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Abstract

This paper is an introduction, in a simplified setting, to Lusztig's theory of character sheaves. It develops a notion of character sheaves on reductive Lie algebras which is more general then such notion of Lusztig, and closer to Lusztig's theory of character sheaves on groups. The development is self contained and independent of the characteristic p of the ground field. The results for Lie algebras are then used to give simple and uniform proofs for some of Lusztig's results on groups.

Introduction

This paper gives a self contained presentation of character sheaves on reductive Lie algebras, independent of the characteristic p of the field (except for the section on the characteristic varieties which have not been defined for $p > 0$). For simplicity, the formulations in the text involve only the D-modules (usually irregular), i.e., the case $p = 0$. However the definitions have "obvious" modifications (in the light of [Lu3]) for perverse sheaves in characteristic $p > 0$ and then the proofs are the same.

The first section studies two Radon transforms and the second defines sheaves monodromic under an action of a vector group. In the third section we linearize Lusztig's construction of character sheaves on reductive groups and show that the resulting character sheaves on Lie algebras are precisely the Fourier transforms of orbital sheaves. In sections 4 and 5 we give an elementary proof of results of [Lu3] (proofs of [Lu3] are for sufficiently large $p > 0$ and use results from [Lu1, Lu2]). The main result is that the Fourier transforms of orbital sheaves are precisely the irreducible constituents of sheaves obtained by inducing from cuspidal sheaves (5.3). For $p = 0$ character sheaves can be described by having a nilpotent characteristic variety and a monodromic behavior in some directions (6.4), this is analogous to a result for groups ([Gi, MV]). The last

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section uses nearby cycles to reprove some results on induction and restriction from [Lu2].

The purpose of this paper is to give a perspective to [Lu3] and an introduction to character sheaves on reductive groups. The original goal was a direct proof in characteristic zero of the fact that the cuspidal sheaves on groups are character sheaves (theorem 6.8). A similar proof was found independently by Ginzburg ([Gi]). Lusztig's original proof is spread though the series of papers ([Lu2]).

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Notation.

We fix an algebraically closed field $\Bbbk$ of characteristic 0. For a $\Bbbk$-variety X, $\mathfrak{m}(X)$ denotes the category of holonomic D-modules on X and $D(X) = D^b[\mathfrak{m}(X)]$ is its bounded derived category, our basic reference for D-modules is [Bo]. If a connected algebraic group A acts on X then $\mathfrak{m}_A(X)$ denotes the subcategory of equivariant sheaves in $\mathfrak{m}(X)$ and $D_A(X)$ is the corresponding triangulated category constructed by Bernstein and Lunts (see [BL, MV]). For any morphism of varieties $X \xrightarrow{\pi} Y$ there are direct image and inverse image functors $f_*, f_!$ and $f^*, f^!$, and the duality functor $\mathbb{D}_X : D(X)^o \to D(X)$ ([Bo]). If the map f is equivariant under a group A the same functors exist on the level of equivariant derived categories ([BL, MV]). We will use a normalized pull-back functor $\pi^\circ \stackrel{\text{def}}{=} \pi^! \,[\dim Y - \dim X]$, and if π is an embedding then $\mathcal{F}|X$ will denote the pull-back $\pi^\circ\mathcal{F}$. For dual vector bundles V and V^* one has the Fourier transform equivalence $\mathcal{F}_V : \mathfrak{m}(V) \to \mathfrak{m}(V^*)$, its basic properties can be found in [Br, KL]).

G will denote a reductive algebraic group over $\Bbbk$ and $P = L \ltimes U$ a Levi decomposition of a parabolic subgroup, while $\mathfrak{g}, \mathfrak{p}, \mathfrak{l}, \mathfrak{u}$ will be the corresponding Lie algebras. We fix an invariant non-degenerate bilinear form on $\mathfrak{g}$ and use it to identify $\mathfrak{g}^*$ and $\mathfrak{g}$, and $\mathfrak{l}^*$ and $\mathfrak{l}$ etc.[1] Now, Fourier transform $\mathcal{F}_{\mathfrak{g}}$ is an autoequivalence of $\mathfrak{m}(\mathfrak{g})$. The adjoint action of $g \in G$ is denoted ${}^g x \stackrel{\text{def}}{=} (\mathrm{Ad}\, g)x$.

1 Grothendieck transform $\mathcal{G}$ and horocycle transform $\mathcal{H}$

1.1. Let $\mathcal{B}$ be the flag variety of $\mathfrak{g}$, viewed as the moduli of Borel subalgebras of $\mathfrak{g}$. Then $\mathfrak{b}^\circ = \{(\mathfrak{b}, x) \in \mathcal{B} \times \mathfrak{g},\ x \in \mathfrak{b}\}$ is the G-homogeneous vector bundle over $\mathcal{B}$ with the fiber $\mathfrak{b}$ at $\mathfrak{b} \in \mathcal{B}$. Similarly, there are vector bundles $\mathfrak{n}^\circ$, $\mathfrak{g}^\circ$, $(\mathfrak{g}/\mathfrak{n})^\circ$ with the fibers $[\mathfrak{b}, \mathfrak{b}]$, $\mathfrak{g}$, $\mathfrak{g}/[\mathfrak{b}, \mathfrak{b}]$ at $\mathfrak{b} \in \mathcal{B}$.

[1] This is just for simplicity of exposition.

1.2. Grothendieck resolution of $\mathfrak{g}$ is the projection $\mathfrak{b}^\circ \xrightarrow{g} \mathfrak{g}$ while Springer resolution of the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$ is its restriction $\mathfrak{n}^\circ = g^{-1}(\mathcal{N}) \xrightarrow{s} \mathfrak{g}$.

1.3. Define Grothendieck and horocycle transforms $D(\mathfrak{b}^\circ) \xleftarrow{\mathcal{G}} D(\mathfrak{g}) \xrightarrow{\mathcal{H}} D([\mathfrak{g}/\mathfrak{n}]^\circ)$, by $\mathcal{G} = g^! = g^\circ$ and $\mathcal{H} = q_* \circ p^\circ$, using the diagram

$$\begin{tikzcd} & \mathcal{B} \times \mathfrak{g} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathfrak{g} & & (\mathfrak{g}/\mathfrak{n})^\circ \end{tikzcd} \qquad \text{defined by} \qquad \begin{tikzcd} & (\mathfrak{b}, x) \arrow[dl] \arrow[dr] & \\ x & & (\mathfrak{b}, x + [\mathfrak{b}, \mathfrak{b}]). \end{tikzcd}$$

Their left adjoints are $\check{\mathcal{G}} = g_! = g_*$ and $\check{\mathcal{H}} = p_* \circ q^\circ$.

1.4. Theorem. (i) Fourier transform interchanges $\mathcal{G}$ and $\mathcal{H}$: $\mathcal{F}_{\mathfrak{b}^\circ} \circ \mathcal{G} = \mathcal{H} \circ \mathcal{F}_{\mathfrak{g}}$, hence also $\check{\mathcal{G}}$ and $\check{\mathcal{H}}$.

(ii) $\mathcal{F}_{\mathfrak{g}}(g_* \mathcal{O}_{\mathfrak{b}^\circ}) = s_* \mathcal{O}_{\mathfrak{n}^\circ}$.

(iii) $\check{\mathcal{G}} \circ \mathcal{G} = g_* \mathcal{O}_{\mathfrak{b}^\circ} \otimes_{\mathcal{O}_{\mathfrak{g}}} -$.

(iv) $\check{\mathcal{H}} \circ \mathcal{H} = s_* \mathcal{O}_{\mathfrak{n}^\circ} * -$.

Remarks. a) (ii) is a result of Kashiwara ([Br]) while (iii) (on G instead of $\mathfrak{g}$) appears in [Gi, MV]. b) The convolution in d) is defined by $A * B = +_*(A \boxtimes B)$ for the addition $+ : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

Proof. (i) The map adjoint to $\mathfrak{g}^\circ \xrightarrow{q} (\mathfrak{g}/\mathfrak{n})^\circ$ is the inclusion $\mathfrak{b}^\circ \overset{j}{\hookrightarrow} \mathfrak{g}^\circ$, so

$$\begin{aligned} \mathcal{F}_{\mathfrak{g}} \circ \mathcal{H} &= \mathcal{F}_{\mathfrak{g}} \circ q_* \circ p^\circ = j^\circ \circ \mathcal{F}_{\mathfrak{g}^\circ} \circ p^\circ \\ &= j^\circ \circ p^\circ \circ \mathcal{F}_{\mathfrak{g}} = g^\circ \circ \mathcal{F}_{\mathfrak{g}} = \mathcal{G} \circ \mathcal{F}_{\mathfrak{g}}. \end{aligned}$$

(ii) The sheaves $g_* \mathcal{O}_{\mathfrak{b}^\circ}$ and $s_* \mathcal{O}_{\mathfrak{n}^\circ}$ on $\mathfrak{g}$, are direct images from $\mathfrak{g}^\circ$ of $(\mathfrak{b}^\circ \overset{j}{\hookrightarrow} \mathfrak{g}^\circ)_* \mathcal{O}_{\mathfrak{b}^\circ}$ and $(\mathfrak{n}^\circ \overset{j}{\hookrightarrow} \mathfrak{g}^\circ)_* \mathcal{O}_{\mathfrak{n}^\circ}$. These two are switched by the Fourier transform since $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{b}^\perp$ for $\mathfrak{b} \in \mathcal{B}$.

(iii) $A \otimes_{\mathcal{O}_{\mathfrak{g}}} g_* \mathcal{O}_{\mathfrak{b}^\circ} = g_*(g^\circ A \otimes_{\mathcal{O}_{\mathfrak{b}^\circ}} \mathcal{O}_{\mathfrak{b}^\circ}) = \check{\mathcal{G}}(\mathcal{G}A)$.

Now (iv) follows since $\mathcal{F}_{\mathfrak{g}}(A * B) = \mathcal{F}_{\mathfrak{g}} A \otimes_{\mathcal{O}_{\mathfrak{g}}} \mathcal{F}_{\mathfrak{g}} B$.

1.5. Corollary. Functors $\check{\mathcal{G}} \circ \mathcal{G}$ and $\check{\mathcal{H}} \circ \mathcal{H}$ both contain identity functors as direct summands.

Proof. It is well known that g is a small resolution hence $g_* \mathcal{O}_{\mathfrak{b}^\circ}$ is a semisimple sheaf and one summand is $\mathcal{O}_{\mathfrak{g}}$. Now the claim for $\mathcal{G}$ follows from (ii) in the theorem and for $\mathcal{H}$ one uses (i).

1.6. Analogous transforms exist for G-equivariant sheaves,

$$\begin{tikzcd} D_G([\mathfrak{g}/\mathfrak{n}]^\circ) \arrow[r, shift right, "\check{\mathcal{H}}"'] & D_G(\mathfrak{g}) \arrow[l, shift right, "\mathcal{H}"'] \arrow[r, shift left, "\mathcal{G}"] & D_G(\mathfrak{b}^\circ) \arrow[l, shift left, "\check{\mathcal{G}}"] \end{tikzcd};$$

and from now on we restrict ourselves to the G-equivariant setting.

1.7. A basic tool in the equivariant setting are Bernstein's induction functors γ_B^A, Γ_B^A (see [MV]). If a group A acts on a smooth variety X then for any subgroup B the forgetful functor $\mathfrak{F}_B^A : D_A(X) \to D_B(X)$ has a left adjoint $\gamma_B^A[\dim A/B]$ which one can describe via the diagram

$$\begin{tikzcd} A \times X \arrow[r, "\nu"] \arrow[d, "p"'] & A \times_B X \arrow[d, "a"] \\ X & X \end{tikzcd}, \quad \text{here} \quad \begin{tikzcd} (g,x) \arrow[r] \arrow[d, mapsto] & \overline{(g,x)} \arrow[d, mapsto] \\ x & gx \end{tikzcd}.$$

For any $\mathcal{A} \in D_B(X)$ there is a unique (up to a unique isomorphism) $\overline{\mathcal{A}} \in D_A(A \times_B X)$ such that $\nu^\circ \overline{\mathcal{A}} = p^\circ \mathcal{A}$ in $D_{A \times B}(A \times X)$. Now $\gamma_B^A \mathcal{A} = a_! \overline{\mathcal{A}}$. Similarly, $\Gamma_B^A \mathcal{A} = a_* \overline{\mathcal{A}}$ is a shift of the right adjoint of the forgetful functor. If A/B is complete then $\gamma_B^A = \Gamma_B^A$.

Since the forgetful functor commutes with all standard functors (including the Fourier transform if X is a vector bundle), the same is also true for Γ_B^A.

1.8. Fix a Borel subalgebra $\mathfrak{b} \in \mathcal{B}$, put $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ and consider the maps

$$(*) \qquad \mathfrak{b}^\circ \overset{r}{\hookleftarrow} \mathfrak{b} \overset{j}{\hookrightarrow} \mathfrak{g} \overset{\pi}{\twoheadrightarrow} \mathfrak{g}/\mathfrak{n} \overset{s}{\hookrightarrow} (\mathfrak{g}/\mathfrak{n})^\circ, \quad r(x) = (\mathfrak{b}, x), \ s(y) = (\mathfrak{b}, y).$$

1.9. Lemma. The following diagram is commutative::

$$\begin{tikzcd} D_G(\mathfrak{b}^\circ) \arrow[r, shift right, "\check{\mathcal{G}}"'] \arrow[d, shift right, "r^\circ"'] & D_G(\mathfrak{g}) \arrow[l, shift right, "\mathcal{G}"'] \arrow[r, shift left, "\mathcal{H}"] \arrow[d, equal] & D_G((\mathfrak{g}/\mathfrak{n})^\circ) \arrow[l, shift left, "\check{\mathcal{H}}"] \arrow[d, shift right, "s^\circ"'] \\ D_B(\mathfrak{b}) \arrow[r, shift right, "\Gamma_B^G \circ j_*"'] \arrow[u, shift right, "\Gamma_B^G \circ r_*"'] & D_G(\mathfrak{g}) \arrow[l, shift right, "j^\circ"'] \arrow[r, shift left, "\pi_*"] & D_B(\mathfrak{g}/\mathfrak{n}) \arrow[l, shift left, "\Gamma_B^G \circ \pi^\circ"] \arrow[u, shift right, "\Gamma_B^G \circ s_*"'] \end{tikzcd}.$$

All vertical arrows are inverse equivalence of categories and they also commute with the Fourier transform.

Proof. Since the diagram $(*)$ is self adjoint, the statements for the first and the second square are exchanged by the Fourier transform. Since $\mathfrak{b}^\circ = G \times_B \mathfrak{b}$, the functors r° and $\Gamma_B^G \circ r_*$ are inverse equivalences by lemma 1.4 in [MV]. The pull-back r° commutes with the Fourier transform, hence so does $\Gamma_B^G \circ r_*$. For commutativity observe that $r^\circ \circ \mathcal{G} = r^\circ \circ g^\circ = j^\circ$, and since g is a G-equivariant map

$$\check{\mathcal{G}} \circ \Gamma_B^G \circ r_* = g_* \circ \Gamma_B^G \circ r_* = \Gamma_B^G \circ g_* \circ r_* = \Gamma_B^G \circ j_*.$$

1.10. In the remainder we use identifications from Lemma 1.9 as definitions. So $\mathfrak{b} \overset{j}{\hookrightarrow} \mathfrak{g} \overset{\pi}{\to} \mathfrak{g}/\mathfrak{n}$ and

$$\begin{tikzcd}
D_B(\mathfrak{b}) \arrow[dr, shift right, "\check{\mathcal{G}}"'] & & D_B(\mathfrak{g}/\mathfrak{n}) \arrow[dl, shift left, "\check{\mathcal{H}}"] \\
& D_G(\mathfrak{g}) \arrow[ul, shift right, "\mathcal{G}"'] \arrow[ur, shift left, "\mathcal{H}"] &
\end{tikzcd}
\qquad
\begin{array}{ll}
\check{\mathcal{H}} = \Gamma_B^G \circ \pi^\circ, & \mathcal{H} = \pi_* \circ \mathfrak{F}_B^G, \\[2ex]
\check{\mathcal{G}} = \Gamma_B^G \circ j_*, & \mathcal{G} = j^0 \circ \mathfrak{F}_B^G.
\end{array}$$

The theorem 1.4 and its corollary still hold in this setting.

2 Monodromic sheaves on vector spaces

Let $V \subseteq U$ be finite dimensional vector spaces over $\Bbbk$.

2.1. Sheaf $0 \neq \mathcal{A} \in \mathfrak{m}(V)$ is said to be a character sheaf if $+^\circ\mathcal{A} \cong \mathcal{A} \boxtimes \mathcal{A}$ for $+ : V \times V \to V$. The equivalent condition on $\mathcal{B} = \mathcal{F}_V\mathcal{A}$ is $\Delta_*\mathcal{B} \cong \mathcal{B} \boxtimes \mathcal{B}$ for the diagonal $\Delta : V^* \hookrightarrow V^* \times V^*$. This implies $(\operatorname{supp} \mathcal{B})^2 \subseteq \Delta(V^*)$ hence $\mathcal{B}$ is supported at a point. Now the condition is equivalent to irreducibility of $\mathcal{B}$. So, the character sheaves on V are precisely the connections $\mathcal{L}_\alpha$, $\alpha \in V^*$, for $\mathcal{L}_\alpha = \mathcal{F}_{V^*}[(\alpha \hookrightarrow V^*)_*\mathcal{O}_{\mathrm{pt}}]$.

2.2. Let $U^* \overset{q}{\to} V^*$ be the quotient map. For any finite subset $\theta \subseteq V^*$ we say that a sheaf $\mathcal{B} \in \mathfrak{m}(U)$ is θ-monodromic if the support of $\mathcal{F}_U\mathcal{B}$ lies in $(-1)\cdot q^{-1}\theta$. Such sheaves form a Serre subcategory $\mathcal{M}_\theta(U)$ of $\mathfrak{m}(U)$. The full triangulated subcategory of $D(U)$ consisting of sheaves with θ-monodromic cohomologies is precisely the derived category of $\mathcal{M}_\theta(U)$. Observe that $\mathcal{M}_\theta(U) = \bigoplus_{\alpha\in\theta} \mathcal{M}_\alpha(U)$. For $\alpha \in V^*$, a sheaf $\mathcal{B} \in \mathfrak{m}(U)$ is α-monodromic if and only if $+^\circ\ \mathcal{B} \cong \mathcal{L}_\alpha \boxtimes \mathcal{B}$.

2.3. The category of V-monodromic sheaves on U is the category $\mathcal{M}(U, V) \overset{\text{def}}{=} \bigoplus_{\alpha\in V^*} \mathcal{M}_\alpha(U)$. A sheaf $\mathcal{B} \in \mathfrak{m}(U)$ is monodromic if and only if the action of V on $\Gamma(U, \mathcal{B})$ (via $V \subseteq U \subseteq \Gamma(U, \mathcal{D}_U)$), is locally finite — this is the requirement that the action of $V \subseteq \Gamma(U^*, \mathcal{O}_{U^*})$ on $\Gamma(U^*, \mathcal{F}_U\mathcal{B})$ is locally finite i.e. that the support of $\mathcal{F}_U\mathcal{B}$ is finite modulo $V^\perp$. So, the category of V-monodromic sheaves on V is semisimple (while this would not be true for a torus).

3 Character sheaves and the orbital sheaves

3.1. Fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ and denote $[\mathfrak{n},\mathfrak{n}]$ and $\mathfrak{h} = \mathfrak{b}/\mathfrak{n}$. Then the diagram

$$\mathfrak{h} \xleftarrow{\nu} \mathfrak{b} \xhookrightarrow{j} \mathfrak{g} \xrightarrow{\pi} \mathfrak{g}/\mathfrak{n} \longleftarrow\!\rhook \mathfrak{h}$$

is self adjoint. For any finite subset $\theta \subseteq \mathfrak{h} = \mathfrak{h}^*$ let $\mathcal{P}_\theta(\mathfrak{b})$ be the category of sheaves on $\mathfrak{b}$ supported on $\nu^{-1}\theta$, so that $\mathcal{M}_\theta(\mathfrak{g}/\mathfrak{n}) = \mathcal{F}_\mathfrak{b}(\mathcal{P}_\theta(\mathfrak{b}))$ is the category of sheaves on $\mathfrak{g}/\mathfrak{n}$ which are θ-monodromic in the direction of $\mathfrak{h} \subset \mathfrak{g}/\mathfrak{n}$. Also, we denote $\mathcal{P}(\mathfrak{b}) = \bigcup_\theta \ \mathcal{P}_\theta(\mathfrak{b})$, $\mathcal{M}(\mathfrak{g}/\mathfrak{n}) = \bigcup_\theta \ \mathcal{M}_\theta(\mathfrak{g}/\mathfrak{n})$ and in the equivariant version $\mathcal{M}_{\theta,B}(\mathfrak{g}/\mathfrak{n}) = \mathcal{M}_\theta(\mathfrak{g}/\mathfrak{n}) \bigcap \mathfrak{m}_B(\mathfrak{g}/\mathfrak{n})$ etc.

3.2. We define character sheaves on $\mathfrak{g}$ as irreducible constituents of the cohomology sheaves of complexes $\check{\mathcal{H}}(\mathcal{A})$ for $\mathcal{A} \in \mathcal{M}_B(\mathfrak{b}) = \bigcup_\theta \mathcal{M}_{\theta,B}(\mathfrak{b})$ (see (1.10)). On the other hand, an irreducible G-equivariant sheaf on $\mathfrak{g}$ is said to be orbital ([Lu3]), if its support is the closure of a single G-orbit.

3.3. Lemma ([Lu1]). a) Let $x = s + n \in \mathfrak{g}$ be a Jordan decomposition with s semisimple and n nilpotent. Let $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{u}$ be a parabolic subalgebra of $\mathfrak{g}$ such that $\mathfrak{l} = Z_\mathfrak{g}(s)$. Then ${}^U x = x + \mathfrak{u}$ for the unipotent subgroup U corresponding to $\mathfrak{u}$.

b) For any $\alpha \in \mathfrak{h}$ the semisimple components of all elements of $\nu^{-1}\alpha$ are conjugate.

c) For any G-orbit α in $\mathfrak{g}$, $\nu(\alpha \cap \mathfrak{b})$ is finite.

Proof. a) appears in the proof of lemma 2.7 in [Lu1]. For b) choose semisimple $s \in v^{-1}\alpha = s + \mathfrak{n}$ and put $\mathfrak{l} = Z_\mathfrak{g}(s)$ and $\mathfrak{p} = \mathfrak{l} + \mathfrak{b} = \mathfrak{l} \ltimes \mathfrak{u}$. Now s is the semisimple component of any element of $s + \mathfrak{l} \cap \mathfrak{n}$ and by a) $(\mathrm{Ad}\, U)(s + \mathfrak{l} \cap \mathfrak{n}) = (s + \mathfrak{l} \cap \mathfrak{n}) + \mathfrak{u} = s + \mathfrak{n}$.

In c) choose $x \in \alpha$. For $g \in G$, ${}^g x \in \mathfrak{b}$ is equivalent to $x \in {}^{g^{-1}}\mathfrak{b}$, and then $\nu({}^g x)$ is the same as the image of x in ${}^{g^{-1}}\mathfrak{b}/{}^{g^{-1}}\mathfrak{n} \cong \mathfrak{h}$. So, $\nu(\alpha \cap \mathfrak{b})$ is the image of the fiber $g^{-1}x$ under the map $\mathfrak{b}^o \to \mathfrak{h}$ given by $(\mathfrak{b}', x) \mapsto x + [\mathfrak{b}', \mathfrak{b}'] \in \mathfrak{b}'/[\mathfrak{b}', \mathfrak{b}'] \cong \mathfrak{h}$. This is a complete subvariety of an affine variety, so it is finite.

3.4. Theorem. (See theorem 5.b) in [Lu3].) Character sheaves are the same as Fourier transforms of orbital sheaves.

3.5. Proposition. Orbital sheaves are exactly the irreducible constituents of complexes $\check{\mathcal{G}}(\mathcal{B})$ with $\mathcal{B} \in \mathcal{P}_B(\mathfrak{b})$.

Proofs. The first claim is the Fourier transform of the second since $\mathcal{M}_B(\mathfrak{g}/\mathfrak{n}) = \mathcal{F}_\mathfrak{b}\ \mathcal{P}_B(\mathfrak{b})$ and $\mathcal{F}_\mathfrak{g} \circ \check{\mathcal{H}} = \check{\mathcal{G}} \circ \mathcal{F}_\mathfrak{g}$ (theorem 1.4). If $\mathcal{C}$ is an irreducible constituent of $\check{\mathcal{G}}(\mathcal{D})$ for some $\mathcal{D} \in \mathcal{P}_B(\mathfrak{b})$, then this $\mathcal{D}$ can be chosen to be irreducible. Now $\mathrm{supp}(\mathcal{D}) \subseteq \nu^{-1}\alpha$ for some $\alpha \in \mathfrak{h}$. Therefore $\mathrm{supp}(\mathcal{C})$ lies in the closure of

$\mathrm{Ad}(G)\cdot\nu^{-1}\alpha$, and this is covered by finitely many G-orbits (Lemma 3.3.b) and $\mathcal{C}$ is orbital.

Conversely, let $\mathcal{C}$ be an orbital sheaf. By the Corollary 1.5, $\mathcal{C}$ is a constituent of $\check{\mathcal{G}}(\mathcal{G}(\mathcal{C}))$, but $\mathcal{G}(\mathcal{C}) = j^0\mathcal{C}$ is in $\mathcal{P}_B(\mathfrak{b})$ by Lemma 3.3.c.

3.6. Remark. Let us say that the L-packet of character sheaves attached to an orbit C in $\mathfrak{g}^*$ consists of Fourier transforms of all orbital sheaves with the support $\overline{C}$. Its elements are in bijection with the irreducible representations of $\pi_0[Z_G(x)]$ for any $x \in C$. If x has Jordan decomposition $s+n$ and $L = Z_G(s)$, this is $\pi_0[Z_L(n)]$.

The semisimple part of C is a semisimple orbit in $\mathfrak{g}^*$ so it corresponds to a Weyl group orbit θ in $\mathfrak{h}^*$. We say that θ is the infinitesimal character of sheaves in the L-packet of C. So a character sheaf $\mathcal{A}$ has infinitesimal character θ if $\mathcal{H}(\mathcal{A}) \in \mathcal{M}_{\theta,B}(\mathfrak{g}/\mathfrak{n})$, or equivalently, if $\mathcal{A}$ is a constituent of some $\check{\mathcal{H}}(\mathcal{B})$ with $\mathcal{B} \in \mathcal{M}_{\theta,B}(\mathfrak{g}/\mathfrak{n})$.

4 Cuspidal sheaves

By observing that the restriction functor commutes with the Fourier transform we obtain an elementary proof of Lusztig's characterization of cuspidal sheaves $\mathcal{C}$ in terms of the support of $\mathcal{C}$ and $\mathcal{F}(\mathcal{C})$.

4.1. Restriction and induction. Let P be a parabolic subgroup of G with the unipotent radical U and $\overline{P} = P/U$. Let $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{u}$ and $\overline{\mathfrak{p}}$ be the corresponding Lie algebras. Restriction (or $\mathfrak{u}$-homology) functor $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}} : D_G(\mathfrak{g}) \to D_{\overline{P}}(\overline{\mathfrak{p}})$ is given in terms of the Cartesian diagram

$$\begin{tikzcd} & \mathfrak{p} \arrow[dl, "i"'] \arrow[dr, "\pi"] & \\ \mathfrak{g} \arrow[dr, "\tau"'] & & \overline{\mathfrak{p}}, \arrow[dl, "j"] \\ & \mathfrak{g}/\mathfrak{u} & \end{tikzcd} \qquad \text{by} \quad \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}} = \pi_* \circ i^! \circ \mathfrak{F}^G_P = j^! \circ \tau_* \circ \mathfrak{F}^G_P.$$

For convenience we have chosen to use the dual of the functor from [Lu3], we will see in (4.7) that this does not matter. The left adjoint of $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}}$ is the induction $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{p}} = \Gamma^G_P \circ i_! \circ \pi^*[\dim G/P] = \Gamma^G_P \circ i_* \circ \pi^\circ$.

4.2. Lemma. Restriction and induction commute with the Fourier transform.

Proof. It suffices to look at the restriction. Since τ and j are adjoints of i and π

$$\mathcal{F}_{\overline{\mathfrak{p}}} \circ (\pi_* \circ i^! \circ \mathfrak{F}_P^G) = j^\circ \circ \tau_*[\dim \mathfrak{u}] \circ \mathfrak{F}_P^G \circ \mathcal{F}_{\mathfrak{g}} = (j^! \circ \tau_* \circ \mathfrak{F}_P^G) \circ \mathcal{F}_{\mathfrak{g}}.$$

The following two observations and their proofs are from Lemma 2.7 in [Lu1].

4.3. Lemma. Let $x = s+n$ be the Jordan decomposition of $x \in \mathfrak{g}$. Let $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{u}$ be a parabolic subalgebra with a Levi factor $\ell = Z_{\mathfrak{g}}(s)$. Then for any $\mathcal{A} \in \mathfrak{m}(\mathfrak{g})$

$$(x \hookrightarrow \mathfrak{l})^! \mathrm{Res}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{A} = (x \hookrightarrow \mathfrak{g})^! \mathcal{A}\ [2 \dim U]$$

Proof. We omit the forgetful functor from the notation and use the base change

$$(x \hookrightarrow \mathfrak{l})^! \mathrm{Res}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{A} = (x + \mathfrak{u} \to x)_* (x + \mathfrak{u} \overset{i}{\hookrightarrow} \mathfrak{g})^! \mathcal{A}.$$

Now $x + \mathfrak{u} = {}^U x$ by (3.3.a) and the U-equivariance gives

$$i^! \mathcal{A} = \mathcal{O}_{x+\mathfrak{u}} \otimes_{\mathbb{C}} (x \hookrightarrow x + \mathfrak{u})^! i^! \mathcal{A} = \mathcal{O}_{x+\mathfrak{u}} \otimes (x \hookrightarrow \mathfrak{g})^! \mathcal{A}[\dim U]$$

So the claim follows from $(\mathfrak{u} \to \mathrm{pt})_* \mathcal{O}_{\mathfrak{u}} = \mathcal{O}_{\mathrm{pt}}[\dim \mathfrak{u}]$.

4.4. Lemma. Any sheaf $\mathcal{A} \in \mathfrak{m}_G(\mathfrak{g})$ such that $\mathrm{Res}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{A} = 0$ for all proper parabolics $\mathfrak{p}$, is supported in $Z(\mathfrak{g}) + \mathcal{N}$.

Proof. For any $0 \neq \mathcal{A} \in \mathfrak{m}_G(\mathfrak{g})$ there is a smooth G-invariant subvariety $S \overset{j}{\hookrightarrow} \mathfrak{g}$ open and dense in the support of $\mathcal{A}$ and such that $j^! \mathcal{A}$ is a connection $\mathcal{E}$ on S. For x in S define s, n, $\mathfrak{l}$, $\mathfrak{p}$ as in (4.3), then $(x \hookrightarrow \mathfrak{l})^! \mathrm{Res}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{A} = (x \hookrightarrow S)^! \mathcal{E}[2 \dim \mathfrak{u}] \neq 0$. Therefore $\mathfrak{p} = \mathfrak{g}$, i.e., $s \in Z(\mathfrak{g})$. So $Z(\mathfrak{g}) + \mathcal{N}$ contains x, and then also S and $\overline{S}$.

4.5. Lemma. Let $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{u}$ be a parabolic subalgebra and $\mathcal{A} \in \mathfrak{m}_G(\mathfrak{g})$. If $\mathrm{supp}(\mathcal{A}) \subseteq \mathcal{N}$ then $\mathrm{supp}(\mathrm{Res}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{A}) \subseteq \mathcal{N} \cap \mathfrak{l}$.

Proof. If $x \in \mathrm{supp}(\mathrm{Res}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{A}) \subseteq \mathfrak{l}$ then $x + \mathfrak{u}$ meets $\mathrm{supp}\, \mathcal{A} \subseteq \mathcal{N}$, hence $x \in \mathcal{N}$.

4.6. An irreducible sheaf $\mathcal{A} \in \mathfrak{m}_G(\mathfrak{g})$ is said to be cuspidal ([Lu3]) if (i) $\mathrm{Res}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{A} = 0$ for any proper parabolic subalgebra $\mathfrak{p}$ and (ii) $\mathcal{A} = \mathcal{L} \boxtimes \mathcal{B}$ for a character sheaf $\mathcal{L}$ on $Z(\mathfrak{g})$ (see 3.1) and some sheaf $\mathcal{B}$ on $[\mathfrak{g}, \mathfrak{g}]$.

4.7. Theorem. (see [Lu3]). Let $\mathfrak{g}$ be a semisimple. An irreducible sheaf $\mathcal{A} \in \mathfrak{m}_G(\mathfrak{g})$ is cuspidal if and only if $\mathcal{A}$ and $\mathcal{F}_{\mathfrak{g}} \mathcal{A}$ are supported in $\mathcal{N}$.

Proof. Since the Fourier transform commutes with the restriction, if $\mathcal{A}$ is cuspidal so is $\mathcal{F}_{\mathfrak{g}} \mathcal{A}$. Then they are both supported in $\mathcal{N}$ by lemma 4.4. Conversely suppose now that $\mathcal{A}$ and $\mathcal{F}_{\mathfrak{g}} \mathcal{A}$ live on $\mathcal{N}$.

For any parabolic $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}$, both $\mathcal{B} = \mathrm{Res}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{A}$ and $\mathcal{F}_{\mathfrak{l}}\mathcal{B} = \mathrm{Res}_{\mathfrak{p}}^{\mathfrak{g}}(\mathcal{F}_{\mathfrak{g}}\mathcal{A})$ are supported in $\mathfrak{l} \cap \mathcal{N} \subseteq [\mathfrak{l}, \mathfrak{l}]$ by the Lemma 4.5. On the other hand if we write $\mathcal{B} = \overline{\mathcal{B}} \boxtimes (0 \hookrightarrow Z(\mathfrak{l}))_* \mathcal{O}_0$ for a sheaf $\overline{\mathcal{B}} \in \mathfrak{m}([\mathfrak{l}, \mathfrak{l}])$, then $\mathcal{F}_{\mathfrak{l}}\mathcal{B} = \mathcal{F}_{[\mathfrak{l},\mathfrak{l}]}\overline{\mathcal{B}} \boxtimes \mathcal{O}_{Z(\mathfrak{l})}$ has a $Z(\mathfrak{l})$-invariant support.

Now if $\mathfrak{p}$ is proper then $Z(\mathfrak{l}) \neq 0$, hence $\mathcal{B} = 0$. So $\mathcal{A}$ is cuspidal.

Remark. A similar proof was found independently by Ginzburg ([Gi]).

4.8. Corollary. Let $\mathfrak{g}$ be semisimple. The set of cuspidal sheaves is finite and invariant under duality. Any cuspidal sheaf is a character sheaf and an orbital sheaf. It is G_m-monodromic and has regular singularities.

Proof. Since G has finitely many orbits in $\mathcal{N}$, any G-equivariant sheaf $\mathcal{B}$ supported on $\mathcal{N}$ is orbital and with regular singularities. Since the fundamental groups of G-orbits are finite there are finitely many irreducible $\mathcal{B}$'s. The Fourier transform $\mathcal{F}_{\mathfrak{g}}(\mathcal{A})$ of a cuspidal sheaf $\mathcal{A}$ is supported in $\mathcal{N}$, so it is an orbital sheaf. So $\mathcal{A}$ is a character sheaf! Since the duality "commutes" with $\mathcal{F}_{\mathfrak{g}}$, the dual of $\mathcal{A}$ is again cuspidal. Finally, by the G_m-invariance of nilpotent orbits $\mathcal{A}$ is smooth on G_m-orbits, i.e., G_m-monodromic.

5 Admissible sheaves

The goal of this section is to identify two approaches to character sheaves: (i) by induction from monodromic sheaves on $\mathfrak{g}/\mathfrak{n}$, and (ii) by induction from cuspidal sheaves on Levi factors. In the standard terminology this is the claim that the classes of *character sheaves* and *admissible sheaves* coincide, and this is essentially the theorem 5 from [Lu3]. One direction, that the admissible sheaves are character sheaves will be essentially obvious by now. The proof of the converse is based on understanding the behavior of character sheaves on the Lusztig strata in $\mathfrak{g}$, i.e., the behavior under equisingular change of the semisimple part of an element of $\mathfrak{g}$. This is stated as: *character sheaves are quasi-admissible*. The proof is essentially from [Lu1], one can simplify it here by proving instead the Fourier transform of the main result (stated in (5.9)), but we use this version in the next section.

5.1. Admissible sheaves are defined as irreducible constituents of all $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{A}$ for parabolic subalgebras $\mathfrak{p}$ and cuspidal sheaves $\mathcal{A}$ on $\overline{\mathfrak{p}}$ ([Lu3]).

5.2. Lemma. If $\mathcal{A}$ and $\mathcal{B}$ are character (resp. orbital) sheaves on $\mathfrak{g}$ and $\overline{\mathfrak{p}}$, then the same holds for all irreducible constituents of $\mathrm{Res}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{A}$ and $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{B}$. The complex $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{B}$ is semisimple.

Proof. Fourier transform reduces the lemma to the orbital case. The semisimplicity of the induced sheaf follows from the decomposition theorem since orbital sheaves are of geometric origin. The rest of the proof is the same as for proposition 3.5.

5.3. Theorem. Admissible sheaves are the same as character sheaves.

5.4. The class of character sheaves contains cuspidal sheaves by (4.8), and is closed under induction by (5.2). So it contains all admissible sheaves. For the converse we will notice that character sheaves are quasi-admissible in (5.6), and use this to show in (5.8) that character sheaves are admissible.

5.5. For a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ we call $Z_r(\mathfrak{l}) = \{x \in \mathfrak{l},\ Z_{\mathfrak{g}}(x) = \mathfrak{l}\}$ the regular part of the center $Z(\mathfrak{l})$ of $\mathfrak{l}$. It appears in the subvarieties $S_{\mathfrak{l},\mathcal{O}} = {}^G[Z_r(\mathfrak{l}) + \mathcal{O}]$ of $\mathfrak{g}$, indexed by nilpotent orbits $\mathcal{O}$ in $\mathfrak{l}$. This gives the Lusztig stratification $\mathfrak{g} = \bigcup\ S_{\mathfrak{l},\mathcal{O}}$ ([Lu1]).

We say that a sheaf $\mathcal{A} \in \mathfrak{m}_G(\mathfrak{g})$ is quasi-admissible if for any $\mathfrak{l}$ and $\mathcal{O}$ all irreducible constituents of $(Z_r(\mathfrak{l}) + \mathcal{O} \hookrightarrow \mathfrak{g})^!\mathcal{A}$ are of the form $\mathcal{L}|Z_r(\mathfrak{l}) \boxtimes \mathcal{E}$ for a character sheaf $\mathcal{L}$ on $Z(\mathfrak{l})$ and a connection $\mathcal{E}$ on $\mathcal{O}$ (see [Lu2]).

5.6. Lemma. Any character sheaf $\mathcal{A}$ is quasi-admissible. In particular it is smooth on the strata $S_{\mathfrak{l},\mathcal{O}}$.

Proof. For a pair $\mathfrak{l}, \mathcal{O}$ choose a parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{u}$. Now the proof of (4.3) gives

$$(*) \qquad [Z_r(\mathfrak{l}) + \mathcal{O} \hookrightarrow \mathfrak{l}]^! \operatorname{Res}^{\mathfrak{g}}_{\mathfrak{p}} \mathcal{A} = [Z_r(\mathfrak{l}) + \mathcal{O} \hookrightarrow \mathfrak{g}]^! \mathcal{A}\ [2 \dim \mathfrak{u}].$$

By (5.2), irreducible constituents of $\operatorname{Res}^{\mathfrak{g}}_{\mathfrak{p}} \mathcal{A}$ are character sheaves so they are of the form $\mathcal{L} \boxtimes \mathcal{F}$ for a character sheaf $\mathcal{L}$ on $Z(\mathfrak{l})$ and $\mathcal{F} \in \mathfrak{m}([\mathfrak{l}, \mathfrak{l}])$. Therefore the constituents of $(*)$ have the needed property.

5.7. Corollary. Let $\mathfrak{g}$ be semisimple. For an irreducible sheaf $\mathcal{A} \in \mathfrak{m}_G(\mathfrak{g})$ the following is equivalent: (i) $\mathcal{A}$ is cuspidal, (ii) $\mathcal{A}$ is a character sheaf supported on $\mathcal{N}$, (iii) $\mathcal{A}$ is both a character sheaf and an orbital sheaf.

Proof. (i) $\Rightarrow$ (ii) is known and (ii) $\Rightarrow$ (iii) is obvious. Finally, if $\mathcal{A}$ is a character sheaf then lemma (5.6) implies that the support of $\mathcal{A}$ is the closure of some stratum $S_{\mathfrak{l},\mathcal{O}}$. If $\mathcal{A}$ is also orbital then $\mathfrak{l} = \mathfrak{g}$ and $\overline{S_{\mathfrak{l},\mathcal{O}}} = \overline{\mathcal{O}} \subseteq \mathcal{N}$. Now (iii) $\Rightarrow$ (i) follows from (4.7) since $\mathcal{F}_{\mathfrak{g}}\mathcal{A}$ is also an orbital character sheaf.

5.8. Lemma. Let $\mathcal{A}$ be an irreducible quasi-admissible sheaf on $\mathfrak{g}$.

(i) $\mathcal{A}$ is the irreducible extension of a connection $\mathcal{E}$ on one of the strata $S_{\mathfrak{l},\mathcal{O}}$.

(ii) Connection $\mathcal{E}_\circ = \mathcal{A}|_{Z_r(\mathfrak{l})+\mathcal{O}}$ is irreducible and its irreducible extension $\mathcal{A}_\circ$ to a sheaf on $\mathfrak{l}$ is a constituent of $\operatorname{Res}^{\mathfrak{g}}_{\mathfrak{p}} \mathcal{A}$.

(iii) $\mathcal{A}$ is a constituent of $\operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{p}}(\mathcal{A}_\circ)$.

(iv) If $\mathcal{A}$ is also a character sheaf then $\mathcal{A}_\circ$ is cuspidal.

Proof. (i) is obvious. Denote $S_\circ = Z_r(\mathfrak{l}) + \mathcal{O}$, then by $(*)$, we have $(S_\circ \hookrightarrow \mathfrak{l})^! \operatorname{Res}^{\mathfrak{g}}_{\mathfrak{p}} \mathcal{A} = \mathcal{E}_\circ$ (up to a shift). So for (ii) it suffices to see that $S_\circ$ is open in $\operatorname{supp}(\operatorname{Res}^{\mathfrak{g}}_{\mathfrak{p}} \mathcal{A})$.

Otherwise there would exist a smooth subvariety T of $\mathfrak{p} - (S_\circ + \mathfrak{u})$ such that $\overline{T}$ meets $S_\circ + \mathfrak{u}$ and $(T \hookrightarrow \mathfrak{l})^! \mathcal{A} \neq 0$. One can make T small enough to lie in some stratum $S_{\tilde{\mathfrak{l}},\tilde{\mathcal{O}}}$. Now $\overline{S_{\tilde{\mathfrak{l}},\tilde{\mathcal{O}}}}$ meets $S_\circ + \mathfrak{u} = {}^U S_\circ \subseteq S_{\mathfrak{l},\mathcal{O}}$ and $(S_{\tilde{\mathfrak{l}},\tilde{\mathcal{O}}} \hookrightarrow \mathfrak{g})^! \mathcal{A} \neq 0$. This gives $S_{\tilde{\mathfrak{l}},\tilde{\mathcal{O}}} = S_{\mathfrak{l},\mathcal{O}}$.

Pick a Cartan subalgebra $\mathfrak{h}_\circ$ of $\mathfrak{l}$ and let W and $W_\mathfrak{l}$ be the Weyl groups of $\mathfrak{g}$ and $\mathfrak{l}$. We can suppose that for some $w \in W$ subvariety T lies in

$$^{PwP}S_\circ = {}^{PwU}S_\circ = {}^{Pw}(S_\circ + \mathfrak{u}),$$

hence in $\mathfrak{p} \cap {}^{Pw}(S_\circ + \mathfrak{u}) = {}^{P}[{}^{w}(S_\circ + \mathfrak{u}) \cap \mathfrak{p}] = T'$. Let $A : \mathfrak{p} \to \mathfrak{h}/W_\mathfrak{l}$ correspond to $\mathbb{C}[\mathfrak{h}]^{W_\mathfrak{l}} \approx \mathbb{C}[\mathfrak{p}]^P \subseteq \mathbb{C}[\mathfrak{p}]$. For any root ϕ of $\mathfrak{h}$ in ${}^w\mathfrak{l}$, the product $\psi = \prod_{u \in W_\mathfrak{l}} u\phi \in \mathbb{C}[\mathfrak{h}/W_\mathfrak{l}]$ vanishes on $A(T')$ since the semisimple parts of elements of ${}^w(S_\circ + \mathfrak{u})$ lie in $Z_r({}^w\mathfrak{l})$. So $\psi \circ A$ vanishes on T and on the non-empty subvariety $\overline{T} \cap (S_\circ + \mathfrak{u})$.

Since for $s \in Z_r(\mathfrak{l})$ and $n \in \mathcal{O} + \mathfrak{u}$ one has $(\psi \circ A)(s + n) = \phi(s)^{|W_\mathfrak{l}|}$, we find that all roots of ${}^w\mathfrak{l}$ vanish at some $s \in Z_r(\mathfrak{l})$. Therefore ${}^w\mathfrak{l} = \mathfrak{l}$. Now the image of T' in $\overline{\mathfrak{p}} = \mathfrak{l}$ is ${}^L[Z_r(\mathfrak{l}) + {}^w\mathcal{O}]$. Its closure can not meet $S_\circ$ — otherwise ${}^w\mathcal{O} \neq \mathcal{O} \subseteq \overline{{}^w\mathcal{O}}$ and $\dim {}^w\mathcal{O} = \dim \mathcal{O}$.

(iii) Let $\tilde{S} = G \times_P (S_\circ + \mathfrak{u}) \xrightarrow{\mu} S_{\mathfrak{l},\mathcal{O}}$ be the conjugation map. Then $(S_{\mathfrak{l},\mathcal{O}} \hookrightarrow \mathfrak{g})^! \operatorname{Ind}_\mathfrak{p}^\mathfrak{g} \mathcal{A}_\circ = \mu_* \mu^\circ \mathcal{E} = \mathcal{E} \otimes \mu_* \mathcal{O}_{\tilde{S}}$. Since by lemma 3.3.a map μ can be identified with the map $G \times_L S_\circ \to G \times_{N_G(L,\mathcal{O})} S_\circ$, we see that $\mu_* \mathcal{O}_{\tilde{S}}$ has $\mathcal{O}_{S_{\mathfrak{l},\mathcal{O}}}$ as a summand. Finally, since $\operatorname{supp}(\operatorname{Ind}_\mathfrak{p}^\mathfrak{g} \mathcal{A}_\circ) = {}^G(\overline{S_\circ + \mathfrak{u}}) = \overline{S_{\mathfrak{l},\mathcal{O}}}$, we are done.

(iv) According to (ii) and (5.2), if $\mathcal{A}$ is a character sheaf then so is $\mathcal{A}_\circ$. Since $\operatorname{supp}(\mathcal{A}_\circ) \subseteq Z(\mathfrak{l})$, corollary 5.7 implies that $\mathcal{A}_\circ$ is cuspidal.

5.9. Corollary. Orbital sheaves are precisely the irreducible constituents of sheaves $\operatorname{Ind}_\mathfrak{p}^\mathfrak{g} (\mathcal{C} \boxtimes \delta_s)$ where $\mathcal{C}$ is a cuspidal sheaf on $[\mathfrak{l},\mathfrak{l}]$ and δ_s is the irreducible sheaf supported at a point $s \in Z(\mathfrak{l})$.

6 Characteristic varieties

In this section we only consider $\Bbbk = \mathbb{C}$ since in positive characteristic there is yet no satisfactory notion of characteristic variety. The characteristic variety of $\mathcal{F} \in D(\mathfrak{g})$ is a subvariety of $T^*(\mathfrak{g}) \approx \mathfrak{g} \times \mathfrak{g}^* \approx \mathfrak{g} \times \mathfrak{g}$. We say that it is nilpotent if it lies in $\mathfrak{g} \times \mathcal{N}$.

6.1. Let $\mathfrak{g}$ be semisimple and consider an irreducible $\mathcal{A} \in \mathfrak{m}_G(\mathfrak{g})$ supported in $\mathcal{N}$. Then $\mathcal{A}$ is G_m-monodromic and regular, hence $\operatorname{pr}_{\mathfrak{g}^*}(\operatorname{Ch} \mathcal{A}) = \operatorname{supp}(\mathcal{F}_\mathfrak{g} \mathcal{A})$ ([Br]). Therefore

$$\mathcal{A} \quad \text{is cuspidal} \quad \Leftrightarrow \quad \operatorname{Ch}\mathcal{A} \quad \text{is nilpotent.}$$

6.2. It is easy to see that the characteristic variety of any character sheaf is nilpotent. For any $\mathcal{B} \in \mathcal{M}_B$ (section 3.1), monodromic property implies that $\mathrm{Ch}(\mathcal{B}) \subseteq T^*(\mathfrak{g}/\mathfrak{n}) = \mathfrak{g}/\mathfrak{n} \times \mathfrak{b}$ actually lies in $\mathfrak{g}/\mathfrak{n} \times \mathfrak{n}$, and therefore $\mathrm{Ch}[(\mathfrak{g} \to \mathfrak{g}/\mathfrak{n})^\circ\mathcal{B}] \subseteq \mathfrak{g}\times\mathfrak{n}$. So it suffices to apply the principle that $\mathrm{Ch}(\Gamma_B^G\mathcal{F}) \subseteq \overline{G \cdot \mathrm{Ch}(\mathcal{F})}$. For $\mathcal{F}$ with regular singularities this is the lemma 1.2 in [MV], however the proof remains correct for irregular $\mathcal{D}$-modules when the homogeneous space G/B is complete (the extension of this lemma to irregular sheaves and arbitrary homogeneous spaces is not true).

6.3. Theorem. An irreducible sheaf $\mathcal{A} \in \mathfrak{m}_G(\mathfrak{g})$ is a character sheaf if and only if $\mathcal{A}$ is quasi-admissible and $\mathrm{Ch}(\mathcal{A})$ is nilpotent.

Proof. It remains to show that a quasi-admissible $\mathcal{A}$ with $\mathrm{Ch}(\mathcal{A}) \subseteq \mathfrak{g} \times \mathcal{N}$ is a character sheaf. Since $\mathcal{A}$ is quasi-admissible, we can use use lemma 5.8 (i)-(iii), its proof and notation. It remains to prove a version of (5.8.iv): if $\mathrm{Ch}(\mathcal{A})$ is nilpotent then $\mathcal{A}_\circ$ is cuspidal. This is equivalent (by (6.1)) to: $\mathrm{Ch}(\mathcal{A}_\circ)$ is nilpotent.

Since $\mathcal{A}$ is quasi-admissible $\mathrm{Ch}(\mathcal{A}_\circ)$ is the union of conormal bundles to $Z(\mathfrak{l}) + \mathcal{O}'$ for some nilpotent L-orbits $\mathcal{O}' \subseteq \overline{\mathcal{O}}$. Let $\mathcal{O}'$ be one of the orbits contributing to $\mathrm{Ch}(\mathcal{A}_\circ)$. Since for $z \in Z_r(\mathfrak{l})$, $n \in \mathcal{O}'$ and $x = z + n$, the conormal space at x is

$$T^*_{S(\mathfrak{l},\mathcal{O}')}(\mathfrak{g})_x = Z(\mathfrak{l})^\perp \cap T_x({}^G x)^\perp = Z(\mathfrak{l})^\perp \cap Z_{\mathfrak{g}}(x) = Z_{[\mathfrak{l},\mathfrak{l}]}(n) = T^*_{Z(\mathfrak{l})+\mathcal{O}'}(\mathfrak{l})_x,$$

it suffices to see that $T^*_{S(\mathfrak{l},\mathcal{O}')}(\mathfrak{g}) \subseteq \mathrm{Ch}(\mathcal{A})$.

Let $\mathfrak{l} \xhookleftarrow{j} Z_r(\mathfrak{l}) + \overline{\mathcal{O}} \xhookrightarrow{i} \mathfrak{g}$ so that $\mathcal{A}_\circ$ is a constituent of $j_* i^\circ\mathcal{A}$, hence $T^*_{Z(\mathfrak{l})+\mathcal{O}'}(\mathfrak{l}) \subseteq \mathrm{Ch}(j_* i^\circ \mathcal{A})$. To compare $\mathcal{A}$ and $i^\circ\mathcal{A}$ we use the radical U_- of the opposite parabolic subgroup and the map $\pi : U_- \times U \times Z_r(\mathfrak{l}) \times \overline{\mathcal{O}} \to \mathfrak{g}$, defined by $\pi(\overline{u}, u, z, n) = {}^{\overline{u}u}(z + n)$. The image S of π is open in $\mathrm{supp}(\mathcal{A})$ since

$$\mathrm{supp}(\mathcal{A}) = \overline{S_{\mathfrak{l},\mathcal{O}}} = \overline{{}^{G/P}(Z_r(\mathfrak{l}) + \mathcal{O} + \mathfrak{u})} = {}^{G/P}(Z(\mathfrak{l}) + \overline{\mathcal{O}} + \mathfrak{u}),$$

and by lemma 3.3.a $S = {}^{U_-}(Z_r(\mathfrak{l}) + \overline{\mathcal{O}} + \mathfrak{u})$.

The map π is finite and any $z \in Z_r(\mathfrak{l})$ has a neighborhood V in $Z_r(\mathfrak{l})$ such that the restriction of π to $U_- \times U \times V \times \overline{\mathcal{O}}$ is an isomorphism onto its image which is open in S. To check this let $d_i = (\overline{u}_i, u_i, z_i, n_i) \in U_- \times U \times Z_r(\mathfrak{l}) \times \overline{\mathcal{O}}$, $i = 1, 2, 3$. If $\pi(d_i) = \pi(d_j)$ then $s_{ij} = u_i^{-1}\overline{u}_i^{-1}\overline{u}_j u_j \in N_G(L)$ since ${}^{s_{ij}}z_j = z_i$. So if $\pi(d_1) = \pi(d_2) = \pi(d_3)$ and $s_{12}L = s_{13}L$, then $d_2 = d_3$. First, $s_{23} \in L$, then $\overline{u}_2^{-1}\overline{u}_3 \in U_- \cap P = 1$, and so lemma 3.3.a gives $d_2 = d_3$.

Since $\mathcal{A}$ is G-equivariant $\pi^\circ\mathcal{A} = \mathcal{O}_{U_-\times U} \boxtimes i^\circ\mathcal{A}$. Now, the above properties of π imply that $T^*_{S(\mathfrak{l},\mathcal{O}')}(\mathfrak{g}) \subseteq \mathrm{Ch}(\mathcal{A})$.

6.4. For $\mathcal{F} \in \mathfrak{m}_G(\mathfrak{g})$ and $x \in \mathfrak{g}$ the fiber $\mathrm{Ch}_x\mathcal{F} = \mathrm{Ch}\,\mathcal{F} \cap T_x^*(\mathfrak{g})$ lies in the conormal space $T^*_{G_x}(\mathfrak{g})_x = Z_{\mathfrak{g}}(x)$, so $\mathrm{Ch}\,\mathcal{F}$ is nilpotent if and only if $\mathrm{Ch}(\mathcal{F}) \subseteq \Lambda$ for $\Lambda = \{(x, y) \in \mathfrak{g} \times \mathcal{N},\ y \in Z_{\mathfrak{g}}(x)\}$. The following is the Lie algebra analogue of Laumon's result for groups ([La]).

6.5. Lemma. Λ is a Lagrangian subvariety. It is actually the union of some of $T^*_{S(\mathfrak{l},\mathcal{O}')}(\mathfrak{g})$.

Proof. The fiber of Λ at $y \in \mathcal{N}$ is $Z_{\mathfrak{g}}(y) = T^*_{G_y}(\mathfrak{g})_y$. So, from the point of view of the projection to $\mathcal{N}$, Λ is the union of conormal bundles to all nilpotent orbits. So Λ is Lagrangian.

Now let $\mathcal{O}$ be a nilpotent orbit in a Levi factor $\mathfrak{l}$ and $s \in Z_r(\mathfrak{l})$, $n \in \mathcal{O}$. Then the tangent space $T_{s+n}(Z_r(\mathfrak{l}) + \mathcal{O}) = Z(\mathfrak{l}) + [n, \mathfrak{l}]$ is orthogonal to $Z_{\mathcal{N}\cap\mathfrak{l}}(n) = Z_{\mathcal{N}}(s+n) = \Lambda \cap T^*_{s+n}(\mathfrak{g})$. So $\Lambda \subseteq \bigcup T^*_{S(\mathfrak{l},\mathcal{O})}(\mathfrak{g})$ and since Λ is Lagrangian it is a union of some of $T^*_{S(\mathfrak{l},\mathcal{O})}(\mathfrak{g})$.

6.6. For sheaves on the group G one defines Res^G_P and cuspidality in an analogous way. By identifying $\mathcal{N} \subseteq \mathfrak{g}$ with the unipotent cone in G we will not cause any confusion since for sheaves supported on $\mathcal{N}$, the exponential map intertwines $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}}$ and Res^G_P. We know that $\mathrm{supp}(\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}} \mathcal{A}) \subseteq \mathfrak{l} \cap \mathcal{N}$ (lemma 4.5) and similarly on the group. In order to calculate $\mathrm{Res}^G_P \mathcal{A}$ (resp. $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}} \mathcal{A}$) we integrate over cosets $e^Y \cdot U$ (resp. $Y + \mathfrak{u}$), for $Y \in \mathcal{N} \cap \mathfrak{l}$. But $e^Y \cdot U = e^{Y+\mathfrak{u}}$.

6.7. Theorem. Any cuspidal sheaf $\mathcal{A}$ on a semisimple group G has a nilpotent characteristic variety. In particular it is a character sheaf.

Remark. Lusztig proved in any characteristic that cuspidal sheaves on a group are character sheaves (theorem 23.1.b in [Lu2]). The above theorem gives a simpler proof but only in characteristic zero. A similar proof was found by Ginzburg ([Gi]).

Proof. The second sentence follows from the first since any irreducible G-equivariant regular D-module on G with a nilpotent characteristic variety is a character sheaf ([Gi, MV]). The first claim is that $\mathrm{Ch}(\mathcal{A}) \subseteq T^*(G) = G \times \mathfrak{g}^* = G \times \mathfrak{g}$ actually lies in $G \times \mathcal{N}$. If $\mathrm{supp}(\mathcal{A}) \subset \mathcal{N}$ then the claim follows from (6.7) and (6.1). To reduce the situation to this case we follow [Lu1].

The pull-back of $\mathcal{A}$ to any cover of G is obviously cuspidal so we can suppose that G is simply connected. Choose S, $\mathcal{E}$ and $x = sn \in S$ as in the proof of Lemma 4.4. Then $H = Z_G(s)$ is connected. It cannot lie in a proper Levi subgroup L of G since we could repeat the proof of (4.4) and get $\mathrm{Res}^G_P \mathcal{A} \neq 0$. Therefore H is semisimple and s is of finite order. So G has finitely many orbits in S and we can assume that $S = {}^G x$ is a single orbit.

Let $\mathcal{O} = {}^H n$ and let $\mathcal{A}_H$ be the irreducible extension of the connection $(s^{-1})^\circ(\mathcal{E}|s\mathcal{O})$ from $\mathcal{O}$ to H. Since $\mathrm{supp}(\mathcal{A}) = \overline{S} = {}^G(s\overline{\mathcal{O}}) \approx G \times_H \overline{\mathcal{O}}$, we see that $\mathrm{Ch}(\mathcal{A})$ is the union of conormal bundles for subvarieties ${}^G(s\mathcal{O}') \subset G$ over all H-orbits $\mathcal{O}'$ in $\overline{\mathcal{O}}$ such that $T^*_{s\mathcal{O}'}(\mathfrak{l}) \subseteq \mathrm{Ch}(\mathcal{A}_H)$. For any $v \in \mathcal{O}'$

$$T^*_{{}^G(s\mathcal{O}')}(G)_{sv} = Z_{\mathfrak{g}}(sv) = Z_{\mathfrak{h}}(v) = T^*_{\mathcal{O}'}(H)_v,$$

so it remains to see that $\mathcal{A}_H$ is cuspidal.

Let $P = L \ltimes U$ be a parabolic subgroup of G such that $P_H = L_H \ltimes U_H$ (for $P_H = P \cap H$ etc.) is a parabolic subgroup of H. Let $v \in \overline{\mathcal{O}} \cap P_H$, $T_H = vU_H \cap \overline{\mathcal{O}}$ and $T = svU \cap {}^G(s\overline{\mathcal{O}})$. The projection to the semisimple part of the Jordan decomposition gives an algebraic map $T \xrightarrow{\alpha} {}^Gs \cap P$ (it is a restriction of the map ${}^G(s\overline{\mathcal{O}}) = G \times_H s\overline{\mathcal{O}} \to G \times_H s = {}^Gs$). By composing α with $P \to \overline{P}$ we find that $\mathrm{Im}(\alpha) \subseteq {}^Gs \cap sU$. Since ${}^Us$ is a connected component of ${}^Gs \cap sU$

$$\alpha^{-1}({}^Us) = {}^U(\alpha^{-1}s) = {}^U(svU \cap s\overline{\mathcal{O}}) = {}^U(sT_H) \approx U \times_{U_H} sT_H$$

is open in T. Therefore the integral of $\mathcal{A}$ over $\alpha^{-1}({}^Us)$ vanishes. By U-equivariance the same is true for the integral $\mathcal{A}$ over sT_H. Finally, since $(sT_H \hookrightarrow G)^! \mathcal{A} = (T_H \hookrightarrow H)^! \mathcal{A}_H$ (up to a shift), $\mathcal{A}_H$ is cuspidal.

7 An application of nearby cycles

Let C be a smooth curve and $O \in C$. Any function $f : X \to C$ defines exact functors of nearby and vanishing cycles $\mathfrak{m}(X) \xrightarrow{\psi_f, \phi_f} \mathfrak{m}_{f^{-1}(0)}(X)$, from D-modules on X to the subcategory of D-modules on X supported in the fiber $f^{-1}(O)$ ([Be]). Suppose that a group G acts on X and fixes the function f.

7.1. Lemma. For any subgroup B such that G/B is complete, functor Γ_B^G commutes with the equivariant versions of ψ_f and ϕ_f.

Proof. Computing Γ_B^G involves inverse images α° for smooth maps α and direct images β_* for proper maps β (see (1.7)), and these commute with ψ_f and ϕ_f.

7.2. As a consequence nearby and vanishing cycles will commute with $\check{\mathcal{H}}$, $\check{\mathcal{G}}$ and induction. This gives a simple proof for the following result of Lusztig.

7.3. Theorem. ([Lu2]). Let $\mathcal{A}$ and $\mathcal{B}$ be character sheaves (resp. orbital sheaves) on $\mathfrak{g}$ and $\mathfrak{l}$. Then

(i) $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{B}$ is a semisimple sheaf,

(ii) $\mathrm{Res}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{A} \in D_{\geqslant 0}(\mathfrak{l})$,

(iii) $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{B}$ and $\mathrm{Res}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{A}$ do not depend on the choice of a parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{u}$.

Proof. As usual it suffices to look at the orbital sheaves. Since $\mathcal{B}$ is an orbital sheaf on $\mathfrak{l}$ we have $\mathcal{B} = \delta_s \boxtimes \mathcal{C}$ for an orbital sheaf $\mathcal{C} \in \mathfrak{m}_L([\mathfrak{l}, \mathfrak{l}])$ and $\delta_s = (s \hookrightarrow \mathfrak{l})_* \mathcal{O}_{\mathrm{pt}}$ for some $s \in Z(\mathfrak{l})$. Suppose that $\mathcal{C}$ is supported in $\mathcal{N} \cap \mathfrak{l}$ and $s \in Z_r(\mathfrak{l})$. Then ${}^G(s + \mathcal{N} \cap \mathfrak{l}) = G \times L(s + \mathcal{N} \cap \mathfrak{l})$ and for any $n \in \mathcal{N} \cap \mathfrak{l}$ we have $s + n + \mathfrak{u} = {}^U(s+n)$. This shows that for $\mathfrak{l} \underset{j}{\overset{p}{\leftrightarrows}} \mathfrak{p} \xhookrightarrow{i} \mathfrak{g}$

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{B} = \Gamma_P^G i_*(p^\circ \mathcal{B}) = \Gamma_P^G i_*(\Gamma_L^P j_* \mathcal{B}) = \Gamma_L^G (ij)_* \mathcal{B}$$

is a sheaf independent of $\mathfrak{p}$. The point was that in this case the parabolic induction is just the naive induction from a Levi factor: $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{B} = \mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}} \mathcal{B}$.

For the general $s \in Z(\mathfrak{l})$ we construct a sheaf on $\mathfrak{l} \times \mathbb{A}^1$ by $\tilde{\mathcal{B}} = \mathcal{C} \boxtimes \delta_K$ for $K = \{(s + c\alpha, c),\, c \in k\} \subseteq Z(\mathfrak{l}) \times \mathbb{A}^1$ such that $s + c\alpha \in Z_r(\mathfrak{l})$ for c in some open dense $U \subseteq \mathbb{A}^1$. By the above calculation, sheaf $\mathrm{Ind}\, \tilde{\mathcal{B}}$ is independent of $\mathfrak{p}$ on $\mathfrak{g} \times U$. Hence so is $\psi(\mathrm{Ind}\, \tilde{\mathcal{B}}) = \mathrm{Ind}(\psi \tilde{\mathcal{B}}) = \mathrm{Ind}\, \mathcal{B}$.

By the transitivity of induction and by corollary (5.9), claim (i) and the induction part of (iii) follow for all orbital sheaves. Since $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}$ and $\mathrm{Res}_{\mathfrak{p}}^{\mathfrak{g}}$ are adjoint functors between full subcategories of $D_G(\mathfrak{g})$ and $D_L(\mathfrak{l})$ consisting of complexes supported on finitely many orbits we also get the restriction part of (iii). The claim (ii) also follows by adjunction since for $i < 0$ we have $0 = \mathrm{Hom}\left(\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{B},\, \mathcal{A}[i]\right) = \mathrm{Hom}\left(\mathcal{B},\, \mathrm{Res}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{A}[i]\right)$, hence $H^j(\mathrm{Res}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{A}) = 0$ for $i < 0$.

7.4. Remarks. (i) The proof is self contained for the class of sheaves supported in nilpotent cones. This can be used for a proof of the Theorem 5.3 which avoids Lemma 5.8. (ii) Actually $\mathrm{Res}_{\mathfrak{p}}^{\mathfrak{g}} \mathcal{A}$ is also a semi-simple sheaf ([Lu2]). A simple proof in characteristic zero was found by Ginzburg ([Gi1]).

7.5. Let $s : \check{\mathcal{N}} \to \mathfrak{g}$ be the Springer resolution of $\mathcal{N}$. Then the proof above shows that the Springer sheaf $s_* \mathcal{O}_{\check{\mathcal{N}}}$ is the limit $\lim\limits_{C \to 0} \delta_C$ of delta-distributions $\delta_C = (C \hookrightarrow \mathfrak{g})_* \mathcal{O}_C$ on regular semisimple conjugacy classes C. More precisely, any regular semisimple conjugacy class C defines a G_m family of D-modules $\delta_{s \cdot C},\ s \in G_m$, and the nearby cycle limit of the family at $s = 0$ is the Springer sheaf.

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