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## Equations For Nilpotent Varieties and Their Intersections With Slodowy Slices

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**EQUATIONS FOR NILPOTENT VARIETIES AND THEIR  
INTERSECTIONS WITH SLOWOWY SLICES**

A Dissertation Presented

by

**BEN JOHNSON**

Submitted to the Graduate School of the  
University of Massachusetts Amherst in partial fulfillment  
of the requirements for the degree of

**DOCTOR OF PHILOSOPHY**

September 2017

Department of Mathematics and Statistics

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## ABSTRACT

### EQUATIONS FOR NILPOTENT VARIETIES AND THEIR INTERSECTIONS WITH SLODOWY SLICES

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This thesis investigates minimal generating sets of ideals defining certain nilpotent varieties in simple complex Lie algebras. A minimal generating set of invariants for the whole nilpotent cone is known due to Kostant. Broer determined a minimal generating set for the subregular nilpotent variety in all simple Lie algebra types. I extend Broer's results to two families of nilpotent varieties, valid in any simple Lie algebra, that include the nilpotent cone, the subregular case, and usually more. In the first part of my thesis I describe a minimal generating set for the ideal of each of these varieties in the coordinate ring of the Lie algebra. My goal in the second part is to describe which images of generators remain necessary when the variety is intersected with a Slodowy slice to a lower orbit and which become redundant, information that can be used to give new proofs of the singularities of minimal degenerations of nilpotent varieties.

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# CHAPTER 1

## INTRODUCTION

Given a nilpotent  $n \times n$  matrix  $E$ , what are the conditions for an  $n \times n$  matrix  $M$  to be conjugate to  $E$ ? At the linear algebra level, one answer is that  $E$  and  $M$  must share the same Jordan form. Although this allows an easy check for a particular matrix  $M$ , this is not a satisfactory answer if we want to describe the set of all such  $M$ . Certainly we know some necessary conditions: all eigenvalues must be zero, and we know that the entries of  $M^k$  must all be zero where  $k$  is the smallest positive integer such that  $E^k = 0$ . Let  $M$  be a generic matrix, so that its entries are  $n^2$  independent coordinates  $\{x_{i,j}\}$ . All eigenvalues being equal to zero is equivalent to the characteristic polynomial of  $M$  being  $t^n$ , with all of the coefficients for lower powers of  $t$ , which are polynomials in the  $x_{i,j}$ , equal to zero. These conditions are also equivalent to the trace of every power of  $M$  being zero.

Taking a more algebraic geometric view, what we have are some polynomials in the entries of a generic matrix  $M$  that are contained in the ideal defining the Zariski closure of the set of matrices conjugate to  $E$ . The Zariski closure of a conjugacy class of nilpotent matrices is one example of a nilpotent variety, whereas the conjugacy class of matrices is an example of a nilpotent orbit. The set of all nilpotent matrices is a nilpotent variety too, called the nilpotent cone. A classical result of Kostant [12] tells us, in this case, that the ideal of the nilpotent cone is generated by the coefficients of the characteristic polynomial of  $M$ , and that this generating set is minimal, i.e. no generators are redundant. In fact, Kostant's result is a minimal generating set for the nilpotent cone in any type of Lie algebra, made up of the fundamental invariants, of which coefficients of the characteristic polynomial and traces of powers are two possible choices. Finding minimal generating sets for ideals of nilpotent varieties is the subject of the first part of this dissertation.

The case we have described above of a conjugacy class of  $n \times n$  matrices was already answered in part by Weyman in two papers [24] [25]. He determined minimal generating sets in the case that the Jordan form of  $E$  has as many blocks as possible of a given size  $k$ , with at most one other block of size  $r < k$ . This condition is usually described by saying that the Jordan form of  $E$  has partition  $[k, \dots, k, r]$  with  $r < k$  and  $n = k + \dots + k + r$ , where the parts of the partition are the sizes of the Jordan blocks. In those cases, the polynomials needed are the first  $k - 1$  coefficients of the

characteristic polynomial for  $M$  and the entries of  $M^k$ . These describe nilpotent varieties in the Lie algebra of all  $n \times n$  matrices  $\mathfrak{gl}_n$ , which is not simple.

As an illustration, consider  $\mathfrak{sl}_3 \subset \mathfrak{gl}_3$ , the Lie algebra of trace-zero  $3 \times 3$  matrices. The nilpotent orbits are the conjugacy classes of nilpotent Jordan forms, of which there are three: the zero matrix itself, which has partition  $[1, 1, 1]$ ,

$$E_{[2,1]} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } E_{[3]} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

If  $M$  is the generic matrix in  $\mathfrak{sl}_n$

$$\begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & -x_{1,1} - x_{2,2} \end{bmatrix},$$

then the non-trace coefficients of the characteristic polynomial are

$$x_{1,1}x_{2,2} - x_{1,2}x_{2,1} + x_{1,1}(-x_{1,1} - x_{2,2}) - x_{1,3}x_{3,1} + x_{2,2}(-x_{1,1} - x_{2,2}) - x_{2,3}x_{3,2} \text{ and}$$

$$x_{1,1}x_{2,2}(-x_{1,1} - x_{2,2}) - x_{1,1}x_{2,3}x_{3,2} - x_{1,2}x_{2,1}(-x_{1,1} - x_{2,2}) + x_{1,2}x_{2,3}x_{3,1} + x_{1,3}x_{2,1}x_{3,2} - x_{1,3}x_{2,2}x_{3,1}$$

In order to be conjugate to the zero matrix, obviously  $M$  must also be the zero matrix, with the ideal of this nilpotent variety generated by all of the entry coordinates. The nilpotent variety for  $E_{[3]}$  is the nilpotent cone, so using Kostant's result, the ideal is generated by the two polynomials above. The minimal generating set proved by Weyman for the nilpotent variety of  $E_{[3]}$  uses the entries of  $M^3$  instead of the determinant. Minimal does not refer to having as few generators as possible among all sets of generators, but that no generator can be removed from that set. For  $E_{[2,1]}$  a minimal generating set consists of the first polynomial above and the entries of  $M^2$ .

We focus on simple Lie algebras of all types and give similar minimal generating sets. In type  $A$ , the Lie algebra of trace-zero  $n \times n$  matrices, our result is a simple corollary to Weyman's result, although our methods are different. His arguments relied on the especially nice properties of type  $A$  that do not carry over easily to other types. Instead, the foundation upon which we have built our work are a pair of papers by Broer [1] [2], giving a minimal generating set for the subregular nilpotent variety, a nilpotent variety that is uniquely defined in each type of simple Lie algebra. In

the  $\mathfrak{sl}_3$  example the subregular is the orbit of  $E_{[3]}$ . We extend this to two related families of nilpotent varieties that contain the subregular.

With minimal generating sets in hand, it becomes possible to get a firm grasp on the structure of a nilpotent variety. Then we can study intersecting nilpotent varieties with other interesting subsets of simple Lie algebras. Specifically, in the second part of this dissertation we consider minimal generating sets for the intersections of nilpotent varieties with Slodowy slices. Returning to our  $n \times n$  matrix scenario, a Slodowy slice is a set of matrices that are formed by adding matrices that commute with a nilpotent matrix  $E$  to another nilpotent matrix  $F$  that is ‘opposite’  $E$  in a particular sense. Geometrically, it could be viewed as taking a subspace of  $n^2$ -dimensional space containing  $E$  and then translating the subspace by  $F$  to get a transverse slice to the conjugacy class of  $E$ .

The intersection of a nilpotent variety and a Slodowy slice to a smaller nilpotent orbit is singular, and the singularity can be described using our minimal generating sets. This allows for an explicit confirmation of the theorems of Fu, Juteau, Levy, and Sommers [9], which inspired our study in the first place. Here we follow the example of Slodowy [19] for whom the slices are named, who studied the case of a slice transverse to the subregular nilpotent orbit. Our exploration makes use of the computer algebra system and programming language Magma. Of course, we have only been able to study intersections with those nilpotent varieties for which we have minimal generating sets.

Returning to the example of  $\mathfrak{sl}_3$ , the Slodowy slice to  $E_{[2,1]}$  consists of all matrices of the form

$$\begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

where  $a, b \in \mathbb{C}$  and the right-hand matrix is  $F$ . Since we know that the ideal of the nilpotent cone is generated by the sum of  $2 \times 2$  diagonal minors and the determinant, we can evaluate those polynomials as written out above on this generic matrix of the slice and get a minimal generating set for the intersection:  $4a$  is the sum of all  $2 \times 2$  diagonal minors, and  $4b$  is the determinant.

## CHAPTER 2

### EQUATIONS FOR NILPOTENT VARIETIES

#### 2.1 Nilpotent varieties

Let  $\mathfrak{g}$  be a simple Lie algebra over the complex numbers  $\mathbb{C}$  with adjoint algebraic group  $G$ . The *adjoint representation* of  $\mathfrak{g}$  is defined as the map  $\text{ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}) = \text{Lie}(G)$  that maps  $E \in \mathfrak{g}$  to the endomorphism  $\text{ad}_E = [E, \cdot] : \mathfrak{g} \rightarrow \mathfrak{g}$ . A *nilpotent element* of  $\mathfrak{g}$  is  $E \in \mathfrak{g}$  such that  $\text{ad}_E$  is a nilpotent endomorphism, i.e. there is some  $d \in \mathbb{Z}_+$  such that  $\text{ad}_E^d = 0$ . A *nilpotent orbit*  $\mathcal{O}_E$  is the  $G$ -orbit of a nilpotent element  $E \in \mathfrak{g}$ . The closure of a nilpotent orbit  $\mathcal{O}_E$  is a *nilpotent variety*, denoted by  $\overline{\mathcal{O}}_E$ . These are singular varieties for nonzero  $E$ , and are not always normal varieties. If  $\mathbb{C}[\mathfrak{g}]$  is the coordinate ring of  $\mathfrak{g}$ , then the *defining ideal* of  $\overline{\mathcal{O}}_E$  is the ideal  $J_E \subset \mathbb{C}[\mathfrak{g}]$  such that  $\mathbb{C}[\mathfrak{g}]/J_E \simeq \mathbb{C}[\overline{\mathcal{O}}_E]$  the coordinate ring of  $\overline{\mathcal{O}}_E$ .

Fix a Borel subgroup  $B \subset G$  with maximal torus  $T$  such that  $\mathfrak{h}$  is the Lie algebra of  $T$ . The *rank* of  $\mathfrak{g}$  is the dimension of the Cartan subalgebra  $\mathfrak{h}$ , which we typically denote by  $n$ . Let  $\Phi \subset X^*(T)$  be the root system of  $G$  within the character group of  $T$  (characters of  $T$  are usually referred to as weights), with positive roots  $\Phi^+$  determined by  $B$ . Let  $\Pi \subset \Phi^+$  be the simple positive roots  $\{\alpha_1, \dots, \alpha_n\}$ , a basis for  $X^*(T)$ . There is a partial ordering on roots defined by  $\lambda \leq \mu$  if  $\mu - \lambda \in \Phi^+$ . For  $\lambda = \sum_{i=1}^n c_i \alpha_i$ , we define the height of weight  $\lambda$  to be  $\text{ht}(\lambda) = \sum_{i=1}^n c_i$ .

Let  $W$  be the Weyl group with respect to  $T$ , also the Weyl group for  $\Phi$ . Fix a nondegenerate  $W$ -invariant inner product  $\langle \cdot, \cdot \rangle$  on  $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ . To each root  $\alpha$  we assign the coroot  $\alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle} \alpha$ , which we view as a vector in the space  $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ . A *dominant weight*  $\lambda$  is one such that  $\langle \lambda, \alpha^\vee \rangle \geq 0$  for all  $\alpha \in \Pi$ . Every weight is  $W$ -conjugate to a unique dominant weight  $\lambda^+$ . There is always a unique dominant root of each root length in  $\Phi$ , and we use the notation  $\theta$  for the dominant long root and  $\phi$  for the dominant short root. These are interchangeable in a simply-laced root system. For a dominant weight  $\lambda$ , we denote by  $V_\lambda$  the simple highest-weight module of highest weight  $\lambda$ .

We denote by  $R \subset \mathbb{C}[\mathfrak{g}]$  the subring of  $G$ -invariants, and by  $R_+ \subset R$  the subring of invariants without constant term. The set of all nilpotent elements in  $\mathfrak{g}$  is the *nilpotent cone*, denoted by  $\mathcal{N}$ , which is the closure of what is called the *regular or principal nilpotent orbit*  $\mathcal{O}_{\text{prin}}$  (the set of nilpotent elements whose stabilizers have dimension  $n$ ). The defining ideal  $J_{\text{prin}}$  is known from Kostant [12] to

be generated by a set of  $n$  algebraically independent,  $G$ -invariant, homogeneous polynomials which we label in order of non-decreasing degree by  $f_1, \dots, f_n$ . These are the *fundamental invariants*, which generate  $R_+$ , so the defining ideal of  $\overline{\mathcal{O}}_{\text{prin}}$  is  $(R_+)$ . Although the choice of these functions is not unique, their degrees  $d_i$  are. More on possible choices of fundamental invariants can be found in Section 3.2, Chapter 3.

In fact, these fundamental degrees indicate the copies of the adjoint representation  $V_\theta$  in  $\mathbb{C}[\mathcal{N}]$ . The *exponents* of  $\mathfrak{g}$  are the integers  $m$  such that

$$\dim \text{Hom}_G(V_\theta, \mathbb{C}^m[\mathcal{N}]) > 0,$$

with multiplicity. The relationship between these and the degrees of the fundamental invariants is straightforward: each degree is one greater than an exponent. This reflects the fact that given a basis  $\{x_i\}$  for  $\mathfrak{g}$  and an invariant  $f \in R_+$ , the partial derivatives  $\{\frac{\partial f}{\partial x_i}\}$  span a copy of the adjoint representation in  $\mathbb{C}[\mathfrak{g}]$ , which we denote by  $[f]$ , and the partial derivatives of the fundamental invariants are a basis for the  $\theta$ -isotypic component of  $\mathbb{C}[\mathcal{N}]$ . We will label the exponents  $m_1, m_2, \dots, m_n$  in non-decreasing order so that  $d_i = m_i + 1$  for  $1 \leq i \leq n$ .

Similarly, we define *generalized exponents* for a weight  $\lambda$  to be the integers  $m$  such that

$$\dim \text{Hom}(V_\lambda, \mathbb{C}^m[\mathcal{N}]) > 0,$$

with multiplicity. To distinguish these, we use the notation  $m_i^\lambda$ , where  $m_i^\theta = m_i$ . The generalized exponents for the short dominant root  $\phi$  are of particular interest, and we will label these as  $m_1^\phi, \dots, m_r^\phi$ . The number of these,  $r$ , is also the number of simple short roots, in the same way that the number of exponents is the number of simple roots.

**Example 2.1.1.** In type  $C_n$ , the exponents are  $1, 3, 5, \dots, 2n - 1$  and the generalized exponents for  $\phi$  are  $2, 4, 6, \dots, 2n - 2$ . So there are  $n$  exponents to match the  $n$  simple roots, and only  $n - 1$  generalized exponents for  $\phi$  to match the  $n - 1$  simple roots which are short.

In the classical simple Lie algebras, nilpotent orbits can be parametrized by partitions. In type  $A_n$ , the partitions are of  $n + 1$ . Types  $B_n, C_n$ , and  $D_n$  have partitions of  $2n$ , with an even number of even parts for  $B_n$  or  $D_n$  and an even number of odd parts for  $C_n$ . In type  $D_n$ , the partitions whose parts are all even describe two distinct nilpotent orbits, the so-called very even nilpotent orbits. The notation  $\mathcal{O}_\mu, J_\mu$ , etc. will sometimes be used to describe the orbits, ideals, etc. corresponding to a

partition  $\mu$ . In the exceptional Lie algebras, nilpotent orbits are classified by Bala-Carter labels [4] instead of partitions, and we will put those in the subscripts instead.

By the Jacobson-Morozov Theorem, to each nilpotent orbit  $\mathcal{O}$  is associated an  $\mathfrak{sl}_2$ -triple  $\{H, E, F\}$  of nonzero elements of  $\mathfrak{g}$  with  $H \in \mathfrak{h}$  and  $E \in \mathcal{O}$  satisfying the bracket relations  $[H, E] = 2E$ ,  $[H, F] = -2F$ , and  $[E, F] = H$ . These triples are unique up to conjugation by  $G$ . For this section, we only need to use the semisimple  $H$  associated to nilpotent  $E$ , but  $\mathfrak{sl}_2$ -triples play a more important role in Chapter 3 when we define a Slodowy slice using the  $E$  and  $F$  of a triple. For now, we use the  $\mathfrak{sl}_2$ -triple element  $H$  to describe the weighted Dynkin diagram of a nilpotent orbit. This is just the Dynkin diagram of the root system of  $\mathfrak{g}$  with the node for the simple root  $\alpha$  labeled by  $\alpha(H)$ . These give another way to classify nilpotent orbits.

Let  $P$  be a parabolic subgroup containing  $B$ , with Lie algebra  $\mathfrak{p}$ . We know that  $\mathfrak{p} = \mathfrak{n}_{\mathfrak{p}} \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \bar{\Theta}} \mathfrak{g}_{\alpha}$  where  $\mathfrak{n}_{\mathfrak{p}}$  is the nilradical of  $\mathfrak{p}$ ,  $\Theta$  is the set of simple roots whose root spaces are not in  $\mathfrak{n}$ , and  $\bar{\Theta}$  is the set of all roots generated by  $\Theta$ . We may then refer to  $\mathfrak{p}$  as the parabolic subalgebra generated by the set  $\Theta$ , denoted by  $\mathfrak{p}_{\Theta}$ . The *Richardson orbit* for  $\mathfrak{p}$  is the unique nilpotent orbit of  $\mathfrak{g}$  that intersects  $\mathfrak{n}_{\mathfrak{p}}$  in an open dense set [5].

## 2.2 Our families of nilpotent varieties

In this thesis, we focus on the nilpotent orbits that are Richardson orbits for parabolic subalgebras generated by pairwise orthogonal simple short roots. The case of one simple short root is the subregular orbit, which was thoroughly studied by Broer [1]. In type  $B_n$ , there are no other simple short roots, so we have nothing to add. In the other types, however, we extend Broer's arguments.

**Lemma 2.2.1.** *Let  $\mathfrak{g}$  be a simple complex Lie algebra of classical type. Orbits whose partitions have two parts are all Richardson orbits for parabolic subalgebras generated by pairwise orthogonal simple short roots. Additionally, in type  $D_n$ , nilpotent orbits whose partitions have the form  $\mu = [2n - 2s + 1, 2s - 3, 1, 1]$  for  $2 \leq s \leq \lfloor \frac{n+2}{2} \rfloor$  are Richardson orbits for parabolic subalgebras defined by orthogonal simple short roots.*

*Proof.* In type  $A_n$ , a two-part partition  $\mu$  has the form  $\mu = [n+1-s, s]$  for some  $1 \leq s \leq \lfloor \frac{n+1}{2} \rfloor$ . Then the transpose partition is  $\mu' = [2^s, 1^{n+1-2s}]$ . We define  $\Theta(\mu') := \{e_1 - e_2, e_3 - e_4, \dots, e_{2s-1} - e_{2s}\}$  and  $\mathfrak{p}(\mu') := \mathfrak{p}_{\Theta(\mu')}$  for ease of notation. We know that  $\mathcal{O}_{\mu}$  is the Richardson orbit for  $\mathfrak{p}(\mu')$  [5], which is generated by  $s$  pairwise orthogonal short roots.

In all other classical types, the weighted Dynkin diagrams for two-part partitions and partitions of the form  $[2n - 2s + 1, 2s - 3, 1, 1]$  consist of twos and zeros, without adjacent zeros. For instance,

in type  $D_5$ , the nilpotent orbit with partition  $[5, 5]$  has weighted Dynkin diagram  $0 - 2 - 0 \begin{smallmatrix} 2 \\ 2 \end{smallmatrix}$  and

the nilpotent orbit with partition  $[7, 1, 1, 1]$  has weighted Dynkin diagram  $2 - 2 - 2 \begin{smallmatrix} 0 \\ 0 \end{smallmatrix}$ . As shown

in the book by Collingwood and McGovern [5, Theorem 7.1.6], these are Richardson orbits for the parabolic subalgebras generated by the simple roots labeled with zeros. It can be easily seen that the zeros are never on long roots in multiply-laced types.  $\square$

For exceptional simple Lie algebras, only type  $E_6$ ,  $E_7$ , and  $E_8$  have multiple orthogonal simple short roots. Broer has completed the subregular case, so we will not work with  $F_4$  or  $G_2$ . We can account for 9 nilpotent orbits in the exceptional Lie algebras that were not already included in Broer's work.

Broer also proved in a second paper [2] that all of these nilpotent varieties are normal, and that there is a birational map  $G \times^P \mathfrak{n}_\mathfrak{p} \rightarrow \overline{\mathcal{O}}$  where  $P$  is a parabolic subgroup of  $G$  generated by orthogonal simple short roots,  $\mathfrak{n}_\mathfrak{p}$  is the nilradical of the Lie algebra of  $P$ , and  $\mathcal{O}$  is the Richardson orbit for the Lie algebra of  $P$ . The method of proof he uses does not easily lead to minimal generating sets for the nilpotent varieties, however. We give a new proof of these same results in the process of our proof of minimal generating sets.

We split all of these Richardson orbits into two families to account for situations where there exist multiple nilpotent orbits in the same Lie algebra that are Richardson for parabolic subalgebras generated by the same number of orthogonal simple short roots. If  $\mathcal{O}$  is a Richardson orbit for a parabolic subalgebra generated by a set  $\Theta$  of  $s$  orthogonal simple short roots, let  $\{H, E, F\}$  be an  $\mathfrak{sl}_2$ -triple for  $\mathcal{O}$  and  $\mathfrak{a} \subset \mathfrak{g}$  be the subalgebra generated by the triple. Our *first family* is the default when there is only one orbit for  $s$ , and is the orbit with minimal  $\dim \mathfrak{g}^\mathfrak{a}$  if there are multiple orbits for  $s$ . In the case of type  $D_n$  with even  $n$ , there are two partitions in the first family for  $s = \frac{n}{2}$ , the very even case. Our *second family*, then, consists of the orbits with maximal  $\dim \mathfrak{g}^\mathfrak{a}$  if there are multiple orbits for  $s$ . More explicitly, the second family contains the nilpotent orbits in type  $D_n$  with partition  $[2n - 2s + 1, 2s - 3, 1, 1]$  and the nilpotent orbit in type  $E_7$  with Bala-Carter label  $E_6$ . We note that the second family only occurs in simply-laced types, where  $r = n$ .

We use the notation  $J_\Theta$  for the ideal of the closure of the nilpotent orbit  $\mathcal{O}_\Theta$  in  $\mathbb{C}[\mathfrak{g}]$ ,  $I_\Theta^{\Theta'}$  for the ideal of  $\overline{\mathcal{O}}_\Theta$  in  $\mathbb{C}[\overline{\mathcal{O}}_\Theta']$  for  $\mathcal{O}_\Theta \subset \overline{\mathcal{O}}_\Theta'$ , and  $\tilde{I}_\Theta^{\Theta'}$  for the ideal in  $\mathbb{C}[\mathfrak{g}]$  generated by generators of  $I_\Theta^{\Theta'}$ . In particular,  $I_\Theta^\emptyset$  is the ideal of  $\overline{\mathcal{O}}_\Theta$  in  $\mathbb{C}[\mathcal{N}]$ . Now we can state the main theorem of this section.





**Theorem 2.2.2.** *The ideal  $I_{\Theta}^{\emptyset}$  where  $|\Theta| = s > 0$  is minimally generated by bases of up to two copies of  $V_{\phi}$  in the following degrees of  $\mathbb{C}[\mathcal{N}]$ :*

- 1.  $m_{r-s+1}^{\phi}$  for nilpotent orbits in the first family or
- 2.  $m_{\lfloor n/2 \rfloor}^{\phi}$  for nilpotent orbits in the second family, and
- $m_{r-s+2}^{\phi}$  for nilpotent orbits with partition  $[2n - 2s + 1, 2s - 3, 1, 1]$  where  $s > 2$  in type  $D_n$ , or Bala-Carter labels  $E_6(a_3)$  in type  $E_6$ ,  $E_7(a_3)$  and  $E_6(a_1)$  in type  $E_7$ ,  $E_8(a_3)$  and  $E_8(a_4)$  in type  $E_8$ .

Moreover, the ideal  $J_{\Theta}$  is minimally generated by pre-images of the generators of  $I_{\Theta}^{\emptyset}$  in  $\mathbb{C}[\mathfrak{g}]$  as well as the following  $n - s$  fundamental invariants:

- $f_1, \dots, f_{n-s}$  for nilpotent orbits in the first family or
- $f_1, \dots, \widehat{f_{\lfloor n/2 \rfloor}}, \dots, f_{n-s+1}$  for nilpotent orbits in the second family.

Following Broer's argument for the subregular case, we use cohomological arguments to determine a minimal generating set for  $I_{\Theta}^{\Theta'}$  where  $\mathcal{O}_{\Theta}$  is the maximal orbit in  $\overline{\mathcal{O}_{\Theta'}} \setminus \mathcal{O}_{\Theta'}$ . Then we need to know which copies of  $V_{\phi}$  in  $\mathbb{C}[\mathcal{N}]$  are already included in  $I_{\Theta}^{\emptyset}$ , which also helps answer the question of which fundamental invariants are needed to generate  $J_{\Theta}$ . Some parts of the proof must be done on a case-by-case level.

### 2.3 Cohomology

Fix a parabolic subgroup  $P \subset G$ . Let  $V$  be a finite dimensional  $P$ -module. Let  $G \times^P V$  be the set of orbits of the right  $P$ -action on  $G \times V$  defined by  $(g, v) \cdot p = (gp, p^{-1} \cdot v)$ . This is the geometric quotient of  $G \times V$  by  $P$ , so the algebra of regular functions on  $G \times^P V$  is isomorphic to the algebra of  $P$ -invariant regular functions on  $G \times V$ ,  $\mathbb{C}[G \times^P V] \simeq \mathbb{C}[G \times V]^P$ .

Using the morphism  $G \times^P V \rightarrow G/P$  defined by taking the  $P$ -orbit of  $(g, v)$  to  $gP$ , we have that  $G \times^P V$  is a vector bundle over  $G/P$  of rank equal to the dimension of  $V$ . We will abuse notation and write the cohomology groups for the associated locally free sheaves on  $G/P$  as  $H^i(G/P, V)$ , neglecting to distinguish between the module and the sheaf. The global sections of the associated locally free sheaf are morphisms  $f : G \rightarrow V$  such that  $f(gp) = p^{-1}f(g)$  for all  $g \in G$  and  $p \in P$ . There is a  $G$ -module structure on  $H^0(G/P, V)$  induced from the  $P$ -module structure on  $V$ .

Using the isomorphism

$$H^i(G/P, V) \simeq H^i(G/B, V) \text{ [11, II.4.6],}$$

the cohomology groups  $H^i(G/P, V)$  can be computed by the Borel-Weil-Bott Theorem when  $V$  is completely reducible. If  $V'$  is another  $P$ -module, the  $G$ -vector bundle  $G \times^P (V \times V')$  can similarly be defined on  $G \times^P V$ . These different types of vector bundle are related in cohomology by

$$H^i(G \times^P V, V') \simeq \bigoplus_{j \geq 0} H^i(G/P, S^j(V^*) \otimes V')$$

where  $S^j(V^*)$  is the  $j$ th symmetric power of the dual  $P$ -module to  $V$ .

There is a natural one-dimensional representation  $\mathbb{C}_\lambda$  of  $B$  for each weight  $\lambda$  (where we recall that a weight is a character of the maximal torus in  $B$ ). For a dominant weight  $\lambda$ , we define  $V_\lambda$  to be the simple  $G$ -module of highest weight  $\lambda$ , which occurs in this cohomological context as  $H^0(G/B, \mathbb{C}_{-\lambda})$ .

Let  $\Theta$  be the set of positive roots associated to a parabolic subgroup  $P_\Theta$ , which has Lie algebra  $\mathfrak{p}_\Theta$ . We use the notation

$$H^i(\lambda)[-m] := \bigoplus_{j \geq 0} H^i(G/B, S^{j-m}(\mathfrak{n}^*) \otimes \mathbb{C}_{-\lambda}) \text{ and } H_\Theta^i(\lambda)[-m] \bigoplus_{j \geq 0} := H^i(G/P_\Theta, S^{j-m}(\mathfrak{n}_\Theta^*) \otimes \mathbb{C}_{-\lambda}),$$

where  $\mathfrak{n}_\Theta$  is the nilradical of  $\mathfrak{p}_\Theta$ .

By identifying the cotangent bundle  $T^*G/P_\Theta$  with  $G \times^{P_\Theta} \mathfrak{n}_\Theta$ , we are able to work with the cohomology groups  $H^i(T^*G/P_\Theta, V)$  where  $V$  is a  $P_\Theta$ -module. Broer studied the line bundles  $\mathbb{C}_\lambda$  on the cotangent bundle and developed several theorems regarding the vanishing of their higher cohomology.

**Theorem 2.3.1.** *[2, Theorem 2.2] Let  $P_\Theta$  be a parabolic subgroup of  $G$  generated by a set of roots  $\Theta$  and  $\lambda$  be a dominant weight such that  $P_\Theta$  stabilizes a one-dimensional subspace in  $V_\lambda$ . Then  $H_\Theta^i(\lambda) = 0$  for  $i > 0$ .*

**Theorem 2.3.2.** *[1, Proposition 2.6, Proposition 3.2] For a dominant weight  $\lambda$ ,  $H^0(\lambda)$  is generated as a  $\mathbb{C}[\mathcal{N}]$ -module by a basis for the dual  $G$ -module to  $V_\lambda$ .*

To each weight  $\lambda$ , we may associate two dominant weights:  $\lambda^+$  and  $\lambda^*$ . The former is the unique dominant weight in the Weyl group orbit of  $\lambda$ , so that for some  $w \in W$ ,  $w \cdot \lambda = \lambda^+$ . The latter is the unique dominant weight that is minimal for the property that  $\lambda \leq \lambda^*$ . The best case for cohomology vanishing is when these are equal.

**Theorem 2.3.3.** [1, Theorem 2.4] Let  $\lambda$  be a weight. The higher cohomology groups  $H^i(\lambda)$ ,  $i > 0$ , vanish if and only if  $\lambda^+ = \lambda^*$ .

The weights that have this vanishing property include dominant weights (in which case  $\lambda = \lambda^+ = \lambda^*$ ) and positive short roots, but not simple long roots. This fact is key to why we focus on sets of orthogonal simple short roots in our results.

*Remark 2.3.4.* While we work over the complex field just as Broer did, Thomsen[22] has proved many of these same results in good characteristic.

To extend these vanishing results, we will use the following cohomological identities, referred to as moves, which originate in work of Demazure [8] and Sommers [20].

**Theorem 2.3.5.** For  $\Theta$  some subset of simple roots,  $\lambda$  a weight, and  $\beta_0$  a simple root such that  $\langle \lambda, \beta_0^\vee \rangle = -1$ ,

1. If  $\langle \beta, \beta_0^\vee \rangle = 0$  for all  $\beta \in \Theta$ , then  $H_{\Theta}^i(\lambda) \simeq H_{\Theta}^i(\lambda + \beta_0)[-1]$  (the  $A_1$  move);
2. If there exists  $\beta_1 \in \Theta$  such that  $\langle \beta_1, \beta_0^\vee \rangle = \langle \beta_0, \beta_1^\vee \rangle = -1$ ,  $\langle \beta, \beta^\vee \rangle = \langle \beta_0, \beta^\vee \rangle = 0$  for all  $\beta \in \Theta \setminus \{\beta_1\}$ , and  $\langle \lambda, \beta_1^\vee \rangle = 0$ , then  $H_{\Theta}^i(\lambda) \simeq H_{\Theta'}^i(\lambda + \beta_0 + \beta_1)[-1]$  where  $\Theta' = \Theta \setminus \{\beta_1\} \cup \{\beta_0\}$  (the  $A_2$  move);
3. If there exist  $\beta_1, \beta_2 \in \Theta$  such that  $\langle \beta_1, \beta_0^\vee \rangle = \langle \beta_2, \beta_0^\vee \rangle = \langle \beta_0, \beta_1^\vee \rangle = \langle \beta_0, \beta_2^\vee \rangle = -1$ ,  $\langle \beta_1, \beta_2^\vee \rangle = 0$ ,  $\langle \beta_1, \beta^\vee \rangle = \langle \beta_0, \beta^\vee \rangle = \langle \beta_2, \beta^\vee \rangle = 0$  for all  $\beta \in \Theta \setminus \{\beta_1, \beta_2\}$ , and  $\langle \lambda, \beta_1^\vee \rangle = \langle \lambda, \beta_2^\vee \rangle = 0$ , then  $H_{\Theta}^i(\lambda) \simeq H_{\Theta}^i(\lambda + \beta_1 + 2\beta_0 + \beta_2)[-2]$  (the  $A_3$  move).

*Remark 2.3.6.* In each move, the weight in parentheses on the right-hand side is in the Weyl group orbit of  $\lambda$ , a fact that will be important in Theorem 2.4.1.

It is also important that we be able to change the set  $\Theta$  with fixed  $\lambda$ :  $H_{\Theta}^i(\lambda) = H_{\Theta'}^i(\lambda)$  when  $\langle \lambda, \alpha^\vee \rangle = 0$  for all  $\alpha \in \Theta \cup \Theta'$ . We will primarily use this when  $\lambda = \phi$  because  $\phi$  is orthogonal to most simple roots.

Because the higher cohomology is known to vanish for dominant weights, we can use these moves to show the same holds for some non-dominant weights dependent on the parabolic subgroup. In particular, we can start with any simple short root and move to the dominant short root, which is the dominant weight in the Weyl group orbit of any simple short root.

## 2.4 Generating the ideal of one variety within another

In this section, we investigate  $I_{\Theta'}^{\Theta'}$ , which we have defined to be the ideal of  $\overline{\mathcal{O}}_{\Theta}$  in  $\mathbb{C}[\overline{\mathcal{O}}_{\Theta}']$  for  $\mathcal{O}_{\Theta} \subset \overline{\mathcal{O}}_{\Theta'}$ . As previously mentioned,  $\mathcal{O}_{\{\beta\}}$  for simple short root  $\beta$  is the subregular orbit. Broer [1,

Theorem 4.9] concluded that in this case  $J_\Theta$  is minimally generated by the first  $n - 1$  fundamental invariants and a basis of a degree  $\text{ht}(\phi)$  copy of  $V_\phi$ . Because  $\text{ht}(\phi) = m_r^\phi$ , that case is complete. So suppose  $\Theta = \{\beta_1, \dots, \beta_s\}$  is a set of  $s$  orthogonal simple short roots where  $s > 1$ . Let  $\Theta' = \Theta \setminus \beta_1$ , which is nonempty because  $s > 1$ . We induct on  $s$ .

**Theorem 2.4.1.**  $I_{\Theta'}^\phi$  is generated by a basis of  $V_\phi$  in degree  $m_{r-s+1}^\phi$

*Proof.* Restricting linear functions on  $\mathfrak{n}_{\Theta'}$  to  $\mathfrak{n}_\Theta$  gives a short exact sequence of  $B$ -modules

$$0 \rightarrow \mathbb{C}_{-\beta_1} \rightarrow \mathfrak{n}_{\Theta'}^* \rightarrow \mathfrak{n}_\Theta^* \rightarrow 0$$

that has Koszul resolution

$$0 \rightarrow S^{\bullet-1}(\mathfrak{n}_{\Theta'}^*) \otimes \mathbb{C}_{-\beta_1} \rightarrow S^\bullet(\mathfrak{n}_{\Theta'}^*) \rightarrow S^\bullet(\mathfrak{n}_\Theta^*) \rightarrow 0$$

which in turn gives a long exact sequence in cohomology as vector bundles over  $G/B$

$$0 \rightarrow H_{\Theta'}^0(\beta_1)[-1] \rightarrow \mathbb{C}[\overline{\mathcal{O}}_{\Theta'}] \rightarrow \mathbb{C}[\overline{\mathcal{O}}_\Theta] \rightarrow H_{\Theta'}^1(\beta_1)[-1] \rightarrow \dots$$

We know that  $H_{\Theta'}^1(\beta_1) = 0$  by Theorem 2.3.1, so  $\mathbb{C}[\overline{\mathcal{O}}_\Theta]$  is a quotient of  $\mathbb{C}[\overline{\mathcal{O}}_{\Theta'}]$ . Using induction on  $s$ , this is sufficient to prove that  $\overline{\mathcal{O}}_\Theta$  is normal and that the map  $G \times^{P_\Theta} \mathfrak{n}_\Theta \rightarrow \overline{\mathcal{O}}_\Theta$  is birational, although those facts are already known through a different argument [2, Theorem 4.1]. All that remains to be understood in the exact sequence is  $H_{\Theta'}^0(\beta_1)[-1]$ .

**Lemma 2.4.2.**  $H_{\Theta'}^i(\beta_1)[-1] \simeq H_\Omega^i(\phi)[-m_{r-s+1}^\phi]$ , where  $\Omega$  is some set of  $s - 1$  orthogonal simple short roots.

In Section 2.8, we prove this lemma case-by-case. In each case, we also see that the roots in  $\Omega$  and  $\Theta'$  are orthogonal to  $\phi$ , so that  $H_\Omega^i(\phi) \simeq H_{\Theta'}^i(\phi)$  anyway.

Next we show that  $H_{\Theta'}^0(\phi)$  is a quotient of  $H^0(\phi)$ , so that generators for the latter are sufficient to generate the former. We induct on  $|\Theta'|$ .

First let  $\Theta' = \{\beta_2\}$ . Restricting linear functions on  $\mathfrak{n}$  to  $\mathfrak{n}_{\Theta'}$  gives short exact sequence of  $B$ -modules

$$0 \rightarrow \mathbb{C}_{-\beta_2} \rightarrow \mathfrak{n}^* \rightarrow \mathfrak{n}_{\Theta'}^* \rightarrow 0$$

that has Koszul resolution

$$0 \rightarrow S^{\bullet-1}(\mathfrak{n}^*) \otimes \mathbb{C}_{-\beta_2} \rightarrow S^{\bullet}(\mathfrak{n}^*) \rightarrow S^{\bullet}(\mathfrak{n}_{\Theta'}^*) \rightarrow 0.$$

Tensoring with the  $B$ -module  $\mathbb{C}_{\phi}$  and then moving to cohomology as vector bundles over  $T^*G/P_{\Theta}$ , we have the exact sequence

$$0 \rightarrow H^0(\phi + \beta_2)[-1] \rightarrow H^0(\phi) \rightarrow H_{\Theta'}^0(\phi) \rightarrow H^1(\phi + \beta_2)[-1] \rightarrow 0$$

where the final zero comes from the fact that  $\phi$  is dominant and so has vanishing higher cohomology. Although  $\phi + \beta_2$  is not necessarily a dominant weight, we have the following lemma that fits this situation.

**Lemma 2.4.3.** *[2, Lemma 3.13] Let  $\lambda$  be a weight and  $\beta$  be a short root orthogonal to  $\lambda$ . Then  $\lambda^+ < (\lambda + \beta)^+$  and if  $\lambda^+ \leq \mu \leq (\lambda + \beta)^+$  for dominant weight  $\mu$ , then  $\mu = \lambda^+$  or  $\mu = (\lambda + \beta)^+$ .*

Then  $(\phi + \beta_2)^+ = (\phi + \beta_2)^*$ , so by Theorem 2.3.3,  $H^i(\phi + \beta_2) = 0$  for  $i \geq 1$ . Hence  $H_{\Theta'}^0(\phi)$  is a quotient of  $H^0(\phi)$  for a set  $\Theta'$  of only one simple short root.

Now let  $|\Theta'| > 1$ . Let  $\Theta'' = \Theta' \setminus \beta_2$ . We have a long exact sequence

$$0 \rightarrow H_{\Theta''}^0(\phi + \beta_2)[-1] \rightarrow H_{\Theta''}^0(\phi) \rightarrow H_{\Theta'}^0(\phi) \rightarrow H_{\Theta''}^1(\phi + \beta_2)[-1] \rightarrow \dots$$

which comes from restriction of linear functions on  $\mathfrak{n}_{\Theta''}$  to  $\mathfrak{n}_{\Theta'}$  as above. By induction we know that  $H_{\Theta''}^0(\phi)$  is a quotient of  $H^0(\phi)$ , so we need only show that  $H_{\Theta''}^1(\phi + \beta_2) = 0$ .

We claim that  $H_{\Theta''}^i(\phi + \beta_2)$  is isomorphic to some shift of  $H_{\Omega}^i(\mu)$ , where  $\mu = (\phi + \beta_2)^+$  is the dominant weight in the Weyl group orbit of  $\phi + \beta_2$  and  $|\Omega| = |\Theta''|$ . Let  $\Phi_{\phi} \subset \Phi$  be the root subsystem of roots orthogonal to  $\phi$ , which contains both  $\beta_2$  and all the roots of  $\Theta''$  as simple roots. The Levi subalgebra for this root subsystem, which we denote by  $\mathfrak{l}_{\phi}$ , will reappear later in Section 3.4 of Chapter 3. Considering only the component of  $\Phi_{\phi}$  containing  $\beta_2$ , and its associated simple component of  $\mathfrak{l}_{\phi}$ , we know from lemma 2.4.2 that we can move from  $\beta_2$  to the dominant short root  $\nu$  of  $\Phi_{\phi}$  (labeling it  $\phi_{\Phi_{\phi}}$  would be more consistent, but confusing). The same moves take

$\phi + \beta_2$  to  $\phi + \nu$ . Because  $\nu$  is a short root in  $\Phi_\phi$ , it can have inner product at worst  $-1$  with simple roots not orthogonal to  $\phi$ . Thus  $\phi + \nu$  is dominant, and must be equal to  $(\phi + \beta_2)^+$  since the moves preserve Weyl group orbit.

Hence we know  $H_{\Theta'}^0(\beta_1)[-1] \simeq H_{\Theta'}^0(\phi)[-m_{r-s+1}^\phi]$  is a quotient of  $H^0(\phi)[-m_{r-s+1}]$ . Then a basis of  $V_\phi$  in degree  $m_{r-s+1}^\phi$  generates the ideal  $I_{\Theta'}^{\Theta'}$ .  $\square$

So now we know a minimal generating set for  $I_{\Theta'}^{\Theta'}$  to be a basis for a copy of  $V_\phi$ . Since this is true for all consecutive subsets of  $\Theta$ , it follows inductively that  $I_\Theta^\emptyset \subset \mathbb{C}[\mathcal{N}]$  is generated by  $s$  copies of  $V_\phi$ . However, we are searching for a minimal generating set, not merely any generating set. The next question to answer is whether all of these copies are needed, or if some are redundant. For that, we must consider the fundamental invariants.

## 2.5 Invariants and copies of representations

As explained in Section 2.1, there is a bijection between copies of the adjoint representation in  $\mathbb{C}[\mathcal{N}]$  and fundamental invariants because the span of partial derivatives of an invariant will be isomorphic to  $V_\theta$ . When these copies are used to generate ideals, we are interested in which other copies are contained in the ideals. The fundamental invariants are algebraically independent by definition, but there is a way to generate fundamental invariants using the partial derivatives of another. We discovered this operation in a paper of De Concini, Papi, and Procesi [6], but it originates with Saito, Yano and Sekiguchi [18].

Let  $\{x_i\}$ ,  $1 \leq i \leq \dim \mathfrak{g}$ , be a basis for  $\mathfrak{g}$  with dual basis  $\{y_i\}$  with respect to the Killing form which gives an isomorphism  $\mathfrak{g} \simeq \mathfrak{g}^*$ . Recall that  $R$  is the ring of  $G$ -invariants in  $\mathbb{C}[\mathfrak{g}]$  and  $R_+$  is the subring of invariants without constant term. Let  $f, g \in R_+$ . We define the operation  $\circ$  on  $R_+ \times R_+$  by

$$f \circ g = \sum_i \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i}.$$

The resulting  $f \circ g$  is also  $G$ -invariant. If  $f$  is homogeneous of degree  $d_f$  and  $g$  is homogeneous of degree  $d_g$ , then  $f \circ g$  is homogeneous of degree  $d_f + d_g - 2$ . We use the notation  $\equiv$  for equivalence modulo  $R_+^2$ , which is significant because a fundamental invariant is a representative of an equivalence class modulo  $R_+^2$ .

Given a homogeneous invariant  $f \in R_+$ , we denote by  $[f]$  the copy of the adjoint representation in  $\mathbb{C}[\mathfrak{g}]$  which is spanned by the partial derivatives  $\frac{\partial f}{\partial x_i}$  or  $\{\frac{\partial f}{\partial y_i}\}$ . Furthermore, the partial derivatives of

$f \circ g$  can be further decomposed to yield another spanning set for a copy of the adjoint representation.

For  $1 \leq j \leq \dim \mathfrak{g}$ , define

$$w_j^{f,g} = \sum_{i=1}^N \frac{\partial^2 f}{\partial x_j \partial x_i} \frac{\partial g}{\partial y_i}.$$

**Lemma 2.5.1.** *The span of  $\{w_j^{f,g}\}$ , if nonzero, is a copy of the adjoint representation in  $\mathbb{C}[\mathfrak{g}]$ .*

*Proof.* Let  $\pi \in \text{Hom}(\mathfrak{g} \otimes \mathfrak{g}, S^2 \mathfrak{g})$  be the defining homomorphism of  $S^2 \mathfrak{g}$ . Because

$$\text{Hom}(\mathfrak{g} \otimes \mathfrak{g}, S^2 \mathfrak{g}) \simeq \text{Hom}(\mathfrak{g}, S^2 \mathfrak{g} \otimes \mathfrak{g}^*),$$

we can consider  $\pi \in \text{Hom}(\mathfrak{g}, S^2 \mathfrak{g} \otimes \mathfrak{g}^*)$ . Consider the map

$$S^2 \mathfrak{g} \otimes \mathfrak{g}^* \simeq S^2 \mathfrak{g}^* \otimes \mathfrak{g}^* \rightarrow S^{d_f+d_g-3} \mathfrak{g}^* \simeq \mathbb{C}^{d_f+d_g-3}[\mathfrak{g}]$$

where the first and last isomorphisms are clear and the map in the center is evaluation at  $f$  and  $g$  respectively. Under this map,  $\pi(\mathfrak{g})$  is taken to the span of  $\{w_j^{f,g}\}$ , so that this is a copy of the adjoint representation.  $\square$

**Theorem 2.5.2.** *Let  $f_l$  and  $f_k$  be distinct fundamental invariants with degrees  $d_l$  and  $d_k$  respectively. Let  $I \subset \mathbb{C}[\mathcal{N}]$  be the ideal generated by the image of  $[f_l]$  in  $\mathbb{C}[\mathcal{N}]$  and  $J \subset \mathbb{C}[\mathfrak{g}]$  be the ideal generated by  $[f_l]$  and fundamental invariants  $f_1, \dots, f_{k-1}$ . Then the following are equivalent:*

1. *There exists  $g \notin R_+^2$  of degree  $d_k - d_l + 2$  such that  $g \circ f_l \equiv f_k$ .*
2. *The image of  $[f_k]$  in  $\mathbb{C}[\mathcal{N}]$  is a subset of  $I$ .*
3.  *$f_k \in J$ .*

*Proof.* (2)  $\implies$  (3). By Euler's homogeneous function theorem,  $f_k = \frac{1}{d_k} \sum_i x_i \frac{\partial f_k}{\partial x_i}$ . If the image of  $[f_k]$  in  $\mathbb{C}[\mathcal{N}]$  is in the ideal generated by the image of  $[f_l]$ , then  $\frac{\partial f_k}{\partial x_i} \in J$  for all  $1 \leq i \leq \dim \mathfrak{g}$ . Thus  $f_k \in J$ .

(1)  $\implies$  (2). Suppose there exists  $g \notin R_+^2$  of degree  $d_k - d_l + 2$  such that  $g \circ f_l \equiv f_k$ . Using lemma 2.5.1, there is a copy of  $V_\theta$  in degree  $d_k - 1$  spanned by  $\{w_j^{g,f_l}\}$ , which is contained in  $[f_l]$  because each  $w_j^{g,f_l}$  is a sum of terms with a partial derivative of  $f_l$  as a factor. So the image of this copy of  $V_\theta$ , and in particular the set  $\{w_j^{g,f_l}\}$  in  $\mathbb{C}[\mathcal{N}]$ , is clearly contained in  $I$ .

We now show that the image of  $[f_k]$  in  $\mathbb{C}[\mathcal{N}]$  is equal to the image of the set spanned by  $\{w_j^{g,f_l}\}$  by showing that the latter is nonzero. Although the images in  $\mathbb{C}[\mathcal{N}]$  are our true concern, we start



by considering copies of  $V_\theta$  in  $\mathbb{C}[\mathfrak{g}]$ . Kostant [12] described  $\mathbb{C}[\mathfrak{g}] \simeq S\mathfrak{g}^*$  as  $R \otimes H$  where  $H$  is a graded subspace of  $\mathbb{C}[\mathfrak{g}]$  isomorphic as a  $G$ -module to  $\mathbb{C}[\mathcal{N}]$ , and together with the fact that the  $\theta$ -isotypic component of  $\mathbb{C}[\mathcal{N}]$  is spanned by the  $[f_j]$ , this implies that every homogeneous copy of  $V_\theta$  in  $\mathbb{C}[\mathfrak{g}]$  is the span of elements of the form

$$v_i = \sum_{j=1}^{\dim \mathfrak{g}} r_j \frac{\partial f_j}{\partial x_i}$$

where  $r_j \in R$  are homogeneous, independent of  $i$ , and unique up to a scalar multiple of the entire set. We see too that such a copy of  $V_\theta$  is contained in the ideal  $(R_+) \subset \mathbb{C}[\mathfrak{g}]$  if and only if the  $r_j$  are all elements of  $R_+$ . Because  $\sum_i x_i v_i = \sum_j d_j r_j f_j$  by Euler's homogeneous function theorem, we have  $\{r_j\} \subset R_+$  if and only if there is a basis  $\{v'_i\}$  of the span of  $\{v_i\}$  such that  $\sum_i x_i v'_i \in R_+^2$ . But  $\sum x_i v'_i$  can only differ from  $\sum x_i v_i$  by a scalar because there is only one copy of the trivial representation in  $\mathfrak{g}^* \otimes \mathfrak{g} \simeq \mathfrak{g} \otimes \mathfrak{g}$ .

The span of  $\{w_j^{g, f_l}\}$  is one such copy of  $V_\theta$  in  $\mathbb{C}[\mathfrak{g}]$ , by lemma 2.5.1. We observe that

$$\begin{aligned} \sum_{j=1}^{\dim \mathfrak{g}} x_j w_j^{g, f_l} &= \sum_{j=1}^{\dim \mathfrak{g}} \sum_{i=1}^{\dim \mathfrak{g}} x_j \frac{\partial g}{\partial x_j} \frac{\partial f_l}{\partial x_i} \frac{\partial f_l}{\partial y_i} \\ &= \sum_{i=1}^{\dim \mathfrak{g}} \sum_{j=1}^{\dim \mathfrak{g}} x_j \frac{\partial}{\partial x_j} \left( \frac{\partial g}{\partial x_i} \right) \frac{\partial f_l}{\partial y_i} \\ &= \sum_{i=1}^{\dim \mathfrak{g}} (d_k - d_l + 1) \frac{\partial g}{\partial x_i} \frac{\partial f_l}{\partial y_i} \text{ by Euler's homogeneous function theorem} \\ &= (d_k - d_l + 1) g \circ f_l. \end{aligned}$$

Then  $\sum_i x_i v_i \notin R_+^2$  since  $g \circ f_l \equiv f_k$  and  $f_k$  is a representative of a nonzero equivalence class modulo  $R_+^2$ . Hence the span of  $\{w_j^{g, f_l}\}$  is not contained in  $(R_+)$  and has nonzero image in  $\mathbb{C}[\mathcal{N}]$ .

(3)  $\implies$  (1). Suppose  $f_k \in J$ . There is a surjection of  $S\mathfrak{g}^*[-m_l] \otimes \mathfrak{g}$  onto  $([f_l])$ , the ideal in  $\mathbb{C}[\mathfrak{g}]$  generated by  $[f_l]$ . Then  $G$ -invariants in  $([f_l])$  come from  $(S\mathfrak{g}^*[-m_l] \otimes \mathfrak{g})^G \simeq \text{Hom}(\mathfrak{g}, S\mathfrak{g}^*[-m_l])$ . So a  $G$ -invariant in  $([f_l])$  takes the form  $\sum_{i=1}^{\dim \mathfrak{g}} v_i \frac{\partial f_l}{\partial y_i}$  where  $v_i$  are as described above in the (1)  $\implies$  (2) step. In particular, there are some  $v_i$ ,  $1 \leq i \leq \dim \mathfrak{g}$  such that  $f_k \equiv \sum_i v_i \frac{\partial f_l}{\partial y_i}$ . We know that  $f_k \notin R_+^2$ , so neither is  $\sum_i v_i \frac{\partial f_l}{\partial y_i}$ , and by the argument in the (1)  $\implies$  (2) step, there must be some  $r_j$  in some  $v_i$  which is degree zero. Therefore  $f_k \equiv \sum_i \frac{\partial g}{\partial x_i} \frac{\partial f_l}{\partial y_i}$  for some homogeneous  $g \notin R_+^2$ .  $\square$

The logical next question would be when the first condition occurs. This has been answered, most clearly in the work of De Concini, Papi, and Procesi [6]. The method to check is to restrict to the Cartan subalgebra. We must be explicit when working with the unusual exceptions in type  $D_n$  when  $n$  is even, and so we introduce notation that  $M$  is a generic matrix in the standard representation

of  $\mathfrak{so}_{2n}$  as skew-symmetric  $2n \times 2n$  matrices, and  $\text{Tr}$  and  $\text{Pf}$  refer to the matrix trace and Pfaffian respectively.

**Theorem 2.5.3.** [6] *Condition (1) is equivalent to  $d_k - d_l + 1$  being an exponent of the Lie algebra except when  $\mathfrak{g}$  is of type  $D_n$  with even  $n$  and one of the following is true:*

- $f_l \equiv c \cdot \text{Pf}(M)$  for some scalar  $c$ , in which case condition (1) holds only for  $d_k = 2n - 2$ , or
- $d_k = n$  and  $f_k - c \cdot \text{Tr}(M^n) \notin R_+^2$  for all scalars  $c$ , in which case condition (1) holds only for  $d_l = 2$ .

We have referred exclusively to the adjoint representation  $V_\theta$  throughout this section, not  $V_\phi$ . In order to determine when copies of  $V_\phi$  generate others for multiply-laced Lie algebras, we use the fact that every multiply-laced Lie algebra is a folding of a simply-laced one and consider the adjoint representation in the simply-laced Lie algebra, as detailed in the next section.

## 2.6 Folding

Suppose  $\mathfrak{g}$  is a simply-laced simple complex Lie algebra with Cartan subalgebra  $\mathfrak{h}$  and root system  $\Phi = \Phi(\mathfrak{g})$  such that there is an involution  $\tau$  on  $\Phi$ . A root system automorphism induces an outer automorphism of  $\mathfrak{g}$ . In particular, for a Chevalley basis  $\{E_\alpha, H_\alpha\}$  of  $\mathfrak{g}$ , define  $\tau^*(E_\alpha) = E_{\tau(\alpha)}$  and  $\tau^*(H_\alpha) = H_{\tau(\alpha)}$ . Let  $\mathfrak{g}$  have Cartan decomposition  $\mathfrak{g}_- \oplus \mathfrak{g}_+$  where  $\mathfrak{g}_+$  is the subalgebra fixed by  $\tau$  and  $\mathfrak{g}_-$  is the  $-1$  eigenspace for  $\tau$ . In this case of  $\tau$  being an outer involution, it follows that  $\mathfrak{g}_+$  is isomorphic to a multiply-laced simple Lie algebra  $\tilde{\mathfrak{g}}_+$ . It is known that types  $B_n, C_n$ , and  $F_4$  are isomorphic to foldings of  $D_{n+1}, A_{2n-1}$ , and  $E_6$  respectively. We denote by  $\pi$  the quotient map on root systems  $\Phi(\mathfrak{g}) \rightarrow \Phi(\mathfrak{g}_+)$ , which induces embedding  $\pi^* : \tilde{\mathfrak{g}}_+ \rightarrow \mathfrak{g}$  with image  $\mathfrak{g}_+$ . Let  $\theta$  denote the highest root of  $\mathfrak{g}$ ,  $\tilde{\theta}$  the highest root of  $\tilde{\mathfrak{g}}_+$ , and  $\tilde{\phi}$  the highest short root of  $\tilde{\mathfrak{g}}_+$ . Let  $\mathcal{N}_+$  be the nilpotent cone in  $\mathfrak{g}_+$ . As general notation, we will use  $\tilde{\alpha}_i$  for the simple roots of  $\tilde{\mathfrak{g}}_+$  and  $\alpha_i$  for simple roots of  $\mathfrak{g}$ .

We observe that  $\mathfrak{g}_+ = (\mathfrak{g}_+ \cap \mathfrak{h}) \oplus \bigoplus_{\alpha \in \Phi(\mathfrak{g}_+)} \mathbb{C}E_\alpha \oplus \bigoplus_{\alpha \notin \Phi(\mathfrak{g}_+)} \mathbb{C}(E_\alpha + E_{\tau(\alpha)})$  and  $\mathfrak{g}_- = (\mathfrak{g}_- \cap \mathfrak{h}) \oplus \bigoplus_{\alpha \notin \Phi(\mathfrak{g}_+)} \mathbb{C}(E_\alpha - E_{\tau(\alpha)})$ . We will denote  $E_\alpha - E_{\tau(\alpha)}$  by  $E_\alpha^-$  and  $E_\alpha + E_{\tau(\alpha)}$  by  $E_\alpha^+$ . Then for  $\alpha, \beta$  roots not fixed by  $\tau$ ,

$$\begin{aligned}
[E_\alpha^+, E_\beta^-] &= [E_\alpha, E_\beta] - [E_{\tau(\alpha)}, E_{\tau(\beta)}] + [E_{\tau(\alpha)}, E_\beta] - [E_\alpha, E_{\tau(\beta)}] \\
&= \lambda_{\alpha+\beta} E_{\alpha+\beta} - \lambda_{\tau(\alpha+\beta)} E_{\tau(\alpha+\beta)} + \lambda_{\tau(\alpha)+\beta} E_{\tau(\alpha)+\beta} - \lambda_{\tau(\tau(\alpha)+\beta)} E_{\alpha+\tau(\beta)} \\
&= \lambda_{\alpha,\beta} E_{\alpha+\beta}^- + \lambda_{\tau(\alpha),\beta} E_{\tau(\alpha)+\beta}^-,
\end{aligned}$$

and for  $\alpha$  root fixed by  $\tau$  and  $\beta$  root not fixed by  $\tau$ ,

$$\begin{aligned} [E_\alpha, E_\beta^-] &= [E_\alpha, E_\beta] - [E_\alpha, E_{\tau(\beta)}] \\ &= \lambda_{\alpha,\beta} E_{\alpha+\beta} - \lambda_{\alpha,\tau(\beta)} E_{\alpha+\tau(\beta)} \\ &= \lambda_{\alpha,\beta} E_{\alpha+\beta}^-, \end{aligned}$$

where  $\lambda_{\alpha,\beta}$  is  $\pm(r+1)$  for  $r$  the greatest integer such that  $\beta - r\alpha$  is a root (because  $\tau$  is an automorphism,  $\tau(\alpha)$  and  $\tau(\beta)$  will have the same  $r$  and hence  $\lambda_{\tau(\alpha),\tau(\beta)} = \lambda_{\alpha,\beta}$ ) and if  $\alpha + \beta$  is not a root,  $E_{\alpha+\beta} = 0$ .

**Lemma 2.6.1.**  $\mathfrak{g}_-$  is a highest-weight  $\tilde{\mathfrak{g}}_+$ -module with highest weight  $\tilde{\phi}$  under the adjoint action of  $\mathfrak{g}_+$ .

*Proof.* Since simple Lie algebras have a complete classification, we need only consider the three possible cases for  $(\mathfrak{g}, \tilde{\mathfrak{g}}_+)$ .

*Case 1:*  $(\mathfrak{g}, \tilde{\mathfrak{g}}_+)$  are of types  $A_{2n-1}, C_n$ .

Here  $\tau(\alpha_i) = \alpha_{2n-i}$  so that

$$\pi(\alpha_i) = \begin{cases} \tilde{\alpha}_n & \text{if } i \leq n \\ \tilde{\alpha}_{2n-i} & \text{if } i \geq n \end{cases}$$

The highest root in type  $A_{2n-1}$  is  $\theta = \sum_{i=1}^{2n-1} \alpha_i$ . In type  $C_n$ , we know that  $\tilde{\theta} = 2\tilde{\alpha}_1 + 2\tilde{\alpha}_2 + \cdots + 2\tilde{\alpha}_{n-1} + \tilde{\alpha}_n$  and  $\tilde{\phi} = \tilde{\theta} - \tilde{\alpha}_1$ , where  $\tilde{\alpha}_n$  is the long simple root. We see that  $\pi(\theta) = \tilde{\alpha}_1 + \tilde{\alpha}_2 + \cdots + \tilde{\alpha}_n + \tilde{\alpha}_{n-1} + \cdots + \tilde{\alpha}_1 = \tilde{\theta}$ . The highest short root  $\tilde{\phi}$  is the image under  $\pi$  of both  $\gamma := \sum_{i=1}^{2n-2} \alpha_i$  and  $\tau(\gamma) = \sum_{i=2}^{2n-1} \alpha_i$  (as expected, since  $\pi \circ \tau = \pi$ ).

Observe that for  $H \in \mathfrak{h}$ ,

$$[H, E_\gamma^-] = \gamma(H)E_\gamma - \tau(\gamma)(H)E_{\tau(\gamma)} = \gamma(H)E_\gamma - \gamma(\tau^*(H))E_{\tau(\gamma)}$$

because  $\tau$  is an involution on  $\mathfrak{g}$ . In particular, if  $\tilde{H} \in (\pi^*)^{-1}(\mathfrak{h})$ , then  $[\pi^*(\tilde{H}), E_\gamma^-] = \gamma(\pi^*(\tilde{H}))E_\gamma^- = \tilde{\phi}(\tilde{H})E_\gamma^-$ . Hence  $E_\gamma^-$  is a weight vector of  $\mathfrak{g}_-$  as a representation of  $\mathfrak{h} \cap \mathfrak{g}_+$ , with weight  $\tilde{\phi}$ .

We show that  $E_\gamma^-$  generates all of  $\mathfrak{g}_-$  under the adjoint action of  $\mathfrak{g}_+$ . Suppose  $\alpha_i$  is a simple root of  $\mathfrak{g}$  not fixed by  $\tau$ . If  $i = 1$ , then  $\gamma - \alpha_i$  is fixed by  $\tau$ , so  $E_{-(\gamma-\alpha_i)} \in \mathfrak{g}_+$  and we have  $[E_{-(\gamma-\alpha_i)}, E_\gamma^-] = E_{\alpha_i}^-$ . Similarly, if  $i = 2n-1$ , then  $E_{-(\tau(\gamma)-\alpha_i)} \in \mathfrak{g}_+$  and  $[E_{-(\tau(\gamma)-\alpha_i)}, E_\gamma^-] = -E_{\alpha_i}^-$ . If  $1 < i < 2n-1$ , then either  $\gamma - \alpha_i$  or  $\tau(\gamma) - \alpha_i$  is a sum of two positive roots that are not fixed by  $\tau$ . Without loss of generality, assume  $\gamma - \alpha_i = \beta + \beta'$ . Then  $[E_{-\beta}^+, [E_{-\beta'}^+, E_\gamma^-]] =$

$[E_{-\beta}^+, E_{\gamma-\beta'}^-] = E_{\alpha_i}^-$ . We know  $\alpha_i - \tau(\alpha_i)$  is not a root because either  $\alpha_i = \tau(\alpha_i)$  or  $\alpha_i$  is orthogonal to  $\tau(\alpha_i)$ . So  $[E_{-\alpha_i}^+, E_{\alpha_i}^-] = H_{\alpha_i} - H_{\tau(\alpha_i)}$ . Finally, for  $\alpha_i$  a simple root not fixed by  $\tau$ ,

$$\begin{aligned} [E_{-\alpha_i}^+, H_{\alpha_i} - H_{\tau(\alpha_i)}] &= [E_{-\alpha_i}, H_{\alpha_i}] + [E_{-\tau(\alpha_i)}, H_{\alpha_i}] - [E_{-\alpha_i}, H_{\tau(\alpha_i)}] - [E_{-\tau(\alpha_i)}, H_{\tau(\alpha_i)}] \\ &= -\alpha_i(H_{\alpha_i})E_{-\alpha_i} - \tau(\alpha_i)(H_{\alpha_i})E_{-\tau(\alpha_i)} + \alpha_i(H_{\tau(\alpha_i)})E_{-\alpha_i} + \tau(\alpha_i)(H_{\tau(\alpha_i)})E_{-\tau(\alpha_i)} \\ &= (-\alpha_i(H_{\alpha_i}) + \alpha_i(H_{\tau(\alpha_i)}))(E_{-\alpha_i} - E_{-\tau(\alpha_i)}). \end{aligned}$$

Because  $\alpha_i$  is orthogonal to  $\tau(\alpha_i)$ , we know that  $-\alpha_i(H_{\alpha_i}) + \alpha_i(H_{\tau(\alpha_i)}) \neq 0$ . For every simple root  $\alpha_i$  not fixed by  $\tau$ , we see that  $E_{\alpha_i}^-, E_{-\alpha_i}^-, H_{\alpha_i} - H_{\tau(\alpha_i)} \in [\mathfrak{g}_+, E_{\gamma}^-]$ . By bracketing with  $E_{\alpha_j}$  or  $E_{\alpha_j}^+$  to add roots fixed or not fixed by  $\tau$  respectively, any  $E_{\alpha}^-$  can be generated, and similarly  $E_{-\alpha}^-$ . Any  $H \in \mathfrak{h} \cap \mathfrak{g}_-$  is a linear combination of  $H_{\alpha_i} - H_{\tau(\alpha_i)}$ . Therefore  $\mathfrak{g}_- = [\mathfrak{g}_+, E_{\gamma}^-]$ .

We prove that  $E_{\gamma}^-$  is maximal in  $\mathfrak{g}_-$ . Let  $\alpha$  be a positive root of  $\mathfrak{g}$ . If  $\alpha = \tau(\alpha)$ , then  $[E_{\alpha}, E_{\gamma}^-] = E_{\gamma+\alpha}^- = 0$  because no such root  $\gamma + \alpha$  exists. If  $\alpha \neq \tau(\alpha)$ , then  $[E_{\alpha}^+, E_{\gamma}^-] = E_{\gamma+\alpha}^- = 0$  because either  $\gamma + \alpha$  is not a root or  $\gamma + \alpha = \theta$  which is fixed by  $\tau$ . Thus  $\mathfrak{g}_-$  is a highest-weight  $\mathfrak{g}_+$ -module generated by  $E_{\gamma}^-$ , which has weight  $\tilde{\phi}$ .

*Case 2:*  $(\mathfrak{g}, \tilde{\mathfrak{g}}_+)$  are of types  $D_{n+1}, B_n$ .

$$\tau(\alpha_i) = \begin{cases} \alpha_i & \text{if } i \leq n-1 \\ \alpha_{n+1} & \text{if } i = n \\ \alpha_n & \text{if } i = n+1 \end{cases} \quad \text{and} \quad \pi(\alpha_i) = \begin{cases} \tilde{\alpha}_i & \text{if } i \leq n \\ \tilde{\alpha}_n & \text{if } i = n+1 \end{cases}.$$

The highest root in type  $D_{n+1}$  is  $\theta = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n + \alpha_{n+1}$ . In type  $B_n$ , we know that  $\tilde{\theta} = \tilde{\alpha}_1 + 2\tilde{\alpha}_2 + \cdots + 2\tilde{\alpha}_{n-1} + 2\tilde{\alpha}_n$  and  $\tilde{\phi} = \sum_{i=1}^n \tilde{\alpha}_i$ . Both  $\gamma := \sum_{i=1}^n \alpha_i$  and  $\tau(\gamma)$  are mapped to  $\tilde{\phi}$  by  $\pi$ .

Observe that for  $H \in \mathfrak{h}$ ,

$$[H, E_{\gamma}^-] = \gamma(H)E_{\gamma} - \tau(\gamma)(H)E_{\tau(\gamma)} = \gamma(H)E_{\gamma} - \gamma(\tau^*(H))E_{\tau(\gamma)}$$

because  $\tau$  is an involution on  $\mathfrak{g}$ . In particular, if  $\tilde{H} \in (\pi^*)^{-1}(\mathfrak{h})$ , then  $[\pi^*(\tilde{H}), E_{\gamma}^-] = \gamma(\pi^*(\tilde{H}))E_{\gamma}^- = \tilde{\phi}(\tilde{H})E_{\gamma}^-$ . Hence  $E_{\gamma}^-$  is a weight vector of  $\mathfrak{g}_-$  as a representation of  $\mathfrak{h} \cap \mathfrak{g}_+$ , with weight  $\tilde{\phi}$ .

We show that  $E_{\gamma}^-$  generates all of  $\mathfrak{g}_-$  under the adjoint action of  $\mathfrak{g}_+$ . Suppose  $\alpha_i$  is a simple root of  $\mathfrak{g}$  not fixed by  $\tau$ . If  $i = n$ , then  $\gamma - \alpha_i$  is fixed by  $\tau$ , so  $E_{-(\gamma-\alpha_i)} \in \mathfrak{g}_+$  and we have  $[E_{-(\gamma-\alpha_i)}, E_{\gamma}^-] = E_{\alpha_i}^-$ . Similarly, if  $i = n+1$ , then  $E_{-(\tau(\gamma)-\alpha_i)} \in \mathfrak{g}_+$  and  $[E_{-(\tau(\gamma)-\alpha_i)}, E_{\gamma}^-] = -E_{\alpha_i}^-$ .

If  $i < n$ , then either  $\gamma - \alpha_i$  is a sum of two positive roots:  $\beta := \sum_{j=1}^{i-1} \alpha_j$  and  $\beta' := \sum_{j=i+1}^n \alpha_j$ . Then  $[E_{-\beta}, [E_{-\beta'}^+, E_{-\gamma}^-]] = [E_{-\beta}, \lambda_{\gamma, -\beta'} E_{-\beta'}^-] = \lambda_{\gamma, -\beta'} \lambda_{\gamma - \beta', -\beta} E_{-\alpha_i}^-$ . We know  $\alpha_i - \tau(\alpha_i)$  is not a root because either  $\alpha_i = \tau(\alpha_i)$  or  $\alpha_i$  is orthogonal to  $\tau(\alpha_i)$ . So  $[E_{-\alpha_i}^+, E_{\alpha_i}^-] = H_{\alpha_i} - H_{\tau(\alpha_i)}$ . Finally, for  $\alpha_i$  a simple root not fixed by  $\tau$ ,

$$\begin{aligned} [E_{-\alpha_i}^+, H_{\alpha_i} - H_{\tau(\alpha_i)}] &= [E_{-\alpha_i}, H_{\alpha_i}] + [E_{-\tau(\alpha_i)}, H_{\alpha_i}] - [E_{-\alpha_i}, H_{\tau(\alpha_i)}] - [E_{-\tau(\alpha_i)}, H_{\tau(\alpha_i)}] \\ &= -\alpha_i(H_{\alpha_i})E_{-\alpha_i} - \tau(\alpha_i)(H_{\alpha_i})E_{-\tau(\alpha_i)} + \alpha_i(H_{\tau(\alpha_i)})E_{-\alpha_i} + \tau(\alpha_i)(H_{\tau(\alpha_i)})E_{-\tau(\alpha_i)} \\ &= (-\alpha_i(H_{\alpha_i}) + \alpha_i(H_{\tau(\alpha_i)}))(E_{-\alpha_i} - E_{-\tau(\alpha_i)}). \end{aligned}$$

Because  $\alpha_i$  is orthogonal to  $\tau(\alpha_i)$ , we know that  $-\alpha_i(H_{\alpha_i}) + \alpha_i(H_{\tau(\alpha_i)}) \neq 0$ . For every simple root  $\alpha_i$  not fixed by  $\tau$ , we see that  $E_{\alpha_i}^-, E_{-\alpha_i}^-, H_{\alpha_i} - H_{\tau(\alpha_i)} \in [\mathfrak{g}_+, E_{-\gamma}^-]$ . By bracketing with  $E_{\alpha_j}$  or  $E_{\alpha_j}^+$  to add roots fixed or not fixed by  $\tau$  respectively, any  $E_{\alpha}^-$  can be generated, and similarly  $E_{-\alpha}^-$ . Any  $H \in \mathfrak{h} \cap \mathfrak{g}_-$  is a linear combination of  $H_{\alpha_i} - H_{\tau(\alpha_i)}$ . Therefore  $\mathfrak{g}_- = [\mathfrak{g}_+, E_{-\gamma}^-]$ .

We prove that  $E_{-\gamma}^-$  is maximal in  $\mathfrak{g}_-$ . Let  $\alpha$  be a positive root of  $\mathfrak{g}$ . If  $\alpha = \tau(\alpha)$ , then  $[E_{\alpha}, E_{-\gamma}^-] = E_{\gamma+\alpha}^- = 0$  because no such root  $\gamma + \alpha$  exists. If  $\alpha \neq \tau(\alpha)$ , then  $[E_{\alpha}^+, E_{-\gamma}^-] = E_{\gamma+\alpha}^- = 0$  because either  $\gamma + \alpha$  is not a root or  $\gamma + \alpha$  is fixed by  $\tau$ .

*Case 3:*  $(\mathfrak{g}, \tilde{\mathfrak{g}}_+)$  are of types  $E_6, F_4$ .

$$\tau(\alpha_i) = \begin{cases} \alpha_{\tau-i} & \text{if } i = 1, 2, 5, 6 \\ \alpha_i & \text{if } i = 3, 4 \end{cases} \quad \text{and } \pi(\alpha_i) = \begin{cases} \tilde{\alpha}_{5-i} & \text{if } i = 3, 4 \\ \tilde{\alpha}_3 & \text{if } i = 2, 5 \\ \tilde{\alpha}_4 & \text{if } i = 1, 6 \end{cases}$$

The highest root in type  $E_6$  is  $\theta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$ . In type  $F_4$ , we have  $\tilde{\theta} = 2\tilde{\alpha}_1 + 3\tilde{\alpha}_2 + 4\tilde{\alpha}_3 + 2\tilde{\alpha}_4$  and  $\tilde{\phi} = \tilde{\alpha}_1 + 2\tilde{\alpha}_2 + 3\tilde{\alpha}_3 + 2\tilde{\alpha}_4$ . Both  $\gamma := \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$  and  $\tau(\gamma)$  are mapped to  $\tilde{\phi}$  by  $\pi$ .

Observe that for  $H \in \mathfrak{h}$ ,

$$[H, E_{-\gamma}^-] = \gamma(H)E_{-\gamma} - \tau(\gamma)(H)E_{\tau(\gamma)} = \gamma(H)E_{-\gamma} - \gamma(\tau^*(H))E_{\tau(\gamma)}$$

because  $\tau$  is an involution on  $\mathfrak{g}$ . In particular, if  $\tilde{H} \in (\pi^*)^{-1}(\mathfrak{h})$ , then  $[\pi^*(\tilde{H}), E_{-\gamma}^-] = \gamma(\pi^*(\tilde{H}))E_{-\gamma}^- = \tilde{\phi}(\tilde{H})E_{-\gamma}^-$ . Hence  $E_{-\gamma}^-$  is a weight vector of  $\mathfrak{g}_-$  as a representation of  $\mathfrak{h} \cap \mathfrak{g}_+$ , with weight  $\tilde{\phi}$ .

We show that  $E_{-\gamma}^-$  generates all of  $\mathfrak{g}_-$  under the adjoint action of  $\mathfrak{g}_+$ . Suppose  $\alpha_i$  is a simple root of  $\mathfrak{g}$  not fixed by  $\tau$ . If  $i = 2$  or  $i = 6$ , then  $\gamma - \alpha_i$  is a root, so that  $[E_{-\gamma+\alpha_i}, E_{-\gamma}^-]$  or  $[E_{-\gamma+\alpha_i}^+, E_{-\gamma}^-]$  is a

nonzero multiple of  $E_{\alpha_i}^-$ . Subtracting any of the other four simple roots from  $\gamma$  gives a sum of positive roots, and through similar procedures as above,  $E_{\alpha_i}^-$  can be found in  $[\mathfrak{g}_+, E_\gamma^-]$ . We know  $\alpha_i - \tau(\alpha_i)$  is not a root because either  $\alpha_i = \tau(\alpha_i)$  or  $\alpha_i$  is orthogonal to  $\tau(\alpha_i)$ . So  $[E_{-\alpha_i}^+, E_{\alpha_i}^-] = H_{\alpha_i} - H_{\tau(\alpha_i)}$ . Finally, for  $\alpha_i$  a simple root not fixed by  $\tau$ ,

$$\begin{aligned} [E_{-\alpha_i}^+, H_{\alpha_i} - H_{\tau(\alpha_i)}] &= [E_{-\alpha_i}, H_{\alpha_i}] + [E_{-\tau(\alpha_i)}, H_{\alpha_i}] - [E_{-\alpha_i}, H_{\tau(\alpha_i)}] - [E_{-\tau(\alpha_i)}, H_{\tau(\alpha_i)}] \\ &= -\alpha_i(H_{\alpha_i})E_{-\alpha_i} - \tau(\alpha_i)(H_{\alpha_i})E_{-\tau(\alpha_i)} + \alpha_i(H_{\tau(\alpha_i)})E_{-\alpha_i} + \tau(\alpha_i)(H_{\tau(\alpha_i)})E_{-\tau(\alpha_i)} \\ &= (-\alpha_i(H_{\alpha_i}) + \alpha_i(H_{\tau(\alpha_i)}))(E_{-\alpha_i} - E_{-\tau(\alpha_i)}). \end{aligned}$$

Because  $\alpha_i$  is orthogonal to  $\tau(\alpha_i)$ , we know that  $-\alpha_i(H_{\alpha_i}) + \alpha_i(H_{\tau(\alpha_i)}) \neq 0$ . For every simple root  $\alpha_i$  not fixed by  $\tau$ , we see that  $E_{\alpha_i}^-, E_{-\alpha_i}^-, H_{\alpha_i} - H_{\tau(\alpha_i)} \in [\mathfrak{g}_+, E_\gamma^-]$ . By bracketing with  $E_{\alpha_j}$  or  $E_{\alpha_j}^+$  to add roots fixed or not fixed by  $\tau$  respectively, any  $E_{\alpha}^-$  can be generated, and similarly  $E_{-\alpha}^-$ . Any  $H \in \mathfrak{h} \cap \mathfrak{g}_-$  is a linear combination of  $H_{\alpha_i} - H_{\tau(\alpha_i)}$ . Therefore  $\mathfrak{g}_- = [\mathfrak{g}_+, E_\gamma^-]$ .

We prove that  $E_\gamma^-$  is maximal in  $\mathfrak{g}_-$ . Let  $\alpha$  be a positive root of  $\mathfrak{g}$ . If  $\alpha = \tau(\alpha)$ , then  $[E_\alpha, E_\gamma^-] = E_{\gamma+\alpha}^- = 0$  because no such root  $\gamma + \alpha$  exists. If  $\alpha \neq \tau(\alpha)$ , then  $[E_\alpha^+, E_\gamma^-] = E_{\gamma+\alpha}^- = 0$  because either  $\gamma + \alpha$  is not a root or  $\gamma + \alpha$  is fixed by  $\tau$ .  $\square$

*Remark 2.6.2.* This implies that as  $\mathfrak{g}_+$ -modules, we have  $\mathfrak{g}_+ \simeq V_{\tilde{\theta}}$ ,  $\mathfrak{g}_- \simeq V_{\tilde{\phi}}$ , and so  $\mathfrak{g} \simeq V_{\tilde{\theta}} \oplus V_{\tilde{\phi}}$  (here and in the following we identify  $\mathfrak{g}_+$  with  $\tilde{\mathfrak{g}}_+$ ).

**Lemma 2.6.3.** *If  $\mathfrak{g}$  is a simple Lie algebra with a folding involution  $\tau$  coming from its Dynkin diagram, then fundamental invariants of  $\mathfrak{g}$  can be chosen such that for every fundamental invariant  $f$ , either  $\tau(f) = f$  or  $\tau(f) = -f$ .*

*Proof.* Suppose we have a complete set  $S$  of fundamental invariants. Let  $f \in S$  be such that  $\tau(f) \neq \pm f$ . We know  $f$  is not a product of lower-degree invariants, so  $\tau(f)$  cannot be a product of lower-degree invariants because  $\tau(\tau(f)) = f$ . Then  $\tau(f)$  is another fundamental invariant of degree  $d$ . Because we started with a full set of fundamental invariants  $S$ , there must be some  $g \in S$  not equal to  $f$  such that  $g \in \text{span}(\{f, \tau(f)\})$ . We replace  $f$  and  $g$  by  $f - \tau(f)$  and  $f + \tau(f)$ , because  $\tau(f - \tau(f)) = -(f - \tau(f))$  and  $\tau(f + \tau(f)) = f + \tau(f)$ . This replacement allows us to get a new complete set of fundamental invariants  $S'$  where every invariant is sent to a multiple of itself by the involution.  $\square$

**Theorem 2.6.4.** *Let  $f_1, \dots, f_n$  be a complete set of fundamental invariants for  $\mathfrak{g}$  such that each fundamental invariant satisfies either  $\tau(f_i) = f_i$  or  $\tau(f_i) = -f_i$ . Let  $S_+ = \{f_i \mid \tau(f_i) = f_i\}$  and  $S_- = \{f_i \mid \tau(f_i) = -f_i\}$ . Then:*

1. The set of restrictions of invariants in  $S_+$  to  $\mathfrak{g}_+$  is a complete set of fundamental invariants for  $\mathfrak{g}_+$ .
2. The restriction of  $[f_i]$  to  $\mathbb{C}[\mathcal{N}_+]$  is isomorphic to either  $V_{\tilde{\theta}}$  or  $V_{\tilde{\phi}}$  depending on whether  $f_i \in S_+$  or  $f_i \in S_-$ , respectively.
3. The exponents of  $\mathfrak{g}_+$  are  $\{m_i \mid f_i \in S_+\}$  and the generalized exponents for  $\tilde{\phi}$  of  $\mathfrak{g}_+$  are  $\{m_i \mid f_i \in S_-\}$ .

*Proof.* Because the Jacobian of the fundamental invariants is rank  $n$  when evaluated at a principal nilpotent element and the principal nilpotent element  $\gamma = \sum_i E_{\alpha_i}$  is contained in  $\mathfrak{g}_+$ , we have that the partial derivatives of the  $f_i$  are linearly independent on the nilpotent cone  $\mathcal{N}_+$  of  $\mathfrak{g}_+$ . If  $f_i \in S_+$ , then partial derivatives of  $f_i$  with respect to coordinates of  $\mathfrak{g}_-$  must be zero. Then not all partial derivatives of  $f_i$  with respect to coordinates of  $\mathfrak{g}_+$  are zero, and these will form a copy of the adjoint representation of  $\mathfrak{g}_+$ . A similar line of reasoning gives us that partial derivatives of  $f_i \in S_-$  with respect to coordinates of  $\mathfrak{g}_-$  span a copy of  $V_{\tilde{\theta}}$ . The dimension of the  $T$ -invariant part of  $V_\lambda$  is equal to the number of generalized exponents for  $\lambda$  [12], thus we have from  $\dim V_{\tilde{\theta}}^T + V_{\tilde{\phi}}^T = \dim \mathfrak{g}_2^T$  that the number of exponents is completely accounted for.  $\square$

As a corollary, we give another proof of a combinatorial method to find generalized exponents for  $\phi$ . Here we use the Kostant-Shapiro formula [12] for the exponents of a simple Lie algebra: Let  $\mu$  be the partition of  $|\Phi^+|$  given by

$$\mu_j = |\{\alpha \in \Phi^+ \mid \text{ht}(\alpha) = j\}|.$$

Then the transpose partition  $\mu'$  gives the exponents:  $m_i = (\mu')_i$ .

**Corollary 2.6.5.** [10, Theorem 4.5][23] *Let  $\Phi_s^+$  be the set of short positive roots of a multiply-laced Lie algebra  $\mathfrak{g}$  which is the folding of a simply-laced Lie algebra  $\mathfrak{g}_2$  by an involution  $\tau$ . Let  $\mu$  be the partition of  $|\Phi_s^+|$  given by*

$$\mu_j = |\{\alpha \in \Phi_s^+ \mid \text{ht}(\alpha) = j\}|.$$

*Then the transpose partition  $\mu'$  gives the generalized exponents for  $\phi$ :  $m_i^\phi = (\mu')_i$ .*

*Proof.* The Kostant-Shapiro formula holds for both  $\mathfrak{g}$  and  $\mathfrak{g}_2$ . We have also seen that there are two positive roots in  $\mathfrak{g}_2$  corresponding to each short positive root in  $\mathfrak{g}$ . If  $\lambda$  is the partition described by the Kostant-Shapiro formula for  $\mathfrak{g}_2$  and  $\nu$  is the partition for  $\mathfrak{g}$ , then  $\lambda_j = 2\mu_j + \nu_j$ . This shows

that the parts of  $\lambda'$  are the parts of  $\nu'$  and the parts of  $\mu'$  in the appropriate order. By the previous theorem, the exponents of  $\mathfrak{g}_2$  are the exponents of  $\mathfrak{g}$  as well as the generalized exponents for  $\phi$  of  $\mathfrak{g}$ . Hence the generalized exponents for  $\phi$  of  $\mathfrak{g}$  are the parts of  $\mu'$ .  $\square$

## 2.7 Copies of representations from matrix entries

### 2.7.1 Type $A_n$

Let  $M$  be the generic matrix in  $\mathfrak{gl}_{n+1}$ . Under the action of conjugation, the entries of  $M$ , which are in  $\mathbb{C}^1[\mathfrak{g}]$ , give a representation of  $\mathfrak{g} = \mathfrak{sl}_{n+1}$  that is isomorphic to  $\mathfrak{g} \oplus \mathbb{C}$ . The same is true for entries of powers of  $M$ , with entries of  $M^i$  in  $\mathbb{C}^i[\mathfrak{g}]$ . As long as  $i \leq n$ , these representations are non-zero on the nilpotent cone  $\mathcal{N}$  because the Jordan block of size  $n+1$  is the principal nilpotent element. Thus the  $n$  independent copies of  $V_\theta$  in  $\mathbb{C}[\mathcal{N}]$  are the images of the representations given by the  $M^i$ .

### 2.7.2 Type $C_n$

Let  $M$  be the generic matrix in  $\mathfrak{g} = \mathfrak{sp}_{2n} \subset \mathfrak{gl}_{2n}$ . It is well-known that  $\mathfrak{g}$  embeds in  $\mathfrak{sl}_{2n}$  (the type  $A_{2n-1}$  Lie algebra). By Theorem 2.6.4 and the above type  $A_n$  explanation, we have that entries of  $M^i$  give a nonzero copy of  $V_\theta$  in  $\mathbb{C}^i[\mathcal{N}]$  for  $i$  odd and less than  $2n$  and a nonzero copy of  $V_\phi$  for  $i$  even and less than  $2n$ .

### 2.7.3 Type $D_n$

Let  $M$  be the generic matrix in  $\mathfrak{g} = \mathfrak{so}_{2n} \subset \mathfrak{gl}_{2n}$ . Here,  $\mathfrak{sl}_{2n}$  is isomorphic as a representation of  $\mathfrak{g}$  to  $V_\theta \oplus V_{2\varpi_1}$  where  $\varpi_1$  is the fundamental weight corresponding to  $\alpha_1$ . We see that the entries of  $M^i$  give a nonzero copy of  $V_\theta$  in  $\mathbb{C}^i[\mathcal{N}]$  for  $i$  odd and less than  $2n-2$ .

### 2.7.4 Type $B_n$

Let  $M$  be the generic matrix in  $\mathfrak{g} = \mathfrak{so}_{2n+1} \subset \mathfrak{gl}_{2n+1}$ . There is a well-known embedding of  $\mathfrak{g}$  into  $\mathfrak{so}_{2n+2}$  corresponding to the folding of  $D_{n+1}$  to  $B_n$ . By Theorem 2.6.4 and the above type  $D_n$  explanation, we see that entries of  $M^i$  give a nonzero copy of  $V_\theta$  in  $\mathbb{C}^i[\mathcal{N}]$  for  $i$  odd and less than  $2n$ . In order to get the nonzero copy of  $V_\phi$  in  $\mathbb{C}^n[\mathcal{N}]$ , we need the derivatives of the Pfaffian of  $\mathfrak{so}_{2n+2}$  along the embedded  $\mathfrak{g}$ , as indicated by Theorem 2.6.4.



## 2.8 Case-by-case results

Let  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  be a faithful representation of  $\mathfrak{g}$  of minimal dimension and let  $M$  be the generic matrix in  $\rho(\mathfrak{g}) \subset \mathfrak{gl}(V)$ . This will allow us to give explicit descriptions of fundamental invariants and copies of the adjoint representation, as explained in Section 3.2 of Chapter 3. Recall that  $[f]$  is the copy of the adjoint representation  $V_\theta$  in  $\mathbb{C}[\mathfrak{g}]$  which is the span of partial derivatives of homogeneous invariant  $f$ . In each case we will first give the explicit set  $\Theta$  in a lemma determining the degree shift of lemma 2.4.2, then refer to it only as  $\Theta$  in a subsequent lemma determining which copies of  $V_\phi$  are needed for a minimal generating set.

### 2.8.1 Type $A_n$

Let  $\mathfrak{g} := \mathfrak{sl}_{n+1}$ , which is type  $A_n$ .

**Lemma 2.8.1.**  $H_{\{\alpha_3, \dots, \alpha_{2s-1}\}}^0(\alpha_1)[-1] \simeq H_{\{\alpha_3, \dots, \alpha_{2s-1}\}}^0(\phi)[-n-1+s]$

*Proof.* We know that  $\phi = \alpha_1 + \alpha_2 + \dots + \alpha_n$ . Using the  $A_2$  move  $s-1$  times on  $\alpha_3, \dots, \alpha_{2s-1}$  consecutively, we get

$$H_{\{\alpha_3, \dots, \alpha_{2s-1}\}}^i(\alpha_1)[-1] = H_{\{\alpha_2, \dots, \alpha_{2s-2}\}}^i\left(\sum_{i=1}^{2s-1} \alpha_i\right)[-s-2].$$

Then the  $A_1$  move can be used to add the  $n - (2s - 1)$  remaining simple roots, giving

$$H_{\{\alpha_3, \dots, \alpha_{2s-1}\}}^i(\alpha_1)[-1] = H_{\{\alpha_3, \dots, \alpha_{2s-1}\}}^i(\phi)[-n-1+s].$$

□

**Lemma 2.8.2.** *The ideal  $I_\Theta^\emptyset$  is generated by a copy of  $V_\phi = V_\theta$  in  $\mathbb{C}^{n-s+1}[\mathcal{N}]$ .*

*Proof.* By the previous lemma and Theorem 2.4.1, the image of  $[f_{n-s+1}]$  in  $\mathbb{C}[\overline{\mathcal{O}}_{\Theta'}]$  generates  $I_{\Theta'}^{\Theta'}$ . By Theorem 2.5.2 and Theorem 2.5.3, the ideal generated by the image of  $[f_{n-s+1}]$  in  $\mathbb{C}[\mathcal{N}]$  also contains the images of  $[f_j]$  in  $\mathbb{C}[\mathcal{N}]$  for  $n-s+1 \leq j \leq n$  because the differences  $d_j - d_{n-s+1} + 1$  are  $1, 2, 3, \dots, s$ , which are exponents of  $\mathfrak{g}$ . Since  $I_\Theta^\emptyset = I_{\Theta'}^{\Theta'} + I_{\Theta'}^\emptyset$ , and by induction  $I_{\Theta'}^\emptyset$  is generated by a copy of  $V_\phi$  in degree  $n-s+2$ , it follows that the image of  $[f_{n-s+1}]$  generates  $I_\Theta^\emptyset$ . □

**Corollary 2.8.3.** *A minimal generating set for  $J_\Theta$  is given by a basis of the entries of  $M^{n-s+1}$  and the fundamental invariants  $\text{Tr}(M^2), \text{Tr}(M^3), \dots, \text{Tr}(M^{n-s})$ .*

## 2.8.2 Type $C_n$

Let  $\mathfrak{g} := \mathfrak{sp}_{2n} \subset \mathfrak{sl}_{2n}$ , which is type  $C_n$ .

**Lemma 2.8.4.**  $H_{\{\alpha_3, \dots, \alpha_{2s-1}\}}^0(\alpha_1)[-1] \simeq H_{\{\alpha_3, \dots, \alpha_{2s-1}\}}^0(\phi)[-2n + 2s]$

*Proof.* We know that  $\phi = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \dots + 2\alpha_{n-1} + \alpha_n$ . The first step in our algorithm is to use  $s - 1$  type  $A_2$  moves on  $\alpha_3, \dots, \alpha_{2s-1}$  consecutively, getting that

$$H_{\{\alpha_3, \dots, \alpha_{2s-1}\}}^i(\alpha_1)[-1] \simeq H_{\{\alpha_2, \dots, \alpha_{2s-2}\}}^i\left(\sum_{i=1}^{2s-1} \alpha_i\right)[-s].$$

Next, we use  $n - 1 - (2s - 1) = n - 2s$  type  $A_1$  moves to add on  $\alpha_{2s}, \dots, \alpha_{n-1}$  consecutively, so that

$$H_{\{\alpha_3, \dots, \alpha_{2s-1}\}}^i(\alpha_1)[-1] \simeq H_{\{\alpha_2, \dots, \alpha_{2s-2}\}}^i\left(\sum_{i=1}^{n-2} \alpha_i\right)[-n + s].$$

Since  $\langle \alpha_{n-1}, \alpha_n^\vee \rangle = -1$ , we get as well

$$H_{\{\alpha_3, \dots, \alpha_{2s-1}\}}^i(\alpha_1)[-1] \simeq H_{\{\alpha_2, \dots, \alpha_{2s-2}\}}^i\left(\sum_{i=1}^n \alpha_i\right)[-n + s - 1].$$

Once that is added, we use another  $n - 2s$  type  $A_1$  moves to add a second copy of each of the roots  $\alpha_{n-1}, \dots, \alpha_{2s}$  consecutively, so that

$$H_{\{\alpha_3, \dots, \alpha_{2s-1}\}}^i(\alpha_1)[-1] \simeq H_{\{\alpha_2, \dots, \alpha_{2s-2}\}}^i\left(\sum_{i=1}^{2s-1} \alpha_i + 2 \sum_{i=2s}^{n-2} \alpha_i + \alpha_{n-1} + \alpha_n\right)[-2n + 3s - 1].$$

Finally, we reverse our original  $s - 1$  type  $A_2$  moves to get to  $\phi$ :

$$H_{\{\alpha_3, \dots, \alpha_{2s-1}\}}^i(\alpha_1)[-1] \simeq H_{\{\alpha_3, \dots, \alpha_{2s-1}\}}^i(\phi)[-2n + 2s].$$

□

**Lemma 2.8.5.** *The ideal  $I_{\Theta}^{\emptyset}$  is generated by a copy of  $V_{\phi}$  in  $\mathbb{C}^{2n-2s}[\mathcal{N}]$ .*

*Proof.* By the previous lemma and Theorem 2.4.1, a copy of  $V_{\phi}$  in  $\mathbb{C}^{2n-2s}[\overline{\mathcal{O}}_{\Theta'}]$  generates  $I_{\Theta}^{\Theta'}$ . By Theorem 2.6.4 we know that copies of  $V_{\theta}$  and  $V_{\phi}$  in  $\mathbb{C}[\mathcal{N}]$  are images of  $[F_i]$  where  $F_1, \dots, F_{2n-1}$  are the fundamental invariants of the type  $A_{2n-1}$  Lie algebra  $\mathfrak{g}_2$  of which  $\mathfrak{g}$  is a folding. In particular, the odd-degree fundamental invariants of  $A_{2n-1}$  are those which contribute copies of  $V_{\phi}$  for  $\mathfrak{g}$ ,

because these are in the  $-1$ -eigenspace of the involution. By Theorem 2.5.2 and Theorem 2.5.3, the ideal generated by the image of  $[F_{2n-2s+1}]$  in  $\mathbb{C}[\mathcal{N}]$  also contains the images of  $[F_{2j-1}]$  in  $\mathbb{C}[\mathcal{N}]$  for  $n-s+1 \leq j \leq n$  because the differences in degrees are  $2j-1 - (2n-2s+1) + 1 = 1, 3, \dots, 2s-1$ , which are exponents of  $\mathfrak{g}$  and  $\mathfrak{g}_2$ . Since  $I_{\Theta}^{\emptyset} = I_{\Theta}^{\Theta'} + I_{\Theta}^{\emptyset}$ , and by induction  $I_{\Theta}^{\emptyset}$  is generated by a copy of  $V_{\phi}$  in degree  $2n-2s+2$ , it follows that the image of  $[F_{2n-2s+1}]$  in  $\mathbb{C}[\mathcal{N}]$  generates  $I_{\Theta}^{\emptyset}$ .  $\square$

**Corollary 2.8.6.** *A minimal generating set for  $J_{\Theta}$  is given by a basis of the entries of  $M^{2n-2s}$  and the fundamental invariants  $\text{Tr}(M^2), \text{Tr}(M^4), \dots, \text{Tr}(M^{2n-2s-2})$ .*

### 2.8.3 Type $D_n$

#### 2.8.3.1 First family, not very even

First we consider nilpotent orbits in type  $D_n$  whose partitions have two unequal parts. Let  $\mathfrak{g} := \mathfrak{so}_{2n} \subset \mathfrak{sl}_{2n}$ , which is type  $D_n$ .

**Lemma 2.8.7.**  $H_{\{\alpha_3, \dots, \alpha_{2s-1}\}}^0(\alpha_1)[-1] \simeq H_{\{\alpha_3, \dots, \alpha_{2s-1}\}}^0(\phi)[-2n+2s+1]$

*Proof.* We know that  $\phi = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ . The first step in our algorithm is to use  $s-1$  type  $A_2$  moves on  $\alpha_3, \dots, \alpha_{2s-1}$  consecutively, getting that

$$H_{\{\alpha_3, \dots, \alpha_{2s-1}\}}^i(\alpha_1)[-1] \simeq H_{\{\alpha_2, \dots, \alpha_{2s-2}\}}^i\left(\sum_{i=1}^{2s-1} \alpha_i\right)[-s].$$

Next, we use  $n-2 - (2s-1) = n-2s-1$  type  $A_1$  moves to add on  $\alpha_{2s}, \dots, \alpha_{n-2}$  consecutively, so that

$$H_{\{\alpha_3, \dots, \alpha_{2s-1}\}}^i(\alpha_1)[-1] \simeq H_{\{\alpha_2, \dots, \alpha_{2s-2}\}}^i\left(\sum_{i=1}^{n-2} \alpha_i\right)[-n+s+1].$$

Both  $\alpha_{n-1}$  and  $\alpha_n$  pair to  $-1$  with this  $\lambda$  and can be added in either order by  $A_1$  moves, giving

$$H_{\{\alpha_3, \dots, \alpha_{2s-1}\}}^i(\alpha_1)[-1] \simeq H_{\{\alpha_2, \dots, \alpha_{2s-2}\}}^i\left(\sum_{i=1}^n \alpha_i\right)[-n+s-1].$$

Once these are added, we use another  $n-2s-1$  type  $A_1$  moves to add a second copy of  $\alpha_{n-2}, \dots, \alpha_{2s}$  consecutively, so that

$$H_{\{\alpha_3, \dots, \alpha_{2s-1}\}}^i(\alpha_1)[-1] \simeq H_{\{\alpha_2, \dots, \alpha_{2s-2}\}}^i\left(\sum_{i=1}^{2s-1} \alpha_i + 2 \sum_{i=2s}^{n-2} \alpha_i + \alpha_{n-1} + \alpha_n\right)[-2n+3s].$$

Finally, we reverse our original  $s-1$  type  $A_2$  moves to get to  $\phi$ :

$$H_{\{\alpha_3, \dots, \alpha_{2s-1}\}}^i(\alpha_1)[-1] \simeq H_{\{\alpha_3, \dots, \alpha_{2s-1}\}}^i(\phi)[-2n+2s+1].$$

□

**Lemma 2.8.8.** *The ideal  $I_{\Theta}^{\emptyset}$  is generated by a copy of  $V_{\phi} = V_{\theta}$  in  $\mathbb{C}^{2n-2s-1}[\mathcal{N}]$ .*

*Proof.* By the previous lemma and Theorem 2.4.1, the image of  $[f_{n-s+1}]$  in  $\mathbb{C}[\overline{\mathcal{O}}_{\Theta'}]$  generates  $I_{\Theta'}^{\Theta'}$ . By Theorem 2.5.2 and Theorem 2.5.3, the ideal generated by the image of  $[f_{n-s+1}]$  in  $\mathbb{C}[\mathcal{N}]$  also contains the images of  $[f_j]$  in  $\mathbb{C}[\mathcal{N}]$  for  $n-s+1 \leq j \leq n$  because the differences  $d_j - d_{n-s+1} + 1$  are  $1, 3, \dots, 2s-1$ , which are exponents of  $\mathfrak{g}$ . Since  $I_{\Theta}^{\emptyset} = I_{\Theta'}^{\Theta'} + I_{\Theta}^{\emptyset}$ , and by induction  $I_{\Theta}^{\emptyset}$  is generated by a copy of  $V_{\phi}$  in degree  $2n-2s+1$ , it follows that the image of  $[f_{n-s+1}]$  generates  $I_{\Theta}^{\emptyset}$ . □

**Corollary 2.8.9.** *A minimal generating set for  $J_{\Theta}$  is given by a basis of the entries of  $M^{2n-2s-1}$  and the fundamental invariants  $\text{Tr}(M^2), \text{Tr}(M^4), \dots, \text{Tr}(M^{2n-2s-2})$  and  $\text{Pf}(M)$ .*

### 2.8.3.2 Second family

Now we consider the nilpotent orbits in type  $D_n$  with partition  $[2n-2s+1, 2s-3, 1, 1]$  for  $2 \leq s \leq \lfloor \frac{n+2}{2} \rfloor$ .

#### 2.8.3.3 Partition $[2n-2s+1, 2s-3, 1, 1]$

**Lemma 2.8.10.**  $H_{\{\alpha_{n-2s+3}, \dots, \alpha_{n-1}\}}^0(\alpha_n)[-1] = H_{\{\alpha_1, \alpha_{n-2s+5}, \dots, \alpha_{n-1}\}}^0(\phi)[-n+1]$

*Proof.* To reach  $\phi$  we must add on  $\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-2} + \alpha_{n-1}$ , but the moves needed to do so vary based on  $s$ . If  $s = 2$ , then using  $n-2$  consecutive  $A_2$  moves, we get that

$$H_{\{\alpha_{n-1}\}}^0(\alpha_n)[-1] \cong H_{\{\alpha_1\}}^0(\phi)[-n+1]$$

If  $s > 2$ , then the first  $s-2$  moves are type  $A_3$ , giving

$$H_{\{\alpha_{n-2s+3}, \dots, \alpha_{n-1}\}}^0(\alpha_n)[-1] = H_{\{\alpha_{n-2s+3}, \dots, \alpha_{n-1}\}}^0(\alpha_{n-2s+3} + 2\alpha_{n-2s+4} + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n)[-2s+3]$$

Then  $n-2s+2$  type  $A_2$  moves are used to add the remaining roots:

$$H_{\{\alpha_{n-2s+3}, \dots, \alpha_{n-1}\}}^0(\alpha_n)[-1] = H_{\{\alpha_1, \alpha_{n-2s+5}, \dots, \alpha_{n-1}\}}^0(\phi)[-n+1]$$

□

**Lemma 2.8.11.** *When  $|\Theta| = s = 2$ , the ideal  $I_{\Theta}^{\emptyset}$  is generated by a copy of  $V_{\phi} = V_{\theta}$  in  $\mathbb{C}^{n-1}[\mathcal{N}]$ .*

*When  $|\Theta| = s > 2$ , the ideal  $I_{\Theta}^{\emptyset}$  is not generated by a single copy of  $V_{\phi} = V_{\theta}$ , but by copies in  $\mathbb{C}^{n-1}[\mathcal{N}]$  and  $\mathbb{C}^{2n-2s+1}[\mathcal{N}]$ .*

*Proof.* By the previous lemma and Theorem 2.4.1, the image of a copy of  $V_\phi$  in  $\mathbb{C}^{n-1}[\overline{\mathcal{O}}_{\Theta'}]$  generates  $I_{\Theta'}^{\Theta'}$ . Considering the Lie algebra  $D_n$  as skew-symmetric  $2n \times 2n$  matrices, we know that the rank of an element in the nilpotent orbit with partition  $[2n - 2s + 1, 2s - 3, 1, 1]$  is  $2n - 4$  by considering its Jordan form. Hence by lemma A.1.1 in Appendix A, the partial derivatives of the Pfaffian  $f_{[n/2]}$  are zero on  $\overline{\mathcal{O}}_{\Theta}$ . By Theorem 2.5.2 and Theorem 2.5.3, the ideal generated by the image of  $[f_{[n/2]}]$  in  $\mathbb{C}[\mathcal{N}]$  also contains the image of  $[f_n]$  in  $\mathbb{C}[\mathcal{N}]$ , but not the image of  $[f_{n-1}]$ . Since  $I_{\Theta}^{\emptyset} = I_{\Theta'}^{\Theta'} + I_{\Theta'}^{\emptyset}$ , and  $I_{\Theta'}^{\emptyset}$  is generated by a copy of  $V_\phi$  in degree  $2n - 1$ , it follows that the images of  $[f_{n-s+1}]$  (if  $s > 2$ ) and  $[f_{[n/2]}]$  generate  $I_{\Theta}^{\emptyset}$ .  $\square$

**Corollary 2.8.12.** *A minimal generating set for  $J'_s$  is given by a basis of the entries of  $M^{2n-2s+1}$ , a basis for  $[\text{Pf}(M)]$ , and fundamental invariants  $\text{Tr}(M^2), \text{Tr}(M^4), \dots, \text{Tr}(M^{2n-2s})$ .*

#### 2.8.3.4 Very even orbits

In the case that  $n = 2k$ , the simple Lie algebra of type  $D_n$  has two distinct nilpotent orbits with partition  $[n, n]$ . These are labeled as  $\mathcal{O}_{\Theta_1}$  and  $\mathcal{O}_{\Theta_2}$ , where the type 1 orbit is Richardson for the parabolic subalgebra generated by the set of orthogonal simple short roots  $\Theta_1 = \{\alpha_1, \alpha_3, \dots, \alpha_{n-3}, \alpha_{n-1}\}$  and the type 2 orbit is Richardson for the parabolic subalgebra generated by the set  $\Theta_2 = \{\alpha_1, \alpha_3, \dots, \alpha_{n-3}, \alpha_n\}$ .

For the first lemma where the shift is found, we show the case for  $\mathcal{O}_{\Theta_1}$ , noting that the other is similar except with all instances of  $\alpha_n$  and  $\alpha_{n-1}$  swapped.

**Lemma 2.8.13.**  $H_{\{\alpha_1, \dots, \alpha_{n-3}\}}^0(\alpha_{n-1})[-1] = H_{\{\alpha_3, \dots, \alpha_{n-3}, \alpha_n\}}^0(\phi)[-n + 1]$

*Proof.* Performing  $\frac{n}{2} - 1$  consecutive type  $A_2$  moves, we add (in pairs)  $\alpha_{n-2}, \alpha_{n-3}, \dots, \alpha_1$ , so that

$$H_{\{\alpha_1, \dots, \alpha_{n-3}\}}^0(\alpha_{n-1})[-1] = H_{\{\alpha_2, \dots, \alpha_{n-2}, \alpha_n\}}^0\left(\sum_{i=1}^{n-1} \alpha_i\right)\left[-\frac{n}{2}\right].$$

Another string of  $\frac{n}{2} - 1$  type  $A_2$  moves adds (in pairs)  $\alpha_n, \alpha_{n-2}, \alpha_{n-3}, \dots, \alpha_2$ , so that

$$H_{\{\alpha_1, \dots, \alpha_{n-3}\}}^0(\alpha_{n-1})[-1] = H_{\{\alpha_3, \dots, \alpha_{n-3}, \alpha_n\}}^0(\phi)[-n + 1].$$

$\square$

Determining the appropriate copy of  $V_\phi = V_\theta$  in degree  $n - 1$  of  $\mathbb{C}[\mathcal{N}]$  is more complicated, as  $\dim \text{Hom}(V_\lambda, \mathbb{C}^{n-1}[\mathcal{N}]) = 2$ . We cannot simply assume that any  $[f_j]$  of the correct degree will be

sufficient, as is possible when the dimension is 1. The full details of the argument are contained in Section A.1 of Appendix A, along with some linear algebra needed to establish the results. Still, we have a statement like the others in this section:

**Corollary 2.8.14.** *For  $N \in \{1, 2\}$ , a minimal generating set for  $J_{\Theta_N}$  is given by a basis of the entries of  $M^{n-1} + (-1)^{N+1}P^T$ , where  $P$  is the matrix whose entries are the derivatives of the Pfaffian, and fundamental invariants  $\text{Tr}(M^2), \text{Tr}(M^4), \dots, \text{Tr}(M^{n-2})$  and  $\text{Pf}(M)$ .*

#### 2.8.4 Type E

There are no partition types to define nilpotent orbits. Instead we use Bala-Carter notation and consider the nilpotent orbits that are Richardson for parabolic subalgebras defined by orthogonal simple short roots. In  $E_6$  these are the subregular  $E_6(a_1)$ ,  $D_5$ , and  $E_6(a_3)$ . In  $E_7$  these are the subregular  $E_7(a_1)$ ,  $E_7(a_2)$ ,  $E_7(a_3)$ ,  $E_6$ , and  $E_6(a_1)$ . In  $E_8$  these are the subregular  $E_8(a_1)$ ,  $E_8(a_2)$ ,  $E_8(a_3)$ , and  $E_8(a_4)$ . Broer has proved the subregular cases already, leaving 9 others. We cover each of these nilpotent orbits individually, specifying both the type of the simple Lie algebra  $\mathfrak{g}$  and the Bala-Carter designation of the orbit. Instead of denoting orbits, ideals, etc. by the set of orthogonal simple short roots generating the parabolic subalgebra from which they are induced, we use the Bala-Carter labels.

Because these three exceptional Lie algebras have only short roots, we have that  $m_j^\phi = m_j$ . The exponents of  $E_6$  are 1, 4, 5, 7, 8, 11. The exponents of  $E_7$  are 1, 5, 7, 9, 11, 13, 17. The exponents of  $E_8$  are 1, 7, 11, 13, 17, 19, 23, 29.

We label the roots with reference to the Dynkin diagrams as follows:

$$1 \ 3 \overset{2}{4} \ 5 \ 6 \ \text{ for } E_6, \quad 1 \ 3 \overset{2}{4} \ 5 \ 6 \ 7 \ \text{ for } E_7, \quad 1 \ 3 \overset{2}{4} \ 5 \ 6 \ 7 \ 8 \ \text{ for } E_8.$$

##### 2.8.4.1 $D_5$ in $E_6$

**Lemma 2.8.15.** *For the  $D_5$  nilpotent orbit in  $E_6$ ,  $H_{\{\alpha_5\}}^0(\alpha_3)[-1] \simeq H_{\{\alpha_4\}}^0(\phi)[-8]$ .*

*Proof.*

$$\begin{aligned} H_{\{\alpha_5\}}^0(\alpha_3)[-1] &\simeq H_{\{\alpha_5\}}^0(\alpha_1 + \alpha_3)[-2] \quad (\text{type } A_1) \\ &\simeq H_{\{\alpha_4\}}^0(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5)[-3] \quad (\text{type } A_2) \\ &\simeq H_{\{\alpha_2\}}^0(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5)[-4] \quad (\text{type } A_2) \\ &\simeq H_{\{\alpha_2\}}^0(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6)[-5] \quad (\text{type } A_1) \\ &\simeq H_{\{\alpha_2\}}^0(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6)[-6] \quad (\text{type } A_1) \end{aligned}$$

$$\begin{aligned} &\simeq H_{\{\alpha_2\}}^0(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6)[-7] \quad (\text{type } A_1) \\ &\simeq H_{\{\alpha_4\}}^0(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)[-8] \quad (\text{type } A_2) \end{aligned}$$

□

**Lemma 2.8.16.** *The ideal  $I_{D_5}^{E_6}$  is generated by a copy of  $V_\phi = V_\theta$  in  $\mathbb{C}^8[\mathcal{N}]$ .*

*Proof.* By the previous lemma and Theorem 2.4.1, the image of  $[f_5]$  in  $\mathbb{C}[\overline{\mathcal{O}}_{E_6(a_1)}]$  generates  $I_{D_5}^{E_6(a_1)}$ . By Theorem 2.5.2 and Theorem 2.5.3, the ideal generated by the image of  $[f_5]$  in  $\mathbb{C}[\mathcal{N}]$  also contains the images of  $[f_j]$  in  $\mathbb{C}[\mathcal{N}]$  for  $j \in \{5, 6\}$  because the differences  $d_j - d_5 + 1$  are 1 and 4 which are exponents of  $\mathfrak{g}$ . Since  $I_{D_5}^{E_6} = I_{D_5}^{E_6(a_1)} + I_{E_6(a_1)}^{E_6}$  and by Broer's subregular argument  $I_{E_6(a_1)}^{E_6}$  is generated by a copy of  $V_\phi$  in degree 11, it follows that the image of  $[f_5]$  generates  $I_{D_5}^{E_6}$ . □

**Corollary 2.8.17.** *The  $D_5$  nilpotent orbit in  $E_6$  is minimally generated by fundamental invariants  $\text{Tr}(M^2), \text{Tr}(M^5), \text{Tr}(M^6), \text{Tr}(M^8)$  and a basis of  $[\text{Tr}(M^9)]$ .*

#### 2.8.4.2 $E_6(a_3)$ in $E_6$

**Lemma 2.8.18.** *For the  $E_6(a_3)$  nilpotent orbit in  $E_6$ ,  $H_{\{\alpha_2, \alpha_5\}}^0(\alpha_3)[-1] \simeq H_{\{\alpha_4, \alpha_6\}}^0(\phi)[-7]$ .*

*Proof.*

$$\begin{aligned} H_{\{\alpha_2, \alpha_5\}}^0(\alpha_3)[-1] &\simeq H_{\{\alpha_2, \alpha_5\}}^0(\alpha_1 + \alpha_3)[-2] \quad (\text{type } A_1) \\ &\simeq H_{\{\alpha_2, \alpha_5\}}^0(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5)[-4] \quad (\text{type } A_3) \\ &\simeq H_{\{\alpha_2, \alpha_6\}}^0(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6)[-5] \quad (\text{type } A_2) \\ &\simeq H_{\{\alpha_2, \alpha_6\}}^0(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6)[-6] \quad (\text{type } A_1) \\ &\simeq H_{\{\alpha_4, \alpha_6\}}^0(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6)[-7] \quad (\text{type } A_2) \end{aligned}$$

□

**Lemma 2.8.19.** *The ideal  $I_{E_6(a_3)}^{E_6}$  is not generated by a copy of  $V_\phi = V_\theta$  in  $\mathbb{C}^8[\mathcal{N}]$ , but by this and a copy of  $V_\phi$  in  $\mathbb{C}^9[\mathcal{N}]$ .*

*Proof.* By the previous lemma and Theorem 2.4.1, the image of  $[f_4]$  in  $\mathbb{C}[\overline{\mathcal{O}}_{D_5}]$  generates  $I_{E_6(a_3)}^{D_5}$ . By Theorem 2.5.2 and Theorem 2.5.3, the ideal generated by the image of  $[f_4]$  in  $\mathbb{C}[\mathcal{N}]$  also contains the images of  $[f_j]$  in  $\mathbb{C}[\mathcal{N}]$  for  $j \in \{4, 6\}$  but not for  $j = 5$  because the differences  $d_4 - d_4 + 1 = 1$ ,  $d_5 - d_4 + 1 = 2$ , and  $d_6 - d_4 + 1 = 4$ . However, we already showed  $I_{D_5}^{E_6}$  contains  $[f_5]$  and  $[f_6]$ . Since  $I_{E_6(a_3)}^{E_6} = I_{E_6(a_3)}^{D_5} + I_{D_5}^{E_6}$ , it follows that the images of  $[f_4]$  and  $[f_5]$  generate  $I_{E_6(a_3)}^{E_6}$ . □

**Corollary 2.8.20.** *The  $E_6(a_3)$  nilpotent orbit in  $E_6$  is minimally generated by fundamental invariants  $\text{Tr}(M^2), \text{Tr}(M^5), \text{Tr}(M^6)$  and bases of  $[\text{Tr}(M^8)]$  and  $[\text{Tr}(M^9)]$ .*

### 2.8.4.3 $E_7(a_2)$ in $E_7$

**Lemma 2.8.21.** *For the  $E_7(a_2)$  nilpotent orbit in  $E_7$ ,  $H_{\{\alpha_4\}}^0(\alpha_6)[-1] \simeq H_{\{\alpha_4\}}^0(\phi)[-13]$ .*

*Proof.*

$$\begin{aligned}
H_{\{\alpha_4\}}^0(\alpha_6)[-1] &\simeq H_{\{\alpha_2\}}^0(\alpha_6 + \alpha_7)[-2] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_5\}}^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)[-3] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_5\}}^0(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)[-4] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_5\}}^0(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)[-5] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_5\}}^0(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)[-6] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_4\}}^0(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7)[-7] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_3\}}^0(\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7)[-8] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_3\}}^0(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7)[-9] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_3\}}^0(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7)[-10] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_3\}}^0(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)[-11] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_4\}}^0(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)[-12] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_4\}}^0(2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)[-13] \quad (\text{type } A_1)
\end{aligned}$$

□

**Lemma 2.8.22.** *The ideal  $I_{E_7(a_2)}^{E_7}$  is generated by a copy of  $V_\phi = V_\theta$  in  $\mathbb{C}^{13}[\mathcal{N}]$ .*

*Proof.* By the previous lemma and Theorem 2.4.1, the image of  $[f_6]$  in  $\mathbb{C}[\overline{\mathcal{O}}_{E_7(a_1)}]$  generates  $I_{E_7(a_2)}^{E_7(a_1)}$ . By Theorem 2.5.2 and Theorem 2.5.3, the ideal generated by the image of  $[f_6]$  in  $\mathbb{C}[\mathcal{N}]$  also contains the images of  $[f_j]$  in  $\mathbb{C}[\mathcal{N}]$  for  $j \in \{6, 7\}$  because the differences  $d_j - d_6 + 1$  are 1 and 5 which are exponents of  $\mathfrak{g}$ . Since  $I_{E_7(a_2)}^{E_7} = I_{E_7(a_2)}^{E_7(a_1)} + I_{E_7(a_1)}^{E_7}$  and by Broer's subregular argument  $I_{E_7(a_1)}^{E_7}$  is generated by a copy of  $V_\phi$  in degree 17, it follows that the image of  $[f_5]$  generates  $I_{E_7(a_2)}^{E_7}$ . □

**Corollary 2.8.23.** *The  $E_7(a_2)$  nilpotent orbit in  $E_7$  is minimally generated by fundamental invariants  $\text{Tr}(M^2), \text{Tr}(M^6), \text{Tr}(M^8), \text{Tr}(M^{10}), \text{Tr}(M^{12})$  and a basis of  $[\text{Tr}(M^{14})]$ .*



#### 2.8.4.4 $E_7(a_3)$ in $E_7$

**Lemma 2.8.24.** *For the  $E_7(a_3)$  nilpotent orbit in  $E_7$ ,  $H_{\{\alpha_2, \alpha_3\}}^0(\alpha_5)[-1] \simeq H_{\{\alpha_5, \alpha_3\}}^0(\phi)[-11]$ .*

*Proof.*

$$\begin{aligned}
H_{\{\alpha_2, \alpha_3\}}^0(\alpha_5)[-1] &\simeq H_{\{\alpha_2, \alpha_3\}}^0(\alpha_5 + \alpha_6)[-2] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_2, \alpha_3\}}^0(\alpha_5 + \alpha_6 + \alpha_7)[-3] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_2, \alpha_3\}}^0(\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)[-5] \quad (\text{type } A_3) \\
&\simeq H_{\{\alpha_1, \alpha_2\}}^0(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)[-6] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_1, \alpha_2\}}^0(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7)[-7] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_1, \alpha_2\}}^0(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7)[-8] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_1, \alpha_4\}}^0(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7)[-9] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_1, \alpha_5\}}^0(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)[-10] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_3, \alpha_5\}}^0(2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)[-11] \quad (\text{type } A_2)
\end{aligned}$$

□

**Lemma 2.8.25.** *The ideal  $I_{E_7(a_3)}^{E_7}$  is not generated by a copy of  $V_\phi = V_\theta$  in  $\mathbb{C}^{11}[\mathcal{N}]$ , but by this and a copy of  $V_\phi$  in  $\mathbb{C}^{13}[\mathcal{N}]$ .*

*Proof.* By the previous lemma and Theorem 2.4.1, the image of  $[f_5]$  in  $\mathbb{C}[\overline{\mathcal{O}}_{E_7(a_2)}]$  generates  $I_{E_7(a_3)}^{E_7(a_2)}$ . By Theorem 2.5.2 and Theorem 2.5.3, the ideal generated by the image of  $[f_5]$  in  $\mathbb{C}[\mathcal{N}]$  also contains the images of  $[f_j]$  in  $\mathbb{C}[\mathcal{N}]$  for  $j \in \{5, 7\}$  but not for  $j = 6$  because the differences  $d_5 - d_5 + 1 = 1$ ,  $d_6 - d_5 + 1 = 3$ , and  $d_7 - d_5 + 1 = 5$ . However, we already showed  $I_{E_7(a_2)}^{E_7}$  contains  $[f_6]$  and  $[f_7]$ . Since  $I_{E_7(a_3)}^{E_7} = I_{E_7(a_3)}^{E_7(a_2)} + I_{E_7(a_2)}^{E_7}$ , it follows that the images of  $[f_5]$  and  $[f_6]$  generate  $I_{E_7(a_3)}^{E_7}$ . □

**Corollary 2.8.26.** *The  $E_7(a_3)$  nilpotent orbit in  $E_7$  is minimally generated by fundamental invariants  $\text{Tr}(M^2), \text{Tr}(M^6), \text{Tr}(M^8), \text{Tr}(M^{10})$  and bases of  $[\text{Tr}(M^{12})]$  and  $[\text{Tr}(M^{14})]$ .*

#### 2.8.4.5 $E_6$ in $E_7$

**Lemma 2.8.27.** *For the  $E_6$  nilpotent orbit in  $E_7$ ,  $H_{\{\alpha_2, \alpha_5\}}^0(\alpha_7)[-1] \simeq H_{\{\alpha_2, \alpha_3\}}^0(\phi)[-9]$ .*

*Proof.*

$$H_{\{\alpha_2, \alpha_5\}}^0(\alpha_7)[-1] \simeq H_{\{\alpha_2, \alpha_6\}}^0(\alpha_5 + \alpha_6 + \alpha_7)[-2] \quad (\text{type } A_2)$$

$$\begin{aligned}
&\simeq H_{\{\alpha_4, \alpha_6\}}^0(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)[-3] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_3, \alpha_6\}}^0(\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)[-4] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_1, \alpha_6\}}^0(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)[-5] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_1, \alpha_5\}}^0(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7)[-6] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_1, \alpha_4\}}^0(\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)[-7] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_1, \alpha_2\}}^0(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)[-8] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_2, \alpha_3\}}^0(2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)[-9] \quad (\text{type } A_2)
\end{aligned}$$

□

**Lemma 2.8.28.** *The ideal  $I_{E_6}^{E_7}$  is generated by a copy of  $V_\phi = V_\theta$  in  $\mathbb{C}^9[\mathcal{N}]$ .*

*Proof.* By the previous lemma and Theorem 2.4.1, the image of  $[f_4]$  in  $\mathbb{C}[\overline{\mathcal{O}}_{E_7(a_2)}]$  generates  $I_{E_6}^{E_7(a_2)}$ . By Theorem 2.5.2 and Theorem 2.5.3, the ideal generated by the image of  $[f_4]$  in  $\mathbb{C}[\mathcal{N}]$  also contains the images of  $[f_j]$  in  $\mathbb{C}[\mathcal{N}]$  for  $j \in \{4, 6, 7\}$  because the differences  $d_j - d_4 + 1$  are 1, 5, 7 which are exponents of  $\mathfrak{g}$ . Since  $I_{E_6}^{E_7} = I_{E_6}^{E_7(a_2)} + I_{E_7(a_2)}^{E_7}$  and  $I_{E_7(a_2)}^{E_7}$  is generated by a copy of  $V_\phi$  in degree 13, it follows that the image of  $[f_4]$  generates  $I_{E_6}^{E_7}$ . □

**Corollary 2.8.29.** *The  $E_6$  nilpotent orbit in  $E_7$  is minimally generated by fundamental invariants  $\text{Tr}(M^2), \text{Tr}(M^6), \text{Tr}(M^8), \text{Tr}(M^{12})$  and a basis of  $[\text{Tr}(M^{10})]$ .*

#### 2.8.4.6 $E_6(a_1)$ in $E_7$

**Lemma 2.8.30.** *For the  $E_6(a_1)$  nilpotent orbit in  $E_7$ ,  $H_{\{\alpha_1, \alpha_2, \alpha_5\}}^0(\alpha_7)[-1] \simeq H_{\{\alpha_2, \alpha_3, \alpha_6\}}^0(\phi)[-9]$ .*

*Proof.*

$$\begin{aligned}
H_{\{\alpha_1, \alpha_2, \alpha_5\}}^0(\alpha_7)[-1] &\simeq H_{\{\alpha_1, \alpha_2, \alpha_6\}}^0(\alpha_5 + \alpha_6 + \alpha_7)[-2] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_1, \alpha_4, \alpha_6\}}^0(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)[-3] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_1, \alpha_4, \alpha_6\}}^0(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7)[-5] \quad (\text{type } A_3) \\
&\simeq H_{\{\alpha_1, \alpha_4, \alpha_6\}}^0(\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)[-7] \quad (\text{type } A_3) \\
&\simeq H_{\{\alpha_1, \alpha_2, \alpha_6\}}^0(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)[-8] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_2, \alpha_3, \alpha_6\}}^0(2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7)[-9] \quad (\text{type } A_2)
\end{aligned}$$

□

**Lemma 2.8.31.** *The ideal  $I_{E_6(a_1)}^{E_7}$  is not generated by a copy of  $V_\phi = V_\theta$  in  $\mathbb{C}^9[\mathcal{N}]$ , but by this and a copy of  $V_\phi$  in  $\mathbb{C}^{11}[\mathcal{N}]$ .*

*Proof.* By the previous lemma and Theorem 2.4.1, the image of  $[f_4]$  in  $\mathbb{C}[\overline{\mathcal{O}}_{E_7(a_3)}]$  generates  $I_{E_6(a_1)}^{E_7(a_3)}$ . By Theorem 2.5.2 and Theorem 2.5.3, the ideal generated by the image of  $[f_4]$  in  $\mathbb{C}[\mathcal{N}]$  also contains the images of  $[f_j]$  in  $\mathbb{C}[\mathcal{N}]$  for  $j \in \{4, 6, 7\}$  because the differences  $d_j - d_4 + 1$  are 1, 5, 7 which are exponents of  $\mathfrak{g}$ . We see that the image of  $[f_5]$  in  $\mathbb{C}[\mathcal{N}]$  is not contained in that ideal since  $d_5 - d_4 + 1 = 3$  is not an exponent of  $\mathfrak{g}$ . However, we already showed  $I_{E_7(a_3)}^{E_7}$  contains  $[f_5]$ . Since  $I_{E_6(a_1)}^{E_7} = I_{E_6(a_1)}^{E_7(a_3)} + I_{E_7(a_3)}^{E_7}$ , it follows that the images of  $[f_4]$  and  $[f_5]$  generate  $I_{E_6(a_1)}^{E_7}$ .  $\square$

**Corollary 2.8.32.** *The  $E_6(a_1)$  nilpotent orbit in  $E_7$  is minimally generated by fundamental invariants  $\text{Tr}(M^2), \text{Tr}(M^6), \text{Tr}(M^8)$  and bases of  $[\text{Tr}(M^{10})]$  and  $[\text{Tr}(M^{12})]$ .*

#### 2.8.4.7 $E_8(a_2)$ in $E_8$

**Lemma 2.8.33.** *For the  $E_8(a_2)$  nilpotent orbit in  $E_8$ ,  $H_{\{\alpha_6\}}^0(\alpha_8)[-1] \simeq H_{\{\alpha_1\}}^0(\phi)[-23]$ .*

*Proof.*

$$\begin{aligned}
H_{\{\alpha_6\}}^0(\alpha_8)[-1] &\simeq H_{\{\alpha_7\}}^0(\alpha_6 + \alpha_7 + \alpha_8)[-2] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_7\}}^0(\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)[-3] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_7\}}^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)[-4] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_7\}}^0(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)[-5] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_7\}}^0(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)[-6] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_7\}}^0(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)[-7] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_7\}}^0(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)[-8] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_7\}}^0(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)[-9] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_7\}}^0(\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)[-10] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_7\}}^0(\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)[-11] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_7\}}^0(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)[-12] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_6\}}^0(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8)[-13] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_5\}}^0(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)[-14] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_4\}}^0(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 4\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)[-15] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_3\}}^0(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)[-16] \quad (\text{type } A_2)
\end{aligned}$$

$$\begin{aligned}
&\simeq H_{\{\alpha_1\}}^0(2\alpha_1 + 2\alpha_2 + 4\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)[-17] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_1\}}^0(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)[-18] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_1\}}^0(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)[-19] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_1\}}^0(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)[-20] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_1\}}^0(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 2\alpha_7 + \alpha_8)[-21] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_1\}}^0(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + \alpha_8)[-22] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_1\}}^0(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8)[-23] \quad (\text{type } A_1)
\end{aligned}$$

□

**Lemma 2.8.34.** *The ideal  $I_{E_8(a_2)}^{E_8}$  is generated by a copy of  $V_\phi = V_\theta$  in  $\mathbb{C}^{23}[\mathcal{N}]$ .*

*Proof.* By the previous lemma and Theorem 2.4.1, the image of  $[f_7]$  in  $\mathbb{C}[\overline{\mathcal{O}}_{E_8(a_1)}]$  generates  $I_{E_8(a_2)}^{E_8(a_1)}$ . By Theorem 2.5.2 and Theorem 2.5.3, the ideal generated by the image of  $[f_7]$  in  $\mathbb{C}[\mathcal{N}]$  also contains the images of  $[f_j]$  in  $\mathbb{C}[\mathcal{N}]$  for  $j \in \{7, 8\}$  because the differences  $d_j - d_7 + 1$  are 1 and 7 which are exponents of  $\mathfrak{g}$ . Since  $I_{E_8(a_2)}^{E_8} = I_{E_8(a_2)}^{E_8(a_1)} + I_{E_8(a_1)}^{E_8}$  and by Broer's subregular argument  $I_{E_8(a_1)}^{E_8}$  is generated by a copy of  $V_\phi$  in degree 29, it follows that the image of  $[f_5]$  generates  $I_{E_8(a_2)}^{E_8}$ . □

**Corollary 2.8.35.** *The  $E_8(a_2)$  nilpotent orbit in  $E_8$  is minimally generated by fundamental invariants  $\text{Tr}(M^2), \text{Tr}(M^8), \text{Tr}(M^{12}), \text{Tr}(M^{14}), \text{Tr}(M^{18}), \text{Tr}(M^{20})$  and a basis of  $[\text{Tr}(M^{24})]$ .*

#### 2.8.4.8 $E_8(a_3)$ in $E_8$

**Lemma 2.8.36.** *For the  $E_8(a_3)$  nilpotent orbit in  $E_8$ ,  $H_{\{\alpha_4, \alpha_6\}}^0(\alpha_8)[-1] \simeq H_{\{\alpha_1, \alpha_4\}}^0(\phi)[-19]$ .*

*Proof.*

$$\begin{aligned}
H_{\{\alpha_4, \alpha_6\}}^0(\alpha_8)[-1] &\simeq H_{\{\alpha_4, \alpha_7\}}^0(\alpha_6 + \alpha_7 + \alpha_8)[-2] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_5, \alpha_7\}}^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)[-3] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_5, \alpha_7\}}^0(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)[-4] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_5, \alpha_7\}}^0(\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)[-5] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_5, \alpha_7\}}^0(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)[-6] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_4, \alpha_7\}}^0(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)[-7] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_4, \alpha_6\}}^0(\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8)[-8] \quad (\text{type } A_2)
\end{aligned}$$

$$\begin{aligned}
&\simeq H_{\{\alpha_3, \alpha_6\}}^0(\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8)[-9] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_3, \alpha_6\}}^0(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8)[-10] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_3, \alpha_5\}}^0(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)[-11] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_3, \alpha_5\}}^0(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)[-13] \quad (\text{type } A_3) \\
&\simeq H_{\{\alpha_3, \alpha_5\}}^0(\alpha_1 + 3\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)[-14] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_1, \alpha_5\}}^0(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)[-15] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_1, \alpha_4\}}^0(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)[-16] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_1, \alpha_4\}}^0(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 2\alpha_7 + \alpha_8)[-17] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_1, \alpha_4\}}^0(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + \alpha_8)[-18] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_1, \alpha_4\}}^0(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8)[-19] \quad (\text{type } A_1)
\end{aligned}$$

□

**Lemma 2.8.37.** *The ideal  $I_{E_8(a_3)}^{E_8}$  is not generated by a copy of  $V_\phi = V_\theta$  in  $\mathbb{C}^{19}[\mathcal{N}]$ , but by this and a copy of  $V_\phi$  in  $\mathbb{C}^{23}[\mathcal{N}]$ .*

*Proof.* By the previous lemma and Theorem 2.4.1, the image of  $[f_6]$  in  $\mathbb{C}[\overline{\mathcal{O}}_{E_8(a_2)}]$  generates  $I_{E_8(a_3)}^{E_8(a_2)}$ . By Theorem 2.5.2 and Theorem 2.5.3, the ideal generated by the image of  $[f_6]$  in  $\mathbb{C}[\mathcal{N}]$  also contains the images of  $[f_j]$  in  $\mathbb{C}[\mathcal{N}]$  for  $j \in \{6, 8\}$  but not for  $j = 7$  because the differences  $d_6 - d_6 + 1 = 1$ ,  $d_7 - d_6 + 1 = 5$ , and  $d_8 - d_6 + 1 = 7$ . However, we already showed  $I_{E_8(a_2)}^{E_8}$  contains  $[f_7]$ . Since  $I_{E_8(a_3)}^{E_8} = I_{E_8(a_3)}^{E_8(a_2)} + I_{E_8(a_2)}^{E_8}$ , it follows that the images of  $[f_6]$  and  $[f_7]$  generate  $I_{E_8(a_3)}^{E_8}$ . □

**Corollary 2.8.38.** *The  $E_8(a_3)$  nilpotent orbit in  $E_8$  is minimally generated by fundamental invariants  $\text{Tr}(M^2), \text{Tr}(M^8), \text{Tr}(M^{12}), \text{Tr}(M^{14}), \text{Tr}(M^{18})$  and bases of  $[\text{Tr}(M^{20})]$  and  $[\text{Tr}(M^{24})]$ .*

#### 2.8.4.9 $E_8(a_4)$ in $E_8$

**Lemma 2.8.39.** *For the  $E_8(a_4)$  nilpotent orbit in  $E_8$ ,  $H_{\{\alpha_1, \alpha_4, \alpha_6\}}^0(\alpha_8)[-1] \simeq H_{\{\alpha_1, \alpha_4, \alpha_6\}}^0(\phi)[-17]$ .*

*Proof.*

$$\begin{aligned}
H_{\{\alpha_1, \alpha_4, \alpha_6\}}^0(\alpha_8)[-1] &\simeq H_{\{\alpha_1, \alpha_4, \alpha_7\}}^0(\alpha_6 + \alpha_7 + \alpha_8)[-2] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_1, \alpha_5, \alpha_7\}}^0(\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)[-3] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_3, \alpha_5, \alpha_7\}}^0(\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)[-4] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_3, \alpha_5, \alpha_7\}}^0(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)[-5] \quad (\text{type } A_1)
\end{aligned}$$

$$\begin{aligned}
&\simeq H_{\{\alpha_3, \alpha_5, \alpha_7\}}^0(\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)[-7] \quad (\text{type } A_3) \\
&\simeq H_{\{\alpha_3, \alpha_5, \alpha_7\}}^0(\alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)[-9] \quad (\text{type } A_3) \\
&\simeq H_{\{\alpha_3, \alpha_5, \alpha_7\}}^0(\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)[-10] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_3, \alpha_5, \alpha_7\}}^0(\alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)[-12] \quad (\text{type } A_3) \\
&\simeq H_{\{\alpha_1, \alpha_5, \alpha_7\}}^0(2\alpha_1 + 2\alpha_2 + 4\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)[-13] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_1, \alpha_5, \alpha_7\}}^0(\alpha_8 + 2\alpha_7 + 3\alpha_6 + 4\alpha_5 + 5\alpha_4 + 3\alpha_2 + 4\alpha_3 + 2\alpha_1)[-14] \quad (\text{type } A_1) \\
&\simeq H_{\{\alpha_1, \alpha_4, \alpha_7\}}^0(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8)[-15] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_1, \alpha_4, \alpha_6\}}^0(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + \alpha_8)[-16] \quad (\text{type } A_2) \\
&\simeq H_{\{\alpha_1, \alpha_4, \alpha_6\}}^0(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8)[-17] \quad (\text{type } A_1)
\end{aligned}$$

□

**Lemma 2.8.40.** *The ideal  $I_{E_8(a_4)}^{E_8}$  is not generated by a copy of  $V_\phi = V_\theta$  in  $\mathbb{C}^{17}[\mathcal{N}]$ , but by this and a copy of  $V_\phi$  in  $\mathbb{C}^{19}[\mathcal{N}]$ .*

*Proof.* By the previous lemma and Theorem 2.4.1, the image of  $[f_5]$  in  $\mathbb{C}[\overline{\mathcal{O}}_{E_8(a_3)}]$  generates  $I_{E_8(a_4)}^{E_8(a_3)}$ . By Theorem 2.5.2 and Theorem 2.5.3, the ideal generated by the image of  $[f_5]$  in  $\mathbb{C}[\mathcal{N}]$  also contains the images of  $[f_j]$  in  $\mathbb{C}[\mathcal{N}]$  for  $j \in \{5, 7, 8\}$  but not for  $j = 6$  because the differences  $d_5 - d_5 + 1 = 1$ ,  $d_6 - d_5 + 1 = 3$ ,  $d_7 - d_5 + 1 = 7$ , and  $d_8 - d_5 + 1 = 11$ . However, we already showed  $I_{E_8(a_3)}^{E_8}$  contains  $[f_6]$ . Since  $I_{E_8(a_4)}^{E_8} = I_{E_8(a_4)}^{E_8(a_2)} + I_{E_8(a_2)}^{E_8}$ , it follows that the images of  $[f_6]$  and  $[f_7]$  generate  $I_{E_8(a_4)}^{E_8}$ . □

**Corollary 2.8.41.** *The  $E_8(a_4)$  nilpotent orbit in  $E_8$  is minimally generated by fundamental invariants  $\text{Tr}(M^2)$ ,  $\text{Tr}(M^8)$ ,  $\text{Tr}(M^{12})$ ,  $\text{Tr}(M^{14})$  and bases of  $[\text{Tr}(M^{18})]$  and  $[\text{Tr}(M^{20})]$ .*

## CHAPTER 3

### INTERSECTIONS WITH SLODOWY SLICES

#### 3.1 Slodowy slices

Let  $\mathfrak{g}$  be a simple complex Lie algebra with adjoint group  $G$ . Given a nilpotent element  $E \in \mathfrak{g}$ , we develop a transverse slice in  $\mathfrak{g}$  to the nilpotent orbit  $\mathcal{O}_E$  at the point  $E$ , which we call the *Slodowy slice*. Our interest lies primarily in the intersections of these Slodowy slices with the nilpotent varieties containing  $\mathcal{O}_E$ . Slodowy slices have significance in other ways, including the fact that they have natural Poisson structures and that their preimages under moment maps  $T^*(G/P) \rightarrow \mathfrak{g}$  are smooth symplectic varieties for parabolic subgroups  $P$ .

Let  $X$  be any variety with a regular action of an algebraic group  $G$ . A *transverse slice* in  $X$  to the orbit of  $x \in X$  at the point  $x$  is a locally closed subvariety  $S \subset X$  containing  $x$  such that the morphism  $G \times S \rightarrow X$  defined by the action of  $G$  is smooth and  $S$  has minimal dimension among subvarieties satisfying these properties. For an affine  $G$ -variety  $X$ , there is a natural construction of a transverse slice: embed  $X$  in a vector space  $V$  with a linear  $G$ -action, choose a subspace  $W \subset V$  complementary to the tangent space  $T_x(G \cdot x) = [\mathfrak{g}, x]$ , and define  $S$  to be the scheme-theoretic intersection of  $X$  with  $x + W$ . In this case,  $\dim S = \operatorname{codim}_X(G \cdot x)$  [16, Section 12.4].

Recall that an  $\mathfrak{sl}_2$ -triple is a set  $\{H, E, F\} \subset \mathfrak{g}$  such that

$$[H, E] = 2E, \quad [H, F] = -2F, \quad \text{and} \quad [E, F] = H.$$

So  $\{H, E, F\}$  span a subalgebra  $\mathfrak{a} \subset \mathfrak{g}$  isomorphic to  $\mathfrak{sl}_2$ . By the Jacobson-Morozov Theorem, such a triple can be found for any nilpotent element  $E$ . The *Slodowy slice* to the nilpotent orbit  $\mathcal{O}_E$  at  $E$  is the affine space  $\mathcal{S}_E := E + \mathfrak{g}^F$ , where  $\mathfrak{g}^F$  is the centralizer of  $F$  in  $\mathfrak{g}$ ,  $\{X \in \mathfrak{g} \mid [X, F] = 0\}$

Under the adjoint action of  $\mathfrak{a}$ , the Lie algebra  $\mathfrak{g}$  decomposes as a direct sum of irreducible  $\mathfrak{sl}_2$ -modules  $\mathfrak{g} = \bigoplus_{j=1}^N V_j$ . Let  $\lambda_j = \dim(V_j) - 1$ . By the well-known representation theory of  $\mathfrak{sl}_2$ , each irreducible  $\mathfrak{sl}_2$ -module  $V_j$  is made up of 1-dimensional  $\operatorname{ad}_H$ -eigenspaces  $V_j(k)$  with integer eigenvalues  $k = -\lambda_j, -\lambda_j + 2, \dots, \lambda_j$  such that  $\operatorname{ad}_E(V_j(k)) \subset V_j(k+2)$  and  $\operatorname{ad}_F(V_j(k)) \subset V_j(k-2)$  where  $V_j(k) = \{0\}$  if  $k > \lambda_j$  or  $k < -\lambda_j$ . Then the subspace of  $V_j$  annihilated by  $\operatorname{ad}_F$  is the  $-\lambda_j$ -eigenspace  $V_j(-\lambda_j)$ .

Now we have that  $\mathfrak{g}^F = \bigoplus_{j=1}^N V_j(-\lambda_j)$ , and from this it can be seen that the dimension of  $\mathfrak{g}^F$  (hence of  $\mathcal{S}_E$ ) is  $N$ , the number of irreducible  $\mathfrak{sl}_2$ -modules in the decomposition of  $\mathfrak{g}$ . This shows that  $\mathfrak{g}^F$  is a subspace of  $\mathfrak{g}$  complementary to  $[\mathfrak{g}, E]$ , so that the Slodowy slice  $\mathcal{S}_E$  is a transverse slice in  $\mathfrak{g}$  to  $\mathcal{O}_E$  at  $E$ , as shown in [19, Section 7.4].

The study of these slices did not originate with Slodowy. Kostant studied the slice to the regular nilpotent orbit [13], and we will make use of some of his results later. Slodowy was primarily concerned with the slice to the subregular orbit, because of some conjectures of Grothendieck about the intersection of this slice with the nilpotent cone. These conjectures were proved by Brieskorn, then extended by Slodowy [19] to all types of simple Lie algebra. In all of this work, the restriction of the adjoint quotient morphism to the slice plays a key role.

A morphism  $F : \mathbb{C}_{x_1, \dots, x_s}^s \rightarrow \mathbb{C}_{y_1, \dots, y_r}^r$  is called *quasihomogeneous of type*  $(d_1, \dots, d_r; w_1, \dots, w_s)$  with  $d_1, \dots, d_r, w_1, \dots, w_s \in \mathbb{Z}$  if every component polynomial  $y_j \circ F$  is a sum of monomials  $a_{p_1, \dots, p_s} x_1^{p_1} \cdots x_s^{p_s}$  with nonzero coefficients  $a_{p_1, \dots, p_s}$  satisfying  $\sum_{i=1}^s p_i w_i = d_j$ . The  $d_j$  are the *quasihomogeneous degrees* of  $F$ , and the  $w_i$  are the *quasihomogeneous weights* of  $F$ . If a  $\mathbb{G}_m$  action is defined on the affine spaces  $\mathbb{C}^s$  and  $\mathbb{C}^r$  of the definition by  $t \cdot (x_1, \dots, x_s) = (t^{w_1} x_1, \dots, t^{w_s} x_s)$  and  $t \cdot (y_1, \dots, y_r) = (t^{d_1} y_1, \dots, t^{d_r} y_r)$ , then  $F$  having quasihomogeneous type  $(d_1, \dots, d_s; w_1, \dots, w_r)$  is equivalent to  $F$  being  $\mathbb{G}_m$ -equivariant.

Let  $\pi : \mathfrak{g} \rightarrow \mathbb{C}^n$  be the adjoint quotient morphism defined by the fundamental invariants,  $\pi = (f_1, \dots, f_n)$ , where we recall that  $n$  is the rank of the Lie algebra  $\mathfrak{g}$  and the degrees of these homogeneous polynomials are given by  $\deg f_i = d_i = m_i + 1$  where the  $m_i$  are the exponents of  $\mathfrak{g}$  in ascending order. Consider the restriction of  $\pi$  to a Slodowy slice  $\mathcal{S}_E$ ,  $\tilde{\pi} : \mathcal{S}_E \rightarrow \mathbb{C}^n$ . We choose nonzero  $Z_j \in V_j(-\lambda_j)$ ,  $1 \leq j \leq N$ , as a basis for  $\mathfrak{g}^F$ . Let  $u_j \in \mathbb{C}[\mathcal{S}_E]$ ,  $1 \leq j \leq N$ , be coordinate functions corresponding to  $Z_j$ , so that  $X \in \mathcal{S}_E$  has the form  $X = E + \sum_{j=1}^N u_j(X) Z_j$  and  $\mathbb{C}[\mathcal{S}_E] = \mathbb{C}[u_1, \dots, u_N]$ . Hence  $\tilde{\pi}$  is a morphism of affine spaces as in the definition above.

Now we must determine the appropriate  $\mathbb{G}_m$ -action, quasihomogeneous degrees and weights that make  $\tilde{\pi} : \mathcal{S}_E \rightarrow \mathbb{C}^n$  a  $\mathbb{G}_m$ -equivariant morphism. There are two natural  $\mathbb{G}_m$  actions: first the scalar action  $\sigma(t) \cdot X = tX$  and second the action coming from the  $\text{ad}_H$  action

$$\tau(t) \cdot (E + \sum_{j=1}^N u_j(X) Z_j) = t^2 E + \sum_{j=1}^N t^{-\lambda_j} u_j(X) Z_j.$$

We combine these into  $\rho(t) = \sigma(t^2)\tau(t^{-1})$ , so that



$$\rho(t) \cdot (E + \sum_{j=1}^N u_j(X)Z_j) = E + \sum_{j=1}^N t^{\lambda_j+2} u_j(X)Z_j.$$

Then

$$f_i(\rho(t) \cdot X) = f_i(\sigma(t^2) \cdot X) = t^{2(m_i+1)} f_i(X)$$

because the polynomials  $f_i$  are  $\text{ad}_H$ -invariant. Thus  $\tilde{\pi}$  is quasihomogeneous of type  $(2(m_1 + 1), \dots, 2(m_n + 1); \lambda_1 + 2, \dots, \lambda_N + 2)$ .

If  $\mathcal{O}$  is another nilpotent orbit such that  $\mathcal{O}_E \subset \overline{\mathcal{O}}$ , we define the nilpotent Slodowy slice  $\mathcal{S}_{E,\mathcal{O}}$  to be the scheme-theoretic intersection  $\mathcal{S}_E \cap \overline{\mathcal{O}}$ . This is also a transverse slice to  $\mathcal{O}_E$  at  $E$ , but in  $\overline{\mathcal{O}}$  instead of  $\mathfrak{g}$ . Later on in Section 3.3 we will discuss explicit equations for these nilpotent Slodowy slices. These can be obtained by restricting the equations defining the nilpotent variety  $\overline{\mathcal{O}}$  to the slice  $\mathcal{S}_E$ .

### 3.2 Choices of explicit fundamental invariants

In order to study the restriction to a Slodowy slice of the equations defining a nilpotent variety, we require explicit fundamental invariants for each type of simple Lie algebra. With an explicit description of a complete set of fundamental invariants, it becomes possible to explicitly describe copies of the adjoint representation in the polynomial ring on the Lie algebra, which can also be restricted to a Slodowy slice.

There are well-known faithful representations of the classical Lie algebra types for which the fundamental invariants can be defined using characteristic polynomial coefficients, which are sums of diagonal minors of appropriate sizes. This method of defining fundamental invariants is not computationally useful for exceptional types of Lie algebra, where the faithful representations have high dimension. Instead, we will define another set of fundamental invariants using traces of powers of generic matrices in these faithful representations. Both sets of explicit fundamental invariants are used in this thesis. In both cases, an exception occurs in type  $D$  Lie algebras.

Our technique for proving that we have a complete set of fundamental invariants of  $\mathfrak{g}$  will make use of the restriction of invariants to the principal Slodowy slice. This significantly reduces the complexity of performing computations without losing any needed information, as will be shown.

As always, let  $\mathfrak{g}$  be a simple complex Lie algebra. Let  $X_1, \dots, X_{\dim \mathfrak{g}}$  be a basis for  $\mathfrak{g}$ . For  $X = \sum \lambda_i X_i$ , we define  $\partial_X = \sum \lambda_i \frac{\partial}{\partial X_i}$ .

Let  $E$  be a principal nilpotent element,  $\{H, E, F\}$  be an  $\mathfrak{sl}_2$ -triple including  $E$ , and  $\mathcal{S} = E + \mathfrak{g}^F$  be the corresponding Slodowy slice. Let  $g_i = \frac{1}{m_i!}(\partial_E)^{m_i} f_i$  for  $i = 1, \dots, n$ , where we recall that the  $m_i$  are the exponents of  $\mathfrak{g}$ . These  $g_i$  are linear functionals on  $\mathfrak{g}$  because  $d_i = m_i + 1$ .

**Theorem 3.2.1.** [13, Theorem 6] *There is a unique basis  $Z_1, \dots, Z_n$  for the centralizer  $\mathfrak{g}^F$  such that*

1.  $Z_j$  has  $\text{ad}_H$  weight  $-m_j$  (i.e.  $\text{ad}_H(Z_j) = -m_j Z_j$ ),

2.  $g_i(Z_j) = \delta_{i,j}$ , and

3.  $\partial_{Z_j} f_i|_{\mathcal{S}} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i < j \end{cases}$ .

Let  $u_1, \dots, u_n \in \mathbb{C}[\mathcal{S}]$  be coordinate functions corresponding to the  $Z_j$ , so that for all  $X \in \mathcal{S}$ ,  $X = E + \sum u_j(X)Z_j$ . Let  $\tilde{\pi} = (\tilde{f}_1, \dots, \tilde{f}_n) : \mathcal{S} \rightarrow \mathbb{C}^n$  be the restriction of the adjoint quotient to the slice  $\mathcal{S}$ .

**Theorem 3.2.2.** [19, Section 7.4, Propositions 1, 2] *The morphism  $\tilde{\pi}$  is quasihomogeneous of type  $(2m_1 + 2, \dots, 2m_n + 2; 2m_1 + 2, \dots, 2m_n + 2)$ .*

**Theorem 3.2.3.** [13, Theorem 7] *For  $i = 1, \dots, n$  there exist polynomials  $p_i$  and  $q_i$  in  $i-1$  variables without constant term such that*

$$\tilde{f}_i = u_i + p_i(u_1, \dots, u_{i-1}) \text{ and } u_i = \tilde{f}_i + q_i(\tilde{f}_1, \dots, \tilde{f}_{i-1}).$$

Hence the map  $R \rightarrow \mathbb{C}[\mathcal{S}]$  defined by  $f_i \mapsto \tilde{f}_i$  is an isomorphism and  $\tilde{\pi} : \mathcal{S} \rightarrow \mathbb{C}^n$  is an isomorphism.

With the isomorphism of  $R$  and  $\mathbb{C}[\mathcal{S}]$ , it becomes possible to use this restriction to the Slodowy slice to the principal nilpotent orbit to study  $R$ . We now give explicit descriptions of our two distinct complete sets of fundamental invariants which can be defined for any simple Lie algebra except for type  $D_n$ .

Let  $\mathfrak{g}$  be a simple complex Lie algebra with rank  $n$  that is not of type  $D_n$ . Let  $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  be a faithful representation of  $\mathfrak{g}$  of minimal dimension and let  $M$  be the generic matrix in  $\rho(\mathfrak{g}) \subset \mathfrak{gl}(V)$ . We define  $f_i^{(a)}$  to be the sum of all  $(m_i + 1) \times (m_i + 1)$  diagonal minors of  $M$  (which are coefficients of the characteristic polynomial of  $M$ ) and we define  $f_i^{(b)} = \text{Tr}(M^{m_i+1})$ , both for  $1 \leq i \leq n$ .

**Proposition 3.2.4.** *Each of the sets  $\{f_1^{(a)}, \dots, f_n^{(a)}\}$  and  $\{f_1^{(b)}, \dots, f_n^{(b)}\}$  is a complete set of fundamental invariants of  $\mathfrak{g}$ .*

*Proof.* First, observe that the proposed  $f_i^{(a)}$  are elements of  $R$  because the corresponding representation of  $G$  in  $GL(V)$  acts on  $\rho(\mathfrak{g})$  by matrix conjugation, and conjugate matrices share the same characteristic polynomial. Similarly, the proposed  $f_i^{(b)}$  are elements of  $R$  because the conjugation-invariance of the trace extends to traces of given powers.

Everything that follows is true regardless of which choice of set of polynomials  $f_i$  is made. Using the previous theorem, it is sufficient to consider the restrictions  $\tilde{f}_i$  to the principal Slodowy slice. Moreover, it is sufficient to show that there exists some polynomial  $p_i$  without constant term such that

$$\tilde{f}_i = u_i + p_i(u_1, \dots, u_{i-1}).$$

Because the  $f_i$  defined does have homogeneous degree  $m_i + 1$ , the corresponding  $\tilde{f}_i$  will have quasihomogeneous degree  $2(m_i + 1)$ . Then no  $u_j$  with  $j > i$  can occur in  $\tilde{f}_i$  because the quasihomogeneous weight of  $u_j$  is  $2m_j + 2$ , and the exponents are strictly increasing. Also, it can be seen that  $\tilde{f}_i$  does not have constant term because evaluating  $\tilde{f}_i$  with all coordinates equal to zero gives  $\tilde{f}_i(E) = f_i(E) = 0$ . All that remains is to verify that  $u_i$  occurs in  $\tilde{f}_i$ . This verification has been performed using Magma code (provided in Appendix B).  $\square$

This shows that although the polynomials  $f_i^{(a)}$  and  $f_i^{(b)}$  are distinct, these two sets of polynomials are equivalent as generators for  $R_+$ . We use both sets of fundamental invariants for different purposes throughout this thesis, specifying our choice when needed.

Neither of the above sets of polynomials contains  $n$  distinct nonzero polynomials in type  $D_n$ . With an appropriate choice of basis for  $V$ , the elements of the type  $D_n$  Lie algebra may be represented as  $2n \times 2n$  skew-symmetric matrices. For odd  $n$ , the defined  $f_i$  of degree  $n$  will be zero, because both the coefficients of odd powers of  $t$  in the characteristic polynomial and traces of odd powers of skew-symmetric matrices are zero. For even  $n$ , the exponent  $n - 1$  has multiplicity two, and both sets of defined polynomials are determined only by the exponents, so would give the same polynomial twice.

In either case,  $\det(M)$  is the square of a polynomial, where  $M$  is the generic matrix in  $\rho(\mathfrak{g})$ . This polynomial is the Pfaffian of  $M$ ,  $\text{Pf}(M)$ , and it is homogeneous of degree  $n$  and invariant under orthogonal change of basis. Recall that the exponent  $m_{\lceil n/2 \rceil}$  is chosen to always be  $n - 1$ , so we can identify the corresponding fundamental invariant with the Pfaffian.

**Proposition 3.2.5.** *Let  $\mathfrak{g}$  be a simple Lie algebra of type  $D_n$ . Let  $M$  be the generic  $2n \times 2n$  skew-symmetric matrix. Then a complete set of fundamental invariants can be defined by  $f_i = f_i^{(a)}$  (or  $f_i = f_i^{(b)}$ ) for  $i \neq \lceil n/2 \rceil$  and  $f_{\lceil n/2 \rceil} = \text{Pf}(M)$ .*

*Proof.* Outside of degree  $n$  everything is the same as the previous proposition. By restricting the given invariants to the principal Slodowy slice, we see using Magma that there are indeed linear terms in the restrictions of both of the degree  $n$  invariants. It is not true that no  $u_j$  with  $j > i$  can occur in the even  $n$  case, as  $u_{\lceil n/2 \rceil}$  can occur in  $f_{\lceil n/2 \rceil - 1}$ . This can be solved by redefining the basis for  $\mathfrak{g}^F$ .  $\square$

### 3.3 Singularities and the subregular slice

A normal surface  $X$  is said to have *rational singularities* if there exists a resolution of singularities  $\pi : Y \rightarrow X$  such that the higher direct images of  $\pi_*$  are trivial,  $R^i \pi_* \mathbb{C}[Y] = 0$  for all  $i > 0$ . This condition is independent of the choice of resolution. Such a resolution of singularities has exceptional subvariety consisting of a union of irreducible curves isomorphic to  $\mathbb{P}^1$ . We consider the minimal resolution  $\pi : Y \rightarrow X$ , which is unique up to isomorphism for the property that none of the irreducible components of the exceptional subvariety have self-intersection index  $-1$ . (The exposition in this section is based on that in chapter 6 of the work by Slodowy [19].)

We associate a graph to each resolution of singularities, based on the structure of the exceptional subvariety. Each of the irreducible curves isomorphic to  $\mathbb{P}^1$  gives a vertex, and every transverse intersection of these curves gives an edge connecting their vertices. If the resulting graph is a Dynkin diagram of type  $\Delta$ , then we say the resolution has type  $\Delta$ .

A *simple singularity* or *rational double point* is a pair  $(X, x)$  where  $X$  is a normal algebraic surface with closed point  $x$  that satisfies any of the equivalent conditions:

1.  $X$  has a rational singularity of embedding dimension 3 at  $x$ ;
2.  $X$  has a rational singularity of multiplicity 2 at  $x$ ;
3.  $X$  is of multiplicity 2 at  $x$  and the singularity can be resolved by successive blow-ups;
4. The minimal resolution of singularities of  $X$  at  $x$  has type  $A$ ,  $D$ , or  $E$ .

If the type of the minimal resolution of a simple singularity is  $\Delta$ , then we say that the simple singularity  $(X, x)$  is of type  $\Delta$ .

**Theorem 3.3.1.** *There is exactly one simple singularity of every type  $A_n$  ( $n \geq 1$ ),  $D_n$  ( $n \geq 4$ ),  $E_6$ ,  $E_7$ , and  $E_8$ , up to isomorphism. Representatives of each class of simple singularity are  $(\mathbf{V}(p), 0)$  where the surfaces  $\mathbf{V}(p) \subset \mathbb{C}^3$  are defined by the following polynomials  $p$ .*

Equation $p$	Dynkin type
$x^{n+1} + yz = 0$	$A_n$
$x^{n-1} + xy^2 + z^2 = 0$	$D_n$ ( $n \geq 4$ )
$x^4 + y^3 + z^2 = 0$	$E_6$
$x^3y + y^3 + z^2 = 0$	$E_7$
$x^5 + y^3 + z^2 = 0$	$E_8$

A *deformation* of a singularity  $(X, x)$  is a flat morphism  $\phi : (Z, z) \rightarrow (Y, y)$  such that  $(\phi^{-1}(y), z)$  is isomorphic to  $(X, x)$ . A deformation  $\phi$  is called *semi-universal* if any deformation  $\phi' : (Z', z') \rightarrow (Y', y')$  can be induced, up to isomorphism, from  $\phi$  by a base change  $\rho : (Z', z') \rightarrow (Z, z)$ , and the differential  $d\rho : T_{z'}Z' \rightarrow T_zZ$  is uniquely determined.

Grothendieck conjectured, and Brieskorn later proved, the following theorem relating these simple singularities to the simply-laced simple Lie algebras of the same types. Recall that  $\mathcal{S}_{E, \mathcal{O}}$  is our notation for the intersection of a Slodowy slice to the nilpotent orbit  $\mathcal{O}_E$  at  $E$  with the closure of another nilpotent orbit  $\mathcal{O}$  such that  $\mathcal{O}_E \subset \overline{\mathcal{O}}$ .

**Theorem 3.3.2.** *Let  $E$  be a subregular nilpotent element of a simply-laced simple Lie algebra  $\mathfrak{g}$  and let  $\mathcal{O}$  be the principal nilpotent orbit in  $\mathfrak{g}$ . Then*

1.  $(\mathcal{S}_{E, \mathcal{O}}, E)$  is a simple singularity of the same type as  $\mathfrak{g}$ .
2. The restriction of the adjoint quotient morphism to the slice  $\mathcal{S}_E$ ,  $\tilde{\pi} : (\mathcal{S}_E, E) \rightarrow (\mathbb{C}^n, 0)$ , is a semi-universal deformation of the singularity  $(\mathcal{S}_{E, \mathcal{O}}, E)$ .

Slodowy made a more thorough investigation into the conjectures and provided a different proof for Brieskorn's theorem. Then he extended this to the multiply-laced types of Lie algebra, using the fact that these can be identified as foldings of simply-laced types, where folding is described in Section 2.6 of Chapter 2.

**Theorem 3.3.3.** *Let  $E$  be a subregular nilpotent element of a multiply-laced Lie algebra  $\mathfrak{g}$  and let  $\mathcal{O}$  be the principal nilpotent orbit in  $\mathfrak{g}$ . Then*

1.  $(\mathcal{S}_{E, \mathcal{O}}, E)$  is a simple singularity whose type is that of the simply-laced Dynkin diagram for which the Dynkin diagram of  $\mathfrak{g}$  is a folding.

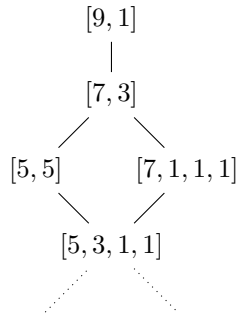
$\mathfrak{g}$	Singularity
$B_n$	$A_{2n-1}$
$C_n$	$D_{n+1}$
$F_4$	$E_6$
$G_2$	$D_4$

2. The restriction of the adjoint quotient morphism to the slice  $\mathcal{S}_E$ ,  $\tilde{\pi} : (\mathcal{S}_E, E) \rightarrow (\mathbb{C}^n, 0)$ , is a semi-universal deformation of the singularity  $(\mathcal{S}_{E, \mathcal{O}}, E)$ .

For every pair of nilpotent element  $E$  and nilpotent orbit  $\mathcal{O}$  in a simple Lie algebra with  $\text{codim}_{\overline{\mathcal{O}}} \mathcal{O}_E = 2$ ,  $(\mathcal{S}_{E, \mathcal{O}}, E)$  is a simple singularity, and the types of these have been classified in the classical types by Kraft and Procesi [15] [16] and in the exceptional types by Fu, Juteau, Levy, and Sommers [9]. In fact, the cited papers go further, classifying all singularities of  $(\mathcal{S}_{E, \mathcal{O}}, E)$  where  $\mathcal{O}_E$  is the maximal nilpotent orbit contained in  $\overline{\mathcal{O}} \setminus \mathcal{O}$ , even when the codimension is greater than 2. Those are no longer simple singularities.

By using our minimal generating sets for the defining ideals of closures of nilpotent orbits, we can determine the simple singularity types in many cases. This is done by restricting the given set of polynomials to the chosen Slodowy slice. We have seen that the adjoint quotient morphism  $\pi$ , which has the fundamental invariants of  $\mathfrak{g}$  as its components, is made quasihomogeneous when restricted to the Slodowy slice. We can use Magma (code provided in Appendix B) to get explicit equations in the coordinates of the Slodowy slice  $\mathcal{S}_E$  for the intersection  $\mathcal{S}_{E, \mathcal{O}}$ , with the only requirement being that  $\mathcal{O}$  is in one of our two families of nilpotent orbits, so that a minimal generating set for the ideal of  $\overline{\mathcal{O}}$  is known.

**Example 3.3.4.** Let  $\mathfrak{g}$  be type  $D_5$ . Based on our findings in Chapter 2, the options for  $\mathcal{O}$  are any of the nilpotent orbits with partitions in the diagram below:



We consider  $\mathcal{O}$  to be the nilpotent orbit with partition  $[5, 3, 1, 1]$  and let  $E$  be in the  $[4, 4, 1, 1]$  nilpotent orbit, which is not in either of our families. There is no nilpotent orbit in the closure ordering between these, and we expect a surface singularity. Here is the output of our Magma program, a Groebner basis for the defining ideal:

```

u[1],
u[2],
u[3],
u[4],
u[5],
u[6],
u[7],
u[8],
u[9],
u[10]*u[13] + 1/4*u[12]^2,
u[11] + 1/2*u[12]

```

Hence this is a type  $A_1$  simple surface singularity.

### 3.4 The subsubregular slice

The principal and subregular nilpotent orbits are always the unique nilpotent orbits of their dimensions, those dimensions being  $\dim \mathfrak{g} - n$  and  $\dim \mathfrak{g} - n - 2$  respectively. It is not the case that there is always a unique nilpotent orbit of dimension  $\dim \mathfrak{g} - n - 4$ , as there are multiple such nilpotent orbits in types  $C$  and  $D$ . In the types where there is a unique nilpotent orbit of this dimension, we name this the subsubregular nilpotent orbit. In types  $C$  and  $D$  we choose to give this name to the nilpotent orbit of this dimension which belongs to our first family of nilpotent orbits, which is the one with a two-part partition.

**Lemma 3.4.1.** *The quasihomogeneous degrees and weights of the restriction of the adjoint quotient morphism to the Slodowy slice to the subsubregular nilpotent orbit are given in the following table.*

	$d_1 = w_1, \dots, d_{n-2} = w_{n-2}$	$d_{n-1}, d_n$	$w_{n-1}, \dots, w_{n+4}$
$A_n, n \geq 3$	$4, 6, \dots, 2n - 2$	$2n, 2n + 2$	$2, 4, n - 1, n - 1, n + 1, n + 1$
$B_n, n \geq 3$	$4, 8, \dots, 4n - 8$	$4n - 4, 4n$	$4, 4, 2n - 4, 2n - 2, 2n - 2, 2n$
$C_3$	4	8, 12	2, 2, 2, 6, 6, 6
$C_n, n \geq 4$	$4, 8, \dots, 4n - 8$	$4n - 4, 4n$	$4, 8, 2n - 6, 2n - 4, 2n - 2, 2n$
$D_n, n \geq 5$	$4, 8, \dots, 4n - 12, 2n$	$4n - 8, 4n - 4$	$4, 8, 2n - 8, 2n - 6, 2n - 4, 2n - 2$
$E_6$	4, 10, 12, 16	18, 24	2, 6, 6, 8, 12, 12
$E_7$	4, 12, 16, 20, 24	28, 36	4, 8, 10, 12, 16, 18
$E_8$	4, 16, 24, 28, 36, 40	48, 60	8, 12, 18, 20, 24, 30
$F_4$	4, 12	16, 24	4, 4, 6, 8, 10, 12

*Proof.* The quasihomogeneous degrees of the morphism are twice the degrees of the fundamental invariants, as explained when the restriction of the adjoint quotient was shown to be quasihomogeneous. That is, the quasihomogeneous degrees are  $d_i = 2(m_i + 1)$  where  $m_i$  are the exponents of the Lie algebra. It is the quasihomogeneous weights that we must determine. We begin by making a list of  $\alpha(H)$  for every root  $\alpha$  in the root system of  $\mathfrak{g}$ , along with  $n$  zeros for  $\mathfrak{h}$ . Each of these is the  $\text{ad}_H$  eigenvalue  $k$  for a 1-dimensional eigenspace  $V_j(k)$  of some irreducible  $\mathfrak{sl}_2$ -submodule  $V_j$  of  $\mathfrak{g}$ . As discussed in Section 3.1, the quasihomogeneous weights are  $-\lambda_j + 2$  where  $\lambda_j = \dim(V_j) - 1$  is the highest weight of  $V_j$ . Our list is thus made up of sets  $\{\lambda_j, \lambda_j - 2, \dots, -\lambda_j\}$ . The simple algorithm to find the  $\lambda_j$  is to choose the highest integer in the list, remove the corresponding set of eigenvalues and repeat.

We will use induction on rank within each type to show the claimed pattern of weights. The base cases are  $n = 3, 4, 5$  for types  $B, C, D$  respectively, and in type  $A$  we require both  $n = 3$  and  $n = 4$  because even and odd rank subsubregular nilpotent orbits have significantly different weighted Dynkin diagrams.

First, consider type  $A_n$  with odd  $n$ . The weighted Dynkin diagram for  $A_3$  is  $0 - 2 - 0$ . This means that  $\alpha_1(H) = 0$ ,  $\alpha_2 = 2$ , and  $\alpha_3 = 0$  where  $\alpha_1 = e_1 - e_2$ ,  $\alpha_2 = e_2 - e_3$ , and  $\alpha_3 = e_3 - e_4$  for  $\{e_1, \dots, e_4\}$  the standard basis of linear functionals for  $\mathfrak{h}^*$ . Because the roots in classical types are sums and differences of no more than two of the basic linear functionals, we find the values of those  $e_i$  evaluated at  $H$ . Thus  $e_1(H) = e_2(H) = 1$  and  $e_3(H) = e_4(H) = -1$ . The  $A_3$  root system consists of  $\pm(e_i - e_j)$  for  $1 \leq i < j \leq 4$ , with positive roots as shown in the table below.



Root $\alpha$	$e_1 - e_4$	$e_1 - e_3$	$e_2 - e_4$	$e_1 - e_2$	$e_2 - e_3$	$e_3 - e_4$
$\alpha(H)$	2	2	2	0	2	0

We see that there are four roots taking the value 2 on  $H$ , so 2 is a highest weight for 4 distinct irreducible  $\mathfrak{sl}_2$ -submodules of  $\mathfrak{g}$ . Each such submodule also has eigenspaces for 0 and  $-2$ , leaving behind only three extra 0s. Hence the highest weights are 0, 0, 0, 2, 2, 2, 2. When the rank is increased by two, a pair of 2s is added, one to each side. Hence

$$e_i(H) = \begin{cases} n - 2i & \text{if } i < \frac{n+1}{2} \\ 1 & \text{if } i = \frac{n+1}{2} \\ -1 & \text{if } i = \frac{n+3}{2} \\ n - 2i + 4 & \text{if } i > \frac{n+3}{2} \end{cases}$$

The new positive roots added when the rank increases by two to  $n$  are  $e_1 - e_j$ ,  $2 \leq j \leq n + 1$ , and  $e_i - e_{n+1}$ ,  $2 \leq i \leq n$ . Evaluated at  $H$ , these give  $2, 4, \dots, 2n - 4$  and  $2, 4, \dots, 2n - 6$  as well as  $n - 3$ ,  $n - 1$ ,  $n - 3$ , and  $n - 1$ . This shows that the new highest weights are  $2n - 4$ ,  $2n - 6$ ,  $n - 1$ , and  $n - 1$ , the latter two replacing previous highest weights  $n - 5$  and  $n - 5$ . Comparing this to our list of highest weights for  $A_3$ , we see that a 0 and a 2 will be left behind as rank increases past  $n = 5$ . After adding 2 to each of these highest weights, we receive the claimed quasihomogeneous weights of type  $A$ .

Now we consider even  $n$ , beginning with  $A_4$ . Here the weighted Dynkin diagram is  $1 - 1 - 1 - 1$ , so that  $(e_i - e_{i+1})(H) = 1$  and  $e_i(H) = 3 - i$  for all  $1 \leq i \leq n$ . In this case, not all highest weights are even, unlike for odd  $n$ . The highest weights here are 0, 1, 1, 2, 2, 3, 3, 4. Increasing the rank by 2 changes the weighted Dynkin diagram in the same way as for odd  $n$ . Hence

$$e_i(H) = \begin{cases} n - 2i & \text{if } i < \frac{n}{2} \\ 1 & \text{if } i = \frac{n}{2} \\ 0 & \text{if } i = \frac{n}{2} + 1 \\ -1 & \text{if } i = \frac{n}{2} + 2 \\ n - 2i + 4 & \text{if } i > \frac{n}{2} + 2 \end{cases}$$

Everything else works exactly as in the other case for type  $A$ , in agreement with our claim that all type  $A$  have the same pattern of quasihomogeneous weights.

The weighted Dynkin diagram for the subsubregular orbit in  $B_3$  is  $0 - 2 - 0$ . The weights of the diagram are the values of the simple roots evaluated on  $H$ :  $(e_1 - e_2)(H) = 0$ ,  $(e_2 - e_3)(H) = 2$ , and  $e_3(H) = 0$ . So  $e_1(H) = e_2(H) = 2$  and  $e_3(H) = 0$ . Roots of the  $B_3$  system are  $\pm e_i \pm e_j$  for  $1 \leq i < j \leq 3$ , and  $\pm e_i$  for  $1 \leq i \leq 3$ . We observe the highest weights are  $0, 2, 2, 2, 2, 2, 4$ . For  $n \geq 4$ , the weighted Dynkin diagram consists of  $n - 3$  2s before the same  $0 - 2 - 0$ . Thus

$$e_i(H) = \begin{cases} 0 & \text{if } i = n \\ 2 & \text{if } i = n - 1, n - 2 \\ 2(n - i - 1) & \text{if } i \leq n - 2 \end{cases}$$

When the rank of root system is increased from  $n - 1$  to  $n$ , the positive roots that are added are  $e_1 \pm e_j$ ,  $2 \leq j \leq n$ , and  $e_1$  itself. Evaluated at  $H$ , these give  $2, 4, \dots, 4n - 10$  as well as  $2n - 6$ ,  $2n - 4$ ,  $2n - 4$ , and  $2n - 2$ . Thus we have highest weights  $4n - 10$  added, with  $2n - 2$  replacing former highest weight  $2n - 6$  and  $2n - 4$  replacing  $2n - 8$ . That implies that  $4k - 2$  for  $1 \leq k \leq n - 2$  and  $2n - 6$ ,  $2n - 4$ ,  $2n - 4$ ,  $2n - 2$  will be highest weights for  $B_n$ . These cover all but two 2s in our base  $B_3$  list, and we observe that those 2s will never be eliminated, as 2 is only  $2n - 6$  or  $2n - 8$  once each. Adding 2 to every highest weight, we get the quasihomogeneous weights as claimed.

For  $C_3$  we have weighted Dynkin diagram  $0 - 2 - 0$ . So  $(e_1 - e_2)(H) = 0$ ,  $(e_2 - e_3)(H) = 2$ , and  $2e_3(H) = 0$ , meaning that  $e_1(H) = e_2(H) = 2$  and  $e_3(H) = 0$ . Knowing that the roots of the  $C_3$  root system are  $\pm e_i \pm e_j$  for  $1 \leq i < j \leq 3$  and  $\pm 2e_i$  for  $1 \leq i \leq 3$ , we see that the highest weights for  $C_3$  are  $0, 0, 0, 2, 4, 4, 4$ . Increasing to the next rank adds a 2 to the right of the weighted Dynkin diagram,  $0 - 2 - 0 - 2$ . Here the values of the basic linear functionals on  $H$  are  $e_1(H) = e_2(H) = 3$  and  $e_3(H) = e_4(H) = 1$ , giving highest weights for  $C_4$  of  $0, 2, 2, 2, 4, 6, 6, 6$ . In general for  $n \geq 4$ , we have weighted Dynkin diagram with  $n - 4$  2s to the left of  $0 - 2 - 0 - 2$  so that

$$e_i(H) = \begin{cases} 1 & \text{if } i = n, n - 1 \\ 3 & \text{if } i = n - 2, n - 3 \\ 2(n - i - 2) + 1 & \text{if } i \leq n - 3 \end{cases}$$

As rank increases by one to  $n$ , we see that the new positive roots are  $e_1 \pm e_j$ ,  $2 \leq j \leq n$ , and  $2e_1$ . Evaluated at  $H$ , these give  $2, 4, \dots, 4n - 10$  as well as  $2n - 8$ ,  $2n - 6$ ,  $2n - 4$ , and  $2n - 2$ . Hence the highest weights that are gained are  $4n - 10$  and  $2n - 2$ , while the previous highest weight  $2n - 10$  is lost. That leads to a pattern whereby  $4k - 2$ ,  $1 \leq k \leq n - 2$ ,  $2n - 8$ ,  $2n - 6$ ,  $2n - 4$ ,  $2n - 2$  are

highest weights for  $C_n$  with  $n \geq 4$ . Comparing to our highest weights for  $C_4$ , we see that there will be an extra 2 and 6 which remain not part of the pattern of lost weights. Adding 2 to every such integer we get the claimed quasihomogeneous weights.

For  $D_5$  the weighted Dynkin diagram is  $0 - 2 - 0 < \frac{2}{2}$  corresponding to simple roots  $e_1 - e_2$ ,  $e_2 - e_3$ ,  $e_3 - e_4$ ,  $e_4 - e_5$ ,  $e_4 + e_5$ . For greater  $n$ , the change is that additional 2s are strung along to the left. The general formula is

$$e_i(H) = \begin{cases} 0 & \text{if } i = n \\ 2 & \text{if } i = n - 1, n - 2 \\ 4 & \text{if } i = n - 3, n - 4 \\ 2(n - i - 2) & \text{if } i \leq n - 4 \end{cases}$$

The highest weights in  $D_5$  are seen to be  $0, 2, 2, 2, 4, 6, 6, 6, 8$  by considering all  $\pm e_i(H) \pm e_j(H)$ ,  $i < j$ . When the rank increases to  $n$ , the positive roots added are  $e_1 \pm e_j$ ,  $2 \leq j \leq n$ , which take values at  $H$  of  $2, 4, \dots, 4n - 14$  as well as  $2n - 10, 2n - 8, 2n - 6, 2n - 4$ , and  $2n - 2$ . Much like with type  $C$ , we add two highest weights ( $4n - 14$  and  $2n - 2$ ) and lose one ( $2n - 12$ ). Also like type  $C$ , there will be a 2 and 6 left over in  $D_5$  when those are accounted for, which will be preserved through all  $D_n$ . As always we add 2 to each weight to get the quasihomogeneous weights.

There is no need for an inductive argument for exceptional types of Lie algebra, as there are only finitely many of these. The process of finding the quasihomogeneous weights starting from the weighted Dynkin diagram is similar to what was done for classical types, except that it is not as useful to find the values of the standard basic linear functionals on  $\mathfrak{h}$ . Instead, positive roots can be written as sums of simple roots so that the weights in the weighted Dynkin diagram are added directly, which could have been done for the classical types as well.  $\square$

*Remark 3.4.2.* From this result, we see that all of our designated subsubregular orbits have quasihomogeneous weights agreeing with the lowest  $n - 2$  quasihomogeneous degrees, with the highest quasihomogeneous weight equal to the quasihomogeneous degree of  $\tilde{f}_{n-2}$ . The other orbits in types  $C$  and  $D$  with dimension equal to  $\dim \mathfrak{g} - n - 4$  do not have these same patterns in weights.

*Remark 3.4.3.* Not every rank of every type of simple Lie algebra is considered in the above lemma. There is no subsubregular nilpotent orbit in type  $A_2$ , because the subregular in that type is also the minimal nontrivial nilpotent orbit. There are unique nilpotent orbits in types  $B_2$  and  $G_2$  which have the correct dimension, but in both cases there is a quasihomogeneous weight agreeing with the

quasihomogeneous degree of  $\tilde{f}_{n-1} = \tilde{f}_1$ , and in  $G_2$  that is not even the highest weight. There are three nilpotent orbits of dimension  $20 = \dim \mathfrak{g} - n - 4$  in type  $D_4$ , but these nilpotent orbits are isomorphic to one another by the triality of  $D_4$ , so there is no canonical choice of subsubregular.

Let  $E$  be an element in the subsubregular orbit of a simple Lie algebra  $\mathfrak{g}$  and let  $\mathcal{O}$  be the principal nilpotent orbit of  $\mathfrak{g}$ . We know that one minimal generating set for the defining ideal of the closure of the subregular nilpotent orbit is comprised of the fundamental invariants  $f_1, \dots, f_{n-1}$  along with a basis for the copy of  $V_\phi$  in degree  $m_r^\phi$  of  $S\mathfrak{g}^*$ . Thus we may determine an explicit minimal generating set for the ideal of  $\mathcal{S}_{E,\mathcal{O}}$  in the coordinate ring  $\mathbb{C}[\mathcal{S}_E]$  as a subset of the restrictions of those polynomials in  $\mathbb{C}[\mathfrak{g}]$  to the slice.

For the simply-laced types of Lie algebra, knowing the quasihomogeneous weights and degrees as well as the minimal generating set of polynomials is sufficient to determine the simple singularity type of  $(\mathcal{S}_{E,\mathcal{O}}, E)$ . These are already known, but we provide a different proof.

**Theorem 3.4.4.** *Let  $E$  be a subsubregular nilpotent element in the simply-laced simple Lie algebra  $\mathfrak{g}$  and  $\mathcal{O}$  be the subregular nilpotent orbit in  $\mathfrak{g}$ . Then the simple singularity  $(\mathcal{S}_{E,\mathcal{O}}, E)$  is of the following type.*

$\mathfrak{g}$	Singularity
$A_n$	$A_{n-2}$
$D_n$	$D_{n-2}$
$E_6$	$A_5$
$E_7$	$D_6$
$E_8$	$E_7$

*Proof.* The pair  $(\mathcal{S}_{E,\mathcal{O}}, E)$  is known to be isomorphic to  $(\mathbf{V}(p), 0)$  where  $\mathbf{V}(p) \subset \mathbb{C}^3$  is the surface defined by a polynomial  $p$  in three variables whose possible forms were listed previously [9, Corollary 5.3]. Since  $\dim \mathcal{S}_E > 3$ , it follows that all but three of the coordinates  $u_1, \dots, u_{n+4}$  must be defined in terms of the remaining three. That requires that each coordinate occur linearly in the restriction of a polynomial in the ideal defining  $\mathcal{S}_{E,\mathcal{O}}$  within  $\mathbb{C}[\mathcal{S}_E]$ .

We know that the defining ideal of  $\overline{\mathcal{O}}$  has a minimal generating set of fundamental invariants  $f_1, \dots, f_{n-1}$  and a basis for a copy of the adjoint representation in  $\mathbb{C}^{m_n}[\mathfrak{g}]$ . A possible choice for that copy is made up of partial derivatives of some choice of the highest-degree fundamental invariant  $f_n$ . When the fundamental invariants in  $\mathbb{C}[\mathfrak{g}]$  are restricted to  $\mathbb{C}[\mathcal{S}_E]$ , the resulting polynomials have the quasihomogeneous degrees listed in lemma 3.4.1.

The first  $n - 2$  quasihomogeneous degrees are also quasihomogeneous weights, so it is plausible that the corresponding coordinates occur linearly in the restrictions of the fundamental invariants. In fact, this is true. Because the subsubregular nilpotent orbit in simply-laced types of Lie algebra are all in our first family of nilpotent orbits, we know that partial derivatives of the first  $n - 2$  fundamental invariants are nonzero when evaluated at  $E$ . Then the partial derivatives of  $\widetilde{f}_i$  with respect to the slice coordinates  $u_j$  are nonzero when evaluated at 0. Hence they do each have a linear term.

All remaining coordinates which occur linearly must do so in the restriction of a partial derivative of  $f_n$ . This would require that the quasihomogeneous weight of the coordinate added to some other quasihomogeneous weight total the quasihomogeneous degree of  $\widetilde{f}_n$ . That is,  $u_k$  occurs linearly in  $\frac{\partial \widetilde{f}_n}{\partial u_j}$  only if  $w_j + w_k = d_n$ , since the quasihomogeneous degree of  $u_k u_j$  must be equal to that of  $\widetilde{f}_n$ .

Eliminating all such weights, a pattern arises. There are, in each case, three quasihomogeneous weights in  $w_{n-1}, \dots, w_{n+4}$  which do not have a corresponding  $w_j = d_n - w_k$  in the list of quasihomogeneous weights. Hence these cannot occur linearly in any polynomial in the restriction of the partial derivatives of  $f_n$  to the slice. These are as follows:

$\mathfrak{g}$	Weights
$A_n$	$2, n - 1, n - 1$
$D_n$	$4, 2n - 8, 2n - 6$
$E_6$	$2, 6, 6$
$E_7$	$4, 8, 10$
$E_8$	$8, 12, 18$

Finally, given three quasihomogeneous weights, there is only ever one of the simple singularity polynomials which will be quasihomogeneous. This allows us to determine the simple singularity type of each of the pairs  $(\mathcal{S}_{E,\mathcal{O}}, E)$  as claimed.  $\square$

*Remark 3.4.5.* The types of these singularities are closely related to (equal except in type  $D_n$ ) the types of the Levi subalgebras  $\mathfrak{l}_\phi$  generated by simple short roots orthogonal to the short dominant root  $\phi$ . In type  $D_n$ ,  $\mathfrak{l}_\phi$  has type  $A_1 + D_{n-2}$ , and it turns out that the other nilpotent orbit in dimension  $\dim \mathfrak{g} - \text{rank } \mathfrak{g} - 4$  will produce a type  $A_1$  singularity.

### 3.5 Type $A_n$ example

To make things even more explicit, we use a different, more accessible representation for type  $A_n$  than what Magma uses (our choices here reflect those in the book by Collingwood and McGovern

[5]) and determine a set of  $n + 2$  equations that generate the ideal for  $\mathcal{S}_{E,\mathcal{O}}$  in  $\mathbb{C}[\mathcal{S}_E]$  where  $E$  is subsubregular and  $\mathcal{O}$  is subregular.

Define  $F \in \mathfrak{g} := \mathfrak{sl}_{n+1}$  to be the nilpotent block diagonal matrix shown:

$$\left[ \begin{array}{cccc|cc} 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 \end{array} \right]$$

Let  $\{H, E, F\}$  be the  $\mathfrak{sl}_2$ -triple in  $\mathfrak{g}$  where  $F$  is the nilnegative element. The corresponding matrix  $H$  is the diagonal matrix with entries  $n - 2, n - 4, \dots, -(n - 2), 1, -1$  from left to right and  $E$  is as follows:

$$\left[ \begin{array}{ccccc|cc} 0 & n-2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 2n-6 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & n-2 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{array} \right]$$

An element of  $\mathfrak{g}^F$  takes the form:

$$\left[ \begin{array}{cccccc|cc} -2u_{n+4} & 0 & \dots & \dots & 0 & 0 & 0 & 0 \\ u_1 & -2u_{n+4} & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ u_{n-3} & u_{n-4} & \ddots & u_1 & -2u_{n+4} & 0 & u_{n+1} & 0 \\ u_{n-2} & u_{n-3} & \dots & u_2 & u_1 & -2u_{n+4} & u_{n+3} & u_{n+1} \\ \hline u_n & 0 & \dots & 0 & 0 & 0 & (n-1)u_{n+4} & 0 \\ u_{n+2} & u_n & \dots & 0 & 0 & 0 & u_{n-1} & (n-1)u_{n+4} \end{array} \right]$$

From this form it can be seen that we have the following basis of  $ad_H$ -weight vectors for  $\mathfrak{g}^F$ , where the subscript on  $Z$  is the weight  $j$  such that  $[H, Z_j] = -jZ_j$ :

$$Z_0 := -2\left(\sum_{i=1}^{n-1} E_{i,i}\right) + (n-1)(E_{n,n} + E_{n+1,n+1}),$$

$$Z_{2j} := \sum_{i=1}^{n-j-1} E_{i+j,i} \text{ for } j = 1, \dots, n-2,$$

$$Z'_2 := E_{n+1,n},$$

$$Z_{n-3} := E_{n,1} + E_{n+1,2} \text{ and } Z'_{n-3} := E_{n-2,n} + E_{n-1,n+1},$$

$$Z_{n-1} := E_{n+1,1} \text{ and } Z'_{n-1} := E_{n-1,n}$$

**Theorem 3.5.1.**  $\text{rank}(d\pi_E) = n - 2$ . [17]

**Proof:** We observe that  $\text{rank}(d\pi_E)$  is the number of linearly independent rows in the Jacobian  $\frac{\partial(\tilde{f}_1, \dots, \tilde{f}_n)}{\partial(u_1, \dots, u_{n+4})}$  evaluated at 0. This depends on which of the variables  $u_j$  appear linearly in the  $\tilde{f}_i$ . Because the  $\tilde{f}_i$  have distinct degrees, it is impossible for any  $u_j$  to appear linearly in more than one polynomial.

First we show that  $u_1, \dots, u_{n-2}$  appear linearly in  $\tilde{f}_1, \dots, \tilde{f}_{n-2}$  respectively. Observing that these variables appear only in the upper left  $(n-1) \times (n-1)$  submatrix, we see that it will suffice to show that these variables appear linearly in appropriately-sized diagonal minors of that submatrix. The terms of the  $\tilde{f}_i$  polynomial come from products of matrix entries  $M_{k_1, l_1} \cdots M_{k_{i+1}, l_{i+1}}$ , where  $\{k_1, \dots, k_{i+1}\}$  and  $\{l_1, \dots, l_{i+1}\}$  are reorderings of some size  $i+1$  subset of  $\{1, \dots, n-1\}$ . For every  $j \in \{i+1, \dots, n-1\}$ , the product  $M_{j, j-i} M_{j-1, j} \cdots M_{j-i, j-i+1}$  will appear in the polynomial. We know that those entries in the first superdiagonal are positive constants coming from  $E$ , and  $M_{j, j-i} = u_i$ . Furthermore, the bottom row of any cut-down square submatrix will contain no constants. This means that if  $M_{j, j-i}$  appears in a term but  $j \neq \max\{k_s\}$ , that term cannot be linear. Hence there can be no cancellation and  $\tilde{f}_i$  will have a linear  $u_i$  term as claimed.

Finally, we show that no  $u_j$  can appear linearly in  $\tilde{f}_{n-1}$  or  $\tilde{f}_n$ , which will prove our claim. As before, a term in one of those polynomials comes from a product of matrix entries, the number of these being one greater than the subscript. We observe that our entire matrix  $M$  has exactly  $n-1$  nonzero constant entries (those of  $E$ ). Already, then, it is impossible to have a linear term in  $\tilde{f}_n$ . We also see that the elimination of any row and corresponding column will eliminate at least one of those nonzero constants. Hence every  $n \times n$  diagonal submatrix has only  $n-2$  nonzero constants. So we also have that  $\tilde{f}_{n-1}$  has no linear terms. Therefore,  $\text{rank}(d\pi_E) = n - 2$ .  $\square$

Now we want equations describing the intersection of the slice  $\mathcal{S}_{ssr} := E + \mathfrak{g}^F$  with the subregular nilpotent orbit  $\mathcal{O}_{sr}$ . It is known that the singularity of  $\bar{\mathcal{O}}_{sr}$  in  $\mathcal{O}_{ssr}$  has type  $A_{n-2}$ , with defining equation  $uv + w^{n-1}$  [15]. By an argument similar to the above, we know that for an element in the subregular nilpotent orbit, the partial derivatives of the determinant should all be zero. This turns out to be sufficient along with the other  $\tilde{f}_i$ .

**Theorem 3.5.2.** *The ideal of polynomials in  $\mathbb{C}[u_1, \dots, u_{n+4}]$  vanishing on  $(E + \mathfrak{g}^F) \cap \mathcal{O}_{sr}$  is generated by*

$$\left\{ \tilde{f}_1, \dots, \tilde{f}_{n-2}, \frac{\partial \tilde{f}_n}{\partial u_{n-2}}, \frac{\partial \tilde{f}_n}{\partial u_{n-1}}, \frac{\partial \tilde{f}_n}{\partial u_{n+2}}, \frac{\partial \tilde{f}_n}{\partial u_{n+3}} \right\}.$$

**Proof:** As we have already seen, the variables  $u_1, \dots, u_{n-2}$  appear linearly in  $\tilde{f}_1, \dots, \tilde{f}_{n-2}$ . We see that the determinant of  $M, \tilde{f}_n$ , contains a term  $M_{\sigma(1), \tau(1)} \cdots M_{\sigma(n+1), \tau(n+1)}$  for every  $\sigma, \tau \in S_{n+1}$ . Considering the form of matrix  $M$ , we can see that there is a nonzero constant in every row except the  $(n-1)$ st and the  $(n+1)$ st, and a nonzero constant in every column except the 1st and the  $n$ th. Two terms of the determinant will include all of those nonzero constant entries, and the remaining pairs of matrix entries are  $M_{n-1,1}$  and  $M_{n+1,n}$  or  $M_{n-1,n}$  and  $M_{n+1,1}$ . Those give quadratic terms that are constant multiples of  $u_{n+2}u_{n+3}$  and  $u_{n-2}u_{n-1}$ , respectively. Thus  $\frac{\partial \tilde{f}_n}{\partial u_{n-2}}$  has a linear  $u_{n-1}$  term,  $\frac{\partial \tilde{f}_n}{\partial u_{n+2}}$  has linear  $u_{n+3}$  term and so on. Using the fact that these polynomials must all be equal to zero at an element in  $\mathcal{O}_{sr}$ , we can give equations for variables  $u_1 \dots, u_{n-1}, u_{n+2}, u_{n+3}$  in terms of only  $u_n, u_{n+1}, u_{n+4}$ .

Notably,  $u_{n-2}$  appears linearly in both  $\tilde{f}_{n-2}$  and  $\frac{\partial \tilde{f}_n}{\partial u_{n-1}}$ . If we take the latter polynomial to eliminate  $u_{n-2}$ ,  $\tilde{f}_{n-2}$  will be transformed into a nonzero polynomial in the three variables  $u_n, u_{n+1}, u_{n+4}$ . This then will be a polynomial of quasihomogeneous type  $(2n-2; n-1, n-1, 2)$ . Relabeling the remaining variables as  $u, v, w$  respectively, we see that the only monomials that can appear in this function are  $u^2, v^2, uv$ , or  $w^{n-1}$ . Because the singularity is isolated, the  $w^{n-1}$  term must appear, and then any nondegenerate quadratic in  $u, v$  is equivalent to  $uv$ , giving us the normal form we expect:  $uv + w^{n-1}$ .  $\square$



## CHAPTER 4

### FURTHER RESEARCH

#### 4.1 Connections to Springer fibers

Aside from intersections with Slodowy slice, we can also consider the intersection of a nilpotent variety with the Cartan subalgebra. As a set, this is only the zero element for any nilpotent variety, but scheme-theoretically these intersections differ. It was found by De Concini and Procesi [7] that in type  $A$ , there is an  $S_n$ -equivariant  $\mathbb{C}$ -algebra isomorphism

$$\mathbb{C}[\overline{\mathcal{O}}_\mu \cap \mathfrak{h}] \simeq H^\bullet(X^{\mu'})$$

where  $\mu$  is a partition of  $n$ ,  $\mu'$  is the transpose partition to  $\mu$ , and  $X^{\mu'}$  is the variety of Borel subalgebras of  $\mathfrak{sl}_n$  containing a fixed element  $E_{\mu'} \in \mathcal{O}_{\mu'}$ . Although not referred to as such in many of these early papers, we note that  $X^{\mu'}$  is the Springer fiber of  $E_{\mu'}$ .

Carrell [3] showed that even though this isomorphism cannot hold for every nilpotent variety in all other types because not all nilpotent orbits have corresponding dual orbits, there is a related surjective homomorphism under some assumptions. Let  $G$  be a semi-simple algebraic group with Borel subgroup  $B$  containing maximal torus  $T$ , and let  $\mathfrak{g}$ ,  $\mathfrak{b}$ , and  $\mathfrak{h}$  be their respective Lie algebras. Let  $W$  be the Weyl group of  $\mathfrak{g}$  with respect to  $\mathfrak{h}$ . Choose a Levi subalgebra  $\mathfrak{l}$  and let  $E$  be a principal nilpotent element in  $\mathfrak{l}$ . Let  $i_E : X^E \rightarrow G/B$  be the inclusion of the Springer fiber of  $E$  (the variety of all Borel subalgebras containing  $E$ ) into the flag variety of all Borel subalgebras of  $\mathfrak{g}$ . Fix a parabolic subgroup  $P$  with Lie algebra  $\mathfrak{p}$  containing  $\mathfrak{l}$  and let  $E'$  be an element of the Richardson orbit for  $\mathfrak{p}$ .

**Theorem 4.1.1.** *[3, Corollary 1] With the setup described, suppose the closure of the Richardson orbit for  $\mathfrak{p}$  is normal and the  $P$  and  $G$  stabilizers of  $E'$  coincide. Then there exists a surjective degree-doubling  $W$ -equivariant  $\mathbb{C}$ -algebra homomorphism*

$$\mathbb{C}[\overline{\mathcal{O}}_{E'} \cap \mathfrak{h}] \rightarrow i_E^* H^\bullet(G/B).$$

This is a corollary in Carrell's paper to the following pair of results relating both sides to the intermediate  $W$ -module  $\text{Ind}_{W_{\mathfrak{l}}}^W(\mathbb{C})$  induced from the trivial representation of the Weyl group  $W_{\mathfrak{l}}$  of the Levi subalgebra  $\mathfrak{l}$ .

**Theorem 4.1.2.** *[3, Theorem 1] There exists a  $W$ -equivariant  $\mathbb{C}$ -algebra isomorphism*

$$\text{Ind}_{W_{\mathfrak{l}}}^W(\mathbb{C}) \simeq i_{E'}^* H^\bullet(G/B).$$

**Theorem 4.1.3.** *[14, Proposition 4] Under the same hypotheses for  $P$  as Theorem 4.1.1,  $\mathbb{C}[\overline{\mathcal{O}}_{E'} \cap \mathfrak{h}]$  contains  $\text{Ind}_{W_{\mathfrak{l}}}^W(\mathbb{C})$ , the  $W$ -module induced from the trivial representation on the Weyl group for  $\mathfrak{l}$ .*

All of the nilpotent varieties for which we have found minimal generating sets of the defining ideals in Chapter 2 satisfy the requirements of the theorem. Because we know how to get explicit polynomials to generate the ideal of one of our nilpotent varieties, we are able to explicitly describe the image of those polynomials in  $\mathbb{C}[\overline{\mathcal{O}}_{E'} \cap \mathfrak{h}]$ , just as we have done for the Slodowy slice intersections. Tanisaki [21] studied the case of partitions with no odd part in type  $C$  using a non-minimal generating set, finding an isomorphism in those cases. His technique relies on a dimension argument, because the dimension of  $\text{Ind}_{W_{\mathfrak{l}}}^W(\mathbb{C})$  is straightforward to compute.

Although it is not apparent in Carrell's work, the nilpotent orbits  $\mathcal{O}_{E'}$  and  $\mathcal{O}_E$  must be Spaltenstein dual in general. In type  $A_n$ , the Spaltenstein dual nilpotent orbit is exactly the one with the transpose partition, but in other types of Lie algebra this can mean that  $E$  and  $E'$  are elements of different Lie algebras. For instance, the Spaltenstein dual nilpotent orbit to the subregular nilpotent orbit of  $B_n$  is the minimal nilpotent orbit in  $C_n$ , and the minimal nilpotent orbit of  $B_n$  is Spaltenstein dual to the subregular nilpotent orbit of  $C_n$ .

In our situation of a Richardson orbit for a parabolic subalgebra generated by  $s$  orthogonal simple short roots, designate the Weyl group of the Levi subalgebra by  $W_s$ . Because of the Levi subalgebra being of type  $A_1 \times A_1 \times \cdots \times A_1$ , we know that  $|W_s| = 2^s$ . Therefore the dimensions of the induced modules are as follows:

Type	$\dim \text{Ind}_{W_s}^W(\mathbb{C})$
$A_n$	$n! \cdot 2^{-s}$
$B_n$	$n! \cdot 2^{n-s}$
$C_n$	$n! \cdot 2^{n-s}$
$D_n$	$n! \cdot 2^{n-s-1}$
$E_6$	$2^{7-s} \cdot 405$
$E_7$	$2^{10-s} \cdot 2835$
$E_8$	$2^{14-s} \cdot 42525$

So every time  $s$  increases by one, the dimension halves. If we can show the same pattern for the coordinate rings  $\mathbb{C}[\overline{\mathcal{O}}_\Theta \cap \mathfrak{h}]$  where  $|\Theta| = s$ , then we would know that for all of these nilpotent varieties, there is in fact an isomorphism between the coordinate ring on the intersection of the nilpotent variety with the Cartan subalgebra and the image of the cohomology of the Springer fiber for the Spaltenstein dual nilpotent orbit.

**Conjecture 4.1.4.** *Let  $\mathcal{O}_\Theta$  be the nilpotent orbit in  $\mathfrak{g}$  that is Richardson for a parabolic subalgebra generated by a set  $\Theta$  of  $s$  orthogonal simple short roots, and let  $E$  be an element of the Spaltenstein dual nilpotent orbit. Then there is a  $W$ -equivariant isomorphism of  $\mathbb{C}$ -algebras*

$$\mathbb{C}[\overline{\mathcal{O}}_\Theta \cap \mathfrak{h}] \simeq i_E^* H^\bullet(G/B).$$

## APPENDIX A

### LINEAR ALGEBRA FOR TYPE $D$

#### A.1 Pfaffian complications

##### A.1.1 Relating Pfaffian and rank

Let  $M$  be the generic skew-symmetric  $2n \times 2n$  matrix, which is to say that  $M_{i,i} = 0$  for all  $i$ ,  $M_{i,j} = m_{i,j}$  and  $M_{j,i} = -m_{i,j}$  for  $i < j$  where  $m_{i,j}$  are indeterminates. Let  $\sigma_r(i_1, \dots, i_{2r})$  be the Pfaffian of the  $2r \times 2r$  submatrix of  $M$  formed from rows and columns  $i_1, \dots, i_{2r}$ . For  $1 \leq r \leq n-1$  let  $\tau_r$  be the sum of the  $2r \times 2r$  diagonal minors of  $M$ , so that  $\{\tau_1, \dots, \tau_{n-1}, \sigma_n\}$  is a complete set of fundamental invariants of type  $D_n$ . We could also define  $\tau_n$  similarly, but that would be the determinant, which we can already denote by  $\sigma_n^2$ .

**Lemma A.1.1.** *For a skew-symmetric  $2n \times 2n$  matrix  $Q$ ,  $\text{rank}(Q) \leq 2n-4$  if and only if  $\partial\sigma_n/\partial m_{j,k}|_Q = 0$  for all  $j < k$ .*

*Proof.* First, suppose  $\text{rank}(Q) \leq 2n-4$ . Then the determinant of every  $(2n-2) \times (2n-2)$  submatrix of  $Q$  is zero. Because a Pfaffian squares to the determinant, we know that every  $\sigma_{n-1}$  is also zero when evaluated at  $Q$ . Since  $\sigma_n = \sum_{j < k} m_{j,k} \sigma_{n-1}(i_1, \dots, \hat{j}, \dots, \hat{k}, \dots, i_{2n})$ , it follows that  $\partial\sigma_n/\partial m_{j,k}|_Q = \sigma_{n-1}(i_1, \dots, \hat{j}, \dots, \hat{k}, \dots, i_{2n})|_Q$ , which is zero.

Now assume that for all  $j < k$  we have  $\partial\sigma_n/\partial m_{j,k}|_Q = 0$ . As above, these partial derivatives are in fact the Pfaffians of the  $(2n-2) \times (2n-2)$  diagonal submatrices of  $Q$ . The determinants of those submatrices are also all zero. If  $\delta$  is the  $2n-2 \times 2n-2$  minor with rows  $a_1, a_2, c_1, \dots, c_{2n-4}$  and columns  $b_1, b_2, c_1, \dots, c_{2n-4}$ , then

$$\delta = \pm \sigma_n \sigma_{n-1}(c_1, \dots, c_{2n-4}) \pm \sum \sigma_{n-1}(i_1, i_2, c_1, \dots, c_{2n-4}) \sigma_{n-1}(j_1, j_2, c_1, \dots, c_{2n-4})$$

where the sum is over orderings of  $\{i_1, i_2\} = \{a_1, a_2\}$  and  $\{j_1, j_2\} = \{b_1, b_2\}$  such that  $\{i_1, i_2, j_1, j_2\}$  is an even permutation of  $\{a_1, b_1, a_2, b_2\}$ . We neglect to specify the signs of these terms because all terms in our situation turn out to be zero. Then  $\text{rank}(Q) < 2n-2$ . Every skew-symmetric matrix has even rank. Therefore  $\text{rank}(Q) \leq 2n-4$ . □

### A.1.2 Alternative Pfaffian

Now we consider instead the matrices skew-adjoint with respect to the quadratic form  $x_1x_{n+1} + x_2x_{n+2} + \cdots + x_nx_{2n}$ . We label our indeterminates such that a generic matrix  $\tilde{M}$  here has

$$\tilde{M}_{i,j} = \begin{cases} m_{i,j} & \text{for } i, j \leq n \text{ or } i > j + n \text{ or } j > i + n \\ -m_{j-n, i-n} & \text{for } i, j > n \\ -m_{j, i-n} & \text{for } n < i < j + n \\ -m_{j-n, i} & \text{for } n < j < i + n \\ 0 & \text{for } i - j = \pm n \end{cases}$$

In this form, we have root spaces easily described as matrices having all  $m_{i,j} = 0$  except for one based on the root. A root  $e_i - e_j$  has  $m_{i,j}$  nonzero, a root  $e_i + e_j$  has  $m_{i,j+n}$  nonzero, and a root  $-e_i - e_j$  has  $m_{i+n,j}$  nonzero.

We define  $\tilde{\tau}_r$  exactly as before, but will need to define  $\sigma_r$  differently. We define

$$\tilde{\sigma}_n := \sum (\text{sgn } s) \tilde{M}_{i_1, j_1} \cdots \tilde{M}_{i_n, j_n}$$

where  $s \in S_n$  is the permutation sending  $\{1, 2, \dots, 2n\}$  to  $\{i_1, j_1 + n, i_2, \dots, i_n, j_n + n\}$  modulo  $2n$  and we require that  $i_1 < i_2 < \cdots < i_n$  and every pair  $i_l, j_l$  is what we will call a positive pair. A positive pair  $i_l, j_l$  satisfies one of the following:  $i_l, j_l \leq n$  or  $i_l > j_l + n$  or  $j_l > i_l + n$ . For ease of notation, indices will henceforth be written modulo  $2n$ .

**Lemma A.1.2.**  $\tilde{\sigma}_n$  is an invariant of degree  $n$  and  $\tilde{\sigma}_n^2 = \det$ .

*Proof.* With our chosen quadratic form, a matrix in  $O_{2n}$  has the form  $Q := \begin{bmatrix} A & B \\ C & D \end{bmatrix}$  where all submatrices are  $n \times n$  and  $Q^{-1} = \begin{bmatrix} D^T & B^T \\ C^T & A^T \end{bmatrix}$ . This replaces the more typical condition that the inverse be the transpose matrix. For a general  $2n \times 2n$  matrix  $Q$ , we use the notation  $Q^\dagger$  for this alternative transpose.

We will show that  $\tilde{\sigma}_n(BQB^\dagger) = \det(B)\tilde{\sigma}_n(Q)$  for  $Q$  a skew-adjoint  $2n \times 2n$  matrix and any invertible  $2n \times 2n$  matrix  $B$  by proving this for elementary matrices  $B$ . Consider the three kinds of elementary transformation:

*Case 1:* Multiplication of row  $i$  by constant  $\lambda \neq 0$ .

Then

$$(BQB^\dagger)_{l,k} = \begin{cases} Q_{l,k} & \text{if } l \neq i \text{ and } k \neq i+n \\ \lambda Q_{l,k} & \text{if } l = i \text{ or } k = i+n \end{cases}.$$

Each term in  $\tilde{\sigma}_n(Q)$  contains exactly one factor  $Q_{i,j}$  or  $Q_{j,i+n}$  for some  $j$  since  $s(i)$  is either a row index or  $n$  away from a column index. Hence  $\lambda$  can be factored out of every term in  $\tilde{\sigma}_n(BQB^\dagger)$ , and  $\tilde{\sigma}_n(BQB^\dagger) = \lambda \tilde{\sigma}_n(Q) = \det(B) \tilde{\sigma}_n(Q)$ .

*Case 2:* Switching row  $i$  with row  $i'$ .

Then  $BQB^\dagger$  is  $Q$  but with rows  $i, i'$  swapped and also columns  $i+n, i'+n$ . We know that exactly one of  $i, i+n$  and exactly one of  $i', i'+n$  occurs as an index in each term of  $\tilde{\sigma}_n$ . If both occur as indices of the same factor of a term in  $\tilde{\sigma}_n(Q)$ , then that term appears with opposite sign in  $\tilde{\sigma}_n(BQB^\dagger)$  because  $\text{sgn}((i \ i')s) = -\text{sgn}(s)$ . Otherwise  $i$  or  $i+n$  and  $i'$  or  $i'+n$  are indices in two different factors of a term. Switching  $i$  and  $i'$  contributes a negative sign as before, but there is also the possibility that the resulting pairs are not positive. This can be resolved by recalling that  $Q_{l,k} = -Q_{k+n, l+n}$ . So if, for instance, a term of  $\tilde{\sigma}_n(Q)$  contains  $Q_{i,j}$  but  $i', j$  is not positive (so that no term containing  $Q_{i',j}$  exists), there will be a term of  $\tilde{\sigma}_n(Q)$  containing  $Q_{j+n, i'+n}$ . The same term containing  $Q_{j+n, i'+n}$  appears in  $\tilde{\sigma}_n(BQB^\dagger)$  but with opposite sign due to two transpositions and the negative on the factor itself. Both of the switched factors not having positive pairs gives  $(-1)^5 = -1$ . Thus every term of  $\tilde{\sigma}_n(BQB^\dagger)$  is the negative of a term of  $\tilde{\sigma}_n(Q)$ , and  $\tilde{\sigma}_n(BQB^\dagger) = \det(B) \tilde{\sigma}_n(Q)$ .

*Case 3:* Adding row  $i'$  into row  $i$ .

Then

$$(BQB^\dagger)_{l,k} = \begin{cases} Q_{i,k} + Q_{i',k} & \text{if } l = i \text{ and } k \neq i+n \\ Q_{l, i+n} + Q_{l, i'+n} & \text{if } k = i+n \text{ and } l \neq i \\ Q_{l,k} & \text{otherwise} \end{cases}.$$

$Q$  term of  $\tilde{\sigma}_n(Q)$  containing factor  $Q_{i, i'+n}$  or  $Q_{i', i+n}$  will be unchanged in  $\tilde{\sigma}_n(BQB^\dagger)$  because  $Q_{i', i'+n} = 0$ . Suppose a term of  $\tilde{\sigma}_n(Q)$  contains as factors  $Q_{i,k}$  where  $k \neq i'+n$  and  $Q_{i', k'}$  where  $k' \neq i+n$ . There is also a term having all the same factors except  $Q_{i,k}$  and  $Q_{i', k'}$  are replaced by  $Q_{i', k}$  and  $Q_{i, k'}$ , and that term has opposite sign. We observe that

$$\begin{aligned} (BQB^\dagger)_{i,k} (BQB^\dagger)_{i',k'} - (BQB^\dagger)_{i,k'} (BQB^\dagger)_{i',k} &= (Q_{i,k} + Q_{i',k}) Q_{i',k'} - (Q_{i,k'} + Q_{i',k'}) Q_{i',k} \\ &= Q_{i,k} Q_{i',k'} - Q_{i,k'} Q_{i',k}. \end{aligned}$$

Similar cancellation occurs for the other possible pairs of indices involving  $i$  or  $i+n$ , which allows us to conclude that  $\tilde{\sigma}_n(BQB^\dagger) = \tilde{\sigma}_n(Q) = \det(B) \tilde{\sigma}_n(Q)$ .

Every invertible  $2n \times 2n$  matrix is a product of elementary matrices, so  $\tilde{\sigma}_n(BQB^\dagger) = \det(B)\tilde{\sigma}_n(Q)$  for all invertible  $B$ . Therefore, if  $B$  is special orthogonal,  $\tilde{\sigma}_n(BQB^{-1}) = \tilde{\sigma}_n(BQB^\dagger) = \det(B)\tilde{\sigma}_n(Q) = \tilde{\sigma}_n(Q)$ . We have that  $\tilde{\sigma}_n$  is a degree  $n$  invariant.

By the Chevalley Theorem, restriction of invariants to the Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  is an isomorphism onto Weyl group invariants in the algebra of functions on  $\mathfrak{h}$ . We have, for a diagonal matrix  $Q$  in  $\mathfrak{g}$ ,  $\tilde{\sigma}_n(Q) = Q_{1,1}Q_{2,2} \cdots Q_{n,n}$  and  $\det(Q) = Q_{1,1}^2 Q_{2,2}^2 \cdots Q_{n,n}^2$ . Because  $\tilde{\sigma}_n^2 = \det$  on  $\mathfrak{h}$  and both are invariants, we have that this equation holds for all of  $\mathfrak{g}$ .  $\square$

### A.1.3 The very even case details

Here are the details referred to in Subsubsection 2.8.3.4 of Chapter 2. In the case that  $n = 2k$ , the simple Lie algebra of type  $D_n$  has two distinct nilpotent orbits with partition  $[n, n]$ . These are labelled as  $\mathcal{O}_{\Theta_1}$  and  $\mathcal{O}_{\Theta_2}$ , where the type 1 orbit is Richardson for the parabolic subalgebra generated by the set of orthogonal simple short roots  $\Theta_1 = \{\alpha_1, \alpha_3, \dots, \alpha_{n-3}, \alpha_{n-1}\}$  and the type 2 orbit is Richardson for the parabolic subalgebra generated by the set  $\Theta_2 = \{\alpha_1, \alpha_3, \dots, \alpha_{n-3}, \alpha_n\}$ . For  $N \in \{1, 2\}$ , let  $I_N$  be the ideal of  $\overline{\mathcal{O}}_{\Theta_N}$  in  $\mathbb{C}[\overline{\mathcal{O}}_{\Theta'}]$  where  $\Theta' = \{\alpha_1, \alpha_3, \dots, \alpha_{n-3}\}$ . Similarly, let  $J_N$  be the ideal of  $\overline{\mathcal{O}}_{\Theta_N}$  in  $\mathbb{C}[\mathfrak{g}]$ .

Let  $P$  be the matrix with

$$P_{i,j} = \begin{cases} \frac{\partial \tilde{\sigma}_n}{\partial \tilde{M}_{i,j}} & \text{for } \tilde{M}_{i,j} \neq 0 \\ 0 & \text{for } \tilde{M}_{i,j} = 0 \end{cases} = \begin{cases} \frac{\partial \tilde{\sigma}_n}{\partial m_{i,j}} & \text{for } i, j \leq n \text{ or } i > j + n \text{ or } j > i + n \\ -\frac{\partial \tilde{\sigma}_n}{\partial m_{j-n, i-n}} & \text{for } i, j > n \\ -\frac{\partial \tilde{\sigma}_n}{\partial m_{j, i-n}} & \text{for } n < i < j + n \\ -\frac{\partial \tilde{\sigma}_n}{\partial m_{j-n, i}} & \text{for } n < j < i + n \\ 0 & \text{for } i - j = \pm n \end{cases}$$

**Proposition A.1.3.**  $J_1$  (respectively  $J_2$ ) is generated by  $\{\tilde{\tau}_1, \dots, \tilde{\tau}_{n/2-1}, \tilde{\sigma}_n\}$  and  $\{(\tilde{M}^{n-1})_{i,j} + P_{j,i} \mid i, j \text{ is a positive pair}\}$  (respectively  $\{(\tilde{M}^{n-1})_{i,j} - P_{j,i} \mid i, j \text{ is a positive pair}\}$ ).

*Proof.* We show the case for  $N = 1$ , noting that the other is similar except with all instances of  $\alpha_n$  and  $\alpha_{n-1}$  swapped. As shown in Theorem 2.4.1 and lemma 2.8.13 of Chapter 2,  $I_1$  is generated by the kernel of the map of coordinate rings  $\mathbb{C}[\overline{\mathcal{O}}_{\Theta_1}] \rightarrow \mathbb{C}[\overline{\mathcal{O}}_{\Theta'}]$ .

Our previous result for  $\overline{\mathcal{O}}_{\Theta'}$  in Section 2.8 of Chapter 2 says that its ideal in  $\mathbb{C}[\mathcal{N}]$  is generated by a copy of  $V_\phi$  in degree  $n + 1$ , our choice of which was the entries of  $\tilde{M}^{n+1}$ . Hence the ideal of  $\overline{\mathcal{O}}_{\Theta_1}$  in  $\mathbb{C}[\mathcal{N}]$  is generated by two copies of  $V_\phi$  in degrees  $n - 1$  and  $n + 1$ .

We now show that the claimed set of matrix entries give an appropriate copy of  $V_\phi$  in degree  $n-1$ . Let  $Q^1$  be the matrix  $\sum_{i=1}^{n-1} E_{i,i+1} - E_{i+1+n,i+n}$ , a sum of root vectors for roots  $\alpha_1, \dots, \alpha_{n-1}$ . This is an element of  $\mathcal{O}_{\Theta_1}$ . Similarly, we define  $Q^2 \in \mathcal{O}_{\Theta_2}$  to be the sum of root vectors for roots  $\alpha_1, \dots, \alpha_{n-2}, \alpha_n$ ,  $(\sum_{i=1}^{n-2} E_{i,i+1} - E_{i+1+n,i+n}) + E_{n-1,2n} - E_{n,2n-1}$ .

From our definition of  $\tilde{\sigma}_n$ , we see that  $\partial\tilde{\sigma}_n/\partial m_{i,j}$  has terms  $\tilde{M}_{i_1,j_1} \cdots \tilde{M}_{i_{n-1},j_{n-1}}$  such that all pairs  $i_l, j_l$  are positive. The sign of a term is the sign of the permutation that takes  $1, \dots, 2n$  to the pairwise rearrangement of  $i_1, j_1+n, \dots, i_{n-1}, j_{n-1}+n, i, j+n$  with row indices increasing. Evaluating on  $Q^1$  or  $Q^2$  we see that there are exactly  $n-1$  nonzero entries whose coordinates are positive pairs. Thus in each case there is exactly one nonzero partial derivative, which has exactly one nonzero term. In  $Q^1$  the nonzero entries are  $Q_{1,2}^1, Q_{2,3}^1, \dots, Q_{n-1,n}^1$ , so the coordinate whose partial derivative is nonzero is  $m_{n,1}$ . In  $Q^2$  the nonzero entries are  $Q_{1,2}^2, Q_{2,3}^2, \dots, Q_{n-2,n-1}^2, Q_{n-1,2n}^2$ , so the coordinate whose partial derivative is nonzero is  $m_{2n,1}$ . The nonzero partial derivatives are -1 for  $Q^1$  and 1 for  $Q^2$ , coming from the signs of the permutations involved.

It is not hard to see that  $(Q^1)^{n-1} = E_{1,n} - E_{2n,n+1}$  and  $(Q^2)^{n-1} = E_{1,2n} - E_{n,n+1}$ . Thus  $(\tilde{M}^{n-1})_{i,j} + P_{j,i}$  evaluated on  $Q^1$  is 0 for every positive pair  $i, j$ , and likewise  $\tilde{M}_{i,j}^{n-1} - P_{j,i}$  evaluated on  $Q^2$ . We know that partial derivatives of a fundamental invariant or entries of a power of  $\tilde{M}$  give a copy of  $V_\phi$ , so the sum or difference of these will also be copies of  $V_\phi$ . Hence  $I_N$  is generated by  $\{(\tilde{M}^{n-1})_{i,j} \pm P_{j,i} \mid i, j \text{ positive pairs}\}$  where the sign depends on  $N$ .

Now let  $J$  be the ideal generated by  $I_N$  and fundamental invariants  $\tilde{\tau}_1, \dots, \tilde{\tau}_{n/2-1}, \tilde{\sigma}_n$ . We show that this is in fact  $J_N$ , the ideal of  $\overline{\mathcal{O}}_{\Theta_N}$  in  $\mathbb{C}[\mathfrak{g}]$ . We need only show that  $\tilde{\tau}_{n/2}$  and the entries of  $\tilde{M}^{n+1}$  are in the ideal generated by  $\tilde{\tau}_1, \dots, \tilde{\tau}_{n/2}$  and  $\tilde{M}_{i,j}^{n-1} - P_{j,i}$  for positive pairs  $i, j$ .

First, we observe that  $\tilde{M}^2(\tilde{M}^{n-1} \pm P^T) = \tilde{M}^{n+1} \pm \tilde{M}^2 P^T$ . We claim that  $\tilde{M} P^T = \tilde{\sigma}_n I_{2n}$ , implying that  $\tilde{M}^2 P^T = \tilde{\sigma}_n \tilde{M}$ , which would mean that entries of  $\tilde{M}^{n+1}$  are in  $J$ . Because of the way  $\tilde{\sigma}_n$  is defined, for any given row  $i$  of  $\tilde{M}$ , each term of  $\tilde{\sigma}_n$  contains exactly one entry  $\tilde{M}_{i,j}$  as a factor. Thus we may think of  $\tilde{\sigma}_n$  as a homogeneous polynomial of degree 1 in the indeterminates occurring in row  $i$  of  $\tilde{M}$ , with all other indeterminates in the coefficient ring. From this perspective, we see that  $(\tilde{M} P^T)_{i,i} = \sum_j \tilde{M}_{i,j} P_{j,i}^T = \sum_j \tilde{M}_{i,j} P_{i,j} = \sum_{\tilde{M}_{i,j} \neq 0} \tilde{M}_{i,j} \frac{\partial \tilde{\sigma}_n}{\partial \tilde{M}_{i,j}} = \tilde{\sigma}_n$ .

For  $i \neq j$ , we have  $(\tilde{M} P^T)_{i,j} = \sum_k \tilde{M}_{i,k} P_{j,k}$ . This is equal to  $\tilde{\sigma}_n$  evaluated on the matrix formed from  $\tilde{M}$  by replacing row  $j$  with a repeated row  $i$  and likewise replacing column  $j+n$  with column  $i+n$ . We previously showed in the lemma above that switching two rows and their corresponding columns would change the sign of  $\tilde{\sigma}_n$ . If two rows and their columns are equal, then it must be that  $\tilde{\sigma}_n$  evaluated on that matrix is 0. Hence  $\tilde{M} P^T = \tilde{\sigma}_n I_{2n}$  as claimed.



This implies that  $J$  already contains the degree  $n+1$  copy of  $V_\phi$ . Finally, the invariant  $\text{Trace}(\tilde{M}^n)$  is in our ideal  $J$ , since  $\tilde{M}^n = \tilde{M}(\tilde{M}^{n-1} \pm P^T) - \tilde{\sigma}_n \tilde{M}$ . The ideal generated by  $\tilde{\tau}_1, \dots, \tilde{\tau}_{n/2-1}$  and  $\text{Trace}(\tilde{M}^n)$  contains  $\tilde{\tau}_{n/2}$ , so  $J$  likewise contains this fundamental invariant, and hence  $J = I_N + J_{\Theta'} = J_N$ .  $\square$

## APPENDIX B

### MAGMA CODE

First, we include the Magma code we used to check our choices of fundamental invariants restricted to the principal Slodowy slice, as explained in Section 3.2 of Chapter 3. This is written not as a function, but as code to be copied in after initializing with a choice of Lie algebra  $L$  and setting  $o$  to be the principal nilpotent orbit. An oddity in the way Magma stores nilpotent orbits means that the principal nilpotent is the first in the sequence for classical types and the last in the sequence for exceptional types.

```
n:=Rank(RootDatum(L));
exp:=Exponents(RootDatum(L));
rho:=StandardRepresentation(L);

TR:= SL2Triple(o);

E:= TR[1];
H:= TR[2];
F:= TR[3];

K:=Centralizer(L,F);
c:=Morphism(K,L);

P<[u]>:=PolynomialRing(Rationals(),Dimension(K));

C:= Matrix(rho(0));
for j:=1 to Dimension(K) do
C:=C+Matrix(rho(c(K.j)))*u[Dimension(K)-j+1];
end for;
```

```

X:=Matrix(rho(E));
M:=C+X;
ch:=CharacteristicPolynomial(M);
k:=#Rows(M);

f1:=[P|];
f2:=f1;
for i:=1 to n do
f1[i]:=Trace(M^(exp[i]+1));
f2[i]:=Coefficient(ch,k-exp[i]-1);
end for;

```

Next we include the Magma code, discussed in Section 3.3 of Chapter 3, for a function which takes in a Lie algebra  $L$ , a nilpotent orbit  $o$  in that Lie algebra to which the Slodowy slice is taken, and two integers ( $s$  and  $family$ ) indicating which nilpotent variety will be intersected. Giving the number of orthogonal simple short roots in  $\Theta$  and the family (1 or 2) is enough to specify any of our nilpotent varieties except for the very even case in even rank  $D_n$ . This function will not give an answer in that case, because as we have seen a different choice of the two fundamental invariants of degree  $n$  is needed. This function also cannot handle types  $B$ ,  $F$ , or  $G$  because we were unable to find a way to embed the multiply-laced types into simply-laced ones.

```

function NilSliceEquations(L,o,s,family)

name:=CartanName(L);
type:=name[1];
n:=StringToInteger(Substring(name,2,#name-1));

rho:=StandardRepresentation(L);

TR:= SL2Triple(o);

E:= TR[1];
H:= TR[2];
F:= TR[3];

```

```

K:=Centralizer(L,F);
c:=Morphism(K,L);

P<[u]>:=PolynomialRing(Rationals(),Dimension(K));

C:= Matrix(rho(0));
for j:=1 to Dimension(K) do
C:=C+Matrix(rho(c(K.j)))*u[Dimension(K)-j+1];
end for;

X:=Matrix(rho(E));
M:=C+X;
exp:=Exponents(RootDatum(L));

f:=[P];
for i:=1 to n do
f[i]:=Trace(M^(exp[i]+1));
end for;

if type eq "A" then
J:=ideal<P|[f[i]:i in [1..(n-s)]],Eltseq(M^(n+1-s))>;
elif type eq "C" then
J:=ideal<P|[f[i]:i in [1..(n-s)]],Eltseq(M^(2*n-2*s))>;
elif type eq "D" then
f[Ceiling(n/2)]:=Factorization(Determinant(M))[1,1];
if family eq 1 then
J:=ideal<P|[f[i]:i in [1..(n-s)]],Eltseq(M^(2*n-2*s-1))>;
elif family eq 2 then
if s ge 3 then
J:=ideal<P|[f[i]:i in [1..(n-s)]],
[Derivative(f[Ceiling(n/2)],i):i in [1..Dimension(K)]],
[Derivative(f[n-s+2],i):i in [1..Dimension(K)]]>;

```

```

elif s lt 3 then
J:=ideal<P|[f[i]:i in [1..(n-s+1)]],
[Derivative(f[Ceiling(n/2)],i):i in [1..Dimension(K)]]>;
end if;
end if;
elif type eq "E" then
if family eq 1 then
if s ge 3 then
J:=ideal<P|[f[i]:i in [1..(n-s)]],
[Derivative(f[n-s+2],i):i in [1..Dimension(K)]],
[Derivative(f[n-s+1],i):i in [1..Dimension(K)]]>;
elif s lt 3 then
J:=ideal<P|[f[i]:i in [1..(n-s)]],
[Derivative(f[n-s+1],i):i in [1..Dimension(K)]]>;
end if;
elif family eq 2 then
J:=ideal<P|[f[i]:i in [1..(n-s+1)]],
[Derivative(f[Ceiling(n/2)],i):i in [1..Dimension(K)]]>;
end if;
end if;

return GroebnerBasis(J);

end function;

```

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