are uniformly mixed so that they cause damage that depends only on the aggregate amount emitted. In this section we specify the fundamental abatement costs and damage functions for the problem, and then characterize optimal policies for one of the pollutants, given the exogenous regulation of its co-pollutant.

2.2.1 Model fundamentals: Abatement costs and damages

Assume that a firm $i$ emits $q_{ij}$ units of the $j^{th}$ pollutant, $j \in \{1, 2\}$, and that its abatement cost function is $C^i (q_{i1}, q_{i2}, u)$. As is standard, the firm’s abatement cost function is the reduction in its profit from reducing its emissions of either or both of
the pollutants. The firm’s abatement cost function is strictly convex in emissions of the two pollutants and random shocks that affect the abatement costs of all firms are captured by changes in \( u \). This random variable is distributed according to the density function \( f(u) \) on support \([u, \bar{u}]\) with zero expectation.\(^6\) Throughout the analysis firms will face prices for emissions of each of the pollutants. These are competitive prices and they are uniform across firms.\(^7\) Consequently, aggregate abatement costs will be minimized, given aggregate levels of the two pollutants and the realization of \( u \). Let aggregate emissions of both pollutants be \( Q_j = \sum_{i=1}^{n} q_{ij} \), for \( j \in \{1, 2\} \). The minimum aggregate abatement cost function for the industry is \( C(Q_1, Q_2, u) \), which is the solution to:

\[
\min_{\{q_{i1}\}_{i=1}^{n}, \{q_{i2}\}_{i=1}^{n},} \sum_{i=1}^{n} C_i(q_{i1}, q_{i2}, u) \\
\text{subject to} \quad Q_j = \sum_{i=1}^{n} q_{ij}, \text{ for } j \in \{1, 2\}.
\] (2.1)

Like nearly all of the literature on price controls for emission trading, we assume a quadratic form of the aggregate abatement cost function so that aggregate marginal abatement costs for both pollutants are linear with the uncertainty in their intercepts. Accordingly, let the aggregate marginal abatement cost function be

\[
C(Q_1, Q_2, u) = a_0 - (a_1 + u)(Q_1 + Q_2) + \frac{a_2}{2} (Q_1^2 + Q_2^2) - wQ_1 Q_2,
\] (2.2)

where \( a_0 \), \( a_1 \), and \( a_2 \) are positive constants.\(^8\)

---

\(^6\)Introducing abatement cost uncertainty via a common random term is a simplification. Yates (2012) shows how to aggregate idiosyncratic uncertainty in individual abatement costs to characterize uncertainty in an aggregate abatement cost function.

\(^7\)Throughout, we assume that government payments or receipts to and from firms via taxes and government purchases or sales of permits are simple transfers with no real effects.

\(^8\)Ambec and Coria (2013) use a similar abatement cost function. It is straightforward to show that the aggregate abatement cost function (2.2) can be derived from individual firms’ abatement
The constant \( w \) in (2.2) determines whether the two pollutants are substitutes or complements in abatement. To see this write the marginal abatement cost of the \( j^{th} \) pollutant; 
\[-C_j = a_1 + u - a_2 Q_j + w Q_k; \quad j \neq k, \]
where \(-C_j = -\partial C/\partial Q_j\). Note that if the two pollutants are complements at the industry level, an increase in aggregate emissions of pollutant \( k \) will increase aggregate marginal abatement costs for pollutant \( j \). If the two pollutants are substitutes in abatement, an increase in emissions of pollutant \( k \) will lead to a decrease in the marginal abatement costs of pollutant \( j \).

Assume that the Hessian matrix of \( C(Q_1, Q_2, u) \) is positive definite so that \( a_2 > 0 \) and \( a_2^2 - w^2 = (a_2 + w)(a_2 - w) > 0 \). This implies that the aggregate abatement cost function is strictly convex and the abatement interaction term \( w \) is limited by \( a_2 - w > 0 \) and \( a_2 + w > 0 \). Moreover, given a realization of \( u \), assume that the minimum of the aggregate abatement cost function occurs at strictly positive emissions, \( Q_j = (a_1 + u)(a_2 + w)/(a_2^2 - w^2) \), for \( j \in \{1, 2\} \). Given \( a_2 + w > 0 \) and \( a_2^2 - w^2 > 0 \), the abatement-cost-minimizing values of aggregate emissions are strictly positive if and only if \( a_1 + u > 0 \).

Let damage from emissions of the two pollutants take the following quadratic forms:

\[
D^1(Q_1) = d_{11} Q_1 + \frac{d_{12}}{2} Q_1^2; \quad (2.3)
\]
\[
D^2(Q_2) = d_{21} Q_2 + \frac{d_{22}}{2} Q_2^2; \quad (2.4)
\]

with constants \( d_{11} > 0, \ d_{12} \geq 0, \ d_{21} > 0, \) and \( d_{22} \geq 0 \). As noted in the introduction, we do not model a potential interaction between the two pollutants in the damage they cause. Both damage functions are convex, though perhaps weakly convex. We costs that are also quadratic with \( u \) affecting the intercepts of the marginal abatement costs of the two pollutants. The derivation of (2.2) from firms’ abatement costs is available upon request.
assume that it will never be optimal to choose policies that produce zero emissions of either pollutant. In part, this requires that the intercept of the marginal abatement cost function is never below either of the intercepts of the marginal damage functions; that is, \( a_1 + u > d_{11} \) and \( a_1 + u > d_{21} \). The damage functions are known with certainty. Alternatively, we could assume that they are imperfectly known, but that the uncertainty only affects the intercepts of the marginal damage functions and that this uncertainty is uncorrelated with the abatement cost uncertainty. In this case, it is well known that damage uncertainty has no bearing on the optimal choice of policy instruments.

### 2.2.2 Optimal policies, given the regulation of a co-pollutant

From here on let us suppose that the primary pollutant in the analysis is pollutant 1, while the co-pollutant is pollutant 2. In this section, we specify optimal regulations for pollutant 1 given the exogenous regulation of pollutant 2. Control of pollutant 2 is either with an emissions tax \( t_2 \) or competitively-traded permits \( L_2 \). Pollutant 1 is controlled by an endogenous tax, tradable permits, or a hybrid. The hybrid is an emissions permit market with a price ceiling and a price floor that was first proposed by Roberts and Spence (1976). Specifically in our case, a hybrid policy for pollutant 1 features \( \lambda_1 \) permits that are distributed to the firms (free-of-charge), the government commits to selling additional pollutant 1 permits at price \( \tau_1 \), and it commits to buying unused permits from firms at price \( \sigma_1 \). Collectively, the hybrid policy is denoted \( h_1 = (\lambda_1, \tau_1, \sigma_1) \). Note that \( \tau_1 \) provides a price ceiling for pollutant 1 permits, while \( \sigma_1 \) provides the price floor. Clearly, the price controls are restricted by \( \tau_1 \geq \sigma_1 \).

The timing of events in the model is as follows. First, the government chooses and commits to a pollutant 1 policy, given that a regulation for pollutant has already been fixed. The uncertainty about aggregate abatement costs is resolved after the pollutant
1 policy is determined. The firms then choose their emissions. If the pollutant 1 policy involves a market, the firms simultaneously choose their permit holdings and the permit market clears. If the pollution 1 policy also includes price controls, any sales of permits to the government or purchases of permits by the government also occur in this final stage.

To calculate the optimal policies for pollutant 1 given the regulation of pollutant 2, we need aggregate emissions responses for all policy combinations. Of course, if the emissions of both pollutants are controlled with tradable permits, they are fixed at $L_j, j = 1, 2$. However, if both pollutants are controlled by prices, say $p_1$ and $p_2$, then the aggregate emissions responses are determined by equating the aggregate marginal abatement costs for each pollutant to these prices; that is,

$$p_j = -C_j(Q_1, Q_2, u), \text{ for } j \in \{1, 2\}. \quad (2.5)$$

Solving these equations simultaneously for $Q_1$ and $Q_2$ yields the emissions responses

$$Q_j(p_j, p_k, u), \text{ for } j, k \in \{1, 2\} \text{ and } j \neq k. \quad (2.6)$$

As one expects, it is straightforward to show that the own-price effect on aggregate emissions is negative but the cross-price effect depends on whether the pollutants are complements or substitutes. In particular, the cross-price effect is negative if the pollutants are complements, and it is positive if the pollutants are substitutes. If one of the pollutants is controlled by a price and the other with a fixed number of tradable permits $L_k$, then the emissions response of the priced pollutant is the solution to

$$p_j = -C_j(Q_j, L_k, u), \text{ for } j, k \in \{1, 2\} \text{ and } j \neq k, \quad (2.7)$$

resulting in

$$Q_j(p_j, L_k, u), \text{ for } j, k \in \{1, 2\} \text{ and } j \neq k. \quad (2.8)$$
In this case, if the pollutants are complements (substitutes), then an increase in the supply of permits of the co-pollutant leads to an increase (decrease) in emissions of the priced pollutant.

We are now ready to specify optimal hybrid policies for pollutant 1, and we will do so first when pollutant 2 is controlled with $L_2$ tradable permits. To specify the expected social cost function in this context we must first specify values of $u$ where the permit supply and the price ceiling bind together, and where the permit supply and the price floor bind together. Denote these values as $u^{\tau_1}$ and $u^{\sigma_1}$, respectively, where $u^{\tau_1} \geq u^{\sigma_1}$. Using (2.7), $u^{\tau_1}$ and $u^{\sigma_1}$ are the solutions to

$$z = -C_1 (\lambda_1, L_2, u^z), \text{ for } z \in \{\tau_1, \sigma_1\}, \quad (2.9)$$

which implicitly define the cut-off values as

$$u^z = u^z(\lambda_1, z, L_2), \text{ for } z \in \{\tau_1, \sigma_1\}. \quad (2.10)$$

These cut-off values are constrained by $u^{\sigma_1} \geq \underline{u}$ and $u^{\tau_1} \leq \bar{u}$. For values of $u < u^{\sigma_1}$ the price floor binds and the pollutant 1 permit price is equal to $\sigma_1$. For values of $u$ between $u^{\sigma_1}$ and $u^{\tau_1}$ the permit supply binds and the permit price is equal to $-C_1 (\lambda_1, L_2, u)$. Values of $u$ above $u^{\tau_1}$ cause the price ceiling to bind so the permit price is equal to $\tau_1$. Given this price schedule, equilibrium pollutant 1 emissions are

$$Q_1 = \begin{cases} 
Q_1 (\tau_1, L_2, u) \text{ for } u \in [u^{\tau_1}, \bar{u}] \\
\lambda_1 \text{ for } u \in [u^{\sigma_1}, u^{\tau_1}] \\
Q_1 (\sigma_1, L_2, u) \text{ for } u \in [\underline{u}, u^{\sigma_1}].
\end{cases} \quad (2.11)$$

Using (2.10) and (2.11), expected social costs are then
\[ W (\lambda_1, \tau_1, \sigma_1, L_2) \]
\[ = \int_{\tau_1}^{\pi} \left[ C (Q_1 (\tau_1, L_2, u), L_2, u) + D^1 (Q_1 (\tau_1, L_2, u)) + D^2 (L_2) \right] f (u) du \]
\[ + \int_{\sigma_1}^{\tau_1} \left[ C (\lambda_1, L_2, u) + D^1 (\lambda_1) + D^2 (L_2) \right] f (u) du \]
\[ + \int_{\sigma_1}^{\tau_1} \left[ C (Q_1 (\sigma_1, L_2, u), L_2, u) + D^1 (Q_1 (\sigma_1, L_2, u)) + D^2 (L_2) \right] f (u) du. \]

The optimal policy for pollutant 1, given \( L_2 \), is the solution to

\[ \min_{\lambda_1, \tau_1, \sigma_1} W (\lambda_1, \tau_1, \sigma_1, L_2), \text{ subject to } \tau_1 \geq \sigma_1, u^{\tau_1} \leq \bar{u}, u^{\sigma_1} \geq \underline{u}. \] (2.13)

To determine whether the pollutant 1 policy should be a tax, a pure market, or a market with price controls (in the next section), we exploit the fact that choosing an optimal emissions market with price controls admits a pure tax and a pure emissions market (one without price controls) as special cases. For example, if the solution to (2.13) produces \( \tau_1 = \sigma_1 \), then the optimal policy is a pure price instrument because there is no chance that the permit supply will bind. In this case, the model cannot distinguish between a policy that effectively subsidizes firms for reducing their emissions at rate \( \sigma_1 \) and a policy that taxes their emissions at rate \( \tau_1 \). This is because there are a fixed number of firms and tax receipts and subsidy payments are transfers with no real effects. However, since a tax would be superior to a subsidy in an extended model (e.g., with an endogenous number of firms or deadweight costs of public funds), we assume that if the optimal policy is a pure price scheme that it is implemented with a tax. In this case, no emissions permits are issued and the optimal policy is a tax denoted as \( t_1^* (L_2) \). Similarly, if the solution to (2.13) produces \( u^{\tau_1} = \bar{u} \) and \( u^{\sigma_1} = \underline{u} \), then there is no chance that either of the price controls will bind and the optimal policy is a pure emissions market. In
this case, the price controls are disabled and the optimal policy is simply $L_1^*(\bar{T}_2)$ tradable permits. If none of the constraints in (2.13) bind at its solution, then there are strictly positive probabilities that the permit supply, the price ceiling and the price floor will bind. In this case the optimal policy is the hybrid emissions market with price controls, $h_1^*(\bar{T}_2) = (\lambda_1^*(\bar{T}_2), \tau_1^*(\bar{T}_2), \sigma_1^*(\bar{T}_2))$, for which each element has a strictly positive probability of being activated.

Given that confusion may arise about the meaning of a hybrid policy, especially since emissions taxes and pure emissions markets can be viewed as special cases, from here on we only use the term hybrid to indicate a market with price controls, each element of which has a strictly positive probability of being activated.

The specification of the optimal policy for pollutant 1, given that pollutant 2 is controlled with the tax $\bar{t}_2$ proceeds in the same way as when pollutant 2 is controlled with tradable permits. Of course, when pollutant 2 emissions are controlled with a tax they depend on the pollutant 1 policy. Therefore, at $u^{\tau_1}$ and $u^{\sigma_1}$ we have

$$z = -C_1(\lambda_1, Q_2(\lambda_1, \bar{t}_2, u^z) , u^z), \text{ for } z \in \{\tau_1, \sigma_1\},$$

(2.14)

which implicitly define $u^{\tau_1}$ and $u^{\sigma_1}$ as

$$u^z = u^z(\lambda_1, z, \bar{t}_2), \text{ for } z \in \{\tau_1, \sigma_1\}.$$  

(2.15)

For values of $u < u^{\sigma_1}$ the price floor binds and the pollutant 1 permit price is equal to $\sigma_1$. For values of $u$ between $u^{\sigma_1}$ and $u^{\tau_1}$ the permit supply binds and the permit price is equal to $-C_1(\lambda_1, Q_2(\lambda_1, \bar{t}_2, u), u)$. Values of $u$ above $u^{\tau_1}$ cause the price ceiling to bind so the permit price is equal to $\tau_1$. Given this price schedule, equilibrium emissions of both pollutants are

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\[(Q_1, Q_2) = \begin{cases} 
(Q_1 (\tau_1, \bar{t}_2, u), Q_2 (\tau_1, \bar{t}_2, u)) & \text{for } u \in [u_{\tau_1}, \bar{u}] 
(\lambda_1, Q_2 (\lambda_1, \bar{t}_2, u)) & \text{for } u \in [u^{\sigma_1}, u_{\tau_1}] 
(Q_1 (\sigma_1, \bar{t}_2, u), Q_2 (\sigma_1, \bar{t}_2, u)) & \text{for } u \in [\underline{u}, u_{\sigma_1}] 
\end{cases} \quad (2.16)\]

Using (2.15) and (2.16), expected social costs are

\[
W (\lambda_1, \tau_1, \sigma_1, \bar{t}_2) = \int_{u_{\tau_1}(\lambda_1, \tau_1, \bar{t}_2)}^{\bar{u}} \left[ C (Q_1 (\tau_1, \bar{t}_2, u), Q_2 (\tau_1, \bar{t}_2, u), u) + D^1 (Q_1 (\tau_1, \bar{t}_2, u)) \right. \\
\left. + D^2 (Q_2 (\tau_1, \bar{t}_2, u)) \right] f (u) \, du \\
+ \int_{u^{\tau_1}(\lambda_1, \tau_1, \bar{t}_2)}^{\bar{u}} \left[ C (\lambda_1, Q_2 (\lambda_1, \bar{t}_2, u), u) + D^1 (\lambda_1) \right. \\
\left. + D^2 (Q_2 (\lambda_1, \bar{t}_2, u)) \right] f (u) \, du \\
+ \int_{u_{\sigma_1}(\lambda_1, \sigma_1, \bar{t}_2)}^{u_{\tau_1}(\lambda_1, \tau_1, \bar{t}_2)} \left[ C (Q_1 (\sigma_1, \bar{t}_2, u), Q_2 (\sigma_1, \bar{t}_2, u), u) + D^1 (Q_1 (\sigma_1, \bar{t}_2, u)) \right. \\
\left. + D^2 (Q_2 (\sigma_1, \bar{t}_2, u)) \right] f (u) \, du \quad (2.17)\]

The optimal policy for pollutant 1 is the solution to

\[
\min_{\lambda_1, \tau_1, \sigma_1} W (\lambda_1, \tau_1, \sigma_1, \bar{t}_2), \text{ subject to } \tau_1 \geq \sigma_1, \; u_{\tau_1} \leq \bar{u}, \; u^{\sigma_1} \geq \underline{u}. \quad (2.18)\]

Again, binding constraints in this problem indicate the optimality of pure instruments. In particular, if the solution to (2.18) involves \(\tau_1 = \sigma_1\), then the optimal policy is the tax \(t_1^*(\bar{t}_2)\). If \(u_{\tau_1} = \bar{u}\) and \(u^{\sigma_1} = \underline{u}\), then the optimal policy is a pure trading policy with \(L_1^*(\bar{t}_2)\) tradable permits. If none of the constraints bind, then the optimal policy is the hybrid \(h_1^*(\bar{t}_2) = (\lambda_1^*(\bar{t}_2), \tau_1^*(\bar{t}_2), \sigma_1^*(\bar{t}_2))\).
2.3 Policy choice in the presence of a co-pollutant

In this section we present the rules for determining optimal choices from among $t_1^*(\bar{x}_2)$, $L_1^*(\bar{x}_2)$, and $h_1^*(\bar{x}_2)$, for pollutant 2 regulations $\bar{x}_2 \in \{\bar{t}_2, \bar{L}_2\}$. The policy choice rules are presented in two propositions, one for when the co-pollutant is regulated with tradable permits and the other when the co-pollutant is regulated with a tax. The proofs of the propositions derive the policy-choice rules by determining the conditions under which the constraints in (2.13) and (2.18) bind. For example, for $\bar{x}_2 \in \{\bar{t}_2, \bar{L}_2\}$, the conditions under which $\tau_1 = \sigma_1$ reveal when the pure tax $t_1^*(\bar{x}_2)$ produces lower social welfare than a policy with markets, $L_1^*(\bar{x}_2)$ or $h_1^*(\bar{x}_2)$. Likewise, the conditions under which $u^{\tau_1} = u$ and $u^{\sigma_1} = u$ reveal when a pure market $L_1^*(\bar{x}_2)$ dominates a policy with fixed prices, $t_1^*(\bar{x}_2)$ or $h_1^*(\bar{x}_2)$. The conditions under which the constraints in (2.13) or (2.18) do not bind tell us when a hybrid emissions market with price controls $h_1^*(\bar{x}_2)$ dominates a pure tax $t_1^*(\bar{x}_2)$ and a pure market $L_1^*(\bar{x}_2)$.

We begin with the policy choice rules for pollutant 1 when pollutant 2 is regulated with tradable permits. The proof of Proposition 1 is in section A.1 in Appendix A.

**Proposition 1:** If a co-pollutant is regulated with a fixed supply of $\bar{L}_2$ tradable permits, then the optimal regulation of the primary pollutant is the emissions tax $t_1^*(\bar{L}_2)$ if $d_{12} = 0$ while the hybrid policy $h_1^*(\bar{L}_2)$ is optimal if $d_{12} > 0$. A pure emissions market is never optimal.

Thus, when a co-pollutant is controlled with a fixed supply of permits, a pure trading scheme is never optimal and a pure tax is optimal if and only if the marginal damage function for pollutant 1 is flat. In all cases in which the marginal damage for pollutant 1 is upward sloping, the optimal policy is a hybrid with tradable permits, a price ceiling and a price floor, each of which has a positive probability of being

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9There are cases in which pure trading program would be optimal if $a_2 = 0$, but we do not consider this possibility in this paper because we would not be able to guarantee the convexity of the aggregate abatement cost function.
activated. As Roberts and Spence (1976) noted many years ago, the reason a hybrid
dominates a pure tax and a pure market in this setting is that the policy produces a
price schedule that approximates the marginal damage function.

Notice in Proposition 1 that the pollutant 1 policy choice rules when pollutant 2
is controlled with tradable permits do not depend on the abatement interaction term
$w$. Consequently, these rules are the same as in the single-pollutant case. The reason
the abatement interaction does not matter in this case is that pollutant 1 regulations
cannot affect pollutant 2 emissions because they are fixed at $L_2$. (Similarly, the
instrument choice rules in Proposition 1 apply when the co-pollutant is controlled
with a set of binding individual emissions standards). However, when the co-pollutant
is regulated with a tax and the pollutants are linked together in abatement, the
regulation of the primary pollutant affects co-pollutant emissions. Consequently, the
policy choice rules for pollutant 1 incorporate features of this dependence. The proof
of Proposition 2 is in section A.2 in Appendix A.

**Proposition 2:** If a co-pollutant is regulated with an emissions tax $t_2$, then the
optimal regulation of the primary pollutant is the emissions tax $t_1^*(t_2)$ if $d_{22}w/a_2 \leq
-d_{12}$; the pure trading scheme with $L_1^*(t_2)$ permits is optimal if $d_{22}w/a_2 \geq a_2 + w$,
and the hybrid policy $h_1^*(t_2)$ is optimal if $d_{22}w/a_2 \in (-d_{12}, a_2 + w)$.

We will explore this proposition in detail, but first notice in both Propositions 1
and 2 that the specific regulations of the co-pollutant do not appear; that is, $L_2$ is
absent from Proposition 1 and $t_2$ is absent from Proposition 2. It is clear from (2.13)
and (2.18) that the form and level of control of pollutant 2 affects the optimal policies
of pollutant 1; that is, $t_2$ or $L_2$ affects the level of the pollutant 1 tax, the number
of tradable permits, and the elements of a hybrid policy. Moreover, Propositions 1
and 2 imply that the form of pollutant 2 regulation affects the policy choice rules
for pollutant 1 when the two pollutants interact in abatement. However, the policy choice rules do not depend on the levels of control of the co-pollutant. Therefore, we have the following corollary.

**Corollary 1:** While the optimal policies for the primary pollutant depend on the form and stringency of the co-pollutant regulations, the choice among alternative policies for the primary pollutant does not depend on the relative efficiency of the co-pollutant regulations.

Corollary 1 may have an important practical implication for the instrument choice problem for the primary pollutant: the choice is simplified because it does not depend on the stringency of co-pollutant regulation.

Since the policy choice rules in Proposition 2 are somewhat complex, it is worthwhile to analyze them in more detail. In the proposition we have written the rules as dependent on the level of $d_{22}w/a_2$ to emphasize the role that impacts on co-pollutant damage play in the policy choice for the primary pollutant. This term captures the effect of variation in emissions of the primary pollutant on the variation in the marginal damage of the co-pollutant and, as such, indicates how variation in pollutant 1 emissions changes expected co-pollutant damage. To understand this, co-pollutant emissions–given its tax, emissions of pollutant 1 and a realization of $u$–is the solution to $t_2 = -C_2(Q_1, Q_2, u)$. With the aggregate abatement cost function (2.2), it is straightforward to calculate $Q_2(Q_1, t_2, u) = (a_1 + u - t_2 + wQ_1)/a_2$. Note how variation in pollutant 1 emissions affects the variation in pollutant 2 emissions directly according to $w/a_2$. In

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10It is straightforward to show that the rules in Proposition 2 collapse to the same rules in Proposition 1 when the two pollutants do not interact in abatement; that is, when $w = 0$. Thus, when there is not an abatement interaction between the two pollutants, the instrument choice rules for the primary pollutant when the co-pollutant is controlled with a tax are the same as when the co-pollutant is controlled with tradable permits. In turn, these instrument choice rules are the same as in the single-pollutant case.
particular, it is straightforward to show that the variance of pollutant 1 emissions and the variance of pollutant 2 emissions move together if the pollutants are complements, while they move in opposite directions if the pollutants are substitutes. Multiplying \( w/a_2 \) by \( d_{22} \) indicates how the variation of pollutant 1 emissions affects variation in co-pollutant marginal damage and, in turn, expected co-pollutant damages. Of course, if the co-pollutant damage function is linear \( (d_{22} = 0) \) then the variance of co-pollutant emissions has no affect on its expected damage. However, with a strictly convex co-pollutant damage function \( (d_{22} > 0) \), expected damage is increasing in the variance of co-pollutant emissions. Thus, if the pollutants are complements a decrease in the variance of pollutant 1 emissions reduces expected co-pollutant damage by causing a reduction in the variance in co-pollutant emissions. On the other hand, if the pollutants are substitutes a decrease in the variance of pollutant 1 emissions increases expected pollutant 2 damage because the variance of pollutant 2 emissions increases. (A formal demonstration of these results is in section A.3 in Appendix A).

The relationship between the variance of pollutant 1 emissions and expected pollutant 2 damage is an important component of the policy choice problem for pollutant 1, because the variance of pollutant 1 emissions is highest under the tax, lower but not zero under a market with price controls that may bind, and zero for a pure trading program. Thus, if the pollutants are complements (substitutes), expected co-pollutant damage decreases (increases) as we move from a simple tax to a hybrid and then to a pure market. Somewhat loosely, we conclude:

**Corollary 2:** When the co-pollutant is controlled with a tax, complementarity between the two pollutants in abatement tends to favor an emissions market for the primary pollutant, perhaps with price controls. On the other hand, substitutability
between the pollutants tends to favor fixed prices for the primary pollutant, either in the form of a simple tax or as price controls under a hybrid policy.\footnote{A similar conclusion appears to hold in Ambec and Coria (2013) analysis of jointly optimal taxes versus quotas. Their Figures 3 and 4 suggest that substitutability tends to favor emissions taxes, either for both pollutants or as part of mixed scheme of a tax for one pollutant and a quota for the other. Likewise, complementarity tends to favor fixed quotas, either for both pollutants or as part of mixed scheme.}

Given the importance of abatement substitutability and complementarity in the policy choice problem, we now present two corollaries of Proposition 2 that summarize the policy choice rules in the two cases separately.

\textbf{Corollary 3:} If the two pollutants are complements in abatement and the co-pollutant is controlled with a tax, then: (1) A tax for the primary pollutant is optimal if and only if both pollutants have constant marginal damages. (2) The regulation of the primary pollutant must involve a market if the marginal damage function of either pollutant is upward sloping. (3) A pure emissions market is optimal for the primary pollutant if the reduction in the expected damage of the co-pollutant from reducing the variation in emissions of the primary pollutant is large enough.

Part (1) of the corollary follows from the result in Proposition 2 that a tax is optimal if \( d_{22}w/a_2 \leq -d_{12} \). Clearly, given that the pollutants are complements so that \( w > 0 \), the only way for this inequality to hold is if \( d_{12} = d_{22} = 0 \). Part (2) of the corollary follows from the fact that the regulation of pollutant 1 must involve tradable permits if \( d_{22}w/a_2 \leq -d_{12} \) does not hold, which occurs if either \( d_{12} \) or \( d_{22} \) are strictly greater than zero. Part (3) of the corollary follows from the result in Proposition 2 that a pure market for the primary pollutant is optimal if \( d_{22}w/a_2 \geq a_2 + w \). Since both sides of this inequality are positive when the pollutants are complements, the inequality holds only if \( d_{22}w/a_2 \) is large enough.
Corollary 3 reveals that there is only one way in which an emissions tax is the optimal policy choice for pollutant 1 when the two pollutants are complements and pollutant 2 is controlled with a tax. That is when the marginal damage functions of both pollutants are flat. In all other cases the control of the primary pollutant must include a market to limit the variation in emissions of both pollutants. This result is important for the control of carbon emissions whose marginal damage is flat. Without considering the impact of control on emissions of co-pollutants, a flat marginal damage is necessary and sufficient to justify control with a carbon tax. However, as noted in the introduction, important carbon co-pollutants like NO\textsubscript{X}, SO\textsubscript{2} and PM are emitted along with carbon. Thus, reducing carbon can also reduce emissions of these pollutants, suggesting that carbon and these co-pollutants are complements in abatement. Corollary 3 suggests that if these co-pollutants have upward sloping marginal damage functions, then the control of carbon must involve a permit market. It seems likely that such a market would involve price controls, but part (3) of Corollary 3 reveals that it is possible that a pure emissions market for the primary pollutant can be optimal if eliminating the variation in pollutant 1 emissions reduces the expected damage from the co-pollutant enough.

**Corollary 4:** If the two pollutants are substitutes in abatement and the co-pollutant is controlled with a tax, then: (1) A tax is optimal for the primary pollutant if its marginal damage is constant. (2) The regulation of the primary pollutant may involve a market if its marginal damage is increasing: however, the regulation must involve a market if in addition the marginal damage of the co-pollutant is constant. (3) A pure emissions market for the primary pollutant is never optimal.

Part (1) of Corollary 4 also follows from the result in Proposition 2 that a tax for the primary pollutant is optimal if \(d_{22}w/a_2 \leq -d_{12}\). Clearly, given \(w < 0\), this inequality holds if \(d_{12} = 0\). For part (2) of the corollary, note that the inequality may
be reversed if \( d_{12} > 0 \). In fact, if \( d_{22} = 0 \) in addition to \( d_{12} > 0 \), then \( d_{22}w/a_2 > -d_{12} \), indicating that the pollutant 1 policy must include a market. Part (3) of Corollary 4 follows from the result in Proposition 2 that a pure market for the primary pollutant is optimal if \( d_{22}w/a_2 \geq a_2 + w \). This inequality can never hold when the two pollutants are substitutes, because the left side is non-positive while the right side is strictly positive.

In contrast to when the pollutants are complements in abatement, there are several ways that a tax can be the optimal pollutant 1 policy when the pollutants are substitutes. The tax for pollutant 1 is optimal if its marginal damage function is flat, but this is only a sufficient condition, not a necessary one. Consequently, a tax may be optimal if the marginal damage of the primary pollutant is upward sloping as long as the reduction in the expected damage of the co-pollutant induced by increasing the variation of emissions of the primary pollutant is large enough. If the reduction in expected co-pollutant damage is not large enough and marginal damage for the primary pollutant is upward sloping, then control of the primary pollutant must involve a market. In this case, the market must also include price controls, because a pure market is never optimal when the pollutants are substitutes in abatement.

We have focused our discussion of Proposition 2 on whether the pollutants are complements or substitutes in abatement and how policy-induced differences in the variation of emissions of the primary pollutant affects the expected damage from co-pollutant emissions. However, it is clear that these features of the problem do not give us all the information we need to pick the correct policy for the primary pollutant—the slope of its marginal damage function and the slope of the marginal abatement cost function are also important elements of the policy choice problem. The standard intuition about the choice between a pure tax and a pure market in the single-pollutant case is that a more steeply sloped marginal damage function tends to
favor a market, while a more steeply sloped marginal abatement cost function tends to favor a tax. Some of this intuition carries into the choice among the pollutant 1 policies we consider when the co-pollutant is controlled with a tax. For example, a more steeply sloped marginal damage function for the primary pollutant in our case also tends to favor a market. The reason is that limiting the variation in emissions of the primary pollutant reduces expected damage from this pollutant by more when $d_{12}$ is larger.

The effects of steeper marginal abatement costs (a higher value of $a_2$) is a bit more complicated. For the choice between a pure market and a hybrid – which is only relevant when the pollutants are complements (Corollary 4 part (3)) – a higher value of $a_2$ tends to favor a hybrid when co-pollutant damage is strictly convex because $d_{22}w/a_2$ is more likely to fall below $a_2 + w$. In this case, a more steeply sloped marginal abatement cost function limits the reduction in the variation of co-pollutant emissions, thereby limiting the benefit of eliminating the variation in pollutant 1 emissions with a pure market. For the choice between a pure tax and a hybrid the steepness of marginal abatement costs only matters if co-pollutant damage is strictly convex so that $d_{22}w/a_2$ is non-zero. Moreover, the value of $a_2$ does not matter when the pollutants are complements and co-pollutant damage is strictly convex because a pure tax is never optimal in the case (Corollary 3 part (2)). However, when the pollutants are substitutes, a higher value of $a_2$ will limit the reduction in expected co-pollutant damage from increasing the variation in the emissions of the primary pollutant. Consequently, a steeper marginal abatement cost will tend to favor a hybrid over a tax when the two pollutants are substitutes.

2.4 An example motivated by the control of greenhouse gases

To gain additional insight into the policy choice problem we examine a specific example in this section. Given the importance of greenhouse gas control in motivating
our work, we assume throughout the section that the marginal damage function for
the primary pollutant is flat. Of course, this implies that the optimal policy for
the primary pollutant is a tax in the single-pollutant case or when the co-pollutant
is controlled with a fixed number of tradable permits. However, the policy choice
problem is not that simple when two pollutants interact in abatement and the co-
pollutant is controlled with a tax. In particular, given a constant marginal damage
for the primary pollutant and an upward sloping marginal damage function for the
co-pollutant, control of the primary pollutant must involve a market when the two
pollutants are complements, but the pure tax is optimal when the pollutants are
substitutes. The example of this section is designed to illustrate these conclusions.

2.4.1 The pollutants are complements in abatement

Part (2) of Corollary 3 tells us that the optimal policy for pollutant 1 must involve
emissions trading when its marginal damage function is a constant, the marginal
damage function for the co-pollutant is upward sloping, and the two pollutants are
complements in abatement. With the help of Figure (2.1) we will illustrate the policy
choice between an emissions tax and a hybrid policy for pollutant 1 in this setting,
and demonstrate the dominance of a hybrid policy.

In each of the panels of Figure (2.1) we have graphed marginal abatement costs
and marginal damage for the primary pollutant on the left and marginal abatement
costs and marginal damage for the co-pollutant on the right. When the pollutants
are controlled with taxes \( t_1^* (\tilde{t}_2) \) and \( \tilde{t}_2 \), their marginal abatement cost functions are:

\[
-C_1(Q_1, Q_2 (\tilde{t}_2, Q_1, u), u) = \frac{(a_1 + u)(a_2 + w) - w\tilde{t}_2 - (a_2^2 - w^2)Q_1}{a_2}, \tag{2.19}
\]

\[
-C_2(Q_1 (t_1^*(\tilde{t}_2), Q_2, u), Q_2, u) = \frac{(a_1 + u)(a_2 + w) - wt_1^*(\tilde{t}_2) - (a_2^2 - w^2)Q_2}{a_2}. \tag{2.20}
\]

These marginal abatement cost functions are decreasing in their own emissions, but
their intercepts shift with changes in the tax on their co-pollutant and the direction
of the shift depends on whether the pollutants are complements or substitutes. The expected values of these functions, labeled $E(-C_1)$ and $E(-C_2)$, are drawn in Figure (2.1a). The corresponding levels of expected emissions are $Q^0_1$ and $Q^0_2$. We need to be clear here that equating $E(-C_1)$ and $E(-C_2)$ with their corresponding marginal damage functions in Figure (2.1a) does not give us the ex ante optimal levels of emissions. This is because the positions of $E(-C_1)$ and $E(-C_2)$ depend on taxes that differ from their optimal levels.

In fact, note in Figure (2.1a) that the co-pollutant tax $t_2$ is below its marginal damage at $Q^0_2$. (We maintain this assumption throughout this section). In contrast the optimal pollutant 1 tax $t^*_1(t_2)$ is above its marginal damage. To understand why $t^*_1(t_2)$ must be above its marginal damage, consider the expected social cost function when both pollutants are taxed,

$$W(t_1, t_2) = E[C(Q_1(t_1, t_2, u), Q_2(t_1, t_2, u)) + D^1(Q_1(t_1, t_2, u)) + D^2(Q_2(t_1, t_2, u))] .$$

The first order condition for minimizing $W(t_1, t_2)$ with respect to $t_1$ given $t_2$ is

$$E \left[ C_1(\cdot) \frac{\partial Q_1}{\partial t_1} + C_2(\cdot) \frac{\partial Q_2}{\partial t_1} + D^1_1(\cdot) \frac{\partial Q_1}{\partial t_1} + D^2_2(\cdot) \frac{\partial Q_2}{\partial t_1} \right] = 0. \quad (2.21)$$

It is straightforward to show that the emissions responses $Q_j(t_1, t_2, u)$, for $j \in \{1, 2\}$, are linear in both prices, so $\partial Q_j/\partial t_1$, for $j \in \{1, 2\}$, are constants. (See equation (A.26) in Appendix A). Substituting $t^*_1(t_2) = -C_1(\cdot)$ and $\bar{t}_2 = -C_2(\cdot)$ into (2.21) and rearranging terms allows us to characterize the optimal pollutant 1 tax as

$$t^*_1(t_2) = D^1_1(E [Q_1(t^*_1(t_2), \bar{t}_2, u)]) - \bar{t}_2 - D^2_2(E [Q_2(t^*_1(t_2), \bar{t}_2, u)]) \frac{\partial Q_2/\partial t_1}{\partial Q_1/\partial t_1} . \quad (2.22)$$

Since $Q^0_j = E [Q_j(t^*_1(t_2), \bar{t}_2, u)]$ for $j \in \{1, 2\}$ in Figure (2.1a) and the marginal damage of the primary pollutant is constant, we can write (2.22a) as
(a) Taxes for both pollutants when they are complements

(b) Optimal emissions following a positive abatement cost shock

(c) A hybrid policy for pollutant 1 may dominate a tax even though its marginal damage is constant

Figure 2.1: Optimal pollutant 1 policies when it has a constant marginal damage function and the pollutants are complements
\[ t^*_1(\bar{t}_2) = D_1^1 - (\bar{t}_2 - D_2^2(Q_2^0)) \frac{\partial Q_2}{\partial t_1}. \] (2.23)

In (2.23), \( \partial Q_1/\partial t_1 < 0 \). Moreover, \( \partial Q_2/\partial t_1 < 0 \) because the pollutants are complements. Then, since \( \bar{t}_2 < D_2^2(Q_2^0) \) in Figure (2.1a), the second term of the right side of (2.23) is strictly positive, which, in turn, implies \( t^*_1(\bar{t}_2) > D_1^1 \) as we have drawn.

Now, like many graphical analyses of the environmental policy choice problem under uncertainty, imagine that there is a positive shock to abatement costs; that is, the realized value of \( u \) is \( u^+ > 0 \). (It is easy to conduct the following analysis under the assumption that there is a negative shock to abatement costs to illustrate the same results). Then, from (2.19) and (2.20), marginal abatement costs for both pollutants shift upward from their expected values by \( u^+(a_2 + w)/a_2 \) in Figure (2.1b). Recall that \( a_2 + w > 0 \) whether the pollutants are complements or substitutes. Notice how the fact that emissions of the pollutants are complements in this example amplifies the effect of the increase in \( u \). Ultimately the abatement cost shock produces marginal abatement costs \( -C_1^+ = -C_1(Q_1, Q_2(\bar{t}_2, Q_1, u^+), u^+) \) and \( -C_2^+ = -C_2(Q_1(t^*_1(\bar{t}_2), Q_2, u^+), Q_2, u^+) \), as well as \( Q_1^+ \) and \( Q_2^+ \) in Figure (2.1b).

Figure (2.1c) adds a hybrid regulation for pollutant 1. This policy consists of permits \( \lambda_1 \) (set equal to \( Q_1^0 \), although this is not necessary), a price ceiling \( \tau_1 \) and a price floor \( \sigma_1 \). (One should not presume that this is an optimal hybrid policy—it is simply used to illustrate the main results of this section). Given the positive policy shock to the marginal abatement cost functions, the price ceiling will be the binding pollutant 1 instrument resulting in emissions \( Q_1^h \) that are lower than under the tax. This illustrates how a hybrid policy limits the variation in pollutant 1 emissions relative to a tax. In addition, using (2.20), the higher price of pollutant 1 emissions shifts the marginal abatement cost for pollutant 2 down by \( -w/a_2 \), resulting in emissions \( Q_2^h \), which are also lower than pollutant 2 emissions under the optimal pollutant 1
Note how limiting the variation in pollutant 1 emissions limits the variation in co-pollutant emissions, because the two pollutants are complements.

Relative to \( t_1^* (\bar{t}_2) \), the pollutant 1 hybrid has countervailing effects on social costs. The shaded area in the left panel of Figure (2.1c) indicates an increase in social costs associated with pollutant 1 of imposing the pollutant 1 hybrid instead of the tax, which occurs because the hybrid reduces pollutant 1 emissions when its marginal abatement cost function is above its marginal damage. The shaded area in the right panel of Figure (2.1c) is the reduction in social costs associated with pollutant 2 of imposing the pollutant 1 hybrid instead of the tax. It is straightforward to show that total abatement costs of \( Q_1^+ \) and \( Q_2^h \) are the same, so the decrease in social costs associated with pollutant 2 is simply the reduction in damage from lower emissions. It is not apparent in Figure (2.1c) whether the pollutant 1 hybrid leads to higher or lower social costs than the pollutant 1 tax. However, we can show that there always exists a hybrid policy that results in lower social costs than the pollutant 1 tax.

To see this, first write the emissions of both pollutants in terms of the price ceiling for pollutant 1 and the tax for pollutant 2 at the realized value of \( u \) as \( Q_1 (\tau_1, \bar{t}_2, u^+) \) and \( Q_2 (\tau_1, \bar{t}_2, u^+) \). Social costs in terms of these emissions are then

\[
C (Q_1 (\tau_1, \bar{t}_2, u^+), Q_2 (\tau_1, \bar{t}_2, u^+), u^+) + D^1 (Q_1 (\tau_1, \bar{t}_2, u^+)) + D^2 (Q_2 (\tau_1, \bar{t}_2, u^+)).
\]

Differentiate this with respect to \( \tau_1 \) and then substitute \( \tau_1 = -C_1 (\cdot) \) and \( \bar{t}_2 = -C_2 (\cdot) \) into the result to obtain

\[
(D_1^1(Q_1 (\tau_1, \bar{t}_2, u^+)) - \tau_1) \frac{\partial Q_1}{\partial \tau_1} + (D_2^2(Q_2 (\tau_1, \bar{t}_2, u^+)) - \bar{t}_2) \frac{\partial Q_2}{\partial \tau_1}.
\]

At the equilibrium described in Figure (2.1c), \( Q_j (\tau_1, \bar{t}_2, u^+) = Q_j^h \), for \( j \in \{1, 2\} \), so we have

\[
(D_1^1 - \tau_1) \frac{\partial Q_1}{\partial \tau_1} + (D_2^2(Q_2^h) - \bar{t}_2) \frac{\partial Q_2}{\partial \tau_1}.
\]

(2.24)
To understand how social costs respond to the relationship between the price ceiling under the hybrid policy and the emissions tax $t_1^*(\mathring{t}_2)$, calculate

$$\left(t_1^*(\mathring{t}_2) - \tau_1\right) \frac{\partial Q_1}{\partial \tau_1} + (D_2^2(Q_2^h) - D_2^2(Q_2^0)) \frac{\partial Q_2}{\partial \tau_1},$$

(2.25)

by combining (2.24) with (2.23).\(^{12}\) Since $\partial Q_1 / \partial \tau_1 < 0$ and $t_1^*(\mathring{t}_2) - \tau_1 < 0$ at the $(Q_1^h, Q_2^h)$ outcome in Figure (2.1c), the first term of (2.25) is positive, which captures the increase in social costs associated with pollutant 1 of imposing the hybrid instead of the tax. For the second term, $\partial Q_2 / \partial \tau_1 < 0$, because the pollutants are complements in abatement. Then, since $D_2^2(Q_2^h) - D_2^2(Q_2^0) > 0$, the second term of (2.24) is negative, which captures the decrease in social costs associated with pollutant 2 of imposing a hybrid on pollutant 1 rather than a tax. The opposite signs of the two terms in (2.25) indicate the trade-off of imposing a hybrid on pollutant 1 rather than a tax in the situation described in Figure (2.1c).

However, there always exists a hybrid with a price ceiling for the pollutant 1 market that is strictly above the optimal tax that results in lower social costs. To understand why, lower the price ceiling $\tau_1$ to $t_1^*(\mathring{t}_2)$. Then emissions of the co-pollutant increase to $Q_2^+$ in Figure (2.1c) and (2.25) becomes

$$(D_2^2(Q_2^+) - D_2^2(Q_2^0)) \frac{\partial Q_2}{\partial \tau_1}.$$  

(2.26)

(2.26) is unambiguously negative because $D_2^2(Q_2^+) - D_2^2(Q_2^0) > 0$ and $\partial Q_1 / \partial \tau_1 < 0$. This implies that implementing a market with a price ceiling above the optimal pollutant 1 tax to limit the potential variation in pollutant 1 emissions reduces social costs associated with a positive abatement cost shock relative to the optimal pollutant

\(^{12}\)More specifically write (2.23) in terms of $\mathring{t}_2$ and substitute the result into (2.24). Then, because the emissions responses to the taxes are linear, use $\partial Q_j / \partial t_1 = \partial Q_j / \partial \tau_1$ for both $j \in \{1, 2\}$ to complete the derivation.
1 tax. This illustrates how a market with price controls for the primary pollutant can outperform a pure tax even though its marginal damage is flat, as long as the pollutants are complements and the co-pollutant has an upward sloping marginal damage function.\footnote{Part (3) of Corollary 3 suggests that it might be optimal to completely eliminate the variation in pollutant 1 emissions by imposing a pure market instead of a hybrid. This outcome could be illustrated in Figure (2.1c) by setting the price ceiling so high that it is never activated.}

What if the co-pollutant marginal damage was constant instead of upward sloping? Part (1) of Corollary 3 tells us that the tax for the primary pollutant dominates the hybrid policy if the marginal damage functions for both pollutants are constants. In this case, (2.25) reduces to \((t_1^*(\bar{t}_2) - \tau_1)(\partial Q_1/\partial \tau_1)\), which is strictly greater than zero. This shows that imposing a hybrid policy rather than a tax when both pollutants have constant marginal damages produces higher social cost for a given abatement cost shock.

### 2.4.2 The pollutants are substitutes in abatement

Now suppose the two pollutants are substitutes in abatement. As with Figure (2.1), we use Figure (2.2) to illustrate the choice between an emissions tax and a hybrid policy for pollutant 1 in this case. Part (1) of Corollary 4 tells us that the optimal pollutant 1 policy in this situation is a tax, so we will illustrate its dominance over a hybrid policy.

In contrast to the complements case, when the two pollutants are substitutes and the co-pollutant tax is too low the optimal tax on the primary pollutant is also lower than its marginal damage. In Figure (2.2a) we again start with the expected outcome \((Q_1^0, Q_2^0)\) at which the co-pollutant tax is below its marginal damage at its expected emissions. Considering (2.23), note that \(\bar{t}_2 < D_2^2(Q_2^0)\), \(\partial Q_1/\partial t_1 < 0\), and \(\partial Q_2/\partial t_1 > 0\) (because the pollutants are substitutes) imply that \(t_1^*(\bar{t}_2) < D_1^1\), which we have drawn in Figure (2.2a).
(a) Taxes for both pollutants when they are substitutes

(b) Optimal emissions following a positive abatement cost shock

(c) A tax on pollutant 1 dominates a hybrid when its marginal damage is constant and the pollutants are substitutes

Figure 2.2: Optimal pollutant 1 policies when it has a constant marginal damage function and the pollutants are substitutes
A positive shock to abatement costs again shifts the marginal abatement costs for both pollutants upward from their expected values by $u^+(a_2 + w)/a_2$. Since $w < 0$ when the pollutants are substitutes, the upward shift is less than if the pollutants were complements or they did not interact in abatement. Ultimately the abatement cost shock produces marginal abatement costs $-C_1^+$ and $-C_2^+$ and emissions $Q_1^+$ and $Q_2^+$ as shown in Figure (2.2b).

Figure (2.2c) adds a hybrid regulation for pollutant 1. Given the positive abatement cost shock, the price ceiling causes lower emissions $Q_1^h$ than under the tax. From (2.20), the higher pollutant 1 price causes a shift of $-w/a_2 > 0$ in the marginal abatement cost of pollutant 2 to $-C_2^h$, which in turn leads to higher emissions $Q_2^h$. Note how limiting the variation in pollutant 1 emissions with a hybrid policy increases the variation in emissions of the co-pollutant when the pollutants are substitutes. Relative to a tax on the primary pollutant, a hybrid decreases social costs associated with pollutant 1 (the shaded area in the left panel Figure (2.2c)) but increases the social cost associated with pollutant 2 (the shaded area in the right panel Figure (2.2c)). We can see this trade-off in the marginal effect of the binding price ceiling on social costs in equation (2.24), where, in the case of Figure (2.2c), the first term is negative indicating the reduction in social costs associated with pollutant 1, while the second term is positive indicating the increase in social costs associated with pollutant 2.

Despite this tradeoff, part (1) of Corollary 4 tells us that the tax on pollutant 1 dominates a hybrid policy in this case. This is clear from equation (2.25), which is strictly positive because $t_1^*(\bar{T}_2) - \tau_1$ and $\partial Q_1/\partial \tau_1$ are both negative, while $D_2^2(Q_1^h) - D_2^2(Q_1^0)$ and $\partial Q_2/\partial \tau_1$ are both positive. Thus, the hybrid increases social costs associated with a positive abatement cost shock relative to an emissions tax when the pollutants are substitutes. (2.25) remains strictly positive if the marginal damage for the co-pollutant is also constant. In this case the second term of (2.25)
is zero because \( D_2^2(Q_h^h) \) would equal \( D_2^2(Q_h^0) \), but the first term would remain strictly positive. This illustrates the result of Corollary 4 that a hybrid for the primary pollutant can never dominate an emissions tax when its marginal damage is constant and the pollutants are substitutes.

2.5 Conclusion

In this chapter we have examined the problem of regulating a pollutant that interacts in abatement with a separately-regulated co-pollutant in a second-best setting. In particular, we have developed rules for determining whether a pollutant should be controlled with a tax, a permit market, or a market with price controls, given the regulation of its co-pollutant. These rules depend on the relative slopes of the damage functions for both pollutants and the aggregate marginal abatement cost function, whether the pollutants are complements or substitutes in abatement, and whether the co-pollutant is controlled with a tax or tradable permits (but not whether the co-pollutant is regulated efficiently). We have stressed how the alternative policies for the primary pollutant affect the expected damage of the co-pollutant through changes in the variance of co-pollutant emissions, and how this effect helps determine the optimal policy for the primary pollutant. In general, the conventional wisdom about the instrument choice problem must be reconsidered when regulation of the primary pollutant affects the variation of emissions of the co-pollutant. For example, we have illustrated how accounting for carbon co-pollutants like \( \text{NO}_X \), \( \text{SO}_2 \) and PM, which are complements with carbon in abatement, can call for a carbon market when a carbon tax would be the efficient choice in the absence of co-pollutants.

There are, of course, many ways to extend our work. For example, important elements of our results depend on how regulation of the primary pollutant changes emissions of the co-pollutant. We have examined two possible cases, one in which
co-pollutant emissions do not change because they are controlled with an exogenous number of fixed permits, and the other in which emissions of the co-pollutant are variable because they are controlled with a tax. However, other regulations of the co-pollutant will allow it to vary as emissions of the primary pollutant vary. One example, among many, is when the co-pollutant is controlled with a performance standard. Therefore, an interesting area for future work is to determine how the rules for instrument choice change with different regulations of the co-pollutant than those we considered in this paper.

Other elements to consider in future research include examining the consequences of multiple co-pollutants with spatially differentiated damages. While we have focused on the regulation of a pollutant with one co-pollutant, a pollutant may have several co-pollutants, some of which may be complements while others are substitutes. Future work can address how the combination of heterogeneous abatement interactions of multiple co-pollutants affects the design of environmental policies, including the instrument choice problem. Moreover, we have assumed that the two pollutants in our model are uniformly mixed pollutants. However, multiple interacting pollutants may cause spatially heterogeneous damages. For example, while carbon is a uniformly mixed pollutant, its co-pollutants NO\(_X\), SO\(_2\) and PM are non-uniformly mixed pollutants. This suggests that efficient regulation of a pollutant that has non-uniformly mixed co-pollutants may have a spatial component. These and other characteristics of co-pollutants are important factors to consider in designing efficient environmental regulation.
CHAPTER 3
SECOND BEST REGULATION IN THE PRESENCE OF CO-POLLUTANTS

3.1 Introduction

This paper examines the effects of the interactions between two pollutants in abatement and damages on the form and performance of optimal price-based policies for a pollutant when its co-pollutant is regulated with either an emissions tax or pure permit trading. The study of regulation in the presence of co-pollutants is motivated by efforts to control greenhouse gases (GHGs) to mitigate the threat of climate change. Most economic activities that emit GHGs also produce other pollutants like NO\textsubscript{X}, SO\textsubscript{2}, and PM simultaneously. Thus, efforts to reduce GHGs can also decrease the emissions of these other pollutants. These ancillary benefits of controlling GHGs are one of the co-benefits of climate policies. The Intergovernmental Panel on Climate Change (IPCC) has reviewed research that assesses various GHG mitigation measures and show the potential co-benefits and adverse side-effects in Chapter 6 of IPCC (2014).\textsuperscript{1} It has concluded that the co-benefits of climate policies from co-pollutants can be significant. There are many studies that focus on evaluating the co-benefits of climate policies (Burtraw et al. (2003); Nemet et al. (2010); Groosman et al. (2011); Parry et al. (2015)).\textsuperscript{2} Although

\textsuperscript{1}Apart from the effects on air pollution and health damages through co-pollutants, IPCC (2014) shows various other co-benefits and side-effects of climate policies like the effects on energy security and access, employment, biodiversity conservation, water use, and food security.

\textsuperscript{2}For instance, Nemet et al. (2010) survey empirical research on the co-benefits of climate policies from co-pollutants and find that the estimated co-benefits have a range of $2 to $196/t\text{CO}_2$ with
the results of these studies often show complementary interactions between multiple pollutants in abatement, this is not always the case. In some cases, the efforts to reduce one pollutant can increase the emissions other pollutants. For instance, using scrubbers to reduce the emissions of SO$_2$ and PM consumes large amount of energy and thus causes increases in emissions of CO$_2$ (Moslener and Requate (2007); Ambec and Coria (2013)). Leightner (1999) finds that increasing the concentration of SO$_2$ by 1% can reduce the average concentration of NO$_X$ by 1.54% to 4.03%, holding inputs and output constant. While this substitution relationship comes from the technical relationship in a single source, the interaction between pollutants can also come from more complicated interactions between multiple sources in a market. For example, Ren et al. (2011) find that increasing biofuel use to replace fossil fuels can reduce the emissions of CO$_2$, but may also increase nitrogen leaching.

Multiple pollutants can interact not only in abatement but also in the damage that they cause. Kortenkamp et al. (2009) show that chemicals combined with each other can produce effects that are larger than the separate effects of each chemical compound. This implies that the benefits of reducing one pollutant can sometimes be much greater than the damage that it causes alone. Like the interaction in abatement, however, the interaction of pollutants in damages can also work in the opposite direction; that is, the interaction among pollutants may reduce damages. For instance, in the case of climate change mitigation, the sulphate aerosol formed from SO$_2$ emissions can have a net cooling effect because they interact with clouds to reflect sunlight back to space (Forster et al. (2007); Ramanathan and Feng (2008); Pleijel (2009); IPCC (2014)). Fuglestvedt et al. (2009) show the possible

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“double warming” effects of controlling SO\textsubscript{2} in shipping sector, one effect is from CO\textsubscript{2} while the other is due to the reduction of SO\textsubscript{2}. Shindell and Faluvegi (2010) find similar results from controlling SO\textsubscript{2} and NO\textsubscript{X} in coal-fired power plants.

The regulation of multiple interacting pollutants has received much attention among researchers. For the efficient control of all pollutants, regulations for each pollutant should be determined jointly to maximize the social net benefits from all pollutants, and thus efficient controls should reflect the interactions of pollutants in abatement and damages. Caplan and Silva (2005) show that joint permit markets for controlling local and global pollutants from a single source can achieve a Pareto optimum. Ambec and Coria (2013) derive the efficient combination of policies for two pollutants that interact in abatement costs and damages. However, since each environmental regulation usually focuses on its own target pollutant, it’s not likely that the interactions of pollutants in abatement and damages are considered on every regulation. This implies that one or both of the regulations may not be set efficiently from the perspective of controlling all pollutants. The theory of the second-best (Lipsey and Lancaster (1956)) implies in this case that the optimal regulation of a pollutant will deviate from its first-best control. We will take this second-best approach to model control of two pollutants that interact in both abatement costs and damages. For environmental regulations, there are many studies which take the second-best approach. Many of them focus on controlling only one pollutant and consider the effects of other distortionary taxes in the economy like labor and capital taxes (Bovenberg and Goulder (1996); Goulder et al. (1999); Bento and Jacobsen (2007); Crago and Khanna (2014)). However, Ren et al. (2011) consider the case where two processes produce the same output but different pollutants. When the tax for one of the two pollutants is not set efficiently, they derive the optimal tax for the other pollutant and show that the optimal tax deviates from its first-best level.
Our model considers two pollutants which interact in abatement costs and damages. In order for efficient outcomes to be achieved, it is required that the regulations of each pollutant be determined jointly. In this case, we derive the ex ante optimal emissions and prices of both pollutants which jointly minimize expected social costs, and we use these ex ante optimal emissions and prices as benchmarks throughout the analysis.

Although there are two regulations, it is unlikely that efficient outcomes are achieved because a single environmental regulation usually focuses only on its target pollutant. Thus, we assume that the regulation of the co-pollutant is exogenously given and it may deviate from its ex ante optimal price or emissions. We consider both an emissions tax and a pure permit market for the exogenous regulation of the co-pollutant. In this situation, we derive the form and performance of optimal price-based regulations for the primary pollutant under asymmetric information about abatement costs. Since the regulation for the co-pollutant is likely to deviate from its ex ante optimal level, we will take the second-best approach to interpret the results of our model. For the regulation of the primary pollutant we consider an emissions tax, pure permit trading, and a hybrid policy which imposes a price ceiling and a price floor on a permit market. We find that, given the regulation of the co-pollutant, all the optimal regulations for the primary pollutant produce the same expected emissions. However, inefficient regulation of the co-pollutant leads the optimal control of the primary pollutant to deviate from its ex-ante optimal level. We show that this deviation depend on 1) the interactions of the two pollutants in abatement costs and damages, 2) the deviation of the regulation of the co-pollutant from its ex ante optimal level, and 3) the form of the regulation for the co-pollutant.

After presenting general results about the how the stringency and form of the regulation of the co-pollutant affects the second-best control of the primary pollutant,
we consider several special cases to illustrate the results. For the simplest case, the co-pollutant is regulated by tradable permits and the interaction of the two pollutants is only in abatement. Then, if the two pollutants are complements in abatement and the number of tradable permits for the co-pollutant is higher (lower) than its ex ante optimal emissions, then the optimal regulations of the primary pollutant produce expected emissions that are higher (lower) than its ex ante optimal emissions. Thus, in this case, the direction of the deviation of the second-best control of the primary pollutant from its ex ante optimal level is the same as the direction of the deviation of control of the co-pollutant from its ex ante optimal level. When, the pollutants are substitutes in abatement the deviations of control from ex ante optimal values of the two pollutant are in opposite directions.

Cases are more complicated when the co-pollutant is regulated by an emissions tax, because its emissions are variable and affected by the emissions of the primary pollutant. This leads to contrary results when the co-pollutant is regulated with tradable permits. When the pollutants interact only in abatement, if the two pollutants are complements and a low (high) tax for the co-pollutant results in the expected emissions that are higher (lower) than its ex ante optimal emissions, the optimal regulations for the primary pollutant produce expected emissions that are lower (higher) than its ex ante optimal emissions as long as its marginal damage function is upward sloping. When the two pollutants are substitutes, the deviations in expected emissions from their ex ante optimal values of the two pollutants work in the same direction. That these results are opposite of the case when the co-pollutant is controlled with tradable permits highlights the importance of the form of regulation of the co-pollutant on the second-best control of the primary pollutant.

Not surprisingly, matters are even more complicated for two pollutants that interact in both abatement and damages. However, our analysis makes it clear that
it is whether the two pollutants are complements or substitutes in social costs that partly determines the qualitative impact of inefficient regulation of the co-pollutant on the second-best control of the primary pollutant, not necessarily the specific interactions in abatement costs and damages.

The remainder of this chapter is organized as follows. In the next section we specify our model and derive the ex-ante optimal emissions and prices for both pollutants. In section 3, we derive the forms of a hybrid price and quantity regulation for the the primary pollutant, given that the co-pollutant is regulated with tradable permits or an emissions tax. Then, as special cases of a hybrid policy, we derive the optimal emissions tax and tradable permits. In section 4, we show the expected performance of all the optimal price-based regulations. We conclude in section 5.

### 3.2 Model

#### 3.2.1 Abatement costs and damage functions

Consider $n$ heterogeneous and risk-neutral firms in an industry. Each firm emits two pollutants which interact in abatement costs and damages. Each pollutant is assumed to be uniformly mixed so that damages depend only on the aggregate emissions of each pollutant. In this subsection, first we will specify the structure of individual firms’ abatement costs, aggregate abatement costs and the damage functions. Then, in the next subsection, we will derive the ex-ante optimal emissions and the ex-ante optimal prices for both pollutants. These values minimize the expected sum of aggregate abatement costs and damages and will be used as benchmarks throughout the analysis.

##### 3.2.1.1 Firm’s abatement cost function

Suppose that firm $i$ emits $q_{ij}$ units of pollutant $j$, $j = 1,2$. We define a firm’s abatement costs as the sacrificed profits from reducing its emissions for either or both
of the pollutants. Following much of the literature, we assume that firm $i$’s abatement cost function, denoted by $C^i(q_{i1}, q_{i2}, u)$, has the following form:

$$C^i(q_{i1}, q_{i2}, u) = c_{i0} - (c_{i1} + u)(q_{i1} + q_{i2}) + \frac{c_{i2}}{2}(q_{i1}^2 + q_{i2}^2) - w_i q_{i1} q_{i2}, \quad (3.1)$$

with the constants $c_{i0} > 0$, $c_{i1} > 0$, and $c_{i2} > 0$. The coefficient $w_i$ represents the interaction of the two pollutants in abatement costs. If $w_i > 0$, then the pollutants are complements in abatement and if $w_i < 0$, then they are substitutes in abatement. Random shocks that affect abatement costs are captured by changes in the random variable $u$, which is distributed according to the probability density function $f(u)$ over the support of $[u, \bar{u}]$ with zero expectation. It is assumed that the random shock $u$ is common to all firms in the industry. Firm $i$’s abatement cost function is strictly convex in the two pollutants and thus, $c_{i2}^2 - w_i^2 > 0$. By the definition of abatement costs, in the absence of regulation, firm $i$’s emissions for both pollutants are determined so that abatement costs are minimized. We assume that this minimum occurs at strictly positive levels of emissions for both pollutants; that is, $q_{i1} = q_{i2} = (c_{i1} + u) / (c_{i2} - w_i) > 0$. To guarantee this last assumption, we assume $c_{i1} + u > 0$ for all realizations of $u$ and $c_{i2} - w_i > 0$. In turn, $c_{i2}^2 - w_i^2 > 0$ implies $c_{i2} + w_i > 0$.

### 3.2.1.2 Aggregate abatement cost function

Firms in an industry will face uniform prices for the emissions of the pollutants, which can have the form of either a competitive permit price or an emissions tax. This guarantees that aggregate abatement costs are minimized for levels of aggregate emissions of both pollutants and a realization of $u$. Let the aggregate emissions of both pollutants be $Q_j = \sum_{i=1}^{n} q_{ij}$, $j = 1, 2$. The (minimum) aggregate abatement cost function, $C(Q_1, Q_2, u)$, is the solution to the following problem:
\[
\min_{\{q_{i1}\}_{i=1}^n, \{q_{i2}\}_{i=1}^n} \sum_{i=1}^n C^i (q_{i1}, q_{i2}, u)
\]

subject to \(Q_j = \sum_{i=1}^n q_{ij}\), for \(j = 1, 2\). \hfill (3.2)

It can be shown that when firms’ abatement cost functions have the form of (3.1), the aggregate abatement cost function has the form

\[
C (Q_1, Q_2, u) = a_0 - (a_1 + u) (Q_1 + Q_2) + \frac{a_2}{2} (Q_1^2 + Q_2^2) - wQ_1Q_2,
\]

(3.3)

with the constants \(a_0 > 0, a_1 > 0, \) and \(a_2 > 0\). In addition, it is straightforward to show that given the structure and assumptions of (3.1), \(a_1 + u > 0, a_2^2 - w^2 > 0, a_2 - w > 0, \) and \(a_2 + w > 0\). Therefore, the aggregate abatement cost function has the same structure as individual firms’ abatement cost functions. First, it is quadratic and strictly convex in both pollutants \((a_2^2 - w^2 > 0)\). Second, a random variable \(u\) that represents an industry-level random shock causes the marginal aggregate abatement cost functions to move up or down in parallel. Next, without any regulations, aggregate emissions of both pollutants are strictly positive for any realization of \(u\) \((Q_1 = Q_2 = (a_1 + u) / (a_2 - w) > 0)\). As before, these two factors limit the interaction parameter \(w\) to \(a_2 - w > 0\) and \(a_2 + w > 0\). Finally, the interaction parameter \(w\) determines whether the two pollutants are complements or substitutes in abatement. To see this write the marginal abatement cost of the \(j^{th}\) pollutant; \(-C_j = a_1 + u - a_2 Q_j + w Q_k, j, k = 1, 2 \) and \(j \neq k\), where \(-C_j = -\partial C / \partial Q_j\). Note that if the two pollutants are complements at the industry level \((w > 0)\), an increase in aggregate emissions of pollutant \(k\) will increase aggregate marginal abatement costs for pollutant \(j\). Thus, given a price for pollutant \(j\), the emissions of pollutant \(j\) will also increase, which implies that the

3The proofs of the structure of the aggregate abatement cost function and our assertions about its characteristics are omitted here to save space. They are available upon request.
emissions of both pollutants tend to move in the same direction if they are complements. If the two pollutants are substitutes in abatement \((w < 0)\), an increase in emissions of pollutant \(k\) will lead to a decrease in the marginal abatement costs of pollutant \(j\). Thus, we can infer that both pollutants will move in the opposite directions.

### 3.2.1.3 Damage function

We assume that a damage function has the following form:

\[
D(Q_1, Q_2) = d_{11}Q_1 + \frac{d_{12}}{2}Q_1^2 + d_{21}Q_2 + \frac{d_{22}}{2}Q_2^2 + vQ_1Q_2, \quad (3.4)
\]

with \(d_{11} > 0, d_{12} \geq 0, d_{21} > 0, \) and \(d_{22} \geq 0\). The parameter \(v\) in (3.4) represents the interaction of the two pollutants in damages. Note that we do not impose any assumptions on the curvature of the damage function at this point. However, since it is required to limit the interaction parameter \(v\) to guarantee the existence of optimal policies, some restrictions will be specified in the next subsection. To understand how the interaction of the two pollutants in damages works, consider the marginal damage function of pollutant \(j\), 

\[
D_j = d_{j1} + d_{j2}Q_j + vQ_k, \quad j, k = 1, 2 \text{ and } j \neq k,
\]

where \(D_j = \partial D/\partial Q_j\). Then, if \(v\) is positive, increases in emissions of pollutant \(k\) will cause the marginal damage of pollutant \(j\) to increase. Thus, it is desirable that the emissions of pollutant \(j\) should be reduced more. In this sense, we can say that if \(v > 0\), the two pollutants are substitutes in damages. On the other hand, for negative values of \(v\), increases in emissions of pollutant \(k\) will decrease the marginal damage of pollutant \(j\). Thus it is permissible to emit more of pollutant \(j\). That is, if \(v < 0\), then the two pollutants are complements in damages.\(^4\)

\(^4\)In terms of abatement, if \(v > 0\), abatement in one pollutant shifts down the marginal damage of the other pollutant because of some tradeoffs between two pollutants. Thus, the marginal benefit of the reduction in the other pollutant decreases. On the other hand, if \(v < 0\), abatement in one
either pollutant cannot be optimal, which requires in part that the intercept of the marginal abatement cost function will never be below either of the intercepts of the marginal damage functions; that is, \( a_1 + u > d_{11} \) and \( a_1 + u > d_{21} \). Finally, we assume that there is no uncertainty about the damage function.

### 3.2.2 Ex ante optimal emissions and prices

In this subsection, we will derive the ex-ante optimal emissions and prices of both pollutants. These values minimize the expected sum of aggregate abatement costs and damages. However, since we focus on the second-best situation where one of the two pollutants is not controlled efficiently, all the main results in this paper will be described by the deviations from these ex ante optimal emissions or prices.

#### 3.2.2.1 Ex ante optimal emissions

Define the ex ante optimal emissions as the amounts of emissions for both pollutants which minimize the expected sum of the aggregate abatement costs and damages and denote them as \( \hat{Q}_1 \) and \( \hat{Q}_2 \). These are the solutions to

\[
\min_{Q_1, Q_2} E \left[ C(Q_1, Q_2, u) + D(Q_1, Q_2) \right].
\] (3.5)

From (3.3) and (3.4), the ex ante optimal emissions of both pollutants are

\[
\hat{Q}_1 = \frac{(a_1 - d_{11})(a_2 + d_{22}) - (a_1 - d_{21})(v - w)}{(a_2 + d_{12})(a_2 + d_{22}) - (v - w)^2};
\] (3.6)

\[
\hat{Q}_2 = \frac{(a_1 - d_{21})(a_2 + d_{12}) - (a_1 - d_{11})(v - w)}{(a_2 + d_{12})(a_2 + d_{22}) - (v - w)^2}.
\] (3.7)

Pollutant shifts up the marginal damage of the other pollutant because of the combined effects of joint abatement. Thus, the marginal benefit of abatement in the other pollutant also increases.
To be sure that (3.6) and (3.7) minimize (3.5), we assume that the denominator of (3.6) and (3.7) is strictly positive, that is:

\[(a_2 + d_{12})(a_2 + d_{22}) - (v - w)^2 > 0.\]

Since this term is the determinant of the Hessian of \(E[C(Q_1, Q_2, u) + D(Q_1, Q_2)]\), this assumption is implied by the strict convexity of \(E[C(Q_1, Q_2, u) + D(Q_1, Q_2)]\). This assumption also limits the interaction parameter \(v\) by

\[w - \sqrt{(a_2 + d_{12})(a_2 + d_{22})} < v < w + \sqrt{(a_2 + d_{12})(a_2 + d_{22})}.\] (3.8)

Moreover, since we want the ex ante optimal emissions to be strictly positive, we add the assumptions that the numerators of (3.6) and (3.7) are strictly positive.

To understand how the damage caused by one pollutant affects the ex ante optimal emissions for both pollutants, consider the effects of a change in the intercept of the marginal damage of one pollutant on the ex ante optimal emissions of the other pollutant:

\[\frac{\partial \hat{Q}_j}{\partial d_{j1}} = \frac{-(a_2 + d_{k2})}{(a_2 + d_{12})(a_2 + d_{22}) - (v - w)^2} < 0, \text{ } j, k = 1, 2, \text{ and } j \neq k; \] (3.9)

\[\frac{\partial \hat{Q}_k}{\partial d_{j1}} = \frac{v - w}{(a_2 + d_{12})(a_2 + d_{22}) - (v - w)^2}, \text{ } j, k = 1, 2, \text{ and } j \neq k. \] (3.10)

(3.9) shows that increases in the marginal damage function of one pollutant decreases its own ex ante optimal emissions. However, (3.10) implies that the effect on the other pollutant depends on the sign of \(v - w\), which represents the net interaction of the two pollutants both in abatement and damages. If \(v - w < 0\), then the two pollutants are complements in social costs and thus an increase in the intercept of the marginal damage function of one pollutant leads to a reduction in the ex ante emissions.
optimal emissions for both pollutants. If the pollutants are substitutes in social costs 
\((v - w > 0)\), an increase in the intercept of the marginal damage function of one 
pollutant leads to a decrease in the ex ante optimal emissions for that pollutant, 
but an increase in the ex ante optimal emissions of its co-pollutant. Note that these 
relationships do not necessarily require that both pollutants should be complements 
(or substitutes) in abatement and damages at the same time. That is, the overall 
interaction effect on social costs can imply complements (or substitutes) even if the 
pollutants are substitutes (or complements) in either abatement costs or damages.

3.2.2.2 Ex ante optimal prices

To derive the ex ante optimal prices for emissions of both pollutants, we need 
to specify how the aggregate emissions of each pollutant respond to changes in their 
prices. For arbitrary prices for emissions of both pollutants, \(P_1\) and \(P_2\), aggregate 
emissions are determined so that the marginal aggregate abatement costs of each 
pollutant are equal to each price. That is,

\[
P_j = -C_j(Q_1, Q_2, u), \quad j = 1, 2.
\] (3.11)

By solving these equations for \(Q_1\) and \(Q_2\) simultaneously, we have

\[
Q_j(P_1, P_2, u) = \frac{(a_1 + u)(a_2 + w) - a_2P_j - wP_k}{a_2^2 - w^2}, \quad j, k = 1, 2, \quad j \neq k.
\] (3.12)

Note that the own-price effect on aggregate emissions is negative but the cross-price 
effect depends on whether the pollutants are complements or substitutes in 
abatement. In particular, the cross-price effect is negative if the pollutants are 
complements in abatement \((w > 0)\), and it is positive if the pollutants are 
substitutes in abatement \((w < 0)\).
Denote the ex ante optimal prices for emissions of both pollutants as \( \hat{P}_1 \) and \( \hat{P}_2 \). Then they are solutions to

\[
\min_{P_1, P_2} E \left[ C \left( Q_1 (P_1, P_2, u), Q_2 (P_1, P_2, u) \right) + D \left( Q_1 (P_1, P_2, u), Q_2 (P_1, P_2, u) \right) \right].
\] (3.13)

Substituting (3.3), (3.4), and (3.12) into (3.13) and solving the problem for \( P_1 \) and \( P_2 \) yields:

\[
\hat{P}_1 = d_{11} + \frac{d_{12} [(a_1 - d_{11})(a_2 + d_{22}) + (a_1 - d_{21}) w] + v [(a_1 - d_{21}) a_2 - (a_1 - d_{11}) (v - w)]}{(a_2 + d_{12}) (a_2 + d_{22}) - (v - w)^2}, \] (3.14)

\[
\hat{P}_2 = d_{12} + \frac{d_{22} [(a_1 - d_{21})(a_2 + d_{12}) + (a_1 - d_{11}) w] + v [(a_1 - d_{11}) a_2 - (a_1 - d_{21}) (v - w)]}{(a_2 + d_{12}) (a_2 + d_{22}) - (v - w)^2}. \] (3.15)

Now consider the relationship between the ex ante optimal emissions and the ex ante optimal prices. First, by substituting (3.14) and (3.15) into (3.12), we can find that, given \( \hat{P}_1 \) and \( \hat{P}_2 \), the expected aggregate emissions are equal to the ex ante optimal emissions:

\[
E \left[ Q_j \left( \hat{P}_1, \hat{P}_2, u \right) \right] = \hat{Q}_j, \ j = 1, 2.
\]

In addition, by rearranging (3.14) and (3.15) and using (3.6) and (3.7), we derive the following relationships:

\[
\hat{P}_j = d_{j1} + d_{j2} \hat{Q}_j + v \hat{Q}_k = E \left[ D_j \left( Q_1 \left( \hat{P}_1, \hat{P}_2, u \right), Q_2 \left( \hat{P}_1, \hat{P}_2, u \right) \right) \right], \ j, k = 1, 2, \ j \neq k,
\]

which implies that the ex ante optimal prices \( \hat{P}_1 \) and \( \hat{P}_2 \) are equal to the expected marginal damages of each pollutant at the ex ante optimal emissions, \( \hat{Q}_1 \) and \( \hat{Q}_2 \).

### 3.3 Optimal price-based regulations

From now on we will denote the primary pollutant as pollutant 1 and the co-pollutant as pollutant 2. In this section we derive the optimal forms of
price-based regulations for pollutant 1 when the regulation of pollutant 2 is given exogenously. While pollutant 2 is regulated through either an emissions tax $\bar{t}_2$ or competitively tradable permits $\bar{L}_2$, pollutant 1 is regulated by an emissions tax, pure permit trading, or a hybrid policy which imposes a price ceiling and a price floor on a permit market that was suggested by Roberts and Spence (1976). Although we can set up the problem for all policy combinations of both pollutants and derive the optimal regulation for pollutant 1, instead we first derive the optimal hybrid policy for pollutant 1 and then derive its optimal emissions tax and its optimal number of permits by exploiting the well-known fact that a hybrid policy can encompass an emissions tax and pure permit trading as special cases.

Under the hybrid policy the government issues total permits $\lambda_1$ and distributes them across regulated firms free of charge. Firms trade their permits in a competitive permit market. When the demand of permits is so high that a competitive permit price would be greater than $\tau_1$, firms can buy additional permits from the government at the price of $\tau_1$. Thus a competitive permit price will bind at the price ceiling $\tau_1$. On the other hand, if the demand of permits is so low that a competitive permit price would be lower than $\sigma_1$, firms will abate more than required and sell unused permits back to the government at the price of $\sigma_1$. Thus a competitive permit price will bind at a price floor $\sigma_1$. We will denote the hybrid policy as $h_1 = (\lambda_1, \tau_1, \sigma_1)$. The price controls are restricted by $\tau_1 \geq \sigma_1$.

To derive the optimal hybrid policy for pollutant 1 given the regulation of pollutant 2, we need to specify how the aggregate emissions of both pollutants respond to all policy combinations. Obviously, if both pollutants are controlled through pure permit markets, aggregate emissions will be fixed at the issued permits for each pollutant. For the case where both pollutants are regulated through emissions taxes, the responses are given by (3.12). So the remaining case is when one pollutant is controlled through a price instrument and the other one is limited by a quantity instrument. Suppose
that pollutant \( j \) is regulated by a price \( P_j \) and pollutant \( k \) is regulated by tradable permits \( L_k \), \( j, k = 1, 2 \) and \( j \neq k \). Then, the aggregate emissions of pollutant \( j \) is the solution to

\[
P_j = -C_j (Q_j, L_k, u), \quad j, k = 1, 2, \ j \neq k.
\]

(3.16)

Applying (3.3) to (3.16) and solving for \( Q_j \) yields

\[
Q_j (P_j, L_k, u) = \frac{a_1 + u - P_j + wL_k}{a_2}, \quad j, k = 1, 2, \ j \neq k.
\]

(3.17)

Note that if both pollutants are complements in abatement \((w > 0)\), increases in permits of pollutant \( k \) will lead to increases in the aggregate emissions of pollutant \( j \). On the other hand, if they are substitutes in abatement \((w < 0)\), they will move in the opposite direction.

### 3.3.1 When the co-pollutant is regulated by tradable permits \( \bar{L}_2 \)

We now derive the optimal hybrid policy for pollutant 1 given tradable permits \( \bar{L}_2 \) for pollutant 2. To specify the expected social costs in this situation, we need to find the cut-off values of the random variable \( u \) where supplied permits \( \lambda_1 \) and either of the price controls bind together. Denote these values of \( u \) as \( u^{\tau_1} \) and \( u^{\sigma_1} \), respectively, with \( u^{\tau_1} \geq u^{\sigma_1} \). From (3.16), \( u^{\tau_1} \) and \( u^{\sigma_1} \) are defined as the solutions to

\[
z = -C_1 (\lambda_1, \bar{L}_2, u^z), \quad \text{for} \ z \in \{\tau_1, \sigma_1\}.
\]

So the cut-off values of \( u \) are

\[
u^z (\lambda_1, z, \bar{L}_2) = -a_1 + z + a_2 \lambda_1 - w\bar{L}_2, \quad \text{for} \ z \in \{\tau_1, \sigma_1\},
\]

(3.18)

and they are restricted by \( u^{\tau_1} \leq u \) and \( u^{\sigma_1} \geq u \). For \( u \leq u^{\sigma_1} \) the competitive permit price binds at the price floor \( \sigma_1 \) and for \( u \geq u^{\tau_1} \) the permit price binds at the price
ceiling \( \tau_1 \). For each case, aggregate emissions of pollutant 1 are determined from (3.17). For \( u^{\sigma_1} \leq u \leq u^{\tau_1} \) a competitive permit price is equal to \(-C_1(\lambda_1, \bar{L}_2, u)\) and aggregate emissions of pollutant 1 are fixed at the supplied permits \( \lambda_1 \). Thus, aggregate emissions of both pollutants can be summarized as

\[
(Q_1, Q_2) = \begin{cases} 
(Q_1(\tau_1, \bar{L}_2, u), \bar{L}_2) & \text{for } u \in [u^{\tau_1}, \bar{u}] \\
(\lambda_1, \bar{L}_2) & \text{for } u \in [u^{\sigma_1}, u^{\tau_1}] \\
(Q_1(\sigma_1, \bar{L}_2, u), \bar{L}_2) & \text{for } u \in [\bar{u}, u^{\sigma_1}] 
\end{cases} 
\]  

(3.19)

Given (3.19), the expected social costs under a hybrid policy for pollutant 1 given tradable permits \( \bar{L}_2 \) for pollutant 2 are

\[
W(\lambda_1, \tau_1, \sigma_1, \bar{L}_2) = \int_{u^{\tau_1}}^{\bar{u}} C(Q_1(\tau_1, \bar{L}_2, u), \bar{L}_2, u) + D(Q_1(\tau_1, \bar{L}_2, u), \bar{L}_2) \ f(u) \ du \\
+ \int_{u^{\sigma_1}}^{u^{\tau_1}} C(\lambda_1, \bar{L}_2, u) + D(\lambda_1, \bar{L}_2) \ f(u) \ du \\
+ \int_{\bar{u}}^{u^{\sigma_1}} C(Q_1(\sigma_1, \bar{L}_2, u), \bar{L}_2, u) + D(Q_1(\sigma_1, \bar{L}_2, u), \bar{L}_2) \ f(u) \ du. 
\]  

(3.20)

The optimal hybrid policy for pollutant 1 given tradable permits \( \bar{L}_2 \) is the solution to

\[
\min_{\lambda_1, \tau_1, \sigma_1} W(\lambda_1, \tau_1, \sigma_1, \bar{L}_2), \ \text{subject to } \tau_1 \geq \sigma_1, \ u^{\tau_1} \leq \bar{u}, \ u^{\sigma_1} \geq \bar{u}. 
\]  

(3.21)

Binding constraints in (3.21) determine the optimal regulation for pollutant 1 given tradable permits \( \bar{L}_2 \). If none of the constraints bind, then a permit market with a price ceiling and a price floor is optimal, denoted by \( h_1^* = (\lambda_1^* (\bar{L}_2), \tau_1^* (\bar{L}_2), \sigma_1^* (\bar{L}_2)) \). However, if the solution satisfies the first constraint with equality, that is, \( \tau_1 = \sigma_1 \), then the optimal policy is a price instrument because the probability that the permit supply will bind becomes zero. In this case, however, it is not possible for our model
to distinguish between subsidy $\sigma_1$ on abatement and emissions tax $\tau_1$. However, since a tax would be preferred to a subsidy in an extended model, we assume that if a price instrument is optimal, then it is implemented in the form of an emissions tax, denoted by $t_1^* (\bar{L}_2)$. Finally, if the last two constraints bind, that is, $u^{\tau_1} = \bar{u}$ and $u^{\sigma_1} = \bar{u}$, then a pure permit market without price controls is optimal, denoted by $L_1^* (\bar{L}_2)$, because the probability that any of the price controls will bind becomes zero.

In section B.1 in Appendix B, we derive the optimal hybrid policy for pollutant 1 given tradable permits $L_2$, $h_1^* = (\lambda_1^* (L_2), \tau_1^* (L_2), \sigma_1^* (L_2))$, as follows:

$$\lambda_1^* (\bar{L}_2) = \hat{Q}_1 - \frac{v - w}{a_2 + d_{12}} \left( \bar{L}_2 - \hat{Q}_2 \right) + \frac{E [u | u^{\sigma_1} \leq u \leq u^{\tau_1}]}{a_2 + d_{12}}; \quad (3.22)$$

$$\tau_1^* (\bar{L}_2) = \hat{P}_1 + \frac{a_2 v + d_{12} w}{a_2 + d_{12}} \left( \bar{L}_2 - \hat{Q}_2 \right) + \frac{d_{12} E [u | u^{\tau_1} \leq u \leq \bar{u}]}{a_2 + d_{12}}; \quad (3.23)$$

$$\sigma_1^* (\bar{L}_2) = \hat{P}_1 + \frac{a_2 v + d_{12} w}{a_2 + d_{12}} \left( \bar{L}_2 - \hat{Q}_2 \right) + \frac{d_{12} E [u | u \leq u \leq u^{\sigma_1}]}{a_2 + d_{12}}, \quad (3.24)$$

where $E [u | u^{\sigma_1} \leq u \leq u^{\tau_1}]$, $E [u | u^{\tau_1} \leq u \leq \bar{u}]$, and $E [u | u \leq u \leq u^{\sigma_1}]$ are conditional expectations of $u$. As explained above, to find the optimal permits for pollutant 1 given tradable permits $\bar{L}_2$, $L_1^* (\bar{L}_2)$, we can set $u^{\tau_1} \geq \bar{u}$ and $u^{\sigma_1} \leq \bar{u}$ to disable the price controls, which implies that $E [u | u^{\sigma_1} \leq u \leq u^{\tau_1}] = 0$. Thus, from (3.22) we find that

$$L_1^* (\bar{L}_2) = \hat{Q}_1 - \frac{v - w}{a_2 + d_{12}} \left( \bar{L}_2 - \hat{Q}_2 \right). \quad (3.25)$$

On the other hand, to derive the optimal emissions tax for pollutant 1 given tradable permits $L_2$, $t_1^* (\bar{L}_2)$, we can set $u^{\tau_1} \leq \bar{u}$ to disable the permit supply, which implies that $E [u | u^{\tau_1} \leq u \leq \bar{u}] = 0$. (We could also disable the permit supply by setting $u^{\sigma_1} \geq \bar{u}$.) Thus, from (3.23) we find that

$$t_1^* (\bar{L}_2) = \hat{P}_1 + \frac{a_2 v + d_{12} w}{a_2 + d_{12}} \left( \bar{L}_2 - \hat{Q}_2 \right). \quad (3.26)$$

\(^5(3.22) \text{ through } (3.24) \text{ are not exact 'solutions', because the right-hand sides of the equations include the optimal policy variables in the conditional expectations.}\)
Note that all policy variables in (3.22) through (3.26) deviate from their ex ante optimal emissions or ex ante optimal price ($\hat{Q}_1$ or $\hat{P}_1$) unless $L_2 = \hat{Q}$. Deviations of these policy variables depend on 1) the interactions of the two pollutants and 2) the deviation of the tradable permits $\bar{L}_2$ for pollutant 2 from its ex ante optimal emissions $\hat{Q}_2$.

To understand how the optimal policy variables deviate from their ex ante optimal values, suppose that tradable permits of pollutant 2 exceed the ex ante optimal emissions; that is, $L_2 > \hat{Q}_2$. First, the deviations of the quantity variables such as $\lambda_1^* (\bar{L}_2)$ and $L_1^* (\bar{L}_2)$ are determined by the sign of $v - w$. As explained in subsection 3.2.2, this term represents the net interaction of both pollutants in abatement and damages together. More specifically, $v - w$ represents parallel movement of the expected marginal social costs of pollutant 1 due to changes in $Q_2$ given emissions of pollutant 1, that is:

$$\frac{\partial E [C_1 (Q_1, Q_2, u) + D_1 (Q_1, Q_2)]}{\partial Q_2} = v - w.$$  

By the definition of the ex ante optimal emissions, both $\hat{Q}_1$ and $\hat{Q}_2$ satisfy

$$E [C_1 (\hat{Q}_1, \hat{Q}_2, u) + D_1 (\hat{Q}_1, \hat{Q}_2)] = 0.$$  

Thus, if two pollutants are complements in social costs ($v - w < 0$), then $L_2 > \hat{Q}_2$ implies that $E [C_1 (\hat{Q}_1, L_2, u) + D_1 (\hat{Q}_1, L_2)] < 0$. In this case, the optimal emissions for pollutant 1 should be adjusted to reflect the inefficiency caused by the over-supply of tradable permits for pollutant 2. The marginal social cost of pollutant 1 is an increasing function of pollutant 1, given emissions of pollutant 2; that is, $\partial^2 [C (Q_1, Q_2, u) + D (Q_1, Q_2)] / \partial Q_1^2 = a_2 + d_{12} > 0$. This implies that the optimal emissions for pollutant 1 should exceed its ex ante optimal emissions $\hat{Q}_1$. By the same reasoning, we can infer that if the two pollutants are substitutes in
social costs \((v - w > 0)\), then \(\lambda_1^* (\bar{L}_2)\) and \(L_1^* (\bar{L}_2)\) should be less than the ex ante optimal emissions \(\hat{Q}_1\).

Next, from (3.23), (3.24), and (3.26) we know that the deviations of the price variables such as \(\tau_1^* (\bar{L}_2)\), \(\sigma_1^* (\bar{L}_2)\), and \(t_1^* (\bar{L}_2)\) are determined by the sign of \(a_2 v + d_{12} w\). This term represents parallel movement of the expected marginal social costs of pollutant 1 due to changes in \(Q_2\), given a price for the emissions of pollutant 1; that is,

\[
\frac{\partial E [C_1 (Q_1 (P_1, Q_2, u), Q_2, u) + D_1 (Q_1 (P_1, Q_2, u), Q_2)]}{\partial Q_2} = \frac{\partial E [D_1 (Q_1 (P_1, Q_2, u), Q_2)]}{\partial Q_2} = \frac{a_2 v + d_{12} w}{a_2}.
\]

When pollutant 1 is regulated by a price instrument, the emissions of pollutant 1 are adjusted so that its expected marginal abatement cost is always equal to the price \(P_1\). Thus, changes in \(Q_2\) can affect only the expected marginal damage of pollutant 1. Interestingly, when the interactions in abatement costs and damages show the same relationship, the two interaction effects on marginal damage work in opposite directions. For instance, suppose that the two pollutants are complements in both abatement and damages; that is, \(w > 0\) and \(v < 0\). Then, increases in the emissions of pollutant 2 affect the marginal damage of pollutant 1 through two channels. First, it will shift down the marginal damage function of pollutant 1 \((v < 0)\). However, increases in the emissions of pollutant 2 lead to increases in the emissions of pollutant 1 because they are complements in abatement \((w > 0)\). As a whole, the marginal damage function of pollutant 1 itself moves down, but the additional emissions of pollutant 1 increases marginal damage. The overall effect is indeterminate and it is determined by the relative magnitudes of each effect. Since

\[
E \left[ C_1 \left( Q_1 \left( \hat{P}_1, \hat{Q}_2, u \right), \hat{Q}_2, u \right) + D_1 \left( Q_1 \left( \hat{P}_1, \hat{Q}_2, u \right), \hat{Q}_2 \right) \right] = 0, \text{ if } a_2 v + d_{12} w > 0,
\]

\(\bar{L}_2 > \hat{Q}_2\) implies that

\[
E \left[ C_1 \left( Q_1 \left( \hat{P}_1, \bar{L}_2, u \right), \bar{L}_2, u \right) + D_1 \left( Q_1 \left( \hat{P}_1, \bar{L}_2, u \right), \bar{L}_2 \right) \right] > 0.
\]
That is, given $L_2 > \hat{Q}_2$, the ex ante optimal price for pollutant 1 cannot be efficient.

Since
\[
\frac{\partial}{\partial P_1} \left[ C_1(Q_1(P_1, Q_2, u), Q_2, u) + D_1(Q_1(P_1, Q_2, u), Q_2) \right] / \partial P_1 = -(a_2 + d_{12}) / a_2 < 0,
\]
the optimal price for pollutant 1 should be greater than its ex ante optimal price for
\[
a_2 v + d_{12} w > 0.
\]
By the same logic, we can infer that the optimal price should be less than its ex ante optimal price for
\[
a_2 v + d_{12} w < 0.
\]

### 3.3.2 When the co-pollutant is regulated by an emissions tax $\bar{t}_2$

Next consider the case in which pollutant 2 is regulated by an emissions tax $\bar{t}_2$. The derivation of the optimal regulations for pollutant 1 given the tax $\bar{t}_2$ will follow the same steps as in subsection 3.3.1. Thus, we begin by obtaining the optimal hybrid policy for pollutant 1, $h_1^*(\bar{t}_2) = (\lambda_1^*(\bar{t}_2), \tau_1^*(\bar{t}_2), \sigma_1^*(\bar{t}_2))$. The cut-off values, $u^{\tau_1}$ and $u^{\sigma_1}$, are solutions to

\[
z = -C_1(\lambda_1, Q_2(\bar{t}_2, \lambda_1, u^z), u^z), \ z \in \{\tau_1, \sigma_1\}.
\]

(3.27)

Explicitly, $u^{\tau_1}$ and $u^{\sigma_1}$ have the following expressions:

\[
u^z(\lambda_1, z, \bar{t}_2) = -a_1 + \frac{a_2 z}{a_2 + w} + (a_2 - w) \lambda_1 + \frac{w \bar{t}_2}{a_2 + w}, \ z \in \{\tau_1, \sigma_1\},
\]

and they are restricted by $u^{\tau_1} \leq u$ and $u^{\sigma_1} \leq u$. For $u \leq u^{\sigma_1} (u^{\tau_1} \leq u)$, the competitive permit price of pollutant 1 will bind at the price floor $\sigma_1$ (the price ceiling $\tau_1$), and the emissions of both pollutants will be determined by (3.12). For $u^{\sigma_1} \leq u \leq u^{\tau_1}$, the competitive permit price is $-C_1(\lambda_1, Q_2(\bar{t}_2, \lambda_1, u), u)$ and the aggregate emissions of pollutant 1 binds at the permit supply $\lambda_1$. Thus, the emissions of both pollutants can be summarized as
Given (3.29), the expected social costs can be expressed as

\[ (Q_1, Q_2) = \begin{cases} 
(Q_1 (\tau_1, \bar{t}_2, u), Q_2 (\tau_1, \bar{t}_2, u)) & \text{for } u \in [u^{\tau_1}, \bar{u}] \\
(\lambda_1, Q_2 (\bar{t}_2, \lambda_1, u)) & \text{for } u \in [u^{\tau_1}, u^{\tau_1}] \cdot \\
(Q_1 (\lambda_1, \bar{t}_2, u), Q_2 (\sigma_1, \bar{t}_2, u)) & \text{for } u \in [u, u^{\sigma_1}] 
\end{cases} \tag{3.29} \]

The optimal policy for pollutant 1, given the tax \( \bar{t}_2 \), are solutions to

\[
\min_{\lambda_1, \tau_1, \sigma_1} W (\lambda_1, \tau_1, \sigma_1, \bar{t}_2) \quad \text{subject to } \tau_1 \geq \sigma_1, u^{\tau_1} \leq \bar{u}, u^{\sigma_1} \geq u. \tag{3.31} \]

In section B.2 in Appendix B, we derive the optimal hybrid policy for pollutant 1, given \( \bar{t}_2 \), \( h_1^*(\bar{t}_2) = (\lambda_1^*(\bar{t}_2), \tau_1^*(\bar{t}_2), \sigma_1^*(\bar{t}_2)) \), as follows:

\[
\lambda_1^*(\bar{t}_2) = \lambda_1^0 + \frac{Y}{a_2X + wY} (\bar{t}_2 - \bar{P}_2) + \frac{[a_2 (a_2 + w) - Y] E [u | u^{\sigma_1} \leq u \leq u^{\tau_1}]}{a_2X + wY}; 
\tag{3.32} \]

\[
\tau_1^*(\bar{t}_2) = \tau_1^0 - \frac{a_2Y + wX}{a_2X + wY} (\bar{t}_2 - \bar{P}_2) + \frac{(a_2 + w) (a_2d_{12} + vw + Y) E [u | u^{\tau_1} \leq u \leq \bar{u}]}{a_2X + wY}; 
\tag{3.33} \]

\[
\sigma_1^*(\bar{t}_2) = \sigma_1^0 - \frac{a_2Y + wX}{a_2X + wY} (\bar{t}_2 - \bar{P}_2) + \frac{(a_2 + w) (a_2d_{12} + vw + Y) E [u | u \leq u \leq u^{\sigma_1}]}{a_2X + wY}, \tag{3.34} \]

where \( X = a_2 (a_2 + d_{12}) + w (v - w), Y = a_2v + d_{22}w, \) and \( a_2X + wY > 0.6 \)

---

6 The strict convexity of \( C (Q_1, Q_2, u) + D (Q_1, Q_2) \) implies \( a_2X + wY > 0 \). To see this, we note that \( a_2X + wY > 0 \) is required for the strict convexity of \( C (Q_1, Q_2 (\bar{t}_2, Q_1, u), u) + D (Q_1, Q_2 (\bar{t}_2, Q_2, u)) \)
The term \( X = a_2(a_2 + d_{12}) + w(v - w) \) represents the effects of \( Q_1 \) on its own marginal social costs:

\[
\frac{\partial[C_1(Q_1,Q_2(\bar{t}_2,Q_1,u),u) + D_1(Q_1,Q_2(\bar{t}_2,Q_1,u))]}{\partial Q_1} = \frac{a_2(a_2 + d_{12}) + w(v - w)}{a_2}. \tag{3.35}
\]

To exclude the case where increasing the emissions of pollutant 1 decreases its own marginal social cost, we assume that \( X = a_2(a_2 + d_{12}) + w(v - w) > 0 \). Then, given the range of \( v \) as (3.8), \( X > 0 \) requires

\[
a_2^2(a_2 + d_{12}) - w^2(a_2 + d_{22}) \geq 0 \tag{3.36}
\]

(We will use this condition in analyzing how the interaction in damage between the two pollutants affects the regulation of pollutant 1, given the tax for pollutant 2). On the other hand, \( Y = a_2v + d_{22}w \) can have any sign but it is limited by \( X > 0 \) and \( a_2X + wY > 0 \).

As we did for the case in which pollutant 2 is regulated by tradable permits \( \bar{L}_2 \), we can derive the optimal number of tradable permits for pollutant 1 under a pure trading scheme, \( L_1^*(\bar{t}_2) \), and the optimal emissions tax, \( t_1^*(\bar{t}_2) \), as special cases of a hybrid policy, resulting in:

\[
L_1^*(\bar{t}_2) = \hat{Q}_1 + \frac{Y}{a_2X + wY}(\bar{t}_2 - \hat{P}_2); \tag{3.37}
\]

\[
t_1^*(\bar{t}_2) = \hat{P}_1 - \frac{a_2Y + wX}{a_2X + wY}(\bar{t}_2 - \hat{P}_2). \tag{3.38}
\]

Moreover, since the range of \( v \) where \( a_2X + wY > 0 \) is larger than (3.8), which is the range of \( v \) where \( C(Q_1,Q_2,u) + D(Q_1,Q_2) \) is strictly convex in \( Q_1 \) and \( Q_2 \), the strict convexity of \( C(Q_1,Q_2,u) + D(Q_1,Q_2) \) implies \( a_2X + wY > 0 \). Throughout, we restrict the range of \( v \) on (3.8).
Note that as the case of given tradable permits \( \bar{L}_2 \), all policy variables in (3.32) through (3.38) will deviate from the ex ante optimal emissions or prices, \( \hat{Q}_1 \) or \( \hat{P}_1 \), unless \( \bar{t}_2 = \hat{P}_2 \). The deviations of optimal policy variables also depend on 1) the interaction effects and 2) the deviation of the emissions tax \( \bar{t}_2 \) from its ex ante optimal price \( \hat{P}_2 \). However, in this case, the multiplier terms of \( Y/(a_2X + wY) \) and \( (a_2Y + wX)/(a_2X + wY) \) include parameters associated with pollutant 2. When pollutant 2 is regulated by a tax, the emissions of pollutant 2 is not fixed and rather it’s affected by the emissions of pollutant 1. This implies that choosing optimal policy variables for pollutant 1 should consider the effects on the marginal social cost of pollutant 2 of changes in the emissions of pollutant 1.

To see how the interactions of the two pollutants and the deviation of the emissions tax \( \bar{t}_2 \) from its ex ante optimal price affects the optimal policy variables for pollutant 1, suppose that the emissions tax for pollutant 2 is set too low compared to its ex ante optimal price; that is, \( \bar{t}_2 < \hat{P}_2 \). First, the deviations of quantity variables such as \( \lambda^*_1(\bar{t}_2) \) and \( L^*_1(\bar{t}_2) \) depend on the sign of \( Y = a_2v + d_{22}w \), which captures the qualitative effects of changes in the price for pollutant 2 on the marginal social cost of pollutant 1, that is,

\[
\frac{\partial}{\partial P_2} \left[ \frac{\partial [C(Q_1, Q_2(P_2, Q_1, u), u) + D(Q_1, Q_2(P_2, Q_1, u))]}{\partial Q_1} \right] = -\frac{a_2v + d_{22}w}{a_2^2}. \tag{3.39}
\]

Note that the term in the inner brackets includes marginal abatement costs and marginal damages of both pollutants, because the emissions of pollutant 2 are also affected by the emissions of pollutant 1. When this term is evaluated at \( \hat{Q}_1 \) and \( \hat{P}_2 \), it becomes zero:

\[
\partial \left[ C(\hat{Q}_1, Q_2(\hat{P}_2, \hat{Q}_1, u), u) + D(\hat{Q}_1, Q_2(\hat{P}_2, \hat{Q}_1, u)) \right] /\partial Q_1 = 0.
\]

Thus, if \( Y = a_2v + d_{22}w > 0 \) and \( \bar{t}_2 < \hat{P}_2 \), at \( \hat{Q}_1 \) and \( \bar{t}_2 \) we have
\[ \frac{\partial}{\partial Q_1} \left[ C \left( \hat{Q}_1, Q_2 \left( \hat{t}_2, \hat{Q}_1, u \right), u \right) + D \left( \hat{Q}_1, Q_2 \left( \hat{t}_2, \hat{Q}_1, u \right) \right) \right] / \partial Q_1 > 0. \]

Since the optimal emissions of pollutant 1 are chosen so that its expected marginal social cost is zero and \( \partial^2 \left[ C \left( Q_1, Q_2 \left( P_2, Q_1, u \right), u \right) + D \left( Q_1, Q_2 \left( P_2, Q_1, u \right) \right) \right] / \partial Q_1^2 = (a_2 X + wY) / a_2 > 0, \) the optimal emissions of pollutant 1 should be less than its ex ante optimal emissions \( \hat{Q}_1. \) On the other hand, if \( Y = a_2 v + d_{22} w < 0 \) and \( \bar{t}_2 < \hat{P}_2, \) the term in the inner bracket becomes negative. Thus the optimal emissions of pollutant 1 should be greater than its ex ante optimal emissions \( \hat{Q}_1. \)

Next, from (3.33), (3.34) and (3.38), we know that the deviations of price variables such as \( \tau^*_1 \left( \bar{t}_2 \right), \sigma^*_1 \left( \bar{t}_2 \right), \) and \( t^*_1 \left( \bar{t}_2 \right) \) are determined by the sign of \( a_2 Y + wX, \) which is implied by the effect of changes in the price for pollutant 2 on the marginal social cost of the price for pollutant 1; that is,

\[ \frac{\partial}{\partial P_2} \left[ \frac{\partial}{\partial P_1} \left[ C \left( Q_1 \left( P_1, P_2, u \right), Q_2 \left( P_1, P_2, u \right), u \right) + D \left( Q_1 \left( P_1, P_2, u \right), Q_2 \left( P_1, P_2, u \right) \right) \right] \right] \]

\[ = \frac{a_2 Y + wX}{(a_2^2 - w^2)^2}. \]

As above, note that the term in the inner brackets includes the marginal abatement cost and marginal damage of both pollutants. By the definition of the ex ante optimal prices, when this term is evaluated at \( \hat{P}_1 \) and \( \hat{P}_2, \) it is zero; that is,

\[ \frac{\partial}{\partial P_1} \left[ C \left( \hat{Q}_1 \left( \hat{P}_1, \hat{P}_2, u \right), Q_2 \left( \hat{P}_1, \hat{P}_2, u \right), u \right) + D \left( Q_1 \left( \hat{P}_1, \hat{P}_2, u \right), Q_2 \left( \hat{P}_1, \hat{P}_2, u \right) \right) \right] = 0. \]

Thus, if \( a_2 Y + wX > 0 \) and \( \bar{t}_2 < \hat{P}_2 \) at \( \hat{P}_1 \) and \( \bar{t}_2, \) we have:

\[ \frac{\partial}{\partial P_1} \left[ C \left( Q_1 \left( \hat{P}_1, \bar{t}_2, u \right), Q_2 \left( \hat{P}_1, \bar{t}_2, u \right), u \right) + D \left( Q_1 \left( \hat{P}_1, \bar{t}_2, u \right), Q_2 \left( \hat{P}_1, \bar{t}_2, u \right) \right) \right] < 0. \]
Since

\[
\frac{\partial^2}{\partial P_2^2} \left[ C \left( Q_1 (P_1, P_2, u), Q_2 (P_1, P_2, u) \right) + D \left( Q_1 (P_1, P_2, u), Q_2 (P_1, P_2, u) \right) \right]
\]

\[
= \frac{a_2 X + w Y}{(a_2^2 - w^2)^2} > 0,
\]

the optimal price for pollutant 1 should be greater than its ex ante optimal price \( \hat{P}_1 \). On the other hand, if \( a_2 Y + w X < 0 \) and \( \bar{t}_2 < \hat{P}_2 \), then the optimal price for pollutant 1 should be less than its ex ante optimal price \( \hat{P}_1 \).

### 3.4 Environmental performance

In this section, we will compare the expected emissions of both pollutants under each of the optimal price-based regulations for pollutant 1, given the regulations of pollutant 2. As we did in previous sections, we will use the ex ante optimal emissions for each pollutant as a benchmark. We will focus on how the deviations of the expected emissions of each pollutant from its own ex ante optimal level are related to each other and how the interactions in abatement and damages affect these relationships. We begin this section with the following findings:

**Finding 1:** Given regulation of pollutant 2, all the optimal price-based regulations for pollutant 1 produce the same expected emissions; that is,

\[
E \left[ Q_1 (h_1^* (\bar{x}), \bar{x}, u) \right] = E \left[ Q_1 (t_1^* (\bar{x}), \bar{x}, u) \right] = L_1^* (\bar{x}), \quad \bar{x} \in \{ \bar{L}_2, \bar{t}_2 \}.
\]

In addition, when pollutant 2 is regulated by an emissions tax, the expected emissions of pollutant 2 are the same among all the optimal price-based regulations for pollutant 1; that is,

\[
E \left[ Q_2 (h_2^* (\bar{t}_2), \bar{t}_2, u) \right] = E \left[ Q_2 (t_2^* (\bar{t}_2), \bar{t}_2, u) \right] = E \left[ Q_2 (L_1^* (\bar{t}_2), \bar{t}_2, u) \right] = E \left[ Q_2 (L_2^* (\bar{t}_2), \bar{t}_2, u) \right].
\]
Derivations of these findings can be found in sections B.3 and B.4 in Appendix B. The first part of Finding 1 implies that, given the regulation for pollutant 2, all the optimal price-based regulations for pollutant 1 produce the same expected emissions for pollutant 1. The only difference among them is the variation around the same expected outcome. This also implies that different regulations of pollutant 2 can cause the expected emissions of pollutant 1 to vary although pollutant 1 is regulated optimally. It is because, depending on the regulation of pollutant 2, the effective channels through which the two pollutants interact with each other are different. Finally, if pollutant 2 is regulated by a tax, then all the optimal price-based regulations for pollutant 1 also produce the same expected emissions of pollutant 2.

To simplify the notation from here on, we denote the expected emissions of both pollutants as:

\[
E(Q^*_1(\bar{x})) = E(Q_1(h^*_1(\bar{x}), \bar{x}, u)) = E(Q_1(t^*_1(\bar{x}), \bar{x}, u)) = L^*_1(\bar{x}), \quad \bar{x} \in \{\bar{L}_2, \bar{t}_2\};
\]

\[
E(Q^*_2(\bar{t}_2)) = E(Q_2(h^*_1(\bar{t}_2), \bar{t}_2, u)) = E(Q_2(t^*_1(\bar{t}_2), \bar{t}_2, u)) = E(Q_2(L^*_1(\bar{t}_2), \bar{t}_2, u)).
\]

Using this notation, we now describe the interaction of the two pollutants in terms of the expected emissions as follows.

**Finding 2:** When pollutant 2 is regulated by tradable permits $\bar{L}_2$, the relationship between the expected emissions of pollutant 1 and its ex ante optimal value can be characterized as

\[
E[Q^*_1(\bar{L}_2)] - \hat{Q}_1 = -\frac{v - w}{a_2 + d_{12}} (\bar{L}_2 - \hat{Q}_2).
\]

On the other hand, when pollutant 2 is regulated by an emissions tax, the relationship between the expected emissions of pollutant 1 and its ex ante optimal value can be characterized as
\[ E \left[ Q_1^* (\bar{t}_2) \right] - \hat{Q}_1 = -\frac{a_2 v + d_{22} w}{a_2 (a_2 + d_{12}) + w (v - w)} \left\{ E \left[ Q_2^* (\bar{t}_2) \right] - \hat{Q}_2 \right\}, \quad (3.42) \]

provided that \( X = a_2 (a_2 + d_{12}) + w (v - w) \neq 0. \)

Derivations of these findings can be found in section B.5 in Appendix B. When tradable permits for pollutant 2 or the expected emissions of pollutant 2 under an emissions tax are equal to the ex ante optimal emissions \( \hat{Q}_2 \), all the optimal price-based regulations make the expected emissions of pollutant 1 equal to its ex ante optimal emissions. However, when the regulation of pollutant 2 fails to achieve its ex ante optimal emissions, the expected emissions of pollutant 1 also deviates from its ex ante optimal level. The deviation of the expected emissions of pollutant 1 depend on 1) the deviation of the expected emissions of pollutant 2 from its ex ante optimal emissions, 2) the interactions of the two pollutants in abatement and damage, and 3) whether pollutant 2 is regulated by tradable permits or by an emissions tax. We will illustrate these relationships in the next two subsections.

**3.4.1 Given tradable permits \( \bar{L}_2 \)**

Throughout the illustration we suppose that the regulation for pollutant 2 is set too leniently compared to its ex ante emissions or prices and thus \( \bar{L}_2 > \hat{Q}_2 \) and \( \bar{t}_2 < \hat{P}_2 \). First, in this subsection we will consider the cases where pollutant 2 is regulated by tradable permits. The case where pollutant 2 is regulated by an emissions tax will be treated in the next subsection.

**3.4.1.1 Interaction in abatement but not in damages**

We begin with the simple case where the interaction between pollutants appears only in abatement, that is, \( w \neq 0 \) and \( v = 0 \). Then the relationship between the
expected emissions of both pollutants can be determined solely by the sign of interaction term $w$:

$$E \left[ Q_1^* \left( \bar{L}_2 \right) \right] - \hat{Q}_1 = \frac{w}{a_2 + d_{12}} \left( \bar{L}_2 - \hat{Q}_2 \right).$$

Note, as shown in Figures (3.1a) and (3.1b), that even when pollutant 2 is regulated with an inefficient number of permits, the expected emissions of pollutant 1 are adjusted to equate its marginal abatement cost and marginal damage. To see why, use the fact that all regulations of pollutant 1 produce the same expected emissions to choose $Q_1$ to minimize $E \left[ C \left( Q_1, \bar{L}_2, u \right) + D \left( Q_1, \bar{L}_2 \right) \right]$. The first order condition is $-E \left[ C_1 \left( Q_1, \bar{L}_2, u \right) \right] = E \left[ D_1 \left( Q_1, \bar{L}_2 \right) \right]$. Expected pollutant 1 emissions under each optimal policy satisfy this marginal condition. (We will see that this condition does not hold for pollutant 1 when pollutant 2 is regulated with an inefficient tax).

When the two pollutants are complements ($w > 0$) and $\bar{L}_2 > \hat{Q}_2$, expected emissions of pollutant 1 exceed its ex ante optimal emissions as shown in Figure (3.1a). Since the emissions of pollutant 2 are too high, the expected marginal abatement cost of pollutant 1 at $\hat{Q}_1$ is higher than its marginal damage. Thus, it is required that the regulation of pollutant 1 results in greater emissions than $\hat{Q}_1$. On the other hand, if the two pollutants are substitutes ($w < 0$) as shown in Figure (3.1b), that the emissions of pollutant 2 are higher than its ex ante optimal level reduces the marginal abatement cost of pollutant 1 at $\hat{Q}_1$ below its marginal damages. Thus, the regulation for pollutant 1 reduces its expected emissions below its ex ante optimal level. In both cases the optimal response of pollutant 1 expected emissions to the inefficiency of the pollutant 2 regulation reduces the wedge between marginal abatement cost and marginal damage for pollutant 2. This occurs because

$$\frac{\partial E \left[ C_2 \left( Q_1, \bar{L}_2, u \right) + D_2 \left( Q_1, \bar{L}_2 \right) \right]}{\partial Q_1} = v - w.$$
Figure 3.1: Environmental performance given tradable permits

- (a) Pollutant 1 for $v = 0$ and $w > 0$
- (b) Pollutant 1 for $v = 0$ and $w < 0$
- (c) Pollutant 2

$\hat{Q}_2 < L_2$
3.4.1.2 Interactions in both abatement and damages

Next consider the case where the interactions between the pollutants appear in both abatement costs and damages. In this case, the relationship between the expected emissions of both pollutants is determined by $-\text{sgn}(v - w)$. As mentioned in section 3.3.1, the sign of this term represents the net interaction in both abatement costs and damages and does not require that both interactions in abatement and damages have the same relationship. If the overall interactions in abatement and damages imply that the two pollutants are complements, that is, $v - w < 0$, then the expected emissions of pollutant 1 deviates from its ex ante optimal emissions in the same direction as the tradable permits for pollutant 2 deviate from its ex ante optimal emissions. Figure (3.2a) shows the case in which both pollutants are complements in both abatement costs and damages. That is, $w > 0$ and $v < 0$, and thus, both pollutants are complements in social costs as well. When supplied permits for pollutant 2 are greater than its ex ante optimal emissions ($\hat{L}_2 > \hat{Q}_2$), the marginal abatement cost function of pollutant 1 moves up and the marginal damage function of pollutant 1 shifts down (drawn by dashed lines). As always, when pollutant 2 is regulated by tradable permits, the expected emissions of pollutant 1 equate its own marginal abatement cost and damage. Thus, in this case the expected emissions of pollutant 1 are greater than its ex ante optimal emissions.

On the other hand, Figure (3.2b) shows the case in which the two pollutants are complements in abatement costs ($w > 0$), they are substitutes in damages ($v > 0$), but the overall interaction implies that the two pollutants are substitutes in social costs ($v - w > 0$). In this case, tradable permits for pollutant 2, which are greater than $\hat{Q}_2$, cause both marginal abatement and damage functions of pollutant 1 to shift up. However, since $v > w$, the marginal damage function moves up more than marginal abatement costs. The difference in parallel movements of these two functions causes the expected emissions of pollutant 1 to move below its ex ante optimal emissions.
optimal emissions. Finally, notice that when both pollutants are complements in social costs, the qualitative deviations of expected emissions from the ex ante optimal values is the same for both pollutants. However, when they are substitutes in social costs, the qualitative deviations of expected emissions from the ex ante optimal values are opposite for the two pollutants.

3.4.2 Given tax $\bar{t}_2$

In this subsection we illustrate the cases where pollutant 2 is regulated by an emission tax that is lower than its ex ante optimal price, that is, $\bar{t}_2 < \hat{P}_2$.

3.4.2.1 Interaction in abatement not in damages

As before, we start from a simple case where the interaction between the two pollutants appears only in abatement. Then, the relationship in Finding 2 can be simplified as

$$E [Q_1^* (\bar{t}_2)] - \hat{Q}_1 = -\frac{d_{22}w}{a_2 (a_2 + d_{12}) - w^2} \left\{ E [Q_2^* (\bar{t}_2)] - \hat{Q}_2 \right\}. \quad (3.43)$$

In this case, the relationship is determined by two factors, $d_{22}$ and $w$. If the marginal damage function of pollutant 2 is flat, then regardless of the regulation for pollutant 2 and the interaction in abatement the ex ante optimal emissions of pollutant 1 can be achieved by any of the optimal price-based regulations. Unless $d_{22} = 0$, the relationship between the expected emissions of both pollutants is determined by the sign of $w$. Notice, however, that the relationship with a given tax for pollutant 2 is opposite of the relationship with given permits for pollutant 2. If the two pollutants are complements ($w > 0$), a low tax for pollutant 2 leads its expected emissions to be greater than its ex ante optimal emissions. Then, (3.43) implies that the optimal regulation of pollutant 1 should result in expected emissions of pollutant 1 in the
Figure 3.2: Environmental performance given tradable permits 2; $\tilde{Q}_2 < \bar{L}_2$. 

- For $v < 0$ and $w > 0$ 
- For $v > 0$, $w > 0$, and $v - w > 0$
opposite direction. That is, the expected emissions of pollutant 1 should be lower than its ex ante optimal emissions.

This is illustrated in Figure (3.3a). Unlike the case in which pollutant 2 is regulated with a fixed number of permits, when pollutant 2 is regulated with a tax the ex ante optimal expected emissions of pollutant 1 does not, in general, equate the marginal abatement cost and marginal damage of pollutant 1. To see this, note that expected emissions of pollutant 1 under each of the optimal policies when pollutant 2 is regulated with a fixed tax is equal to $Q_1$ that minimizes $C(Q_1, Q_2(\bar{t}_2, Q_1, u), u) + D(Q_1, Q_2(\bar{t}_2, Q_1, u))$. The first order condition for this minimization is

$$E[C_1(Q_1, Q_2(\bar{t}_2, Q_1, u), u) + D_1(Q_1, Q_2(\bar{t}_2, Q_1, u))]$$

$$+ E[C_2(Q_1, Q_2(\bar{t}_2, Q_1, u), u) + D_2(Q_1, Q_2(\bar{t}_2, Q_1, u))] \frac{\partial Q_2(\bar{t}_2, Q_1, u)}{\partial Q_1} = 0.$$ 

Since $-C_2(Q_1, Q_2(\bar{t}_2, Q_1, u), u) = \bar{t}_2$ and $\frac{\partial Q_2(\bar{t}_2, Q_1, u)}{\partial Q_1} = \frac{w}{a_2}$, the first order condition can be rewritten as

$$-E[C_1(Q_1, Q_2(\bar{t}_2, Q_1, u), u)]$$

$$= D_1(Q_1, EQ_2(\bar{t}_2, Q_1, u)) - [\bar{t}_2 - D_2(Q_1, EQ_2(\bar{t}_2, Q_1, u))] \frac{w}{a_2}.$$ 

The facts that $w > 0$ in this example and $\bar{t}_2 < D_2(Q_1^*(\bar{t}_2), EQ_2(\bar{t}_2, Q_1, u))$ in Figure (3.3a) results in the marginal abatement cost exceeding the marginal damage for pollutant 1 at $Q_1^*(\bar{t}_2)$. To understand the intuition behind this result, note that if the tax for pollutant 2 is so low that its expected emissions will be greater than its ex ante optimal level, then the marginal abatement cost of pollutant 1 at its ex ante optimal level becomes higher than its marginal damages. If this was the case when pollutant
2 is controlled by given tradable permits, the optimal regulation for pollutant 1 would induce the expected emissions of pollutant 1 to be greater than its ex ante optimal level (as shown in Figure (3.1a)). However, when pollutant 2 is regulated by a tax, if regulation of pollutant 1 also caused its expected emissions to exceed its ex ante optimal value, then the expected marginal abatement cost of pollutant 2 also moves up due to greater emissions of pollutant 1. This, in turn, would cause the expected emissions of pollutant 2 to move further away from its ex ante optimal level and increase the wedge between its marginal abatement cost and its marginal damage. Therefore to minimize the efficiency loss from the low tax for pollutant 2, the optimal regulation for pollutant 1 should be stricter than its ex ante optimal emissions as shown in Figure (3.3a).

The opposite is true if the two pollutants are substitutes in abatement ($w < 0$). In this case, if the tax for pollutant 2 is too low, then the optimal regulation for pollutant 1 will also be lenient so that the expected emissions of pollutant 1 exceeds its ex ante optimal emissions $\hat{Q}_1$ as shown in Figure (3.3b). Interestingly, these mechanisms can help us understand the effects of a flat marginal damage function for pollutant 2. When the marginal damage function of pollutant 2 is flat, the difference between the marginal abatement cost and marginal damage of pollutant 2 is constant and the wedge between them cannot be reduced by adjusting the emissions of pollutant 1. Thus, in this case, the optimal regulation for pollutant 1 results in its ex ante optimal emissions.

### 3.4.2.2 Interactions in both abatement and damages

Next consider the case in which the interactions between the two pollutants appear in both abatement and damages. As one might expect these situations are more complex than the cases discussed above. First, since pollutant 2 is regulated by a tax,
Figure 3.3: Environmental performances given emissions tax 1; $\tilde{t}_2 < \hat{P}_2$
the emissions of pollutant 2 are not only variable but also affected by the emissions of pollutant 1 via the interaction in abatement. Second, the marginal damage of one pollutant is affected by the emissions of the other pollutant via the interaction in damages. These lead the multiplier in (3.42) to have a more complex form than when pollutant 2 is regulated by tradable permits $\bar{L}_2$:

$$E [Q_1^* (\bar{t}_2)] - \hat{Q}_1 = -\frac{a_2 v + d_{22} w}{a_2 (a_2 + d_{12}) + w (v - w)} \left\{ E [Q_2^* (\bar{t}_2)] - \hat{Q}_2 \right\}.$$  

As shown (3.35) and (3.39) in subsection 3.3.2, $X = a_2 (a_2 + d_{12}) + w (v - w)$ represents the effects of changing the emissions of pollutant 1 on its own marginal social cost, and $Y = a_2 v + d_{22} w$ represents the effects of changing the price for pollutant 2 on the marginal social cost of pollutant 1. Since we assume $X > 0$ to avoid cases in which increasing emissions of pollutant 1 reduce its marginal social cost, the relationship between the expected emissions of both pollutants are determined by the sign of $Y = a_2 v + d_{22} w$.

Recall that without the interaction in damage, the optimal regulation for pollutant 1 given an inefficient emissions tax for pollutant 2 results in expected emissions of pollutant 1 that reduces the inefficiency of the regulation of pollutant 2. When there exist interactions in both abatement and damage, we focus on how the effect of the optimal regulations for pollutant 1 on the reduction in the inefficiency from pollutant 2 can be restricted or magnified by the interaction in damages. To see this, we first look at how the multiplier $-Y/X$ in (3.42) changes over $v$. Given (3.36), which is the condition for $X > 0$ given (3.8), we have

$$-\frac{\partial}{\partial v} \left( \frac{Y}{X} \right) = -\frac{a_2^2 (a_2 + d_{12}) - w^2 (a_2 + d_{22})}{X^2} \leq 0,$$

which implies that

$$-Y/X|_{v<0} \geq -Y/X|_{v=0} \geq -Y/X|_{v>0}.$$  

(3.44)
That is, the multiplier $-Y/X$ decreases (weakly) as $v$ increases. We illustrate the effects of this result in the following examples.

First, Figure (3.4a) shows the case in which the two pollutants are complements in both abatement and damages ($v < 0$ and $w > 0$) and $Y = a_2v + d_{22}w > 0$. Then, when it is expected that a low tax for pollutant 2 ($\bar{t}_2 < \hat{P}_2$) results in expected pollutant 2 emissions that are higher than its ex ante optimal emissions $\hat{Q}_2$, (3.42) implies that the expected emissions of pollutant 1, $EQ^*_1(\bar{t}_2)$, is lower than its ex ante optimal emissions $\hat{Q}_1$. Since $w > 0$, we know that $-Y/X|_{v=0} < 0$, which implies that without the interaction in damages ($v = 0$), the expected emissions of pollutant 1 would be lower than its ex ante optimal emissions (marked as $\tilde{Q}_1$ in Figure (3.4a)). However, from (3.44), we know that the complementary interaction in damages ($v < 0$) increases the multiplier. In addition, for $Y = a_2v + d_{22}w > 0$, the multiplier is still negative. Thus, we have

$$-\frac{Y}{X}|_{v=0} = -\frac{d_{22}w}{a_2(a_2 + d_{12}) - w^2} < -\frac{a_2v + d_{22}w}{a_2(a_2 + d_{12}) + w(v - w)} = -\frac{Y}{X}|_{v<0} < 0.$$  

Therefore, the expected emissions of pollutant 1 deviate less from its ex ante optimal value than when the pollutants only interact in abatement (marked as $EQ^*_1(\bar{t}_2)$ in Figure (3.4a)).

The reason why the expected emissions of pollutant 1 deviates less with the interaction in both abatement and damages is as follows. We already know that when a low tax for pollutant 2 results in higher expected emissions of pollutant 2 than its ex ante optimal emissions, the optimal regulations for pollutant 1 work to reduce the wedge between the expected marginal abatement cost and marginal damage of pollutant 2, and thus the expected emissions of pollutant 1 are lower than its ex ante optimal emissions. However, when there is a complementary interaction in damages ($v < 0$), pollutant 1 emissions that are lower than its ex ante optimal emissions will shift up the marginal damage of pollutant 2 (marked as
$D_2\left(Q_2, \tilde{Q}_1\right)$ and $D_2\left(Q_2, EQ^*_1(\tilde{t}_2)\right)$ in Figure (3.4a)). If the optimal regulation for pollutant 1 produced $\tilde{Q}_1$, it is possible that the wedge between the expected marginal abatement cost and damage of pollutant 2 can increase conversely (the difference between $D_2\left(Q_2, \tilde{Q}_1\right)$ and $-EC_2\left(Q_2, \tilde{Q}_1\right)$ at $\tilde{Q}_2$ in Figure (3.4a)). Therefore, although the expected emissions of pollutant 1 deviate from its ex ante optimal emissions, it deviates less than when there is no interaction in damage.

Next, Figure (3.4b) shows the case where the two pollutants are substitutes in abatement ($w < 0$) but complements in damage ($v < 0$) and thus $Y = a_2v + d_{22}w < 0$. Then, when it is expected that a low tax for pollutant 2 ($\tilde{t}_2 < \tilde{P}_2$) results in higher expected emissions than its ex ante optimal emissions $\bar{Q}_2$, (3.42) in Finding 2 implies that the expected emissions of pollutant 1, $E\left(Q^*_1(\tilde{t}_2)\right)$, is also greater than its ex ante optimal emissions $\bar{Q}_1$. Unlike the above example, the complementary interaction in damage ($v < 0$) magnifies the deviation of $E\left(Q^*_1(\tilde{t}_2)\right)$ from $\bar{Q}_1$, because from (3.44) we have

$$0 < -\left.\frac{Y}{X}\right|_{v=0} = -\frac{d_{22}w}{a_2(a_2 + d_{12}) - w^2} < -\frac{a_2v + d_{22}w}{a_2(a_2 + d_{12}) + w(v - w)} = -\left.\frac{Y}{X}\right|_{v<0}.$$ 

That is, the expected emissions of pollutant 1 will deviate more from $\bar{Q}_1$ than when there is no interaction in damages ($v = 0$). Considering the expected emissions of pollutant 1 without the interaction in damages (marked as $\bar{Q}_1$ in Figure (3.4b)), both the marginal damage and the marginal abatement cost of pollutant 2 move down (marked as $D_2\left(Q_2, \bar{Q}_1\right)$ and $-EC_2\left(\bar{Q}_1, Q_2\right)$ in Figure (3.4b)). However, increasing the emissions of pollutant 1 to $EQ^*_1(\tilde{t}_2)$ from $\bar{Q}_1$ can further reduce the wedge between the marginal damage and the marginal abatement cost of pollutant 2 by $D_2\left(Q_2, EQ^*_1(\tilde{t}_2)\right)$ and $-EC_2\left(EQ^*_1(\tilde{t}_2), Q_2\right)$.

In sum, we have shown that the interaction in damage can affect the response of the optimal regulation for pollutant 1 to the inefficient tax for pollutant 2. Although the optimal regulation for pollutant 1 still works to reduce the wedge
between marginal abatement cost and damage for pollutant 2, the adjustment can
be restricted or magnified depending on the interactions in both abatement and
damage. Interestingly, when the two interactions imply the same relationship—for
instance, the two pollutants are either complements or substitutes in both
abatement and damages—the expected emissions of pollutant 1 will deviate less from
its ex ante optimal value than without the interaction in damage. On the other
hand, when the interactions in abatement and damage are opposite of each
other—for instance, there is a complementary interaction in abatement but the two
pollutants are substitutes in damage, or vice versa— the expected emissions of
pollutant 1 deviate more from its ex ante optimal value than without the interaction
in damages.

3.5 Conclusion

In this chapter, we investigated the second-best, price-based regulation for a
pollutants when it interacts with another pollutant in abatement costs and/or
damages. Given that the co-pollutant is controlled by either an emissions tax or
tradable permits, we derived the optimal forms of price-based regulation for the
primary pollutants. We consider an emissions tax, a pure permit market, and a
hybrid policy which is a permit market with price controls. Inefficient regulation of
the co-pollutant leads the optimal regulation for the primary pollutant to deviate
from its ex-ante optimal emissions and price, but all optimal regulations of the
primary pollutant produce the same expected emissions. Our main results reveal
that the deviation of the expected emissions of the primary pollutant from its ex
ante optimal value is determined by: 1) the interactions in abatement costs and
damages, in particular the substitutability and complementarity of the pollutants;
(a) For $v < 0$, $w > 0$ and $a_2 v + d_2 w > 0$

(b) For $v < 0$, $w < 0$ and $a_2 v + d_2 w < 0$

Figure 3.4: Environmental performances given emissions tax 2; $t_2 < \hat{P}_2$
2) the deviation of the regulation for the co-pollutant from its ex ante optimal level, and 3) the form of the regulation of the co-pollutant.

We examined several special cases of our main results. In the simplest case the co-pollutant is regulated by tradable permits and the pollutants interact only in abatement. In this case, if the two pollutants are complements in abatement, then if the number of tradable permits for the co-pollutant is higher (lower) than ex ante optimal emissions, then the optimal regulations for the primary pollutant produce expected emissions that are higher (lower) than its ex ante optimal emissions. When we allow the pollutants to interact in damages as well we find that if the interaction in damages has the same (different) relationship with the interaction in abatement, then the expected emissions of the primary pollutant deviates more (less) from its ex ante optimal emissions than without the interaction in damages.

When the co-pollutant is regulated by an emissions tax, alternative cases are more complicated than when pollutant 2 is regulated with tradable permits, because the emissions of the co-pollutant are affected by the emissions of the primary pollutant. This can lead to results that are opposite of the case when the co-pollutant is regulated with tradable permits. If the two pollutants are complements and they interact only in abatement, a co-pollutant tax that produces expected emissions that are higher (lower) than its ex ante optimal emissions results in regulations for the primary pollutant that produce expected emissions that are lower (higher) than its ex ante optimal emissions (as long as co-pollutant damage is strictly convex). Thus, the expected emissions of the primary pollutant deviates from its ex ante optimal emissions in the opposite direction than in the case of tradable permits for the co-pollutant. We also show interesting results from the effects of the interaction in damages. Unlike the case with tradable permits for the co-pollutant, if the interactions in abatement and damages have the same (different)
relationship, then the expected emissions of the primary pollutant will deviate less (more) than its ex ante optimal emissions than without the interaction in damages.

In this paper we were able to examine how the expected performance of the optimal price-based regulations for the primary pollutant can be affected by interactions in abatement and damages with another pollutant, inefficient regulations for the co-pollutant, and the form of the regulation of the co-pollutant. However, there are still many elements that future research should consider. First, since we have used a static model, we ignore dynamic properties of pollutants. Although Moslener and Requate (2007) already showed how the interaction in abatement affects the optimal dynamic paths of multiple pollutants that interact in abatement, they modeled all pollutants as stock pollutants. However, some co-pollutants that are emitted along with CO$_2$ such as NO$_X$, SO$_2$ and PM are flow pollutants. In addition, research about climate change shows that some flow pollutants such as NO$_X$ and SO$_2$ can have a net cooling effect. Thus, it would be interesting to combine our model and Moslener and Requate (2007) to include these factors and to show how these factors affect the optimal paths of the pollutants.

Next, although we have considered only two pollutants to simplify the model, one pollutant can have multiple co-pollutants. It is unlikely that all of them have the same interaction in abatement. In addition, while some co-pollutants may be regulated, others may not be. To derive more general results about the effects of the interactions in abatement and damages of multiple pollutants, we need to consider multiple co-pollutants that may be regulated differently.
CHAPTER 4
IRREVERSIBLE INVESTMENTS IN EMISSIONS CONTROL UNDER A HYBRID PRICE AND QUANTITY REGULATION

4.1 Introduction

Regulation of pollutants with markets for emissions permits have been examined both theoretically as well as practically. For example, the European Union has implemented the world’s largest Emissions Trading System (ETS) to reduce greenhouse gases (GHGs). Recently, South Korea has introduced its own ETS for GHGs. China, which is the world’s largest GHG emitter, has been operating a pilot program of ETS. At the sub-national level, California has implemented its own cap-and-trade program to reduce GHGs emissions, and nine northeast and mid-Atlantic states have participated in the Regional Greenhouse Gas Initiative (RGGI) to reduce CO$_2$ from power plants. Of course, the U.S. Environmental Protection Agency has succeeded in limiting SO$_2$ and NO$_X$ with emissions markets.

Despite its theoretical efficiency, implementing an emissions permit market has brought about some practical concerns. Containing uncertain costs has been one of those concerns. Unexpectedly high abatement costs can make it difficult for firms to comply, and unexpectedly low abatement costs deter investments in new abatement technologies (Fell et al. (2011)). In addition, high uncertainty in abatement costs can reduce the incentive to invest in new abatement capital (Zhao (2003)). The most commonly proposed measures to deal with uncertain abatement costs are allowing firms to bank and borrow permits and implementing hybrid policies that typically impose price controls on a permit market. Banking and borrowing permits allow firms
to adjust emissions and abatement over time depending on current and expected permit prices and abatement costs. A firm can save permits for future use when abatement costs are low and a firm can borrow permits from future years when abatement costs are high. On the other hand, a hybrid policy imposes a price ceiling and a price floor directly on a permit market. When abatement costs are high, the market price binds at the ceiling and firms are allowed to increase their emissions beyond the emissions permit cap. When the market price is low, a price floor provides incentives for firms to reduce their emissions below the permit cap.

Another important concern about permit trading has been the extent to which regulations induce investments in abatement capital or technology. Many studies of investments in cost-reducing abatement capital focus on comparing incentives from different policy instruments and ranking them (Jung et al. (1996); Montero (2002); Van Soest (2005); Requate (2005)). As concerns about cost containment have increased, the effects of cost containment policies on abatement investments have gained attention among researchers (Phaneuf and Requate (2002); Burtraw et al. (2010); Nemet (2010); Park (2012)). Our paper contributes to this literature by studying the effects of a hybrid policy on investment in abatement capital. The benefits of investing in abatement capital is the reduction in abatement costs. Although high abatement costs impose a heavy burden on regulated firms, they also provide an incentive for investment in abatement capital. Thus, cost-containment measures which suppress potentially high abatement costs may reduce the incentive for investment. Another factor which affects investment is uncertainty in the benefits of new abatement capital, which can come from price volatility or uncertainty in abatement costs. A typical conclusion from the real-option literature is that higher uncertainty in the benefits of investment reduces the incentive to invest and delays investments (Dixit and Pindyck (1994); Pindyck (2006)). In the environmental regulation context, higher uncertainty in abatement costs can reduce
the incentive to invest in a permit market (Zhao (2003)). Since cost containment measures are designed to limit the variations in price and compliance costs, these measures may have some positive effects on investment. As a whole, limiting the potential for high permit prices and the volatility of compliance costs can have opposite effects on investment in abatement capital.\footnote{For the case of bankable permits (without the ability to borrow permits), Phaneuf and Requate (2002) show that banking and investments in abatement capital are substitutes in reducing abatement costs, and thus positive banking reduces these investments. Interestingly, Park (2012), who extends Phaneuf and Requate (2002) by using a real-option model, argues that permit banking could generate a greater incentive to invest than a permit market without banking. In a real option model, holding an option to invest creates an option value which acts as an opportunity cost of exercising the option. Permit banking lowers the opportunity cost of exercising the investment option. Thus, when banking is profitable, more permits can be obtained through investment. Park (2012) shows that for sufficiently large uncertainty in abatement costs, permit banking can give a larger incentive to invest than permit trading without banking. For low uncertainty, the investment incentive in a permit market without banking is still larger than in a market with banking.}

Other authors have examined the impact of price controls on abatement investments. For example, Nemet (2010) shows that a price ceiling (a so-called safety valve) reduces the incentive to invest in clean technologies because suppressing high permit prices eliminates the higher payoff from investment. Burtraw et al. (2010) show that imposing a symmetric price ceiling and price floor (also known as a symmetric price collar) can amend this problem, because the price floor can motivate greater investment when abatement costs are low. Both Burtraw et al. (2010) and Nemet (2010) focus only on the effects of price controls on the expected benefits from investment. Hence, solely suppressing potentially high prices reduces the expected price of permits and thus reduces the incentive for investment. However, imposing the symmetric price ceiling and floor curtails high and low tails of the price distribution and thus the expected price can stay at a similar level as without the price ceiling and floor.

Our approach to the problem is to construct a dynamic stochastic model of the investment decision under an emissions market with price controls. This approach
allows us to consider the irreversibility of investments in abatement capital, which has been shown to delay investment in other contexts (Dixit and Pindyck (1994); Pindyck (2006)). The model is an extension of Zhao (2003), who considered the differences in investment under a pure emissions market and an emissions tax. In contrast, we consider investment decisions in an emissions market with price controls, and compare these to the decisions in a market without price controls.\(^2\) We find that the effect of abatement-cost uncertainty on the expected marginal value of new investment in a period is non-monotonic under a hybrid policy. This contrasts sharply with the results of Zhao (2003) who found that the expected marginal values of new investment is strictly increasing in positive abatement cost shocks under pure permit trading, while the expected marginal value of new investment is strictly decreasing under an emissions tax. We show that a large positive shock to aggregate abatement costs in a period may trigger additional investment under the pure emissions market, but not under the hybrid scheme. On the other hand, additional investment in abatement capital may occur in a period with low abatement costs that would not occur under the pure market scheme. Moreover, we show that there is an upper bound of the abatement capital stock such that no additional investment occurs under the hybrid policy, but no such upper bound exists for the pure emissions market. Finally, in contrast to Burtraw et al. (2010) who argue that symmetric price controls cancel each other out so that they do not affect the expected benefits from investment, we

\(^2\)A related work is Weber and Neuhoff (2010) who examine the effects of technological innovation on the optimal regulation of a pure tax, pure permit trading, and a hybrid price and quantity regulation. Using a static model, they show how optimal regulations depend on innovation effectiveness, a measure of the increase in abatement from innovation, given an emissions price. The goal of their work is to find optimal regulations, given firms’ investment behavior, in a static environment. In contrast, we explore firms’ investment decisions given a hybrid price and quantity regulation in a stochastic dynamic model with irreversible investments in technology. In addition, Weber and Neuhoff (2010) focus on investment in technology innovation which has an uncertain impact on abatement costs, while we assume that investments in new abatement capital automatically reduce abatement costs non-stochastically although abatement costs are uncertain.
demonstrate that the effects of the price controls on abatement investments may not be symmetric.

The rest of this chapter is organized as follows. In section 2, we construct a model of aggregate incremental investment in irreversible abatement capital under a permit market with price controls. In section 3, we derive the investment decision rule under a hybrid policy. Section 4 contains a comparison of investment decisions under a hybrid policy and under pure permit trading. In section 5, we will show comparative static results with respect to policy instruments. We conclude in section 6.

4.2 Model

In this section we present a model of abatement capital investment under a hybrid policy. We specify an aggregate abatement cost function for an industry, the permit market equilibrium under a hybrid policy, and optimal aggregate investments in abatement capital. The model is a stochastic dynamic model with uncertainty in aggregate abatement costs and an infinite time horizon.

4.2.1 Abatement cost function

Throughout we consider an industry under a hybrid price and emissions regulation. Let the aggregate abatement cost function for the industry at a moment \( t \) be

\[
C(a, K, \theta) = \frac{1}{2} c(K) \theta a^2,
\]

where \( a \geq 0 \) is the aggregate abatement level, \( c(K) > 0 \) is a stationary function that partially captures the slope of the aggregate marginal abatement cost function and is a function of the aggregate abatement capital stock \( K \geq 0 \) in a period. \(^3\) Assume

---

\(^3\)Our model is a dynamic model, so the variables \( a, K, \) and \( \theta \) have a time index. However, for notational brevity, we omit the time index except when it is required. We use \( t \) to indicate a certain time. For instance, \( K(t) \) represents firm aggregate capital stock at time \( t \) and \( \theta(t) \) is the realized value of a random shock at time \( t \).
that \( c' (K) < 0 \) and \( c'' (K) > 0 \) so that increasing the aggregate capital stock reduces total and marginal abatement costs at decreasing rates. The variable \( \theta \) captures random shocks that affects aggregate abatement costs. It is assumed that \( \theta \) follows a geometric Brownian motion

\[
\frac{d\theta(t)}{\theta(t)} = \alpha dt + \sigma dz(t), \tag{4.2}
\]

with \( E [dz(t)] = 0 \) and \( Var [dz(t)] = dt \). \( \alpha \) represents the expected growth rate of \( \theta \), which may have any sign, and \( \sigma \) is the volatility rate of \( \theta \). To make sure that the expected present value of aggregate abatement costs over some time period is bounded, we assume that

\[
r - \alpha > 0 \text{ and } r - \sigma^2 + \alpha > 0, \tag{4.3}
\]

for a given interest rate \( r \).

### 4.2.2 Instantaneous market equilibrium under a hybrid policy

Suppose that at every moment \( t \) a government issues a total of \( L \) permits to regulate emissions. Let \( Q \) be unregulated aggregate emissions from the industry. Then, a target abatement level which the government aims to achieve in every time period is

\[
\bar{a} = Q - L. \tag{4.4}
\]

For a given permit price \( p \), the aggregate abatement level is determined so that the permit price is equal to aggregate marginal abatement costs; that is,

\[
a (p, K, \theta) = \frac{p}{c(K) \theta}. \tag{4.5}
\]
Then, when the target abatement level $\bar{a}$ is achieved, the equilibrium permit price is

$$p = c(K) \theta \bar{a}. \quad (4.6)$$

Note that the price changes proportionately to the random shock $\theta$ and it is decreasing as the capital stock $K$ increases.

A hybrid price and quantity regulation imposes a price ceiling $\tau$ and a price floor $s$, with $\tau > s$, on the permit market. Firms are allowed to buy additional permits from the government at price $\tau$, and they can sell unused permits back to the government at price $s$. To specify how firms’ aggregate behavior changes at the price controls, we must specify cutoff values of $\theta$ where $p$ binds at either $\tau$ or $s$ without any additional purchases of permits from the government or sales to the government. Denote these values by $\theta_\tau$ and $\theta_s$, respectively. Then, using (4.6), it can be shown that:

$$\theta_\tau(K) = \left. \frac{\tau}{c(K) \bar{a}} \right|; \quad (4.7)$$

$$\theta_s(K) = \left. \frac{s}{c(K) \bar{a}} \right|. \quad (4.8)$$

For $\theta \geq \theta_\tau(K)$ and thus $p = \tau$, firms can buy additional permits from the government at the price ceiling $\tau$. Then, aggregate abatement at the price ceiling is

$$a(\tau, K, \theta) = \left. \frac{\tau}{c(K) \theta} \right|, \quad (4.9)$$

which implies that aggregate abatement level is decreasing as $\theta$ increases. From (4.8), it can be shown that

$$a(\tau, K, \theta) = \left. \frac{\tau}{c(K) \theta} \right| \leq \left. \frac{\tau}{c(K) \theta_\tau(K)} \right| = \bar{a},$$

which implies that aggregate abatement can be less than the target abatement level as firms make use of the price ceiling.
For $\theta \leq \theta_s(K)$ and thus $p = s$, firms can sell permits back to the government at the price floor $s$. Then, aggregate abatement at the price floor is

$$a(s, K, \theta) = \frac{s}{c(K)\theta}. \tag{4.10}$$

From (4.7), it can be shown that

$$a(s, K, \theta) = \frac{s}{c(K)\theta} \geq \frac{s}{c(K)\theta_s(K)} = \bar{a},$$

which implies that aggregate abatement can exceed the target abatement. At the price floor $s$, aggregate abatement increases as $\theta$ decreases. Since abatement levels cannot go beyond the unregulated aggregate emissions $Q$, we need to define one more cutoff value of $\theta$ where abatement at the price floor $s$ completely eliminates unregulated emissions $Q$. Denote this value by $\theta_F$, which from (4.10), can be expressed as

$$\theta_F(K) = \frac{s}{c(K)Q}. \tag{4.11}$$

Combining our results, aggregate abatement under a hybrid price and quantity regulation can be summarized as:

$$a^*(K, \theta; \bar{a}, Q, s, \tau) = \begin{cases} 
Q & \text{for } \theta \leq \theta_F \\
 s/(c(K)\theta) & \text{for } \theta_F \leq \theta \leq \theta_s \\
 \bar{a} & \text{for } \theta_s \leq \theta \leq \theta_r \\
\tau/(c(K)\theta) & \text{for } \theta_r \leq \theta.
\end{cases} \tag{4.12}$$

We note that three cutoff values are increasing functions of $K$. That is,
\[
\frac{\partial \theta_s (K)}{\partial K} = -\frac{sc' (K)}{c(K)^2 a} = -\theta_s \frac{c' (K)}{c(K)} > 0; \tag{4.13}
\]
\[
\frac{\partial \theta_r (K)}{\partial K} = -\frac{\tau c' (K)}{c(K)^2 a} = -\theta_r \frac{c' (K)}{c(K)} > 0; \tag{4.14}
\]
\[
\frac{\partial \theta_F (K)}{\partial K} = -\frac{sc' (K)}{c(K)^2 Q} = -\theta_F \frac{c' (K)}{c(K)} > 0. \tag{4.15}
\]

4.2.3 Investments in irreversible abatement capital

For \( s < p < \tau \), while an increase in \( \theta \) does not change the aggregate abatement level, it increases aggregate compliance costs because \( c(K) \theta \) increases. At the price ceiling, \( p = \tau \), allowing purchases of additional permits and fixing the price at the price ceiling (which is lower than it would be without the price ceiling) can reduce firms’ compliance costs for high values of \( \theta \). However, compliance costs under the price ceiling still increase as \( \theta \) increases. Thus, there may still be incentives for firms to invest in more abatement capital stock for high values of \( \theta \), although these investments are less attractive than under pure permit trading where the permit price would be higher without the price ceiling. With a price floor, \( p = s \), firms may have greater incentive to invest because the permit price is kept higher than under a pure permit trading policy and because firms can sell unused permits to the government.

We assume that the industry’s aggregate capital stock is adjusted instantaneously when firms are motivated to invest. However, irreversibility prohibits firms from decreasing their abatement capital stock. Investment in abatement capital is measured by increases in the aggregate capital stock \( \Delta K \) with unit costs equal to the constant \( w \). Thus, total investment costs in a period are \( w\Delta K \). For simplicity we assume that the capital stock does not depreciate.

4.2.4 Optimal industry investment

Because competitive behavior under a hybrid scheme will minimize aggregate compliance costs (Roberts and Spence (1976)), to specify the industry’s aggregate investment strategy and optimal capital expansion path we follow Lucas and
Prescott (1971), Baldursson and Karatzas (1996) and Zhao (2003) by appealing to a fictitious social planner who minimizes expected industry costs over time. Under a pure permit trading program, aggregate compliance costs are just equal to aggregate abatement costs. However, under a hybrid price and quantity regulation, they can be different depending on the realized value of a random shock, because purchases of additional permits or sales of unused permits are allowed. At a point in time, with given abatement capital $K$ and realized value of $\theta$, the planner determines the aggregate abatement level ($a$) and potential purchases of additional permits from the government at the price ceiling $\tau$ (denoted by $L^c$) or permit sales back to the government at the price floor $s$ (denoted by $L^f$). That is, the planner solves:

$$
S(K, \theta; \bar{a}, s, \tau, Q) \equiv \min_{a,L^c,L^f} C(a, K, \theta) + \tau L^c - s L^f \\
\text{s.t. } \bar{a} = a + L^c - L^f \\
L^c \geq 0, \ L^f \geq 0 \\
a \leq Q.
$$

(4.16)

From (4.1) and (4.12), the aggregate compliance cost function $S(\cdot)$ can be expressed as

$$
S(K, \theta; \bar{a}, s, \tau, Q) =
\begin{cases}
\frac{1}{2} c(K) \theta Q^2 - s (Q - \bar{a}) & \text{for } \theta \leq \theta_F \\
-\frac{1}{2} s^2 / (c(K) \theta) + s \bar{a} & \text{for } \theta_F \leq \theta \leq \theta_s \\
\frac{1}{2} c(K) \theta \bar{a}^2 & \text{for } \theta_s \leq \theta \leq \theta_\tau \\
-\frac{1}{2} \tau^2 / (c(K) \theta) + \tau \bar{a} & \text{for } \theta_\tau \leq \theta.
\end{cases}
$$

(4.17)

Since $S(\cdot)$ already reflects optimal abatement at every moment in time, the only intertemporal choices are capital investments. Then, the planner’s problem is to
minimize the present value of the stream of the industry’s expected compliance costs plus its capital investments. Formally, this problem is the following:

\[
V(K, \theta) \equiv \max_K -E \int_0^{\infty} S(K, \theta; \bar{a}, s, \tau, Q) e^{-rt} dt - \sum_h w\Delta K(h) e^{-rh} \\
\text{s.t. } \frac{d\theta(t)}{\theta(t)} = \alpha dt + \sigma dz(t) \\
\Delta K(h) \geq 0 \forall h \in H.
\] (4.18)

In (4.18): \( h \) is the moment in time when investment in additional capital occurs; \( H \) is a set of these moments in time; \( \Delta K(h) \) is aggregate investment at \( h \), and \( w\Delta K(h) \) is the cost of this investment. The non-negativity constraint on \( \Delta K(h) \) captures the irreversibility of the capital stock. The objective function of (4.18) consists of two parts. The first part is the expected present value of the stream of compliance costs, and the second part is the present value of the costs of all investments occurring at moments in time \( h \in H \). The timing of an investment at \( h \) is determined simultaneously with the optimal investment strategy.

Based on Bellman’s Principle of Optimality (Dixit and Pindyck (1994, Ch.11)), we will decompose (4.18) into an instantaneous flow from the current state and the expected maximized value over the remaining time horizon. Suppose that \((K, \theta)\) is the current state. Consider an arbitrarily short interval of time \( dt \). Since we are considering decisions made continuously, we will take the limit as \( dt \) goes to zero. Since (4.16) is current aggregate compliance costs, the flow of compliance costs during the interval \( dt \) is \( S(K, \theta; \bar{a}, s, \tau) dt \). Suppose that an increase in the capital stock is required at the end of this interval to \( K' \). Total investment costs which are incurred at the end of \( dt \) are \( w(K' - K) = w\Delta K \). While the random shock changes over the interval from \( \theta \) to \( \theta + d\theta \), at current time \( t = 0 \) it is not known what \( d\theta \) will be. All that is known is the distribution of \( d\theta \) from (4.2). Thus, the state at the end of \( dt \) is \((K', \theta + d\theta)\), and the value at this state should be calculated as the expected value
conditional on the current information \( \theta \). Of course, aggregate investment costs and the expected maximum value \( E[V(K', \theta + d\theta)] \) at the end of \( dt \) should be discounted over this length of time. Then, we can obtain the following transformed problem of the planner:

\[
V(K, \theta) = \max_{K'} -S(K, \theta; \bar{a}, s, \tau, Q) dt + e^{-rdt} \{ E[V(K', \theta + d\theta)] - w(K' - K) \}
\]

s.t. \[
\frac{d\theta(t)}{\theta(t)} = \alpha dt + \sigma dz(t)
\]

\[
K' \geq K
\]

(4.19)

The planner chooses \( K' \) to solve (4.19). For the maximum to exist, we require the concavity of \( V(K, \theta) \) in \( K \). Following Dixit and Pindyck (1994), the concavity of \( V(K, \theta) \) is guaranteed if \( -S(K, \theta; \bar{a}, s, \tau) \) is concave in \( K \). For this purpose, we assume that each function in (4.17) is strictly convex in \( K \) for any realized value of \( \theta \), which requires

\[
\frac{\partial^2}{\partial K^2} \left( \frac{1}{c(K)} \right) = \frac{2c'(K)^2 - c(K)c''(K)}{c(K)^3} < 0,
\]

(4.20)

in addition to \( c''(K) > 0 \).

Given that \( V(K, \theta) \) is strictly concave in \( K \), the solution of (4.19) can be identified by the Kuhn-Tucker conditions:

\[
e^{-rdt} \{ E[V_K(K', \theta + d\theta)] - w \} \leq 0, \ K' - K \geq 0, \ \{ E[V_K(K', \theta + d\theta)] - w \} (K' - K) = 0.
\]

As mentioned above, the expectation operator is present in these conditions because \( d\theta \) is uncertain at time \( t \). However, as \( dt \) goes to zero, \( e^{-rdt} \) goes to 1 and \( d\theta \) goes to
zero with probability one. Therefore, $E \left[ V_K (K', \theta + d\theta) \right] - w$ goes to $V_K (K', \theta) - w$. Thus, the Kuhn-Tucker conditions can be expressed as

$$V_K (K', \theta) - w \leq 0, \ K' - K \geq 0, \ [V_K (K', \theta) - w] (K' - K) = 0. \quad (4.21)$$

Given capital stock $K$ and the realized value of the random shock, denoted by $\theta'$, investment occurs if $V_K (K, \theta') > w$ and the new capital stock $K'$ must satisfy $V_K (K', \theta') = w$. Given the current state $(K, \theta')$, investment does not occur in the period if $V_K (K, \theta') \leq w$. As $\theta$ changes over time, investment prevents a state $(K, \theta)$ such that $V_K (K, \theta) > w$. In this sense, $V_K (K, \theta) = w$ can be considered as a barrier to additional investment. In addition, this investment barrier implicitly defines the boundaries of the investment and non-investment (or inaction) intervals of $\theta$, given $K$.

### 4.2.5 The expected marginal value function

The Kuhn-Tucker conditions (4.21) tells us whether the industry will increase its abatement capital stock in a period, given the realized value of $\theta$. However, $V_K (K, \theta)$ has four different expressions depending on where $\theta$ falls. That is,

$$V_K (K, \theta) = \begin{cases} V_K^F (K, \theta) & \text{for } \theta \leq \theta_F \\ V_K^s (K, \theta) & \text{for } \theta_F \leq \theta \leq \theta_s \\ V_K^m (K, \theta) & \text{for } \theta_s \leq \theta \leq \theta_r \\ V_K^r (K, \theta) & \text{for } \theta_r \leq \theta. \end{cases}$$

In section C.1 of Appendix C, we derive the explicit forms of these expressions as follows:
\( V^F_K(K, \theta) = -\frac{\beta_1 - 1}{\beta_1} \frac{c'(K) \theta Q^2}{2(r - \alpha)}; \) (4.22)

\[
V^s_K(K, \theta) = \frac{1}{\beta_1} \left( \frac{\beta_1 + 1}{r - \sigma^2 + \alpha} - \frac{\beta_1 - 1}{r - \alpha} \right) c'(K) Q s \left( \frac{\theta}{\theta_F} \right)^{\beta_2} - \frac{\beta_1 + 1}{\beta_1} \frac{c'(K) s^2}{2(r - \sigma^2 + \alpha) c(K)^2 \theta}; \] (4.23)

\[
V^m_K(K, \theta) = \frac{1}{\beta_1} \left( \frac{\beta_1 + 1}{r - \sigma^2 + \alpha} - \frac{\beta_1 - 1}{r - \alpha} \right) \left[ c'(K) Q s \left( \frac{\theta}{\theta_F} \right)^{\beta_2} - \frac{c'(K) s^2}{2c(K)} \left( \frac{\theta}{\theta_s} \right)^{\beta_2} \right]
- \frac{\beta_1 - 1}{\beta_1} \frac{c'(K) \theta \bar{a}^2}{2(r - \alpha)}; \] (4.24)

\[
V^\tau_K(K, \theta) = \frac{1}{\beta_1} \left( \frac{\beta_1 + 1}{r - \sigma^2 + \alpha} - \frac{\beta_1 - 1}{r - \alpha} \right) \left[ c'(K) Q s \left( \frac{\theta}{\theta_F} \right)^{\beta_2} - \frac{c'(K) \bar{a}s}{2c(K)} \left( \frac{\theta}{\theta_s} \right)^{\beta_2} + \frac{c'(K) \bar{a}\tau}{2c(K)} \left( \frac{\theta}{\theta_\tau} \right)^{\beta_2} \right]
- \frac{\beta_1 + 1}{\beta_1} \frac{c'(K) \tau^2}{2(r - \sigma^2 + \alpha) c(K)^2 \theta}. \] (4.25)

Given capital stock \( K \), \( V_K(K, \theta) \) is connected at the cutoff values of \( \theta \). That is, \( V^F_K(K, \theta_F) = V^s_K(K, \theta_F) \), \( V^s_K(K, \theta_s) = V^m_K(K, \theta_s) \) and \( V^m_K(K, \theta_\tau) = V^\tau_K(K, \theta_\tau) \). Thus, (4.22) through (4.25) fully describes the continuous function \( V_K(K, \theta) \) over the entire range of \( \theta \).

We can observe that each expression of \( V_K(K, \theta) \) consists of two terms except in the case of full abatement (i.e., for (4.22)). \( V^F_K(K, \theta) \) and the last terms in (4.23), (4.24), and (4.25) are linked with the expected present value of the stream of marginal benefits from investment, which are equal to the reductions in expected aggregate abatement costs, assuming the current policy scheme is maintained. (4.22) and the last term of (4.24) are increasing in \( \theta \) while the last terms in (4.23) and (4.25) are decreasing in \( \theta \). The fact that the effects of \( \theta \) on these terms alternate implies that the first terms in (4.23), (4.24), and (4.25) can be interpreted as possible gains and loss from changes in policy schemes, which occur when \( \theta \) goes beyond either of the cutoff values. For example, assume that the realized value of \( \theta \) is between \( \theta_F \) and \( \theta_s \),
so consider \( V_s^K(K, \theta) \). Note that the last term of \( V_s^K(K, \theta) \) increases as \( \theta \) decreases, indicating that the present value of the stream of benefits from investment increases as \( \theta \) decreases in this interval. However, as \( \theta \) decreases further and goes below \( \theta_F \), the effect of \( \theta \) on the benefits of investment changes; in particular, lower values of \( \theta \) below \( \theta_F \) reduce the benefits of investment. Thus, from the perspective of the price floor, arriving at and converting to zero-emissions implies a possible loss in the benefits of investment. Interestingly, the cutoff value of \( \theta \) at the price ceiling (\( \theta_r \)) does not appear in \( V_K(K, \theta) \) below the price ceiling. This result is consistent with the results in Dixit (1991) and Roques and Savva (2009), who examine the effects of a price ceiling on the irreversible investment in a competitive market and an oligopolistic market, respectively. Both show that if the output price that triggers investment is less than the price ceiling, this price is not affected by the ceiling.

The boundaries of investment and non-investment intervals of \( \theta \) are determined as solutions to \( V_K(K, \theta) = w \). However, note that \( V_K(K, \theta) \) is non-linear in \( \theta \) over its entire range, which makes it very difficult to find explicit solutions of \( \theta \). Therefore, to examine the boundaries of the investment and non-investment intervals of \( \theta \), we need to investigate the structure of (4.22), (4.23), (4.24), and (4.25) in more detail.

### 4.3 Investment decision rules under a hybrid policy

Since the marginal value function at each interval includes the effects of the other intervals, it has proven to be difficult to find explicit expressions of the boundary of investment and non-investment intervals of \( \theta \). Instead, to specify intervals of \( \theta \) when investment occurs and how these intervals changes as \( K \) increases, we use how \( V_K(K, \theta) \) changes with respect to \( \theta \). In this section, we present 1) the conditions that determine the shape of \( V_K(K, \theta) \) and 2) how the investment and non-investment intervals of \( \theta \) change as \( K \) increases. Then, in section 4 we will compare the investment
decision under a hybrid price and quantity regulation with the decision under a pure permit trading program.

4.3.1 Shape of the marginal value function $V_K(K, \theta)$

The expected marginal value function is a complicated function of $\theta$, which has important implications for whether new investments occur with realizations of $\theta$. The following lemma provides a qualitative summary of the potential forms of $V_K(K, \theta)$. It is proved in section C.2 in Appendix C.

**Lemma 1:** $V_K(K, \theta)$ has at least one and at most two local maxima with respect to $\theta$. All maxima occur at the price controls. In addition,

$$\lim_{\theta \to 0} V_K(K, \theta) = \lim_{\theta \to \infty} V_K(K, \theta) = 0.$$  

4.3.2 Investment decisions

The form of $V_K(K, \theta)$ depends on the policy parameters $(\bar{a}, s, \tau)$ and the unregulated emissions level $Q$ given $r$ and $(\alpha, \sigma^2)$. In this subsection we examine how alternative shapes of $V_K(K, \theta)$ as specified in Lemma 1 determine optimal investment decisions, and how these decisions change as the abatement capital stock increases.

4.3.2.1 $V_K(K, \theta)$ has two local maxima

$V_K(K, \theta)$ has two local maxima at the price floor and ceiling, respectively, if the following conditions hold:

$$1 - \left( \frac{Q}{\bar{a}} \right)^{\beta_2+1} > \frac{1}{2} \left( 1 + \frac{1}{\beta_2} \right);$$  

$$\left[ 1 - \left( \frac{Q}{\bar{a}} \right)^{\beta_2+1} \right] \left( \tau \frac{s}{\bar{a}} \right)^{\beta_2-1} < \frac{1}{2} \left( 1 + \frac{1}{\beta_2} \right).$$
Derivation of these conditions are in the proof of Lemma 1 in subsection C.2.2 in Appendix C. When $\beta_2 < -1$, the right side of (4.26) is strictly less than 1/2. Thus, (4.26) is more likely to hold if the value of $(Q/{\bar{a}})^{\beta_2+1}$ is lower. Since $\beta_2 + 1 < 0$, this implies that the target abatement $\bar{a}$ is set relatively low compared to unregulated emissions $Q$. Thus, $\theta_F(K) = s/c(K)Q$ and $\theta_s(K) = s/c(K)\bar{a}$ are not too close for the benefits from increasing abatement at the price floor to be reflected in $V_K(K, \theta)$. Given (4.26), for (4.27) to hold true at the same time, $(\tau/s)^{\beta_2-1}$ must be small enough to change the direction of the inequality in (4.27). Since $\beta_2 - 1 < 0$, this requires that $\tau/s$ is not close to 1, or rather the price controls are not close to each other. When the interval between the price controls $(s, \tau)$, and thus the interval of $(\theta_s(K), \theta_{\tau}(K))$ are not short, neither the price ceiling or the floor binds easily. Consequently, the effects of the market when neither price control binds is reflected in $V_K(K, \theta)$.

To understand the shape of $V_K(K, \theta)$ when the intervals of $[\theta_F, \theta_s]$ and $[\theta_s, \theta_{\tau}]$ have sufficient lengths, we will examine how prices and abatement levels vary over $\theta$. Figure (4.1) shows prices, abatement levels, and the marginal value function under a hybrid policy when it has two local maxima and the global maximum is at the price floor. In $\theta \in (0, \theta_F]$, the permit price binds at the price floor $\sigma$ and the abatement level is fixed at the unregulated level of emissions $Q$. That is, all emissions are abated. In this interval, $V_K(K, \theta)$ is linearly increasing in $\theta$. Higher levels of $\theta$ (i.e., higher marginal abatement costs), given full abatement, increases the marginal value of investments in reducing abatement costs.

In the interval $\theta \in [\theta_F, \theta_s]$, the price still binds at the price floor. Abatement is decreasing in $\theta$, but it is still greater than target abatement $\bar{a}$. This is because when realized values of $\theta$ are low (marginal abatement costs are low) firms can abate more to sell unused permits back to the government. At the start of this interval, there is short part where $V(K, \theta)$ is increasing in $\theta$. This is due to the full-abatement constraint. However, over most of the interval $[\theta_F, \theta_s]$, decreasing abatement as $\theta$
increases causes the marginal value from investment to decrease over most of the interval.

In \( \theta \in [\theta_s, \theta_t] \), neither of price controls binds and thus price is linearly increasing in \( \theta \), but the abatement level is fixed at target abatement level \( \bar{a} \). Given the target abatement level, increasing values of \( \theta \) increase marginal abatement cost and the permit price, and thus the marginal value from investment also increases over much of \([\theta_s, \theta_t]\). However, for lower values of \( \theta \) in \([\theta_s, \theta_t]\), the chance of falling into the interval \([\theta_F, \theta_s]\) increases, which can cause the decreasing part of \( V_K(K, \theta) \) at the beginning of \([\theta_s, \theta_t]\).

Finally, in \( \theta \in [\theta_t, \infty) \) the permit price binds at a price ceiling \( \tau \), and abatement is decreasing in \( \theta \). Decreasing abatement causes the marginal value from investment to decrease over most of the interval. However, at the beginning of this interval there exists a short part where \( V_K(K, \theta) \) is increasing because of the potential benefits of decreasing \( \theta \) and falling into \([\theta_s, \theta_t]\) dominates the effects of decreasing abatement.

The facts that \( V_K(K, \theta) \) is non-monotonic in \( \theta \) and has at least one maxima highlight the importance of the unit cost of capital \( w \). If the global maximum of \( V_K(K, \theta) \) is strictly less than \( w \), then investment will not occur at any realized value of \( \theta \). Suppose that the unit cost of capital stock is \( w' \), which is greater than the global maximum of \( V_K(K, \theta) \) at initial state as shown in Figure (4.2a). Since \( V_K(K, \theta) < w' \) for any \( \theta \), investment can’t ever occur. We can’t exclude this situation as the initial state regardless of the shape of \( V_K(K, \theta) \). However, to explain the dynamics of the capital stock and the investment intervals, suppose that the unit cost of capital stock is \( w \), which is strictly less than the global maximum at the initial state as shown in Figure (4.2a).

**Global maximum at the price floor** When \( V_K(K, \theta) \) has two local maxima, the location of the global maximum determines its shape. Consider the case where the global maximum of \( V_K(K, \theta) \) occurs at the price floor. Suppose at first that the
Figure 4.1: Price, abatement, and marginal value function 1
current capital stock is \( K_1 \) such that \( V_K^{n_1}(K_1, \theta) > w \) for all \( \theta_s(K_1) < \theta < \theta_r(K_1) \). Then, the investment interval given \( K_1 \) is \((\theta^{x1}(K_1), \theta^{x2}(K_1))\) as shown in Figure (4.2a). If the realized value of \( \theta \) is between \((\theta^{x1}(K_1), \theta^{x2}(K_1))\), then \( V_K(K_1, \theta) > w \) and thus investment occurs immediately to increase the capital stock. However, if the realized value of \( \theta \) is either below \( \theta^{x1}(K_1) \) or above \( \theta^{x2}(K_1) \), then investment does not occur and the current capital stock \( K_1 \) is maintained for that time period. Note that with \( K_1 \), additional investment can occur if the realized value of \( \theta \) results in full abatement \((\theta \leq \theta_F(K_1))\), the price floor binds \((\theta \leq \theta_s(K_1))\), when neither price control binds \((\theta \in (\theta_s(K_1), \theta_r(K_1)))\), and when the price ceiling binds \((\theta \geq \theta_r(K_1))\).

With this low level of capital stock, investment does not occur only when \( \theta \) is very low under full abatement or very high when the price ceiling binds.

Now suppose that investment occurs at \( K_1 \) and thus the current capital stock increases to \( K_2 \). Since \( V_K(K, \theta) < 0 \), the reduction in \( V_K(K, \theta) \) due to this investment can cause the investment interval to contract to two disjoint intervals, \((\theta^{x1}(K_2), \theta^{x2}(K_2))\) and \((\theta^{x3}(K_2), \theta^{x4}(K_2))\) as shown in Figure (4.2b). In this case, there are values of \( \theta \) that do not motivate additional investment when neither price control binds. However, if further investment occurs say to \( K_3 \), then because the global maximum of \( V_K(K, \theta) \) occurs at the price floor, \( V_K(K_3, \theta) \) might be such that investment beyond \( K_3 \) only occurs for a small interval at the price floor as shown in Figure (4.2c). Since \( V_K(K, \theta) \) has the global maximum at the price floor, the investment interval around the local maximum at the price ceiling disappears first. Since \( V_K(K, \theta) \) is strictly decreasing in \( K \), eventually investment in abatement capital will cease because the global maximum of \( V_K(K, \theta) \) becomes equal to \( w \) and the values of \( V_K(K, \theta) \) fall strictly below \( w \) for all \( \theta \) except at the global maximum.

\[ \text{4It can be shown that each expression in (4.22)-(4.25) satisfies } V_{KK}(K, \theta) < 0. \text{ The demonstrations are omitted because they are too long. The demonstrations are available upon request.} \]
This is the case for capital stock $K_4$ shown in Figure (4.2d). That there is an upper bound on abatement capital investment under the hybrid scheme is important in comparing investment under a pure emissions market.

**Global maximum at the price ceiling.** Now consider the case where $V_K(K, \theta)$ has two local maxima, and the global maximum occurs at the price ceiling. The shape of $V_K(K, \theta)$ for this case is depicted in Figure (4.3). Consider first the capital stock $K_1$ in Figure (4.3a) which is very low so that the investment interval given $K_1$ is $(\theta^{*1}(K_1), \theta^{*2}(K_1))$. Unless the realized value of $\theta$ is either too low or too high, investment occurs immediately to increase the abatement capital stock. Suppose that additional investment occurs and the current capital stock increases to $K_2$ as in Figure (4.3b). Then, the investment interval contracts to two disjoint intervals, $(\theta^{*1}(K_2), \theta^{*2}(K_2))$ and $(\theta^{*3}(K_2), \theta^{*4}(K_2))$. Each investment interval is formed around the local maxima at the price floor and ceiling. If additional investment takes place to increase the capital stock, say to $K_3$ such that the local maximum at the price floor falls below $w$, then the investment interval appears only around the global maximum at the price ceiling as shown in Figure (4.3c). In contrast, recall from Figure (4.2) that when $V_K(K, \theta)$ has two maxima and the global maximum occurs at the price floor, higher levels of capital eliminates the investment intervals of $\theta$ around the price ceiling first. In all cases, however, no further investment occurs for higher levels of capital. This is depicted in Figure (4.3d).

**4.3.2.2 $V_K(K, \theta)$ has a single maximum at the price floor.**

$V_K(K, \theta)$ has a single maximum point at the price floor if the following conditions hold true:
Figure 4.2: Investment decisions when $V_K(K, \theta)$ has its global maximum at the price floor; $K_1 < K_2 < K_3 < K_4$
Figure 4.3: Investment decisions when $V_K(K, \theta)$ has its global maximum at the price ceiling; $K_1 < K_2 < K_3 < K_4$
\[ 1 - \left( \frac{Q}{\bar{a}} \right)^{\beta_2 + 1} > \frac{1}{2} \left( 1 + \frac{1}{\beta_2} \right); \quad (4.28) \]
\[ \left[ 1 - \left( \frac{Q}{\bar{a}} \right)^{\beta_2 + 1} \right] \left( \frac{\tau}{s} \right)^{\beta_2 - 1} > \frac{1}{2} \left( 1 + \frac{1}{\beta_2} \right). \quad (4.29) \]

From the discussion above, we know that (4.28) occurs if the target level of abatement \( \bar{a} \) is not too close to unregulated emissions \( Q \). Thus, the interval of \((\theta_F (K), \theta_s (K))\) is of sufficient length for the positive effects of the price floor on investment to occur. (4.29) depends on the stringency of the price controls. Given (4.28), for (4.29) to hold true at the same time, \((\tau/s)^{\beta_2 - 1}\) must be large enough to maintain the inequality in (4.29). Since \(\beta_2 - 1 < 0\), this requires that \(\tau/s\) is close to 1, or rather the price controls are close to each other. When the interval between the price controls \((s, \tau)\), and thus the interval of \((\theta_s (K), \theta_r (K))\) are short, either the price ceiling or the floor binds easily. Consequently, the effects of the market when neither price control binds is not reflected in \(V_K (K, \theta)\).

Figure (4.4) shows prices, abatement levels, and the marginal value function under a hybrid policy when \(V_K (K, \theta)\) has a single maximum at the price floor. The qualitative effects of \(\theta\) on prices and abatement levels are the same as in Figure (4.1). However, the shape of marginal value function \(V_K (K, \theta)\) is different. Especially, we can see that the increasing part of \(V_K (K, \theta)\) in the interval of \([\theta_s, \theta_r]\) in Figure (4.1) does not appear in Figure (4.4).

Next, consider how the marginal value function \(V_K (K, \theta)\) and the investment intervals change as the capital stock \(K\) increases when \(V_K (K, \theta)\) has a single maximum at the price floor. As aggregate abatement capital increases, the investment interval contracts as shown in Figure (4.5b). Finally, at a further higher level of capital stock, the single maximum value at the price floor becomes equal to the unit capital costs \(w\) and the investment interval vanishes as shown in Figure (4.5c).
Figure 4.4: Price, abatement, and marginal value function 2
Figure 4.5: Investment decisions when $V_K(K, \theta)$ has a single maximum at the price floor; $K_1 < K_2 < K_3$
4.3.2.3 $V_K (K, \theta)$ has a single maximum at the price ceiling.

$V_K (K, \theta)$ has a single maximum at the price ceiling if the following relationship holds:

$$1 - \left( \frac{Q}{\bar{a}} \right)^{\beta_2 + 1} < \frac{1}{2} \left( 1 + \frac{1}{\beta_2} \right).$$ \hspace{1cm} (4.30)

When $\beta_2 < -1$, the right side of this relationship is strictly less than $1/2$. Thus, the condition holds if $(Q/\bar{a})^{\beta_2 + 1}$ is sufficiently close to 1. Since $\beta_2 + 1 < 0$, this implies that the target level of abatement $\bar{a}$ is set relatively high compared to unregulated emissions $Q$. Thus, $\theta_F (K) = s/c (K) Q$ and $\theta_s (K) = s/c (K) \bar{a}$ are too close for the benefits from increasing abatement at the price floor to be reflected in $V_K (K, \theta)$.

As shown in Figure (4.6), the target abatement level $\bar{a}$ is very high relative to the unregulated emissions $Q$, which makes the interval $[\theta_F, \theta_s]$ very short. As a result we can see that although prices and abatement levels move as usual with $\theta$, the marginal value function $V_K (K, \theta)$ is different from the previous cases in Figures (4.1) and (4.4). $V_K (K, \theta)$ has a single maximum at the price floor, which means that the decreasing part of $V_K (K, \theta)$ in the interval of $[\theta_F, \theta_s]$ in Figures (4.1) and (4.4) does not appear in Figure (4.6).

Next, consider how the marginal value function $V_K (K, \theta)$ and investment intervals change as the capital stock $K$ increases when $V_K (K, \theta)$ has a single maximum at the price ceiling. As aggregate abatement capital increases, the investment interval shrinks as shown in Figure (4.7b). In Figures (4.7a) and (4.7b), investment can only occur when the price ceiling binds, but the price ceiling still works to limit investment for high values of $\theta$. In addition, it is possible that the lower bound of the investment interval occurs when the price ceiling does not bind. As in all cases, there is no further investment once the capital stock reaches higher levels like $K_3$ in Figure (4.7c).
Figure 4.6: Price, abatement, and marginal value function 3
Figure 4.7: Investment decisions when $V_K(K, \theta)$ has the single maximum at the price ceiling: $K_1 < K_2 < K_3$. 

(a) With a capital stock $K_1$

(b) With a capital stock $K_2$

(c) With a capital stock $K_3$
4.4 Investments in abatement capital under a hybrid policy and a pure permit market

4.4.1 Derivation of investment decisions under a pure permit market

To further examine the effects of a hybrid price and quantity regulation on investment in aggregate abatement capital, we will compare the investment intervals under a hybrid policy with the investment intervals under a pure permit trading. We begin by deriving the investment intervals under pure permit trading. Aggregate compliance costs under the pure permit trading are just equal to the aggregate abatement costs of achieving the target abatement $a$, because unlike a hybrid price and quantity regulation, there are no additional purchases or sales of permits. Thus, aggregate compliance costs under a pure permit market are

$$S^{pp}(K, \theta; a) = \frac{1}{2} c(K) \theta a^2.$$  

For the comparison between the hybrid policy and a pure permit market, we assume that the random shock $\theta$ follows the same stochastic process under both regulations and we impose the parameter restrictions of the random shock (4.3). Then, by the same steps we used to derive $V_K(K, \theta)$, which is shown in section C.3 in Appendix C, the expected marginal value function under the pure market, denoted by $V^{pp}_K(K, \theta)$, is

$$V^{pp}_K(K, \theta) = -\frac{\beta_1 - 1}{\beta_1} \frac{c'(K) \theta a^2}{2(r - \alpha)},$$  

(4.31)

where $\beta_1 > 1$ is a positive solution to $\frac{1}{2} \sigma^2 \beta (\beta - 1) + \alpha \beta - r = 0$. (4.31) consists of two parts: an option value coefficient $(\beta_1 - 1)/\beta_1 < 1$ and the present value of the stream of marginal benefits from investment, $-c'(K) \theta a^2 / 2(r - \alpha)$. Since (4.31) is linearly increasing in $\theta$, given $K$ there is a single boundary for the non-investment and investment intervals of $\theta$, which is the solution to $V^{pp}_K(K, \theta) = w$. Denote this
boundary as $\theta^*(K)$. Since (4.31) is monotonically increasing in $\theta$, additional investment occurs for realizations of $\theta$ above $\theta^*(K)$.

**4.4.2 Comparison of $V_K(K, \theta)$ and $V_{KPP}(K, \theta)$**

The following proposition shows the relationships between the expected marginal values of further investment under the hybrid scheme and the pure market. It is proved in section C.4 in Appendix C. In the next subsection, we will use this proposition to help us compare how the investment and non-investment intervals of $\theta$ differ under the two regulations.

**Proposition 1:** 1) Given aggregate abatement capital $K$, there is a single value of $\theta > 0$, call it $\theta_I(K)$, such that $V_K(K, \theta_I(K)) = V_{KPP}(K, \theta_I(K))$.

2) In addition, $\theta_I(K) > \theta_T(K)$.

3) Moreover, the following relationships always hold:

$$V_K(K, \theta) > V_{KPP}(K, \theta) \text{ for } 0 < \theta < \theta_I(K)$$

and

$$V_K(K, \theta) < V_{KPP}(K, \theta) \text{ for } \theta > \theta_I(K).$$

**4.4.2.1 Comparison of the investment and non-investment intervals**

Proposition 1 allows us to compare the investment and non-investment intervals of $\theta$ under a hybrid policy and a pure permit trading. In addition, we look for how these intervals respond to increases in the abatement capital stock. The overall results from these comparisons are as follows:

**Finding 1:** Given $K$, if there exists an investment interval (or intervals) of $\theta$ under the hybrid scheme, there is an upper bound of $\theta$ after which no additional investment
occurs. This upper bound is due to the price ceiling. However, there is no such upper bound of $\theta$ under a pure permit market. Thus, in a period, high realizations of the random shock to abatement costs may trigger additional investment under the pure emissions market, but not under the hybrid scheme.

**Finding 2:** Under the hybrid regulation there exists an upper bound of the abatement capital stock such that no additional investment will occur. No such upper bound exists for the pure emissions market.

**Finding 3:** Given $K$, investment intervals of $\theta$ under the hybrid policy may exist for low realizations of $\theta$ that do not exist under the pure market regulation. Thus, there may be investment opportunities for low realizations of abatement costs under the hybrid scheme that are not available under the pure market.

To demonstrate these results, we’ll compare investment intervals under a hybrid scheme and under a pure market using the case where $V_K(K, \theta)$ has two local maxima and the global maximum occurs at the price floor, which is depicted in Figure (4.8). As following Proposition 1, in every panel the expected marginal value function under a hybrid policy and under pure permit trading, which are marked by $V_K(K, \theta)$ and $V_{Kp}^{pp}(K, \theta)$, respectively, intersect only once after $\theta_\tau(K)$ except when $\theta = 0$. Moreover, for any level of capital stock, $V_K(K, \theta)$ is higher than $V_{Kp}^{pp}(K, \theta)$ before the intersection while $V_K(K, \theta)$ falls below $V_{Kp}^{pp}(K, \theta)$ after the intersection. These relationships hold regardless of the specific shape of $V_K(K, \theta)$.

Figure (4.8a) considers $V_K(K, \theta)$ and $V_{Kp}^{pp}(K, \theta)$ for an initial low level of capital stock like $K_1$, such that $V_K(K_1, \theta) > w$ for all $\theta \in [\theta_s(K_1), \theta_\tau(K_1)]$. Investment

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5In Figure (4.8), we can see that $V_K(K, \theta)$ in the interval of $[\theta_s, \theta_\tau]$ and $V_{Kp}^{pp}(K, \theta)$ have very similar structures. However, according to Proposition 1, $V_K(K, \theta)$ is higher than $V_{Kp}^{pp}(K, \theta)$ in this interval. Under a hybrid policy, there is a chance that low values of $\theta$ fall into the interval of $[\theta_F, \theta_s]$ and thus firms can increase abatement. This potential benefit under a hybrid policy makes $V_K(K, \theta)$ higher than $V_{Kp}^{pp}(K, \theta)$. However, as $\theta$ increases, the probability that $\theta$ falls into $[\theta_F, \theta_s]$ decreases, which makes the gap between $V_K(K, \theta)$ and $V_{Kp}^{pp}(K, \theta)$ decrease.
under a hybrid policy will occur except when the realized value of $\theta$ is either too low or too high. While investment is triggered only by higher values of $\theta$ under pure permit trading, investment under a hybrid policy can occur for lower values of $\theta$ (Finding 3). However, investment does not occur under a hybrid policy for higher values of $\theta$ because of the price ceiling, but would occur under pure permit trading (Finding 1). Suppose that investment occurs at $K_1$ and capital stock increases to $K_2$. Then, as shown in Figure (4.8b), a hybrid policy have two disjoint investment intervals. The left one for smaller values of $\theta$ lies in the non-investment interval under a pure permit trading (Finding 3). The right investment interval under a hybrid policy starts from slightly lower values of $\theta$ (which is indicated by the red lines) than the investment interval under pure permit trading (which is indicated by black lines). This does not always occur; that is, it is also possible that the lower boundary of the upper investment interval is equal to or greater than the lower boundary of the investment interval under pure permit trading. However, the investment interval under the hybrid policy still has an upper bound but there is no upper bound under pure permit trading (Finding 1).

Suppose again that investment occurs at $K_2$ and capital stock increases to $K_3$ such that the local maximum at the price ceiling under the hybrid policy falls below $w$, but the global maximum at the price floor is still higher than $w$. Then, the left investment interval in Figure (4.8c) still remains but is smaller, while the right interval disappears. Finally, as in all cases, for high level of capital stock like $K_4$ in Figure (4.8d), the global maximum also falls below $w$ and thus there is no further investment under the hybrid policy. However, under the pure permit trading there are still possible realizations of $\theta$ for which investment occurs under pure permit trading (Finding 2).
Figure 4.8: Comparison when $V_K(K, \theta)$ has the global maximum at the price floor; $K_1 < K_2 < K_3 < K_4$
4.4.2.2 Comparison the levels of investment

As shown in Figure (4.8), the investment interval(s) under each policy can have some overlapping intervals even when a hybrid policy has two disjoint investment intervals. Thus, given the abatement capital stock, investments can occur under both policies for some values of a random shock. From Proposition 1, we can compare increases in the capital stock under each policy when the realized value of $\theta$ falls into the overlapping investment interval and thus investments occur under both policies.

**Corollary 1:** Given a capital stock $K$ and the realized value of a random shock $\theta'$ that satisfy both $V_K(K, \theta') > w$ and $V_{pp}^p(K, \theta') > w$, investments occur to increase the capital stock such that $V_K(K', \theta') = w$ under a hybrid policy and $V_{pp}^p(K'', \theta') = w$ under a pure permit market, respectively. Then,

1) If $\theta' > \theta_I(K)$, then $K' - K < K'' - K$. That is, if the realized value of the random shock $\theta'$ is greater than the intersection of $V_K(K, \theta)$ and $V_{pp}^p(K, \theta)$, then investment under pure permit trading is greater than under a hybrid policy.

2) If $\theta' < \theta_I(K)$, then $K' - K > K'' - K$. If the realized value of the random shock $\theta'$ is less than the intersection of $V_K(K, \theta)$ and $V_{pp}^p(K, \theta)$, then investment under a hybrid policy is greater than under pure permit trading.

**Proof.** We’ll prove only the part (1) of Corollary 1. Part (2) can be derived using the same steps. Suppose that $\theta' > \theta_I(K)$. From Proposition 1, we know that $V_K(K, \theta') < V_{pp}^p(K, \theta')$. Let the increased capital stock under a hybrid policy be $K'$ such that $V_K(K', \theta') = w$. Then,

$$w = V_K(K', \theta') < V_{pp}^p(K', \theta').$$
under pure permit trading implies over-investment because the capital stock increases more than the level that is required to restore the equality. Therefore, such that \( V_{K}^{pp} (K'', \theta') = w \) must be less than \( K' \). □

Proposition 1 shows that given \( K \), \( V_{K} (K, \theta) \) and \( V_{K}^{pp} (K, \theta) \) intersects only at the price ceiling. Thus, if investment occurs at the price ceiling, it is likely that the realized value of \( \theta \) is greater than the intersection of \( V_{K} (K, \theta) \) and \( V_{K}^{pp} (K, \theta) \). In turn, from Corollary 1, investments at the price ceiling are likely to be less than investments under pure permit trading.

4.5 Comparative statics and numerical examples

The parameters in our model can be separated into groups: 1) the policy variables \((\bar{a}, s, \tau)\) and 2) the parameters of the distribution of \( \theta \), \((\alpha, \sigma^2)\). In the following two subsections, we will show the effects of changing each of these parameters on the investment decision. To do this, we have to find how \( V_{K} (K, \theta) \) is affected by changes in each parameter. However, since the marginal value function \( V_{K} (K, \theta) \) has complicated expressions depending on \( \theta \), sometimes it is not possible to obtain analytical expressions for changes in \( V_{K} (K, \theta) \). For these cases, we will add numerical examples. For numerical examples we need to assume the functional form of \( c (K) \).

Following Zhao (2003) and Park (2012), we assume that

\[
c (K) = c_0 K^{-\gamma}.
\]

For our base case we assume that \( Q = 200, \bar{a} = 60, s = 25, \tau = 75, r = 0.075, \alpha = 0, \sigma = 0.1, K_0 = 100, \gamma = 0.04, \) and \( c_0 = 1 \). For the initial value of a random shock, we assume that \( \theta (0) = 1 \). Then, with these parameters, the expected permit price at the initial period is \( p (0) = c (K_0) \theta (0) \bar{a} \approx 50 \). Note that we assume that the price
floor and ceiling are symmetric around the initial expected price of permits. Finally, we assume that $w = 8$ is the unit cost of capital.

### 4.5.1 Comparative statics with respect to policy parameters

#### 4.5.1.1 Changes in $\bar{a}$

In our model, the target abatement level $\bar{a}$ affects: 1) the two cutoff values $\theta_s(K)$ and $\theta_r(K)$ and 2) $V^m_K(K, \theta)$ and $V^r_K(K, \theta)$. From (4.7), (4.8), (4.24), and (4.25), we can find:

$$\frac{\partial \theta_r}{\partial \bar{a}} = -\frac{\tau}{c(K) \bar{a}^2} < -\frac{s}{c(K) \bar{a}^2} = \frac{\partial \theta_s}{\partial \bar{a}} < 0,$$

$$\frac{\partial V^m_K(K, \theta)}{\partial \bar{a}} = \beta_1 - 1 \frac{1}{r - \alpha} \frac{c'(K)}{c(K)} s \left[ \left( \frac{\theta}{\theta_s} \right)^{\beta_2} - \left( \frac{\theta}{\theta_s} \right) \right] > 0,$$

$$\frac{\partial V^r_K(K, \theta)}{\partial \bar{a}} = \beta_1 - 1 \frac{1}{r - \alpha} \frac{c'(K)}{c(K)} \left( \frac{\theta}{\theta_r} \right)^{\beta_2} \left[ \left( \frac{\theta}{\theta_s} \right)^{\beta_2-1} - 1 \right] > 0.$$

Thus, increases in the target abatement level $\bar{a}$ decrease both $\theta_s$ and $\theta_r$. Since $\theta_r$ is reduced more than $\theta_s$, in addition, the interval $(\theta_s, \theta_r)$ where neither of the price controls binds also decreases. Since $V^m_K(K, \theta)$ and $V^r_K(K, \theta)$ increase, we can expect that the investment intervals also increase as $\bar{a}$ increases. For instance, Figure (4.9) displays how the marginal value function $V_K(K, \theta)$ and thus the investment intervals change for different values of $\bar{a}$. Both graphs are drawn with the parameters of the base case which are explained above and $w = 8$. Holding the other parameters constant, only the values of the target abatement level $\bar{a}$ are changed. For the upper panel of Figure (4.9), we use $\bar{a}_1 = 40$, $\bar{a}_2 = 60$, and $\bar{a}_3 = 80$. For $\bar{a}_1 = 40$, there exists the investment interval only around the global maximum at the price floor because the local maximum at the price ceiling is strictly below $w = 8$. However, as $\bar{a}$ increases to $\bar{a}_2 = 60$ and then to $\bar{a}_3 = 80$, $V^m_K(K, \theta)$ and $V^r_K(K, \theta)$ both increase and thus the local maximum at the price ceiling can rise above $w$. This causes a disjoint investment interval to appear around the local maximum at the price ceiling.
The investment interval around the local maximum at the price floor is unlikely to increase because \( V_F(K, \theta) \) and \( V^s_K(K, \theta) \) are not affected by changes in \( \bar{a} \).

In the bottom panel of Figure (4.9), we illustrate the investment intervals as the target abatement level \( \bar{a} \) is changed from 20 to 120. Given \( Q = 200 \), the rates of the target abatement vary from 10% to 60%. Lower boundaries of the investment intervals are denoted by \( \theta_1 \) and \( \theta_3 \) with \( \theta_1 < \theta_3 \) and upper boundaries of the investment intervals are marked by \( \theta_2 \) and \( \theta_4 \) with \( \theta_2 < \theta_4 \). In addition, dashed line marked by \( \theta_{pp} \) represents the lower boundary of the investment interval under a pure permit trading. (There is no upper boundary of this interval). Given a target abatement level \( \bar{a} \), the investment intervals of lower and higher values of \( \theta \) are denoted by \( II_L(\bar{a}) \) and \( II_H(\bar{a}) \), respectively. For lower values of \( \bar{a} \), investment occurs only for small values of \( \theta \), in \( II_L(\bar{a}) \). For \( \bar{a}_1 = 40 \), there is only one investment interval. However, as \( \bar{a} \) becomes greater than about 50, the local maximum at the price ceiling rises above \( w = 8 \) and thus the disjoint investment intervals consisting of higher values of \( \theta \) appear such as \( II_H(\bar{a}_2) \) and \( II_H(\bar{a}_3) \). As \( \bar{a} \) increases, \( \theta_3 \) decreases while \( \theta_4 \) keeps increasing. Thus, as \( \bar{a} \) gets larger and larger the investment intervals, especially \( (\theta_3, \theta_4) \), expand. If the target abatement level \( \bar{a} \) is greater than about 100, then the local minimum value of \( V_{mK}(K, \theta) \) goes beyond \( w \), which implies that \( \theta_2 \) and \( \theta_3 \) become equal and thus disjoint investment intervals are connected. So \( (\theta_1, \theta_4) \) becomes the sole investment interval. Increases in the target abatement level increase the marginal value of the capital stock and so the investment intervals expand.

### 4.5.1.2 Changes in \( s \)

When the price floor \( s \) changes, the following things are affected: 1) \( \theta_F(K) \) and \( \theta_s(K) \), and 2) \( V_F^s(K, \theta) \), \( V_{mK}(K, \theta) \), and \( V_{TK}(K, \theta) \). From (4.8), (4.11), (4.23), and (4.24), we can find:
Figure 4.9: Changes in the target abatement level: $\bar{a}_1 < \bar{a}_2 < \bar{a}_3$
\[
\frac{\partial \theta_s}{\partial s} = \frac{1}{c(K)\bar{a}} > \frac{1}{c(K)Q} = \frac{\partial \theta_F}{\partial s} > 0; \\
\frac{\partial V_K^s(K, \theta)}{\partial s} = \frac{\beta_1 + 1}{\beta_1} \frac{1}{r - \sigma^2 + \alpha} \frac{c'(K)Q}{c(K)} \left[ \left( \frac{\theta}{\theta_F} \right)^{\beta_2} - \left( \frac{\theta}{\theta_F} \right)^{-1} \right] > 0; \\
\frac{\partial V_K^m(K, \theta)}{\partial s} = \frac{\partial V_K^s(K, \theta)}{\partial s} = \frac{\beta_1 + 1}{\beta_1} \frac{1}{r - \sigma^2 + \alpha} \frac{c'(K)}{c(K)} \left( \frac{c(K)}{s} \right)^{\beta_2} (Q^{\beta_2 + 1} - \bar{a}^{\beta_2 + 1}) > 0.
\]

Thus, increases in the price floor \( s \) increase both \( \theta_F \) and \( \theta_s \). Since \( \theta_s \) increases more than \( \theta_F \), the interval \( (\theta_F, \theta_s) \) also increases, which makes it more likely that the price floor binds. Since \( V_K^s(K, \theta) \), \( V_K^m(K, \theta) \) and \( V_K^s(K, \theta) \) move upward, we can expect that the investment intervals also increase as \( s \) increases.

For instance, the top panel of Figure (4.10) shows how the marginal value function \( V_K^s(K, \theta) \) moves for different price floors \( s \) and the bottom panel describes how the investment intervals change with \( s \). Both graphs are drawn with the parameters of the base case and \( w = 8 \). With the other parameters fixed, only the level of the price floor \( s \) is changed. For the upper graph, we choose \( s_1 = 15 \), \( s_2 = 25 \), and \( s_3 = 35 \) and for the lower one we vary the level of the price floor from 5 to 50. As depicted in the top panel of Figure (4.10), for \( s_1 = 15 \), there is one investment interval around the global maximum at the price ceiling. However, as \( s \) increases to \( s_2 = 25 \) and \( s_3 = 35 \), \( V_K^s(K, \theta) \), and especially \( V_K^s(K, \theta) \) and \( V_K^s(K, \theta) \), increase which can cause the local maximum at the price floor to rise above \( w \). This creates a new disjoint investment interval appear around the local maximum at the price floor. Note that unlike the previous case of the target abatement level \( \bar{a} \), \( V_K^s(K, \theta) \) and \( V_K^m(K, \theta) \) increase at the same time, which increases the investment interval around the local maximum at the price floor.

Like the previous case, both \( \theta_1 \) and \( \theta_3 (\theta_1 < \theta_3) \) are lower boundaries of the investment intervals and both \( \theta_2 \) and \( \theta_4 (\theta_2 < \theta_4) \) are upper boundaries of the investment intervals. The dashed line marked by \( \theta_{pp} \) is the lower boundary of the investment interval under a pure permit trading. Finally, given a price floor \( s \), the
investment intervals of lower and higher values of \( \theta \) are denoted by \( II_L(s) \) and \( II_H(s) \), respectively. For lower values of \( s \), there exists the investment interval with higher values of \( \theta \) such as \( II_H(s_1) \). However, around \( s = 18 \), the investment interval with lower values of \( \theta \) appears, \( II_L(s) \). Note that \( II_L(s) \) expands as \( s \) increases. In addition, as \( s \) increases \( \theta_3 \) decreases while \( \theta_4 \) increases and thus the investment interval of higher values of \( \theta \), \( II_H(s) \), also increases. Therefore, as \( s \) is increased, the investment intervals tend to expand. For \( s \geq 40 \), the local minimum value of \( V^m_K(K,\theta) \) is greater than \( w = 8 \), which makes \( \theta_2 \) and \( \theta_3 \) equal and thus the two investment intervals are connected to become \( (\theta_1,\theta_4) \).

4.5.1.3 Changes in \( \tau \)

When the price ceiling \( \tau \) changes, the following are affected: 1) \( \theta_\tau \), and 2) \( V^\tau_K(K,\theta) \). From (4.7) and (4.25), we can find:

\[
\frac{\partial \theta_\tau}{\partial \tau} = \frac{1}{c(K) \bar{a}} > 0; \\
\frac{\partial V^\tau_K(K,\theta)}{\partial \tau} = \frac{\beta_1 + 1}{\beta_1} \frac{1}{r - \sigma^2 + \alpha} \frac{c'(K) \bar{a}}{c(K)} \left[ \left( \frac{\theta}{\theta_\tau} \right)^{\beta_2} - \left( \frac{\theta}{\theta_\tau} \right)^{-1} \right] > 0.
\]

As the price ceiling \( \tau \) increases, only \( \theta_\tau \) increases and thus the interval \( (\theta_s,\theta_\tau) \) where neither of the price controls binds also increases. Changes in \( \tau \) affect only \( V^\tau_K(K,\theta) \). As depicted in Figure (4.11), since \( \partial V^\tau_K(K,\theta)/\partial \tau > 0 \), we can expect that the investment interval will increase for higher levels of \( \tau \). However, this only occurs for the investment interval around the local maximum at the price ceiling. Because \( V^E_K(K,\theta), V^a_K(K,\theta), \) and \( V^m_K(K,\theta) \) are not affected by the level of the price ceiling \( \tau \), the lower investment interval around the local maximum at the price floor is not affected at all. Given the base case parameters, we change the values of the price ceiling \( \tau \) from 50 to 90 in the bottom panel of Figure (4.11). For lower values of \( \tau \)
Figure 4.10: Changes in a price floor: $s_1 < s_2 < s_3$
less than 59.4, the local maximum at the price ceiling is below \( w \) and thus the only investment interval exists for low values of \( \theta \), or \( II_L(\tau) \). However, as \( \theta \) increases, the investment interval of high values of \( \theta \), \( II_H(\tau) \), appears and expands.

### 4.5.1.4 Symmetric changes in \((s, \tau)\)

So far we have examined how changes in only one of the two price controls affect the investment intervals. We found that increases in the price floor \( s \) make investment at lower values of \( \theta \) more likely while increases in the price ceiling \( \tau \) cause investment to occur at higher values of \( \theta \). However, since in our base case we assume that the price floor and ceiling are symmetric around the initial expected price, we now change the price floor and ceiling at the same time keeping them symmetric around \( p(0) \approx 50 \).

The symmetric price floor and ceiling can be expressed by

\[
s = (1 - \delta) p(0); \quad \tau = (1 + \delta) p(0).
\]

We will now change \( \delta \) and the effects on the marginal value function \( V_K(K, \theta) \) and the investment intervals will be examined using numerical examples. Given the base case parameters, we change \( \delta \) from 10\% to 80\%, that is, \( s \) decreases from 45 to 10 while \( \tau \) increases from 55 to 90. The top panel of Figure (4.12) shows the marginal value functions for \( \delta_1 = 15\% \), \( \delta_2 = 50\% \), and \( \delta_3 = 80\% \). For low values of \( \delta \), \( s \) is relatively high while \( \tau \) is relatively low. Thus, the global maximum occurs at the price floor. However, as \( \delta \) increases, \( V^s_K(K, \theta) \) and \( V^m_K(K, \theta) \) move downward while \( V^c_K(K, \theta) \) moves upward. As a result, for higher value of \( \delta \), the location of the global maximum moves from the price floor to the price ceiling. We can expect that the investment interval with lower values of \( \theta \), marked as \( II_L(\delta) \) in the bottom panel of Figure (4.12), will shrink and finally disappear. However, the investment interval with higher values of \( \theta \), marked as \( II_H(\delta) \) in the bottom panel, will expand as \( \delta \) increases.
Figure 4.11: Changes in a price ceiling: $\tau_1 < \tau_2 < \tau_3$
For $\delta \leq 0.2$, the investment interval $(\theta_1, \theta_4)$ has relatively low values of $\theta$ compared to the investment interval $(\theta_{pp}, \infty)$ under pure permit trading. For $0.2 < \delta \leq 0.64$, the investment interval of low values of $\theta$, $II_L(\delta)$, keeps falling while the investment interval of high values of $\theta$, $II_H(\delta)$, keep increasing. Finally, for $\delta > 0.64$, the investment interval of low values of $\theta$ disappears and thus investment will occur only at high values of $\theta$.

4.5.2 Comparative statics with respect to $(\alpha, \sigma)$

4.5.2.1 Changes in $\sigma$

Since the marginal value function under pure permit trading, $V^{PP}_K(K, \theta)$, is monotonically decreasing in $\sigma$, the lower bound of the investment interval under a pure permit trading is strictly increasing in $\sigma$. This implies that higher values of $\theta$ are required to stimulate investment under greater uncertainty. However, changes in $V_K(K, \theta)$ under a permit market with price controls are not monotonic in $\sigma$. Moreover, depending on the values of $\theta$, it is possible that $V_K(K, \theta)$ can increase as $\sigma$ increases. Given $r$ and $\alpha$, the values of $\sigma$ must be $\sigma^2 < r + \alpha$ from the second condition of (4.3). With the base case parameters of $r = 0.075$ and $\alpha = 0$, we examine values of $\sigma$ from 0.01 to 0.25. The results of this numerical exercise are shown in Figures (4.13) and (4.14).

At first, from Figure (4.13), we see the two distinguishing movements of $V_K(K, \theta)$ as $\sigma$ increases: the two local maximum values shift down while the local minimum value increases, causing $V_K(K, \theta)$ to become flatter. Decreases in $V_K(K, \theta)$ are likely to reduce the investment interval. However, increases in the local minimum value of $V_K^m(K, \theta)$ may cause the investment intervals to expand.
Figure 4.12: Symmetric changes in the interval of $(s, \tau)$; $\delta_1 < \delta_2 < \delta_3$
Figure 4.13: Changes in $V_K(K, \theta; \sigma)$ for $\sigma_1 < \sigma_2 < \sigma_3$

The effects of changes in $\sigma$ on the investment intervals can differ depending on the relative size of unit cost $w$ and capital stock $K$. Given the base case parameters with $w = 8$, changes in the investment intervals over $\sigma$ are shown in Figure (4.14a). The lower boundaries of the investment intervals $\theta_1$ and $\theta_3$ keep increasing as $\sigma$ increases. However, the upper boundaries $\theta_2$ and $\theta_4$ increase for $\sigma \leq 0.19$ and then decrease for $\sigma > 0.19$. Thus, the investment intervals start to expand for lower values of $\sigma$. However, after around $\sigma = 0.19$, they shrink. As a benchmark, the lower bound of the investment interval under pure permit trading keeps increasing as $\sigma$ increases, which implies that higher values of $\theta$ are required to stimulate investments. Under a hybrid policy, $\theta_3$ plays a similar role. As $\sigma$ increases, both $\theta_3$ and $\theta_{pp}$ become larger. However, increases in $\theta_3$ are less than increases in $\theta_{pp}$. This implies that under a hybrid policy it is a little easier for investment to occur than under pure permit trading. However, a hybrid policy still has a upper bound of the investment interval,
Moreover, this upper boundary becomes smaller which reduces the investment intervals of high values of $\theta$ as $\sigma$ increases.

For low unit cost $w = 7$, as depicted in Figure (4.14b), expansion of the investment intervals is enhanced compared to the case of $w = 8$. Given the range of $\sigma$, the investment intervals under a hybrid policy never shrink. However, for the case of $w = 9$, as shown in Figure (4.14c) another interesting thing happens. Because of the downward movement of $V_K(K, \theta)$ over $\sigma$, all of $V_K(K, \theta)$ falls below $w$ around $\sigma = 0.215$ and thus the investment intervals disappear. Similar effects occur for the case of $w = 8$ and a higher initial capital stock $K_0 = 107.5$ (Figure (4.14d)). In this example, the marginal rate of return to investment is reduced. As a result, after $\sigma = 0.24$, $V_K(K, \theta)$ falls below $w$ and the investment intervals vanish.

4.5.2.2 Changes in $\alpha$

Like the case of changes in $\sigma$, the marginal value function under pure permit trading, $V_{KP}^P(K, \theta)$, is monotonically increasing in $\alpha$. Therefore, the lower bound of the investment interval under pure permit trading strictly decreasing in $\alpha$. However, the marginal value function under a permit market with price controls, $V_{K}(K, \theta)$, is not monotonic in $\alpha$. Moreover, depending on the values of $\theta$, it is possible that $V_K(K, \theta)$ can decrease as $\alpha$ increases. Given $r$, the values of $\alpha$ must be $r > \alpha$ and $\alpha > \sigma^2 - r$ from (4.3). With the base case parameters of $r = 0.075$ and $\sigma = 0.1$, we vary $\alpha$ from -0.03 to 0.05 in Figure (4.15). From the upper panel of Figure (4.15), we can see that the maximum values of $V_K(K, \theta)$ tend to move upward as $\alpha$ increases while the interior minimum of $V_K(K, \theta)$ tends to shift down. Changes in the investment intervals as $\alpha$ changes are illustrated in the lower panel of Figure (4.15). Note that it is not clear how the lengths of the investment intervals change with $\alpha$. 

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Figure 4.14: Changes in the investment intervals over $\sigma$

(a) $w = 8$ and $K_0 = 100$
(b) $w = 7$ and $K_0 = 100$
(c) $w = 9$ and $K_0 = 100$
(d) $w = 8$ and $K_0 = 107.5$
Figure 4.15: Changes in $\alpha$: $\alpha_1 < \alpha_2 < \alpha_3$
4.5.3 Capital expansions under a hybrid policy and a pure permit trading

Although we derived the investment decision rule under a hybrid policy and compared it with the investment decision rule under pure permit trading, we did not show which policy induces greater investments in abatement capital. In this subsection, we demonstrate expansion paths of the capital stock based on the investment decision rules under both policies and compare them. To execute this numerical example, first we generate 10,000 sample paths of a random shock for 30 years. Then, we apply the two investment decision rules to each sample path. In each period of a sample path, the realized value of a random shock \( \theta' \) is observed. Then given capital stock \( K \) we can determine whether investment is required or not. If it is required, then we can calculate the increase in the capital stock that will satisfy \( V_K(K', \theta') = w \) under the hybrid policy, and \( V_K^{pp}(K', \theta') = w \) under pure permit trading, where \( K' = K + \Delta K \). Note that \( K' \) will likely differ under the two policies.

Except for the parameters of the distribution of \( \theta (\alpha, \sigma^2) \), the other parameters of the model are the same as explained above: \( Q = 200, \bar{a} = 60, s = 25, \tau = 75, r = 0.075, K_0 = 100, \gamma = 0.04, c_0 = 1 \) and \( w = 8 \). We take the value of \( \alpha \) from \( \{-0.03, -0.02, -0.01, 0, 0.01, 0.02\} \) so that \( r > \alpha \) because of (4.3). Then, the permissible values of \( \sigma \) are taken from \( \{0.05, 0.10, 0.15, 0.20, 0.25\} \) given \( r \) and \( \alpha \) so that the second assumption of (4.3), \( r + \alpha > \sigma^2 \), is satisfied. After applying the decision rules to all sample paths, we calculate the average level of capital stock at each period over all sample paths. The results are shown in Figures (4.16) and (4.17). For \( \alpha = -0.03 \), the capital stock is eventually greater under the hybrid policy than under pure permit trading, regardless of the level of volatility \( \sigma \) (as shown in the top-left panel of Figure (4.16)). For \( \alpha = -0.02 \) and low levels of \( \sigma \) such as 0.10 and 0.15, the hybrid policy motivates greater investment than pure permit trading. However, for a high level of volatility (\( \sigma = 0.20 \)), the average level of capital stock under pure
permit trading at the end period is greater than under a hybrid policy (as depicted in the top-right panel of Figure 134). When the expected growth rate of a random shock is greater than or equal to $-0.01$ ($\alpha \geq -0.01$), for any level of volatility $\sigma$, a hybrid policy results in lower investment than pure permit trading, except the case of $\alpha = -0.01$ and $\sigma = 0.1$ where both show almost the same average levels of the capital stock (as shown in the bottom panels of Figure (4.16) and all panels of Figure (4.17)).

Another interesting phenomenon is evident in the cases of the positive expected growth rate $\alpha \geq 0.01$. Under a pure permit trading, as $\sigma$ increases, the average level of the capital stock also increases. However, under a hybrid policy, as $\sigma$ increases total investment in the capital stock decreases. That is, as shown in Figure (4.17), the average level of the capital stock at the end period under a hybrid policy decreases as $\sigma$ increases from 0.05 to 0.15 and then to 0.25. In the previous subsection, we found that as $\sigma$ increases, $V_K(K,\theta)$ can fall below the unit cost $w$ and thus the investment intervals may disappear depending on the relative amount of capital $K$ and unit cost $w$. When $\sigma$ is high, it’s more likely for $V_K(K,\theta)$ to fall below the unit cost $w$ for small increases in the capital stock $K$. For instance, when the expected growth rate is $\alpha = 0.01$, the maximum capital stock level for $\sigma = 0.05$ is 157.63 while the maximum capital stock for $\sigma = 0.25$ is 109.24. Thus, for $\sigma = 0.05$, small increases in the capital stock $K$ still allow additional investment. On the other hand, for $\sigma = 0.25$, small increases in the capital stock $K$ makes most of the marginal value function $V_K(K,\theta)$ falls below $w$, and thus there is not much additional investment after that.

### 4.6 Conclusions

As a cost containment measure, a hybrid policy imposes a price ceiling and a price floor directly on a permit market. It has been already shown that a hybrid policy can outperform a pure permit market in a static setting. Another criterion to
Figure 4.16: Average expansion paths of the capital stock

(a) $\alpha = -0.03$ and $\sigma \in \{0.05, 0.10, 0.15\}$

(b) $\alpha = -0.02$ and $\sigma \in \{0.10, 0.15, 0.20\}$

(c) $\alpha = -0.01$ and $\sigma \in \{0.10, 0.15, 0.20\}$

(d) $\alpha = 0$ and $\sigma \in \{0.10, 0.15, 0.20\}$
Figure 4.17: Average expansion paths of the capital stock 2
evaluate environmental regulation is how much a regulation can induce technological innovations. Considering investment in new abatement capital which reduces total and marginal abatement costs as one aspect of technological innovations, we have investigated the effects of a hybrid price and quantity regulation on irreversible and incremental investment in abatement capital under uncertainty about abatement costs.

We have derived the explicit investment decision rule under a hybrid policy. Investments are determined by an expected marginal value function and unit costs of new investment. We found that the marginal value function under a hybrid policy is not monotonic in shocks to abatement costs. Moreover, the marginal value function has a global maximum over abatement cost shocks and goes to zero as the shocks get large. This implies that there is an upper bound on total investment in the abatement capital stock under the hybrid policy. No such upper bound exists under pure permit trading: there always exists the possibility of further investment under this policy. This is our most important, policy-relevant result.

In addition, we found that the expected marginal value function under a hybrid policy can take four different shapes depending on the levels of the policy instruments, that is, the target abatement level, the price ceiling and the price floor. The investment intervals usually locate around the local maxima and they shrink as investments occur and the capital stock increases. Then, the investment interval around the global maximum can persist longer than the investment interval around the other local maximum as the capital stock increases. If the target abatement level is set high relative to unregulated emissions, then the interval where the price floor binds without achieving full abatement becomes too short for the benefits from increasing abatement at the price floor. Therefore, the marginal value function has the single maximum at the price ceiling. Provided that the target abatement level is not too close to unregulated emissions, the marginal value function has a single
maximum at the price floor if the price ceiling and floor are close to each other. The interval where neither the price ceiling or the price floor bind is so short that either of the price controls can bind easily. Provided that the target abatement level is not too close to the unregulated level of emissions and that the price ceiling and the price floor are not too close, the marginal value function has two local maxima.

By comparing the marginal value functions under a hybrid policy and under pure permit trading, we found that they intersect only one time at the price ceiling. In addition, before they intersect, the marginal value function under a hybrid policy is greater than the marginal value function under pure permit trading. However, after the intersection, the relationship is reversed. As a cost containment measure, the goal of a hybrid policy is to restrict variation in abatement costs. This implies that a hybrid policy keeps abatement costs high when they could be lower without the price floor, and the policy keeps abatement costs low when they could be higher without the price ceiling. These properties flatten the marginal value function under a hybrid policy compared to the marginal value function under a pure emissions permit trading.

If there exists an investment interval (or intervals) of abatement costs shocks under the hybrid policy, there is an upper bound of this shock after which no additional investment occurs. This upper bound is due to the price ceiling. However, there is no such upper bound under a pure permit trading. Thus, a high realization of a random shock to abatement costs may trigger additional investment under a pure emissions permit trading, but not under a hybrid policy. On the other hand, investment intervals of abatement cost shocks under a hybrid policy may exist for low realizations of \( \theta \) that do not exist under a pure market regulation. Thus, there may be investment opportunities for low realizations of abatement costs under the hybrid scheme that are not available under the pure market.
From the comparative statics with respect to the volatility of the abatement cost shocks, we found that increases in uncertainty can have opposite effects on the investment intervals. First, as volatility increases, the investment intervals can increase especially when either the current level of the capital stock or the unit costs of capital stock are low. However, when either the current level of the capital stock or the unit cost of investment are high, the investment intervals can decrease and can even disappear for higher volatility. Numerical examples that demonstrate the expansion paths of the capital stock shows the significant effects of the latter case.

In this paper, by deriving the explicit investment decision rule under a hybrid price and quantity regulation, we were able to compare that with investment decisions under a pure emissions permit trading. However, to show which policy is superior to the other policy in inducing investment in abatement capital is not the goal of this paper. In addition, without calculating a probability that investment occurs given the current state, we cannot argue which one is better than the other one. While we examined the effects of a hybrid policy on irreversible and incremental investment in abatement capital, there are some limits in our model. First, we considered only the case where a random shock follows a geometric Brownian motion to derive explicit solutions of the model. However, it would be useful to consider other cases. Next, we assumed irreversible and incremental investment in abatement capital. However, some types of investment can be exercised only once. This type of investment may characterize the adoption of a better technology which can reduce abatement costs. It would be useful to examine whether a hybrid policy has similar effects in this situation. Finally, we did not consider changes in the levels of policy variables over time. For example, most implemented and proposed carbon policies increase the target abatement level over time. This implies that to keep the price controls symmetric, the levels of price controls also increase at the same rate with which the target abatement increases.
A.1 Proof of Proposition 1

The first task in the proof of the proposition is to characterize the optimal emissions market with price controls, given pollutant 2 is controlled with \( L_2 \) tradable permits and that the constraints in (2.13) do not bind. We then develop the policy choice rules in the proposition by determining when the constraints will bind.

A.1.1 Characterization of a pollutant 1 market with price controls, given \( L_2 \).

We begin by using the definitions of the cut-off values \( u^{\tau_1} \) and \( u^{\sigma_1} \) in (2.9) to specify the following relationships:

\[
Q_1(\tau_1, L_2, u^{\tau_1}) = Q_1(\sigma_1, L_2, u^{\sigma_1}) = \lambda_1; \quad (A.1)
\]

\[
-C_1(Q_1(z, L_2, u), L_2, u) = z, \text{ for } z \in \{\tau_1, \sigma_1\}. \quad (A.2)
\]

(A.1) states that at \( u^{\tau_1} \) the price ceiling \( \tau_1 \) and the permit supply \( \lambda_1 \) bind together, and at \( u^{\sigma_1} \) the price floor \( \sigma_1 \) and the permit supply bind together. (A.2) states that at the price controls, pollutant 1 emissions are such that aggregate marginal abatement...
costs given \( L_2 \) are equal to the price controls. With these results, the first order conditions for the unconstrained version of (2.13) can be written as:

\[
\frac{\partial W (\lambda_1, \tau_1, \sigma_1, L_2)}{\partial \lambda_1} = \int_{u_1}^{u_1^{\tau_1}} \left[ C_1(\lambda_1, L_2, u) + D_1^1(\lambda_1) \right] f(u) \, du = 0;
\]

\[
\frac{\partial W (\lambda_1, \tau_1, \sigma_1, L_2)}{\partial \tau_1} = \int_{u_1}^{u_1^{\tau_1}} \left[ -\tau_1 + D_1^1(Q_1(\tau_1, L_2, u)) \right] \frac{\partial Q_1(\tau_1, L_2, u)}{\partial \tau_1} f(u) \, du = 0;
\]

\[
\frac{\partial W (\lambda_1, \tau_1, \sigma_1, L_2)}{\partial \sigma_1} = \int_{u_1}^{u_1^{\sigma_1}} \left[ -\sigma_1 + D_1^1(Q_1(\sigma_1, L_2, u)) \right] \frac{\partial Q_1(\sigma_1, L_2, u)}{\partial \sigma_1} f(u) \, du = 0.
\]

(A.3)

To write these conditions in terms of the functional forms for abatement costs and the damage functions, we need to specify the emissions response of pollutant 1 to its own price and the quantity of emissions of pollutant 2. Using the aggregate abatement cost function (2.2), note that the explicit form of \( p_1 = -C_1(Q_1, L_2, u) \) is

\[
p_1 = a_1 + u - a_2 Q_1 + w L_2,
\]

which gives us the pollutant 1 emissions response at the price ceiling and price floor:

\[
Q_1(z, L_2, u) = \frac{a_1 + u - z + w L_2}{a_2}, \text{ for } z \in \{\tau_1, \sigma_1\}, \quad (A.5)
\]

with

\[
\frac{\partial Q_1(z, L_2, u)}{\partial z} = -\frac{1}{a_2}, \text{ for } z \in \{\tau_1, \sigma_1\}. \quad (A.6)
\]

Given (A.5) and the marginal damage functions, we also have

\[
D_1^1(Q_1(z, L_2, u)) = d_{11} + d_{12} a_1 + u - z + w L_2, \text{ for } z \in \{\tau_1, \sigma_1\}. \quad (A.7)
\]

Moreover,

\[
D_1^1(\lambda_1) = d_{11} + d_{12} \lambda_1, \quad (A.8)
\]

and

\[
C_1(\lambda_1, L_2, u) = -(a_1 + u) + a_2 \lambda_1 - w L_2. \quad (A.9)
\]
Substitute (A.5) through (A.9) into (A.3) to obtain:

\[
\frac{\partial W(\lambda_1, \tau_1, \sigma_1, L_2)}{\partial \lambda_1} = \int_{u_1}^{u_{\tau_1}} \left[ -a_1 + d_{11} + (a_2 + d_{12}) \lambda_1 - wL_2 - u \right] f(u) \, du = 0;
\]

\[
\frac{\partial W(\lambda_1, \tau_1, \sigma_1, L_2)}{\partial \tau_1} = \int_{u_1}^{u_{\tau_1}} \left( -a_{1d_{12}} + a_{2d_{11}} + a_2 + d_{12} \right) \frac{\tau_1}{a_2^2} - \frac{wd_{12}L_2 - d_{12}}{a_2^2} \, u \right] f(u) \, du = 0;
\]

\[
\frac{\partial W(\lambda_1, \tau_1, \sigma_1, L_2)}{\partial \sigma_1} = \int_{u}^{u_{\sigma_1}} \left( -a_{1d_{12}} + a_{2d_{11}} + a_2 + d_{12} \right) \left( -\frac{\sigma_1}{a_2^2} + \frac{wd_{12}L_2 - d_{12}}{a_2^2} \right) f(u) \, du = 0,
\]

which can be rearranged to obtain:

\[
\lambda_1^* (L_2) = \frac{a_1 - d_{11} + wL_2 + E[u|u_{\sigma_1}^* \leq u \leq u_{\tau_1}^*]}{a_2 + d_{12}}, \quad (A.10)
\]

\[
\tau_1^* (L_2) = \frac{a_1d_{12} + a_2d_{11} + wd_{12}L_2 + d_{12}E[u|u_{\tau_1}^* \leq u \leq \bar{u}]}{a_2 + d_{12}}, \quad (A.11)
\]

\[
\sigma_1^* (L_2) = \frac{a_1d_{12} + a_2d_{11} + wd_{12}L_2 + d_{12}E[u|u \leq u_{\sigma_1}^*]}{a_2 + d_{12}}. \quad (A.12)
\]

In (A.10) through (A.12), \(E[u|u_{\sigma_1}^* \leq u \leq u_{\tau_1}^*]\), \(E[u|u_{\tau_1}^* \leq u \leq \bar{u}]\), and \(E[u|u \leq u_{\sigma_1}^*]\) are conditional expectations of \(u\). Note how the regulation of pollutant 2 and the abatement interaction between the two pollutants affects the choices of the policy variables. Also note that (A.10) through (A.12) are not the solutions for \(\lambda_1^* (L_2)\), \(\tau_1^* (L_2)\) and \(\sigma_1^* (L_2)\), because these variables also appear on the right sides of (A.10) through (A.12).

It is straightforward to calculate the optimal pure tax \(t_1^* (L_2)\) and the optimal number of permits under a pure market \(L_1^* (L_2)\) from (A.10) through (A.12). We do not do so here, because their specific values are not required for the proofs of this paper. The derivations of \(t_1^* (L_2)\) and \(L_1^* (L_2)\) are available upon request.

We are now ready to examine the optimal choices from among the alternative pollutant 1 policies.
A.1.2 When is the tax the preferred policy?

Recall that the optimal policy is the tax \( t_1^* (\bar{L}_2) \) if and only if the solution to (2.13) yields \( \tau_1^* (\bar{L}_2) \leq \sigma_1^* (\bar{L}_2) \). To determine the conditions under which this is true subtract (A.12) from (A.11) to obtain

\[
\tau_1^* (\bar{L}_2) - \sigma_1^* (\bar{L}_2) = \frac{d_{12}}{a_2 + d_{12}} \{ E [u | u^{\tau_1^*} \leq u \leq \bar{u}] - E [u | u \leq u \leq u^{\sigma_1^*}] \} \quad (A.13)
\]

The denominator of (A.13) is strictly positive. Moreover, \( E(u) = 0 \) implies that \( E [u | u \leq u \leq u^{\sigma_1^*}] \leq 0 \) and \( E [u | u^{\tau_1^*} \leq u \leq \bar{u}] \geq 0 \), but both cannot be zero simultaneously. Therefore, \( E [u | u^{\tau_1^*} \leq u \leq \bar{u}] - E [u | u \leq u \leq u^{\sigma_1^*}] > 0 \). Consequently, \( \tau_1^* (\bar{L}_2) \leq \sigma_1^* (\bar{L}_2) \iff d_{12} \leq 0 \). However, since \( d_{12} \geq 0 \), \( \tau_1^* (\bar{L}_2) \leq \sigma_1^* (\bar{L}_2) \Leftrightarrow d_{12} = 0 \), which reveals that the \( t_1^* (\bar{L}_2) \) is the optimal policy if and only if \( d_{12} = 0 \), which is the desired result.

A.1.3 The pure emissions market is never the optimal choice.

Recall from the discussion following (2.13) that a simple emissions market is optimal if and only if

\[
u^{\tau_1^*} = u^{\tau_1^*} (\lambda^*_1 (\bar{L}_2), \tau_1^* (\bar{L}_2), \bar{L}_2) \geq \bar{u} \quad \text{and} \quad u^{\sigma_1^*} = u^{\sigma_1^*} (\lambda^*_1 (\bar{L}_2), \sigma_1^* (\bar{L}_2), \bar{L}_2) \leq \bar{u}.
\] (A.14)

Toward specifying \( u^{\tau_1^*} \) and \( u^{\sigma_1^*} \), recall from (2.9) that the cut-off values of \( u \) are implicitly defined by \( z = -C_1 (\lambda_1, \bar{L}_2, wz) \), for \( z \in \{ \tau_1, \sigma_1 \} \). Explicitly,

\[
u^z = z - a_1 + a_2 \lambda_1 - w \bar{L}_2, \quad \text{for} \quad z \in \{ \tau_1, \sigma_1 \}.
\] (A.15)

At the unconstrained solution to (2.13):

\[
u^{\tau_1^*} (\lambda^*_1 (\bar{L}_2), \tau_1^* (\bar{L}_2), \bar{L}_2) = \tau_1^* (\bar{L}_2) - a_1 + a_2 \lambda_1 (\bar{L}_2) - w \bar{L}_2;
\] (A.16)
\[
u^{\sigma_1^*} (\lambda^*_1 (\bar{L}_2), \tau_1^* (\bar{L}_2), \bar{L}_2) = \sigma_1^* (\bar{L}_2) - a_1 + a_2 \lambda_1 (\bar{L}_2) - w \bar{L}_2.
\] (A.17)
To demonstrate that pure emissions trading cannot be the optimal policy choice, we show that (A.14) cannot hold. After substituting (A.10) and (A.11) into (A.16), it is possible to write

\[ u_{\tau_1} (\lambda_1^* (L_2), \tau_1^* (L_2), L_2) = \frac{d_{12}}{a_2 + d_{12}} E[u | u_{\tau_1}^i \leq u \leq \bar{u}] + \frac{a_2}{a_2 + d_{12}} E[u | u_{\sigma_1}^i \leq u \leq u_{\tau_1}^i]. \]

(A.18)

Now evaluate the conditional expectations in (A.18) under the assumption that a pure market is optimal; that is, \( u_{\tau_1}^i \geq \bar{u} \) and \( u_{\sigma_1}^i \leq \underline{u} \). Since \( E(u) = 0 \) and the support of \( u \) is \( [\underline{u}, \bar{u}] \), \( u_{\tau_1}^i \geq \bar{u} \) and \( u_{\sigma_1}^i \leq \underline{u} \) imply \( E[u | u_{\sigma_1}^i \leq u \leq u_{\tau_1}^i] = 0 \). Therefore, the second term on the right side of (A.18) is equal to zero. To evaluate the first term, note that we cannot directly evaluate the conditional expectation

\[ E[u | u_{\tau_1}^i \leq u \leq \bar{u}] = \int_{u_{\tau_1}^i}^{\bar{u}} u f(u) \, du \int_{u_{\tau_1}^i}^{\bar{u}} f(u) \, du, \]

given \( u_{\tau_1}^i \geq \bar{u} \), but we can use l’Hopital’s rule to determine

\[ \lim_{u_{\tau_1}^i \to \bar{u}} E[u | u_{\tau_1}^i \leq u \leq \bar{u}] = \bar{u}. \]

This implies

\[ u_{\tau_1}^i = u_{\tau_1} (\lambda_1^* (L_2), \tau_1^* (L_2), L_2) = \frac{d_{12} \bar{u}}{a_2 + d_{12}} < \bar{u}. \]

(A.19)

The inequality follows because \( d_{12} / (a_2 + d_{12}) < 1 \). (A.19) implies that (A.14) cannot hold, and therefore, a pure trading scheme for pollutant 1 cannot be the optimal policy. (We could have proved the same by showing that \( u_{\sigma_1} (\lambda_1^* (L_2), \sigma_1^* (L_2), L_2) \leq \underline{u} \) is also not possible).

A.1.4 When is the hybrid preferred to the tax?

Since the tax is optimal if and only if \( d_{12} = 0 \) and a pure trading scheme is never optimal, it must be true that the hybrid policy is optimal if and only if \( d_{12} > 0 \).
A.2 Proof of Proposition 2

The proof of Proposition 2 proceeds in the same way as the proof of Proposition 1. We first characterize the optimal emissions market with price controls when pollutant 2 is controlled with an emissions tax $t_2$, given that the constraints in (2.18) do not bind. We then determine when these constraints do bind to derive the instrument choice rules of the proposition.

A.2.1 Characterization of a pollutant 1 market with price controls, given $\bar{t}_2$.

We begin by using the definitions of the cut-off values $u^{\tau_1}$ and $u^{\sigma_1}$ in (2.14) to specify the following relationships:

\begin{align*}
Q_1 (\tau_1, \bar{t}_2, u^{\tau_1}) &= Q_1 (\sigma_1, \bar{t}_2, u^{\sigma_1}) = \lambda_1; \tag{A.20}
Q_2 (\tau_1, \bar{t}_2, u^{\tau_1}) &= Q_2 (\lambda_1, \bar{t}_2, u^{\tau_1}); \tag{A.21}
Q_2 (\sigma_1, \bar{t}_2, u^{\sigma_1}) &= Q_2 (\lambda_1, \bar{t}_2, u^{\sigma_1}). \tag{A.22}
\end{align*}

We also have:

\begin{align*}
-C_1 \left( Q_1 (z, \bar{t}_2, u), Q_2 (z, \bar{t}_2, u), u \right) &= z, \quad \tag{A.23}
-C_2 \left( Q_1 (z, \bar{t}_2, u), Q_2 (z, \bar{t}_2, u), u \right) &= \bar{t}_2 \text{ for } z \in \{\tau_1, \sigma_1\}. \tag{A.24}
\end{align*}

With (A.20) through (A.24), the first order conditions for the unconstrained version of (2.18) are:
\[
\frac{\partial W (\lambda_1, \tau_1, \sigma_1, \bar{t}_2)}{\partial \lambda_1} = \int_{u_1}^{u_{11}} \left\{ C_1 (\lambda_1, Q_2 (\lambda_1, \bar{t}_2, u), u) + D_1^1 (\lambda_1) + [-\bar{t}_2 + D_2^2 (Q_2 (\lambda_1, \bar{t}_2, u))] \frac{\partial Q_2 (\lambda_1, \bar{t}_2, u)}{\partial \lambda_1} \right\} f (u) du = 0;
\]

\[
\frac{\partial W (\lambda_1, \tau_1, \sigma_1, \bar{t}_2)}{\partial \tau_1} = \int_{u_1}^{u_{11}} \left\{ [-\tau_1 + D_1^1 (Q_1 (\tau_1, \bar{t}_2, u))] \frac{\partial Q_1 (\tau_1, \bar{t}_2, u)}{\partial \tau_1} + [-\bar{t}_2 + D_2^2 (Q_2 (\tau_1, \bar{t}_2, u))] \frac{\partial Q_2 (\tau_1, \bar{t}_2, u)}{\partial \tau_1} \right\} f (u) du = 0;
\]

\[
\frac{\partial W (\lambda_1, \tau_1, \sigma_1, \bar{t}_2)}{\partial \sigma_1} = \int_{u}^{u_{11}} \left\{ [-\sigma_1 + D_1^1 (Q_1 (\sigma_1, \bar{t}_2, u))] \frac{\partial Q_1 (\sigma_1, \bar{t}_2, u)}{\partial \sigma_1} + [-\bar{t}_2 + D_2^2 (Q_2 (\sigma_1, \bar{t}_2, u))] \frac{\partial Q_2 (\sigma_1, \bar{t}_2, u)}{\partial \sigma_1} \right\} f (u) du = 0.
\]

(A.25)

To write these conditions in terms of the functional forms for abatement costs and damages, we require the emissions responses of both pollutants when they are controlled by prices. In this case the explicit forms of \( p_j = -C_j (Q_1, Q_2, u) \), for \( j \in \{1, 2\} \), are
\[
p_j = a_1 + u - a_2 Q_j + w Q_k, \text{ for } j \in \{1, 2\} \text{ and } j \neq k.
\]

Solving these equations for \( Q_1 \) and \( Q_2 \) produces the explicit forms of the emissions responses of both pollutants when they are controlled by prices:
\[
Q_j (p_j, p_k, u) = \frac{(a_1 + u)(a_2 + w) - a_2 p_j - w p_k}{a_2^2 - w^2}, \text{ for } j \in \{1, 2\} \text{ and } j \neq k. \quad (A.26)
\]

At the price ceiling and price floor we have:
\[
Q_1 (z, \bar{t}_2, u) = \frac{(a_1 + u)(a_2 + w) - a_2 z - w \bar{t}_2}{a_2^2 - w^2}, \text{ for } z \in \{\tau_1, \sigma_1\}; \quad (A.27)
\]
\[
Q_2 (z, \bar{t}_2, u) = \frac{(a_1 + u)(a_2 + w) - a_2 \bar{t}_2 - w z}{a_2^2 - w^2}, \text{ for } z \in \{\tau_1, \sigma_1\}. \quad (A.28)
\]
We also need to specify the pollutant 2 emissions response to its tax when the quantity of pollutant 1 is fixed at its supply of permits; that is, \( Q_2 (\lambda_1, \bar{t}_2, u) \). The explicit form of the condition \( \bar{t}_2 = -C_2(\lambda_1, Q_2, u) \) is \( \bar{t}_2 = a_1 + u - a_2 Q_2 + w \lambda_1 \), with solution

\[
Q_2 (\lambda_1, \bar{t}_2, u) = \frac{(a_1 + u) - \bar{t}_2 + w \lambda_1}{a_2}.
\]  

(A.29)

Note that:

\[
\frac{\partial Q_1 (z, \bar{t}_2, u)}{\partial z} = -\frac{a_2}{a_2^2 - w^2}, \text{ for } z \in \{\tau_1, \sigma_1\}; \quad (A.30)
\]

\[
\frac{\partial Q_2 (z, \bar{t}_2, u)}{\partial z} = -\frac{w}{a_2^2 - w^2}, \text{ for } z \in \{\tau_1, \sigma_1\}; \quad (A.31)
\]

\[
\frac{\partial Q_2 (\lambda_1, \bar{t}_2, u)}{\partial \lambda_1} = \frac{w}{a_2}. \quad (A.32)
\]

With (A.27), (A.28), and (A.29) and the functional forms of the marginal damage and marginal abatement cost functions we have:

\[
C_1 (\lambda_1, Q_2 (\lambda_1, \bar{t}_2, u), u) = -(a_1 + u) + a_2 \lambda_1 - w \frac{(a_1 + u) - \bar{t}_2 + w \lambda_1}{a_2}; \quad (A.33)
\]

\[
D_1^1 (\lambda_1) = d_{11} + d_{12} \lambda_1; \quad (A.34)
\]

\[
D_2^2 (Q_2 (\lambda_1, \bar{t}_2, u)) = d_{21} + d_{22} \frac{a_1 + u - \bar{t}_2 + w \lambda_1}{a_2}; \quad (A.35)
\]

\[
D_1^1 (Q_1 (z, \bar{t}_2, u)) = d_{11} + d_{12} \frac{(a_1 + u)(a_2 + w) - a_2 z - w \bar{t}_2}{a_2^2 - w^2}, \text{ for } z \in \{\tau_1, \sigma_1\}; \quad (A.36)
\]

\[
D_2^2 (Q_2 (z, \bar{t}_2, u)) = d_{21} + d_{22} \frac{(a_1 + u)(a_2 + w) - a_2 \bar{t}_2 - wz}{a_2^2 - w^2}, \text{ for } z \in \{\tau_1, \sigma_1\}. \quad (A.37)
\]

Substitute (A.30) through (A.37) into (A.25) to rewrite the first order conditions for the unconstrained version of (2.18) as:
\[ \frac{\partial W(\lambda_1, \tau_1, \sigma_1, \bar{\tau}_2)}{\partial \lambda_1} \]

\[= \int_{u^{(1)}}^{u^{(2)}} \left[ - (a_1 + u) + a_2 \lambda_1 - w \frac{(a_1 + u) - \bar{\tau}_2 + w \lambda_1}{a_2} + d_{11} + d_{12} \lambda_1 \right. \\
+ \left. \left( - \bar{\tau}_2 + d_{21} + d_{22} \frac{(a_1 + u) - \bar{\tau}_2 + w \lambda_1}{a_2} \right) \frac{(w)}{a_2} \right] f(u) \, du = 0; \]

\[ \frac{\partial W(\lambda_1, \tau_1, \sigma_1, \bar{\tau}_2)}{\partial \tau_1} \]

\[= \int_{u^{(1)}}^{u^{(2)}} \left[ - \tau_1 + d_{11} + d_{12} \left( a_1 + u \right) \left( a_2 + w \right) - a_2 \tau_1 - w \bar{\tau}_2 \right] \left( \frac{-a_2}{a_2^2 - w^2} \right) \\
+ \left[ - \bar{\tau}_2 + d_{21} + d_{22} \left( a_1 + u \right) \left( a_2 + w \right) - a_2 \bar{\tau}_2 - w \tau_1 \right] \left( \frac{w}{a_2^2 - w^2} \right) \right\} f(u) \, du = 0; \]

\[ \frac{\partial W(\lambda_1, \tau_1, \sigma_1, \bar{\tau}_2)}{\partial \sigma_1} \]

\[= \int_{u^{(1)}}^{u^{(2)}} \left[ - \sigma_1 + d_{11} + d_{12} \left( a_1 + u \right) \left( a_2 + w \right) - a_2 \sigma_1 - w \bar{\tau}_2 \right] \left( \frac{-a_2}{a_2^2 - w^2} \right) \\
+ \left[ - \bar{\tau}_2 + d_{21} + d_{22} \left( a_1 + u \right) \left( a_2 + w \right) - a_2 \bar{\tau}_2 - w \sigma_1 \right] \left( \frac{-w}{a_2^2 - w^2} \right) \right\} f(u) \, du = 0. \]

Collecting common terms for \( u, \lambda_1, \tau_1, \sigma_1 \) and \( \bar{\tau}_2 \) and rearranging them allows us to characterize the optimal hybrid policy for pollutant 1 as follows:

\[ \lambda_1^* (\bar{\tau}_2) = \frac{a_2^2 (a_1 - d_{11}) + w (a_2 (a_1 - d_{21}) - d_{22} (a_1 - \bar{\tau}_2)) + (a_2 (a_2 + w) - wd_{22}) E[u|w^* \leq u \leq w^*]}{A} \] \hspace{1cm} (A.38)

\[ \tau_1^* (\bar{\tau}_2) = \frac{B - (a_2^2 - w^2 + a_2 (d_{12} + d_{22})) w \bar{\tau}_2 + (a_2 + w) (a_2 d_{12} + d_{22} w) E[u|w^* \leq u \leq w^*]}{A} \] \hspace{1cm} (A.39)

\[ \sigma_1^* (\bar{\tau}_2) = \frac{B - (a_2^2 - w^2 + a_2 (d_{12} + d_{22})) w \bar{\tau}_2 + (a_2 + w) (a_2 d_{12} + d_{22} w) E[u|u \leq u \leq w^*]}{A} \] \hspace{1cm} (A.40)

where
\[ A = a_2 \left( a_2^2 - w^2 \right) + a_2^2 d_{12} + d_{22} w^2; \]  
(A.41)

\[ B = a_1 (a_2 + w) (a_2 d_{12} + d_{22} w) + \left( a_2^2 - w^2 \right) (a_2 d_{11} + d_{21} w), \]  
(A.42)

and \( E \left[ u \mid u^\sigma_1 \leq u \leq u^\tau_1 \right], E \left[ u \mid u^\tau_1 \leq u \leq \bar{u} \right], \) and \( E \left[ u \mid u \leq u \leq u^\sigma_1 \right] \) are conditional expectations of \( u. \)

Again, it is straightforward to calculate the optimal pure tax \( t_1^* (\bar{t}_2) \) and the optimal number of permits under a pure market \( L_1^* (\bar{t}_2) \) from (A.38) through (A.40). These calculations are available upon request.

### A.2.2 When is the tax the preferred policy?

The optimal policy is the tax \( t_1^* (\bar{t}_2) \) if and only if the solution to (2.18) yields \( \tau_1^* (\bar{t}_2) \leq \sigma_1^* (\bar{t}_2). \) To determine the conditions under which this holds, subtract (A.40) from (A.39) to obtain

\[
\tau_1^* (\bar{t}_2) - \sigma_1^* (\bar{t}_2) = \frac{(a_2 + w) (a_2 d_{12} + wd_{22})}{a_2 (a_2^2 - w^2) + a_2^2 d_{12} + w^2 d_{22}} \left\{ E \left[ u \mid u^\tau_1 \leq u \leq \bar{u} \right] - E \left[ u \mid u \leq u \leq u^\sigma_1 \right] \right\}. \tag{A.43}
\]

The denominator of (A.43) and \( a_2 + w \) are strictly positive. Moreover, since \( E(u) = 0, \) \( E \left[ u \mid u \leq u \leq u^\sigma_1 \right] \leq 0 \) and \( E \left[ u \mid u^\tau_1 \leq u \leq \bar{u} \right] \geq 0, \) but they both cannot be zero simultaneously. Therefore, \( E \left[ u \mid u^\tau_1 \leq u \leq \bar{u} \right] - E \left[ u \mid u \leq u \leq u^\sigma_1 \right] > 0. \)

Consequently, \( \tau_1^* (\bar{t}_2) \leq \sigma_1^* (\bar{t}_2) \) if and only if \( a_2 d_{12} + wd_{22} \leq 0 \) which reveals that the tax \( t_1^* (\bar{t}_2) \) is the optimal policy if and only if \( d_{22} w / a_2 \leq -d_{12}, \) which is the desired result.

### A.2.3 When is the pure emissions market the preferred policy?

A simple emissions market is optimal if and only if:

\[ u^\tau_1 = u^\tau_1 \left( \lambda_1^* (\bar{t}_2), \tau_1^* (\bar{t}_2), \bar{t}_2 \right) \geq \bar{u} \text{ and } u^\sigma_1 = u^\sigma_1 \left( \lambda_1^* (\bar{t}_2), \sigma_1^* (\bar{t}_2), \bar{t}_2 \right) \leq \bar{u}. \]
Toward specifying $u^{τ*}$ and $u^{σ*}$, recall from (2.14) that the cut-off values of $u$ are implicitly defined by

$$z = -C_1 (λ_1, Q_2 (λ_1, t_2, u^z), u^z), \text{ for } z \in \{τ_1, σ_1\}.$$  

Explicitly,

$$u^z = z - a_1 + a_2 λ_1 - w Q_2, \text{ for } z \in \{τ_1, σ_1\}. \quad (A.44)$$

Given that pollutant 2 is controlled with the tax $t_2$, emissions of the co-pollutant at the price controls satisfy

$$t_2 = -C_2 (λ_1, Q_2, u^z) = a_1 + u^z - a_2 Q_2 + w λ_1, \text{ for } z \in \{τ_1, σ_1\};$$

that is,

$$Q_2 (λ_1, t_2, u^z) = \frac{a_1 + u^z - t_2 + w λ_1}{a_2}, \text{ for } z \in \{τ_1, σ_1\}. \quad (A.45)$$

Substitute $Q_2 (λ_1, t_2, u^z)$ in for $Q_2$ in (A.44) to obtain

$$u^z (λ_1, z, t_2) = -a_1 + (a_2 - w) λ_1 + \frac{a_2 z}{a_2 + w} + \frac{w t_2}{a_2 + w}, \text{ for } z \in \{τ_1, σ_1\}. \quad (A.45)$$

At the unconstrained solution to (2.18):

$$u^{τ_1} (λ_1^*, (t_2), τ_1^* (t_2), t_2) = -a_1 + (a_2 - w) λ_1^* (t_2) + \frac{a_2 τ_1^* (t_2)}{a_2 + w} + \frac{w t_2}{a_2 + w}; \quad (A.46)$$

$$u^{σ_1} (λ_1^* (t_2), σ_1^* (t_2), t_2) = -a_1 + (a_2 - w) λ_1^* (t_2) + \frac{a_2 σ_1^* (t_2)}{a_2 + w} + \frac{w t_2}{a_2 + w}. \quad (A.47)$$

After substituting (A.38) through (A.40) into (A.46) and (A.47), it is possible to show:
\[ u^{\tau_1} \left( \lambda_1^* \left( \bar{t}_2 \right), \tau_1^* \left( \bar{t}_2 \right), \bar{t}_2 \right) = \frac{(a_2 - w) \left( a_2^2 + a_2 w - wd_{22} \right)}{a_2 \left( a_2^2 - w^2 \right) + a_2^2 d_{12} + d_{22} w^2} E \left[ u \mid u^{\tau_1} \leq u \leq u^{\tau_1} \right] \]

\[ + \frac{a_2 \left( a_2 d_{12} + d_{22} w \right)}{a_2 \left( a_2^2 - w^2 \right) + a_2^2 d_{12} + d_{22} w^2} E \left[ u \mid u^{\tau_1} \leq u \leq \bar{u} \right]; \quad (A.48) \]

\[ u^{\sigma_1} \left( \lambda_1^* \left( \bar{t}_2 \right), \sigma_1^* \left( \bar{t}_2 \right), \bar{t}_2 \right) = \frac{(a_2 - w) \left( a_2^2 + a_2 w - wd_{22} \right)}{a_2 \left( a_2^2 - w^2 \right) + a_2^2 d_{12} + d_{22} w^2} E \left[ u \mid u^{\sigma_1} \leq u \leq u^{\sigma_1} \right] \]

\[ + \frac{a_2 \left( a_2 d_{12} + d_{22} w \right)}{a_2 \left( a_2^2 - w^2 \right) + a_2^2 d_{12} + d_{22} w^2} E \left[ u \mid u \leq u \leq u^{\sigma_1} \right]; \quad (A.49) \]

The necessary condition (or conditions) for a pure market to be the optimal policy is found by evaluating the conditions under which \( u^{\tau_1} = u^{\tau_1} \left( \lambda_1^* \left( \bar{t}_2 \right), \tau_1^* \left( \bar{t}_2 \right), \bar{t}_2 \right) \geq \bar{u} \) and \( u^{\sigma_1} = u^{\sigma_1} \left( \lambda_1^* \left( \bar{t}_2 \right), \sigma_1^* \left( \bar{t}_2 \right), \bar{t}_2 \right) \leq \bar{u} \). To do so, we must evaluate the conditional expectations in (A.48) and (A.49) under these conditions. First, since \( E(u) = 0 \) and the support of \( u \) is \([u, \bar{u}]\), \( u^{\tau_1} \geq \bar{u} \) and \( u^{\sigma_1} \leq \bar{u} \) imply \( E \left[ u \mid u^{\sigma_1} \leq u \leq u^{\tau_1} \right] = 0 \). Next, we cannot directly evaluate

\[ E[u|u^{\tau_1} \leq u \leq \bar{u}] = \frac{\int_{u^{\tau_1}}^{\bar{u}} u f(u) \, du}{\int_{u^{\tau_1}}^{\bar{u}} f(u) \, du}, \]

given \( u^{\tau_1} \geq \bar{u} \), but we can use l’Hopital’s rule to determine

\[ \lim_{u^{\tau_1} \to \bar{u}} E[u|u^{\tau_1} \leq u \leq \bar{u}] = \bar{u}. \]

Similarly,

\[ \lim_{u^{\sigma_1} \to \bar{u}} E[u|u \leq u \leq u^{\sigma_1}] = \bar{u}. \]

Substitute these limiting values and \( E \left[ u \mid u^{\sigma_1} \leq u \leq u^{\tau_1} \right] = 0 \) into (A.48) and (A.49) to obtain:

\[ u^{\tau_1} = u^{\tau_1} \left( \lambda_1^* \left( \bar{t}_2 \right), \tau_1^* \left( \bar{t}_2 \right), \bar{t}_2 \right) \geq \bar{u} \implies \frac{a_2 \left( a_2 d_{12} + d_{22} w \right) \bar{u}}{a_2 \left( a_2^2 - w^2 \right) + a_2^2 d_{12} + d_{22} w^2} \geq \bar{u}; \]

\[ u^{\sigma_1} = u^{\sigma_1} \left( \lambda_1^* \left( \bar{t}_2 \right), \sigma_1^* \left( \bar{t}_2 \right), \bar{t}_2 \right) \leq \bar{u} \implies \frac{a_2 \left( a_2 d_{12} + d_{22} w \right) \bar{u}}{a_2 \left( a_2^2 - w^2 \right) + a_2^2 d_{12} + d_{22} w^2} \leq \bar{u}. \]
Since \( a_2 (a_2^2 - w^2) + a_2^2 d_{12} + d_{22} w^2 \) and \( a_2 \) are strictly positive, these inequalities hold if and only if

\[
\frac{a_2 (a_2 d_{12} + w d_{22})}{a_2 (a_2^2 - w^2) + a_2^2 d_{12} + w^2 d_{22}} \geq 1,
\]

which simplifies to \( w d_{22}/a_2 \geq a_2 + w \): therefore, this is a necessary condition for a simple emission market to be the optimal policy.

To prove that \( d_{22} w/a_2 \geq a_2 + w \) is also sufficient for emissions trading to be the optimal policy, suppose toward a contradiction that \( d_{22} w/a_2 \geq a_2 + w \), but that pure trading is not optimal; that is, \( u^\tau = u^\tau (\lambda^*_1 (\bar{t}_2), \tau^*_1 (\bar{t}_2), \bar{t}_2) < \bar{u} \) or \( u^{\sigma} = u^{\sigma} (\lambda^*_1 (\bar{t}_2), \sigma^*_1 (\bar{t}_2), \bar{t}_2) > u \). Note first that since \( u^\tau < \bar{u} \),

\[
E [u|u^\tau \leq u \leq \bar{u}] > u^\tau (\lambda^*_1 (\bar{t}_2), \tau^*_1 (\bar{t}_2), \bar{t}_2).
\]

Using (A.48),

\[
E [u|u^\tau \leq u \leq \bar{u}] > \frac{(a_2 - w)(a_2^2 + a_2 w - w d_{22})}{a_2 (a_2^2 - w^2) + a_2^2 d_{12} + d_{22} w^2} E [u|u^{\sigma} \leq u \leq u^\tau] + \frac{a_2 (a_2 d_{12} + d_{22} w)}{a_2 (a_2^2 - w^2) + a_2^2 d_{12} + d_{22} w^2} E [u|u^\tau \leq u \leq \bar{u}],
\]

which implies,

\[
\frac{(a_2 - w)(a_2(a_2 + w) - w d_{22})}{a_2 (a_2^2 - w^2) + a_2^2 d_{12} + w^2 d_{22}} \{E [u|u^{\sigma} \leq u \leq u^\tau] - E [u|u^\tau \leq u \leq \bar{u}]\} < 0.
\]

(A.50)

The first term of the left side of (A.50) is less than or equal to zero because

\[
a_2 (a_2^2 - w^2) + a_2^2 d_{12} + w^2 d_{22} > 0, \quad a_2 - w > 0, \quad \text{and} \quad a_2(a_2 + w) - w d_{22} \leq 0 \quad \text{by assumption.}
\]

The second term involving the conditional expectations is strictly negative because

\[
E [u|u^{\sigma} \leq u \leq u^\tau] \leq u^\tau < E [u|u^\tau \leq u \leq \bar{u}] .
\]

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Since the first term of (A.50) is weakly negative and the second is strictly negative, the inequality cannot hold and we have obtained our contradiction. Thus, \( d_{22}w/a_2 \geq a_2 + w \) is also a sufficient condition for the optimal policy to be a pure emissions market. As an aside, we could have obtained a similar contradiction by showing that \( u^\sigma_1 = u^\sigma_1(\lambda_1^*(\bar{t}_2), \sigma_1^*(\bar{t}_2)) > u \) cannot hold if \( d_{22}w/a_2 \geq a_2 + w \).

**A.2.4 When is the hybrid regulation the preferred policy?**

Since the tax is the optimal policy pollutant 1 if and only if \( d_{22}w/a_2 \leq -d_{12} \) and the pure trading program is optimal if and only if \( d_{22}w/a_2 \geq a_2 + w \), then it must be the case that the hybrid policy is optimal if and only if \( d_{22}w/a_2 \in (-d_{12}, a_2 + w) \).

**A.3 The impact of variation in pollutant 1 emissions on expected pollutant 2 damage**

Given \( \bar{t}_2 \) and emissions of pollutant 1, emissions of pollutant 2 can be written as

\[
Q_2(Q_1, \bar{t}_2, u) = \frac{a_1 + u - \bar{t}_2 + wQ_1}{a_2}. \tag{A.51}
\]

From this we can write pollutant 2 emissions as \( Q_2(Q_1, \bar{t}_2, u) = E(Q_2) + u/a_2 \), where \( u/a_2 \) is random variation of \( Q_2 \) around its expected value. If pollutant 1 is also controlled with a tax, we can write its emissions in the same fashion, that is, \( Q_1(t_1, Q_2, u) = E(Q_1) + u/a_2 \). Now let us introduce a new variable \( \gamma \in [0, 1] \) so that \( Q_1(\cdot) = E(Q_1) + u\gamma/a_2 \). The introduction of \( \gamma \) allows us to modify the variation of pollutant 1 emissions around its expected value. For alternative policies that produce the same expected pollutant 1 emissions, \( \gamma = 1 \) when the pollutant is controlled with a tax, \( \gamma \in (0, 1) \) when the pollutant is controlled with a hybrid and \( \gamma = 0 \) when the pollutant is controlled with a pure emissions market. Given the specification of pollutant 1 emissions, calculate its variance as
\[ \text{Var}(Q_1) = E(u^2)\gamma^2/a_2^2. \] Clearly, the variance of pollutant 1 emissions is declining as \( \gamma \) is reduced.

To show how variation in pollutant 1 emissions affects the variation in pollutant 2 emissions when the latter is controlled with a tax, substitute \( Q_1(\cdot) = E(Q_1) + u\gamma/a_2 \) into (A.51) to obtain

\[
Q_2(Q_1,\overline{t}_2,u) = \frac{a_1 + u - \overline{t}_2 + w(E(Q_1) + u\gamma/a_2)}{a_2}. \tag{A.52}
\]

This substitution does not affect the expected value of pollutant 2 emissions, but its variance is

\[
\text{Var}(Q_2) = \frac{E(u^2)(1 + w\gamma/a_2)^2}{a_2^2}.
\]

The effect of a reduction in the variation of pollutant 1 emissions on the variance of pollutant 2 emissions is

\[
\frac{\partial \text{Var}(Q_2)}{\partial \gamma} = \left[ \frac{w}{a_2} \right] \left[ \frac{E(u^2)(1 + w\gamma/a_2)}{a_2^2} \right]. \tag{A.53}
\]

Since \( a_2 + w > 0 \) and \( \gamma \in [0,1] \), \( 1 + w\gamma/a_2 > 0 \), which implies that the second term of (A.53) in hard brackets is strictly positive. Thus, the qualitative effect of the variance of pollutant 1 emissions on the variance of pollutant 2 emissions depends on whether the pollutants are substitutes or complements. In particular, a decrease in the variance of pollutant 1 emissions also reduces the variance of pollutant 2 emissions if the pollutants are complements in abatement, while the variance of pollutant 2 emissions increases if the pollutants are substitutes.

To show how the variation of pollutant 1 emissions affects the expected damage of pollutant 2 emissions substitute (A.52) into (2.4) and collect terms to obtain

\[
D^2(Q_2) = d_{21} \left( E(Q_2) + \frac{u(1 + w\gamma/a_2)}{a_2} \right) + \frac{d_{22}}{2} \left( E(Q_2) + \frac{u(1 + w\gamma/a_2)}{a_2} \right)^2.
\]
Take the expectation to obtain

\[ E(D^2(Q_2)) = d_{21}E(Q_2) + \frac{d_{22}}{2} [E(Q_2)]^2 + \frac{d_{22}}{2} \left[ \frac{E(u^2)(1 + w\gamma/a_2)^2}{a_2^2} \right], \]

and differentiate with respect to \( \gamma \) to obtain

\[ \frac{\partial E(D^2(Q_2))}{\partial \gamma} = \left( \frac{d_{22}w}{a_2} \right) \left[ \frac{E(u^2)(1 + w\gamma/a_2)}{a_2^2} \right] = d_{22} \frac{\partial \text{Var}(Q_2)}{\partial \gamma}. \]

Of course, if pollutant 2 damage is a linear function (so that \( d_{22} = 0 \)) then changes in the variance of pollutant 2 emissions induced by changes in the variance of pollutant 1 emissions have no impact on expected co-pollutant damage. However, if the pollutant 2 damage function is strictly convex, then, when the pollutants are complements in abatement, a decrease in the variance of pollutant 1 emissions reduces expected co-pollutant damage through a reduction in the variance of pollutant 2 emissions. If the pollutants are substitutes, a decrease in the variance of pollutant 1 emissions increases expected co-pollutant damage by causing an increase in the variation of pollutant 2 emissions.
B.1 Optimal hybrid policy for pollutant 1 given tradable permits \( \bar{L}_2 \)

In this section, we will characterize the optimal hybrid policy for pollutant 1 given tradable permits for pollutant 2, which are presented by equations (3.22) through (3.24) in subsection 3.3.1. Given expected social costs (3.20), the optimal hybrid policy for pollutant 1 is the solution to (3.21). Since we focus at first on the characterization of the optimal hybrid policy, we assume that each of the constraints in (3.21) do not bind.

Given the definitions of \( u^\tau_1 \) and \( u^\sigma_1 \), we have the following:

\[
Q_1 (z, \bar{L}_2, u^z) = \lambda_1, \quad z \in \{ \tau_1, \sigma_1 \}.
\]

With this result the first order conditions for the unconstrained version of (3.21) can be written as

\[
\frac{\partial W (\lambda_1, \tau_1, \sigma_1, \bar{L}_2)}{\partial \lambda_1} = \int_{u^\tau_1} \left[ C_1 (\lambda_1, \bar{L}_2, u) + D_1 (\lambda_1, \bar{L}_2) \right] f (u) \, du = 0; \quad (B.1)
\]

\[
\frac{\partial W (\lambda_1, \tau_1, \sigma_1, \bar{L}_2)}{\partial \tau_1} = \int_{u^\tau_1} \left[ C_1 (Q_1 (\tau_1, \bar{L}_2, u), \bar{L}_2, u) + D_1 (Q_1 (\tau_1, \bar{L}_2, u), \bar{L}_2) \right] \frac{\partial Q_1 (\tau_1, \bar{L}_2, u)}{\partial \tau_1} f (u) \, du = 0; \quad (B.2)
\]
\[ \frac{\partial W (\lambda_1, \tau_1, \sigma_1, \bar{L}_2)}{\partial \sigma_1} = \int_u^{\sigma_1} [C_1(Q_1(\sigma_1, \bar{L}_2, u), \bar{L}_2, u) + D_1(Q_1(\sigma_1, \bar{L}_2, u), \bar{L}_2)] \frac{\partial Q_1(\sigma_1, \bar{L}_2, u)}{\partial \sigma_1} f(u) du = 0. \]  

(B.3)

The following always holds true:

\[ -C_1(Q_1(z, \bar{L}_2, u), \bar{L}_2, u) = z, \quad z \in \{\tau_1, \sigma_1\}, \]  

(B.4)

and from (3.17) we have

\[ \frac{\partial Q_1(z, \bar{L}_2, u)}{\partial z} = -\frac{1}{a_2}, \quad z \in \{\tau_1, \sigma_1\}. \]  

(B.5)

With (3.17), (B.4), and (B.5), applying the functional forms of the marginal abatement cost and marginal damage functions and collecting common terms for \( u \) allows us to write (B.1) through (B.3) as:

\[ \frac{\partial W (\lambda_1, \tau_1, \sigma_1, \bar{L}_2)}{\partial \lambda_1} = - [(a_1 - d_{11}) - (a_2 + d_{12}) \lambda_1 + (w - v) \bar{L}_2] \int_{u_1}^{u_1} f(u) du - \int_{u_1}^{u_1} uf(u) du = 0; \]

\[ \frac{\partial W (\lambda_1, \tau_1, \sigma_1, \bar{L}_2)}{\partial \tau_1} = \left[ \frac{a_1 d_{12} + a_2 d_{11} - (a_2 + d_{12}) \tau_1 + (a_2 v + \omega d_{12}) \bar{L}_2}{a_2} \right] \int_{u_1}^{u_1} f(u) du + \frac{d_{12}}{a_2} \int_{u_1}^{u_1} uf(u) du = 0; \]

\[ \frac{\partial W (\lambda_1, \tau_1, \sigma_1, \bar{L}_2)}{\partial \sigma_1} = \left[ \frac{a_1 d_{12} + a_2 d_{11} - (a_2 + d_{12}) \sigma_1 + (a_2 v + \omega d_{12}) \bar{L}_2}{a_2} \right] \int_{u}^{u} f(u) du + \frac{d_{12}}{a_2} \int_{u}^{u} uf(u) du = 0, \]

which can be rearranged to obtain the following.
\[ \lambda_1^* (L_2) = \frac{a_1 - d_{11} + (w - v) \bar{L}_2 + E \left[ u | u^{\tau_1} \leq u \leq u^{\sigma_1} \right]}{a_2 + d_{12}}; \]

\[ \tau_1^* (L_2) = \frac{a_1 d_{12} + a_2 d_{11} + (a_2 v + w d_{12}) \bar{L}_2 + d_{12} E \left[ u | u^{\tau_1} \leq u \leq \bar{u} \right]}{a_2 + d_{12}}; \]

\[ \sigma_1^* (L_2) = \frac{a_1 d_{12} + a_2 d_{11} + (a_2 v + w d_{12}) \bar{L}_2 + d_{12} E \left[ u | u^{\sigma_1} \leq u \leq \bar{u} \right]}{a_2 + d_{12}}. \]

Using (3.6), (3.7), and (3.14), we have

\[ \frac{a_1 - d_{11} + (w - v) \bar{L}_2}{a_2 + d_{12}} = \hat{Q}_1 - \frac{v - w}{a_2 + d_{12}} \left( \bar{L}_2 - \hat{Q}_2 \right); \]

\[ \frac{a_1 d_{12} + a_2 d_{11} + (a_2 v + w d_{12}) \bar{L}_2}{a_2 + d_{12}} = \hat{P}_1 + \frac{a_2 v + d_{22} w}{a_2 + d_{12}} \left( \bar{L}_2 - \hat{Q}_2 \right). \]

With these results, (B.6) can be simplified further to obtain equations (3.22) through (3.24).

### B.2 Optimal hybrid policy for pollutant 1 given emissions tax \( \bar{t}_2 \)

In this section we derive the characterizations of the optimal hybrid policy for pollutant 1 when pollutant 2 is regulated by an emissions tax, which are presented by equations (3.32) through (3.34) in subsection 3.3.2. We apply the same approach that we took to derive the optimal hybrid policy given tradable permits for pollutant 2. Given expected social costs (3.30), the optimal hybrid policy for pollutant 1 given emissions tax \( \bar{t}_2 \) is the solution to (3.31).

From the definitions of \( u^{\tau_1} \) and \( u^{\sigma_1} \), we have the following:

\[ Q_1 (z, \bar{t}_2, u^z) = \lambda_1, \; z \in \{ \tau_1, \sigma_1 \}; \]

\[ Q_2 (z, \bar{t}_2, u^z) = Q_2 (\lambda_1, \bar{t}_2, u^z), \; z \in \{ \tau_1, \sigma_1 \}. \]
In addition, the following always holds:

\[-C_1(Q_{1}(z, \bar{t}_2, u), Q_{2}(z, \bar{t}_2, u), u) = z, \quad z \in \{\tau_1, \sigma_1\};\]

\[-C_2(Q_{1}(z, \bar{t}_2, u), Q_{2}(z, \bar{t}_2, u), u) = \bar{t}_2, \quad z \in \{\tau_1, \sigma_1\};\]

\[-C_2(\lambda_1, Q_{2}(\lambda_1, \bar{t}_2, u), u) = \bar{t}_2.\]

With these results the first order conditions of the unconstrained version of (3.31) are:

\[
\begin{aligned}
\frac{\partial W(\lambda_1, \tau_1, \sigma_1, \bar{t}_2)}{\partial \lambda_1} &= \int_{u^\uparrow} \left\{ [C_1(\lambda_1, Q_{2}(\lambda_1, \bar{t}_2, u), u) + D_1(\lambda_1, Q_{2}(\lambda_1, \bar{t}_2, u))] \\
&\quad + [-\bar{t}_2 + D_2(\lambda_1, Q_{2}(\lambda_1, \bar{t}_2, u))] \frac{\partial Q_{2}(\lambda_1, \bar{t}_2, u)}{\partial \lambda_1} \right\} f(u) \, du = 0; \\
\frac{\partial W(\lambda_1, \tau_1, \sigma_1, \bar{t}_2)}{\partial \tau_1} &= \int_{u^\uparrow} \left\{ [-\tau_1 + D_1(Q_{1}(\tau_1, \bar{t}_2, u), Q_{2}(\tau_1, \bar{t}_2, u))] \frac{\partial Q_{1}(\tau_1, \bar{t}_2, u)}{\partial \tau_1} \\
&\quad + [-\bar{t}_2 + D_2(Q_{1}(\tau_1, \bar{t}_2, u), Q_{2}(\tau_1, \bar{t}_2, u))] \frac{\partial Q_{2}(\tau_1, \bar{t}_2, u)}{\partial \tau_1} \right\} f(u) \, du = 0; \\
\frac{\partial W(\lambda_1, \tau_1, \sigma_1, \bar{t}_2)}{\partial \sigma_1} &= \int_{u^\uparrow} \left\{ [-\sigma_1 + D_1(Q_{1}(\sigma_1, \bar{t}_2, u), Q_{2}(\sigma_1, \bar{t}_2, u))] \frac{\partial Q_{1}(\sigma_1, \bar{t}_2, u)}{\partial \sigma_1} \\
&\quad + [-\bar{t}_2 + D_2(Q_{1}(\sigma_1, \bar{t}_2, u), Q_{2}(\sigma_1, \bar{t}_2, u))] \frac{\partial Q_{2}(\sigma_1, \bar{t}_2, u)}{\partial \sigma_1} \right\} f(u) \, du = 0.
\end{aligned}
\]

From (3.17) we have

\[
\frac{\partial Q_{2}(\lambda_1, \bar{t}_2, u)}{\partial \lambda_1} = \frac{w}{a_2}, \quad (B.7)
\]

and from (3.12), we have
\[
\frac{\partial Q_1 (z, \bar{r}_2, u)}{\partial z} = -\frac{a_2}{a_2^2 - w^2}, \quad z \in \{\tau_1, \sigma_1\}; \\
\frac{\partial Q_2 (z, \bar{r}_2, u)}{\partial z} = -\frac{w}{a_2^2 - w^2}, \quad z \in \{\tau_1, \sigma_1\}. 
\]

(B.8)

Applying (3.12), (3.17), (B.7), (B.8) and the functional forms of the marginal abatement cost and marginal damage functions and collecting common terms for \(u, \lambda_1, \tau_1, \sigma_1,\) and \(\bar{r}_2\) allow us to write the first order conditions as:

\[
\frac{\partial W (\lambda_1, \tau_1, \sigma_1, \bar{r}_2)}{\partial \lambda_1} = \left\{ \frac{- (a_2 v + d_{22} w) \bar{r}_2}{a_2^2} + \frac{a_2 [a_2 (a_2 + d_{12}) - w(w - 2v) + d_{22}w^2]}{a_2^2} \lambda_1}{\int_{u_{\tau_1}}^{u_{\tau_1}} f(u) \, du} \\
+ \frac{a_2 [a_2 d_{11} - a_1 (a_2 + w - v) + d_{21}w]}{a_2^2} \lambda_1 \\
\int_{u_{\tau_1}}^{u_{\tau_1}} f(u) \, du = 0;
\]

\[
\frac{\partial W (\lambda_1, \tau_1, \sigma_1, \bar{r}_2)}{\partial \tau_1} = \left\{ \frac{[a_2 (d_{12} + d_{22}) w + a_2^2 (v + w) + w^2 (v - w)] \bar{r}_2}{(a_2^2 - w^2) \lambda_1} \right\} \int_{u_{\tau_1}}^{u_{\tau_1}} f(u) \, du = 0;
\]

\[
\frac{\partial W (\lambda_1, \tau_1, \sigma_1, \bar{r}_2)}{\partial \sigma_1} = \left\{ \frac{[a_2 (d_{12} + d_{22}) w + a_2^2 (v + w) + w^2 (v - w)] \bar{r}_2}{(a_2^2 - w^2) \lambda_1} \right\} \int_{u_{\sigma_1}}^{u_{\sigma_1}} f(u) \, du = 0.
\]

These equations can be rearranged for \(\lambda_1, \tau_1,\) and \(\sigma_1\) to obtain the following:

\[
\lambda_1 (\bar{r}_2) = \frac{a_2 [a_1 (a_2 + w - v) - a_2 d_{11} - d_{21}w] - a_1 d_{22}w + (a_2 v + d_{22}w) \bar{r}_2}{a_2 [a_2 (a_2 + d_{12}) - w(w - 2v)] + d_{22}w^2} \\
+ \frac{[a_2 (a_2 + w - v) - d_{22}w] E \{u|u^2 \leq u \leq u^\tau\}}{a_2 [a_2 (a_2 + d_{12}) - w(w - 2v)] + d_{22}w^2}; 
\]

(B.9)
Using (3.6), (3.14), and (3.15), the first term of \( \lambda_1^* (\bar{t}_2) \) in (B.9) and the first two terms of \( \tau_1^* (\bar{t}_2) \) and \( \sigma_1^* (\bar{t}_2) \) in (B.10) and (B.11) can be rearranged as:

\[
\begin{align*}
\tau_1^* (\bar{t}_2) &= \frac{(a_2 + w) [(a_2 - w) (a_2 d_{11} + d_{21} w) + a_1 (a_2 (d_{12} + v) + w (d_{22} + v))]}{a_2 [a_2 (a_2 + d_{12}) - w(w - 2v)] + d_{22} w^2} \\
&\quad - \frac{[a_2 (d_{12} + d_{22}) w + a_2^2 (v + w) + w^2 (v - w)] \bar{t}_2}{a_2 [a_2 (a_2 + d_{12}) - w(w - 2v)] + d_{22} w^2} \\
&\quad + \frac{(a_2 + w) [a_2 (d_{12} + v) + w (d_{22} + v)] E [u | u^* \leq u \leq \bar{u}]}{a_2 [a_2 (a_2 + d_{12}) - w(w - 2v)] + d_{22} w^2}; \\
\sigma_1^* (\bar{t}_2) &= \frac{(a_2 + w) [(a_2 - w) (a_2 d_{11} + d_{21} w) + a_1 (a_2 (d_{12} + v) + w (d_{22} + v))]}{a_2 [a_2 (a_2 + d_{12}) - w(w - 2v)] + d_{22} w^2} \\
&\quad - \frac{[a_2 (d_{12} + d_{22}) w + a_2^2 (v + w) + w^2 (v - w)] \bar{t}_2}{a_2 [a_2 (a_2 + d_{12}) - w(w - 2v)] + d_{22} w^2} \\
&\quad + \frac{(a_2 + w) [a_2 (d_{12} + v) + w (d_{22} + v)] E [u | u \leq u \leq u^*]}{a_2 [a_2 (a_2 + d_{12}) - w(w - 2v)] + d_{22} w^2}.
\end{align*}
\]  

which results in equations (3.32) through (3.34).

### B.3 Expected emissions of pollutant 1

Finding 1 in section 3.4 reveals that all the optimal price-based regulations for pollutant 1 produce the same expected emissions. In this subsection we demonstrate this result, first when pollutant 2 is regulated with tradable permits and then when the pollutant is regulated with an emissions tax.
B.3.1 Given tradable permits $\bar{L}_2$

Here we will calculate the expected emissions of pollutant 1 under the optimal hybrid policy given tradable permits for pollutant 2. We then show that this value is equal to the optimal number of tradable permits for pollutant 1, $L^*_1(\bar{L}_2)$, and the expected emissions of pollutant 1 under the optimal emissions tax, $t^*_1(\bar{L}_2)$.

Under the optimal hybrid policy $h^*_1(\bar{L}_2) = (\lambda^*_1(\bar{L}_2), \tau^*_1(\bar{L}_2), \sigma^*_1(\bar{L}_2))$, the expected emissions of pollutant 1 can be expressed as

$$E[Q_1(h^*_1(\bar{L}_2), \bar{L}_2, u)] = \int_{u_1^*}^{\bar{u}} Q_1(\tau^*_1(\bar{L}_2), \bar{L}_2, u) f(u) du + \int_{u_1^*}^{\bar{u}} \lambda^*_1(\bar{L}_2) f(u) du + \int_{\bar{u}}^{\bar{u}^*} Q_1(\sigma^*_1(\bar{L}_2), \bar{L}_2, u) f(u) du. \quad (B.12)$$

Applying (3.17) to (B.12) yields

$$E[Q_1(h^*_1(\bar{L}_2), \bar{L}_2, u)] = \int_{u_1^*}^{\bar{u}} \left( \frac{a_1 + u - \tau^*_1(\bar{L}_2) + w\bar{L}_2}{a_2} \right) f(u) du + \int_{u_1^*}^{\bar{u}^*} \lambda^*_1 f(u) du + \int_{\bar{u}}^{\bar{u}^*} \left( \frac{a_1 + u - \sigma^*_1(\bar{L}_2) + w\bar{L}_2}{a_2} \right) f(u) du.$$

Substituting the optimal hybrid policy variables (3.22), (3.23), and (3.24) into this result and manipulating terms produces

$$E[Q_1(h^*_1(\bar{L}_2), \bar{L}_2, u)] = \hat{Q}_1 - \frac{v - w}{a_2 + d_{12}} (\bar{L}_2 - \hat{Q}_2) + \frac{G}{a_2 + d_{12}},$$

where

$$G = \int_{u_1^*}^{\bar{u}} u f(u) du + \int_{u_1^*}^{\bar{u}^*} u f(u) du + \int_{\bar{u}}^{\bar{u}^*} u f(u) du.$$

Clearly, $G = 0$ under the optimal hybrid policy $h^*_1(\bar{L}_2) = (\lambda^*_1(\bar{L}_2), \tau^*_1(\bar{L}_2), \sigma^*_1(\bar{L}_2))$, because $E(u) = 0$. Moreover, it is zero under the optimal tax $t^*_1(\bar{L}_2)$ because $u^* \leq u$. It is also zero under the optimal trading program $L^*_1(\bar{L}_2)$ because $u^* \leq u$. 

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and $u^\tau \geq \bar{u}$. Therefore, the expected emissions of pollutant 1 are the same under optimal price-based policies when pollutant 2 is regulated with a fixed number of tradable permits.

### B.3.2 Expected emissions of pollutant 1 given emissions tax $\bar{t}_2$

Now we demonstrate that the optimal price-based regulations for pollutant 1 produces the same expected emissions, given an emissions tax for pollutant 2. We will take the same steps that we took in subsection B.3.1. We first derive the expected emissions of pollutant 1 under the optimal hybrid policy $h^*_1(\bar{t}_2) = (\lambda^*_1(\bar{t}_2), \tau^*_1(\bar{t}_2), \sigma^*_1(\bar{t}_2))$ and then we show that this value is the same under the optimal emissions tax $t^*_1(\bar{t}_2)$ and under optimal pure permit trading with $L^*_1(\bar{t}_2)$ permits.

Under the optimal hybrid policy $h^*_1(\bar{t}_2) = (\lambda^*_1(\bar{t}_2), \tau^*_1(\bar{t}_2), \sigma^*_1(\bar{t}_2))$, the expected emissions of pollutant 1 can be written as

$$E[Q_1(h^*_1(\bar{t}_2), \bar{t}_2, u)] = \int^{u^\tau}_u Q_1(\tau^*_1(\bar{t}_2), \bar{t}_2, u) f(u) \, du + \int^{u^\tau}_u \lambda^*_1(\bar{t}_2) f(u) \, du$$
$$+ \int_u^{u^\tau} Q_1(\sigma^*_1(\bar{t}_2), \bar{t}_2, u) f(u) \, du. \tag{B.13}$$

Applying (3.12) to (B.13) produces

$$E[Q_1(h^*_1(\bar{t}_2), \bar{t}_2, u)] = \int^{u^\tau}_u \left( \frac{(a_1 + u)(a_2 + w) - a_2 \tau^*_1(\bar{t}_2) - w \bar{t}_2}{a_2^2 - w^2} \right) f(u) \, du$$
$$+ \int_{u^\tau}^{u^\tau} \lambda^*_1(\bar{t}_2) f(u) \, du$$
$$+ \int_u^{u^\tau} \left( \frac{(a_1 + u)(a_2 + w) - a_2 \sigma^*_1(\bar{t}_2) - w \bar{t}_2}{a_2^2 - w^2} \right) f(u) \, du.$$

Substituting (3.32), (3.33), and (3.34) into this result and manipulating terms yields

$$E[Q_1(h^*_1(\bar{t}_2), \bar{t}_2, u)] = \hat{Q}_1 + \frac{Y}{a_2X + wY} \left( \bar{t}_2 - \hat{P}_2 \right) + \frac{[a_2(a_2 + w) - Y]}{a_2X + wY} H \tag{B.14}$$
where \( X = a_2 (a_2 + d_{12}) + w (v - w), \) \( Y = a_2 v + d_{22} w, \) and

\[
H = \int_{u^*}^{\bar{u}} uf(u) du + \int_{u^*}^{\bar{u}} uf(u) du + \int_{u}^{u^*} uf(u) du.
\]

By the same logic given above, \( H = 0 \) under \( h^*_1 (\mathcal{T}_2), t^*_1 (\mathcal{T}_2), \) and \( L^*_1 (\mathcal{T}_2). \) Therefore, the expected emissions of pollutant 1 are the same among the optimal price-based regulations for pollutant 1 when pollutant 2 is regulated by an emissions tax.

### B.4 Expected emissions of pollutant 2

In this section, we demonstrate the last part of Finding 1: the optimal price-based regulations for pollutant 1 also produce the same expected emissions for pollutant 2, given the emissions tax for pollutant 2.

When pollutant 2 is regulated by a tax \( \bar{t}_2 \) and pollutant 1 is regulated by the optimal hybrid policy \( h^*_1 (\bar{t}_2) = (\lambda^*_1 (\bar{t}_2), \tau^*_1 (\bar{t}_2), \sigma^*_1 (\bar{t}_2)) \), the expected emissions of pollutant 2 can be expressed as

\[
E [Q_2 (h^*_1 (\bar{t}_2), \bar{t}_2, u)] = \int_{u^*}^{\bar{u}} Q_2 (\tau^*_1 (\bar{t}_2), \bar{t}_2, u) f(u) du + \int_{u^*}^{\bar{u}} Q_2 (\lambda^*_1 (\bar{t}_2), \bar{t}_2, u) f(u) du
+ \int_{u}^{u^*} Q_2 (\sigma^*_1 (\bar{t}_2), \bar{t}_2, u) f(u) du. \tag{B.15}
\]

Applying (3.12) to (B.15) produces

\[
E [Q_2 (h^*_1 (\bar{t}_2), \bar{t}_2, u)] = \int_{u^*}^{\bar{u}} \left( \frac{(a_1 + u)(a_2 + w) - a_2 \bar{t}_2 - w \tau^*_1 (\bar{t}_2)}{a_2^2 - w^2} \right) f(u) du
+ \int_{u^*}^{\bar{u}} \frac{a_1 + u - \bar{t}_2 + w \lambda^*_1 (\bar{t}_2)}{a_2} f(u) du
+ \int_{u}^{u^*} \left( \frac{(a_1 + u)(a_2 + w) - a_2 \bar{t}_2 - w \sigma^*_1 (\bar{t}_2)}{a_2^2 - w^2} \right) f(u) du.
\]
Applying (3.32), (3.33), and (3.34) to this result and manipulating terms yields

\[
E [Q_2 (h^*_1(\tilde{t}_2), \tilde{t}_2, u)] = \hat{Q}_2 - \frac{X}{a_2 X + w Y} (\tilde{t}_2 - \hat{P}_2) + \frac{a_2 (a_2 + w) + a_2 d_{12} + w v I}{a_2 X + w Y}.
\]

(B.16)

where \( X = a_2 (a_2 + d_{12}) + w (v - w), \) \( Y = a_2 v + d_{22} w, \) and \( I = \int_{u_{\gamma}}^{u} u f (u) du + \int_{u_{\gamma}}^{u} u f (u) du + \int_{u_{\gamma}}^{u} u f (u) du. \) Then, by the same arguments in subsections B.3.1 and B.3.2, we can show that \( I = 0 \) under \( h^*_1(\tilde{t}_2), t^*_1(\tilde{t}_2), \) and \( L^*_1(\tilde{t}_2). \)

### B.5 Relationship between the expected emissions of both pollutants given emissions tax \( \bar{t}_2 \)

In this section, we demonstrate Finding 2 in section 3.4. When pollutant 2 is controlled by tradable permits \( \bar{L}_2, \) the emissions of pollutant 2 are fixed at \( \bar{L}_2. \) Thus, the first relationship in Finding 2 is clear from (3.25) and Finding 1. Here we focus on the case where pollutant 2 is regulated by an emissions tax \( \bar{t}_2. \)

From the notations defined by (3.40) and (3.41) and the results of (B.14) and (B.16), we have

\[
E [Q^*_1 (\bar{t}_2)] = \hat{Q}_1 + \frac{a_2 (a_2 + d_{12}) + w (v - w)}{a_2 X + w Y} (\bar{t}_2 - \hat{P}_2); \quad \text{(B.17)}
\]

\[
E [Q^*_2 (\bar{t}_2)] = \hat{Q}_2 - \frac{X}{a_2 X + w Y} (\bar{t}_2 - \hat{P}_2). \quad \text{(B.18)}
\]

As explained in subsection 3.3.2, \( X = a_2 (a_2 + d_{12}) + w (v - w) \) represents the effects of \( Q_1 \) on its own marginal social costs. Combining (B.17) and (B.18) gives us

\[
E [Q^*_1 (\bar{t}_2)] - \hat{Q}_1 = -\frac{Y}{X} \left\{ E [Q^*_2 (\bar{t}_2)] - \hat{Q}_2 \right\},
\]

which, upon substitution of \( X = a_2 (a_2 + d_{12}) + w (v - w) \) and \( Y = a_2 v + d_{22} w, \) yields (3.42).
APPENDIX C
APPENDICES FOR CHAPTER 4

C.1 Derivation of the marginal value function, (4.24) through (4.25)

C.1.1 The value function

The Kuhn-Tucker conditions (4.21) in section 4.2 tells us whether the industry will increase its abatement capital stock in a period, given the realized value of \( \theta \). To derive the explicit form of the decision rule, we need to derive the explicit form of \( V(K, \theta) \). However, this is not an easy task. Instead, we will derive certain characteristics of \( V(K, \theta) \) that will allow us to derive the explicit marginal value function in subsection C.1.3.

To begin, suppose that in state \((K, \theta)\) no investment occurs. Then, (4.19) is

\[
V(K, \theta) = -S(K, \theta; \bar{a}, s, \tau, Q) \, dt + e^{-rdt} \{ E[V(K, \theta + d\theta)] \},
\]

which can be written as

\[
V(K, \theta) = -S(K, \theta; \bar{a}, s, \tau, Q) \, dt + (1 - rdt) \{ V(K, \theta) + E[dV] \} \quad \text{(C.1)}
\]

By applying Ito’s Lemma, the expected change in the value function can be expressed as

\[
E[dV] = \left[ \alpha \theta V_{\theta} + \frac{1}{2} \sigma^2 \theta^2 V_{\theta\theta} \right] \, dt. \quad \text{(C.2)}
\]
Substituting (C.2) into (C.1), collecting terms and dropping terms involving \((dt)^2\) allows us to derive the differential equation which \(V(K, \theta)\) must satisfy:

\[
\frac{1}{2}\sigma^2 \theta^2 V_{\theta\theta}(K, \theta) + \alpha \theta V_\theta(K, \theta) - r V(K, \theta) - S(K, \theta; \bar{a}, s, \tau, Q) = 0. \tag{C.3}
\]

Since \(\theta\) has three cutoff values \(\theta_F(K), \theta_s(K)\) and \(\theta_r(K)\), and \(S(K, \theta; \bar{a}, s, \tau, Q)\) has a different form in each interval, we will derive the value function for each interval and stitch them together at the cutoff values. Then, the general solution of (C.3) consists of:

\[
V(K, \theta) = \begin{cases} 
A_1(K) \theta^{\beta_1} + A_2(K) \theta^{\beta_2} - \frac{c(K)\bar{a}Q^2}{2(r-\alpha)} + \frac{(Q-\bar{a})s}{r} & \text{for } \theta \leq \theta_F \\
B_1(K) \theta^{\beta_1} + B_2(K) \theta^{\beta_2} + \frac{s^2}{2(r-\sigma^2+\alpha)c(K)\bar{a}} - \frac{\bar{a}^2}{r} & \text{for } \theta_F \leq \theta \leq \theta_s \\
D_1(K) \theta^{\beta_1} + D_2(K) \theta^{\beta_2} - \frac{c(K)\bar{a}^2}{2(r-\alpha)} & \text{for } \theta_s \leq \theta \leq \theta_r \\
E_1(K) \theta^{\beta_1} + E_2(K) \theta^{\beta_2} + \frac{r^2}{2(r-\sigma^2+\alpha)c(K)\bar{a}} - \frac{\bar{a}r}{r} & \text{for } \theta_r \leq \theta.
\end{cases}
\]

where \(\beta_1\) and \(\beta_2\) are solutions to \(\frac{1}{2} \sigma^2 \beta (\beta - 1) + \alpha \beta - r = 0\). Given the assumptions that \(r - \alpha > 0\) and \(r - \sigma^2 + \alpha > 0\) (equation (4.3)), it must be that \(\beta_1 > 1\) and \(\beta_2 < -1\). \(^{1}\)

As \(\theta\) goes to zero, we assume that all emissions are abated because abatement costs go to zero (see (4.1)), that is, \(a = Q\). Thus, all issued permits \(L\) will be sold back to the government. Therefore, it is unlikely that investment in extra capital stock will occur as \(\theta\) goes to zero. Therefore, we impose a boundary condition:

\[
\lim_{\theta \to 0} V(K, \theta) = \frac{s(Q-\bar{a})}{r} = \frac{sL}{r}.
\]

\(^{1}\)For \(r - \alpha > 0\) and \(r - \sigma^2 + \alpha > 0\), it can be shown that \(\beta_1 = \left(-2\alpha + \sigma^2 + \sqrt{8r\sigma^2 + (\sigma^2 - 2\alpha)^2}\right)/2\sigma^2 > 1\) and \(\beta_2 = \left(-2\alpha + \sigma^2 - \sqrt{8r\sigma^2 + (\sigma^2 - 2\alpha)^2}\right)/2\sigma^2 < -1\). The demonstrations are omitted to save spaces.
Since $\beta_2 < -1$, $A_2 (K) \theta^{\beta_2}$ goes to either $\infty$ or $-\infty$ as $\theta$ goes to zero. This implies that for this boundary condition to be satisfied, it should be that $A_2 (K) = 0$. Then, the value function $V(K, \theta)$ is expressed as:

\[
V^F(K, \theta) = A_1(K) \theta^{\beta_1} - \frac{c(K) Q^2}{2 (r - \alpha)} + \frac{s (Q - \bar{a})}{r} \quad \text{for } \theta \leq \theta_F \tag{C.4}
\]

\[
V^s(K, \theta) = B_1(K) \theta^{\beta_1} + B_2(K) \theta^{\beta_2} + \frac{s^2}{2 (r - \sigma^2 + \alpha)} \frac{c(K)}{c(K)} \theta - \frac{\bar{a}s}{r} \quad \text{for } \theta_F \leq \theta \leq \theta_s \tag{C.5}
\]

\[
V^m(K, \theta) = D_1(K) \theta^{\beta_1} + D_2(K) \theta^{\beta_2} - \frac{c(K) \bar{a}^2}{2 (r - \alpha)} \quad \text{for } \theta_s \leq \theta \leq \theta_\tau \tag{C.6}
\]

\[
V^\tau(K, \theta) = E_1(K) \theta^{\beta_1} + E_2(K) \theta^{\beta_2} + \frac{\tau^2}{2 (r - \sigma^2 + \alpha)} \frac{c(K)}{c(K)} \theta - \frac{\bar{a}\tau}{r} \quad \text{for } \theta_\tau \leq \theta. \tag{C.7}
\]

At the three cutoff values $\theta_F$, $\theta_s$ and $\theta_\tau$, we can derive six equations to specify coefficients $A_1$, $B_1$, $B_2$, $D_1$, $D_2$, $E_1$, and $E_2$. In addition, we have two more equations from the optimality conditions to find the investment barrier. Although we can’t specify some of coefficients explicitly, we can obtain enough information to derive investment decision rules which define the barrier in each interval. Expressions of some coefficients and relationships among all coefficients are found in the process of stitching together (C.4), (C.5), (C.6), and (C.7). The demonstration is given in the next subsection.

**C.1.2 Connecting the value function for all regimes**

In the previous subsection, we show that the value function has the expressions of (C.4), (C.5), (C.6) and (C.7) depending on where the realized value of $\theta$ falls. Now we will derive the explicit expressions of some coefficients and the relationships between the other coefficients in (C.4), (C.5), (C.6) and (C.7). These results will be used when we derive the investment decision rules in the next subsection. There are three cutoff values of $\theta$; $\theta_F (K)$, $\theta_s (K)$ and $\theta_\tau (K)$. From Dixit (1993), the value function for each
regime (C.4), (C.5), (C.6) and (C.7) must be tangent at the cutoff values. Thus, at 
\( \theta = \theta_F(K) \):

\[
V^F(K, \theta_F) = V^s(K, \theta_F) ;
\]
\[
V^F_{\theta}(K, \theta_F) = V^s_{\theta}(K, \theta_F) .
\]

By solving these equations using (C.4) and (C.5), we can express \( B_2(K) \) and \( A_1(K) - B_1(K) \) as follows

\[
B_2(K) = \frac{\theta_F^\beta_2}{\beta_1 - \beta_2} \left( \frac{1 - \beta_1}{r - \alpha} - \frac{1 + \beta_1}{r - \sigma^2 + \alpha} + \frac{2\beta_1}{r} \right) \frac{sQ}{2} ; \quad (C.8)
\]
\[
A_1(K) - B_1(K) = \frac{\theta_F^\beta_1}{\beta_1 - \beta_2} \left( \frac{1 - \beta_2}{r - \alpha} - \frac{1 + \beta_2}{r - \sigma^2 + \alpha} + \frac{2\beta_2}{r} \right) \frac{s\bar{a}}{2} .
\]

At \( \theta = \theta_s(K) \), we need

\[
V^s(K, \theta_s) = V^m(K, \theta_s) ;
\]
\[
V^s_{\theta}(K, \theta_s) = V^m_{\theta}(K, \theta_s) .
\]

From (C.5) and (C.6), we can derive the expressions of \( B_1(K) - D_1(K) \) and \( B_2(K) - D_2(K) \) as follows

\[
B_1(K) - D_1(K) = -\frac{\theta_s^\beta_1}{\beta_1 - \beta_2} \left( \frac{1 - \beta_2}{r - \alpha} - \frac{1 + \beta_2}{r - \sigma^2 + \alpha} + \frac{2\beta_2}{r} \right) \frac{s\bar{a}}{2} ;
\]
\[
B_2(K) - D_2(K) = \frac{\theta_s^\beta_2}{\beta_1 - \beta_2} \left( \frac{1 - \beta_1}{r - \alpha} - \frac{1 + \beta_1}{r - \sigma^2 + \alpha} + \frac{2\beta_1}{r} \right) \frac{s\bar{a}}{2} . \quad (C.9)
\]

Finally, at \( \theta = \theta_r(K) \), we need

\[
V^m(K, \theta_r) = V^\tau(K, \theta_r) ;
\]
\[
V^m_{\theta}(K, \theta_r) = V^\tau_{\theta}(K, \theta_r) .
\]
From (C.6) and (C.7), we can derive the expression of $D_1(K) - E_1(K)$ and $D_2(K) - E_2(K)$ as follows

$$D_1(K) - E_1(K) = \frac{\theta - \beta_1}{\beta_1 - \beta_2} \left( \frac{1 - \beta_1}{r - \alpha} - \frac{1 + \beta_1}{r - \sigma^2 + \alpha} + \frac{2\beta_1}{r} \right) \frac{\tau \bar{a}}{2};$$

$$D_2(K) - E_2(K) = -\frac{\theta - \beta_2}{\beta_1 - \beta_2} \left( \frac{1 - \beta_1}{r - \alpha} - \frac{1 + \beta_1}{r - \sigma^2 + \alpha} + \frac{2\beta_1}{r} \right) \frac{\tau \bar{a}}{2}. \quad (C.10)$$

Solving (C.8), (C.9), and (C.10) gives us explicit expressions of $B_2(K)$, $D_2(K)$, and $E_2(K)$. However, we are unable to find the explicit expressions of $A_1(K)$, $B_1(K)$, $D_1(K)$, and $E_1(K)$. They will be replaced with relationships that are obtained from solving the optimal investment strategy.

C.1.3 Derivation of the marginal value function

In this subsection, we will derive the expression of $V_K(K, \theta)$ in each interval of $\theta$ as shown by (4.22) through (4.25) in section 4.2.

C.1.3.1 When aggregate abatement reaches $Q$.

If a new investment in abatement capital is required, when the price floor $s$ binds and aggregate abatement achieves the unregulated emissions $Q$, given $K$, a trigger point of $\theta^*$ should be less than the cutoff value $\theta_F$ ($\theta^* \leq \theta_F$). Then, following Dixit (1991), Sarkar (2009), and Hagspiel et al. (2012), (C.4) will be used to make an investment decision. The optimality conditions at a trigger point $\theta^*$ are

$$V_K^F(K, \theta^*) = w; \quad (C.11)$$

$$V_{K\theta}^F(K, \theta^*) = 0. \quad (C.12)$$

(C.11) is known as the value-matching condition and it implies that marginal value of investment should be equal to the unit cost of investment. (C.12) is the smooth-pasting condition which implies at the boundary $\theta^*$, two functions $V_K^F(K, \theta^*)$ and $w$
should be tangent with respect to a random shock $\theta$. By applying (C.11) and (C.12) to (C.4), we have

$$\frac{\partial A_1(K)}{\partial K} (\theta^*)^{\beta_1} - \frac{1}{r - \alpha} \frac{1}{2} c'(K) \theta^* Q^2 = w; \quad (C.13)$$

$$\beta_1 \frac{\partial A_1(K)}{\partial K} (\theta^*)^{\beta_1 - 1} - \frac{1}{r - \alpha} \frac{1}{2} c'(K) Q^2 = 0. \quad (C.14)$$

By isolating $\frac{\partial A_1(K)}{\partial K} (\theta^*)^{\beta_1}$ in (C.14), we have

$$\frac{\partial A_1(K)}{\partial K} (\theta^*)^{\beta_1} = \frac{1}{\beta_1} \frac{1}{r - \alpha} \frac{1}{2} c'(K) \theta^* Q^2.$$

By substituting this result into (C.13) we obtain (4.22); that is,

$$V_K^F(K, \theta) = -\frac{\beta_1 - 1}{\beta_1} \frac{1}{r - \alpha} \frac{1}{2} c'(K) \theta Q^2 = w.$$

**C.1.3.2 At the price floor.**

If investment to increase capital stock is required when the price floor $s$ binds but aggregate abatement does not reach the unregulated emissions $Q$, a trigger point $\theta^*$ must be between two cutoff values $\theta_F$ and $\theta_s$ ($\theta_F \leq \theta^* \leq \theta_s$). Then, (C.7) is used to make the investment decision. The optimality conditions at $\theta^*$ are:

$$V_K^m(K, \theta^*) = w; \quad (C.15)$$

$$V_{K\theta}^m(K, \theta^*) = 0. \quad (C.16)$$

Applying (C.15) and (C.16) to (C.5) gives us the following equations to solve:

$$\frac{\partial B_1(K)}{\partial K} (\theta^*)^{\beta_1} + \frac{\partial B_2(K)}{\partial K} (\theta^*)^{\beta_2} - \frac{1}{r - \sigma^2} \frac{1}{2} c'(K) s^2 = w; \quad (C.17)$$

$$\beta_1 \frac{\partial B_1(K)}{\partial K} (\theta^*)^{\beta_1 - 1} + \beta_2 \frac{\partial B_2(K)}{\partial K} (\theta^*)^{\beta_2 - 1} + \frac{1}{r - \sigma^2} \frac{1}{2} c'(K) s^2 (\theta^*)^2 = 0. \quad (C.18)$$
By multiplying (C.18) by \( \theta^* \) and isolating \( \frac{\partial B_1(K)}{\partial K} (\theta^*)^{\beta_1} \), we have

\[
\frac{\partial B_1(K)}{\partial K} (\theta^*)^{\beta_1} = \frac{\beta_2}{\beta_1} \frac{\partial B_2(K)}{\partial K} (\theta^*)^{\beta_2} - \frac{1}{\beta_1} \frac{1}{r - \sigma^2 + \alpha 2c(K)^2 \theta^*} c'(K) s^2.
\]

Substituting this result into the left-side of (C.17) yields

\[
\frac{\partial B_1(K)}{\partial K} (\theta^*)^{\beta_1} + \frac{\partial B_2(K)}{\partial K} (\theta^*)^{\beta_2} - \frac{1}{\beta_1} \frac{1}{r - \sigma^2 + \alpha 2c(K)^2 \theta^*} c'(K) s^2 = w. \tag{C.19}
\]

Since we integrated the two optimality conditions, the last step to derive the investment decision rule for \( \theta_F \leq \theta \leq \theta_s \) is to find the expressions of \( \frac{\partial B_2(K)}{\partial K} \) in (C.19). Then, from (4.15) and (C.8), we have

\[
\frac{\partial B_2(K)}{\partial K} = \frac{\beta_2 \theta_F^{-\beta_2}}{\beta_1 - \beta_2} \left( \frac{1 - \beta_1}{r - \alpha} - \frac{1 + \beta_1}{r - \sigma^2 + \alpha} + \frac{2\beta_1}{r} \right) sQ c'(K) \frac{c(K)}{2}. 
\]

Substituting this result into (C.19) gives us

\[
\frac{\beta_1 - \beta_2}{\beta_1} \frac{\partial B_2(K)}{\partial K} (\theta^*)^{\beta_2} - \frac{\beta_1 + 1}{\beta_1} \frac{1}{r - \sigma^2 + \alpha 2c(K)^2 \theta^*} c'(K) s^2 = w.
\]

Since

\[
\frac{\beta_2}{\beta_1} \left( \frac{1 - \beta_1}{r - \alpha} - \frac{1 + \beta_1}{r - \sigma^2 + \alpha} + \frac{2\beta_1}{r} \right) c'(K)Qs \left( \frac{\theta^*}{\theta_F} \right)^{\beta_2} - \frac{\beta_1 + 1}{\beta_1} \frac{1}{r - \sigma^2 + \alpha 2c(K)^2 \theta^*} c'(K)s^2 = w.
\]

Since

\[
\frac{\beta_2}{\beta_1} \left( \frac{1 - \beta_1}{r - \alpha} - \frac{1 + \beta_1}{r - \sigma^2 + \alpha} + \frac{2\beta_1}{r} \right) = \frac{1}{\beta_1} \left( \frac{\beta_1 + 1}{r - \sigma^2 + \alpha} - \frac{\beta_1 - 1}{r - \alpha} \right) > 0,
\]

we obtain (4.23), that is,

\[
V^s(K, \theta) = \frac{1}{\beta_1} \left( \frac{\beta_1 + 1}{r - \sigma^2 + \alpha} - \frac{\beta_1 - 1}{r - \alpha} \right) c'(K)Qs \left( \frac{\theta}{\theta_F} \right)^{\beta_2} - \frac{\beta_1 + 1}{\beta_1} \frac{1}{r - \sigma^2 + \alpha 2c(K)^2 \theta} c'(K)s^2 = w.
\]

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None of price controls bind.

When $\theta_s \leq \theta^* \leq \theta_r$, (C.6) is used to make an investment decision. In this case the value matching and smoothing pasting conditions are

\begin{align*}
V^m_K(K, \theta^*) &= w; \quad \text{(C.20)} \\
V^m_{K\theta}(K, \theta^*) &= 0. \quad \text{(C.21)}
\end{align*}

By applying (C.20) and (C.21) to (C.6), we have the following:

\begin{align*}
\frac{\partial D_1(K)}{\partial K} (\theta^*)^{\beta_1} + \frac{\partial D_2(K)}{\partial K} (\theta^*)^{\beta_2} - \frac{1}{r - \alpha} \frac{1}{2} c'(K) \theta^* \bar{a}^2 &= w; \quad \text{(C.22)} \\
\beta_1 \frac{\partial D_1(K)}{\partial K} (\theta^*)^{\beta_1 - 1} + \beta_2 \frac{\partial D_2(K)}{\partial K} (\theta^*)^{\beta_2 - 1} - \frac{1}{r - \alpha} \frac{1}{2} c'(K) \bar{a}^2 &= 0. \quad \text{(C.23)}
\end{align*}

By multiplying (C.23) by $\theta^*$ and isolating $\frac{\partial D_1(K)}{\partial K} (\theta^*)^{\beta_1}$ we find

\begin{align*}
\frac{\partial D_1(K)}{\partial K} (\theta^*)^{\beta_1} &= -\frac{\beta_2}{\beta_1} \frac{\partial D_2(K)}{\partial K} (\theta^*)^{\beta_2} + \frac{1}{\beta_1} c'(K) \theta^* \bar{a}^2 \\
&= \frac{\beta_1 - \beta_2}{\beta_1} \frac{\partial D_2(K)}{\partial K} (\theta^*)^{\beta_2} - \frac{\beta_1 - 1}{\beta_1} \frac{1}{r - \alpha} \frac{1}{2} c'(K) \theta^* \bar{a}^2 = w. \quad \text{(C.24)}
\end{align*}

Then, by substituting this result into (C.22), we have

\begin{align*}
\frac{\partial D_1(K)}{\partial K} (\theta^*)^{\beta_1} + \frac{\partial D_2(K)}{\partial K} (\theta^*)^{\beta_2} - \frac{1}{r - \alpha} \frac{1}{2} c'(K) \theta^* \bar{a}^2 &= w \\
&= \frac{\beta_1 - \beta_2}{\beta_1} \frac{\partial D_2(K)}{\partial K} (\theta^*)^{\beta_2} - \frac{\beta_1 - 1}{\beta_1} \frac{1}{r - \alpha} \frac{1}{2} c'(K) \theta^* \bar{a}^2 = w. \quad \text{(C.24)}
\end{align*}

Using (4.13), (4.15), (C.8), and (C.9), we have

\begin{align*}
\frac{\partial D_2(K)}{\partial K} &= \frac{\beta_2}{\beta_1 - \beta_2} \left( \frac{1 - \beta_1}{r - \alpha} - \frac{1 + \beta_1}{r - \sigma^2 + \alpha} + \frac{2\beta_1}{r} \right) \left( \frac{sQ c'(K)}{2} \theta^* \beta_2 - \frac{s a c'(K)}{2} c(K) \theta^* \beta_2 \right).
\end{align*}
Substituting this result into (C.24) gives us (4.24), that is,

\[ V^m_K(K, \theta) = \frac{1}{\beta_1} \left( \frac{\beta_1 + 1}{r - \sigma^2 + \alpha} - \frac{\beta_1 - 1}{r - \alpha} \right) \left[ \frac{sQ}{2} \left( \frac{\theta}{\theta_F} \right)^{\beta_2} - \frac{s \bar{a}}{2} \left( \frac{\theta}{\theta_s} \right)^{\beta_2} \right] - \frac{\beta_1 - 1}{\beta_1} \frac{1}{2} \frac{c'(K) \theta \bar{a}^2}{r - \sigma^2 + \alpha} = w. \]

C.1.3.4 At the price ceiling.

When \( \theta^* \geq \theta_\tau \), the value-matching and smooth-pasting conditions are:

\[ V^r_K(K, \theta^*) = w; \quad (C.25) \]
\[ V^T_{K\theta}(K, \theta^*) = 0. \quad (C.26) \]

Applying (C.25) and (C.26) to (C.7) gives us

\[ \frac{\partial E_1(K)}{\partial K} (\theta^*)^{\beta_1} + \frac{\partial E_2(K)}{\partial K} (\theta^*)^{\beta_2} - \frac{1}{r - \sigma^2 + \alpha} \frac{c'(K) \tau^2}{2c(K)^2 \theta^*} = w; \quad (C.27) \]
\[ \frac{\beta_1}{\partial K} \frac{\partial E_1(K)}{\partial K} (\theta^*)^{\beta_1 - 1} + \beta_2 \frac{\partial E_2(K)}{\partial K} (\theta^*)^{\beta_2 - 1} + \frac{1}{r - \sigma^2 + \alpha} \frac{c'(K) \tau^2}{2c(K)^2 (\theta^*)^2} = 0. \quad (C.28) \]

By multiplying (C.28) by \( \theta^* \) and isolating \( \frac{\partial E_1(K)}{\partial K} (\theta^*)^{\beta_1} \) we have

\[ \frac{\partial E_1(K)}{\partial K} (\theta^*)^{\beta_1} = - \beta_2 \frac{\partial E_2(K)}{\partial K} (\theta^*)^{\beta_2} - \frac{1}{\beta_1} \frac{1}{r - \sigma^2 + \alpha} \frac{c'(K) \tau^2}{2c(K)^2 \theta^*}. \]

Substituting this result into (C.27) gives us

\[ \frac{\beta_1}{\beta_1} \frac{\partial E_1(K)}{\partial K} (\theta^*)^{\beta_1} - \frac{\beta_2}{\beta_1} \frac{\partial E_2(K)}{\partial K} (\theta^*)^{\beta_2} - \frac{1}{\beta_1} \frac{1}{r - \sigma^2 + \alpha} \frac{c'(K) \tau^2}{2c(K)^2 \theta^*} = w. \quad (C.29) \]
From (4.13), (4.14), (4.15), (C.8), (C.9), and (C.10), we have

\[
\frac{\partial E_2(K)}{\partial K} = \frac{\beta_2}{\beta_1 - \beta_2} \left( \frac{1 - \beta_1}{r - \alpha} - \frac{1 + \beta_1}{r - \sigma^2 + \alpha} + \frac{2\beta_1}{r} \right) \left( \frac{sQ c'(K)}{2 c(K)} \theta_F^{-\beta_2} - \frac{s\bar{a} c'(K)}{2 c(K)} \theta_s^{-\beta_2} + \frac{\tau\bar{a} c'(K)}{2 c(K)} \theta_r^{-\beta_2} \right).
\]

Substituting this into (C.29) gives us (4.25), that is,

\[
V^\tau_K(K, \theta) = \frac{1}{\beta_1} \left( \frac{\beta_1 + 1}{r - \sigma^2 + \alpha} - \frac{\beta_1 - 1}{r - \alpha} \right) \left[ sQ c'(K) \left( \frac{\theta}{\theta_F} \right)^{\beta_2} - s\bar{a} c'(K) \left( \frac{\theta}{\theta_s} \right)^{\beta_2} + \tau\bar{a} c'(K) \left( \frac{\theta}{\theta_r} \right)^{\beta_2} \right]
\]

\[
- \frac{\beta_1 + 1}{\beta_1} \frac{1}{r - \sigma^2 + \alpha} \frac{c'(K)}{2c(K)^2 \theta}
\]

\[
= w.
\]

C.2 Proof of Lemma 1

In this section, we will prove Lemma 1 in subsection 4.3.1, which characterizes the shape of $V_K(K, \theta)$ with respect to $\theta$. First, in parts (a) to (d) of subsection C.2.1, we will show that three of the four expressions for $V_K(K, \theta)$ ((4.23), (4.24), and (4.25)) have stationary points and two of them are local maxima and the other one is a local minimum. However, these stationary points are not always manifested in our model. In order for these stationary points to show up in our model, they must be located in the relevant interval where the matched expression of $V_K(K, \theta)$ is defined. Thus, in parts (e) to (g) of subsection C.2.2, we will show the relationships among these stationary points from which we can infer the shape of $V_K(K, \theta)$. The proof will be completed by examining the limitd of $V_K(K, \theta)$ as $\theta$ goes to 0 and $\infty$ in part (h) of subsection C.2.2.
C.2.1 Stationary points of $V_K(K, \theta)$

Part (a). When aggregate abatement reaches $Q$.

From (4.22) we have

$$\frac{\partial V^F_K(K, \theta)}{\partial \theta} = -\frac{\beta_1 - 1}{\beta_1} c'(K) Q^2 - \frac{1}{2} (r - \alpha) > 0; \quad \frac{\partial^2 V^F_K(K, \theta)}{\partial \theta^2} = 0,$$

which implies that $V^F_K(K, \theta)$ is monotonically increasing as $\theta$ increases in the interval of $(0, \theta_F]$ and the slope of $V^F_K(K, \theta)$ with respect to $\theta$ is constant, given the capital stock $K$.

Part (b). At the price floor.

By differentiating (4.23) with respect to $\theta$, we have

$$\frac{\partial V^s_K(K, \theta)}{\partial \theta} = \left[\frac{\beta_2}{\beta_1} \left( \frac{\beta_1 + 1 - \beta_1}{r - \sigma^2 + \alpha} \frac{1}{\theta - \alpha} \right) \left( \frac{\theta}{\theta_F} \right)^{\beta_2 + 1} + \frac{\beta_1 + 1}{\beta_1} \frac{1}{r - \sigma^2 + \alpha} \right] c'(K) s^2 2c'K s^2 2cK s^2 \theta^2. \quad \text{(C.30)}$$

Let the first part of $\partial V^s_K(K, \theta) / \partial \theta$ in square brackets be

$$g^s(\theta) = \frac{\beta_2}{\beta_1} \left( \frac{\beta_1 + 1 - \beta_1}{r - \sigma^2 + \alpha} \frac{1}{\theta - \alpha} \right) \left( \frac{\theta}{\theta_F} \right)^{\beta_2 + 1} + \frac{\beta_1 + 1}{\beta_1} \frac{1}{r - \sigma^2 + \alpha}.$$

Denote a stationary point of $V^s_K(K, \theta)$ as $\theta^*_0$; that is, $\partial V^s_K(K, \theta^*_0) / \partial \theta = 0$. Then, since the last term of (C.30) is always strictly negative; that is, $(c'(K) s^2) / (2cK s^2 \theta^2) < 0$, $\theta^*_0$ is a solution to $g^s(\theta) = 0$, which implies

$$\theta^*_0 = \theta_F \left[ -\frac{\beta_1 + 1}{\beta_1} \frac{r - \sigma^2 + \alpha}{\beta_2 \theta_F} \right]^{\frac{1}{\beta_2 + 1}} = \theta_F \left[ \frac{1}{2} \left( 1 - \frac{1}{\beta_2} \right) \right]^{\frac{1}{\beta_2 + 1}} > \theta_F. \quad \text{(C.31)}$$

The last inequality in (C.31) always holds because $\beta_2 < -1$. 

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Since the last term of (C.30) is always negative the sign of $\partial V_s^K (K, \theta) / \partial \theta$ is completely determined by the sign of $g^s (\theta)$; that is,

$$\text{sign} \left[ \frac{\partial V_s^K (K, \theta)}{\partial \theta} \right] = \text{sign} [-g^s (\theta)].$$

Since $\partial g^s (\theta) / \partial \theta > 0$ for all $\theta$, $g^s (\theta)$ is a strictly increasing function of $\theta$ and becomes zero at $\theta = \theta_0^s$, which implies that

$$g^s (\theta) \begin{cases} < 0 & \text{for } \theta < \theta_0^s \\ = 0 & \text{for } \theta = \theta_0^s \\ > 0 & \text{for } \theta > \theta_0^s, \end{cases}$$

and thus,

$$\frac{\partial V_s^K (K, \theta)}{\partial \theta} \begin{cases} > 0 & \text{for } \theta < \theta_0^s \\ = 0 & \text{for } \theta = \theta_0^s \\ < 0 & \text{for } \theta > \theta_0^s. \end{cases} \quad (C.32)$$

Therefore, $\theta_0^s$, the stationary point of $V_s^K (K, \theta)$, is a local maximum.

**Part (c). Neither of price controls binds.**

By differentiating (4.24) with respect to $\theta$, we have

$$\frac{\partial V_K^m (K, \theta)}{\partial \theta} = \left\{ \frac{\beta_2}{\beta_1} \left( \frac{\beta_1 + 1}{r - \sigma^2 + \alpha} - \frac{\beta_1 - 1}{r - \alpha} \right) \left[ \left( \frac{Q}{\bar{a}} \right)^2 \left( \frac{\theta}{\bar{a}_f} \right)^{\beta_2 - 1} - \left( \frac{\theta}{\bar{a}_s} \right)^{\beta_2 - 1} \right] - \frac{\beta_1 - 1}{\beta_1} \frac{1}{r - \alpha} \right\} c'(K) \bar{a}^2.$$ 

Let the first part of $\frac{\partial V_K^m (K, \theta)}{\partial \theta}$ in braces be

$$g^m (\theta) = \frac{\beta_2}{\beta_1} \left( \frac{\beta_1 + 1}{r - \sigma^2 + \alpha} - \frac{\beta_1 - 1}{r - \alpha} \right) \left[ \left( \frac{Q}{\bar{a}} \right)^{\beta_2 + 1} - \left( \frac{\theta}{\bar{a}_s} \right)^{\beta_2 - 1} \right] - \frac{\beta_1 - 1}{\beta_1} \frac{1}{r - \alpha}.$$
Since $\beta_2 < -1$ and $Q/\bar{a} > 1$, $(Q/\bar{a})^{\beta_2 + 1} - 1 < 0$, which implies that $g^m(\theta)$ is a decreasing function of $\theta$. Denote a stationary point of $V^m_K(K, \theta)$ as $\theta^m_0$; that is, $\partial V^m_K(K, \theta^m_0) / \partial \theta = 0$. Then, $\theta^m_0$ is a solution to $g^m(\theta) = 0$; which implies

$$\theta^m_0 = \theta_s \left[ \frac{\beta_1 - 1}{\beta_1} \frac{1}{r - \alpha} \right]^{1/\beta_2} = \theta_s \left[ \frac{1}{2} \left( 1 + \frac{1}{\beta_2} \right) \frac{1}{1 - (Q/\bar{a})^{\beta_2 + 1}} \right]^{1/\beta_2 - 1}. \quad (C.33)$$

Since $\ell'(K) Q^2/2 < 0$, the sign of $\partial V^m_K(K, \theta) / \partial \theta$ is determined by the sign of $g^m(\theta)$; that is, $\text{sign} [\partial V^m_K(K, \theta) / \partial \theta] = \text{sign} [-g^m(\theta)]$. Since $g^m(\theta)$ is a decreasing function of $\theta$ and becomes zero at $\theta = \theta^m_0$,

$$g^m(\theta) \begin{cases} > 0 & \text{for } \theta < \theta^m_0 \\ = 0 & \text{for } \theta = \theta^m_0 \\ < 0 & \text{for } \theta > \theta^m_0, \end{cases}$$

and thus $\partial V^m_K(K, \theta) / \partial \theta$ is

$$\frac{\partial V^m_K(K, \theta)}{\partial \theta} \begin{cases} < 0 & \text{for } \theta < \theta^m_0 \\ = 0 & \text{for } \theta = \theta^m_0 \\ > 0 & \text{for } \theta > \theta^m_0. \end{cases}$$

Therefore, $\theta^m_0$, the stationary point of $V^m_K(K, \theta)$, is a local minimum.

**Part (d). At the price ceiling.**

The derivative of (4.25) with respect to $\theta$ is
\[
\frac{\partial V^\tau_K (K, \theta)}{\partial \theta} = \frac{\beta_2}{\beta_1} \left( \frac{\beta_1 + 1}{r - \sigma^2 + \alpha} - \frac{\beta_1 - 1}{r - \alpha} \right) \left[ \frac{sQ}{2\theta_F} \left( \frac{\theta}{\theta_F} \right)^{\beta_2 - 1} \frac{1}{\theta_F} - \frac{s\bar{a}}{2} \left( \frac{\theta}{\theta_s} \right)^{\beta_2 - 1} \frac{1}{\theta_s} + \frac{\tau\bar{a}}{2} \left( \frac{\theta}{\theta_\tau} \right)^{\beta_2 - 1} \frac{1}{\theta_\tau} \right] c'(K) \frac{c(K)}{c(K)} + \beta_1 + 1 \frac{1}{r - \sigma^2 + \alpha} \frac{c'(K) \tau^2}{2c(K)^2 \theta^2}.
\]

By using the definition of \( \theta_F, \theta_s, \) and \( \theta_\tau \) from (4.7), (4.8), and (4.11) and collecting common terms, we have,

\[
\frac{\partial V^\tau_K (K, \theta)}{\partial \theta} = \beta_2 \left( \beta_1 + 1 \right) \left[ \left( \frac{Q}{\bar{a}} \right)^{\beta_2 + 1} - 1 \right] \left( \frac{\tau}{s} \right)^{\beta_2 - 1} + 1 \left( \frac{\theta}{\theta_\tau} \right)^{\beta_2 + 1} + \beta_1 + 1 \frac{1}{r - \sigma^2 + \alpha} \frac{c'(K) \tau^2}{2c(K)^2 \theta^2}.
\]

Let the last part of \( \frac{\partial V^\tau_K (K, \theta)}{\partial \theta} \) in braces be

\[
g^\tau(\theta) = \frac{\beta_2}{\beta_1} \left( \beta_1 + 1 \right) \left[ \left( \frac{Q}{\bar{a}} \right)^{\beta_2 + 1} - 1 \right] \left( \frac{\tau}{s} \right)^{\beta_2 - 1} + 1 \left( \frac{\theta}{\theta_\tau} \right)^{\beta_2 + 1} + \beta_1 + 1 \frac{1}{r - \sigma^2 + \alpha} \frac{c'(K) \tau^2}{2c(K)^2 \theta^2}.
\]

Denote a stationary point of \( V^\tau_K (K, \theta) \) as \( \theta_0^\tau \), that is, \( \frac{\partial V^\tau_K (K, \theta_0^\tau)}{\partial \theta} = 0 \). Then, \( \theta_0^\tau \) is a solution to \( g^\tau(\theta) = 0 \), which implies

\[
\theta_0^\tau = \theta^\tau \left[ \frac{1}{2} \left( 1 - \frac{1}{\beta_2} \right) \left[ 1 - \left( \frac{Q}{\bar{a}} \right)^{\beta_2 + 1} \left( \frac{\tau}{s} \right)^{\beta_2 - 1} \right] \right]^{\frac{1}{\beta_2 + 1}}.
\]

(C.34)

Since \( (c'(K) \tau^2) / (2c(K)^2 \theta^2) < 0 \), the sign of \( \partial V^\tau_K (K, \theta) / \partial \theta \) is determined by the sign of \( g^\tau(\theta) \); that is, \( \text{sign} \left[ \frac{\partial V^\tau_K (K, \theta)}{\partial \theta} \right] = \text{sign} \left[ -g^\tau(\theta) \right] \). \( g^\tau(\theta) \) is an increasing function of \( \theta \) because \( \left( \frac{Q}{\bar{a}} \right)^{\beta_2 + 1} - 1 \left( \frac{\tau}{s} \right)^{\beta_2 - 1} + 1 > 0 \):
\[ g^r(\theta) \begin{cases} < 0 & \text{for } \theta < \theta_0^r \\ = 0 & \text{for } \theta = \theta_0^r \\ > 0 & \text{for } \theta > \theta_0^r, \end{cases} \]

which implies that
\[ \frac{\partial V_K^r(K, \theta)}{\partial \theta} \begin{cases} > 0 & \text{for } \theta < \theta_0^r \\ = 0 & \text{for } \theta = \theta_0^r \\ < 0 & \text{for } \theta > \theta_0^r. \end{cases} \]

(C.35)

Therefore, \( \theta_0^r \), the stationary point of \( V_K^r(K, \theta) \), is a local minimum.

Through parts (a) to (d) we have shown that the expressions of (4.23), (4.24), and (4.25) have the stationary points: \( \theta_s^0 \), \( \theta_m^0 \), and \( \theta_0^r \), respectively. In addition, \( \theta_s^0 \) and \( \theta_0^r \) are local maxima and \( \theta_m^0 \) is a local minimum. However, for these stationary points to be stationary points of \( V_K^r(K, \theta) \), they should lie in the relevant intervals. That is, if \( \theta_F < \theta_s^0 < \theta_s < \theta_m^0 < \theta_r < \theta_0^r \), then \( V_K^r(K, \theta) \) has two local maxima. However, there are other possibilities if these relationships do not hold. In the next subsubsection C.2.2, we derive conditions to determine the locations of these stationary points from which we can complete the proof of Lemma 1.

C.2.2 Conditions for the shape of \( V_K^r(K, \theta) \)

Part (e). Relationship between \( \theta_s^0 \), \( \theta_s \), and \( \theta_m^0 \).

To derive the relationship between \( \theta_s^0 \), \( \theta_s \), and \( \theta_m^0 \), suppose at first that \( \theta_s^0 \leq \theta_s \).

Then, from (C.31), we have

\[ \left[ \frac{1}{2} \left( 1 - \frac{1}{\beta_2} \right) \right] \frac{1}{\beta_2 + 1} \leq \frac{\theta_s}{\theta_F} = \frac{Q}{a}, \quad (C.36) \]

which can be rearranged to be
\[ \frac{1}{2} \left( 1 + \frac{1}{\beta_2} \right) \leq 1 - \left( \frac{Q}{a} \right)^{\beta_2+1}. \]  
(C.37)

Since \(0 < (Q/\bar{a})^{\beta_2+1} < 1\), applying this result to (C.33) yields the following:

\[ \left( \frac{\theta_0^m}{\theta_s} \right)^{\beta_2-1} = \frac{1}{2} \left( 1 + \frac{1}{\beta_2} \right) \frac{1}{1 - \left( \frac{Q}{a} \right)^{\beta_2+1}} \leq 1, \]  
(C.38)

which implies that

\[ \theta_0^m = \theta_s \left[ \frac{1}{2} \left( 1 + \frac{1}{\beta_2} \right) \frac{1}{1 - \left( \frac{Q}{a} \right)^{\beta_2+1}} \right]^{\frac{1}{\beta_2-1}} \geq \theta_s, \]  
(C.39)

because \(\beta_2 - 1 < 0\). In addition, (C.36) through (C.39) hold with equality only when \(\theta_0^s = \theta_s\). Therefore, if \(\theta_0^s \leq \theta_s\), then \(\theta_0^m \geq \theta_s\). On the other hand, if \(\theta_0^s \geq \theta_s\), then the directions of all the inequalites in (C.36) through (C.39) are reversed, which implies that \(\theta_0^m \leq \theta_s\). We can take the same step from the assumption of \(\theta_0^m \geq (\leq) \theta_s\) to derive \(\theta_0^s \leq (\geq) \theta_s\). With these results, we can show that the conditions that specify the locations of \(\theta_0^s, \theta_s, \) and \(\theta_0^m\) are:

\[ \frac{1}{2} \left( 1 + \frac{1}{\beta_2} \right) < 1 - \left( \frac{Q}{a} \right)^{\beta_2+1} \Rightarrow \theta_0^s < \theta_s < \theta_0^m; \]

\[ \frac{1}{2} \left( 1 + \frac{1}{\beta_2} \right) > 1 - \left( \frac{Q}{a} \right)^{\beta_2+1} \Rightarrow \theta_0^s > \theta_s > \theta_0^m; \]

\[ \frac{1}{2} \left( 1 + \frac{1}{\beta_2} \right) = 1 - \left( \frac{Q}{a} \right)^{\beta_2+1} \Rightarrow \theta_0^s = \theta_s = \theta_0^m. \]  
(C.40)

**Part (f). Relationship between \(\theta_0^m, \theta_r, \) and \(\theta_0^r\).**

To derive the relationship between \(\theta_0^m, \theta_r, \) and \(\theta_0^r\), suppose at first that \(\theta_0^m \leq \theta_r\). Then, from (C.33) we have

\[ \left[ \frac{1}{2} \left( 1 + \frac{1}{\beta_2} \right) \frac{1}{1 - \left( \frac{Q}{a} \right)^{\beta_2+1}} \right]^{\frac{1}{\beta_2-1}} \leq \frac{\theta_r}{\theta_s} = \frac{\tau}{s}, \]  
(C.41)
which can be rearranged to be
\[
\frac{1}{2} \left( 1 + \frac{1}{\beta_2} \right) \geq \left[ 1 - \left( \frac{Q}{\bar{a}} \right)^{\beta_2+1} \right] \left( \frac{T}{s} \right)^{\beta_2-1}.
\] (C.42)

This result can be manipulated further to be
\[
\frac{1}{2} \left( 1 - \frac{1}{\beta_2} \right) \leq 1 - \left( 1 - \left( \frac{Q}{\bar{a}} \right)^{\beta_2+1} \right) \left( \frac{T}{s} \right)^{\beta_2-1}.
\] (C.43)

Since \(0 < 1 - \left( 1 - (Q/\bar{a})^{\beta_2+1} \right) (\tau/s)^{\beta_2-1} < 1\), applying this result to (C.34) gives us the following
\[
\left( \frac{\theta_0^r}{\theta_r} \right)^{\beta_2+1} = \frac{1}{2} \left( 1 - \frac{1}{\beta_2} \right) \frac{1}{1 - \left( 1 - (Q/\bar{a})^{\beta_2+1} \right) (\tau/s)^{\beta_2-1}} \leq 1,
\] (C.44)

which implies that
\[
\theta_0^r = \theta_r \left[ \frac{1}{2} \left( 1 - \frac{1}{\beta_2} \right) \frac{1}{1 - \left( 1 - (Q/\bar{a})^{\beta_2+1} \right) (\tau/s)^{\beta_2-1}} \right]^{\frac{1}{\beta_2+1}} \geq \theta_r.
\] (C.45)

(C.41) through (C.45) hold as equalities only when \(\theta_0^m = \theta_r\). Therefore, if \(\theta_0^m \leq \theta_r\), then \(\theta_0^r \geq \theta_r\). On the other hand, if \(\theta_0^m \geq \theta_r\), then the directions of all the inequalities in (C.41) through (C.45) are reversed, which implies \(\theta_0^r \leq \theta_r\). We can take the same steps from the assumption of \(\theta_0^r \leq (\geq) \theta_r\) to derive \(\theta_0^r \geq (\leq) \theta_r\). With these results, we can show that the conditions that specify the locations of \(\theta_0^m, \theta_r, \) and \(\theta_0^r\) are:

\[
\begin{align*}
\frac{1}{2} \left( 1 + \frac{1}{\beta_2} \right) &> \left[ 1 - \left( \frac{Q}{\bar{a}} \right)^{\beta_2+1} \right] \left( \frac{T}{s} \right)^{\beta_2-1} \Rightarrow \theta_0^m < \theta_r < \theta_0^r; \\
\frac{1}{2} \left( 1 + \frac{1}{\beta_2} \right) &< \left[ 1 - \left( \frac{Q}{\bar{a}} \right)^{\beta_2+1} \right] \left( \frac{T}{s} \right)^{\beta_2-1} \Rightarrow \theta_0^m > \theta_r > \theta_0^r; \\
\frac{1}{2} \left( 1 + \frac{1}{\beta_2} \right) &= \left[ 1 - \left( \frac{Q}{\bar{a}} \right)^{\beta_2+1} \right] \left( \frac{T}{s} \right)^{\beta_2-1} \Rightarrow \theta_0^m = \theta_r = \theta_0^r.
\end{align*}
\] (C.46)
Part (g). First part of Lemma 1

From the results of (C.40) and (C.46), possible combinations for the locations of \((\theta^s_0, \theta^m_0, \theta^\tau_0)\) and \((\theta_s, \theta^\tau)\) are as follows:

1) If \(\theta^s_0 \geq \theta_s\), then \(\theta^m_0 \leq \theta_s < \theta^\tau\), which implies that \(\theta^s_0\) and \(\theta^m_0\) are not stationary points of \(V_K(K, \theta)\). Thus, \(\theta^\tau_0\) is the only stationary point, implying that \(V_K(K, \theta)\) has a single maximum at the price ceiling.

2) If \(\theta^s_0 < \theta_s\) and \(\theta^m_0 \geq \theta^\tau\), then \(\theta^\tau_0 \leq \theta^\tau\). This implies further \(\theta^m_0\) and \(\theta^\tau_0\) are not stationary points of \(V_K(K, \theta)\). Thus, \(\theta^s_0\) is the only stationary point, implying that \(V_K(K, \theta)\) has a single maximum at the price floor.

3) If \(\theta^s_0 < \theta_s\) and \(\theta^m_0 < \theta^\tau\), then \(\theta^s_0\), \(\theta^m_0\), and \(\theta^\tau_0\) are all stationary points of \(V_K(K, \theta)\). Therefore, \(V_K(K, \theta)\) has two local maxima, one at the price floor and the other at the price ceiling.

Having shown that there is at least one local maximum and at most two local maxima of \(V_K(K, \theta)\), the last part of Lemma 1 describes the limit of \(V_K(K, \theta)\) as \(\theta\) approaches either 0 or \(\infty\), which we prove in part (h).

Part (h). The last part of Lemma 1

As \(\theta\) approaches 0, the limit of \(V_K(K, \theta)\) is determined by (4.22):

\[
\lim_{\theta \to 0} V_K(K, \theta) = \lim_{\theta \to 0} V^E_K(K, \theta) = \lim_{\theta \to 0} \left( -\frac{\beta_1 - 1}{\beta_1} \frac{c'(K) \theta Q^2}{2 (r - \alpha)} \right) = 0.
\]

On the other hand, as \(\theta\) goes to \(\infty\), the limit of \(V_K(K, \theta)\) is determined by (4.25). Since \(\beta_2 < -1\), all terms associated with \(\theta\) in (4.25) converge to zero:

\[
\lim_{\theta \to \infty} \left[ \frac{c'(K) Q s}{2 c(K)} \left( \frac{\theta}{\theta_F} \right)^{\beta_2} - \frac{c'(K) \bar{a}s}{2 c(K)} \left( \frac{\theta}{\theta_s} \right)^{\beta_2} + \frac{c'(K) \bar{a}\tau}{2 c(K)} \left( \frac{\theta}{\theta^\tau} \right)^{\beta_2} \right] = 0;
\]

\[
\lim_{\theta \to \infty} \frac{c'(K) \tau^2}{2 (r - \sigma^2 + \alpha) c(K)^2 \theta} = 0.
\]
Therefore,
\[
\lim_{\theta \to \infty} V_K(K, \theta) = \lim_{\theta \to \infty} V^r_K(K, \theta) = 0. \tag{C.47}
\]
This completes the proof of Lemma 1.

C.3 Derivation of the marginal value function under pure permit trading, (4.31)

In this section, we show the derivation of the marginal value function under pure permit trading, \(V^{pp}_K(K, \theta)\) ((4.31) in subsection 4.4.1). The derivation is accomplished in the same steps as the derivation of \(V_K(K, \theta)\) in section C.1. Since we assume that a random shock \(\theta\) follows the same stochastic process described by (4.2), a value function under pure permit trading (denoted as \(V^{pp}(K, \theta)\)) must satisfy the differential equation,

\[
\frac{1}{2} \sigma^2 \theta^2 V^{pp}_{\theta \theta}(K, \theta) + \alpha \theta V^{pp}_{\theta}(K, \theta) - r V^{pp}(K, \theta) - S^{pp}(K, \theta; \bar{a}) = 0.
\]

The general solution to this differential equation is

\[
V^{pp}(K, \theta) = E_1(K) \theta^{\beta_1} + E_2(K) \theta^{\beta_2} - \frac{c(K) \theta \bar{a}^2}{2(r - \alpha)}, \tag{C.48}
\]

where \(\beta_1 > 1\) and \(\beta_2 < -1\) are solutions to \(\frac{1}{2} \sigma^2 \beta (\beta - 1) + \alpha \beta - r = 0\). Since aggregate compliance costs go to zero as \(\theta\) goes to zero, we impose the boundary condition

\[
\lim_{\theta \to 0} V^{pp}(K, \theta) = 0.
\]

For this boundary condition to hold, \(E_2(K)\) in (C.48) must be zero because \(\beta_2 < 0\). Thus, the value function under the pure permit trading is

\[
V^{pp}(K, \theta) = E_1(K) \theta^{\beta_1} - \frac{c(K) \theta \bar{a}^2}{2(r - \alpha)}. \tag{C.49}
\]
The explicit form of $V_{pp}^p(K, \theta)$ satisfies the value-matching condition $V_{pp}^p(K, \theta) = w$ and the smooth-pasting condition $V_{pp}^p(K, \theta) = 0$ at the boundary of the investment and non-investment interval, $\theta^*(K)$. With these two conditions, we can find (4.31):

$$V_{pp}^p(K, \theta) = -\frac{\beta_1 - 1}{\beta_1} \frac{c'(K) \theta Q^2}{2(r - \alpha)}.$$

### C.4 Proof of Proposition 1

The proof of Proposition 1 in subsection 4.4.2 proceeds in the following way.

Part (a): First we show that $V_K(K, \theta)$ is always greater than $V_{pp}^p(K, \theta)$ for $0 < \theta \leq \theta_F(K)$. Part (b): Then we demonstrate that there is at least one value of $\theta$ at which $V_K(K, \theta) = V_{pp}^p(K, \theta)$. Part (c): Parts (a) and (b) imply that every point at which $V_K(K, \theta) = V_{pp}^p(K, \theta(K))$ occurs for $\theta > \theta_F(K)$. Part (d): We then demonstrate that there is only one value of $\theta$ at which $V_K(K, \theta) = V_{pp}^p(K, \theta)$. This completes parts 1) and 2) of the proposition. Part (e): Finally, we show that part 3) of the proposition follows from previous elements of the proof.

**Part (a).**

Since $V_K(K, \theta)$ takes a different expression in each interval, we will compare $V_K(K, \theta)$ and $V_{pp}^p(K, \theta)$ in a piecewise fashion. We begin with comparing $V_F^F(K, \theta)$ and $V_{pp}^m(K, \theta)$ to $V_{pp}^p(K, \theta)$, respectively, because $V_F^F(K, \theta)$ and $V_{pp}^m(K, \theta)$ have a similar structure with $V_{pp}^p(K, \theta)$. In addition, the results from these comparisons will be used in comparing $V_K^p(K, \theta)$ and $V_{pp}^p(K, \theta)$.

**Part (a-i).**

In the interval $0 < \theta \leq \theta_F(K)$, from (4.22), we know that $V_K(K, \theta)$ is expressed as

$$V_F^F(K, \theta) = -\frac{\beta_1 - 1}{\beta_1} \frac{c'(K) \theta Q^2}{2(r - \alpha)}.$$
Since \( c'(K) < 0 \) and \( Q > \bar{a} \), with (4.31), we have

\[
V_K^F(K, \theta) - V_K^{pp}(K, \theta) = -\frac{\beta_1 - 1}{\beta_1} \frac{c'(K) \theta Q^2}{2(\alpha - r)} + \frac{\beta_1 - 1}{\beta_1} \frac{c'(K) \theta \bar{a}^2}{2(\alpha - r)} = -\frac{\beta_1 - 1}{\beta_1} \frac{c'(K) \theta (Q^2 - \bar{a}^2)}{2(\alpha - r)} > 0 \text{ for } 0 < \theta \leq \theta_F(K).
\]

Therefore, \( V_K(K, \theta) \) is strictly greater than \( V_K^{pp}(K, \theta) \) in \( 0 < \theta \leq \theta_F(K) \). Clearly, there is no \( \theta \) where \( V_K(K, \theta) = V_K^{pp}(K, \theta) \) in this interval.

**Part (a-ii).**

In the interval \( \theta_s(K) \leq \theta \leq \theta_r(K) \), from (4.24), we know that \( V_K(K, \theta) \) is expressed as

\[
V_K^m(K, \theta) = \frac{1}{\beta_1} \left( \frac{\beta_1 + 1}{r - \sigma^2 + \alpha} - \frac{\beta_1 - 1}{r - \alpha} \right) \left[ \frac{c'(K) Qs}{2c(K)} \left( \frac{\theta}{\theta_F} \right)^{\beta_2} - \frac{c'(K) \bar{a}s}{2c(K)} \left( \frac{\theta}{\theta_s} \right)^{\beta_2} \right] - \frac{\beta_1 - 1}{\beta_1} \frac{c'(K) \theta \bar{a}^2}{2(\alpha - r)}.
\]

(Note that we are skipping the interval \( \theta_F(K) \leq \theta \leq \theta_s(K) \) for a moment). The difference between \( V_K^m(K, \theta) \) and \( V_K^{pp}(K, \theta) \) is

\[
V_K^m(K, \theta) - V_K^{pp}(K, \theta) = \frac{1}{\beta_1} \left( \frac{\beta_1 + 1}{r - \sigma^2 + \alpha} - \frac{\beta_1 - 1}{r - \alpha} \right) \left[ \frac{c'(K) Qs}{2c(K)} \left( \frac{\theta}{\theta_F} \right)^{\beta_2} - \frac{c'(K) \bar{a}s}{2c(K)} \left( \frac{\theta}{\theta_s} \right)^{\beta_2} \right]. \tag{C.50}
\]

With our assumptions on the parameters of the distribution of \( \theta \) (i.e, equation (4.3)), it can be shown that

\[
\frac{1}{\beta_1} \left( \frac{\beta_1 + 1}{r - \sigma^2 + \alpha} - \frac{\beta_1 - 1}{r - \alpha} \right) > 0.
\]
Moreover, the term in the hard brackets in (C.50),

\[
\frac{c'(K) Q_s}{2c(K)} \left( \frac{\theta}{\theta_F} \right)^{\beta_2} - \frac{c'(K) \bar{a}s}{2c(K)} \left( \frac{\theta}{\theta_s} \right)^{\beta_2} = \left[ \frac{Q}{\bar{a}} \left( \frac{\theta_s}{\theta_F} \right)^{\beta_2} - 1 \right] \frac{c'(K) \bar{a}s}{2c(K)} \left( \frac{\theta}{\theta_s} \right)^{\beta_2}
\]

\[
= \left[ \left( \frac{Q}{\bar{a}} \right)^{\beta_2+1} - 1 \right] \frac{c'(K) \bar{a}s}{2c(K)} \left( \frac{\theta}{\theta_s} \right)^{\beta_2} > 0,
\]

is also positive because \((Q/\bar{a})^{\beta_2+1} < 1\) for \(Q/\bar{a} > 1\) and \(\beta_2 + 1 < 0\). Therefore,

\[
V^m_K (K, \theta) > V^{pp}_K (K, \theta) \text{ for } \theta_s(K) \leq \theta \leq \theta_\tau(K);
\]

that is, \(V_K (K, \theta)\) is strictly greater than \(V^{pp}_K (K, \theta)\) in the interval \(\theta_s(K) \leq \theta \leq \theta_\tau(K)\).

**Part (a-iii).**

In the interval \(\theta_F(K) \leq \theta \leq \theta_s(K)\), from (4.23), we know that \(V_K (K, \theta)\) is expressed as

\[
V^*_K (K, \theta) = \frac{1}{\beta_1} \left( \frac{\beta_1 + 1}{r - \sigma^2 + \alpha} - \frac{\beta_1 - 1}{r - \alpha} \right) c'(K) \frac{Q_s}{2c(K)} \left( \frac{\theta}{\theta_F} \right)^{\beta_2} - \frac{\beta_1 + 1}{\beta_1} \frac{c'(K) s^2}{2(r - \sigma^2 + \alpha) c(K)^2 \theta}.
\]

Because \(V_K (K, \theta)\) is continuous at the cutoff values \(\theta_F(K), \theta_s(K),\) and \(\theta_\tau(K)\), from the results for the intervals of \((0, \theta_F(K)]\) and \([\theta_s(K), \theta_\tau(K)]\), we know that

\[
V^F_K (K, \theta_F (K)) = V^*_K (K, \theta_F (K)) > V^{pp}_K (K, \theta_F (K)); \quad (C.51)
\]

\[
V^m_K (K, \theta_s (K)) = V^*_K (K, \theta_s (K)) > V^{pp}_K (K, \theta_s (K)). \quad (C.52)
\]
Suppose that $V_s^s(K, \theta)$ meets $V_{pp}^p(K, \theta)$ between $\theta_F(K)$ and $\theta_s(K)$ and define $\theta_I(K)$ as the first point such that $V_s^s(K, \theta) = V_{pp}^p(K, \theta)$; that is,

1) $V_s^s(K, \theta) > V_{pp}^p(K, \theta)$ for $\theta_F(K) \leq \theta < \theta_I(K)$; (C.53)

2) $V_s^s(K, \theta_I(K)) = V_{pp}^p(K, \theta_I(K))$.

Note that the direction of the inequality in (C.53) defines $\theta_I(K)$ as the first point at which $V_s^s(K, \theta) = V_{pp}^p(K, \theta)$ and the direction of the inequality is determined by (C.51).

Part (a-iii-1). Let us now determine whether $\theta_I(K)$ occurs before or after the stationary point $\theta_s^0(K)$. First, consider the case for $\theta_I(K) \geq \theta_s^0(K)$. As shown in subsection C.2.1, $\theta_s^0(K)$ is a possible stationary point of $V_s^s(K, \theta)$. For this point to be a local maximum of $V_K(K, \theta)$ it must fall into $[\theta_F(K), \theta_s(K)]$. If $\theta_s^0(K) > \theta_s(K)$, then it is not a maximum point. Moreover, between $\theta_F(K)$ and $\theta_s(K)$ there is no point of $\theta$ where $V_s^s(K, \theta) = V_{pp}^p(K, \theta)$ because $\theta_s(K) < \theta_s^0(K) \leq \theta_I(K)$ and $V_s^s(K, \theta)$ is defined on $[\theta_F(K), \theta_s(K)]$. Therefore, we focus on the case for $\theta_s^0(K) \leq \theta_I(K) \leq \theta_s(K)$. For $\theta \geq \theta_I(K)$, the slopes of $V_s^s(K, \theta)$ and $V_{pp}^p(K, \theta)$ are

$$
\frac{\partial V_s^s(K, \theta)}{\partial \theta} \leq 0 < \frac{\partial V_{pp}^p(K, \theta)}{\partial \theta},
$$

because $V_s^s(K, \theta)$ is decreasing in $\theta$ after its maximum point $\theta_s^0(K)$ (as shown in (C.32)) and $V_{pp}^p(K, \theta)$ is strictly increasing in $\theta$. That is, after the first intersection point $\theta_I(K)$, $V_s^s(K, \theta)$ is decreasing while $V_{pp}^p(K, \theta)$ is increasing as $\theta$ increases. Therefore, for $\theta_I(K) < \theta \leq \theta_s(K)$ there cannot be another point of $\theta$ at which $V_s^s(K, \theta) = V_{pp}^p(K, \theta)$. In addition, at $\theta_s(K)$, $V_s^s(K, \theta)$ and $V_{pp}^p(K, \theta)$ are such that

$$
V_s^s(K, \theta_s(K)) < V_{pp}^p(K, \theta_s(K)).
$$
However, this is a contradiction with (C.52). Therefore, $\theta_I(K)$ cannot exist after $\theta_0^s(K)$.

Part (a-iii-2). Next, consider the case for $\theta_F(K) < \theta_I(K) < \theta_0^s(K)$. From (C.32) in subsection C.2.1, we know that for $\theta < \theta_0^s(K)$, $V_K^s(K,\theta)$ has a positive slope with respect to $\theta$. Then, due to the definition of $\theta_I(K)$, at $\theta_I(K)$ the slope of $V_K^s(K,\theta)$ should be less than or equal to the slope of $V_K^{pp}(K,\theta)$. Otherwise, $\theta_I(K)$ can’t be the first point where $V_K^s(K,\theta)$ and $V_K^{pp}(K,\theta)$ meet. In addition, $V_K^s(K,\theta)$ is strictly increasing in $\theta$ before its maximum point $\theta_0^s(K)$. That is,

$$0 < \frac{\partial V_K^s(K,\theta_I(K))}{\partial \theta} \leq \frac{\partial V_K^{pp}(K,\theta_I(K))}{\partial \theta}.$$

If both $\partial V_K^s(K,\theta)/\partial \theta$ and $\partial V_K^{pp}(K,\theta)/\partial \theta$ have the same slope at $\theta_I(K)$, then for $\theta < \theta_I(K)$, the slope of $V_K^s(K,\theta)$ should be greater than the slope of $V_K^{pp}(K,\theta)$ because for $\theta_F(K) \leq \theta < \theta_0^s(K)$,

$$\frac{\partial^2 V_K^s(K,\theta)}{\partial \theta^2} = \frac{\partial^2 V_K^{pp}(K,\theta)}{\partial \theta^2} = 0.$$

That is,

$$\frac{\partial V_K^s(K,\theta)}{\partial \theta} > \frac{\partial V_K^{pp}(K,\theta)}{\partial \theta} \quad \text{for} \quad \theta < \theta_I(K).$$

Since $V_K^s(K,\theta_I(K)) = V_K^{pp}(K,\theta_I(K))$, this implies that

$$V_K^s(K,\theta) < V_K^{pp}(K,\theta) \quad \text{for} \quad \theta < \theta_I(K).$$

However, this is a contradiction with (C.53) in the definition of $\theta_I(K)$. Therefore, at $\theta_I(K)$, the slope of $V_K^s(K,\theta)$ is strictly less than the slope of $V_K^{pp}(K,\theta)$; that is.
Because of (C.54), the slope of $V_s^s(K, \theta)$ decreases after $\theta_f(K)$ and becomes zero at $\theta^*_0(K)$. Moreover, after $\theta^*_0(K)$, $V_s^s(K)$ is strictly decreasing in $\theta$. On the other hand, $V_{pp}^p(K, \theta)$ maintains the same positive slope at $\theta_f(K)$. That is, for $\theta \geq \theta_f(K)$,

$$\frac{\partial V_s^s(K, \theta)}{\partial \theta} < \frac{\partial V_{pp}^p(K, \theta)}{\partial \theta}.$$

Therefore, after $\theta_f(K)$, there is no point of \( \theta \) where $V_s^s(K, \theta) = V_{pp}^p(K, \theta)$. In addition, this implies that at $\theta^*(K)$, $V_s^s(K, \theta)$ and $V_{pp}^p(K, \theta)$ are such that

$$V_s^s(K, \theta_s(K)) < V_{pp}^p(K, \theta_s(K)).$$

However, this contradicts (C.52). Therefore, the first point of $\theta$ where $V_s^s(K, \theta) = V_{pp}^p(K, \theta)$ can’t exist before $\theta^*_0(K)$.

The results from cases for $\theta_f(K) \geq \theta^*_0(K)$ and $\theta_f(K) < \theta^*_0(K)$ show that if $\theta^*_0(K)$ is a stationary point in the interval $\theta_f(K) \leq \theta \leq \theta_s(K)$, then $\theta_f(K)$ cannot lie before and after $\theta^*_0(K)$ in the interval. As a result, there is no point of $\theta$ where $V_s^s(K, \theta) = V_{pp}^p(K, \theta)$ for $\theta_f(K) \leq \theta \leq \theta_s(K)$. Moreover, this implies that $V_s^s(K, \theta)$ is greater than $V_{pp}^p(K, \theta)$ for $\theta_f(K) \leq \theta \leq \theta_s(K)$. To see why, suppose that there is a point of $\theta$ where $V_s^s(K, \theta) < V_{pp}^p(K, \theta)$ and denote this point as $\theta'$. Then, from (C.51), we have the following relationships:

$$V_s^s(K, \theta_f(K)) - V_{pp}^p(K, \theta_f(K)) > 0 \text{ and } V_s^s(K, \theta') - V_{pp}^p(K, \theta') < 0.$$  

Because both $V_s^s(K, \theta)$ and $V_{pp}^p(K, \theta)$ are continuous in $\theta$, from the intermediate value theorem, in the interval of $[\theta_f(K), \theta']$, there should be a point of $\theta$ where
\( V^s_K (K, \theta) - V^{pp}_K (K, \theta) = 0 \). However, we have already shown that such a point cannot exist. Therefore, there is no point of \( \theta \) where \( V^s_K (K, \theta) < V^{pp}_K (K, \theta) \).

In summary, there is no point of \( \theta \) where \( V_K (K, \theta) = V^{pp}_K (K, \theta) \) and \( V_K (K, \theta) \) is greater than \( V^{pp}_K (K, \theta) \) in \( \theta_F (K) \leq \theta \leq \theta_s (K) \).

**Part (b). Existence of the intersection:**

Thus far, we have proved that \( V_K (K, \theta) > V^{pp}_K (K, \theta) \) for \( 0 < \theta \leq \theta_r (K) \).

Therefore, if there exists a \( \theta \) where \( V_K (K, \theta) \) meets \( V^{pp}_K (K, \theta) \), this point must be strictly greater than \( \theta_r (K) \). From (4.25), we know that for \( \theta \geq \theta_r \), \( V_K (K, \theta) \) is expressed as

\[
V^\tau_K (K, \theta) = \frac{1}{\beta_1} \left( \frac{\beta_1 + 1}{r - \sigma^2 + \alpha} \right) \left[ \frac{c'(K) Qs}{2c(K)} \left( \frac{\theta}{\theta_F} \right)^{\beta_2} - \frac{c'(K) \bar{a} s}{2c(K)} \left( \frac{\theta}{\theta_s} \right)^{\beta_2} + \frac{c'(K) \bar{a} \tau}{2c(K)} \left( \frac{\theta}{\theta_r} \right)^{\beta_2} \right] - \frac{1}{\beta_1} \frac{\beta_1 + 1}{r - \sigma^2 + \alpha} \frac{c'(K) \tau^2}{2c(K)^2 \theta}.
\]

Let the maximum point of \( V^\tau_K (K, \theta) \) be \( \theta^*_0 (K) \). Given \( K \), \( V^\tau_K (K, \theta) \) is continuous in \( \theta \), strictly decreasing in \( \theta \) after \( \theta^*_0 (K) \) and \( \lim_{\theta \to \infty} V^\tau_K (K, \theta) = 0 \) from (C.47). Thus, there is a point \( \theta' \) after \( \theta^*_0 (K) \) such that

\[
0 < V^\tau_K (K, \theta') < V^{pp}_K (K, \theta_r (K)) < V^\tau_K (K, \theta_r (K)).
\]

Then, at \( \theta_r (K) \) and \( \theta' \), we have the following relationships:

\[
V^\tau_K (K, \theta_r (K)) = V^{pp}_K (K, \theta_r (K)) > V^{pp}_K (K, \theta_r (K));
\]

\[
V^\tau_K (K, \theta') < V^{pp}_K (K, \theta_r (K)) < V^{pp}_K (K, \theta').
\]
The first inequality is obtained from the results of part (a), and the second inequality is derived from the facts that $\theta' > \theta_\tau(K)$ and $V_K^{pp}(K, \theta)$ is increasing in $\theta$. Then, the difference between $V_K^\tau(K, \theta)$ and $V_K^{pp}(K, \theta)$ have opposite signs at $\theta_\tau(K)$ and $\theta'$, that is,

\[
V_K^\tau(K, \theta_\tau(K)) - V_K^{pp}(K, \theta_\tau(K)) > 0; \quad V_K^\tau(K, \theta') - V_K^{pp}(K, \theta') < 0.
\]

Since both $V_K^\tau(K, \theta)$ and $V_K^{pp}(K, \theta)$ are continuous in $\theta$, by the intermediate value theorem, there exists a point of $\theta$ between $\theta_\tau(K)$ and $\theta'$ such that

\[
V_K^\tau(K, \theta) - V_K^{pp}(K, \theta) = 0.
\]

Therefore, there exists a point of $\theta > \theta_\tau(K)$ where $V_K^\tau(K, \theta) = V_K^{pp}(K, \theta)$.

**Part (c).**

In part (a) we showed that $V_K(K, \theta) > V_K^{pp}(K, \theta)$ for $0 < \theta \leq \theta_\tau(K)$. In part (b), we showed that there is at least one point of $\theta > \theta_\tau(K)$ where $V_K^\tau(K, \theta) = V_K^{pp}(K, \theta)$. Therefore, any point at which $V_K(K, \theta_I(K)) = V_K^{pp}(K, \theta_I(K))$ occurs for $\theta > \theta_\tau(K)$.

**Part (d).**

In this part, we will show that $V_K(K, \theta)$ and $V_K^{pp}(K, \theta)$ intersect only once. To complete parts 1) and 2) of the proposition, we need to show that there is only one point of $\theta$ at which $V_K^\tau(K, \theta) = V_K^{pp}(K, \theta)$. Define $\theta_I(K)$ as the first point of $\theta$ where $V_K^\tau(K, \theta) = V_K^{pp}(K, \theta)$. From the results of parts (a) and (b), $\theta_\tau(K) < \theta_I(K)$; that is,

\[
\begin{align*}
V_K^\tau(K, \theta) &> V_K^{pp}(K, \theta) \quad \text{for} \quad \theta_\tau(K) \leq \theta < \theta_I(K); \\
V_K^\tau(K, \theta_I(K)) & = V_K^{pp}(K, \theta_I(K)).
\end{align*}
\]
Note that the direction of the inequality defines $\theta_I(K)$ as the first point at which $V_K^\tau(K, \theta) = V_K^{pp}(K, \theta)$ and the direction of the inequality is determined by $V_K(K, \theta_I(K)) > V_K^{pp}(K, \theta_I(K))$. Then, we will analyze cases depending on whether $\theta_I(K)$ is greater than $\theta_0^\tau(K)$ or not.

**Part (d-i).**

Cases for $\theta_0^\tau(K) \leq \theta_I(K) < \theta_0^\tau(K)$ and $\theta_0^\tau(K) < \theta_I(K) \leq \theta_I^\tau(K)$: Suppose that $\theta_0^\tau(K) \leq \theta_I(K)$. Since $V_K^\tau(K)$ is strictly decreasing in $\theta$ after $\theta_0^\tau(K)$, and $V_K^{pp}(K, \theta)$ is monotonically increasing in $\theta$, $V_K^\tau(K, \theta)$ and $V_K^{pp}(K, \theta)$ cannot meet again after $\theta_I(K)$. Thus, $\theta_I(K)$ is the single intersection between $V_K^\tau(K, \theta)$ and $V_K^{pp}(K, \theta)$ when $\theta_0^\tau(K) \leq \theta_I(K)$. In addition, combined with the definition of $\theta_I(K)$, this implies that when $\theta_0^\tau(K) \leq \theta_I(K)$:

$$V_K^\tau(K, \theta) > V_K^{pp}(K, \theta) \quad \text{for} \quad \theta_I(K) \leq \theta < \theta_I(K);$$

$$V_K^\tau(K, \theta) < V_K^{pp}(K, \theta) \quad \text{for} \quad \theta > \theta_I(K).$$

**Part (d-ii).**

Case for $\theta_0^\tau(K) < \theta_I(K) < \theta_0^\tau(K)$: Suppose that $\theta_I(K)$ occurs strictly before $\theta_0^\tau(K)$. Then, by the definition of $\theta_I(K)$, we have

$$V_K^\tau(K, \theta) > V_K^{pp}(K, \theta) \quad \text{for} \quad \theta_0^\tau(K) < \theta < \theta_I(K) < \theta_0^\tau(K);$$

$$V_K^\tau(K, \theta_I(K)) = V_K^{pp}(K, \theta_I(K)).$$

Since $\theta_I(K)$ is the first intersection point, the slopes of $V_K^\tau(K, \theta)$ and $V_K^{pp}(K, \theta)$ at $\theta_I(K)$ should be

$$\frac{\partial V_K^\tau(K, \theta_I(K))}{\partial \theta} < \frac{\partial V_K^{pp}(K, \theta_I(K))}{\partial \theta}. \quad (C.56)$$
The case where $V_{\tau}^c(K, \theta)$ and $V_{pp}^p(K, \theta)$ are tangent at $\theta_I(K)$ can be ruled out by the same logic as Part (a-iii-2) because for $\theta_{\tau}(K) < \theta \leq \theta_0^\tau(K)$,

\[
\frac{\partial^2 V_{\tau}^c(K, \theta)}{\partial \theta^2} = \left\{ \frac{\beta_2(\beta_2 - 1)}{\beta_1} \left( \frac{\beta_1 + 1}{r - \sigma^2 + \alpha} - \frac{\beta_1 - 1}{r - \alpha} \right) \left[ \left( \frac{Q}{\bar{a}} \right)^{\beta_2+1} - 1 \right] \left( \frac{\tau}{s} \right)^{\beta_2-1} + 1 \right\} \left( \frac{\theta}{\theta_{\tau}} \right)^{\beta_2+1}
\]

\[-\frac{\beta_1 + 1}{\beta_1} \left( \frac{2}{r - \sigma^2 + \alpha} \right) \frac{\theta_{\tau}^2 c'(K) a_\bar{a}^2}{2} < 0, \tag{C.57}
\]

\[
\frac{\partial^2 V_{pp}^p(K, \theta)}{\partial \theta^2} = 0.
\]

If there exists another intersection point after $\theta_I(K)$, at that point the slope of $V_{\tau}^c(K, \theta)$ should be greater than or equal to the slope of $V_{pp}^p(K, \theta)$. Because of (C.57), the slopes of $V_{\tau}^c(K, \theta)$ are positive but decreasing, become zero at $\theta_0^\tau(K)$, and become negative after $\theta_0^\tau(K)$ while the slope of $V_{pp}^p(K, \theta)$ remains the same. Since the slope of $V_{\tau}^c(K, \theta)$ cannot become greater than or equal to the slope of $V_{pp}^p(K, \theta)$ after $\theta_I(K)$, there is no $\theta$ where $V_{\tau}^c(K, \theta) = V_{pp}^p(K, \theta)$ other than $\theta_I(K)$. Therefore, $\theta_I(K)$ is the only intersection between $V_{\tau}^c(K, \theta)$ and $V_{pp}^p(K, \theta)$ when $\theta_I(K) \leq \theta_0^\tau(K)$.

The results of parts (d-i) and (d-ii) show that $\theta_I(K)$ is the first point of $\theta$ at which $V_{\tau}^c(K, \theta) = V_{pp}^p(K, \theta)$ and it is actually the only such a point. As a result, we have completed the proof for parts 1) and 2) of the proposition.

**Part (e).**

In part (a), we showed that $V_{\tau}^c(K, \theta) > V_{pp}^p(K, \theta)$ for $0 < \theta \leq \theta_{\tau}(K)$. In parts (b) and (d), we showed that the only point of $\theta$ at which $V_{\tau}^c(K, \theta) = V_{pp}^p(K, \theta)$ is strictly greater than $\theta_{\tau}(K)$. To complete part 3) of the proposition, we need to show that $V_{\tau}^c(K, \theta) > V_{pp}^p(K, \theta)$ for $\theta_{\tau} < \theta < \theta_I(K)$ and $V_{\tau}^c(K, \theta) < V_{pp}^p(K, \theta)$ for $\theta > \theta_I(K)$. 

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Suppose that there is a point of $\theta$ in $[\theta_{\tau}(K),\theta_{I}(K)]$ such that $V_K(K,\theta) < V_{K}^{pp}(K,\theta)$, and denote this point as $\theta'$. Because both $V_{K}^{\tau}(K,\theta)$ and $V_{K}^{pp}(K,\theta)$ are continuous in $\theta$, by the intermediate value theorem, this requires another point of $\theta$ in $[\theta_{\tau}(K),\theta']$ at which $V_{K}^{\tau}(K,\theta) = V_{K}^{pp}(K,\theta)$. However, this contradicts the results of part (d) which showed $\theta_{I}(K)$ is the only point of $\theta$ at which $V_{K}^{\tau}(K,\theta) = V_{K}^{pp}(K,\theta)$. Therefore, there is no point of $\theta$ in $[\theta_{\tau}(K),\theta_{I}(K)]$ such that $V_K(K,\theta) < V_{K}^{pp}(K,\theta)$. As a result, combined with the results of part (a), we have

$$V_K(K,\theta) > V_{K}^{pp}(K,\theta) \text{ for } 0 < \theta < \theta_{I}(K).$$

Since $\theta_{I}(K)$ is the only point of $\theta$ at which $V_K(K,\theta) = V_{K}^{pp}(K,\theta)$, (C.56) implies that

$$V_K(K,\theta) < V_{K}^{pp}(K,\theta) \text{ for } \theta > \theta_{I}(K).$$

Therefore, part 3) of the proposition is proved. $\square$
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