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A CATEGORICAL FORMULATION OF ALGEBRAIC GEOMETRY

A Dissertation Presented

by

BRADLEY M. WILLOCKS

Submitted to the Graduate School of the

University of Massachusetts Amherst in partial fulfillment

of the requirements for the degree of

DOCTOR OF PHILOSOPHY

September 2017

Mathematics

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ABSTRACT

A CATEGORICAL FORMULATION OF ALGEBRAIC GEOMETRY

SEPTEMBER 2017

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We construct a category, Ω, of which the objects are pointed categories and the arrows are pointed correspondences. The notion of a "spec datum" is introduced, as a certain relation between categories, of which one has been given a Grothendieck topology. A "geometry" is interpreted as a sub-category of Ω, and a formalism is given by which such a subcategory is to be associated to a spec datum, reflecting the standard construction of the category of schemes from the category of rings by affine charts.

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INTRODUCTION

We construct in the present work a category Ω of equivalence classes of pointed correspondences between pointed categories. It serves as a common category for various geometries, with the intuition that a geometric object should be one of the form $(\mathcal{O}_X, (Sh((X,\tau_X),\mathcal{S})^{opp})$, being a distinguished structure sheaf in a category of sheaves of some type (for technical reasons the opposite categories $(Sh((Y,\tau_Y),\mathcal{O}_Y))^{opp}$ are taken). The notion of a "spec datum" is defined, consisting of a pair of functors $sp : \mathcal{R} \to \mathcal{T}, \mathcal{O} : \mathcal{R} \to (\mathcal{S})^{opp}$, with the category $\mathcal{T}$ being equipped with a Grothendieck topology and an admissibility structure. Formally imitating the construction of the structure sheaf of an affine scheme from topological spec functor $Spec : \mathfrak{Ring}^{opp} \longrightarrow \mathfrak{Top}$, we construct a functor $\tilde{sp} : \mathcal{R} \to \Omega$, representing the derivation of the full spec functor from the topological spec. Classical morphisms of schemes are given by composition of $(((f_*)^{opp})^{opp} \times_{\mathfrak{Cat}} (id)^{opp})$ with the Hom functor $(Sh((Y,\tau_Y),\mathfrak{Ring})^{opp} \times Sh((Y,\tau_Y),\mathfrak{Ring}) \longrightarrow \mathfrak{Set}$, where f_* denotes the usual pushforward of sheaves, with the usual $f_\sharp : f_*(\mathcal{O}_X) \to \mathcal{O}_Y$ being the distinguished object. Besides merely containing the usual morphisms of schemes, there are Ω arrows $f : (Sh((X,\tau_X),\mathfrak{Ring}),\mathcal{O}_X) \to (Sh((Y,\tau_Y),\mathfrak{Ring}),\mathcal{O}_Y)$ for any pair of functors $Sh((X,\tau_X),\mathfrak{Ring}) \xrightarrow{F} C \xleftarrow{G} Sh((Y,\tau_Y),\mathfrak{Ring})$. This allows for arrows which are "purely (co)homological," in addition to the usual morphisms, in that C may be taken to be an Abelian category, with the functors F and G given by a functor sending a sheaf of rings to the (co)homology group assigned to the affine scheme over the base given by the tensoring a given sheaf of rings with the distinguished sheaf of rings (over the sheafification of the constant sheaf), and sending this to whatever (co)homology group would be associated to the affine scheme over the base scheme given by the given sheaf of rings over the distinguished sheaf (e.g., given primes $l \neq p, q$, suppose that $e : H^{et}_{l-ad}(X) \to H^{et}_{l-ad}(X \times_{Spec(\mathbb{Z})} Spec(\mathbb{Z}/p\mathbb{Z}))$ and $f : H^{et}_{l-ad}(X) \to H^{et}_{l-ad}(X \times_{Spec(\mathbb{Z})} Spec(\mathbb{Z}/q\mathbb{Z}))$ are arrows of $\mathbb{Z}_l$-modules given by the fibred product, and $e', f' \in Arr(\mathbb{Z}_l - \mathfrak{Mod})$ are such that $e \cdot e' = id$ and $f \cdot f' = id$ are the identities on their respective objects. Then $f \cdot e'$ and $e \cdot f'$ can be considered to be the distinguished element for either of (i) an Ω-arrow from X to itself, the functor being given by tensoring with $\mathbb{Z}/p\mathbb{Z}$ (or $\mathbb{Z}/q\mathbb{Z}$), tensoring with the distinguished base, and applying H^{et}_{l-ad}, or (ii) an Ω-arrow $X_p \leftrightarrow X_q$). Section 4 of the present work concerns conditions under which $\tilde{sp}$ is faithful, and defines the analogue to the category of locally "affine" spaces, i.e. schemes, the coincidence of concepts in this case following from faithfulness of $\tilde{spec}$.

Section 2 concerns categorical preliminaries, namely (sk)-limits and weakly enriched categories. The former is a generalization of the concept of a limit, intended for situations in which the strict limit either does not exist, or is too restrictive, e.g. limits of categories, for which diagrams are not specified by equalities, but by natural isomorphisms. In future work this would be used in the foreshadowed operation of gluing spec functors, i.e. taking colimits in the arrow category of categories in such cases as those in which each object is a spec functor. Weak enrichments and $\mathfrak{Cat}$-limits are to be used in defining n-categories, and in defining (weak) sheaves of categories. The latter would be used to define homotopy and (co)homology, either on subcategories of Ω directly or induced on the subcategories from $\mathfrak{Cat}$-sheaves defined on the domains of the spec functors. $\mathfrak{Cat}$-sheaves defined directly on subcategories of Ω would be used to guide the choice of a collection of spec functors to be glued together.

0.1 Axioms - ZFC with Universes

In principle, all definitions, propositions, etc. are to be written in the language of first order logic. I.e., once the set of primitive statements is defined, the set of all statements is defined to be the closure of the set of primitive statements under the operations of quantification, negation, and conjunction. Any variable symbol is brought into being by its presence here, and each variable symbol is associated to quantifiers denoted "$\forall$" and "$\exists$," which may modify any statement. A statement may be regarded as meaningful, (in the sense in which a definite attempt should be regarded as a possibility, that this statement should, by repetition of itself, be found to state $TRUE$, the trivially true statement) only so far as every variable symbol appears only within such statements as are subjected to the quantifier corresponding to that symbol. Negation is a statement modifier, formally denoted by "$\neg$," with the usual interpretations (with Excluded Middle). It is presumed, that a certain perception of the metalanguage is maintained, by which the symbols appearing within a statement may be differentiated from the aspects inherent to it, in the sense of being meant to differentiate that statement from another a priori. This is, in some sense, equivalent to the presumption that the quantifiers are rightly applied, or interpreted, i.e. that quantified statements appear where they are meant to appear. Quotes "$\ulcorner$" and "$\urcorner$" appear around any non-negated statement, acting effectively as parentheses.

The language should be of one type, with a binary equality relation, "$=$," for it, so that statements of the form $\ulcorner x = y \urcorner$ are primitive.

In particular, since there is a universal equality sign, used to compare any two elements of the domain of this discourse, one can construct finite sets of sequences of sets inductively by requiring that each set in the the $(n+1)^{th}$ set should be equal to a set constructed in a particular fashion from the sets in the n^{th} set together with other constructions. One can describe sets in an implicit fashion, by their use within a statement.

0.1.1 Element

There is a binary relation, "$\in$," so that that statements of the form $\ulcorner x \in y \urcorner$ are primitive.

0.1.2 Empty Set

The symbol $\emptyset$ denotes the "empty set" (is a 0-ary function symbol), such that $\ulcorner \forall x, \neg \ulcorner x \in \emptyset \urcorner \urcorner$.

0.1.3 Singleton

The unary function symbol "$\{(\ )\}$" denotes the singleton construction, so that

$$\ulcorner \forall x, \ulcorner \forall y, \ulcorner \ulcorner x \in \{y\} \urcorner \iff \ulcorner x = y \urcorner \urcorner \urcorner \urcorner$$

0.1.4 Union

The binary function symbol "$(\ ) \cup (\ )$" denotes the union construction, so that

$$\ulcorner \forall x, \ulcorner \forall y, \ulcorner \forall z, \ulcorner \ulcorner x \in y \cup z \urcorner \iff \ulcorner \ulcorner x \in y \urcorner \text{ or } \ulcorner x \in z \urcorner \urcorner \urcorner \urcorner \urcorner \urcorner \urcorner$$

0.1.5 Power Set

The unary function symbol "$2^{(\ )}$" denotes the power set construction, so that

$$\ulcorner \forall x, \ulcorner \forall y, \ulcorner \ulcorner x \in 2^{y} \urcorner \iff \ulcorner \forall z, \ulcorner \ulcorner z \in x \urcorner \implies \ulcorner z \in y \urcorner \urcorner \urcorner \urcorner \urcorner \urcorner \urcorner$$

0.1.6 Extension

Extension, that a set is determined by the elements contained therein

$$\ulcorner \forall x, \ulcorner \forall y, \ulcorner \ulcorner x = y \urcorner \iff \ulcorner \forall z, \ulcorner \ulcorner z \in x \urcorner \iff z \in y \urcorner \urcorner \urcorner \urcorner \urcorner \urcorner$$

0.1.7 Comprehension Schema

A schema is here used, since there is no separate type for statements of the language. By this we mean that the following is to be understood as determining an infinite list of function symbols and corresponding statements.

For any statement $\Phi(x)$ of the language in which x appears as an open $(un-quantified)$ variable, there is a unary function symbol "$\{(\ ); \Phi(x)\}$". This sends a set z to the set of all elements $y \in z$,satisfying the statement $\Phi(x)$, generally denoted by "$\{y \in z; \Phi(y)\}$". Formally, we require that

$$\ulcorner \forall z, \ulcorner \forall y, \ulcorner, \ulcorner \ulcorner y \in \{z; \Phi(x)\} \urcorner \iff \ulcorner \ulcorner y \in z \urcorner \text{ and } \Phi(y) \urcorner \urcorner \urcorner \urcorner.$$

0.1.8 Codomain

For any set x which defines a function on a set w, there exists a set w' such that for any $(z_1, z_2) \in x$, $z_2 \in w'$. Formally,

$$\ulcorner \forall w, \ulcorner \forall x,$$
$$\ulcorner\ulcorner\ulcorner \forall y, \ulcorner\ulcorner y \in x \urcorner \Longrightarrow \ulcorner \exists z_1, z_1,$$
$$\ulcorner\ulcorner z_1 \in w \urcorner \text{ and } \ulcorner y = \{\{z_1\}\} \cup \{\{z_1\} \cup \{z_2\}\} \urcorner \text{ and}$$
$$\ulcorner \forall z_2', \ulcorner\ulcorner \{\{z_1\}\} \cup \{\{z_1\} \cup \{z_2'\}\} \in x \urcorner \Longrightarrow \ulcorner z_2' = z_2 \urcorner\urcorner\urcorner\urcorner\urcorner\urcorner\urcorner \text{ and}$$
$$\ulcorner \forall z_1, \ulcorner\ulcorner z_1 \in w \urcorner \Longrightarrow \ulcorner \exists z_2, \ulcorner \{\{z_1\}\} \cup \{\{z_1\} \cup \{z_2\}\} \in x \urcorner\urcorner\urcorner\urcorner\urcorner$$
$$\Longrightarrow \ulcorner \exists w', \ulcorner \forall z_1, z_2, \ulcorner\ulcorner \{\{z_1\}\} \cup \{\{z_1\} \cup \{z_2\}\} \in x \urcorner \Longrightarrow \ulcorner z_2 \in w' \urcorner\urcorner\urcorner\urcorner\urcorner\urcorner\urcorner$$

0.1.9 Foundation

Every set must have a minimal element with respect to the relation $\in$, of (0.1.1.1).

$$\ulcorner \forall x, \ulcorner \exists y, \ulcorner\ulcorner y \in x \urcorner \text{ and } \ulcorner \forall z, \ulcorner\ulcorner z \in x \urcorner \Longrightarrow \neg \ulcorner z \in y \urcorner\urcorner\urcorner\urcorner\urcorner\urcorner$$

0.1.10 Natural Numbers

There is given a natural numbers object, i.e. nullary function symbols $\mathbb{N}$, $+_{\mathbb{N}}$, $\times_{\mathbb{N}}$, 0, and 1, such that $+_{\mathbb{N}}, \times_{\mathbb{N}} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ are commutative and distributive, with units 0 and 1 respectively, such that $\mathbb{N}, \{\{\{x\}\} \cup \{\{x\} \cup \{+_{\mathbb{N}}(x, 1)\}\} \in 2^{2^{\mathbb{N}}}; \ulcorner x \in \mathbb{N} \urcorner\}, 0)$ is a model of Peano arithmetic, translated into the present language for set theory.

0.1.11 Choice

This is assumed in the form of Zorn's Lemma, i.e. that every inductively ordered set has a maximal element.

0.1.12 Definition of a Universe

For any set U, U is defined to be a *universe* iff the tuple

$$(U, = |_U, \in |_U, \emptyset, \{(\ )\}|_U, (\ ) \cup (\ )|_U, 2^{(\ )}|_U, (\mathbb{N}, +_{\mathbb{N}}, \times_{\mathbb{N}}, 0, 1))$$

satisfies the previous axioms, (0.1.2) - (0.1.13). It is assumed that

$$\ulcorner \forall x, \ulcorner \exists U, \ulcorner\ulcorner U \text{ is a universe} \urcorner \text{ and } \ulcorner x \in U \urcorner\urcorner\urcorner\urcorner$$

Within this setting, we define the notion of a category.

0.1.13 Definition of Pairs "(x, y)"

$\ulcorner\forall x, \ulcorner\forall y, \ulcorner(x, y) := \{\{x\}\} \cup \{\{x\} \cup \{y\}\}\urcorner\urcorner\urcorner$

0.1.14 Definition of Products "$x \times_{\mathfrak{Set}} y$"

$\ulcorner\forall x, \ulcorner\forall y, \ulcorner x \times_{\mathfrak{Set}} y :=$
$\{(x', y') \in 2^{2^{2^{x \cup y}}}; \ulcorner\ulcorner x' \in x\urcorner \text{ and } \ulcorner y' \in y\urcorner\urcorner\}\urcorner\urcorner\urcorner$

0.1.15 Definition of a Function "$f : x \to y$"

$\ulcorner\forall x, \ulcorner\forall y, \ulcorner\forall f, \ulcorner\ulcorner f : x \to y\urcorner \iff$
$\ulcorner\ulcorner f \subseteq x \times_{\mathfrak{Set}} y\urcorner \text{ and } \ulcorner\forall x', \ulcorner\ulcorner x' \in x\urcorner \implies \ulcorner\exists! y', \ulcorner\ulcorner y' \in y\urcorner \text{ and } \ulcorner(x', y') \in f\urcorner\urcorner\urcorner\urcorner\urcorner\urcorner\urcorner\urcorner\urcorner\urcorner$

0.1.16 Definition of Associating Functions "a"

$\ulcorner\forall x, \ulcorner\forall y, \ulcorner\forall z, \ulcorner\forall a, \ulcorner\ulcorner a \text{ associates}(x, y, z)\urcorner$
$\iff \ulcorner\ulcorner a : ((x \times_{\mathfrak{Set}} y) \times_{\mathfrak{Set}} z) \to (x \times_{\mathfrak{Set}} (y \times_{\mathfrak{Set}} z))\urcorner \text{ and}$
$\ulcorner\forall x', \ulcorner\forall y', \ulcorner\forall z', \ulcorner(((x, y), z), (x, (y, z))) \in a\urcorner\urcorner\urcorner\urcorner\urcorner\urcorner\urcorner\urcorner$

0.1.17 Definition of a Category

$\ulcorner\forall O, \ulcorner\forall A, \ulcorner\forall H, \ulcorner\forall \circ \ulcorner\forall id,$
$\ulcorner\ulcorner(O, A, H, \circ, id) \text{ is a category}\urcorner \iff$
$\ulcorner\ulcorner H : O \times_{\mathfrak{Set}} O \to 2^{A}\urcorner \text{ and}$
$\ulcorner\forall f \in A, \ulcorner\exists! x, y \in O, \ulcorner f \in H(x, y)\urcorner\urcorner\urcorner \text{ and}$
$\ulcorner\circ : O \times_{\mathfrak{Set}} O \times_{\mathfrak{Set}} O \to 2^{(A \times_{\mathfrak{Set}} A) \times \mathfrak{Set}}\urcorner \text{ and}$
$\ulcorner\forall x, y, z \in O, \ulcorner\circ(x, y, z) : H(y, z) \times_{\mathfrak{Set}} H(x, y) \to H(x, z)\urcorner\urcorner \text{ and}$
$\ulcorner\forall w, x, y, z \in O, \forall((\phi, \psi), \chi) \in (H(y, z) \times_{\mathfrak{Set}} H(x, y)) \times_{\mathfrak{Set}} H(w, x),$
$\ulcorner\circ(w, x, z)(\circ(x, y, z)(\phi, \psi), \chi) = \circ(w, y, z)(\phi, \circ(w, x, y)(\psi, \chi))\urcorner\urcorner \text{ and}$
$\ulcorner\forall x, y \in O, \ulcorner\ulcorner\forall f \in H(x, y), \ulcorner\circ(x, x, y)(f, id(x)) = f\urcorner\urcorner \text{ and}$
$\ulcorner\forall g \in H(y, x), \ulcorner\circ(y, x, x)(id(x), f) = f\urcorner\urcorner\urcorner\urcorner\urcorner\urcorner\urcorner\urcorner\urcorner\urcorner\urcorner$

0.2 Notation

Parentheses indicate expected arguments.

0.2.1

Temporary definitions are denoted by $:_t=$. They are valid only within the definition, proposition, paragraph, etc. in which they appear. Definitions denoted by $:=$ are valid for all subsequent definitions, propositions, etc.

0.2.2

"U" will generally denote a universe, i.e. a model of set theory, and U' will denote some universe containing U, i.e. $U \in U' \in U'' \in ....$

0.2.3

If U is some universe, then $U - \mathfrak{Cat}$ is the category of $U - small$ categories, i.e. its objects are categories $C = (O, A, H, \circ, id)$ for which $O, A \in U$, i.e. the set of arrows is U-small, and its arrows are functors $F : C \to D$. We usually denote $U - \mathfrak{Cat}$ just by $\mathfrak{Cat}$ and unless otherwise stated "category" means a "U-category".

0.2.4

"$a \circ b$" denotes "$a \circ_C b$", the composition of arrows in a category, as well as the composition of functions, where the context should eliminate the ambiguity.

0.2.5 $F_{(0)}$ and $F_{(1)}$

If F is a functor, $F_{(0)}$ is the map on objects and $F_{(1)}$ is the map on arrows. Often the subscripts will be omitted, and "F" will refer to either the object map or the arrow map, e.g. $F(c), F(\phi)$. The reader used to the standard notation can ignore the subscripts.

0.2.6

For any category C, $Ob(C)$ is the class of objects of the category, and $Arr(C)$ is the class of arrows of C.

0.2.7

The functions $dom_{(C)}, codom_{(C)} : Arr(C) \longrightarrow Ob(C)$ are the domain and codomain maps for the category C.

0.2.8

The category $\star$ is the terminal category, having one arrow.

0.2.9

The dual category functor $()^{opp} : \mathfrak{Cat} \longrightarrow \mathfrak{Cat}$ sends a category to its opposite. It may carry an index based on (i) when one wants to differentiate between the map on objects, and the map on arrows. Then it would be written as ${}_{()}()^{opp}$.[1]

0.2.10 Functorial Products

Let "$\times$" denote the product of two categories (i.e. the objects of $C \times D$ are pairs of objects (c, d) from the component categories and the arrows are pairs $(f, g) : (c, d) \to (c', d')$ of arrows from the component categories). Then there are two canonical product functors, one for categories and one for sets, defined as follows.

For sets,

$\times_{U-\mathfrak{Set}} : (U - \mathfrak{Set})^{opp} \times U - \mathfrak{Set} \longrightarrow U - \mathfrak{Set}$

sends a pair of sets to their product, $(a, b) \mapsto \{\pi \in Hom_{(U-\mathfrak{Set})}(\{a, b\}, a \cup b); \pi(a) \in a \text{ and } \pi(b) \in b\}$ and a pair of functions to their product, so that $(f, g) \in Arr(U - \mathfrak{Set} \times U - \mathfrak{Set})$ is sent to the map from $a \times_{U-\mathfrak{Set}} b$ to $a' \times_{U-\mathfrak{Set}} b'$ determined by applying f to the a component and g to the b component, i.e. $(x, y) \mapsto (f(x), g(y))$.

For categories,

$\times_{U-\mathfrak{Cat}} : (U - \mathfrak{Cat})^{opp} \times U - \mathfrak{Cat} \longrightarrow U - \mathfrak{Cat}$

sends a pair of categories to their product i.e. (a U-small restriction of $\times$ above) and sends a pair of functors to their product, $(F, G) \mapsto$
$(((a, b) \mapsto (F_{(0)}(a), G_{(0)}(b))_{a \in Ob(dom(F)), b \in Ob(dom(G))},$
$((f, g) \mapsto (F_{(1)}(f), G_{(1)}(g)))_{f \in Arr(dom(F)), g \in Arr(dom(G))}).$

Note that $\times_{U-\mathfrak{Set}}$ and $\times_{U-\mathfrak{Cat}}$ do *not* denote fibred products in the category of U'-categories.

0.2.11

For any category C, for any $\phi, \psi, f, g \in Arr(C)$, we define a relation $(f, g) fibres_{(C)}(\phi, \psi)$ ("f and g fibre ϕ and ψ in C") iff $codom(\phi) = codom(\psi)$ and $dom(f) = dom(g)$ and f, g form a fibred product of ϕ, ψ (as in [1]). Informally, the C may be omitted if it is understood.

---

[1] We move the subscript to the left because the aesthetic sense resists the appearance of a subscript of a superscript. The attachment of the subscript should be to the functor ^{opp}, rather than to the category or functor on which it acts, and there is a fear that placement of the subscript after the term would suggest its attachment to the category or functor in the argument.

0.2.12

The hom functor map $Hom_{()}$ is a map of large sets which assigns to a category C the functor $(C)^{opp} \times C \longrightarrow \mathfrak{Set}$ which sends a pair of objects (a, b) to their hom set, i.e. $(a, b) \mapsto Hom_C(a, b)$, and sends a pair of arrows (f, g) to the map of hom sets given by composition, i.e. $(g, f) \mapsto (h \mapsto f \circ h \circ g)_{h \in Hom_C(codom(g), dom(f))}$, which lies in $Hom_{\mathfrak{Set}}(Hom_C(codom(g), dom(f)), Hom_C(dom(g), codom(f)))$. In the standard notation it is written "Hom_C." We include the parentheses from the point of view that "Hom_C" is the function $Hom_{(\ )}$ evaluated at C. [2]

0.2.13 Category $\downarrow_{()} (,)$ of arrows (in a third category) between two categories

.

If $A \xrightarrow{F} C$ and $B \xrightarrow{G} C$ are functors with the same codomians $codom_{(\mathfrak{Cat})}(F) = C = codom_{(\mathfrak{Cat})}(G)$, then $\downarrow_{(C)} (F, G)$ is the category of arrows from A to B in C with respect to F and G. So. the objects are triples (a, ϕ, b) where $a \in Ob(dom(F))$ and $b \in Ob(dom(G))$ and $\phi \in Hom_C(F(a), G(b))$, and the arrows $(a, \psi, b) \longrightarrow (a', \phi, b')$ are pairs (f, g) where $f : a \to a'$ and $g : b \to b'$ such that $G(g) \cdot \psi = \phi \cdot F(f)$.

0.2.14

The symbol $ob_{()}()$ assigns to any category C and an object $c \in Ob(C)$ the "object functor," $ob_{(C)}(c)$, which sends the category with one arrow $\star = (\{\emptyset\}, \{\emptyset\}, ...) \xrightarrow{ob_{(C)}(c)} C$ to C by mapping $\emptyset \mapsto c$ on objects and $\emptyset \mapsto id_c$ on arrows.

0.2.15

The *domain object functor* $dob \downarrow_{()} (,)$. If again $A \xrightarrow{F} C \xleftarrow{G} B$, then

$$\downarrow_{(C)} (F, G) \xrightarrow{dob\downarrow_{(C)}(F,G)} dom_{(\mathfrak{Cat})}(F)$$

is defined by sending any triple (a, ϕ, b) in $\downarrow_{(C)} (F, G)$ to a, i.e., one only remembers the domain of the arrow.

---

[2] The hom map can be thought of as a single map (that on objects, assigning the hom set to a pair of objects), or a pair of maps (as given, a functor). Strictly speaking, the former is part of the data which determines a category, while the latter exists only after a definite category is constructed

0.2.16

The *codomain object functor* $cob \downarrow_{()} (,)$. If again $A \xrightarrow{F} C \xleftarrow{G} B$, then

$$\downarrow_{(C)} (F,G) \xrightarrow{cob\downarrow_{(C)}(F,G)} dom_{(\mathfrak{Cat})}(G)$$

is defined by sending a triple (a,ϕ,b) in $\downarrow_{(C)} (F,G)$ to b, i.e. one remembers only the codomain of the arrow.

0.2.17 Symbol $\langle()\rangle_{Full()}$

Here, $\langle S\rangle_{Full(C)}$ means the full subcategory of C generated from $S \subseteq Ob(C)$.

0.2.18

$\langle()\rangle_{CatA()}$

$\langle S\rangle_{CatA(C)}$ generates a subcategory from $S \subseteq Arr(C)$

0.2.19

$\langle()\rangle_{Equiv()}$

$\langle R\rangle_{Equiv(S)}$ is the equivalence relation on S generated by $R \subseteq S \times S$.

0.2.20

$[]_{()}$

$[f]_{(R)}$ is the equivalence class of f with respect to the equivalence relation R.

0.2.21

$\mathfrak{TCat}$

is the *category of tensor categories*. Objects are pairs $(A, \otimes : A \times A \to A)$, and arrows are pairs $(F : A \to B, \rho : \otimes_B \circ (F \times F) \to F \cdot \otimes_A)$, where ρ is a natural transformation of functors (see [1]). Note that we do not require that ρ should be an isomorphism, or that $(A, \otimes)$ should be equipped with an associator or a unit.

0.2.22

$\mathfrak{ATCat}$

is the *category of associative tensor categories.* Objects are triples $(A,\otimes,\alpha)$, where $(A,\otimes) \in Ob(\mathfrak{TCat})$ is a tensor category and $\alpha : \otimes\cdot(\otimes\times_{\mathfrak{Cat}} id_A) \to \otimes\cdot(id_A\times_{\mathfrak{Cat}}\otimes)\cdot\alpha_{\mathfrak{Cat}}(A,A,A)$ is a natural isomorphism, where $\alpha_{\mathfrak{Cat}}(A,A,A) \in Hom_{\mathfrak{Cat}^2}((A\times_{\mathfrak{Cat}} A)\times_{\mathfrak{Cat}} A, A\times_{\mathfrak{Cat}} (A\times_{\mathfrak{Cat}} A)))$ is the usual associator for the product category functor $\times_{\mathfrak{Cat}} : \mathfrak{Cat}\times\mathfrak{Cat} \longrightarrow \mathfrak{Cat}$. We refer to the natural isomorphism α as an "associator" for $(A,\otimes)$.[3]

0.2.23 The Functor $Hom_{U-\mathfrak{Cat}^2}$

We do not define $U-\mathfrak{Cat}^2$, the 2-category of U-small categories, itself, but it appears as part of the symbol $Hom_{U-\mathfrak{Cat}^2}$, since this functor essentially constitutes the enrichment data associated to $U-\mathfrak{Cat}^2$. The "enrichment" of $U-\mathfrak{Cat}$ over itself is given by a functor

$$Hom_{U-\mathfrak{Cat}^2} := (Hom_{U-\mathfrak{Cat}^2(0)}, Hom_{U-\mathfrak{Cat}^2(1)}) : \ U-\mathfrak{Cat}^{opp}\times U-\mathfrak{Cat} \longrightarrow U-\mathfrak{Cat},$$

defined by the two functions, $Hom_{U-\mathfrak{Cat}^2(i)}$, for $i = 0,1$, defined below. [4]

The functor $Hom_{U-\mathfrak{Cat}^2}$ on objects is the function $Hom_{U-\mathfrak{Cat}^2(0)}$ that sends a pair of categories (C,D) to the U-category $Hom_{U-\mathfrak{Cat}^2(0)}(C,D)$ of functors $C\to D$. (Its set of objects is the set of functors $F : C \longrightarrow D$ and the set of arrows is the set of natural transformations $F \xrightarrow{\alpha} G$; see [1] or [2].)

On arrows the functor $Hom_{U-\mathfrak{Cat}^2}$ is given by the function $Hom_{U-\mathfrak{Cat}^2(1)}$ which sends an arrow $(G,F) \in Arr(U-\mathfrak{Cat}^{opp}\times_{U'-\mathfrak{Cat}} U-\mathfrak{Cat})$, given by a pair of functors $(C' \xrightarrow{G} C) \in (U-\mathfrak{Cat})^{opp}$ and $(D \xrightarrow{F} D') \in U-\mathfrak{Cat}$, to a functor $Hom_{U-\mathfrak{Cat}^2(1)}(G,F) : Hom_{U-\mathfrak{Cat}^2(0)}(C,D) \longrightarrow Hom_{U-\mathfrak{Cat}^2(0)}(C',D')$.

We define $Hom_{U-\mathfrak{Cat}^2(1)}(G,F)$ on objects by forwards and backwards composition, i.e. the object map is given by $(H \mapsto F\circ H\circ G)_{H\in Hom_{U-\mathfrak{Cat}}(C,D)}$.

We define the arrow map $Hom_{U-\mathfrak{Cat}^2(1)}(G,F)$ as follows. For a natural transformation $\alpha : H_1 \to H_2$ between functors $H_1, H_2 : C \longrightarrow D$, we define

$$Hom_{U-\mathfrak{Cat}^2(1)}(G,F)(\alpha) := (i \mapsto F(\alpha(G(i))))_{i\in Ob(C')} \in Arr(Hom_{U-\mathfrak{Cat}^2(0)}(C',D')),$$

so that $Hom_{U-\mathfrak{Cat}^2(1)}(G,F)(\alpha) : F\circ H_1\circ G \to F\circ H_2\circ G$.

---

[3] In contrast to the usual definition, we do not require a self-consistency condition for the associativity constraint α (the pentagon condition which requires that the two ways of "associating" the tensor product of four objects should be the same). Nether do we require unital structures or properties.

[4] Note also that the term "enrichment" has not yet been defined, and is not yet necessary. This will be done in section (2.2).

0.2.24

Suppose that I and C are categories. Then the diagonal functor

$\Delta_{(I,C)} : C \to Hom_{U-\mathfrak{Cat}^2(0)}(I,C)$

sends an object $c \in Ob(C)$ to its constant functor, which sends every object in I to c and every arrow to the identity arrow of c. It sends an arrow $\phi : c_1 \to c_2$ to the natural transformation $\Delta_{(I,A)(0)}(c_1) \to \Delta_{(I,A)(0)}(c_2)$ which assigns to every $i \in Ob(I)$ the arrow $\phi : c_1 \to c_2$. I.e., it is defined on objects by

$\forall c \in Ob(C), \forall f \in Arr(I),\ \Delta_{(I,C)(0)}(c)_{(1)}(f) = id_c$

and on arrows by the following

$\forall \phi \in Arr(A),\ \Delta_{(I,C)(1)}(\phi) := (i \mapsto \phi)_{i \in Ob(I)}.$

0.2.25

For any category $C \in Ob(U - \mathfrak{Cat})$, the Yoneda functors

$$Yo_{(C)} : C^{opp} \hookrightarrow Hom^{(1)}_{\mathfrak{Cat}^2}(C, U - \mathfrak{Set})$$

$$Yo^{opp}_{(C)} : C^{opp} \hookrightarrow Hom^{(1)}_{\mathfrak{Cat}^2}(C, U - \mathfrak{Set})$$

send an object $c \in Ob(C)$ to the functors given by $Yo_{(C)(0)}(c) : x \mapsto Hom_C(x,c)$ and $Yo^{opp}_{(C)(0)}(c) : x \mapsto Hom_C(c,x)$. [5]

[5] I imagine that it may be desirable to restrict the codomain of the Yoneda functors to a U-small sub-category, that their use might not force the unnecessary invocation of higher universes in certain circumstances.

CHAPTER 1

LIMITS AND ENRICHMENTS

In this section we define a variation of the concept of a limit of a functor. Given functors $J \xrightarrow{e} I \xrightarrow{F} A \xrightarrow{sk} B$, the $(sk, e) - limit$ of F is defined as follows.

One first constructs a certain category, denoted by "$P = P(sk, e)$." Its objects are essentially pairs (a, α), where $a \in Ob(A)$ and $\alpha : Ob(J) \to Arr(A)$ is a natural transformation [1] $\Delta_{(J,A)}(a) \xrightarrow{\alpha} F \circ e$ which lifts to a natural transformation $\tilde{\alpha} : \Delta_{(I,B)}(sk(a)) \to sk \circ F$. This is to say that, given the notation of (1.2.23), the pullback of $\tilde{\alpha}$, the natural transformation $e^*(\tilde{\alpha}) = Hom_{U-\mathfrak{Cat}^2(1)}(e, id_A)_{(1)}(\tilde{\alpha}) : \Delta_{(J,B)}(sk(a)) \to sk \circ F \circ e$, is equal to the pushforward of α, the natural transformation $sk_*(\alpha) = Hom_{U-\mathfrak{Cat}^2(1)}(id_J, sk)_{(1)}(\alpha) : \Delta_{(J,B)}(sk(a)) \to sk \circ F \circ e$. An arrow $\phi : (a, \alpha) \to (b, \beta)$ in the category P is an arrow $f : a \to b$ in the category A for which $\alpha(j) = \beta(j) \cdot f$ for any $j \in Ob(J)$.

We now define the (sk, e)-limit of F to be the colimit of the functor $P \longrightarrow A$ defined by $(a, \alpha) \mapsto a$. A "colimit" is a pair (l, λ) consisting of an object l and a natural transformation λ from the functor in question to the constant functor of l, satisfying a universal property.

In the case in which $A = U - \mathfrak{Cat}$, the functor sk should be the quotient functor $Skel : U - \mathfrak{Cat} \longrightarrow U - \mathfrak{SCat}$ (see 1.1.7), which identifies isomorphic pairs of functors, and the functor $e : J \longrightarrow I$ should be that which includes the subcategory of I consisting of all of the identity arrows. Then the objects of the category $P(Skel, e)$ are pairs (C, f) of a category C and a family of functors $f_i : C \to F(i)$ for $i \in Ob(I)$, which is a "transformation" of functors from a constant functor $\Delta_{I,} : I \to \mathfrak{Cat}$ with value C, to F, which is "natural up to isomorphism". The meaning of this is that for any arrow $g : i \to j$ in I, the functors $F(g) \circ f(i)$ and $f(j)$ from C to $F(j)$ are isomorphic. Then the (sk, e)-limit of F is the colimit of all such categories C.

We define in (2.2), for each tensor category $(A, \otimes)$ and functor $sk : A \longrightarrow B$ a notion of a category $WE_{(A,\otimes)(sk)}$ of weakly enriched sets, by which one replaces hom sets by hom

[1] Recall that for any categories $C, D \in Ob(\mathfrak{Cat})$, given any object $d \in Ob(D)$, we denote by $\Delta_{(C,D)}(d) : C \longrightarrow D$ the constant functor, which sends every arrow in C to id_d; see (1.2.24).

objects, which are objects of A. As weakly enriched sets S, T in ... can be thought of as "categores" (i.e., they are certain approximations of categories), the hom-set Hom... consists of "functors", i.e.,certain approximationsof functors. We give in (2.3) a construction by which one can enrich the set of "functors" $Hom_{WE(A,\otimes)(sk)}(S,T)$ to an enriched set.

We define a notion $(U,n)-\mathfrak{Cat}$ of the category of n-categories in (2.4).[2] By the lemmas of the "Enrichments" section (2.3), $(U,n)-\mathfrak{Cat}$ is naturally enriched over itself. In (2.5) we define a sort of co-simplicial structure on the category $(U,n)-\mathfrak{Cat}$ of n-categories. As usual denote by Δ the category of finite ordered sets, its arrows are order-preserving maps. Let $[n] = \{0 < \cdots < n-1\} \in Ob(\Delta)$ and denote by $\Delta_{[n]/}$ the category of arrows under the ordered set $[n]$. For each n we construct a functor $\rho : \Delta_{[n]/} \longrightarrow U'-\mathfrak{Cat}_{(U,n)-\mathfrak{Cat}/}$ (denoted also $\downarrow_{(U'-\mathfrak{Cat})} (ob_{(U'-\mathfrak{Cat})}(`(U.n)-\mathfrak{Cat}), id_{U'-\mathfrak{Cat}}))$.

The arrow $f_k : [m+1] = \{..., k, k+1, ...\} \to [m] = \{..., k, ...\}$ which identifies k and $k+1$ is sent to the functor which "collapses all $(n-k)$-categories into their component $(n-k-1)$-categories".[3]

The arrow $g_k : [m] \to [m+1]$ which omits k, i.e., "skips the k^{th} step", replaces all $(m-k)$-categories by their classifying $(m-k+1)$-categories.[4].

1.1 A Variation on Limits ((sk, e)-limits)

1.1.1 The Use of $dob \downarrow_{(-)}$

Recall that $\Delta_{(J,A)} : A \longrightarrow Hom_{U-\mathfrak{Cat}^2(0)}(J, A)$ by sending an object c to the c-valued constant functor, and $ob_{(Hom_{U-\mathfrak{Cat}^2(0)}(J,A))}(F \cdot e) : \star \longrightarrow Hom_{U-\mathfrak{Cat}^2(0)}(J, A)$ by sending the one arrow in $\star$ to $id_{F \cdot e}$. Recall also that the category

$$\downarrow_{(Hom_{U-\mathfrak{Cat}^2(0)}(J,A))} (\Delta_{(J,A)}, ob_{Hom_{U-\mathfrak{Cat}^2(0)}(J,A)}(F \cdot e))$$

of arrows is defined so that its objects are triples $(a, \alpha, \emptyset)$, where $a \in Ob(A)$, $\alpha : \Delta_{(J,A)}(a) \to F \cdot e$ is a natural transformation, and $\emptyset$ is the object in the category $\star$

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[2]This is not one of the standard notions of an n-category. In particular, $(U,n)-\mathfrak{Cat}$ is not an $(n+1)$-category, i.e. an object of $(U', n+1)-\mathfrak{Cat}$.

[3]By the collapse of a category C to a set we mean the set of all morphisms $\coprod_{(a,b)\in Ob(C)^2} C(a,b)$. This generalizes for any $(m+1)$-category C, the m-category $\rho(f_k)(C)$ is such that every k-category $h(a,b)$ appearing in the structure of C is replaced by the $(k-1)$-category $\coprod_{a',b'\in Ob(h(a,b))} h_{h(a,b)}(a',b')$.

[4]having a trivial underlying set of objects and the same $(m-k)$-category as the only hom object, i.e. for any m-category D, the $(m+1)$-category $\rho(g)(D)$ is such that every k-category $h(a,b)$ appearing in the structure of D is replaced by the $(k+1)$-category whose underlying set $\{\emptyset\}$, is the singleton, so that $h(\emptyset,\emptyset) = h(a,b)$. Composition is trivial, being given by projection to the left (or right) component

(the category with one arrow). An arrow between $(a_1, \alpha_1, \emptyset) \xrightarrow{(\phi, id_\emptyset)} (a_2, \alpha_2, \emptyset)$ is a pair of arrows $((a_1 \xrightarrow{\phi} a_2), id_\emptyset) \in Arr(A) \times Arr(\star)$ for which $\alpha_2 \cdot \Delta_{(J,A)(1)}(\phi) = \alpha_1$.

An isomorphic category is given by forgetting both the $\emptyset$ symbol, and the a term (since for any $j \in Ob(J)$, $a = dom(\alpha(j))$, so that a is determined by α), so that its objects are natural transformations α, where $a \in Ob(A)$ and $\alpha : \Delta_{(J,A)}(a) \to F \cdot e$. If $\alpha_1 : \Delta_{(J,A)}(a_1) \to F \cdot e$ and $\alpha_2 : \Delta_{(J,A)}(a_2) \to F \cdot e$, then an arrow $\alpha_1 \xrightarrow{\phi} \alpha_2$ is an arrow $(a_1 \xrightarrow{\phi} a_2) \in Arr(A)$ for which $\alpha_2 \cdot \Delta_{(J,A)(1)}(\phi) = \alpha_1$

1.1.2 Definition of an (sk, e)-Limit

Consider functors $J \xrightarrow{e} I \xrightarrow{F} A \xrightarrow{sk} B$.

Consider the set of maps $\alpha : Ob(I) \to Arr(A)$ such that $sk \cdot \alpha$ defines a natural transformation from a diagonal functor to $sk \cdot F$. Let

$$\mathcal{C} :_t = \downarrow_{(Hom^{(1)}_{U-\mathfrak{Cat}^2(0)}(I,B))} (\Delta_{(I,B)}, ob_{(Hom^{(1)}_{U-\mathfrak{Cat}^2}(I,B))}(sk \cdot F))$$

and

$$\mathcal{D} :_t = \downarrow_{(Hom^{(1)}_{U-\mathfrak{Cat}^2(0)}(J,A))} (\Delta_{(J,A)}, ob_{(Hom^{(1)}_{U-\mathfrak{Cat}^2}(J,A))}(F \circ e))$$

and

$$\mathcal{E} :_t = \downarrow_{(Hom^{(1)}_{U-\mathfrak{Cat}^2(0)}(J,B))} (\Delta_{(J,B)}, ob_{(Hom^{(1)}_{U-\mathfrak{Cat}^2}(J,B))}(sk \circ F \circ e))$$

Let $\varepsilon : P :_t = \mathcal{C} \times_{\mathcal{E}} \mathcal{D} \longrightarrow \mathcal{D}$ be one of the arrows of a fibred product, an arrow in $U' - \mathfrak{Cat}$. If For is the functor which takes the object a from an arrow $\Delta_{(J,A)}(a) \longrightarrow F \circ e$, an (sk, e)-limit is a colimit of $For \circ \varepsilon$.

This is explained in the following sections.

1.1.2.1. Let P be the full sub-category of the category

$$\downarrow_{(Hom_{U-\mathfrak{Cat}^2(0)}(J,A))} (\Delta_{(J,A)}, ob_{Hom_{U-\mathfrak{Cat}^2(0)}(J,A)}(F \circ e)) \subseteq Hom_{U-\mathfrak{Cat}^2(0)}(J, A)_{/F \circ e}$$

whose objects are natural transformations α, such that for some $a \in Ob(A)$ we have $\alpha : \Delta_{(J,A)}(a) \to F \cdot e$, such that there exists a natural transformation $\tilde{\alpha} : \Delta_{(I,B)}(sk(a)) \to sk \cdot F$ such that the natural transformation $Hom_{U-\mathfrak{Cat}^2(1)}(id_J, sk)(\alpha) : \Delta_{(J,B)}(sk(a)) \to sk \cdot F \cdot e$ given by sending $j \in Ob(J)$ to $sk(\alpha(j)) : sk(a) \to sk(F(e(j)))$ is equal to the natural transformation $Hom_{U-\mathfrak{Cat}^2(1)}(e, id_B)(\tilde{\alpha}) : \Delta_{(J,B)}(sk(a)) \to sk \cdot F \cdot e$ given by sending $j \in Ob(J)$ to $\tilde{\alpha}(e(j)) : sk(a) \to sk(F(e(j)))$.

1.1.2.2. Denote by $\varepsilon : P \longrightarrow \downarrow_{(Hom_{U-\mathfrak{Cat}^2(0)}(J,A))} (\Delta_{(J,A)}, ob_{(Hom_{U-\mathfrak{Cat}^2(0)}(J,A))}(F \cdot e))$ the inclusion, given by $\alpha \mapsto (a, \alpha, \emptyset)$. Denote also by p the functor,

$$p :_t= dob \downarrow_{(Hom_{U-\mathfrak{Cat}^2(0)}(J,A))} (\Delta_{(J,A)}, ob_{(Hom_{U-\mathfrak{Cat}^2(0)}(J,A))}(F \cdot e) \cdot \varepsilon : P \longrightarrow A.$$

Thus, p is given by sending $\alpha \mapsto a$, and the fibre of p over any given $a \in Ob(A)$ is the set of natural transformations $\Delta_{(J,A)}(a) \xrightarrow{\alpha} F \cdot e$ such that for some natural transformation $\tilde{\alpha} : sk \cdot \Delta_{(I,A)}(a) = \Delta_{(I,B)}(sk(a)) \longrightarrow sk \cdot F$, one has

$$Hom_{U-\mathfrak{Cat}^2(1)}(id_B, e)(\tilde{\alpha}) = Hom_{U-\mathfrak{Cat}^2(1)}(id_J, sk)(\alpha).$$

In other words,

$$P \subseteq \downarrow_{(Hom_{U-\mathfrak{Cat}^2(0}(J,A))} (\Delta_{(J,A)}, ob_{(Hom_{U-\mathfrak{Cat}^2(0)}(J,A))}(F \cdot e)$$

is the full subcategory which contains all objects α such that the image of α in $Hom_{U-\mathfrak{Cat}^2(0)}(J, B)$ under the functor $Hom_{U-\mathfrak{Cat}^2(1)}(id_J, sk)$ has a lift to $Hom_{U-\mathfrak{Cat}^2(0)}(I, B)$ by the functor $Hom_{U-\mathfrak{Cat}^2(1)}(e, id_B)$ (we denote this lift by $\tilde{\alpha}$).

1.1.2.3. Then the (sk, e)-limit of F is the colimit of p, i.e., for any pair $(l, \lambda) \in Ob(A) \times Arr(Hom_{U-\mathfrak{Cat}^2(0)}(P, A))$ for which $\lambda : p \to \Delta_{(P,A)}(l)$, we say that (l, λ) is an $(sk) - limit(F)$ iff (l, λ) is a $colimit(p)$ (in the sense in which (λ, l) is a universal arrow in $Hom_{U-\mathfrak{Cat}^2(0)}(P, A)$, from p to the constant functor $\Delta_{(P,A)} : A \longrightarrow Hom_{U-\mathfrak{Cat}^2(0)}(P, A))$.

1.1.3 Example

In the above, if either e or sk is an identity functor, then the (sk)-limit of F is the limit (l, λ) of F, if the latter exists.

If $e = id_I$, then P consists of all natural transformations $\alpha : \Delta_{(I,A)}(a) \to F \circ e = F$ for which there exists some lift $\tilde{\alpha} : \Delta_{(I,B)}(sk(a)) \to sk \circ F$. But $\tilde{\alpha} = Hom_{U-\mathfrak{Cat}^2(1)}(sk, id_I)(\alpha)$ would be such a lift. Therefore any $\alpha : \Delta_{(I,A)}(a) \to F$ has a lift. Furthermore, for any $i \in Ob(I)$, $\alpha(i) : a \to F(i)$, and for any $\beta : \Delta_{(I,A)}(b) \to F$, and any arrow $\phi : \alpha \to \beta$ in P, by definition of P, we have that $\beta(i) \circ \phi = \alpha(i)$. Therefore, by the definition of a colimit, there exists a unique $\alpha_l(i) : l \to F(i)$ such that for any $(\Delta_{(I,A)}(a) \xrightarrow{\alpha} F) \in Ob(P)$, $\alpha(i) = \alpha_l(i) \circ \lambda(\alpha)$. Since each colimit arrow $\alpha_l(i)$ is determined by the arrows $\alpha(i)$ which come from natural transformations α, the assignment $\alpha_l = (i \mapsto \alpha_l(i))_{i \in Ob(I)}$ determines a natural transformation $\Delta_{(I,A)}(l) \to F$. Therefore $\alpha_l \in Ob(P)$. If the limit of F exists, then it is isomorphic to a terminal object in P, $\alpha_t \in Ob(P)$. But by the above argument, this terminal object α_t determines a colimit arrow $\lambda(\alpha_t) : a_t \to l$, and being a terminal object in P there is a unique arrow $e_l : l \to a_t = dom(\alpha_t(i))$ in P. By the definition of terminal objects, $e_l \circ \lambda(\alpha_t) = id_{a_t}$

1.1.4 Lemma

(Inclusion, via right exactness) Given $sk, F, e : J \to I$, $\varepsilon : P \to$
$\downarrow_{(Hom_{(U-\mathfrak{Cat}^2)(0)}(J,A))} (\Delta_{(J,A)}, ob_{(Hom_{(U-\mathfrak{Cat}^2)(0)}(J,A))}(F \circ e))$, l, and λ as above, suppose further that sk is right exact, and $Ob(I) = Ob(J)$. For each $i \in Ob(I) = Ob(J)$, consider the arrow induced from the colimit l to $F(i)$ by $\alpha \mapsto \alpha(i)$, where $(p, \alpha, \cdot) \in Ob(P)$ is an object in P. Then this assignment determines an object $(l, \alpha_l, \cdot) \in Ob(P)$.

I.e. the (sk)-limit determines an object,

$$(l, (j \mapsto \lambda(((b, f, \cdot) \mapsto f(j))_{(b,f,\cdot) \in Ob(P)}))_{j \in Ob(J)}, \cdot) \in Ob(P)$$

in P.

Proof. If the colimit (l, λ) is sent to the colimit of the forward composition by sk of the $dob \downarrow$ diagram on P, then arrows from l to it are yet determined by their pullbacks to the components of the forward composition of the colimit diagram, which commute after forward composition. □

1.1.5 Lemma

(Uniqueness, via monic) For any $sk : A \to B, F : I \to A \in Arr(U - \mathfrak{Cat})$, for any $(l, \lambda) \in Ob(A) \times Arr(Hom_{U-\mathfrak{Cat}^2(0)}(P, A))$, ($P$ being as above) if the arrow from the (sk)-limit(F) to the product $\prod_{j \in J} F(j)$ induced by the arrows from the previous lemma, i.e. $\lambda_{\prod}((j \mapsto \lambda(((b, f, \cdot) \mapsto f(j))_{(b,f,\cdot) \in Ob(P)})))_{j \in Ob(J)})) : l \to \prod_{j \in Ob(J)} F_{(0)} \circ e_{(0)}(j)$, is monic, then for each $(b, f, \cdot) \in Ob(P)$, the coproduct arrow $b \to l$ is the unique arrow λ' for which $\lambda(((b, f, \cdot) \mapsto f(j))_{(b,f,\cdot) \in Ob(P)}) \circ \lambda' = f(j)$.

Proof. Trivial. □

1.1.6 Remark

This is the uniqueness of factorization usually associated to limits.

1.1.7 Definition of the Skeleton Functor

Define the category $U - \mathfrak{SCat} \in Ob(U' - \mathfrak{Cat})$ so that

$$Ob(U - \mathfrak{SCat}) = Ob(U - \mathfrak{Cat})$$

and for any $x, y \in Ob(U - \mathfrak{SCat})$, $Hom_{(U-\mathfrak{SCat})}(x, y)$ is the the set

$$Hom_{(U-\mathfrak{SCat})}(x, y) := \{[F] \subseteq Ob(Hom_{U-\mathfrak{Cat}^2(0)}(x, y)); F \in Hom_{U-\mathfrak{Cat}}(x, y)\}$$

of isomorphism classes of functors $x \xrightarrow{F} y$, where $[F] = [G]$ iff $F \cong G$, i.e. iff there exists an isomorphism $F \xrightarrow{\alpha} G$ of functors.

Define the functor

$$Skel : U - \mathfrak{Cat} \to U - \mathfrak{SCat}$$

so that $Skel$ is the identity map on the objects and the quotient map $F \mapsto [F]$ on the arrows.

1.1.8 Example

Consider $\phi, \psi \in Arr(U - \mathfrak{Cat})$, with the same codomain. The $(Skel)$-limit of the diagram is the subcategory L of $dom(\phi) \times_{U-\mathfrak{Cat}} dom(\psi)$ such that $Arr(L) = \{f \in Arr(dom(\phi) \times_{U-\mathfrak{Cat}} dom(\psi); \exists u, v \in Arr(codom(\phi)), u, v$ are isomorphisms and $u \circ \pi_\phi(f) = \pi_\psi(f) \circ v\}$. Any category with such functors into the two domain categories that the composition of functors on one side is isomorphic to the composition of functors on the other side factors through L via the compositions of the projections with the embedding into the product. By the monic lemma the factorization is unique. However the conclusion of the inclusion lemma might not apply to it, i.e. the two compositions $L \to dom(\phi) \to codom(\phi)$ and $L \to dom(\psi) \to codom(\psi) = codom(\phi)$ might not be isomorphic, since I might imagine having two different pairs of arrows (f_1, g_1), and (f_2, g_2), such that the isomorphisms $u_1, v_1 \in Arr(codom(\phi))$ which form the commuting square $u_1 \circ f_1 = g_1 \circ v_1$ differ from the isomorphisms $u_2, v_2 \in Arr(codom(\phi))$ which form the commuting square $u_2 \circ f_2 = g_2 \circ v_2$.

1.1.9 Lemma, for Reduction to the Standard Limit

If

$$(l, (j \mapsto \lambda(((b, f, \cdot) \mapsto f(j))_{(b,f,\cdot) \in Ob(P)}))_{j \in Ob(J)}, \cdot) \in Ob(P)$$

and the limit arrows are unique then this is the usual limit.

Proof. Trivial. □

1.1.10 Functoriality

An arrow of functors $F \circ e \to G \circ e$ which lifts to an arrow of functors $sk \circ F \to sk \circ G$ (i.e. an arrow in the fibred product of the two functors $Hom_{U-\mathfrak{Cat}^2(1)}(e, id_B)$ and $Hom_{U-\mathfrak{Cat}^2(1)}(id_J, sk)$) induces a map from the (sk, e)-limit of F to that of G, using the colimit map. I.e. $\alpha : F \to G$ implies that $\alpha(dom(\phi)) \circ F(\phi) = G(\phi) \circ \alpha(codom(\phi))$, so that for any arrow $\beta : \Delta_{(J,C)(0)}(c) \to F \circ e$ associated to $(a, \beta, \emptyset) \in Ob(P)$ (notation as in the first definition), $Hom_{U-\mathfrak{Cat}^2(1)}(e, id_A)(\alpha) \circ \beta$ also commutes after applying sk (i.e. comes from an arrow in $Hom_{U-\mathfrak{Cat}^2(0)}(I, B)$). Therefore each such a has an arrow into the

sk-limit of G from the colimit diagram of the definition, which induces a map from the colimit diagram which determines the sk-limit of F.

1.1.10.1. Given a diagram $F : I' \longrightarrow Hom_{U-\mathfrak{Cat}^2(0)}(I, A)$, and a choice of an (sk, e)-limit $(l(i), \lambda(i))$ for any object $i \in Ob(I')$, the construction of (1.1.10) determines a function $Arr(I') \longrightarrow Arr(A)$

1.1.10.2. If for any $i \in Ob(I')$, the (sk, e)-limit $(l(i), \lambda(i))$ is included in $P(i)$ ($P(i)$ being as in the definition of the (sk, e)-limit for $F(i)$) then (1.1.10.1) determines a functor $I' \longrightarrow A$.

1.1.11 Remark

Roughly speaking, one takes the colimit of the domains of all limit diagrams on the trivial category which, when forwards composed with sk, are the backwards composition by e of an actual limit diagram of $sk \circ F$. Definition (2.3) following this remark is dual to Definition (2.1).

1.1.12 Definition of the (sk, e)-Colimit

Consider functors $J \xrightarrow{e} I \xrightarrow{F} A \xrightarrow{sk} B$.

1.1.12.1. Let P be the full sub-category of the category

$$\downarrow_{(Hom_{U-\mathfrak{Cat}^2}(J,A))} (ob_{Hom_{U-\mathfrak{Cat}^2}(J,A)}(F \circ e), \Delta_{(J,A)}) \subseteq Hom_{U-\mathfrak{Cat}^2(0)}(J, A)_{\backslash F \circ e}$$

of arrows, whose objects are given by natural transformations from functor $F \circ e$ to functor $\Delta_{(J,A)}(a)$, i.e. triples $(\emptyset, \alpha, a)$ for varying $a \in Ob(A)$, such that there exists a natural transformation $\tilde{\alpha}$ from functor $sk \circ F$ to functor $\Delta_{(I,B)}(a)$ such that the natural transformation from $sk \circ F$ to $\Delta_{(J,B)}(sk(a))$ is equal to the natural transformation given by sending $j \in Ob(J)$ to $\tilde{\alpha}(e(j))$, i.e. by the set

$$\{\alpha : \Delta_{(J,A)}(p) \xrightarrow{\alpha} F \circ e;$$
$$\exists \tilde{\alpha} : sk \circ \Delta_{(I,A)}(p) = \Delta_{(I,B)}(sk(p)) \longrightarrow sk \circ F, Hom^{(1)}_{U-\mathfrak{Cat}^2(1)}(id_B, e)(\tilde{\alpha}) = \alpha\},$$

so as to be given by the category of arrows from the diagonal functor to the object functor of $F \circ e$ in the category of functors from J to A.

1.1.12.2. Suppose that $\varepsilon : P \longrightarrow \downarrow_{(Hom_{U-\mathfrak{Cat}^2}(J,A))} (\Delta_{(J,A)}, ob_{(Hom_{U-\mathfrak{Cat}^2}(J,A))}(F \circ e))$ is the inclusion.

1.1.12.3. For any $sk : A \to B, F : I \to A \in Arr(U - \mathfrak{Cat})$, $codom(F) = dom(sk)$ implies that any pair $(l, \lambda) \in Ob(A) \times Arr(Hom_{(U-\mathfrak{Cat}^2)(0)}(A, U - \mathfrak{Set}))$, (l, λ) is a $(sk, e) - colimit(F)$ iff (l, λ) is a limit $(cob \downarrow_{(Hom_{(U-\mathfrak{Cat}^2)(0)}(J,A))}$

$(ob_{(Hom_{(U-\mathfrak{Cat}^2)(0)}(J,A))}(F \circ e), \Delta_{(J,A)}) \circ \varepsilon_c)$.

1.1.13 Lemma

(Inclusion via exactness) Dual to the above.

1.1.14 Lemma

(Uniqueness via epic) Dual to the above.

1.1.15 Example

Consider ϕ,$\psi \in Arr(U-\mathfrak{Cat})$, with the same domain. The $(Skel)$-colimit of the diagram is the category L such that its set of objects is the disjoint union of the objects of the codomain categories and the arrows are the formal compositions of the disjoint union of arrows in $Arr(codom(\phi))$, $Arr(codom(\psi))$, and arrows $e_a : \phi_{(0)}(a) \to \psi_{(0)}(a)$, $e_a^{-1} : \psi_{(0)}(a) \to \phi_{(0)}(a)$ formally added for each $a \in Ob(dom(\phi)) = Ob(dom(\psi))$, with the relation generated by requiring that $\forall f \in Arr(dom(\phi)), \phi_{(1)}(f) \circ e_{dom(f)} = e_{codom(f)} \circ \psi_{(1)}(f)$. If $l_\phi : codom(\phi) \to L$ and $l_\psi : codom(\psi) \to L$ are given by the $U-\mathfrak{Set}$ coproduct maps then for any $l'_\phi, l'_\psi \in Arr(U-\mathfrak{Cat})$ such that $l'_\phi \circ \phi \cong l'_\psi \circ \psi$, there is an arrow $q : L \to codom(l'_\phi) = codom(l'_\psi)$ such that $l'_\phi = q \circ l_\phi$ and $l'_\psi = q \circ l_\psi$. If an isomorphism $\alpha : l'_\phi \circ \phi \to l'_\psi \circ \psi$ is specified (or vice versa), then there is a unique $q : L \to codom(l'_\phi)$ such that $Hom_{(U-\mathfrak{Cat}^2)(1)}((id_{dom(\phi)}, q))_{(1)}((a \mapsto e_a)_{a \in Ob(dom(\phi))}) = \alpha$ (and vice versa).

1.1.16 Lemma

(Reduction) Dual to the above.

1.1.17 Lemma

(Functoriality) An arrow of functors $F \to G$ induces a map from the (sk)-colimit of F to that of G, using the limit map.

1.1.18 Remark

Products and coproducts are not affected by sk.

1.2 Definitions regarding Enrichments

We will define weak enrichment of sets and categories. Sets will be enriched over tensor categories $(A, \otimes)$ and categories over triples $(A, \otimes, F)$ where tensor category $(A, \otimes)$ comes with a tensor functor $F : (A, \otimes) \longrightarrow (\mathfrak{Set}, \times)$.

A weak enrichment of a set s over $(A, \otimes)$ adds to s a category-like structure, a version of Hom which has values in A (rather than in sets) but without any associativity or unital requirements. We later introduce, for each functor $sk : A \longrightarrow B$, a category of weakly enriched sets, "associative up to sk," in that the associativity diagrams are commutative after the functor sk is applied to them. A weak enrichment of a category C over $(A, \otimes)$ is a weak enrichment of the set $Ob(C)$ which is compatible with the Hom_C, this compatibility is formulated in terms of the functor F.

1.2.1 Definition of a Weakly Enriched Set

A weak enrichment of a set $s \in Ob(U-\mathfrak{Set})$ over a tensor category $(A, \otimes) \in Ob(U-\mathfrak{TCat})$ is a pair of a map $h : s^2 \to Ob(A)$ and a "composition map" $\circ : s^3 \to Arr(A))$ such that for any $a, b, c \in s$,

$$\circ(a, b, c) : h(a, b) \otimes h(b, c) \longrightarrow h(a, c)$$

1.2.2 Definition the Category of Weak Enrichments

For any $(A, \otimes) \in Ob(U - \mathfrak{TCat})$, the category of $(A, \otimes)$-enriched sets $VWE(A, \otimes) \in Ob(U - \mathfrak{Cat})$ has as objects weak enrichements if sets $S = (s, h_S, \circ_S)$, and for two weak enrichments S and T an arrow $f : S = (s, h_S, \circ_S) \to T = (t, h_T, \circ_T)$ is a pair of functions $f = (f_1 : s \to t, f_2 : s^2 \to Arr(A))$ such that the following hold.

1.2.2.1. $\forall a, b \in s, \quad f_2(a, b) : h_S(a, b) \longrightarrow h_T(f_1(a), f_1(b))$, and

1.2.2.2. $\forall a, b, c \in s$,

$$\circ_T(f_1(a), f_1(b), f_1(c)) \circ (f_2(a, b) \otimes f_2(b, c)) = f_2(a, c) \circ \circ_S(a, b, c),$$

i.e. the compositions commute with the arrows defining a "functor from S to T".

1.2.3 Lemma

One can construct a functor $VWE_{\mathfrak{Cat}} : U - \mathfrak{TCat} \to U - \mathfrak{Cat}$ from the category of tensor categories to the category of categories by the following. For any functor of tensor categories $(F, \rho) : (A, \otimes_A) \to (B, \otimes_B)$ (see the notation section 0.2.21) define a functor $VWE(F, \rho) : VWE(A, \otimes_A) \to VWE(B, \otimes_B)$ from the category of very weak enrichments over $(A, \otimes_A)$ to that of $(B, \otimes_B)$ as follows.

1.2.3.1. It sends an object $S = (s, h, \circ)$ of $VWE(A, \otimes_A)$ to the triple $F(S) = (s, h', \circ')$ where for $a, b, c \in s$,

$$h'(a, b) = F(h(a, b)) \text{ and } \circ'(a, b, c) = F(\circ(a, b, c)) \circ \rho(h(b, c), h(a, b)).$$

1.2.3.2. It sends an arrow $\phi : S = (s, h_s, \circ_s) \to (t, h_t, \circ_t) = T$ in $VWE(A, \otimes_A)$ (here $s^2 \ni (a,b) \mapsto \phi(a,b) \in Arr(A)$) to the arrow $F(\phi) : F(S) \to F(T)$ that sends $(a,b) \in s^2$ to $F(\phi(a,b))) \in Arr(B)$.

1.2.4 Definition of an Weakly Enriched Category

A weak enrichment of a category C with respect to a tensor functor $(A, \otimes) \xrightarrow{(F,\rho)} (U - \mathfrak{Set}, \times_{U-\mathfrak{Set}})$ is a quadruple $(C, h, \circ, \phi)$ such that $C \in Ob(U - \mathfrak{Cat})$ is a category, h and $\circ$ define a weak enrichment of the set $Ob(C)$, and $\phi : Ob(C)^2 \to Arr(U - \mathfrak{Set})$ is a function, such that

1.2.4.1. For any $a, b \in Ob(C)$, $\phi(a,b) : F(h(a,b)) \to Hom_C(a,b)$ is an isomorphism;

1.2.4.2. For any $a, b, c \in Ob(C)$, the composition

$$\circ_C(a,b,c) : Hom_C(b,c) \times_{U-\mathfrak{Set}} Hom_C(a,b) \to Hom_C(a,c)$$

of hom sets in C is given by the weak enrichment, i.e.

$$\circ_C(a,b,c) = \phi(a,c)^{-1} \circ F(\circ(a,b,c)) \circ \rho(h(b,c), h(a,b)) \circ (\phi(b,c) \times_{U-\mathfrak{Set}} \phi(a,b))$$

1.2.5 Definition of the Category of Weakly Enriched Categories

The category $WE_{\mathfrak{Cat}0}(F, \rho)$ of categories weakly enriched over a tensor category $(A, \otimes)$ with respect to a tensor functor $(F, \rho) : (A, \otimes) \to (U - \mathfrak{Set}, \times_{U-\mathfrak{Set}})$, has objects which are categories $(C, h, \circ, \phi)$ weakly enriched over $(A, \otimes)$. An arrow $f : (C, h_C, \circ_C, \phi) \to (D, h_D, \circ_D, \psi)$ consists of a functor $(f_0, f_1) : C \longrightarrow D$ and a function $f_2 : Ob(C)^2 \longrightarrow Arr(A)$, such that

1.2.5.1. $(f_0, f_2) : (Ob(C), h_C, \circ_C) \to (Ob(D), h_D, \circ_D)$ is an arrow of weak enrichments of sets;

1.2.5.2. For any $a, b \in Ob(C)$,

$$F_1(f_2(a,b)) = \psi(f_0(a), f_0(b)) \circ F(f_2(a,b)) \circ \phi(a,b)^{-1}$$

i.e. the functor agrees with that implied by the enrichment.

1.2.6

One can construct a functor from the category of tensor categories over the tensor category of sets $U - \mathfrak{TCat}_{/(U-\mathfrak{Set}, \times_{U-\mathfrak{Set}})}$ to the category of categories, i.e.

$$WE_{\mathfrak{Cat}}() := (WE_{\mathfrak{Cat}0}(), WE_{\mathfrak{Cat}1}()) : U - \mathfrak{TCat}_{/(U-\mathfrak{Set}, \times_{U-\mathfrak{Set}})} \longrightarrow U' - \mathfrak{Cat}$$

in analogue to the construction of Lemma 1.2.3, as follows. For any arrow $(\Phi, \rho) : (F, \rho_F) \longrightarrow (G, \rho_G)$ of tensor categories $(F, \rho_F) : (A, \otimes_A) \longrightarrow (U - \mathfrak{Set}, \times_{U-\mathfrak{Set}})$ and

$(G, \rho_G) : (B, \otimes_B) \longrightarrow (U-\mathfrak{Set}, \times_{U-\mathfrak{Set}})$ over $(\mathfrak{Set}, \times_{U-\mathfrak{Set}})$ define a functor $WE_{\mathfrak{Cat}0}(F, \rho_F) \longrightarrow WE_{\mathfrak{Cat}0}(G, \rho_G)$.

1.2.6.1. It is defined on an object $(C, h, \circ, \phi) \in Ob(WE_{\mathfrak{Cat}0}(F, \rho))$ by

$$(C, h, \circ, \phi) \longmapsto (C, \Phi_{(0)} \circ h, ((a, b, c) \mapsto \Phi_{(1)}(\circ(a, b, c)) \circ \rho(h(b, c), h(a, b)))_{a,b,c \in Ob(\mathcal{C})}, \phi).$$

1.2.6.2. It is deifined on arrows $(F, F_2) : (C, h_C, \circ_C, \phi) \to (D, h_D, \circ_D, \psi)$ by

$$(F, F_2) \mapsto WE_{\mathfrak{Cat}1}(\Phi, \rho)(F, F_2) := (F, \Phi_{(1)} \circ F_2).$$

1.2.7 Definition of Two Forgetful Functors

Define the following two functors.

1.2.7.1. For any tensor functor $(F, \rho) : (A, \otimes) \longrightarrow (U-\mathfrak{Set}, \times_{U-\mathfrak{Set}})$, the forgetful functor $For^{WE(F,\rho)}_{VWE(dom(F,\rho))} : WE_{\mathfrak{Cat}}(A, \otimes, F) \longrightarrow VWE_{\mathfrak{Cat}}(A, \otimes)$ from the category of weakly enriched categories with respect to (F, ρ) to weakly enriched sets with respect to $(A, \otimes)$ is the functor given by passing from a category C to its set of objects $Ob(C)$. More precisely, it is defined on an object $(C, h, \circ, \phi) \in Ob(WE_{\mathfrak{Cat}}(F, \rho))$ by

$$(C, h, \circ, \phi) \mapsto (Ob(C), h, \circ)$$

and on an arrow $(f, f_2) \in Arr(WE_{\mathfrak{Cat}}(F, \rho))$ by

$$(f, f_2) \mapsto (f_{(0)}, f_2)$$

1.2.7.2. The forgetful functor from the category of weakly enriched categories to the category of categories $For^{WE(F,\rho)}_{\mathfrak{Cat}} : WE_{\mathfrak{Cat}}(F, \rho) \longrightarrow U-\mathfrak{Cat}$ is the functor which forgets the enrichment structure, returning the underlying category. I.e. it sends a weakly enriched category $(C, h, \circ, \phi)$ to C.

1.2.8 Definition of (sk)-Associativity

Consider an associative tensor category $(A, \otimes, \alpha) \in Ob(U-\mathfrak{ATCat})$ A weakly $(A, \otimes)$-enriched set $(S, h, \circ) \in Ob(WE(A, \otimes))$, is said to be *(sk)-associative* for a functor $sk : A \to B$ (an arrow in $U-\mathfrak{Cat}$) if for any $a, b, c, d \in S$,

$$sk_{(1)}(\circ(a, b, d) \circ (id_{h(a,b)} \otimes \circ(b, c, d)) \circ \alpha(h(a, b), h(b, c), h(c, d))) =$$

$$sk_{(1)}(\circ(a, c, d) \circ (\circ(a, b, c) \otimes id_{h(c,d)})),$$

i.e. the standard self-consistency diagram (pentagram) for the enriched composition $\circ$ is required to commute after applying the functor sk

$$\begin{array}{ccc}
h(c,d)\otimes(h(b,c)\otimes h(a,b)) & \xrightarrow{\alpha(h(c,d),h(b,c),h(a,b))} & (h(c,d)\otimes h(b,c))\otimes h(a,b) \\
{\scriptstyle id_{h(c,d)}\otimes\circ(a,b,c)}\downarrow & & {\scriptstyle \circ(b,c,d)\otimes id_{h(a,b)}}\downarrow \\
h(c,d)\otimes h(a,c) & & h(b,d)\otimes h(a,b) \\
{\scriptstyle \circ(a,c,d)}\downarrow & & {\scriptstyle \circ(a,b,d)}\downarrow \\
h(a,d) & \xrightarrow{=} & h(a,d).
\end{array}$$

1.2.9 Definition of the Category $WE_{(A,\otimes)(sk)}$

For any $sk : A \longrightarrow B \in Arr(Cat)$, define the category $WE_{(A,\otimes)(sk)} \in Ob(\mathfrak{Cat})$ of $((A,\otimes), sk)$-enriched sets.

1.2.9.1. Its objects are sets enriched over A.

1.2.9.2. The hom sets

$$Hom_{WE_{(A,\otimes)(sk)}}((S,h_S,\circ),(T,h_T,\circ_T)) =$$

are the pairs of maps of sets $(F_0,F_1) \in Arr(\mathfrak{Set})^2$ such that $F_0 : S \to T$ and $F_1 : S^2 \to Arr(A)$ and

1.2.9.2.1. For any $a,b \in S, F_1(a,b) \in Hom_A(h_S(a,b), h_T(F_0(a),F_0(b))$.

1.2.9.2.2. F_1 respects composition after applying sk.

1.2.10 Remark

Roughly speaking, F_0 is the map between objects of enriched sets, and $F_1 : h_S \to h_T \circ F$ is the "natural transformation of hom functors," (there are no non-trivial arrows in S). This means that applying the "functor," (F_0,F_1), then composing in T, versus composing in S and then applying the functor, gives two arrows in A, such that sk of one arrow is equal to sk of the other.

1.2.11 Lemma

$WE_{(A,\otimes)(sk)}$ is a category.

Proof. The issue is composition. Given composible arrows

$$((S, h_S, \circ_S) \xrightarrow{(F_0,F_1)} (T, h_T, \circ_T)), ((T, h_T, \circ_T) \xrightarrow{(G_0,G_1)} (U, h_U, \circ_U)) \in Arr(WE_{(A,\otimes)(sk)})$$

Starting from the result of application of the functor sk to the arrow which uses the composition $\circ_U$,

$$\begin{gathered} sk(\\ \circ_U(G_0 \circ F_0(a), G_0 \circ F_0(b), G_0 \ \circ \ F_0(c)) \\ \circ((G_1(F_0(a), F_0(b)) \circ F_1(a,b)) \otimes (G_1(F_0(b), F_0(c)) \circ F_1(b,c))) \\ ) = \end{gathered}$$

by functoriality of $\otimes$

$$\begin{gathered} sk(\\ \circ_U(G_0 \circ F_0(a), G_0 \circ F_0(b), G_0 \circ F_0(c)) \circ \\ (G_1(F_0(a), F_0(b)) \otimes G_1(F_0(b), F_0(c))) \circ \\ (F_1(a,b) \otimes F_1(b,c)) \\ ) = \end{gathered}$$

by functoriality of sk

$$\begin{gathered} sk(\circ_U(G_0 \circ F_0(a), G_0 \circ F_0(b), G_0 \circ F_0(c))) \circ \\ sk((G_1(F_0(a), F_0(b)) \otimes G_1(F_0(b), F_0(c)))) \circ \\ sk((F_1(a,b) \otimes F_1(b,c))) = \end{gathered}$$

by $(G_0, G_1) \in Arr(WE_{(A,\otimes)(sk)})$,

$$sk(G_1(F_0(a), F_0(c))) \circ sk(\circ_T(F_0(a), F_0(b), F_0(c))) \circ sk((F_1(a,b) \otimes F_1(b,c))) =$$

by $(F_0, F_1) \in Arr(WE_{(A,\otimes)(sk)})$,

$$sk(G_1(F_0(a), F_0(c))) \circ sk(F_1(a,c)) \circ sk(\circ_S(a,b,c))$$

□

1.2.12 Lemma

If $(A, \otimes)$ has products, then so does $WE_{Set(sk)}(A, \otimes)$. The product is functorial.

1.3 Enrichment of $Hom_{WE_{(A,\otimes)(sk)}}(I,C)$

Consider a tuple of functors $\{p_i : I_i \longrightarrow A\}_{i=1}^n$. Suppose that for each $i \in \{1,...,n\}$, the colimit $colim\ p_i \in Ob(A)$ exists, with universal arrows $e_{i(x_i)} : p_i(x_i) \to colim\ p_i$. Suppose that the colimit of the functor $\otimes_{i=1}^n p_i : \prod_{i=1}^n I_i \longrightarrow A$ defined by $(x_i)_{i=1}^n \mapsto \otimes_{i=1}^n p_i(x_i)$ is also an object in A. Consider the arrow $(colim \otimes_{i=1}^n p_i \to \otimes_{i=1}^n colim\ p_i) \in Arr(A)$ induced by $(x_i)_{i=1}^n \mapsto \otimes_{i=1}^n e_{i(x_i)}$; i.e. by tensoring the universal arrows together. The following lemma states that under certain conditions on the p_i, the above defines a natural transformation with respect to arrows of functors $\phi_i : p_i \to q_i$.

The $(A,\otimes)$-enrichment of the hom-sets in $WE_{(A,\otimes)(sk)}$ involves such colimits, and the definition of the composition requires that the above arrows should be isomorphisms. This means that the "forward and backward composition functors" to be introduced in lemma 1.3.7 below are determined by the arrows between products $\prod h_S(...) \to \prod h_T(...)$.

1.3.1 Lemma on the Naturality of τ

Suppose that $\{F_i, G_i : I_i \longrightarrow A\}_{i=1}^n$ are functors, and $\{(F_i \xrightarrow{\phi_i} G_i)\}_{i=1}^n$ are arrows of functors. Suppose that for each $i \in \{1,...,n\}$, $P_{F_i} \subseteq\downarrow_{(Hom^{(1)}_{U-\mathfrak{Cat}^2}(J_i,A))} (\Delta_{(J_i,A)}, F_i \circ \varepsilon_i)$ and $P_{G_i} \subseteq\downarrow_{(Hom^{(1)}_{U-\mathfrak{Cat}^2}(J_i,A))} (\Delta_{(J_i,A)}, G_i \circ \varepsilon_i)$ are subcategories, where $\varepsilon_i : J_i \subseteq I_i$ is the subcategory with only identity arrows.

1.3.1.1. Suppose that the functors $p_{F_i} : P_{F_i} \longrightarrow A$ and $p_{G_i} : P_{G_i} \longrightarrow A$ are as in the conditions of the limit inclusion lemma (i.e. $colim(p_{F_i})$ determines an object in P_{F_i}, with the analogue holding for G_i)

1.3.1.2. Define an arrow of sets $\tau_{(\ )} : \prod_{i=1}^n Ob(Hom^{(1)}_{U-\mathfrak{Cat}^2}(I_i,A)) \to Arr(A)$ so that for any $(H_i)_{i=1}^n \in \prod_{i=1}^n Ob(Hom^{(1)}_{U-\mathfrak{Cat}^2}(I_i,A))$, $\tau_{((H_i)_{i=1}^n))} : colim(\bigotimes_{i=1}^n p_{H_i}) \to \bigotimes_{i=1}^n colim(p_{H_i})$ is the universal arrow for the colimit induced by the assignment (where $\lambda_{(i)}$ is the natural transformation defining the colimit of p_{H_i})

$$((a_{(i)}, f_{(i)})_{i=1}^n \mapsto \otimes_{i=1}^n \lambda_{(i)}((a_{(i)}, f_{(i)}))\ )_{(a_{(i)},f_{(i)})_{i=1}^n \in \prod_{i=1}^n P_{H_i}}$$

1.3.1.3. If $u = u_{(p)} : colim(\bigotimes_{i=1}^n p_{F_i}) \to colim(\bigotimes_{i=1}^n p_{G_i})$ is the universal arrow for the colimit induced by the assignment

$$((a_{(\ )}, f_{(\ )}) \mapsto \lambda'_G((\otimes_{i=1}^n a_{(i)}, \otimes_{i=1}^n \phi_i \circ f_{(i)}))\ )_{(a_{(\ )},f_{(\ )}) \in Ob(\prod_{i=1}^n P_{F_i})}$$

then

$$\otimes_{i=1}^n \lambda_{G_j}((a, \phi_j \circ f)) \circ \tau_{F_{(\ )}} = \tau_{G_{(\ )}} \circ u$$

I.e., $\tau : colim(\bigotimes_{i=1}^n p_i) \to \bigotimes_{i=1}^n colim(p_i)$ is "natural at $(\phi_i)_{i=1}^n$."

Proof. By the monic arrow condition, the arrows involved are situated above the products, $\prod_{a\in Ob(I_i)} F_i(a)$, so that the arrows $\otimes_{i=1}^n a_{(i)} \to \otimes_{i=1}^n l_{G_i}$ are pure tensors respecting the arrows ϕ_i. $\square$

The following lemma defines a weak enrichment of the set $Hom_{WE_{(A,\otimes)(sk)}}(C,D)$. To any "$(A,\otimes)$-functors" $\Phi,\Psi : C \to D$, one attaches a category P, and defines the hom object between Φ and Ψ to be a colimit of a certain functor $P \longrightarrow A$. Roughly speaking, P keeps track of all arrows into to the product $\prod_{x\in Ob(C)} h_D(\Phi(x),\Psi(x))$ which respect the composition with any arrows "coming from some $h_C(x,y)$," after one applies sk. P is a full sub-category of the category of arrows over $\prod_{x\in Ob(C)} h_D(\Phi(x),\Psi(x))$. The objects of P are all arrows $(a \xrightarrow{\pi} \prod_{x\in Ob(C)} h_D(\Phi(x),\Psi(x))$, such that for any $x,y \in Ob(C)$, for any $(t_0 \xrightarrow{t} h_C(x,y)) \in Arr(A)$, tensoring π with t, projecting to the y-component $\prod_{x\in Ob(C)} h_D(\Phi(x),\Psi(x)) \to h_D(\Phi(y),\Psi(y))$, and composing in D is (sk)-equal to tensoring t with π, projecting to the x-component, and composing in D. One defines $p : P \longrightarrow A$ to be the functor which remembers the domain of a given arrow. One associates to Φ and Ψ the object $colim(p) \in Ob(A)$ (assuming that the colimit exists).

One composes, i.e. defines, for all $(A,\otimes)$-functors Φ,Ψ,Ξ, an arrow

$$(h(\Phi,\Psi) \otimes h(\Psi,\Xi) \xrightarrow{\circ} h(\Phi,\Xi)) \in Arr(A)$$

by taking the inverse of the arrow $colim\ (P_{\Phi,\Psi} \otimes P_{\Psi,\Xi}) \to (colim\ P_{\Phi,\Psi}) \otimes (colim\ P_{\Psi,\Xi})$ (that this is an isomorphism is assumed), and recognizing $colim\ (P_{\Phi,\Psi} \otimes P_{\Psi,\Xi})$ as an object in $P_{\Phi,\Xi}$ by using the composition in D and the projection for the products to define arrows $P_{\Phi,\Psi}(x) \otimes P_{\Psi,\Xi}(y) \to \prod_{x\in Ob(C)} h_D(\Phi(x),\Xi(x))$. As an object in $P_{\Phi,\Xi}$, $colim\ (P_{\Phi,\Psi} \otimes P_{\Psi,\Xi})$ has assigned to it an arrow into $colim\ P_{\Phi,\Xi}$, which is defined to be the hom object assigned to Φ and Ξ. One composes this colimit arrow with the inverse of the first arrow to define the composition arrow.

1.3.2 Lemma on the Enrichment of $Hom_{WE_{(A,\otimes)(sk)}}(C,D)$

Suppose that $(A,\otimes)$ has a symmetrizer and associator for the tensor.

1.3.2.1. For any $\Phi,\Psi \in Hom_{WE(A,\otimes)(sk)}(C,D)$, define

$$P \subseteq\downarrow_{(A)} (id_A, ob_{(A)}(\prod_{x\in Ob(C)} h_D(\Phi(x),\Psi(x))))$$

to be the full subcategory generated by objects (i.e. arrows $a \xrightarrow{\pi} \prod_{x\in Ob(C)} h_D(\Phi(x),\Psi(x))$ in A) such that for any $x,y \in Ob(C), (t_0 \xrightarrow{t} h_C(x,y)) \in Arr(A)$,

$$sk(\circ_D \circ (id_{h_D(\Phi(y),\Psi(y))} \otimes \Psi(x,y)) \circ ((\pi \circ \pi_y) \otimes t)) =$$

$$sk(\circ_D \circ (\Phi(x,y) \otimes id_{h_D(\Phi(x),\Psi(x))}) \circ \sigma \circ ((\pi \circ \pi_x) \otimes t))$$

Then define $\bar{h}_{WE(A,\otimes)(sk)1}(C,D)(\Phi,\Psi)$ to be the colimit of the domain object functor $p : P \longrightarrow A$ defined by $(a,f) \mapsto a$.

1.3.2.2. Suppose that the compostion on D is (sk)-associative, and the arrows $u = u_{(p)} : colim \otimes_{i=1}^n \; p_i \rightarrow \otimes_{i=1}^n colim\ p_i$ are isomorphisms, defined as in the previous lemma, and $p = \{(1, p_{\Phi,\Psi})\} \cup \{(2, p_{\Psi,X})\} : \{1,2\} \rightarrow Arr(\mathfrak{Cat})$. Then define the composition $\circ_{WE(A,\otimes)(sk)}(C,D)(\Phi,\Psi,X) \in Arr(A)$ by taking it to be the composition of the colimit arrow $e : colim(-_1 \otimes -_2) \rightarrow h_{WE(A,\otimes)(sk)}(C,D)(\Phi,X)$ associated to the object $(colim(-_1 \otimes -_2), \phi) \in Ob(P_{\Phi,X})$ determined by the arrow

$$\phi : colim(-_1 \otimes -_2) \rightarrow \prod_{a \in Ob(C)} h_D(\Phi(a), X(a))$$

induced by sending any given $((a,f),(b,g)) \in Ob(P_{\Phi,\Psi} \times_{\mathfrak{Cat}} P_{\Psi,X})$ to the product arrow given to the assignment

$$a \mapsto \circ_D(\Phi(a),\Psi(a),X(a)) \circ (\pi_{(\Phi,\Psi)a} \circ f \otimes \pi_{(\Psi,X)a} \circ g)$$

with u^{-1}, I.e.

$$\circ_{WE(A,\otimes)(sk)}(C,D)(\Phi,\Psi,X) := e \circ u^{-1}$$

1.3.2.3. Define $\bar{h}_{WE(A,\otimes)(sk)}(C,D) :=$

$$(Hom_{WE_{(A,\otimes)(sk)}}(C,D), \bar{h}_{WE(A,\otimes)(sk)1}(C,D)(\ ,\ ), \circ_{WE(A,\otimes)(sk)}(C,D)(\ ,\ ,\ )) \in Ob(WE(A,\otimes))$$

i.e. part i. gives the hom objects and part ii. gives the composition.

1.3.2.4. $\bar{h}_{WE(A,\otimes)(sk)}(C,D)$ is (sk)-associative. If $(A,\otimes)$ has a unit I such that $\circ_D$ is $(Yo_{(0)}^{opp}(I))$-associative, then so does $\bar{h}_{WE(A,\otimes)(sk)}(C,D)$.

1.3.3 Remark

The enrichment on $Hom_{WE_{(A,\otimes)(sk)}}(C,D)$, i.e. the objects $h(\Phi,\Psi)$ defined in the previous lemma for $(A,\otimes)$-functors Φ and Ψ, were initially constructed as (sk)-equalizers. I believe that the present construction can also be realized as an (sk)-equalizer, but by use of a diagram containing arrows of the form $[(Homfun(A) \circ ((- \otimes J) \times id_A), \circ \circ (\pi \otimes id_J))] \in Arr(\Omega)$, and with restrictions on A.

1.3.4 Definition of the Enriched Arrows Functor

If $(A, \otimes) \in Ob(\mathfrak{TCat})$ has coproducts, then define

$$\bar{Arr}_{(A,\otimes)} : WE_{(A,\otimes)} \longrightarrow A$$

by $(S, h, \circ) \mapsto \coprod_{s,t \in S} h(s,t)$ and $(F_0, F_1) \mapsto \coprod_{s,t \in S} F_1(s,t)$.

1.3.5 Remark

The functor sk is not referred to in this definition. $\bar{Arr}_{(A,\otimes)}$ is the "enriched arrow functor."

1.3.6 Lemma

$\bar{Arr}_{(A,\otimes)}$ is faithful.

The following lemma concerns the self-enrichment of the category $WE_{(A,\otimes)(sk)}$. The enriched hom set defined in the previous lemma is denoted by "$\bar{h}_{WE(A,\otimes)(sk)}(B, C)$." Part (i) of the following lemma defines the "forward composition/pushforward functor," $\bar{h}_{WE(A,\otimes)(sk)}(B, C) \rightarrow \bar{h}_{WE(A,\otimes)(sk)}(B, D)$. Part (ii) defines the "backward composition / pullback functor," $\bar{h}_{WE(A,\otimes)(sk)}(C, D) \rightarrow \bar{h}_{WE(A,\otimes)(sk)}(B, D)$. Part (iii) states that one can use these to define an arrow

$$(\bar{h}_{WE(A,\otimes)(sk)}(B, C) \times \bar{h}_{WE(A,\otimes)(sk)}(C, D) \rightarrow \bar{h}_{WE(A,\otimes)(sk)}(B, D)) \in Arr(WE_{(A,\otimes)(sk)})$$

which gives the enriched composition in $WE_{(A,\otimes)(sk)}$.

1.3.7 Lemma on Composition Functors

Given $(sk) \in Arr(Cat)$, for any $(F : C \rightarrow D)$, $(G : B \rightarrow C) \in Arr(WE_{(A,\otimes)(sk)})$,

1.3.7.1.

$$(F_* : \bar{h}_{WE(A,\otimes)(sk)}(B, C) \rightarrow \bar{h}_{WE(A,\otimes)(sk)}(B, D)) \in Arr(WE_{(A,\otimes)(sk)})$$

is induced by $\prod_{a \in Ob(B)} h_C(\Psi_1(a), \Psi_2(a)) \rightarrow \prod_{a \in Ob(B)} h_D(F \circ \Psi_1(a), F \circ \Psi_2(a))$, which induces a functor $P_{\bar{h}_{WE(A,\otimes)(sk)}(B,C)(\Psi_1,\Psi_2)} \longrightarrow P_{\bar{h}_{WE(A,\otimes)(sk)}(B,D)(F\circ\Psi_1,F\circ\Psi)}$, so that an arrow is induced from the colimit of the first diagram $(p_{\bar{h}_{WE(A,\otimes)(sk)}(B,C)(\Psi,\Psi_2)}$ to the colimit of the second $p_{\bar{h}_{WE(A,\otimes)(sk)}(B,D)(\mathbb{F}\circ\Psi_1,F\circ\Psi)}$.

1.3.7.2.

$$(G^* : \bar{h}_{WE(A,\otimes)(sk)}(C, D) \rightarrow \bar{h}_{WE(A,\otimes)(sk)}(B, D)) \in Arr(WE_{Set(sk)}(A, \otimes))$$

is induced by $\prod_{a \in Ob(C)} h_D(\Phi_1(a), \Phi_2(a)) \rightarrow \prod_{a \in Ob(B)} h_D(\Phi_1 \circ G(a), \Psi_2 \circ G(a))$, which is the product map induced by the assignment $(a \mapsto \pi_{G(a)})$.

These are analogues to the usual forward and backward functors associated to composition on either end of a functor category $Hom(B, C)$.

1.3.7.3. From an arrow of functors $\alpha : \times_A \to \otimes$, the previous two constructions, and the product structure, construct an arrow in $WE_{Set(A,\otimes)}$

$$\bar{h}_{WE(A,\otimes)(sk)}(B, C) \times_{WE_{(A,\otimes)(sk)}} \bar{h}_{WE(A,\otimes)(sk)}(C, D) \longrightarrow \bar{h}_{WE(A,\otimes)(sk)}(B, D)$$

(Not unique. Corresponding to the choice of the path $F_1 \circ G_1 \to F_1 \circ G_2 \to F_2 \times G_2$)

1.3.7.4. Defining $\bar{\circ} : Ob(WE_{Ass(A,\otimes)(sk)})^3 \mapsto Arr(WE_{Ass(A,\otimes)(sk)})$ by sending $(B, C, D) \in Ob(WE_{Ass(A,\otimes)(sk)}$ to the arrow in (iii),

$$(Ob(WE_{Assoc(sk)}(A, \otimes)), \bar{h}_{WE_{(sk)}(A,\otimes)}, \bar{\circ})$$

is an $(WE_{Ass(sk)}(A, \otimes), \times_{WE_{Assoc(sk)}}(A, \otimes)$-enriched set, whose composition is (Ob)-associative and $(sk \circ \bar{Arr}_{(A,\otimes)})$-associative.

Proof. Parts i. and ii. consist only in checking for (sk)-commutativity so that the constructions can be made. Part iii., states that for any $C, D, E \in Ob(WE(A, \otimes))$, for any $\Phi_1, \Phi_2, \Phi_3 \in Ob(\bar{h}_{WE(A,\otimes)(sk)}(C, D))$, for any $\Psi_1, \Psi_2, \Psi_3 \in Ob(\bar{h}_{WE(A,\otimes)(sk)}(D, E)$,

$$\circ_{\bar{h}_{(C,E)}}(\Psi_1 \circ \Phi_1, \Psi_2 \circ \Phi_2, \Psi_3 \circ \Phi_3)\circ$$

$$(\circ_{\bar{h}_{(C,E)}}(\Psi_1 \circ \Phi_1, \Psi_1 \circ \Phi_2, \Psi_2 \circ \Phi_2) \otimes \circ_{\bar{h}_{(C,E)}}(\Psi_2 \circ \Phi_2, \Psi_2 \circ \Phi_3, \Psi_3 \circ \Phi_3))\circ$$
$$((\Phi_2^* \otimes \Psi_{1*}) \otimes (\Phi_3^* \otimes \Psi_{2*})) \circ \sigma_{1*} =_{(sk)}$$
$$\circ_{\bar{h}_{(C,E)}}(\Psi_1 \circ \Phi_1, \Psi_1 \circ \Phi_3, \Psi_3 \circ \Phi_3) \circ (\Phi_3^* \otimes \Psi_{1*}) \circ (\circ_{\bar{h}_{(D,E)}}(\Psi_1, \Psi_2, \Psi_3) \otimes \circ_{\bar{h}_{(C,D)}}(\Phi_1, \Phi_2, \Phi_3)))$$

given that $colim(p) \in P$ with a monic arrow into the relevant product, and that $\forall f, g : x \to \prod_{i\in I} y_i, \forall i \in I, sk(\pi_i \circ f) = sk(\pi \circ g) \Longrightarrow sk(f) = sk(g)$.

All arrows between the objects $\bar{h}_{WE(A,\otimes)(sk)}(C, D) \to \bar{h}_{WE(A,\otimes)(sk)}(C', D')$ commute with monic arrows $\bar{h}_{WE(A,\otimes)(sk)}(C, D) \to \prod_{c\in Ob(C)} h_D(F(c), G(c))$.

After taking the inverse of the isomorphism $\otimes colim\ p_i \leftarrow colim \otimes\ p_i$ (that this is an isomorphism is assumed), these maps are determined by the arrows Ψ_{i*} and Φ_i^*. On the components of the product Φ_i^* come from identity arrows and Ψ_{i*} from $\Psi(a, b)$.

Diagram with two arrows,

$$\prod_{a\in Ob(D)} h_E(\Psi_1(a), \Psi_2(a)) \otimes \prod_{a\in Ob(D)} h_E(\Psi_2(a), \Psi_3(a))\otimes$$

$$\prod_{a\in Ob(C)} h_D(\Phi_1(a), \Psi_2(a)) \otimes \prod_{a\in Ob(C)} h_D(\Phi_2(a), \Phi_3(a))$$

$$\to h_E(\Psi_1 \circ \Phi_1(a), \Psi_3 \circ \Phi_3(a))$$

(one side is $\Phi_3^* \otimes \Phi_3^* \otimes \Psi_{1*} \otimes \Psi_{1*}$ and the other is $\Phi_2^* \otimes \Phi_3^* \otimes \Psi_{1*} \otimes \Psi_{2*}$). The $\Phi_3^* \otimes \Phi_3^* \otimes \Psi_{1*} \otimes \Psi_{1*}$ side is

$$\Pi \to$$

$$h_E(\Psi_1 \circ \Phi_3(a), \Psi_2 \circ \Phi_3(a)) \otimes h_E(\Psi_2, \Phi_3(a), \Psi_3 \circ \Phi_3(a)) \otimes h_D(\Phi_1(a), \Phi_2(a)) \otimes h_D(\Phi_2(a), \Phi_3(a))$$

$$\xrightarrow{id \otimes id \otimes \Psi_1(\Phi_1(a), \Phi_2(a)) \otimes \Psi_1(\Phi_2(a), \Phi_3(a))}$$

$$h_E(\Psi_1 \circ \Phi_3(a), \Psi_2 \circ \Phi_3(a)) \otimes h_E(\Psi_2 \circ \Phi_3(a), \Psi_3 \circ \Phi_3(a)) \otimes$$

$$h_E(\Psi_1 \circ \Phi_1(a), \Psi_1 \circ \Phi_2(a)) \otimes h_E(\Psi_1 \circ \Phi_2(a), \Psi_1 \circ \Phi_3(a))$$

$$\xrightarrow{\circ_E} h_E(\Psi_1 \circ \Phi_1(a), \Psi_3 \circ \Phi_3(a))$$

The $\Phi_2^* \otimes \Phi_3^* \otimes \Psi_{1*} \otimes \Psi_{2*}$ side is

$$\Pi \to$$

$$h_E(\Psi_1 \circ \Phi_2(a), \Psi_2 \circ \Phi_2(a)) \otimes h_E(\Psi_2, \Psi_3(a), \Psi_3 \circ \Phi_3(a)) \otimes h_D(\Phi_1(a), \Phi_2(a)) \otimes h_D(\Phi_2(a), \Phi_3(a))$$

$$\xrightarrow{id \otimes id \otimes \Psi_1(\Phi_1(a), \Phi_2(a)) \otimes \Psi_2(\Phi_2(a), \Phi_3(a)) \circ \sigma}$$

$$h_E(\Psi_2 \circ \Phi_3(a), \Psi_3 \circ \Phi_3(a)) \otimes h_E(\Psi_1 \circ \Phi_2(a), \Psi_2 \circ \Phi_2(a)) \otimes$$

$$h_E(\Psi_2 \circ \Phi_2(a), \Psi_2 \circ \Phi_3(a)) \otimes h_E(\Psi_1 \circ \Phi_1(a), \Psi_1 \circ \Phi_1(a))$$

$$\xrightarrow{\circ_E} h_E(\Psi_1 \circ \Phi_1(a), \Psi_3 \circ \Phi_3(a))$$

By definition of $P_{\bar{h}_{(C,D)}(\Psi_1, \Psi_2)}$, in particular, "commutativity" of the composition with any arrow going through a hom object of C, the two arrows are (sk)-equal.$\square$

1.3.8 Remark

On underlying "objects" this is the usual composition (e.g. 1-composition, of functors).

1.3.9 n-Categories

An n-Category is defined inductively as an object in the category of (n-1)-enriched categories.

The refutations of this approach (that it returns strict n-categories) which I've read referred only to enrichments associative in the strict sense. I therefore expect that requiring only (sk)-associativity (in a sense to be made precise below) should sidestep this. n-Categories with their basic structures are inductively defined, referring to each other (and therefore inseparable).

1.3.10 The inductive construction of n-categories

We define, inductively and simultaneously, the

1. "forgetful functors" ("objects functors") $F(n)$,
2. natural transformations $\rho(n)$,
3. the "associators" $\alpha(n)$,
4. the "product functors" $\times(n)$,
5. "symmetrizers" $\sigma(n)$,
6. "unit objects" $I(n)$,
7. right and left unit arrows $\rho_u(n)$, $\lambda_u(n)$,
8. $(n) - \textit{equivalence}$ of (n)-categories,
9. (n)-equivalence of (n)-functors,
10. the (n)-skeleton functor $sk(n)$, and
11. the U'-category $U(n) - \mathfrak{Cat}$.

Here, for any $n \in \mathbb{N}$ the category of (n)-categories $U(n) - \mathfrak{Cat}$ is the category of sets that are weakly enriched over the category $U(n-1) - \mathfrak{Cat}$ of $(n-1)$-categories.

1.3.11 Definition of n-Category

Assuming that we have defined these objects for all integers $\leq n$ we define them for $n+1$.

1.3.11.1. The "forgetful", or "objects" functor is defined on n-categories and it takes an n-category (an enriched set) to the underlying set

$$F(n+1) := Yo^{opp}_{((U,n+1)-\mathfrak{Cat})(0)}(I(n+1)) : (U, n+1) - \mathfrak{Cat} \longrightarrow U - \mathfrak{Set}$$

1.3.11.2. Define a natural transformation

$$\rho(n) : \times_{U'-\mathfrak{Set}} \circ (F(n) \times_{U'-\mathfrak{Cat}} F(n)) \to F(n) \circ \times(n)$$

by the identity maps.

1.3.11.3. Define the (n)-associator

$\alpha(n+1) : \times(n+1)\circ(\times(n+1)\times_{U'-\mathfrak{Cat}} id_{U'-\mathfrak{Cat}}) \to \times(n+1)\circ(id_{(U,n+1)-\mathfrak{Cat}} \times_{U'-\mathfrak{Cat}} \times(n+1))$

as arrow of functors $((U,n+1)-\mathfrak{Cat})^3 \longrightarrow (U,n+1)-\mathfrak{Cat}$, defined on objects by the associator and on hom objects by $\alpha(n)$.

1.3.11.4. Define the (n)-product functor

$$\times(n+1) : (U,n+1)-\mathfrak{Cat} \times_{U'-\mathfrak{Cat}} (U,n+1)-\mathfrak{Cat} \longrightarrow (U,n+1)-\mathfrak{Cat}$$

on objects by the usual product functor (arrow in $U'-\mathfrak{Cat}$), defined on objects by the usual product functor and on hom objects by $\times(n)$, $\sigma(n)$, and $\alpha(n)$.

1.3.11.5. Define the symmetrizing transformation

$$\sigma(n+1) : \times(n+1) \to \times\sigma_{U'-\mathfrak{Cat}}$$

on underlying objects by the usual symmetrizer and on hom objects by $\times(n)$ and $\sigma(n)$.

1.3.11.6. Define the unit object

$$I(n+1) \in Ob((U,n+1)-\mathfrak{Cat})$$

having $\{\emptyset\}$ as its underlying set, and $I(n)$ for the hom object.

1.3.11.7. Define the left and right unit arrows

$$\rho_u(n+1), \lambda_u(n+1) \in Arr(Hom^{(1)}_{U'-\mathfrak{Cat}^2}((U,n+1)-\mathfrak{Cat}, (U,n+1)-\mathfrak{Cat}))$$

$$\rho_u(n+1) : -\times I(n+1) \to Id$$

$$\lambda_u(n+1) : I(n+1)\times - \to Id$$

By the usual units on objects and $\rho_u(n)$ and $\lambda_u(n)$ on the hom objects.

1.3.11.8. Define, for any $C, D \in Ob((U,n+1)-\mathfrak{Cat})$, the statement

(C,D) are $(n+1)equivalent_0 \iff$

There exist $F : C \to D$, $G : D \to C$, such that for any $(c_1,c_2) \in Ob(C)$, $(d_1,d_2) \in Ob(D)$, $F_{(1)}(c_1,c_2)$ and $G_{(1)}(d_1,d_2)$ are (n)-equivalences, and $(G\circ F, id_C), (F\circ G, id_D)$ are $(n+1)$-equivalent$_1$.

1.3.11.9. Define, for any $C, D \in Ob((U,n+1)-\mathfrak{Cat})$, for any $F, G \in Hom_{(U,n+1)-\mathfrak{Cat})}(C,D)$, the statement

(F,G) are $(n+1)-equivalent_1 \iff$

There exist

$$\phi \in F(n)(\bar{h}_{WE((U,n)-\mathfrak{Cat},\times(n))(sk(n))}(C,D)(F,G))$$
$$\psi \in F(n)(\bar{h}_{WE((U,n)-\mathfrak{Cat},\times(n))(sk(n))}(C,D)(G,F))$$

,

such that the various arrows

$$\bar{h}_{WE((U,n)-\mathfrak{Cat},\times(n))(sk(n))}(C,D)(F,F) \to \bar{h}_{WE((U,n)-\mathfrak{Cat},\times(n))(sk(n))}(C,D)(F,G)$$
$$\bar{h}_{WE((U,n)-\mathfrak{Cat},\times(n))(sk(n))}(C,D)(G,G) \to \bar{h}_{WE((U,n)-\mathfrak{Cat},\times(n))(sk(n))}(C,D)(G,F)$$

given by the composition of $\bar{\circ}$, the arrow $I(\bar{n}) \to \bar{h}_{WE((U,n)-\mathfrak{Cat},\times(n))(sk(n))}(C,D)(F,G)$ associated to ϕ or ψ (see 1.3.2 part(iv).) and a unit arrow ($\lambda_u(n)$ or $\rho_u(n)$), are (n)-equivalences$_0$ (Slightly loose usage. Adapt part (8).) (i.e. they $(sk(n))$-invert one another).

Roughly speaking there are $(sk(n))$-natural transformations between F and G, which induce forward and backward compostion functors by the unit and enrichment lemma, which are (n)-equivalences, and such that $\phi \circ \psi$ and $\psi \circ \phi$ induce (n)-equivalent functors to the identities for the respective hom objects.

1.3.11.10. Define the (n)-skeleton functor $sk(n)$ as a quotient functor

$$sk(n) : (U,n) - \mathfrak{Cat} \longrightarrow Q$$

where Q is the category defined by

$$Ob(Q) = Ob((U,n) - \mathfrak{Cat})$$
$$Hom_{(Q)}(C,D) :_t= \{[F]_{(n)eq} \in \mathbf{2}^{(Hom_{((U,n)-\mathfrak{Cat})}(C,D))}; F \in Hom_{((U,n)-\mathfrak{Cat})}(C,D)\}$$

where $[F]_{(n)eq} = [G]_{(n)eq}$ iff (F,G) are (n)-equivalent.

1.3.11.11. Define the category of $(n+1)$-categories

$$(U, n+1) - \mathfrak{Cat} := WE_{Ass((U,n)-\mathfrak{Cat},\times(n)(sk(n),\alpha(n))}$$

to be the category of sets $(sk(n))$-associatively enriched over over the category of (n)-categories

1.3.12

Parts (ii) and (iii) of the following lemma give construction for limits and colimits in $WE(A,\otimes)$, to be applied to the (co)limits appearing in the construction of the enriched hom sets.

1.3.13 Lemma on Limits and Colimits in $WE(A, \otimes)$

For any $(A, \otimes) \in Ob(\mathfrak{T}\mathfrak{Cat})$,

1.3.13.1. $For : WE_{(sk)}(A, \otimes) \longrightarrow WE_{(term \circ sk)}(A, \otimes)$ is faithful, where $term : codom(sk) \longrightarrow \star$ is the functor whose codomain is the terminal category. I.e. one forgets that one had had a composition requirement.

1.3.13.2. The limit of $F : I \longrightarrow WE(A, \otimes)$ can be constructed by the limit of the underlying sets and $(a_i, b_i)_{i \in I} \mapsto limF'_1$, where $F'_1 : I \longrightarrow A$ is defined on objects by

$$F'_{1(0)} : j \mapsto h_{F_{(0)}(j)}(a_j, b_j))$$

1.3.13.3. For any $F : I \longrightarrow WE_{(sk)}(A, \otimes)$, if $\tau : colim \circ \otimes \to \otimes \circ (colim \times_{\mathfrak{Cat}} colim)$ is an isomorphism where hom objects $h_{F_{(0)}(i)}(x, y)$ are concerned, then the colimit can be similarly constructed, by

$$([(a, i)], [(b, j)]) \mapsto colimF'_1 \circ cob \downarrow_{(Hom^{(1)}_{\mathfrak{Cat}^2}((\{1,2\}, \ldots), I)} (ob_{(I)}(i) \cup ob_{(I)}(j), \Delta_{(\{1,2\})})$$

i.e. taking the colimit of all hom objects below both i and j. Define composition by the arrow induced by tensoring the colimit arrows assigned to $([(a, i)], [(b, j)])$ and $([(b, j)], [(c, k)])$, composed with the inverse of τ.

1.3.14 Remark

The explicit description of limits and (co)limits is applied to verify in the following lemma the isomorphism required for part (ii) of 1.3.2.

1.3.15 Lemma

$\forall n \in \mathbb{N}$, $colim \circ \times(n) \to \times(n) \circ (colim \times_{\mathfrak{Cat}} colim)$ is an isomorphism.

Proof. On the level of sets, this is the isomorphism given by $[(a_i, b_j)] \mapsto ([a_i], [b_j])$. By the previous lemma the product of enriched sets is given by taking the products of their hom objects, so that $\tau_{n+1} : colim \circ \times(n+1) \to \times(n+1) \circ (colim \times_{\mathfrak{Cat}} colim)$ is determined by τ_0 on underlying set and τ_n on hom objects. By induction, τ_n is for any n an isomorphism. $\square$

The "meaning" of the following theorem consists in the special cases of parts (iii) and (iv) of 1.3.7.

1.3.16 Theorem on $(U, n) - \mathfrak{Cat}$

The category $(U, n) - \mathfrak{Cat}$ is weakly enriched over itself. I.e.

$$((U, n+1) - \mathfrak{Cat}, \bar{h}_{WE((U,n)-\mathfrak{Cat},\times(n))(sk(n)}, \bar{\circ}(n)) \in Ob(WE((U, n+1) - \mathfrak{Cat}, \times(n+1)).$$

The hom set agrees with that given by applying the objects functor $Ob = F(n)$ to the hom n-category, i.e. $Ob \circ \bar{Hom}_{(U,n)-\mathfrak{Cat}} \cong Hom_{(U,n)-\mathfrak{Cat}}$.

Proof. One must check that the constructions of 1.3.2 (see part(ii)) and 1.3.7 can be applied at each step.

$sk(n)$-associativity is part of the definition of $(U, n) - \mathfrak{Cat}$. The isomorphism of the previous lemma is the only other requirement. □

1.3.17 Remark

The restriction of $WE((U, n) - \mathfrak{Cat}, \times(n))$ to the subcategory of $(sk(n))$-associative enrichments is necessary for the construction of the hom set enrichment, which is necessary for the definition of the next skeleton functor, $sk(n + 1))$.

1.3.18 Remark

That $(U, n + 1) - \mathfrak{Cat}$ as an enriched set is $sk(n + 1)$-associative (and therefore properly an $(n + 2)$-category) was expected, but not yet clear to me. By part (iv) of 1.3.7 it is associative with respect to the objects functor and $sk(n) \circ \bar{Arr}_{((U.n)-\mathfrak{Cat},\times(n))}$, i.e. it is $sk(n)$-associative with respect to each hom object (n-category). The difficulty seems to be in inferring, from the arrows giving the equivalences within the hom objects, arrows giving equivalences from without. I suspect that this should be easier to do for particular types of n-categories.

1.3.19 Example

$(2) - \mathfrak{Cat} \in Ob(WE((2) - \mathfrak{Cat}, \times(2)))$. The skeleton is used at the level of the hom objects, so that only the usual skeleton, $sk(1)$, is seen in this case. The objects are enriched sets.

$$O = Ob((2) - \mathfrak{Cat}) = \{\bar{C} = (C, h, \circ)\}$$

where the composition is (sk)-associative, where $sk = sk(1) : \mathfrak{Cat} \longrightarrow Q$ is the quotient functor determined by identifying isomorphic arrows (functors). The arrows are arrows of enriched sets

$$\Phi = (\Phi_0, \Phi_1) : (C, h_C, \circ_C) \to (D, h_D, \circ_D)$$

respecting composition after the application of (sk).

By the Hom-enrichment construction one associates to any $C, D \in Ob((2)-\mathfrak{Cat})$, $\Phi, \Psi \in Hom_{((2)-\mathfrak{Cat})}(C, D)$, the category $P_{\Phi,\Psi}$ of all arrows $(x \xrightarrow{f} \prod_{c \in Ob(C)} h_D(\Phi(c), \Psi(c))$ satisfying the (sk)-commutativity requirement. $p : P_{\Phi,\Psi} \longrightarrow \mathfrak{Cat}$ is the functor defined by $((x, f) \mapsto x)_{(x,f) \in Ob(P_{\Phi,\Psi})}$. By definition $\bar{h}_{2-\mathfrak{Cat}}(a, b)(\Phi, \Psi) := colimP_{\Phi,\Psi}$

The description of the enrichment on $(2)-\mathfrak{Cat}$ requires, for any $(C, D, E) \in O$, an arrow

$$(\bar{h}_{2-\mathfrak{Cat}}(C, D) \times \bar{h}_{2-\mathfrak{Cat}}(D, E) \xrightarrow{\circ} \bar{h}_{2-\mathfrak{Cat}}(C, E)) \in Arr((2)-\mathfrak{Cat})$$

representing composition. That the above is an arrow in $(2)-\mathfrak{Cat}$, interpreted, means that for any $\Phi_1, \Phi_2, \Phi_3 \in Hom_{((2)-\mathfrak{Cat})}(C, D)$, $\Psi_1, \Psi_2, \Psi_3 \in Hom_{((2)-\mathfrak{Cat})}(D, E)$,

$$F \cong G \in Hom_{\mathfrak{Cat}}($$

$$\bar{h}_{(2)-\mathfrak{Cat}}(C, D)(\Psi_1, \Psi_2) \times \bar{h}_{(2)-\mathfrak{Cat}}(C, D)(\Psi_2, \Psi_3)) \times (\bar{h}_{2-\mathfrak{Cat}}(D, E)(\Phi_1, \Phi_2) \times \bar{h}_{2-\mathfrak{Cat}}(\Phi_2, \Phi_3)),$$
$$\bar{h}_{2-\mathfrak{Cat}}(C, E)(\Phi_1 \circ \Psi_1, \Phi_3 \circ \Psi_3))$$

where

$$F :_t= \bar{\circ}(C, E) \circ (\Phi_{1*} \times \Psi_3^*) \circ (\bar{\circ}(C, D) \times \bar{\circ}(D, E))$$
$$G :_t= \bar{\circ}(C, E) \circ (\bar{\circ}(C, E) \times \bar{\circ}(C, E)) \circ ((\Phi_{1*} \times \Psi_2^*) \times (\Phi_{2*} \times \Psi_3^*)) \circ \sigma$$

where $\bar{\circ}(C, D)$ denotes the enriched composition in $\bar{h}_{2-\mathfrak{Cat}}(C, D)$. I.e., there is a function (arrow of sets) $\alpha : Ob(dom(F)) = Ob(dom(G)) \to Arr(\mathfrak{Cat})$ defining a natural isomorphism between the functors F and G.

1.3.20 Proposition

If the P-colimit inclusion condition is satisfied for $(U, n)-\mathfrak{Cat}$, regarding the construction of the hom enrichment, then it is satisfied for $(U, n+1)-\mathfrak{Cat}$ as well. I.e., the two arrows $colim\ p \otimes e_0 \to \prod_{c \in Ob(C)} h_D(\Phi(c), \Psi(c))$, one from right composition and the other from left composition, are $(n+1)$-equivalent.

Proof. The forgetful functor is at each step given by the objects functor. In this case, P is given by all arrows $(a \xrightarrow{\pi} \prod_{c \in Ob(C)} h_D(\Phi(c), \Psi(c))) \in Arr((U, n)-\mathfrak{Cat})$, such that for any arrow $(e_0 \xrightarrow{e} h_C(x,y)) \in Arr((U,n)-\mathfrak{Cat})$ into a hom object in C, the two arrows (if $\otimes = \times(n)$)

$$r_{(a)}, l_{(a)} : a \otimes e_0 \to \bar{h}_D(\Phi(x), \Psi(y))$$

one given by composition with e_0 on one side and the other by composition on the other, are $sk(n)$-equivalent. Therefore a choice of an $(n+1)$-equivalence of $(n+1)$-functors is still a choice of

$$\phi \in F(n)(\bar{h}_{WE((U,n)-\mathfrak{Cat},\times(n))(sk(n))}(r,l))$$
$$\psi \in F(n)(\bar{h}_{WE((U,n)-\mathfrak{Cat},\times(n))(sk(n))}(l,r))$$

where $\bar{h}_{WE((U,n)-\mathfrak{Cat},\times(n))(sk(n))}(r,l)$ is itself by construction a colimit of the domain object functor

$$p = dob \downarrow_{((U,n)-\mathfrak{Cat})} (id_{(U,n)-\mathfrak{Cat}}, ob_{((U,n)-\mathfrak{Cat})}(\prod_{x \in Ob(t_0)} h_D(r_{(0)}(x), l_{(0)}(x)))) \circ \varepsilon :$$

$$P \longrightarrow (U,n)-\mathfrak{Cat}.$$

By the inclusion condition for the n case the hom object assigned to r and l has a monic arrow into the product of hom objects $h_D(r_{(0)}(x), l_{(0)}(x))$. By the isomorphism of the previous lemma and the construction of the colimit in $WE_{(A,\otimes)(sk)}$ in the lemma before that, an arrow of functors $\phi \in Ob(\bar{h}_{WE((U,n)-\mathfrak{Cat},\times(n))(sk(n))}(l_{colim\ p}, r_{colim\ p})))$ is a map of sets

$$\phi : Ob(colim\ p \times (n) e_0) \cong Ob(colim\ p) \times Ob(e_0) =$$
$$\{(a,\pi); a \in dom(\pi) \text{ and } (dom(\pi),\pi) \in Ob(P)\} \times Ob(e_0)$$
$$\to \bigcup Ob(h_D(l_{colim\ p}, r_{colim\ p}))$$

Claim - That a choice argument implies the existence of a natural isomorphism ϕ from the natural isomorphism ϕ_i. $\square$

1.4 Addresses

We introduce the notion of an address, which is sequence of hom objects, each nested within the previous by the n-categorical enrichment. It is essentially a book-keeping tool, meant to record the "location of a k-arrow within an n-category."

1.4.1 Definition of the Empty n-Category

$\emptyset_{U(1)-\mathfrak{Cat}} := (\emptyset,\emptyset,\emptyset,\emptyset,\emptyset) \in Ob(U-\mathfrak{Cat}) = Ob(U(1)-\mathfrak{Cat})$ is the empty category, and $\forall n \in \mathbb{N}, \emptyset_{U(n+2)-\mathfrak{Cat}} := (\emptyset_{U(1)-\mathfrak{Cat}},\emptyset,\emptyset) \in Ob(U(n+2)-\mathfrak{Cat})$ is the empty (n+2)-category.

1.4.2 Definition of Addresses

We define two address functions, one for objects in $(U, n) - \mathfrak{Cat}$ and one for arrows.

1.4.2.1. For any $n \in \mathbb{N}$, $fAdd_{U(n)0} : Ob(U(n) - \mathfrak{Cat}) \to U'$ is defined to be the function which sends an n-category $x \in Ob(U(n)) - \mathfrak{Cat})$ to the set of functions $\alpha : \{1, ..., j\} \to U'$ such that for any $k \in \{1, ..., j\}$, where $j \in \{0, ..., n\}$,

$$\alpha(k) = (a(k), b(k), C(k), h(k), \circ(k)$$

$$h(k)(a(k), b(k)) = (C(k+1), h(k+1), \circ(k+1))$$
$$a(k), b(k) \in Ob(C(k))$$
$$x = (C(0), h(0), \circ(0))$$

For any $n \in \mathbb{N}$, $Add_{U(n)0} : Ob(U(n) - \mathfrak{Cat}) \to U'$ is the function which sends an n-category x as above to the set of functions $\alpha : \{0, ..., j\} \to U'$ such that there exist $a, b, C, h, \circ$ for which $\alpha = (a(k), b(k))_{k \in \{0,...,j\}}$ and $(a(k), b(k), C(k), h(k), \circ(k)) \in fAdd_{U(n)0}(x)$.

These assign to each n-category its set of "(full) addresses," being sequences $(a(i), b(i), \mathcal{C}(i), h(i), \circ(i))$ such that $(a(i+1), b(i+1))$ is a pair of objects in the base category $\mathcal{C}(i)$ of the $(n-i-1)$-category associated to the previous pair $(a(i), b(i))$ by the enrichment. $fAdd$ refers to the former list and Add to the truncated latter.

The "length," $|\alpha| = |(a, b)|$, will denote its order as a set.

1.4.2.2. For any $n \in Ob(\mathbb{N})$, $Add_{U(n)1} : Arr(U(n) - \mathfrak{Cat}) \to U'$ is defined to be the function which sends $\phi \in Arr(U(n) - \mathfrak{Cat})$ to a function

$$S : Add_{U(n)0}(dom(\phi)) \to \bigcup_{k \in \mathbb{N}} Arr(U(k) - \mathfrak{Cat})$$

defined inductively, by requiring that

$$S : \emptyset \mapsto \phi$$

and that for any $(a, b) \in Add_{U(n)0}(dom(\phi))$, for any $\bar{\phi} \in Arr(U(n - |(a, b)|) - \mathfrak{Cat})$, $S(a, b) := \bar{\phi}$ iff there exists $(a_0, b_0) \in Add_{U(n)0}(dom(\phi))$ such that

$$|(a_0, b_0)| + 1 = |(a, b)| \text{ and } (a, b)|_{0,...,|(a_0,b_0)|-1} = (a_0, b_0)$$

and there exists $\psi = ((f_0, f_1), f_2) \in Arr(U(n - |(a, b)| + 1) - \mathfrak{Cat})$, such that

$$\psi = S(a_0, b_0) \text{ and } f_2(a(|(a, b)|), b(|(a, b)|)) = \bar{\phi}$$

This associates to every arrow of n-categories a function which sends an address for the domain category to the arrow of $(n-k)$-categories assigned to it by the original arrow.

1.4.3 Remark

That the above definition consists of two maps, one for n-categories and the other for arrows of n-categories, suggests some functor giving an alternate description of n-categories.

1.4.4 Definition of the Functors $Inc_{U(n)-\mathfrak{Cat}}^{U(m)-\mathfrak{Cat}}$ and $For_{U(n)-\mathfrak{Cat}}^{U(m)-\mathfrak{Cat}}$

For any $n, m \in \mathbb{N}\backslash\{0\}$ such that $n < m$, define functors $Inc_{U(n)-\mathfrak{Cat}}^{U(m)-\mathfrak{Cat}} : U(n) - \mathfrak{Cat} \longrightarrow U(m) - \mathfrak{Cat}$ and $For_{U(n)-\mathfrak{Cat}}^{U(m)-\mathfrak{Cat}} : U(m) - \mathfrak{Cat} \longrightarrow U(n) - \mathfrak{Cat}$ inductively, by the following.

1.4.4.1. For any $x = (C, h, \circ) \in Ob(U(n+1) - \mathfrak{Cat})$,

$$Inc_{0U(n+1)-\mathfrak{Cat}}(x) :_t= (C, Inc_{0U(n)-\mathfrak{Cat}} \circ h, Inc_{1U(n)-\mathfrak{Cat}} \circ \circ)$$

and for any $\phi = (\phi_0, \phi_2) \in Arr(U(n+1) - \mathfrak{Cat})$,

$$Inc_{1U(n+1)-\mathfrak{Cat}}(\phi) :_t= (\phi_0, Inc_{U(n)-\mathfrak{Cat}}(\phi_2))$$

so that $Inc_{U(n+1)-\mathfrak{Cat}} := (Inc_{0U(n+1)-\mathfrak{Cat}}, Inc_{1U(n+1)-\mathfrak{Cat}}) : U(n+1)-\mathfrak{Cat} \longrightarrow U(n+2)-\mathfrak{Cat}$.

Now temporarily define $Inc_{U(1)-\mathfrak{Cat}} : U - \mathfrak{Cat} \longrightarrow U(2) - \mathfrak{Cat}$ to be the functor which sends a category C to the 2-category with enrichment $h_C(a, b) := (Hom_C(a, b), \{id_f; f \in Hom_C(a, b)\}, ...)$ given by attaching only identity arrows. Define

$$Inc_{U(n)-\mathfrak{Cat}}^{U(m+1)-\mathfrak{Cat}} := Inc_{U(m)-\mathfrak{Cat}} \circ Inc_{U(n)-\mathfrak{Cat}}^{U(m)-\mathfrak{Cat}}, \text{ and}$$

$$Inc_{U(1)-\mathfrak{Cat}}^{U(2)-\mathfrak{Cat}} := Inc_{U(1)-\mathfrak{Cat}}$$

1.4.4.2. Similarly, for any $x = (C, h, \circ) \in Ob(U(m+1) - \mathfrak{Cat})$,

$$For_0^{U(m+1)-\mathfrak{Cat}}(x) :_t= (C, For_0^{U(m)-\mathfrak{Cat}} \circ h, For_1^{U(m)-\mathfrak{Cat}} \circ \circ)$$

and for any $\phi = (\phi_0, \phi_2) \in Arr(U(m+1) - \mathfrak{Cat})$,

$$For_1^{U(m+1)-\mathfrak{Cat}}(\phi) :_t= (\phi_0, For_1^{U(m)-\mathfrak{Cat}}(\phi_2))$$

so that $For^{U(m+1)-\mathfrak{Cat}} :_t= (For_0^{U(m+1)-\mathfrak{Cat}}, For^{U(m+1)-\mathfrak{Cat}})_1) : U(m+1) - \mathfrak{Cat} \longrightarrow U(m) - \mathfrak{Cat}$.

Now temporarily define $For^{U(2)-\mathfrak{Cat}} : U(2) - \mathfrak{Cat} \longrightarrow U - \mathfrak{Cat}$ to be the functor which forgets the enrichment. Define

$$For_{U(n)-\mathfrak{Cat}}^{U(m+1)-\mathfrak{Cat}} := For_{U(n)-\mathfrak{Cat}}^{U(m)-\mathfrak{Cat}} \circ For^{U(m+1)-\mathfrak{Cat}} \text{ and}$$

$$For_{U(n)-\mathfrak{Cat}}^{U(n)-\mathfrak{Cat}} := id_{U(n)-\mathfrak{Cat}}$$

1.4.5 Lemma

$\forall n \in \mathbb{N}$, $U(n+1) - \mathfrak{Cat}$ has products and coproducts.

Proof. For products, by induction on n. At the base take the usual product category. For any tuple $(x_i)_{i \in S}$, $(y_i)_{i \in S}$, use the inductive step to take the product $\prod_{i \in S} h_{\mathcal{C}_i}(x_i, y_i)$.

For coproducts, at the base take the usual coproduct category (objects are the disjoint union. $Hom_{\coprod_{i \in S}}((a,j),(b,k))$ is for $j \neq k$, and $Hom_{\mathcal{C}_j}(a,b)$ for $j = k$). If $n \geq 1$, then for the enrichment, $h_{\coprod_{i \in S} \mathcal{C}_i}((a,j),(b,k))$ is $\emptyset_{U(n)-\mathfrak{Cat}}$ for $j \neq k$, and $h_{\mathcal{C}_j}(a,b)$ for $j = k$. $\square$

1.4.6 Definition of Products and Coproducts

$\prod_{U(n)-\mathfrak{Cat}}$ and $\coprod_{U(n)-\mathfrak{Cat}}$ will be functions $\bigcup_{S \in U} Hom_{U'-\mathfrak{Set}}(S, Ob(U(n) - \mathfrak{Cat})) \longrightarrow Ob(U(n) - \mathfrak{Cat})$, the canonical constructions described in the previous lemma's proof.

1.4.7 Definition of the Restricted Simplicial Sets

Define $\Delta \in Ob(U - mathfrakSet)$ to to be the simplicial category, i.e. its objects are finite ordered sets and its arrows are order-preserving functions.

For any $n \in Ob(U - \mathfrak{Set})$, define the category $\Delta_{(n)} := \Delta_{\backslash(\{j \in \mathbb{N}; j \leq n-1\}, \leq_{\mathbb{N}})} = \downarrow_{(\Delta)} (ob_{(\Delta)}((\{j \in \mathbb{N}; j \leq n-1\}, \leq_{\mathbb{N}})), id_{(U-\mathfrak{Cat})(\Delta)})$. This is the arrow category under the set with n elements.

1.4.8 Definition of Primitive Arrows

$\forall n \in \mathbb{N}$, $\forall f \in Arr(\Delta)$, f is primitive iff $||dom(f)| - |codom(f)|| = 1$. $\forall \phi = (f, e, id_{\circ}) \in Arr(\Delta_{(n)})$, ϕ is primitive iff f is primitive.

1.4.9 Lemma

Any arrow in Δ or $\Delta_{(n)}$ is a composition of primitive arrows.

1.4.10 Lemma on a Pseudo-Simplicial Structure on $(U.n) - \mathfrak{Cat}$

For any $n \in \mathbb{N}$, there exists a unique

$$\rho \in Hom_{(U''-\mathfrak{Cat})}(\Delta_{(n)}, \downarrow_{(U'-\mathfrak{Cat})} (ob_{(U'-\mathfrak{Cat})}(U(n) - \mathfrak{Cat}), id_{U'-\mathfrak{Cat}}))$$

such that for any $\phi = (f, id_{(\{1,\ldots,n\},\leq)}) \in Arr(\Delta_{(n)})$, f is primitive implies the following.

1.4.10.1. If f injective, then $\rho_{(1)}(\phi) : U(|dom(f)|) - \mathfrak{Cat} \longrightarrow U(|codom(f)|) - \mathfrak{Cat}$ is defined on objects by

$$\rho_{(1)}(\phi)_{(0)} : (C, h, \circ) \mapsto (D, \bar{h}, \bar{\circ})$$

iff

$$For^{U(|codom(f)|)-\mathfrak{Cat}}_{U(|dom(f)|-1)-\mathfrak{Cat}}(D) = For^{U(|dom(f)|)-\mathfrak{Cat}}_{U(|dom(f)|-1)-\mathfrak{Cat}}(C)$$

and for any full address $\alpha = (a, b, C_\alpha, h_\alpha, \circ_\alpha) \in fAdd_{U(|dom(f)|)}((C, h, \circ))$, for any $k \in \{1, ..., |dom(f)|\}$, $f(k+1) = f(k) + 2$ implies

$$Ob(h_\alpha(k)(a(k), b(k))) = \{\emptyset\} \text{ and}$$

$$\alpha \in fAdd_{U(|codom(f)|)}((D, \bar{h}, \bar{\circ})) \text{ and}$$

$$\forall \bar{\alpha} = (\bar{a}, \bar{b}, C_{\bar{\alpha}}, h_{\bar{\alpha}}, \circ_{\bar{\alpha}}) \in fAdd_{U(|codom(f)|)}((D, \bar{h}, \bar{\circ})),$$

$$\ulcorner |\bar{\alpha}| = |\alpha| + 1 \text{ and } \bar{\alpha}_{\{0,...,k\}} = \alpha \urcorner \Longrightarrow h_{\bar{\alpha}}(|\bar{\alpha}|)(\emptyset, \emptyset) = h_\alpha(|\alpha|)(a(k), b(k))$$

The functor $\rho_1(\phi)$ is defined on arrows by

$$\rho_1(\phi)_{(1)} : F = ((F_0, F_1), F_2) \mapsto ((G_0, G_1), G_2) = G$$

iff

$$For^{U(|codom(f)|)-\mathfrak{Cat}}_{U(|dom(f)|-1)-\mathfrak{Cat}}(G) = For^{U(|dom(f)|)-\mathfrak{Cat}}_{U(|dom(f)|-1)-\mathfrak{Cat}}(F)$$

and for any address $\alpha = (a, b) \in Add_{U(|codom(f)|)}(dom(G))$, for any $k \in \{1, ..., |dom(f)|\}$,

$$\ulcorner f(k+1) = f(k) + 2 \text{ and } |\alpha| = k + 1 \urcorner \Longrightarrow$$

$$Add_{U(|codom(f)|)1}(G)(\alpha) = Add_{U(|dom(f)|)1}(F)(\alpha|_{\{0,...,k\}})$$

1.4.10.2. If f is surjective, then $\rho_1(\phi) : U(|dom(f)|) - \mathfrak{Cat} \longrightarrow U(|codom(f)|) - \mathfrak{Cat}$ is defined on objects by

$$\rho_1(\phi)_{(0)} : (C, h, \circ) \mapsto (D, \bar{h}, \bar{\circ})$$

iff

$$For^{U(|dom(f)|)-\mathfrak{Cat}}_{U(|codom(f)|-1)-\mathfrak{Cat}}(C) = For^{U(|codom(f)|)-\mathfrak{Cat}}_{U(|codom(f)|-1)-\mathfrak{Cat}}(D)$$

and for any full address $\alpha = (a, b, C_\alpha, h_\alpha, \circ_\alpha) \in fAdd_{U(|codom(f)|)0}(D)$, for any $k \in \mathbb{N}$, $f(k+1) = f(k)$ implies

$$(C_\alpha(k), h_\alpha(k), \circ_\alpha(k)) = \coprod_{\bar{\alpha} \in S} (C_{\bar{\alpha}}(k+1), h_{\bar{\alpha}}(k+1), \circ_{\bar{\alpha}}(k+1))$$

where

$$S = \{\bar{\alpha} = (\bar{a}, \bar{b}, C_{\bar{\alpha}}, h_{\bar{\alpha}}, \circ_{\bar{\alpha}}) \in fAdd_{U(|dom(f)|)0}(C);$$
$$(\bar{a}, \bar{b})_{\{0,\ldots,k-1\}} = (a,b)_{\{0,\ldots,k-1\}} \text{ and } |\bar{\alpha}| = k+1\}.$$

The functor $\rho_1(\phi)$ is defined on arrows by

$$\rho_1(\phi) : F = ((F_0, F_1), F_2) \mapsto ((G_0, G_1), G_2) = G$$

iff

$$For^{U(|dom(f)|)-\mathfrak{Cat}}_{U(|codom(f)|-1)-\mathfrak{Cat}}(F) = For^{U(|codom(f)|)-\mathfrak{Cat}}_{U(|codom(f)|-1)-\mathfrak{Cat}}(G)$$

and for any full address $\alpha = (a, b, C_\alpha, h_\alpha, \circ_\alpha) \in fAdd_{U(|codom(f)|)0}(C)$, for any $k \in \mathbb{N}$, $f(k+1) = f(k)$ and $|\alpha| = k+1$ imply

$$Add_{U(|codom(f)|)1}(G)(\alpha) = \coprod_{\bar{\alpha} \in S} Add_{U(|dom(f)|)1}(F)(\bar{\alpha})$$

I.e. if f is injective, delete the k-th step, replacing it with the coproduct of all n-k-1 categories appearing in the enriched homs there. If surjective, add a step, a base category with only one object, leaving its enriched hom as that which had preceded it.

1.4.11 Lemma on Representing the k-Arrows Functor

Adopt the notation of (1.4.10). Then for any arrow $f \in Arr(\Delta_{(n)})$ if the functor $R : (U,n)-\mathfrak{Cat} \longrightarrow (U,|f|)-\mathfrak{Cat}$ given by requiring that $\rho(f) = (\cdot, R, (U, |codom(f)|) - \mathfrak{Cat})$, then the functor

$$F_{(U,n)-\mathfrak{Cat}} \circ \rho(f) : (U,n) - \mathfrak{Cat} \longrightarrow U' - \mathfrak{Set}$$

is representable.

1.4.12 Remark

I expect there to be some enriched version of this.

1.4.13 Lemma

Adjunction of functors given to opposite pairs of primitive arrows by ρ.

1.4.14 Conjecture on (sk)-associativity for a Subcategory of $n - Cat$

For any $n \in \mathbb{N}$, for any $\bar{B} = (B, h_B, \bar{\circ}) \in Ob((WE(U,n) - \mathfrak{Cat}, \times(n)))$, if there exists $C \in Ob(U - \mathfrak{Cat})$, and

$$(Add(\bar{B}) \xrightarrow{\Phi} Arr(U - \mathfrak{Cat})), (Add(\bar{B}) \xrightarrow{\varepsilon} Arr(U - \mathfrak{Cat}) \in Arr(U' - \mathfrak{Set})$$

satisfying the following, properties, then $\bar{B}$ is $sk(n)$-associative.

1.4.14.1. C has colimits.

1.4.14.2. For any address $\beta = (B_i, h_i, \bar{\circ}_i, a_i, b_i)_{i \in \{1,\ldots,k\}} \in Add(\bar{B})$, for some $c, d \in Ob(C)$, the functor

$$\varepsilon(\beta) : E \longrightarrow C_{/c} =\downarrow_{(C)} (id_{(C)}, ob_{(C)}(c))$$

is faithful, and

$$\Phi(\beta) : For_{U-\mathfrak{Cat}}^{(U,n-|\beta|)-\mathfrak{Cat}}(B_{|\beta|}) \longrightarrow Hom^{(1)}_{U-\mathfrak{Cat}^2}(E, C_{/d})$$

is an equivalence of categories, where "$|\beta|$" denotes the order of β as a set of pairs, i.e. the number of categories or pairs of objects appearing in the sequence.

1.4.14.3. The functors $\Phi(\beta)$ agree with the composition given by the Hom enrichment lemma, (1.3.7), up to natural isomorphism. Explanation follows.

1.4.14.3.1. Let there be three addresses $\beta, \beta_1, \beta_2 \in Add(\bar{B})$, such that

$$|\beta_1| = |\beta_2| = |\beta_3| = |\beta| + 1 \text{ and } \beta = \beta_1 \cap \beta_2 \cap \beta_3$$

and

$$b_{1(|\beta|+1)} = a_{2(|\beta|+1)} \text{ and } a_{3(|\beta|+1)} = a_{1(|\beta|+1)} \text{ and } b_{3(|\beta|+1)} = b_{2(|\beta|+1)}$$

i.e. the addresses β_1 and β_2 correspond to a triple $a_{1(k+1)}, b_{1(k+1)} = a_{2(k+1)}, b_{2(k+1)} \in Ob(C_k)$ in the underlying category for one of the hom objects, composed to yield β_3.

1.4.14.3.2. Then there is a natural isomorphism of functors

$$\bar{\circ}_{\mathfrak{Cat}} \circ (\Phi(\beta_1) \times \Phi(\beta_2)) \cong \Phi(\beta_3) \circ For_{U-\mathfrak{Cat}}^{(U,n-|\beta|)-\mathfrak{Cat}}(\bar{\circ}),$$

where $\bar{\circ}_{\mathfrak{Cat}}$ is that of (Enrichment, 1.3.7) for $(U, 2) - \mathfrak{Cat}$.

CHAPTER 2

THE CATEGORY Ω OF POINTED CORRESPONDENCES AND THE SPEC FUNCTORS

We define a category, Ω, intended to contain several "categories of geometric objects," allowing for arrows between them. A spec functor is a formalization of the usual affine chart construction, which determines a subcategory of Ω.

2.1 Definitions Regarding the Category of Pointed Correspondences

We denote by Ω a category of pointed categories whose arrows are pointed correspondences. There is, for every category C, a canonical functor $\kappa_C : C \longrightarrow \Omega$. A functor $F : C' \to C''$ between two categories induces a a natural transformation between the two canonical functors $\kappa_{C'} \to \kappa_{C''}$.

It is intended that Ω should serve as a common locale for several types of geometries. Here, "geometry" is used in the sense of categories such that their objects have some local structure (e.g. those of schemes or manifolds). This is formalized by requiring any "geometry" Π to be a sub-category of Ω generated in Ω from a subcategory interpreted as "inclusion of the affine objects". The objects in these "geometric" subcategories of Ω are given by pairs $(Sh, \mathcal{O}_x)$, where Sh is the category of sheaves on some Grothendieck topology and $\mathcal{O}_x \in Ob(Sh)$.

It is intended that Ω-arrows, are given by correspondences between the categories of sheaves associated to "schemes". They therefore should allow for morphisms beside the usual ones (i.e. those constructed from functors beside the usual pushforward/pullback functors), such as algebraic correspondences.

2.1.1 Definition of Ω

For any two universes $U \in U'$, the *omega category* for (U, U') is a U' category Ω whose objects are pointed U-categories and arrows are the isomorphism classes of pointed U-correspondences.

2.1.1.1. The set of objects is the set of categories with a distinguished object, i.e. an object is a pair (C, x), where $C \in Ob(U - \mathfrak{Cat})$ and $x \in Ob(C)$.

2.1.1.2. Suppose that $C, D \in U - \mathfrak{Cat}$ are categories, with $x \in Ob(C)$, and $y \in Ob(D)$. Consider the set of all pairs (F, ϕ), where $F : C^{opp} \times D \longrightarrow U - \mathfrak{Set}$ is a functor and $\phi \in F(x, y)$. Define an equivalence relation on this set by requiring that

$$(F, \phi) \equiv (G, \psi)$$

iff there exists an isomorphism of functors $\Phi : F \longrightarrow G$, such that $\Phi(x, y)(\phi) = \psi$.

Then the hom set $Hom_{\Omega}((C, x), (D, y))$ is defined to be the set of all equivalence classes $[(F, \phi)]$ of such pairs.[1]

2.1.1.3. *(The relation $R'(F, G, x, z)$).* In order to define the composition in Ω, we define, for any functors

$$F : C^{opp} \times D \longrightarrow U - \mathfrak{Set} \longleftarrow D^{opp} \times E : G$$

and for any $x \in Ob(C)$ and $z \in Ob(E)$, an equivalence relation $R'(F, G, x, z)$ on the set

$$\coprod_{y \in Ob(D)} F(x, y) \times G(y, z)$$

It is that which is generated by requiring that for any $y, y' \in Ob(D)$, for any $(\phi, \psi) \in F(x, y) \times G(y, z)$, for any $(\phi', \psi') \in F(x, y') \times G(y', z)$,

$$(\phi, \psi) \cong_{R'(F,G,x,z)} (\phi', \psi')$$

if there exists an arrow $(y \xrightarrow{f} y') \in Arr(D)$ such that

$$G(f, id_z)(\psi') = \psi \text{ and } F(id_x, f)(\phi) = \phi'$$

This is to say roughly that we require (ϕ, ψ) and (ϕ', ψ') to be equivalent if they can be related by an arrow $(y \xrightarrow{f} y')$ in the middle category D, in the sense that $\phi' = F(f) \circ \phi$ and $\psi = \psi' \circ G(g)$.

For convenience the relation will be written $R(F, G, x, z) = R(F, G)$ when x and z are understood from the context.

[1] Formally, they should carry the information of their intended domain and codomain, so that the hom sets should be disjoint. This is a peculiarity of the definition here employed of a category, which requires that a set of arrows should be specified.

2.1.1.4. For any composable arrows $(C,x) \xrightarrow{[(F,\phi)]} (D,y) \xrightarrow{[(G,\psi)]} (E,z)$ in Ω, the composition is defined by

$$[(G,\psi)] \circ [(F,\phi)] := [(F *_\Omega G, [(\phi,\psi)]_{R'(F,G,x,z)})]$$

where the functor $F *_\Omega G : C^{opp} \times E \longrightarrow U - \mathfrak{Set}$ is defined to send a pair $(x',z') \in Ob(C^{opp} \times E)$ to the set of equivalence classes with respect to $R'(F,G,x',z')$ of pairs $(\psi',\phi') \in F(x',y') \times G(y',z')$ over varying y', and the distinguished equivalence class of pairs is $[(\psi,\phi)]_{R'(F,G,x,z)}$.

2.1.1.5. For $(C,x) \in Ob(\Omega)$, we define the identity arrow $id_{(C,x)} : (C,x) \to (C,x)$ to be

$$id_{(C,x)} := [(Hom_C, id_{(C)(x)})]_R$$

i.e. it is defined by the pair consisting of the hom functor $Hom_C : C^{opp} \times C \longrightarrow U - \mathfrak{Set}$ and $id_x \in Hom_C(x,x)$.

2.1.2 Proposition

Every universe within a universe $U \in U'$ has a unique category $\Omega \in Ob(U' - \mathfrak{Cat})$ as constructed above.

Proof. The details of the proof are in the following subsections. As a set, Ω is determined by the definition. It remains to show that it is a category, in particular, we will show in the following that the composition defined in (2.1.1.4) is associative (see 2.1.2.2) and that the identity properties hold (see 2.1.2.3). Associativity comes from the usual natural isomorphisms associating two different product functors, $((-_1 \times -_2) \times -_3) \to (-_1 \times (-_2 \times -_3))$. The identity is given by $id_{\Omega((C,x))} := [(Hom_C, id_x)]$. The required natural isomorphism for the identity is found by applying (2.1.1.3) to any pair $(f,g) \in Hom_C(a,x) \times F(x,y)$ to show that the arrow of functors given by $(f,g) \mapsto F_{(1)}((f,id_y))(g)$ is injective. Surjectivity is obvious.

2.1.2.1. Recall the functors $F *_\Omega G$ constructed in (2.1.1.4), with functors F and G as in that section.

2.1.2.2. (*Associativity*). For any composible arrows $((B,c) \xrightarrow{[(F,\phi)]} (C,c) \xrightarrow{[(G,\psi)]} (D,d) \xrightarrow{[(H,\chi)]} (E,e)) \in Arr(\Omega)$, define the associator α, the isomorphism of functors required in (2.1.1.2), by sending each object $(b,e) \in Ob(B^{opp} \times E)$ to the map of sets

$$\alpha(b,e) : ((F *_\Omega G) *_\Omega H)(b,e) \to (F *_\Omega (G *_\Omega H))(b,e)$$

defined by

$$\alpha(b,e) : [(\phi', [(\psi',\chi')]_{R'(F,G)})]_{R'(F *_\Omega G,H)} \mapsto [([(\phi',\psi')]_{R'(G,H)}, \chi')]_{R'(F,G *_\Omega H)})$$

Obviously α is an isomorphism.

2.1.2.3. (*Identity*). For any $(C,x) \xrightarrow{[(F,\phi)]} (D,y) \in Arr(\Omega)$, one must construct natural isomorphisms $F *_\Omega Hom_C \to F$, for the left unit, and $Hom_D *_\Omega F \to F$, for the right unit (See 2.1.1.4).

Define the left unit isomorphism by sending an object $(x',y') \in Ob(C^{opp} \times D)$ to the map of sets

$$u_l(x',y') : (Hom_D *_\Omega F)(x',y') \to F(x',y')$$

defined by

$$u_l(x',y') : [(\psi',\phi')]_{R'(Hom_D,F)} \mapsto F(\psi', id_{(dom_{(C)}(\phi'))})(\phi').$$

The following gives the right unit isomorphism, mapping $(x',y') \in Ob(C^{opp} \times D)$ to the map of sets

$$u_r(x',y') : (F *_\Omega Hom_C)(x',y') \to F(x',y')$$

defined by

$$u_r(x',y') : [(\psi',\phi')]_{R'(F,Hom_C)} \mapsto F(id_{(codom_{(D)}(\psi'))}, \phi')(\psi').$$

2.1.2.3.1. One must show that u_l and u_r are isomorphisms of functors. The argument is symmetric, and so only the right side (involving u_r) will be explicitly written. First injectivity.

Consider any objects

$$x' \in Ob(C), y', y'', y''' \in Ob(D),$$

any arrows

$$\psi'' \in Hom_D(y'',y') \text{ and } \psi''' \in Hom_D(y''',y'),$$

and any elements

$$\phi'' \in F(x',y'') \text{ and } \phi''' \in F(x',y''').$$

Suppose that u_l sends two elements (equivalence classes of pairs) to the same element:

$$u_r(x',y')([(\phi'',\psi'')]_{R'(Hom_D,F)}) = u_r(x',y')([(\phi''',\psi''')]_{R'(Hom_D,F)}).$$

Then the two elements of $F(x',y')$ are the same:

$$F(id_{x'},\psi'')(\phi'') = F(id_{x'},\psi''')(\phi''').$$

Therefore the relation of (2.1.1.3) implies an equality:

$$[(\phi'',\psi'')]_{R'(Hom_D,F)} = [(F(id_{x'},\psi'')(\phi''), id_{y'})]_{R'(Hom_D,F)} =$$

$$[(F(id_{x'},\psi''')(\phi'''), id_{y'})]_{R'(Hom_D,F)} = [(\phi''',\psi''')]_{R'(Hom_D,F)}.$$

2.1.2.3.2. Surjectivity of the map u_l is straightforward. Given any objects

$$x' \in Ob(C) \text{ and } y' \in Ob(D),$$

for any element $\phi' \in F_{(0)}(x', y')$,

$$u_r(x', y')([(\phi', id_{y'})]_{R'(Hom_D, F)}) = \phi'.$$

2.1.2.3.3. So is the compatibility of the distinguished elements,

$$u_r(x, y)([(\phi, id_y)]_{R'(Hom_D, F)}) = \phi \circ id_y = \phi.$$

2.1.2.3.4. A symmetric argument should be given for u_l and the left identity)

This completes the argument for the identity. □

2.1.3 Convention

If a universe U is understood, then "Ω" should refer to the above category. If not, then "Ω_U" will.

2.1.4 Remark

All hom sets in Ω are non-empty. To see this, let T be the terminal category, i.e. that having only one object, $\emptyset \in Ob(T)$, and one arrow, $id_\emptyset$. Denote, for any objects $(C, x), (D, y) \in Ob(\Omega)$, their terminal functors by $C \xrightarrow{t_C} T \xleftarrow{t_D} D$. Then $[(Hom_T \circ (t_C^{opp} \times t_D), id_\emptyset)] \in Hom_\Omega((C, x), (D, y))$.

2.1.5 Definition of the Canonical Functor κ

For any category C, define the functor $\kappa_C : C \longrightarrow \Omega$ by sending objects x to (C, x), i.e. so that x is distinguished within C, and arrows ϕ to $[(Hom_C, \phi)]$, i.e. so that ϕ is distinguished within the hom set given by C.

2.1.6 Remark

That $\kappa_{C(0)}$ (the map on objects) is injective is trivial.

2.1.7 Lemma

For any $(I, i) \in Ob(\Omega)$, for any $C \in Ob(U - \mathfrak{Cat})$, the functor $Yo_\Omega(I, i) \circ \kappa_C^{opp} : C^{opp} \longrightarrow U' - \mathfrak{Set}$ (where $Yo_\Omega : \Omega \hookrightarrow Hom^{(1)}_{U''-\mathfrak{Cat}}(\Omega^{opp}, U' - \mathfrak{Set})$ is the Yoneda functor), sends an object $c \in Ob(C)$ to the set of pairs consisting of isomorphism classes of functors $F : C^{opp} \longrightarrow Hom^{(1)}_{U-\mathfrak{Cat}^2}(I, U - \mathfrak{Set})$ with distinguished elements of the set $F(c)(i)$ given by i.

2.1.8 Lemma

For any category C, κ_C is faithful iff the only automorphism of the identity functor $Id_C : C \longrightarrow C$ is the identity, i.e. $Aut_{End^{(1)}_{U-\mathfrak{Cat}^2}(C)}(Id_C) = \{id_{Id_C}\}$.

Proof. The failure of κ_C to be faithful would mean that there are distinct arrows $\phi, \psi : x \to y \in Arr(C)$ such that there is an isomorphism

$$(Hom_C \xrightarrow{\Phi} Hom_C) \in Arr(Hom^{(1)}_{U-\mathfrak{Cat}^2}(C^{opp} \times C, U - \mathfrak{Set}))$$

such that $\Phi(x, y)(\phi) = \psi$. Consider the map of sets $f : Ob(C) \to Arr(C)$ given by $f : c \mapsto \Phi(c, c)(id_c)$. By a Yoneda-type argument, f is a non-trivial automorphism of the functor Id_C.

Now the reverse. If $g : Id_C \to Id_C$ is an natural isomorphism of functors, then define an arrow of functors, $\Phi : Hom_C \to Hom_C$, by requiring that for any $x, y \in Ob(C)$, for any $\phi \in Hom_C(x, y)$,

$$\Phi(x, y) : \phi \mapsto \phi \circ g(x) = g(y) \circ \phi$$

Naturalness on either side follows from the naturalness of g. Since g is invertible, Φ can be inverted by

$$\Phi^{-1}(x, y) : \phi \mapsto \phi \circ g(x)^{-1} = g(y)^{-1} \circ \phi,$$

which is similarly also a natural transformation, since g^{-1} is natural. $\square$

2.1.9 Lemma

Suppose that a category C is generated by objects $G \subseteq Ob(C)$, i.e. that $\prod_{g \in G} Yo^{opp}(g) : C \longrightarrow U - \mathfrak{Set}$ is faithful.

2.1.9.1. The functor κ_C is faithful iff the set of automorphisms of each generator $g \in G$ is trivial.

2.1.9.2. If a category C has an initial object e, then for any index category I the category of functors from I to C has a set of generators, given by the left Kan extensions of the diagonal functors $\Delta_{(I_{i/}, C)(0)}(g)$ for $g \in G$, along the inclusions in I

$$i : I_{i/} = \downarrow_{(I)} (ob_{(I)}(i), id_{(I)}) \xrightarrow{cob} I$$

of the arrow categories under varying objects in I, where *cob* is the functor which remembers the codomain object of an arrow.

2.1.10 Corollary

For any universes $U \in U' \in U''$, for any $D \in Ob(U - \mathfrak{Cat})$, κ_C for U'' is faithful if C is either $Hom^{(1)}_{U'-\mathfrak{Cat}^2}(D, U - \mathfrak{Set})$, $Hom^{(1)}_{U'-\mathfrak{Cat}^2}(D, U - \mathfrak{Cat})$ or some category of diagrams in either.

2.1.11 Corollary

Every $U - \mathfrak{Cat}$ object has a faithful functor into the omega of U''.

2.1.12 Definition of Natural Transformations $\kappa_{tw,F}$ and κ_{cotw}

For any $(C \xrightarrow{F} D) \in Arr(U - \mathfrak{Cat})$, define the $\kappa - twist$ of F to be the function $\kappa_{tw,F} : Ob(C) \to Arr(\Omega)$ given by composing the Hom functor Hom_D with F^{opp} for one of the arguments, with the identity arrow $id_{F(c)}$ in the codomain of F,

$$\kappa_{tw,F} := (c \mapsto [(Hom_D \circ (F^{opp} \times_{U-\mathfrak{Cat}} id_D), id_{F(c)})])_{c \in Ob(C)}.$$

The $\kappa - cotwist$ of F is similarly given by F and $id_{F(c)}$, being a function $\kappa_{cotw,F} : Ob(C) \to Arr(\Omega)$, so that

$$\kappa_{cotw,F} := (c \mapsto [(Hom_D \circ ((id_D^{opp} \times_{U-\mathfrak{Cat}} F), id_{F(c)})])_{c \in Ob(C)}.$$

2.1.13 Lemma

Given a functor F as above, the κ-twist and κ-cotwist of F are natural transformations of functors,

$$\kappa_C \xrightarrow{\kappa_{tw,F}} \kappa_D \circ F,$$

$$\kappa_D \circ F \xrightarrow{\kappa_{cotw,F}} \kappa_C,$$

i.e. arrows in the category $Hom^{(1)}_{U'-\mathfrak{Cat}^2}(C, \Omega)$ of functors.

2.2 Ω-Lifts

An Ω-lift is essentially the construction of a structure sheaf associated to the topological spec functor $sp : \mathcal{R} \longrightarrow \mathcal{T}$ between categories, when the codomain of the concerned functor has been given a Grothendieck topology, and a functorial notion of an open immersion.

An admissibility structure is defined to be some sub-functor of a "fibre functor," which essentially assignes to each object in $\mathcal{T}$ a "category of open subsets." For any functor

$\mathcal{O} : \mathcal{R} \longrightarrow \mathcal{S}$ one constructs a sheaf on the the category assigned to each object $sp(x) \in Ob(\mathcal{T})$, by sheafifying the Kan extension of $\mathcal{O}$ restricted to the pre-image category of the functor which sends the arrow category in $\mathcal{R}$ over x to the arrow category in $\mathcal{T}$ over $sp(x)$.

2.2.1 Definition *of a Fibre Functor*

For any universes $U \in U'$, for any category $C \in Ob(U - \mathfrak{Cat})$ a functor $(C^{opp} \xrightarrow{Fib} U - \mathfrak{Cat})$ is a fibre functor if the following, (2.2.1.1) and (2.2.1.2) hold.

2.2.1.1. For any $c \in Ob(C)$, the category $Fib(c)$ is the category $C_{/c}$ of objects in C that lie over c, i.e., of morphisms $x \to c$.

2.2.1.2. For any $(c_1 \xrightarrow{f} c_2) \in Arr(C)$, the functor $Fib(f) : C_{/c_2} \longrightarrow C_{/c_1}$ is the pullback functor, i.e., it sends objects $x \xrightarrow{a} c_2$ to their fibred products by f, i.e.

$$Fib(f) : (x \xrightarrow{a} c_2) \mapsto (c_1 \times_{c_2} x \xrightarrow{b} c_1);$$

and it sends an arrow $\phi : (x \xrightarrow{a} c_2) \to (y \xrightarrow{b} c_2)$ (i.e. $\phi : x \to y$ with $b \circ \phi = a$), to its pullback $\phi \times_{c_2} 1_{c_1} : x \times_{c_2} c_1 \to y \times_{c_2} c_1$.

2.2.2 Lemma

Consider any arrow $x \xrightarrow{\phi} y \in Arr(C)$.

2.2.2.1. Define a category $L_\phi \in Ob(U - \mathfrak{Cat})$ by the following.

2.2.2.1.1. Its objects are diagrams consisting of triples of arrows $(u, f, v) \in Arr(C)^3$ such that $v \circ f = \phi \circ u$

2.2.2.1.2. Its arrows are natural transformations (given by certain pairs of arrows $(a, b) \in Arr(C)$).

2.2.2.2. Define the functor $For_c : L_\phi \longrightarrow Fib(y)$, on objects by $(u, f, v) \mapsto v$, and on arrows by $(a, b) \mapsto b$.

2.2.2.3. Define the functor $For_d : L_\phi \longrightarrow Fib(x)$, on objects by $(u, f, v) \mapsto u$, and on arrows by $(a, b) \mapsto a$.

2.2.2.4. Suppose that G is a right adjoint functor to For_c.

2.2.2.5. Then $Fib(\phi) \cong For_d \circ G$.

Proof. For any $v' \in Ob(Fib(y))$, $(u, f, v) \in Ob(L_\phi)$, $u', f' \in Arr(C)$,

$$(u', f') fibres(\phi, v') \Longrightarrow$$

$$Hom_{L_\phi}((u, f, v), G(v')) \cong Hom_{Fib(y)}(For_c((u, f, v)), v') =$$

$$Hom_{Fib(y)}(v,v') \cong Hom_{L_\phi}((u,f,v),(u',f',v')),$$

where the last isomorphism $F_{(u,f,v)} : Hom_{Fib(y)}(v,v') \to Hom_{L_\phi}((u,f,v),(u',f',v'))$ is given, for any $b \in Hom_{Fib(y)}(v,v')$, by

$$F_{(u,f,v)} : b \mapsto (a,b) \Longleftrightarrow f' \circ a = b \circ f \text{ and } u' \circ a = u,$$

so that

$$F^{-1}_{(u,f,v)} : (a,b) \mapsto b.$$

By the Yoneda lemma the right adjoint functor is determined to be the functor which sends an arrow over y to its fibred product with ϕ. $\Box$

2.2.3 Remark

A fibre functor is a functor determined by fibre diagrams. The previous lemma (2.2.2) expresses the notion that the data of the arrows f opposite to the arrows ϕ in the fibre diagrams by which one constructs Fib are implicitly retained. The arrows f are to be used in the following as "localizations" of ϕ.

2.2.4 Definition of an Admissibility Structure.

An admissibility structure ε is a subfunctor of a fibre functor, with an identity object. I.e. it is a natural transformation

$$\varepsilon : E \to Fib$$

of $\mathfrak{Cat}$-valued functors, so that the following hold.

2.2.4.1. Fib is a fibre functor(C).

2.2.4.2. For any $c \in Ob(C)$, $\varepsilon(c) : E(c) \hookrightarrow Fib(c)$ is faithful.

2.2.4.3. For any $c \in Ob(C)$, there exists some $e \in Ob(E(c))$ such that e is terminal, and $\varepsilon(c)(e) \cong (c, id_c, \circ)$ (i.e. e is terminsl, and sent to the terminal object in the arrow category, given by the identity arrow).

2.2.5 Definition of a Spec Datum.

For any universes $U \in U'$, a spec datum(U,U') is a tuple $(sp, \mathcal{O}, \tau, \varepsilon)$, consisting of two functors $(\mathcal{S} \xleftarrow{\mathcal{O}} \mathcal{R} \xrightarrow{sp} \mathcal{T}) \in Arr(U-\mathfrak{Cat})$ with the same domain, a Grothendieck topology τ on $codom(sp)$, and an admissibility structure $\varepsilon \in Arr(Hom^{(1)}_{U'-\mathfrak{Cat}^2}(codom(sp), U-\mathfrak{Cat}))$ on $\mathcal{T}$.

2.2.6 Remark

I believe that there ought to be a definition for a category, whose object set is the set of spec data. I'd imagined at first the category of functors from the category with two arrows with the same domain to the category of categories, with the codomain of one functor having a topology and admissibility structure. But I am uncertain of which functors should be allowed between the categories with the topology and admissibility, and the "correct" direction of functors on the common domain category. I might imagine a path category generated by all possibilities.

2.2.7 Definition of Ω-lifts of arrows in $\mathcal{R}$

For any spec datum $(sp : \mathcal{R} \to \mathcal{T}, \mathcal{O} : \mathcal{R} \to \mathcal{S}, \tau, \varepsilon)$, for any $(\phi : x \longrightarrow y) \in Arr(\mathcal{R})$, we say that an arrow $\tilde{\phi} \in Arr(\Omega)$ is an Ω-lift of ϕ with respect to $(sp, \mathcal{O}, \tau, \varepsilon)$ iff $\tilde{\phi}$ is constructed from the sheafification construction of a left Kan extension of $\mathcal{O}$ along a "localization" of sp, i.e. iff $\tilde{\phi}$ can be constructed by the following procedure.

To an object $x \in Ob(\mathcal{R})$ we associate an object $\tilde{x}$ in Ω. The category component is the product of the categories $\mathcal{T}$ and the category of $\mathcal{S}$-valued sheaves on the category $E(sp(x))$ assigned by the admissibility to $sp(x)$,

$$\tilde{x} = (\mathcal{T} \times Sh((E(sp(x)), \tau_x), \mathcal{S}^{opp})^{opp}, (sp(x), \mathcal{O}_x)) \in Ob(\Omega).$$

Consider the functor between arrow categories, $\sigma : \mathcal{R}_{/x} \longrightarrow \mathcal{T}_{/sp(x)}$ induced by sp, and the functor given by the admissibility $\varepsilon(sp(x)) : E(sp(x)) \longrightarrow \mathcal{T}_{/sp(x)}$. The pre-image (under sp) category $\mathcal{R}_{/x} \times_{\mathcal{T}_{/sp(x)}} E(sp(x))$ is the domain of the fibred product of the functors σ and $\varepsilon(sp(x))$, and the codomain is the category $\mathcal{T}_{/sp(x)}$. The distinguished object $(sp(x), \mathcal{O}_x)$ is roughly the sheafification of the left Kan extension of the functor $\mathcal{O}_0 : \mathcal{R}_{/x} \times_{\mathcal{T}_{/sp(x)}} E(sp(x)) \longrightarrow \mathcal{S}$ given by $\mathcal{O}$.

The functor $\mathcal{O}_0$ is a restriction of $\mathcal{O}$, which sends an object $u : U \to x$ over x, to $\mathcal{O}(U) \in Ob(\mathcal{S})$.

Let the functor $\tilde{\mathcal{O}}_0 : E(sp(x)) \longrightarrow \mathcal{S}$ be the left Kan extension of $\mathcal{O}_0$ along the pullback, $\tilde{\sigma} : \mathcal{R}_{/x} \times_{\mathcal{T}_{/sp(x)}} E(sp(x)) \longrightarrow E(sp(x))$, of the functor σ. Let the functor ${\tilde{\mathcal{O}}_0}^{opp} : E(sp(x))^{opp} \longrightarrow \mathcal{S}^{opp}$, be the opposite of $\tilde{\mathcal{O}}_0$.

The functor $\mathcal{O}_x : E(sp(x))^{opp} \longrightarrow \mathcal{S}^{opp}$ is the sheafification of ${\tilde{\mathcal{O}}_0}^{opp}$ with respect to the pullback topology τ_x on $E(sp(x))$ of the topology τ on $\mathcal{T}$.

Generally, an arrow $\tilde{\phi} = [(F, \phi')] \in Arr(\Omega)$ consists of a functor F and some $\phi' \in F(x', y')$, where x' and y' are the distiguished objects in the domain and codomain respectively. For any arrow $\phi : x \to y$ in $\mathcal{R}$, any Ω-lift

$$(\mathcal{T} \times_{U-\mathfrak{Cat}} Sh((E(sp(x)), \tau_x), \mathcal{S}^{opp})^{opp}, (sp(x), \mathcal{O}_x)) \xrightarrow{\tilde{\phi}}$$

$$(\mathcal{T} \times_{U-\mathfrak{Cat}} Sh((E(sp(y)), \tau_y), \mathcal{S}^{opp})^{opp}, (sp(y), \mathcal{O}_y))$$

has the corresponding functor

$$F : (\mathcal{T} \times Sh((E(sp(x)), \tau_x), \mathcal{S}^{opp})^{opp})^{opp} \times (\mathcal{T} \times Sh((E(sp(y)), \tau_y), \mathcal{S}^{opp})^{opp}) \longrightarrow U - \mathfrak{Set}$$

(so that it is determined by ϕ). It is the composition of the hom functor of the category $\mathcal{T} \times Sh((E(sp(y)), \tau_y), \mathcal{S}^{opp})^{opp}$, with the pushforward functor

$$id_{\mathcal{T}} \times sp(\phi)_* : \mathcal{T} \times Sh((E(sp(x)), \tau_x), \mathcal{S}^{opp})^{opp} \longrightarrow \mathcal{T} \times Sh((E(sp(y)), \tau_y), \mathcal{S}^{opp})^{opp}$$

given by the admissibility structure, i.e. the functor $sp(\phi)_*$ sends a sheaf $\mathcal{F}$ to the composition $\mathcal{F} \circ E(sp(\phi)))$, so that

$$F = Hom_{\mathcal{T} \times Sh((E(sp(y)), \tau_y), \mathcal{S}^{opp})^{opp}} \circ ((id_{\mathcal{T}} \times sp(\phi)_*^{opp})^{opp} \times (id_{\mathcal{T} \times Sh((E(sp(y)), \tau_y), \mathcal{S}^{opp})^{opp}})).$$

The second part of the arrow $\tilde{\phi}$, the distinguished element $(sp(\phi), \phi_\sharp)$, is a natural transformation $\phi_\sharp : \mathcal{O}_y \to sp(\phi)_*(\mathcal{O}_x)$ in the category $Sh(E(sp(y)), \tau_y), \mathcal{S}^{opp})$. We view it as an arrow in the opposite direction $sp(\phi)_*(\mathcal{O}_x) \xrightarrow{\phi_\sharp} \mathcal{O}_y$ in $Sh(E(sp(y)), \tau_y), \mathcal{S}^{opp})^{opp}$. We require that the arrow $\phi_\sharp$ is locally given by a choice of fibres of the arrow $\mathcal{O}(\phi)$ in $\mathcal{S}$. This is to say, that for any $u \in Ob(E(sp(y)))$, there exists a cover $\Gamma \subseteq Arr(E(sp(y)))$ of u such that for any arrow $(u' \xrightarrow{v} u) \in \Gamma$, the arrow $\phi_\sharp(u') \in Arr(\mathcal{S})$, assigned to u' by the natural transformation $\phi_\sharp$, is the fibre along $t \circ \mathcal{O}_y(u')$, of the arrow $(\mathcal{O}(x) \xrightarrow{\mathcal{O}(\phi)} \mathcal{O}(y)) \in Arr(\mathcal{S})$, where t is the composition of the universal arrows associated to the Kan extension, from applying the isomorphism

$$Hom_{Hom^{(1)}_{U-\mathfrak{Cat}^2}(\mathcal{R}_{/y} \times \mathcal{T}_{/sp(y)} E(sp(y)), \mathcal{S}^{opp})}(Hom^{(1)}_{U-\mathfrak{Cat}^2}(Id_{\mathcal{S}^{opp}}, i)(\tilde{\mathcal{O}_{y0}}), \mathcal{O}_{y0})) \cong$$

$$Hom_{Hom^{(1)}_{U-\mathfrak{Cat}^2}(E(sp(y)), \mathcal{S}^{opp})}(\tilde{\mathcal{O}_{y0}}, \tilde{\mathcal{O}_{y0}})$$

to $id_{\tilde{\mathcal{O}_{y0}}}$, and sheafification, from applying the isomorphism

$$Hom_{Sh((E(sp(y)), \tau_y), \mathcal{S}^{opp})}(\tilde{\tilde{\mathcal{O}}}_{y0}, \tilde{\tilde{\mathcal{O}}}_{y0}) \cong$$

$$Hom_{Hom^{(1)}_{U-\mathfrak{Cat}^2}(E(sp(y)), \mathcal{S}^{opp})}(\tilde{\mathcal{O}_{y0}}, For^{Sh((E(sp(y)), \tau_y), \mathcal{S}^{opp})}_{Hom^{(1)}_{U-\mathfrak{Cat}^2}(E(sp(y)), \mathcal{S}^{opp})}(\tilde{\tilde{\mathcal{O}}}_{y0})).$$

to $id_{\tilde{\tilde{\mathcal{O}}}_{y0}}$.

All together, one can write $\tilde{\phi} =$

$[(Hom_{\mathcal{T}\times Sh((E(sp(y)),\tau_y),\mathcal{S}^{opp})^{opp}} \circ ((id_{\mathcal{T}} \times sp_{\phi *}^{opp})^{opp} \times (id_{\mathcal{T}\times Sh((E(sp(y)),\tau_y),\mathcal{S}^{opp})^{opp}})), (sp(\phi), \phi_{\sharp}))]$.

2.2.7.1. Define a functor

$$\sigma_d : \mathcal{R}_{/x} \longrightarrow \mathcal{T}_{/sp(x)}$$

between arrow categories over objects corresponding to the domain of ϕ, induced by sp, by $(U \xrightarrow{u} x) \mapsto (sp(U) \xrightarrow{sp(u)} sp(x))$.

2.2.7.2. The functor σ_c on arrow categories over objects corresponding to the codomain of ϕ is constructed in a similar fashion.

2.2.7.3. Functors p_d, p_c, q_d, q_c are chosen so as to be fibred products of the above functors σ_d and σ_c with the sub-category arrows

$$\varepsilon(sp(x)) : E(sp(x)) \longrightarrow Fib(sp(x))$$

and

$$\varepsilon(sp(y)) : E(sp(y)) \longrightarrow Fib(sp(y))$$

given to the respective objects by the admissibility structure. I.e.
$(p_d, q_d) fibres(\sigma_d, \varepsilon(sp(x)))$ and $(p_c, q_c) fibres(\sigma_c, \varepsilon(sp(y)))$.

2.2.7.4. Denote by

$$dob_d = dob \downarrow_{(\mathcal{R})} (id_{(U-\mathfrak{Cat})(\mathcal{R})}, ob_{(\mathcal{R})}(x)) : \mathcal{R}_{/x} \longrightarrow \mathcal{R}$$

and

$$dob_c = dob \downarrow_{(\mathcal{R})} (id_{(U-\mathfrak{Cat})(\mathcal{R})}, ob_{(\mathcal{R})}(y)) : \mathcal{R}_{/y} \longrightarrow \mathcal{R}$$

the domain object functors, defined by sending an arrow to its domain object. Recall that, for any category $C \in Ob(\mathfrak{Cat})$, we denote by

$$Yo_C^{opp} : C^{opp} \hookrightarrow Hom^{(1)}_{\mathfrak{Cat}^2}(C, \mathfrak{Set})$$

the Yoneda embedding. Choose morphisms between $\mathfrak{Set}$-valued functors

$$\Phi_d : Yo^{opp}_{Hom^{(1)}_{U-\mathfrak{Cat}^2}(E(sp(x)),\mathcal{S})}(\tilde{\mathcal{O}}_d) \to Yo^{opp}_{Hom^{(1)}_{U-\mathfrak{Cat}^2}(dom(p_d),\mathcal{S})}(\mathcal{O} \circ dob_d \circ p_d) \circ Hom^{(1)}_{U-\mathfrak{Cat}^2}(q_d, id_{\mathcal{S}})$$

and

$$\Phi_c : Yo^{opp}_{Hom^{(1)}_{U-\mathfrak{Cat}^2}(E(sp(y)),\mathcal{S})}(\tilde{\mathcal{O}}_c) \to Yo^{opp}_{Hom^{(1)}_{U-\mathfrak{Cat}^2}(dom(p_c),\mathcal{S})}(\mathcal{O} \circ dob_c \circ p_c) \circ Hom^{(1)}_{U-\mathfrak{Cat}^2}(q_c, id_{\mathcal{S}})$$

so that they should be isomorphisms of hom sets, determining the left Kan extensions $\tilde{\mathcal{O}}_d$ of $\mathcal{O} \circ dob_d \circ p_d$ along q_d and $\tilde{\mathcal{O}}_c$ of $\mathcal{O} \circ dob_c \circ p_c$ along q_c, respectively.

2.2.7.5. Choose arrows of functors $(s_d : \tilde{\tilde{\mathcal{O}}}_d \to \tilde{\mathcal{O}}_d)$ and $(s_c : \tilde{\tilde{\mathcal{O}}}_c \to \tilde{\mathcal{O}}_c)$ so that they are universal arrows going from the sheafification of the Kan extension to the Kan extension in the opposites of their respective categories of sheaves (originating from the adjunction to the forgetful functor from sheaves to presheaves, the arrows having been reversed by the self-adjunction of ^{opp} so as to agree with the direction in $\mathcal{R}$).

2.2.7.6. Define the Grothendieck topology $\tau_y \subseteq \mathbf{2}^{Arr(E(sp(y)))}$ to be that induced from the topology τ on $\mathcal{T}$ given by the spec datum. I.e., for any set of arrows $\Gamma \in \mathbf{2}^{Arr(E(sp(y)))}$, $\Gamma \in \tau_y$ iff the image of Γ in $\mathcal{T}$ by the domain object functor composed with the admissibility functor is a cover in $\mathcal{T}$, i.e. iff

$$\{dob \circ \varepsilon(sp(y))(c) \in Arr(\mathcal{T}); c \in \Gamma\} \in \tau$$

where

$$dob = dob \downarrow_{(\mathcal{T})} (id_{\mathcal{T}}, ob_{\mathcal{T}}(sp(y))) : \mathcal{T}_{/sp(y)} \longrightarrow \mathcal{T}$$

is the domain object functor.

2.2.7.7. Suppose that of the maps of sets

$$e_c : Ob(E(sp(y))) \to Arr(E(sp(y)))$$

and

$$e_d : Ob(E(sp(x))) \to Arr(E(sp(x)))$$

each sends an object u in its respective category to its terminal arrow $u \to t_d, t_c$ for some fixed objects $t_d \in Ob(E(sp(x)))$ and $t_c \in Ob(E(sp(y)))$ satisfying (2.2.4.3).

2.2.7.8. Composition with the sub-category functor given by the admissibility structure, $E(sp(\phi))$, gives a functor from the category of sheaves over x to that of sheaves over y (the "pushforward"). Let

$$\phi_\sharp : sp(\phi)_*(\tilde{\tilde{\mathcal{O}}}_d) \to \tilde{\tilde{\mathcal{O}}}_c$$

be any arrow which satisfies the requirement, that for every object $u \in Ob(E(sp(y)))$ in the admissibility structure on the codomain, there is a cover $\Gamma \in \tau_y$ of u such that for each $v \in \Gamma$, the arrows in $\mathcal{S}^{opp}$ given by ϕ and the terminal arrows from each domain of an arrow of the cover, composed with s_d and s_c, form with $\phi_\sharp$ a fibre diagram in $\mathcal{S}$, i.e.

$$(\phi_\sharp(v), \Phi_d(\tilde{\mathcal{O}}_x)(id_{\tilde{\mathcal{O}}_x}) \circ s_d(t_d) \circ \tilde{\tilde{O}}_x(e_d(E(sp(\phi))(v))))$$

$$fibres_{(\mathcal{S})}$$

$$(\Phi_c(\tilde{\mathcal{O}}_y)(id_{\tilde{\mathcal{O}}_y}) \circ s_c(t_c) \circ \tilde{\tilde{O}}_y(e_c(v)), \mathcal{O}(\phi))$$

(which uses (2.2.4.3), to determine arrows from the Kan extensions on the terminal object to the original objects $\mathcal{O}(x)$, $\mathcal{O}(y)$).

2.2.7.9. If the functor F is given by the product of the hom functor of $\mathcal{T}$ with the hom functor on the category of sheaves over the codomain composed with the product of the the pushforward with the identity functor for the category of sheaves over the codomain, i.e. by

$$F \cong Hom_{\mathcal{T}\times Sh((E(sp(y)),\tau_y),\mathcal{S})} \circ (id_{\mathcal{T}} \times sp(\phi)_*)$$

then $\tilde{\phi} = [(F, (sp(\phi), \phi_\sharp))] \in Arr(\Omega)$.

2.2.7.10. The object markers (i.e. the domain and codomain) are $(sp(x), \tilde{\tilde{\mathcal{O}}}_x)$ and $(sp(y), \tilde{\tilde{\mathcal{O}}}_y)$.

2.2.8 Remark

The arrows $\tilde{sp}$ being determined by a product of functors and categories, two forgetful functors suggest themselves, one being to the image of $\kappa_{\mathcal{T}}$, and the other being to a subcategory of Ω containing objects $(Sh(...), sp(x))$, for the various topologies on categories given to objects of $\mathcal{R}$ by the admissibility structure.

2.2.9 Lemma

For any universes $U \in U'$, for any spec datum(U, U'), $(sp, \mathcal{O}, \tau, \varepsilon)$, for a given $\tilde{\tilde{\mathcal{O}}}_d$ and $\tilde{\tilde{\mathcal{O}}}_c$.as above, if the cover Γ of the previous definition an be chosen so that for all $v \in \Gamma$ with associated terminal arrow e_v, the arrow $\tilde{\tilde{\mathcal{O}}}_c(e_v)$ is monic, then $\tilde{\phi}$ is unique.

2.2.10 Example

Localization of rings is epic, and so monic in the opposite category.

2.2.11 Definition of an Ω-lift Functor

For any universes $U \in U'$, for any spec datum, $(sp, \mathcal{O}, \tau, \varepsilon)$, we say that a functor $\tilde{sp} : \mathcal{R} \longrightarrow \Omega$, is an Ω-lift of $(sp, \mathcal{O}, \tau, \varepsilon)$ iff for any arrow $\phi \in Arr(\mathcal{R})$, the arrow $\tilde{sp}(\phi)$ is an Ω-lift of ϕ with respect to $(sp, \mathcal{O}, \tau, \varepsilon)$.

2.2.12 Proposition

There is an isomorphism of functors between any two Ω-lifts having the same spec datum.

2.2.13 Remark

One might define a sheaf of categories, by the Ω lift of the spec datum differing from the original only in replacing $\mathcal{O}$ with a fibre functor on $\mathcal{S}$ composed with $\mathcal{O}$. If each object in the restrictions of such a sheaf over a given object $sp(x)$ gives a different manifestation of the resulting sheafification, then by the preceding proposition I'd imagine that the functor category of Ω-lifts should correspond to a group (if the preceding lemma holds, and a fibre functor on $\mathcal{S}$ taken to be the neutral element, then the elements of the group should be given by the objects), functorially determined on $\mathcal{R}$, which for each $x \in Ob(\mathcal{R})$ has elements in a one-to-one correspondence with the set of Ω-lifts of sp restricted to the arrow category over x.

2.2.14 Lemma

Given a spec datum $\bar{sp} = (sp, \mathcal{O} : \mathcal{R} \to \mathcal{S}, \tau, \varepsilon)$ an functor $F : \mathcal{S} \longrightarrow \mathcal{S}'$ induces an arrow of functors, one from the omega-lift of $\bar{sp}' = (sp, F \circ \mathcal{O}, \tau, \varepsilon)$ to that of the original

$$x \longmapsto \kappa_{tw}(i^{Sh(\ldots,\mathcal{S})}_{Hom^{(1)}_{U-\mathfrak{Cat}}(dom(\varepsilon)(sp(x),\mathcal{S})}) \circ \kappa_{cotw}(Hom^{(1)}_{U-\mathfrak{Cat}^2}(id_{\mathcal{T}} \times Hom^{(1)}_{U-\mathfrak{Cat}^{(2)}}(F, id_{\mathcal{R}}))) \circ$$

$$\kappa_{Hom^{(1)}_{U-\mathfrak{Cat}}(dom(\varepsilon)(sp(x),\mathcal{S}')}(u) \circ \kappa_{cotw}(i^{Sh(\ldots,\mathcal{S}')}_{Hom^{(1)}_{U-\mathfrak{Cat}}(dom(\varepsilon)(sp(x),\mathcal{S}')}).$$

Where u is the universal arrow from the sheafification.[2]

2.3 $End_{End^{(1)}_{(U-\mathfrak{Cat}^2)}(C)}(Id_C)$ and Π_{Sp}

In the present section we define a notion of the global sections functor $\Omega \supseteq \Pi \longrightarrow \mathcal{S}$, appropriate to such sub-categories of Ω as contain the images of a particular Ω-lift. It is defined with respect to a compatible collection of functors, whose domains are the various categories of sheaves which appear in the objects defining the sub-category. We first define, for every functor $F : I \longrightarrow U - \mathfrak{Cat}$, a subcategory $\Pi_\Gamma(F) \subseteq \Omega$, whose arrows are those in Ω generated by the compositions of hom functors of various categories (the codomains of the functors $F(i)$) with the functors $F(i)$.

2.3.1 Definition of $\Pi_\Gamma(F)$

Let $F : I \longrightarrow U - \mathfrak{Cat}$ be a functor.

---

[2]Some smaller version might be desirable here

2.3.1.1. Define the "category generated by F," $\Pi_\Gamma(F) \subseteq \Omega$ to be the subcategory of Ω generated by arrows whose functors factor through the hom functor of some category and some functor in the diagram F, i.e. are given by arrows of the form

$$[(Hom_{codom(F(i))} \circ (F(i)^{opp} \times id_{codom(F(i))}), \phi)] \in Arr(\Omega),$$

where $i \in Arr(I)$ and $\phi \in Arr(codom(F(i)))$.

2.3.1.2. Dually, define the "category co-generated by F," $\Pi_{co\Gamma(F)} \subseteq \Omega$ to be the subcategory generated by arrows of the form

$$[(Hom_{codom(F(i))} \circ (id^{opp}_{codom(F(i))} \times F(i)), \phi)] \in Arr(\Omega),$$

where $i \in Arr(I)$ and $\phi \in Arr(codom(F(i)))$.

2.3.2 Example

For any category $C \in Ob(U - \mathfrak{Cat})$, if F is the constant functor $\Delta_{(I,U-\mathfrak{Cat})}(C) : I \longrightarrow U - \mathfrak{Cat}$, which sends every $i \in Arr(I)$ to the identity functor Id_C, then the category $\Pi_\Gamma(F)$ is the image in Ω of the functor $\kappa_C : C \longrightarrow \Omega$.

2.3.3 Example

For any $\mathcal{S} \in Ob(U - \mathfrak{Cat})$, for any category $\mathcal{T} \in Ob(U - \mathfrak{Cat})$ with an admissibility structure (inclusion of $U - \mathfrak{Cat}$-valued functors) $\varepsilon : E \hookrightarrow Fib$ on $\mathcal{T}$, define a functor $F : \mathcal{T} \longrightarrow U - \mathfrak{Cat}$, on objects by sending an object $x \in Ob(\mathcal{T})$ to the category of $\mathcal{S}$-valued sheaves on the category $E(x)$, assigned by the admissibility structure to x.

$$x \mapsto Sh((E(x), \tau_X), \mathcal{S}),$$

and on arrows by the usual pushforward,

$$f \mapsto Hom^{(1)}_{U-\mathfrak{Cat}^2}(E(f)^{opp}, Id_{\mathcal{S}}) \circ e,$$

being defined by the composition of any given sheaf with the arrow $E(f)$ given by the subfunctor given by the admissibility structure, where where e is the inclusion of the category of sheaves into the category of presheaves. In this case $\Pi_\Gamma(F)$ can be thought of as the category of spaces with $\mathcal{S}$-valued sheaves, i.e. the objects are given by pairs $(x, \mathcal{O}_x)$, where $x \in Ob(\mathcal{T})$ and $\mathcal{O}_x : E(x)^{opp} \longrightarrow \mathcal{S}$ is a sheaf on $E(x)$.

2.3.4 Definition of a Global Sections Datum

For any category $I \in Ob(U - \mathfrak{Cat})$, let $I_0 \in Ob(U - \mathfrak{Cat}$ be the subcategory which consists of the identity arrows of I, and $\varepsilon : I_0 \longrightarrow I$ be the inclusion functor. A *global sections datum*(I) is defined to be an element of the set $Ob(\Gamma)$, where $\Gamma \in Ob(U - \mathfrak{Cat})$ is defined to be $dom(p) = dom(q)$, where $p, q \in Arr(U - \mathfrak{Cat})$ and $(p, q) fibres(\varepsilon^*, sk_*)$, where

$$\varepsilon^* : \downarrow_{(Hom^{(1)}_{U-\mathfrak{Cat}^2}(I, U-\mathfrak{SCat}))} (id_{(\ldots)}, id_{(\ldots)}) \longrightarrow \downarrow_{(Hom^{(1)}_{U-\mathfrak{Cat}^2}(I_0, U-\mathfrak{SCat}))} (id_{(\ldots)}, id_{(\ldots)})$$

and

$$sk_* : \downarrow_{(Hom^{(1)}_{U-\mathfrak{Cat}^2}(I_0, U-\mathfrak{SCat}))} (id_{(\ldots)}, id_{(\ldots)}) \longrightarrow \downarrow_{(Hom^{(1)}_{U-\mathfrak{Cat}^2}(I_0, U-\mathfrak{SCat}))} (id_{(\ldots)}, id_{(\ldots)})$$

where the functor $sk : U - \mathfrak{Cat} \longrightarrow U - \mathfrak{SCat}$ is the quotient functor defined by equating naturally isomorphic functors (arrows of categories).

I.e. it is a sk-natural transformation of $(U - \mathfrak{Cat})$-valued functors.

2.3.5 Definition of a Global Sections Functor, Transformation

Adopt the notation of (2.3.4). For any sk-natural transformation $(F \xrightarrow{t} G) \in Ob(P)$ for which F is injective on objects (we essentially require the $Hom^{(1)}_{U-\mathfrak{Cat}^2}(I_0, U - \mathfrak{Cat})$ data from the fibred product), we make the following constructions.

2.3.5.1. Define a functor

$$\gamma_t : \Pi_\Gamma(F) \longrightarrow \Pi_\Gamma(G)$$

on objects by the κ-twist, i.e. for any $i \in Ob(I)$,

$$\gamma_t : (F(i), x) \mapsto (G(i), t(x)) = codom(\kappa_{tw,t(i)}) = (F(i), x)$$

and on arrows, so that for any $f \in Arr(I)$,

$$\gamma_t : [(Hom_{codom(F(f))} \circ (F(f)^{opp} \times id_{codom(F(f))}), \phi)] \mapsto$$
$$[(Hom_{codom(G(f))} \circ (G(f)^{opp} \times id_{codom(G(f))}), t(codom(f))(\phi))],$$

2.3.5.2. Let $\Pi_\Gamma(F) \xrightarrow{\varepsilon_F} \Omega$ and $\Pi_\Gamma(G) \xrightarrow{\varepsilon_G} \Omega$ be the inclusion functors.

Define an arrow of functors

$$\kappa_{\Pi-tw(t)} : \varepsilon_{\Pi_\Gamma(F)} \to \varepsilon_{\Pi_\Gamma(G)} \circ \gamma_t$$

by the various κ-twists, i.e. for any $i \in Ob(I)$, $\kappa_{\Pi-tw(t)}(i) = \kappa_{tw,t(i)}$.

2.3.6 Proposition

The previous constructions are valid, i.e.

2.3.6.1. The map (2.3.5.2) defines a natural transformation.

2.3.6.2. Suppose that for any $i \in Ob(dom(F))$, for any automorphism $\phi \in Aut_{End_{U-\mathfrak{Cat}^2}(F(i))}(Id_{F(i)})$, there exists an automorphism $\tilde{\phi} \in Aut_{End_{U-\mathfrak{Cat}^2}(F(i))}(Id_{F(i)})$ such that

$$Hom^{(1)}_{U-\mathfrak{Cat}^2}(t(i), Id_{G(i)})(\tilde{\phi}) \cong Hom^{(1)}_{U-\mathfrak{Cat}^2}(Id_{F(i)}, t(i))(\phi),$$

i.e. the automorphisms of $Id_{F(i)}$ "lift" to automorphisms of $Id_{G(i)}$. Then the map (2.3.5.1) defines a functor.[3]

2.3.7

Suppose that $F : \mathcal{T} \longrightarrow U-\mathfrak{Cat}$ is as in (2.3.3). Consider, for any $x \in Ob(\mathcal{T})$, the "global sections functor,"

$$t(x) : Sh((E(x), \varepsilon(x)^*(\tau)), \mathcal{S}) \longrightarrow \mathcal{S}$$

defined by $\mathcal{O} \mapsto \mathcal{O}(e)$, where $e \in Ob(E(x))$ is the terminal object. Then the map of sets $t : Ob(\mathcal{T}) \to Arr(U - \mathfrak{Cat})$ is an sk-natural transformation of functors, $t : F \to \Delta_{(\mathcal{T},U-\mathfrak{Cat})}(\mathcal{S})$.

2.3.8

If the canonical functor $\kappa_{\mathcal{S}}$ is faithful, then there exists a functor $\gamma' : \Pi_\Gamma(F) \to \mathcal{S}$, unique such that $\kappa_{\mathcal{S}} \circ \gamma' = \gamma_t$.

2.3.9 Proposition on Diagrams in $\Pi_\Gamma(F)$

For any functor $F : I \longrightarrow U - \mathfrak{Cat}$ and any category $J \in Ob(U - \mathfrak{Cat}$, there exists a functor $\bar{F} : \bar{I} \longrightarrow U - \mathfrak{Cat}$ such that there is an equivalence of categories

$$Hom^{(1)}_{U-\mathfrak{Cat}^2}(J, \Pi_\Gamma(F)) \cong \Pi_\Gamma(\bar{F})$$

given by the following.

[3]One might slightly weaken this, by requiring only that those automorphisms should lift which might equate two different arrows under the $\kappa_{F(i)}$ functor.

2.3.9.1. Suppose that

$$e : J \longrightarrow U - \mathfrak{Cat}$$

is the functor defined on objects by sending $j \in Ob(J)$ to the category of arrows over it,

$$e : j \mapsto J_{/j} = \downarrow_{(J)} (id_J, ob_{(J)}(j))$$

and on arrows by composition, so that for any arrow $(j \xrightarrow{f} k) \in Arr(J)$, the functor

$$e(f) : J_{/j} \longrightarrow J_{/k}$$

is defined on objects, so that for any $g \in Ob(J_{/j})$, map $(g \mapsto f \circ g)$. The map on arrows is essentially the expected identity map, i.e. one sends a triangle $g_2 \circ \phi = g_1$ to the triangle $f \circ g_2 \circ \phi = f \circ g_1$.

2.3.9.2. Define a function

$$p : Hom_{U'-\mathfrak{Cat}}(J, \Pi_\Gamma(F)) \to Hom_{U'-\mathfrak{Cat}}(J, U - \mathfrak{Cat})$$

by sending a functor $X : J \longrightarrow \Pi_\Gamma(F)$ to the functor $p(X) : J \longrightarrow U - \mathfrak{Cat}$ defined by sending an arrow $X(f)$ to the functor $F(\tilde{f})$ associated to it, i.e. it is defined by requiring that, for any $f \in Arr(J)$, and for any $\tilde{f} \in Arr(I)$,

$$p(X) : f \mapsto F(\tilde{f})$$

iff there exists $\phi \in Arr(codom(F(f)))$, such that

$$X(f) = [(Hom_{codom(F(\tilde{f}))} \circ (F(\tilde{f})^{opp} \times_{U-\mathfrak{Cat}} Id_{codom(F(\tilde{f}))}), \phi)].$$

Define a function

$$p_1 : Arr(Hom^{(1)}_{U'-\mathfrak{Cat}^2}(J, \Pi_\Gamma(F))) \to Arr(Hom^{(1)}_{U'-\mathfrak{Cat}^2}(J, U - \mathfrak{Cat}))$$

similarly, by sending a natural transformation of $\mathfrak{Cat}$-valued functors $t : X \xrightarrow{t} Y$ to the tuple of functors associated to the given tuple of arrows in Ω, i.e. for any $j \in Ob(J)$, and for any $\tilde{f} \in I$, require that $p_1(t)(j) = F(\tilde{f})$ iff there exists $\phi \in Arr(codom(F(f)))$, such that

$$t(j) = [(Hom_{codom(F(\tilde{f}))} \circ (F(\tilde{f})^{opp} \times_{U-\mathfrak{Cat}} Id_{codom(F(tildef))}), \phi)].$$

2.3.9.3. Define the functor

$$\bar{F} : \bar{I} := Hom^{(1)}_{U-\mathfrak{Cat}^2}(J, \Pi_\Gamma(F)) \longrightarrow U - \mathfrak{Cat}$$

on objects by sending a functor $X : J \longrightarrow \Pi_\Gamma(F)$ to the category

$$\bar{h}_{WE(U-\mathfrak{Cat}, \times_{U-\mathfrak{Cat}})}(L(J), U - \mathfrak{Cat})(e, p(X))$$

where $L(J)$ is the category J with the trivial $U - \mathfrak{Cat}$ enrichment, where a given hom category is the category whose objects are given by the hom set in J, with only identity arrows. Define $\bar{F}$ on arrows by sending a natural transformation $(X \xrightarrow{t} Y) \in Arr(Hom^{(1)}_{U'-\mathfrak{Cat}^2}(J, \Pi_\Gamma(F)))$ to the functor given by the enriched composition of the diagram category, i.e.

$$\bar{h}_{WE(U'-\mathfrak{Cat},\times_{U'-\mathfrak{Cat}})}(L(J), U-\mathfrak{Cat})(e, p(X))$$

$$\xrightarrow{\rho} \circ \xrightarrow{Id \times_{U'-\mathfrak{Cat}} p_1(t)} \circ \xrightarrow{\bar{\circ}_{WE(U-\mathfrak{Cat},\times_{U-\mathfrak{Cat}})}}$$

$$\bar{h}_{WE(U'-\mathfrak{Cat},\times_{U'-\mathfrak{Cat}})}(L(J), U-\mathfrak{Cat})(e, p(Y)),$$

where ρ is the right unit isomorphism $Id \to Id \times_{U'-\mathfrak{Cat}} I'$, the functor $Id \times_{U'-\mathfrak{Cat}} p_1(t)$ is given by the representation by the unit I' of the objects functor, and $\bar{\circ}_{WE(U-\mathfrak{Cat},\times_{U-\mathfrak{Cat}})}$ is the enriched composition.

2.3.10 Conjecture on Enrichments

For any functor $F : I \longrightarrow (A, \otimes) - \mathfrak{Cat}$ and any tensor functor $(For, \rho) : (A, \otimes) \longrightarrow (U - \mathfrak{Set}, \times_{U-\mathfrak{Set}})$, the category $\Pi_\Gamma(For^{(For,\rho)-\mathfrak{Cat}}_{U-\mathfrak{Cat}}(F))$ carries a natural enrichment over $(A, \otimes)$.

2.3.11 Remark

Whether the functor κ_C is left or right exact in its image seems to depend upon the compatibility of automorphisms of the identity over various objects in C. In particular, given a functor $F : I \longrightarrow C$, with a limit $(l, \lambda) \in Ob(C) \times Arr(Hom^{(1)}_{U'-\mathfrak{Cat}}(Hom^{(1)}_{U-\mathfrak{Cat}^2}(I, C)^{opp}, U-\mathfrak{Set}))$, one might associate to any arrow of functors $\phi : \Delta_{(I,Im(\kappa_C))}(a) \to \kappa_C \circ F$ an arrow of functors $\tilde{\phi} : \Delta_{(I,C)}(a) \to \kappa_C$ by an explicit choice of arrows in C representing the arrows appearing in ϕ, hoping that the image by κ_C of the limit arrow $\lambda(\bar{\phi})$ should be the limit arrow associated to ϕ. But such a choice seems difficult, since each arrow in I could a priori be associated to its own automorphism $Id_C \to Id_C$, whereas one would like to associate to each object in I such an automorphism. Furthermore, once the candidate limit arrow is chosen, uniqueness seems to rely upon the idea that an assignment of objects in I to automorphisms of Id_C, such as should relate the component arrows, would determine an automorphism of Id_C suitable for relating the limit arrows.

2.3.12 Definition of Π_{Sch}

For any given spec datum $\bar{sp} = (\mathcal{T} \xleftarrow{sp} \mathcal{R} \xrightarrow{\mathcal{O}} \mathcal{S}, \tau, \varepsilon)$, for any $\Omega - lift(U, \bar{sp})$, $\tilde{sp} : \mathcal{R} \longrightarrow \Omega$, let $F : \mathcal{T} \longrightarrow U - \mathfrak{Cat}$ be as in (2.3.3), and let

$$\pi : \Delta_{(\mathcal{T},U-\mathfrak{Cat})}(\mathcal{T}) \times F \to \Delta_{(\mathcal{T},U-\mathfrak{Cat})}(\mathcal{T})$$

be the projection. Define a subcategory $\Pi_{Sch}(\bar{sp}, \tilde{sp}) \subseteq \Pi_\Gamma(\Delta_{(\mathcal{T},U-\mathfrak{Cat})}(\mathcal{T}) \times F) \subset \Omega$ so that its arrows are those arrows $(x \xrightarrow{\phi} y) \in Arr(\Pi_\Gamma(\Delta_{(\mathcal{T},U-\mathfrak{Cat})}(\mathcal{T}) \times F))$, such that subsets $C_x, C_y \subset Arr(\Pi_\Gamma(\Delta_{(\mathcal{T},U-\mathfrak{Cat})}(\mathcal{T}) \times F))$ exist such that the following hold.

2.3.12.1. The image of C_x under $\gamma_{\pi(x)}$ is a cover of $\gamma_{\pi(x)}(x)$, and the image of C_y under $\gamma_{\pi(y)}$ is a cover of $\gamma_{\pi(y)}$, in $(Im(\kappa_{\mathcal{T}}), \tau_{Im(\kappa_{\mathcal{T}})})$, where $\tau_{Im(\kappa_{\mathcal{T}})}$ is the topology generated by all the images of all elements of τ (i.e. covers in $(\mathcal{T}, \tau)$).

2.3.12.2. Each of the arrows in C_x, when composed with ϕ, yields an arrow factoring through an omega lift of some arrow in $\mathcal{R}$, i.e., for any $u \in C_x$, there exists $\phi' \in Arr(\mathcal{R})$, such that there exists $v \in C_y$, such that $v \circ \tilde{sp}(\phi') = \phi \circ u$.

2.3.13 Remark

The functor ${}^{opp} : \mathfrak{Cat} \longrightarrow \mathfrak{Cat}$ is its own adjoint, implying a fortiori that it preserves limits, and therefore that $(\mathcal{T} \times Sh((dom(\varepsilon)(sp(y)), (\varepsilon(sp(y)))^* \circ$

$(dob \downarrow_{(\mathcal{T})} (id_{\mathcal{T}}, ob_{(\mathcal{T})}(sp(y))))^*(\tau)), \mathcal{R}))^{opp} \cong \mathcal{T}^{opp} \times (Sh((dom(\varepsilon)(sp(y)),$

$(\varepsilon(sp(y)))^* \circ (dob \downarrow_{(\mathcal{T})} (id_{\mathcal{T}}, ob_{(\mathcal{T})}(sp(y))))^*(\tau)), \mathcal{R}))^{opp}$. Thus isomorphisms of functors which identify representations of arrows in Ω might be thought of in a piecewise fashion.

2.3.14 Proposition on Finite Diagrams

For any spec datum $\bar{sp} = \ldots$ with an Ω-lift $\tilde{sp}$, for any category $I \in Ob(U - \mathfrak{Cat})$ such that the set $Arr(I)$ is finite, there exists a spec datum

$$\bar{sp}' = (\circ \xleftarrow{Hom^{(1)}_{U-\mathfrak{Cat}^2}(Id_I, \mathcal{O})} Hom^{(1)}_{U-\mathfrak{Cat}^2}(I, \mathcal{R}) \xrightarrow{Hom^{(1)}_{U-\mathfrak{Cat}^2}(Id_I, sp)} \circ, \ldots)$$

with an Ω-lift $\tilde{sp}'$, such that $Hom^{(1)}_{U'-\mathfrak{Cat}^2}(I, \Pi_{Sch}(\bar{sp}, \tilde{sp})) \cong \Pi_{Sch}(\bar{sp}', \tilde{sp}')$.

2.3.15 Lemma

For any ring k, $Aut_{End^{(1)}_{(U-\mathfrak{Cat}^2)}(k-\mathfrak{Alg})}(Id_{k-\mathfrak{Alg}}) = \{id_{Id_{k-\mathfrak{Alg}}}\}$

Proof. If $k = \{0\}$, then the category of k-algebras is the category with one arrow, and the result is trivial. If k has at least two elements, then consider the free ring in one variable $k[x]$. But if there were a non-trivial automorphism of $Id_{k-\mathfrak{Alg}}$ as in the preceding lemma, then there would exist $A \in Ob(k - \mathfrak{Alg})$ such that the arrow $\alpha(A) : A \to A$ associated to A would be other than the identity. Therefore there would exist $a \in A$ such that $\alpha(A)(a) \neq a$, implying that, if $f : k[x] \to A$ were the arrow given by $x \mapsto a$, $\alpha(k[x]) = id_{k[x]}$ would imply that $a = f(x) = f \circ \alpha(k[x])(x) = \alpha(A) \circ f(x) = \alpha(A)(a) \neq a$, a contradiction. Therefore $\alpha(k[x]) \neq id_{k[x]}$. Therefore the arrow assigned by the natural transformation to $k[x]$ would be other than the identity, and by naturality, it would have to commute with every other arrow $k[x] \mapsto k[x]$. The only automorphisms are $x \mapsto ax + b$, composed with automorphisms of k. But if k has at least two elements, then a map $x \mapsto cx + d$ can be found which should make the required commuting square impossible. $\square$

2.3.16 Remark

I've a more general notion of a free object x over an object k with respect to a "forgetful" functor F, being that $Yo^{opp}(x) \cong Yo^{opp}(k) \times F$. I get the sense that the corollary might be extended. At any rate, I've the sense that the notion of freedom ought to be generalized and made precise, so far as possible, the inducement being thoughts of expressing "finite type" in a recursive fashion, and examining the relationship between it (an "algebraic" notion) and compactness, a property referring only to the topology. $codom(F)$ seems somewhat flexible, assuming that the category in question is enriched over $codom(F)$, so that the functor category should inherit the enrichment. I suspect that this should involve the vague hopes on constructing spec functors described below.

2.3.17 Corollary

For any ring k, $\kappa_{k-\mathfrak{Alg}}$ is faithful.

2.3.18 Corollary

If $\bar{spec} = (\mathfrak{Top} \xleftarrow{spec} \mathfrak{Ring} \xrightarrow{Id_{\mathfrak{Ring}}} \mathfrak{Ring},)$ is the usual topological spectrum data and $\tilde{spec}$ is its Ω-lift, then the following hold.

2.3.18.1. The functor $\tilde{spec}$ is faithful.

2.3.18.2. The left Kan extension of the usual inclusion functor $\mathfrak{Ring} \longrightarrow \mathfrak{Sch}$ along the natural factor of the Ω-lift $\tilde{spec}' : \mathfrak{Ring} \longrightarrow \Pi(\bar{spec}, \tilde{spec})$ through its image (i.e. the functor such that $\varepsilon_{\Pi(\bar{spec},\tilde{spec})} \circ \tilde{spec}' = \tilde{spec}$), is an equivalence of categories $\Pi(\bar{spec}, \tilde{spec}) \longrightarrow \mathfrak{Sch}$.

Proof. In this case $\mathcal{O} = Id_{U-k-\mathfrak{Alg}}$, so that the global sections functor exists, and by the usual arguments $\Gamma \circ \tilde{spec} \cong Id_{k-\mathfrak{Alg}}$. An arrow of schemes is uniquely determined by its open affine fibres. $\square$

2.4 Separateness and Extensions of $\Pi(sp, \mathcal{O}, ...)$

We interpret separateness as "the sufficiency of the sheaf category component of $\tilde{x} = (\mathcal{T} \times Sh(X), (x, \mathcal{O}_x))$ to determine points," by the following conjecture.

2.4.1 Definition of Separateness

For any sk-natural transformation of functors $F, G : I \longrightarrow \Omega$, $t : F \to G$, for any $x \in Ob(\Pi_\Gamma(F))$, we say that x is *separated*(t) iff for any $y \in Ob(\Pi_\Gamma(F))$, the function $\gamma_t(y, x) : Hom_{\Pi_\Gamma(F)}(y, x) \to Hom_{\Pi_{\Gamma(G)}}(\gamma_t(y), \gamma_t(x))$ is injective.

2.4.2 Remark

By (2.3.9) and (2.3.14) there are at least two ways in which a category of diagrams in some $\Pi_{Sch}(\bar{sp}, \tilde{sp})$ can be realized as a subcategory of some $\Pi_\Gamma(\bar{F})$. The former is intrinsic to Π_Γ, and essentially involves tuples of functors $e(j) \longrightarrow X(j)$ (in this case F of that construction would be the product of the functor in (2.3.3) with $Id_\mathcal{T}$). The latter essentially restricts the spec functor to appropriate diagrams in $\mathcal{R}$, and uses the product of the functor of (2.3.3) with $Id_{Hom^{(1)}_{U-\mathfrak{Cat}^2}(I,\mathcal{T})}$. In either case, I expect that the natural transformation determining separateness on the diagram category in $\Pi_{Sch}(\bar{sp}, \tilde{sp})$ should be given by the "projection which forgets the constant $\mathcal{T}$ component."

2.4.3 Conjecture on Separateness

Adopt the conditions and notation of (2.3.3). Let

$$\pi : \Delta_{(\mathcal{T}, U-\mathfrak{Cat})}(\mathcal{T}) \times F \to F$$

be the projection. For any $x \in Ob(\Pi(\bar{sp}, \tilde{sp}))$, x is *separated*$(\bar{sp}, \tilde{sp})$ iff for any $y \in Ob(\Pi(\bar{sp}, \tilde{sp}))$, $\gamma_\pi(y, x)$ is injective.

Suppose that $(X, X \xrightarrow{f} S, \emptyset) \in Ob(\mathfrak{Sch})$ is a scheme over Y. Then the notions of separateness coincide, i.e. $X \xrightarrow{\Delta} X \times_S X$ is closed, an immersion iff X is separated$(\bar{sp}, \tilde{sp})$.

2.4.4 Remarks on Correspondences

If $\tilde{sp}$ is an Ω-lift of $\bar{sp}$, S is separated$(\bar{sp}, \tilde{sp})$, and if $COLIM$ denotes a functor $Hom^{(1)}_{U-\mathfrak{Cat}^2}(I, A) \longrightarrow A$ which sends a functor to its colimit (in this case, $I = A$), and $\bar{Yo}_{opp}$ denotes a $\mathcal{O}_{X \times_S Y}-\mathfrak{Mod}$-enriched hom functor (i.e. for any $\mathcal{M} \in Ob(\mathcal{O}_{X \times_S Y}-\mathfrak{Mod})$, $\bar{Yo}_{opp}(\mathcal{M}) : a \mapsto Hom^\sharp(M, a)$, so that $Yo = (A \xrightarrow{\bar{Yo}} Hom^{(1)}_{U-\mathfrak{Cat}^2}(A, A) \xrightarrow{Hom^{(1)}_{U-\mathfrak{Cat}^2(1)}(id_A, For)} \circ)$, then it is expected that the set of arrows of the form

$$[(For_{U-\mathfrak{Set}}^{COLIM(\mathcal{O}_{X\times_S Y})-\mathfrak{Mod}} \circ COLIM \circ COLIM \circ \bar{Yo}$$
$$\circ For_{\mathcal{O}_{X\times_S Y}-\mathfrak{Mod}}^{Sh((X\times_S Y,\tau_{X\times_S Y}),\mathfrak{Ring})} \circ \otimes_{Sh((X\times_S Y,\tau_{X\times_S Y}),\mathfrak{Ring})} \circ ((p^*)^{opp} \times_{\mathfrak{Cat}} q^*), \phi)] \in Arr(\Omega),$$

where $(p,q)fibres_{(\mathfrak{Sch})}(X \xrightarrow{f} S, Y \xrightarrow{g} S)$, should be isomorphic to the set of correspondences from X to Y. [4]

We also expect that restricting the "sub-object diagram" by replacing $COLIM \circ \bar{Yo}$ with $COLIM \circ Hom^{(1)}_{U-\mathfrak{Cat}^2}(\varepsilon_i, id_{\mathcal{O}_{X\times_S Y}-\mathfrak{Mod}}) \circ \bar{Yo}$ in the above arrow in Ω, where $\varepsilon_i \hookrightarrow \mathcal{O}_{X\times_S Y} - \mathfrak{Mod}$ is the inclusion of modules with an epic map $\bigoplus_{j=1}^{i} \mathcal{O}_{X\times_S Y} \longrightarrow \mathcal{M}$, yields the set of correspondences given by subschemes $Z \hookrightarrow X \times_S Y$ of codimension i.

However they would appear here as arrows $(Sh(X)^{opp}, \mathcal{O}_X) \to (Sh(Y), \mathcal{O}_Y)$ in Ω, rather than having $(Sh(X), \mathcal{O}_X)$ as the domain, since the tensor functor $\otimes$ is covariant in each variable.

2.5 Addenda

2.5.1 Remark

For any category I, consider a limit (i,l) of some diagram $D : J \longrightarrow I$, i.e. $i \in Ob(I)$ and $l : \Delta_{(J)}(i) \to D$ is a universal arrow of functors, where $\Delta_{(J)} : C \longrightarrow Hom^{(1)}_{U-\mathfrak{Cat}^2}(J, C)$ sends an object to its constant functor, so that any $f : \Delta_{(J)}(i') \to D$ factors uniquely through some $\Delta_{(J)}(\tilde{f}) : \Delta_{(J)}(i') \to \Delta_{(J)}(i)$. Then for any arrow $[(F,\phi)] \in Arr(\Omega)$, the distinguished element of the set, $\phi \in F(c,i)$, determines compatible elements $F(id_c, l(j))(\phi) \in F(c, D(j))$, where $j \in Ob(J)$. This is to say that one can also distinguish collections of elements in various sets by using a limit object $i \in Ob(I)$ for the distinguished element in the object (I,i) which determines a representable functor in Ω.

2.5.2 Remark

I expect that the representatives of many sorts of functors $\Omega \supseteq \Pi \longrightarrow \mathcal{S}$, in Π may be sought by the colimits of the relevant $dob \downarrow_{(\Omega)}$ functors from arrow categories in Ω over

[4] The forgetful functor, from sheaves of rings over (in opp) the pullback of the base structure sheaf to modules over the structure sheaf of the fibre, seems suspect. I believe that correspondences are usually in the literature restricted to smooth schemes over a field (and projective, I imagine so that $X \times_S Y \times_S Z \xrightarrow{\pi} X \times_S Z$ should be closed for the sake of composition, but I debate whether this is necessary in the above formulation), and I wonder therefore whether one should replace For with $Diff : Sh((X \times_S Y, \tau_{X\times_S Y}), \mathfrak{Ring})_{\longrightarrow \mathcal{O}_{X\times_S Y}-\mathfrak{Mod}}$.

(I, i), i.e. by constructing right adjoints to inclusion functors $\Pi \hookrightarrow \bar{\Pi}$, where $\bar{\Pi}$ is Π with a single extra object.

CHAPTER 3

(WEAK) $U - \mathfrak{Cat}$-VALUED SHEAVES AND STEPS TOWARD HOMOTOPY GROUPS, AND (CO)-HOMOLOGY

We intend, in future work, to derive at least "the" homotopy groups from admissibility structures, by imitating the description of π_1 in terms of covering spaces and adopting the notion that $\pi_{n+1}(x) \cong \pi_n(\Lambda(x))$ for some endofunctor $\Lambda : \Pi \longrightarrow \Pi$.

It is intended that (1.4.14) should be applied to such admissibility structures as appear in the lemma (3.1.14) below, them being in the place of the $\varepsilon(\beta)$ of (1.4.14.2), to define functors $\Pi \longrightarrow n - \mathfrak{Cat}$, so that "the n-categery attached to an object should contain its homotopy data." To this end we define, in (3.2), a certain functor $\mathfrak{Cat} \longrightarrow \mathfrak{PreGroup}$.

3.1 Augmented Admissibility Structures

It is expected that one should desire to consider $\mathfrak{Cat}$-valued pseudo-functors attached to a space, such that composition might be respected only up to isomorphism. We use the κ-twists attached to the concatenation functors $Path(\mathcal{T}) \longrightarrow \mathcal{T}$ to construct, from an admissibility structure on $\mathcal{T}$, an $\mathfrak{Cat}$-valued functor on a category $\bar{\mathcal{T}}^{opp}$, where $\mathcal{T} \subseteq \bar{\mathcal{T}} \subseteq \Omega$. If $\kappa_{\mathcal{T}}$ is faithful, then the restriction of this functor to $\mathcal{T}^{opp}$ is a subfunctor of a fibre functor, so that the categories of functors with domains given the new "admissibility" should be equivalent to categories of pseudo-functors with domains given by the old admissibility.

3.1.1 Definition of the Category of Pre-Categories

Define the category of pre-categories to be the category of sets with hom set assignments, but no composition structure. Explanation follows.

3.1.1.1. For any universe U, define the category $U - \mathfrak{PreCat} \in Ob(U' - \mathfrak{Cat})$ to be the category of U-small pre-categories. Its set of objects is the set of triples (O, A, h), where

O and A are sets and $h : O^2 \to 2^A$ assigns to every pair of "objects" a "hom set," so that distinct objects have disjoint hom sets.

$$\{(O, A, h) \in U^3; \ulcorner\ulcorner h \in Hom_{(U-\mathfrak{Set})}(O^2, 2^A)\urcorner \text{ and}$$
$$\ulcorner \forall a, b, c, d \in O, \ulcorner\ulcorner (a, b) \neq (c, d)\urcorner \Longrightarrow \ulcorner h((a, b)) \cap \hbar((c, d)) = \emptyset\urcorner\urcorner\urcorner\urcorner\}$$

Its set of arrows is the set of "pre-functors," pairs $F = (F_0, F_1)$ between the object sets and the arrow sets which are compatible with the hom sets, i.e. for any two pre-categories $(O_C, A_C, h_C), (O_D, A_D, h_D) \in Ob(U - \mathfrak{PreCat})$, define

$$Hom_{U-\mathfrak{PreCat}}((O_C, A_C, h_C), (O_D, A_D, h_D)) :=$$
$$\{(F_0, F_1) \in U; F_0 : O_C \to O_D \text{ and } F_1 : A_C \to A_D \text{ and}$$
$$\forall c_1, c_2 \in O_C, Im(F_1|_{h_C(c_1,c_2)}) \subseteq h_C(F_0(c_1), F_0(c_2))\}$$

3.1.1.2. Define the functor $For^{U-\mathfrak{Cat}}_{U-\mathfrak{PreCat}} : U - \mathfrak{Cat} \longrightarrow U - \mathfrak{PreCat}$ to be the expected forgetful functor, with the arrow map given by the identity.

3.1.2 Definition of the Functor "$Path$"

Define the functor $Path : U - \mathfrak{Cat} \longrightarrow U - \mathfrak{Cat}$ as follows.

For any category $C \in Ob(U - \mathfrak{Cat})$, define the category $Path(C)$ to be the path category of C, so that its objects $Ob(Path(C)) = Ob(C)$, are the same, and arrows are composible sequences of arrows in C,

$$Hom_{Path(C)}(x, y) = \{\emptyset\} \cup \coprod_{n\in\mathbb{N}^+} \coprod_{z:\{1,\dots,n\}\to Ob(C); z(1)=x \text{ and } z(n)=y} \prod_{i=1}^{n-1} Hom_C(z(i), z(i + 1)),$$

with composition given by concatenation.

For any functor $F : C \longrightarrow D$, define the functor $Path(F) : Path(C) \longrightarrow Path(D)$ so that for any objects $x, y \in Ob(C)$,

$$Path(F)|_{Hom_{Path(C)}(x,y)} = id_{\{\emptyset\}} \cup \coprod_{n\in\mathbb{N}^+} \coprod_{z:\{1,\dots,n\}\to Ob(C); z(1)=x \text{ and } z(n)=y} \prod_{i=1}^{n-1} F_{Hom_C(z(i),z(i+1))}$$

3.1.3 Lemma

$For^{U-\mathfrak{Cat}}_{U-\mathfrak{PreCat}}$ has a left adjoint L, which when composed with the forgetful functor yields $Path \cong L \circ For^{U-\mathfrak{Cat}}_{U-\mathfrak{PreCat}}$.

3.1.4 Lemma

For any category $C \in Ob(U - \mathfrak{Cat})$, $\kappa_{Path(C)}$ is faithful.

3.1.5 Remark on a Generalization

There should be a version of this for $WE_{(A,\otimes)}$ for any associative tensor category $((A, \otimes), \alpha, \sigma)$, which satisfies a pentagonal requirement for α and σ, i.e. that the associator is canonical for larger tensors.

3.1.6 Proposition

For any $C \in Ob(U' - \mathfrak{Cat})$, $\kappa_{Hom^{(1)}_{U'-\mathfrak{Cat}^2}(Path(C), U-\mathfrak{Cat})}$ is faithful.

3.1.7 Lemma

Suppose that $T \in Ob(U-\mathfrak{Cat})$ is a category, such that κ_T is faithful. If $P : Path_{(0)}(T) \longrightarrow T$ is the concatenation functor (universal arrow from adjunction), then for any admissibility structure ε on any $T \in Ob(U - \mathfrak{Cat})$, there is a unique admissibility structure ε' on the subcategory of Ω generated by the image of κ_T such that for any $x \in Ob(T)$, the category $dom(\varepsilon')(x) \subseteq \Omega_{/\kappa_T(x)}$ is that whose objects are the set

$$\{[(Hom_T \circ (P^{opp} \times_{U-\mathfrak{Cat}} Id_T), u)] \in Arr(\Omega); u \in Im(\varepsilon(x))\} \cup$$

$$\{[(Hom_T, u)] \in Arr(\Omega); u \in Im(\varepsilon(x)\}$$

and whose arrows are the set

$$\{[(Hom_{Path(T)}, u)] \in Arr(\Omega); P(u) \in Im(\varepsilon(x))\} \cup$$

$$\{[(Hom_T \circ (P^{opp} \times_{U-\mathfrak{Cat}} Id_T), u)] \in Arr(\Omega); u \in Im(\varepsilon(x))\} \cup$$
$$\{[(Hom_T, u)] \in Arr(\Omega); u \in Im(\varepsilon(x)\}$$

3.1.8 Corollary on Iterated "*Path* Augmentation"

Suppose that $\Pi \in Ob(U - \mathfrak{Cat})$ has an admissibility structure ε_0. Then there is an admissibility structure ε on $Im(\kappa_{\Omega(\Pi)}) \subseteq \Omega$ such that for every $n \in \mathbb{N} \cup \{\infty\}$, there exists $\varepsilon' \subseteq \varepsilon$, such that for any $x \in Im(\kappa_{\Omega(\Pi)}), dom(\varepsilon')(x) \cong Path^n(dom(\varepsilon_0)(x))$

3.1.9 Remark

In light of the fact that $\kappa_{Path(C)}$ is faithful, this domain for $\mathfrak{Cat}$-valued functors is naturally attached in Ω by arrows to a domain for $\mathfrak{Cat}$-valued pseudo-functors. In particular, the set of objects of the category of functors $Hom^{(1)}_{U'-\mathfrak{Cat}^2}(Path(C), U-\mathfrak{Cat})$ is by the adjunction isomorphic to the set $Hom_{U'-\mathfrak{PreCat}}(For^{U-\mathfrak{Cat}}_{U-\mathfrak{PreCat}}(C), For^{U'-\mathfrak{Cat}}_{U'-\mathfrak{PreCat}}(U-\mathfrak{Cat}))$ of pseudo-functors .

3.1.10 Remark

For any spec datum $(sp : R \to T, \mathcal{O}, \tau, \varepsilon)$, if $\kappa_{\Omega(codom_{(U-\mathfrak{Cat})}(sp))}$ were faithful then it would be an isomorphism onto its image, and the latter could be given structures so that $(\kappa_{\Omega(T)} \circ sp, \mathcal{O}, \tau', \varepsilon')$ should be a spec datum as well.

3.1.11 Remark

Since the subcategory denoted by Π associated to Ω-lifts has objects of the form $(\mathcal{T} \times_{U-\mathfrak{Cat}(0)} (Sh((X,\tau_X),\mathcal{R}))^{opp}, (X,\mathcal{O}_X))$ the previous lemma can be used to extend the admissibility structure on Π in a component-wise fashion.

3.1.12 Lemma

Suppose that $F : I \longrightarrow U-\mathfrak{Cat}$ and (l,λ) is a $(\mathfrak{Skel})-limit(F)$.

Suppose that $c : l \longrightarrow \prod_{i\in Ob(I)} F(i)$ is the map induced by the definition of the $(sk)-limit$ as a colimit (as in lemma on sk-limits having unique maps into them when the arrow from the (sk)-limit to the product is monic) and that c is monic (faithful). Then $\forall G \subseteq Ob(l), \ulcorner \forall i \in Ob(I), \ulcorner\{\pi_i \circ c(g); g \in G\}$ generates $F(i)\urcorner\urcorner \Longrightarrow \ulcorner G$ generates $l\urcorner$

$x \in Ob(dom_{(U-\mathfrak{Cat})}(\lambda_i))$ from those of the components, such that $\forall i \in Ob(I), \lambda_{(i)}(x) \cong g_i$, g_i being some generator of $F(i)$

3.1.13 Corollary

$\kappa_{\Omega(lim_{i\in\mathbb{N}}Path^{(i)}(C))}$ is faithful.

3.1.14 Lemma on Gluing Admissibility Structures

If categories $\Pi_1, \Pi_2 \in Ob(U-\mathfrak{Cat})$, such that κ_{Π_1} and κ_{Π_2} are faithful, have admissibility structures $\varepsilon_1, \varepsilon_2$ respectively, then a functor $(\Pi_1 \xrightarrow{\Lambda} \Pi_2) \in Arr(U-\mathfrak{Cat})$ determines an admissibility structure on the subcategory of Ω generated by arrows of the form $\kappa_{cotw(\Lambda)}(x)$, where $x \in Ob(\Lambda_1)$, or contained in the image of either κ_{Π_1} or κ_{Π_2}. Explanation follows.

3.1.14.1. Let $\Lambda : \Pi_1 \longrightarrow \Pi_2$ be a functor between U-categories.

3.1.14.2. Suppose that κ_{Π_i} is faithful, for $i \in \{1, 2\}$.

3.1.14.3. Let ε_i be an admissibility structure on Π_i, for $i \in \{1, 2\}$.

3.1.14.4. Let $\Pi \subseteq \Omega$ be the subcategory generated by the union of the images of the κ-functors, with the set of arrows appearing in κ-cotwists, i.e. the set

$$Im(\kappa_{\Pi_1 1}) \cup Im(\kappa_{\Pi_2 1}) \cup \{\kappa_{cotw(\Lambda)}(x) \in Arr(\Omega); x \in Ob(\Pi_1)\}$$

.

3.1.14.5. Then there is a unique admissibility structure $\varepsilon \in Arr(Hom^{(1)}_{U'-\mathfrak{Cat}^2}(\Pi^{opp}, U-\mathfrak{Cat}))$ formed from gluing the arrows appearing in the original admissibility structures by the κ-twist arrows, i.e. such that the following hold.

3.1.14.5.1. For any object $x \in Ob(\Pi_1)$, the objects of the category assigned to $(\Pi_1, x) \in Ob(\Pi)$ are compositions of arrows appearing in the original admissibility structures, i.e.

$$\begin{gathered} Ob(dom(\varepsilon)(x)) = \{[(Hom_{\Pi_2} \circ (Id^{opp}_{\Pi_2} \times_{U-\mathfrak{Cat}} \Lambda), u)] \circ [(Hom_{\Pi_1}, v)] \in Arr(\Omega); \\ u \in Im(\varepsilon_1(x)) \text{ and } v \in Im(\varepsilon_2(\Lambda(dom(u))))\} \cup \\ \{[(Hom_T, u)] \in Arr(\Omega); u \in Im(\varepsilon_1(x)\}, \end{gathered}$$

and the set of arrows of the category assigned to $(\Pi_1, x) \in Ob(\Pi)$ is the pre-image of the original admissibility structures, i.e. arrows of the form $[(Hom_{\Pi_2}, v)]$, where $v \in Im(\varepsilon_2(\Lambda(x)))$ or the form $[Hom_{\Pi_1}, u)]$, where $u \in Im(\varepsilon_1(x))$.

3.1.14.5.2. For any object $x \in Ob(\Pi_2)$, define the category assigned to the object $(\Pi_2, x) \in Ob(\Omega)$ so as to agree with the original admissibility structure, i.e.

$$dom(\varepsilon)((\Pi_2, x)) \cong dom(\varepsilon_2)(x)$$

3.2 Remarks toward Invariants from Categories

We define a functor $U - \mathfrak{Cat} \longrightarrow \mathfrak{PreGrp}$, intended to send the category of functors $Hom_{\mathfrak{Cat}^2}(E(x), E(x))$ to a pre-group (without inversion) containing the fundamental group $\pi_1(x)$.

3.2.1 Remark

If $dom_{(U'-\mathfrak{Cat})}(P)$ has U-small limits, then P has a left adjoint and if it has U-small colimits, and P preserves them, then it has a right adjoint. This is a general categorical construction. I think of the standard example of an admissibility structure, the open subsets in a topological space, when considering the feasibility of the existence of the right adjoint.

3.2.2 Definition of Sweep

Define the "sweep functor" by sending a category C to the set of equivalence classes of ordered lists of arrows in C, where the equivalence is that generated by allowing the compositions of consecutive pairs of arrows in the list. Composition is given by concatenation.

3.2.3 Lemma

Sweep is functorial.

3.2.4 Remark

We hope, in future work, to associate to each spec datum $\bar{sp} = (..., \varepsilon : E \hookrightarrow Fib)$ a $\mathfrak{Cat}$ valued functor, which would attach, to each $x \in Ob(\mathcal{T})$, the category of endofunctors of $E(x)$. The expectation is that any two Ω-lifts of the same arrow in $\mathcal{R}$ should be related by automorphisms of the fibres used to determine the Ω-lifts, which would determine an isomorphism in the endofunctor category.

APPENDIX

MISCELLANEOUS VARIATIONS

A.1 An Older Definition of n-Categories

This construction is simpler than that which appears in the main text, but does not involve defining a sequence of skeleton functors $sk(n) : n - \mathfrak{Cat} \longrightarrow \cdot$ to weaken the required equalities (e.g. associativity).

The following five definitions

$$For^{U(n)-\mathfrak{Cat}}_{U-\mathfrak{Cat}} : U(n) - \mathfrak{Cat} \longrightarrow U - \mathfrak{Cat}$$

$$\alpha_{U(n)-\mathfrak{Cat}} Ob(U(n) - \mathfrak{Cat})^3 \to Arr(U(n) - \mathfrak{Cat})$$

$$\times_{U(n+1)-\mathfrak{Cat}} : U(n+1) - \mathfrak{Cat} \times_{U'-\mathfrak{Cat}} U(n+1) - \mathfrak{Cat} \longrightarrow U(n+1) - \mathfrak{Cat}$$

$$\sigma_{U(n+1)-\mathfrak{Cat}} : Ob(U(n+1) - \mathfrak{Cat})^2 \to Arr(U(n+1) - \mathfrak{Cat})$$

$$U(n) - \mathfrak{Cat} \in Ob(U' - \mathfrak{Cat})$$

are simultaneously made, for any $n \in \mathbb{N}\backslash\{0\}$,[1] so that the category of (n+1)-categories (the fifth of these definitions) should be defined as the category of weakly $(n - \mathfrak{Cat})$-enriched categories. [2]

For any pair of universes $U \in U'$, for any tuple $S = (S_1, S_2, S_3, S_4, S_5) \in Ob(U' - \mathfrak{Set})$, we say that S defines higher categories in $U \in U'$ iff

and

[1]The expected case of sets, for $n = 0$, is excluded, since a set does not seem naturally to be an enriched set $(S, h, \circ)$, unless it takes the enrichment over $\mathfrak{Set}$ given by the trivial category functor $\mathfrak{Set} \longrightarrow \mathfrak{Cat}$.

[2]The definition applies the unary map of the comprehension schema, attached to the open statement described below, to the universe U'. It makes use of the absolute "=" sign, a relation on the one type, to compare the sets S_i below, a priori existing, though uncharacterized, to certain sets constructed from them.

$$S_1 = \{(n, (F_1 = \{((C,h,\circ),C) \in U'; \exists c \in Ob(U' - \mathfrak{Cat}), (C,h,\circ) \in Ob(c) \text{ and } (n,c) \in S_5\},$$

$$F_2 = \{(((f_0,f_1),f_2),(f_0,f_1)) \in U';$$
$$\exists c \in Ob(U' - \mathfrak{Cat}), ((f_0,f_1),f_2) \in Arr(c) \text{ and } (n,c) \in S_5\}))$$
$$\in (\mathbb{N}\backslash\{0\}) \times U'\}$$

For any $n \in \mathbb{N}\backslash\{0\}$, $For^{U(n)-\mathfrak{Cat}}_{U-\mathfrak{Cat}} : U(n) - \mathfrak{Cat} \longrightarrow U - \mathfrak{Cat}$ is given by $(C,h,\circ) \mapsto C$, the underlying category functor or the n-forgetful functor i.e. a pair $For^{U(n)-\mathfrak{Cat}}_{U-\mathfrak{Cat}} = For^{WE(F,id)}_{\mathfrak{Cat}}$. The underlying objects functor for n, $F_{U(n)-\mathfrak{Cat}} : U(n) - \mathfrak{Cat} \longrightarrow U - \mathfrak{Set}$ is given by $(C,h,\circ) \mapsto Ob(C)$.

(ii) The set S_2 consists of all pairs of the form $(n+1, As)$, where $n \in \mathbb{N}$ and As is a U'-function (set of pairs), consisting of all pairs of the form $((a,b,c),(F,F_2))$, where there exists some U'-category C such that $(n+1,C) \in S_5$ (i.e. C can be identified with the category of $n+1$-categories), $a = (a_0,h_a,\circ_a), b = (b_0,h_b,\circ_b), c = (c_0,h_c,\circ_c) \in Ob(C)$, and if $a',b',c' \in Ob(U - \mathfrak{Cat})$ are such that $(a,a'),(b,b'),(c,c') \in s$ and $(n+1,s) \in S_1$ for some $s \in U'$, (i.e. s provides an $(n+1)$-forgetful functor), then F_2 is a function which sends a pair of triples $((x_a,x_b),x_c),((y_a,y_b),y_c) \in Ob((a' \times b') \times c')$ to α'' iff there exists $\alpha' \in U'$ such that $(n,\alpha') \in S_2$ (i.e. α' provides an n-associator), and α'' is assigned by α' to the triple $(h_a(x_a,y_a), h_b(x_b,y_b), h_c(x_c,y_c))$.

$$S_2 = \{(n+1, \{((a,b,c),(F,F_2)) \in U';$$
$$\exists C \in Ob(U' - \mathfrak{Cat}), (n+1,C) \in S_5 \text{ and } a,b,c \in Ob(C) \text{ and}$$
$$\exists a',b',c' \in Ob(U - \mathfrak{Cat}), \exists s,$$
$$(n+1,s) \in S_1 \text{ and } (a,a'),(b,b'),(c,c') \in s \text{ and } F = \alpha(a',b',c') \text{ and}$$
$$\exists \alpha' \in U', (n,\alpha') \in S_2 \text{ and } \forall x_a,y_a \in Ob(a'), \forall x_b,y_b \in Ob(b'), \forall x_c,y_c \in Ob(c'), \forall \alpha'' \in U',$$
$$((((x_a,x_b),x_c),((y_a,y_b),y_c)),\alpha'') \in F_2 \text{ iff}$$
$$((h_a(x_a,y_a), h_b(x_b,y_b), h_c(x_c,y_c)),\alpha'') \in \alpha'\}) \in U'\}$$

The associator $\alpha_{U(n)-\mathfrak{Cat}} : Ob(U(n) - \mathfrak{Cat})^3 \to Arr(U(n) - \mathfrak{Cat})$ is inductively defined so that for any $n \in \mathbb{N}$ for which $\alpha_{U(n)-\mathfrak{Cat}}$ and $\times_{U(n+1)-\mathfrak{Cat}}$ are defined, for any $a,b,c \in Ob(U(n+1) - \mathfrak{Cat})$

$$\alpha_{U(n+1)-\mathfrak{Cat}}(a,b,c) : (a \times_{U(n+1)-\mathfrak{Cat}} b) \times_{U(n+1)-\mathfrak{Cat}} c \longrightarrow a \times_{U(n+1)-\mathfrak{Cat}} (b \times_{U(n+1)-\mathfrak{Cat}} c)$$

is given by the usual associator $\alpha_{U-\mathfrak{Cat}}$ on the underlying category

$$For^{U(n+1)-\mathfrak{Cat}}_{U-\mathfrak{Cat}}((a \times_{U(n+1)-\mathfrak{Cat}} b) \times_{U(n+1)-\mathfrak{Cat}} c) =$$

$$(For_{U-\mathfrak{Cat}}^{U(n+1)-\mathfrak{Cat}}(a) \times_{U-\mathfrak{Cat}} For_{U-\mathfrak{Cat}}^{U(n+1)-\mathfrak{Cat}}(b)) \times_{U-\mathfrak{Cat}} For_{U-\mathfrak{Cat}}^{U(n+1)-\mathfrak{Cat}}(c)$$

and

$$\alpha_{U(n)-\mathfrak{Cat}}(h_a(x_a, y_a), h_b(x_b, y_b), h_c(x_c, y_c))$$

on each of the hom n-categories.

(iii) The set S_3 is defined to consist of pairs $(n+1, (F_0, F_1))$, so that F_0 and F_1 are functions, defined so that (a). F_0 consists of pairs associating to the pair $((\bar{C} = (C, h_C, \circ_C), \bar{D} = (D, h_D, \circ_D))$, where there exists $Ca \in Ob(U' - \mathfrak{Cat})$, $(n+1, Ca) \in S_5$ (i.e. Ca is identified with the category of $n+1$-categories) such that $\bar{C}, \bar{D} \in Ob(Ca)$, the product category $C \times D$, enriched by taking the n-products of each hom object, and (b). F_1 associates to a pair $\Phi = (\phi, \phi_2), \Psi = (\psi, \psi_2) \in Arr(Ca)$ the product functors on the underlying categories, paired with the function which sends a pair of pairs $((x_c, x_d), (y_c, y_d))$ to the n-product of ϕ_2 and ψ_2.

$$S_3 = \{(n+1, (\{((\bar{C} = (C, h_C, \circ_C), \bar{D} = (D, h_D, \circ_D)),$$
$$(C \times_{U-\mathfrak{Cat}} D, \{(((a,b),(c,d)), \times(h_C(a,c), h_D(b,d))); \exists \times \in U', (n, \times) \in S_3\}, \circ));$$
$$\exists Ca \in Ob(U' - \mathfrak{Cat}), (n+1, Ca) \in S_5 \text{ and } \bar{C}, \bar{D} \in Ob(Ca)\}, \{Arrows\})) \in U'\}$$

The product $\times_{U(n+1)-\mathfrak{Cat}} : U(n+1) - \mathfrak{Cat} \times_{U'-\mathfrak{Cat}} U(n+1) - \mathfrak{Cat} \longrightarrow U(n+1) - \mathfrak{Cat}$ is defined, for any $n \in \mathbb{N}$ such that $\times_{U(n)-\mathfrak{Cat}}$, $\alpha_{U(n)-\mathfrak{Cat}}$, and $\sigma_{U(n)-\mathfrak{Cat}} : Ob(U(n) - \mathfrak{Cat})^2 \to Arr(U(n) - \mathfrak{Cat})$ are already defined so as to be given by

$$((a, h_a, \circ_a), (b, h_b, \circ_b)) \mapsto (a \times_{U-\mathfrak{Cat}} b, (((x_a, x_b), (y_a, y_b)) \mapsto h_a(x_a, y_a) \times_{U(n)-\mathfrak{Cat}} h_b(x_b, y_b)), ...)$$

with the composition defined component-wise (using the n-symmetrizer and associator maps).

(iv). The set S_4 is defined so as to consist of pairs of the form $(n+1, Sym)$, where Sym is a set consisting of pairs $((\bar{C}, \bar{D}), (F, F_2))$, where for some $Ca \in Ob(U' - \mathfrak{Cat})$ such that $(n+1, Ca) \in S_5$, $\bar{C}, \bar{D} \in Ob(Ca)$, F is the usual symmetrizer, and F_2 sends a pair of pairs $((x_c, x_d), (y_c, y_d))$ to the n-symmetrizer.

$$S_4 = \{(n+1, \{((\bar{C} = (C, h_C, \circ_C), \bar{D} = (D, h_D, \circ_D)), (F, F_2)); F = \sigma(C, D) \text{ and}$$
$$F_2 = \{(((x_c, x_d), (y_c, y_d)), \sigma') \in U'; \exists \sigma'' \in U', (n, \sigma'') \in S_4 \text{ and}$$
$$((h_{(C)}(x_c, y_c), h_D(x_d, y_d)), \sigma') \in \sigma''\}\}) \in U'; \exists Ca \in Ob(U' - \mathfrak{Cat}), (n+1, Ca) \in S_5 \text{ and}$$
$$\bar{C}, \bar{D} \in Ob(Ca)\}$$

The symmetrizer $\sigma_{U(n+1)-\mathfrak{Cat}} : Ob(U(n+1)-\mathfrak{Cat})^2 \to Arr(U(n+1)-\mathfrak{Cat})$ is defined for any $n \in \mathbb{N}$ for which $\sigma_{U(n)-\mathfrak{Cat}}$ and $\times_{U(n+1)-\mathfrak{Cat}}$ are defined, using the usual associator on the underlying categories and $\sigma_{U(n)-\mathfrak{Cat}}$ on the hom n-categories, so that

$$\sigma_{U(n+1)-\mathfrak{Cat}}(a,b) : a \times_{U(n+1)-\mathfrak{Cat}} b \longrightarrow b \times_{U(n+1)-\mathfrak{Cat}} a$$

for any $a, b \in Ob(U(n+1)-\mathfrak{Cat})$.

(v). The set S_5 consists of all pairs of the for $(n+1, D)$, where there exist $D' \in Ob(U' - \mathfrak{Cat})$, $F, \times \in U'$, for which $(n, D') \in S_5$, $F = For_{U(n)-\mathfrak{Cat}}$ is the underlying objects functor for n, $\times$ is the n-product (i.e. $(n, \times) \in S_3$), $\rho : (F \circ \times) \to \times_{U'-\mathfrak{Set}} \circ (F \times_{U'-\mathfrak{Cat}} F)$ is an isomorphism of functors, given by the identity on each set, and D is the category of weakly $(D', F, \rho, \times)$-enriched categories.

$$S_5 = \{(n+1, WE_{\mathfrak{Cat}}((Ca, F), \times)) \in U'; (n, Ca) \in S_5 \text{ and } (n, F) \in S_1 \text{ and } (n, \times) \in S_3\} \cup$$

$$\{(1, U - \mathfrak{Cat})\}$$

For any $n \in \mathbb{N}$ for which $U(n) - \mathfrak{Cat}$ and $\times_{U(n)-\mathfrak{Cat}}$ are defined, $U(n+1) - \mathfrak{Cat}$ is defined to be the category of all categories weakly enriched over $U(n) - \mathfrak{Cat}$, i.e. $U(n+1) - \mathfrak{Cat} := WE((U(n) - \mathfrak{Cat}, \times_{U(n)-\mathfrak{Cat}}), F_{U(n)-\mathfrak{Cat}})$.

A.2 Lemma

$F_{U()-\mathfrak{Cat}}$, $\alpha_{U()-\mathfrak{Cat}}$, $\times_{U()-\mathfrak{Cat}}$, $\sigma_{U()-\mathfrak{Cat}}$, and $U() - \mathfrak{Cat}$ are well defined functions $\mathbb{N} \longrightarrow U'$.

Proof. If $U(n+1) - \mathfrak{Cat}$ is uniquely determined, then $\alpha_{U(n+1)-\mathfrak{Cat}}$, $\times_{U(n+1)-\mathfrak{Cat}}$, and $\sigma_{U(n+1)-\mathfrak{Cat}}$ are defined directly from $\alpha_{U(n)-\mathfrak{Cat}}$, $\times_{U(n)-\mathfrak{Cat}}$, and $\sigma_{U(n)-\mathfrak{Cat}}$. $U(n+1) - \mathfrak{Cat}$ is defined from $\alpha_{U(n)-\mathfrak{Cat}}$, $\times_{U(n)-\mathfrak{Cat}}$, $\sigma_{U(n)-\mathfrak{Cat}}$, and $U(n) - \mathfrak{Cat}$. Apply induction. $F_{U(n)-\mathfrak{Cat}}$ is straightforward, using only $U(n) - \mathfrak{Cat}$ and the usual object functor. $\square$

A.3 Inducing Enrichments of $Hom^{(1)}_{\mathfrak{Cat}^2}(I, C)$ from Enrichments of C

The following is an older version of the hom set enrichment which appears in the main body. The exact relation should be worked out.

A.4 Lemma

(Hom-Cat Enrichment)

Suppose that the amnesia $F : A \to \mathfrak{Set}$ of a weak $(sk : A \to B)$-associative enrichment $(C, h, \circ)$ over $((A, \otimes), \alpha)$ is co-represented by a unit object

$$\Phi : F \longrightarrow Yo^{opp}_{(A)(0)}(I)$$

with unit arrows (natural transformations)

$$u_l : Id_A \longrightarrow I \otimes - \text{ and } u_r : Id_A \longrightarrow - \otimes I$$

Then for any category J one can construct a very weak enrichment of $Hom^{(1)}_{U-\mathfrak{Cat}^2}(J, C)$ over $(A, \otimes)$ by the following.

(i). Choose for each pair $G, H \in Ob(Hom^{(1)}_{U-\mathfrak{Cat}^2}(J, C))$ of functors, products $Dia :_t= \prod_{\phi \in Arr(J)} h(G(dom(\phi)), H(codom(\phi)))$ and $Hor :_t= \prod_{a \in Ob(J)} h(G(a), H(a))$.

(ii). Choose, for each $G, H \in Ob(Hom^{(1)}_{U-\mathfrak{Cat}^2}(J, C))$ arrows $f, g : Hor \to Dia$ in A, determined by the construction of Dia as a product, so that f is given by associating $\phi \in Arr(J)$ with

$$Hor \xrightarrow{\pi} h(G(dom(\phi)), H(dom(\phi))) \xrightarrow{u_l} I \otimes h(G(dom(\phi)), H(dom(\phi))) \xrightarrow{id \otimes \Phi(H(\phi))}$$
$$h(H(dom(\phi)), H(codom(\phi))) \otimes h(G(dom(\phi)), H(dom(\phi))) \xrightarrow{\circ}$$
$$h(G(dom(\phi)), H(codom(\phi)))$$

and g is given by associating $\phi \in Arr(J)$ with

$$Hom \xrightarrow{\pi} h(G(codom(\phi)), H(codom(\phi))) \xrightarrow{u_r} h(G(codom(\phi)), H(codom(\phi))) \otimes I \xrightarrow{\Phi(G(\phi)) \otimes id}$$
$$h(G(codom(\phi)), H(codom(\phi))) \otimes h(G(dom(\phi)), G(codom(\phi))) \xrightarrow{\circ}$$
$$h(G(dom(\phi)), H(codom(\phi)))$$

(iii). Let $l : Ob(Hom^{(1)}_{U-\mathfrak{Cat}^2}(J, C))^2 \to Ob(A)$ assign to each pair (F, G) the (sk, ε_0)-limit of the diagram $D(F, G)$ consisting of the two arrows f and g, where ε_0 is the inclusion of the sub-category of $dom(D(F, G))$ with the same objects, and only identity arrows.

(iv). Suppose that for each $G_1, G_2, G_3 \in Ob(Hom^{(1)}_{U-\mathfrak{Cat}^2}(J, C))$,

$$s(G_1, G_2, G_3) : colim(\otimes \circ (p_{D(G_2,G_3)} \times p_{D(G_1,G_2)})) \to colim(p_{D(G_2,G_3)}) \otimes colim(p_{D(G_1,G_2)})$$

is defined to be that which is induced by sending $((a, \alpha), (b, \beta)) \in Ob(P_{D(G_2,G_3)} \times P_{D(G_1,G_2)})$ to the tensor of its colimit arrows, $e_\alpha \otimes e_\beta : a \otimes b \to colim(p_{D(G_2,G_3)}) \otimes colim(p_{D(G_1,G_2)})$, and that $s(G_1, G_2, G_3)$ is an isomorphism.

(v). Suppose that for each $G_1, G_2 \in Ob(Hom^{(1)}_{U-\mathfrak{Cat}^2}(J, C))$, for any $(a, \alpha) \in Ob(P_{D(G_1,G_2)})$, the colimit arrow $\lambda_{p_{D(G_1,G_2)}}(a, \alpha) : a \to colimit(p_{D(G_1,G_2)})$ is the unique arrow for which $\pi_i \circ \lambda_\pi \circ \lambda_{p_{D(G_1,G_2)}}(a, \alpha) = \alpha(i)$ for each $i \in Ob(dom(D(G_1, G_2)))$, where

$$\lambda_\pi : colim(p_{D(G_1,G_2)}) \to \prod_{i \in Ob(dom(D(G_1,G_2)))} D(G_1, G_2)(i)$$

is induced by the arrows $\alpha(i)$ for $(a, \alpha) \in Ob(P_{D(G_1,G_2)})$.[3]

(vi). Define a function $t : Ob(Hom^{(1)}_{U-\mathfrak{Cat}^2}(J, C))^3 \to Arr(A)$ so that for any $G_1, G_2, G_3 \in Ob(Hom^{(1)}_{-\mathfrak{Cat}^2}(J, C))$,

$$t(G_1, G_2, G_3) : colim(\otimes \circ (p_{D(G_2,G_3)} \times p_{D(G_1,G_2)})) \to colim(p_{D(G_1,G_3)})$$

is the colimit arrow assigned to $(colim(\otimes \circ (p_{D(G_2,G_3)} \times p_{D(G_1,G_2)}), \alpha) \in Ob(P_{D(G_1,G_3)})$, where if $Ob(dom(D(G_1, G_2))) = Ob(dom(D(G_2, G_3))) = Ob(dom(D(G_1, G_3))) = \{c, d, m\}$ so that

$$c \text{ corresponds to } \prod_{\phi \in Arr(J)} h(G_i(codom(\phi)), G_j(codom(\phi))) \text{ and}$$

$$d \text{ corresponds to } \prod_{\phi \in Arr(J)} h(G_i(dom(\phi)), G_j(dom(\phi))) \text{ and}$$

$$m \text{ corresponds to } \prod_{\phi \in Arr(J)} H(G_i(dom(\phi)), G_j(codom(\phi))),$$

then $\alpha : \Delta_{(dom(\varepsilon_0),A)}(colim(\otimes \circ (p_{D(G_2,G_3)} \times p_{D(G_1,G_2)})) \to D(G_1, G_3) \circ \varepsilon_0$ is given by

$$\alpha(c) : colim(\otimes \circ (\otimes \circ (p_{D(G_2,G_3)} \times p_{D(G_1,G_2)})) \to \prod_{\phi \in Arr(J)} h(G_1(codom(\phi)), G_3(codom(\phi)))$$

corresponding to the assignment $(\phi \mapsto \circ \circ \otimes((\pi_\phi \circ \beta(c), \pi_\phi \circ \gamma(c))))_{\phi \in Arr(J)}$ for any $(b, \beta) \in Ob(P_{D(G_2,G_3)})$ and any $(c, \gamma) \in Ob(P_{D(G_1,G_2)})$,

$$\alpha(d) : colim(\otimes \circ (\otimes \circ (p_{D(G_2,G_3)} \times p_{D(G_1,G_2)})) \to \prod_{\phi \in Arr(J)} h(G_1(dom(\phi)), G_3(dom(\phi)))$$

corresponding to the assignment $(\phi \mapsto \circ \circ \otimes((\pi_\phi \circ \beta(d), \pi_\phi \circ \gamma(d))))_{\phi \in Arr(J)}$ for any $(b, \beta) \in Ob(P_{D(G_2,G_3)})$ and any $(c, \gamma) \in Ob(P_{D(G_1,G_2)})$,

[3]This condition is as in the conclusion of the uniqueness via monic lemma of II.1

and any

$$\alpha(m) : colim(\otimes \circ (\otimes \circ (p_{D(G_2,G_3)} \times p_{D(G_1,G_2)})) \to \prod_{\phi \in Arr(J)} h(G_1(dom(\phi)), G_3(codom(\phi)))$$

for which $(colim(\otimes \circ (p_{D(G_2,G_3)} \times p_{D(G_1,G_2)}), \alpha) \in Ob(P_{D(G_1,G_3)})$. (Herein is a choice required in the construction)

(vii). Then $\bar{\circ} :_t= ((G_1, G_2, G_3) \mapsto t(G_1, G_2, G_3) \circ s(G_1, G_2, G_3)^{-1})_{G_1,G_2,G_3 \in Ob(Hom^{(1)}_{U-\mathfrak{Cat}^2}(J,C))}$ implies that $(Ob(Hom^{(1)}_{U-\mathfrak{Cat}^2}(J, C)), l, \bar{\circ})$ is a very weak A-enrichment.

(viii). If the enrichment is F-associative, then the above construction yields a weak enrichment over the category whose objects are functors $J \longrightarrow C$ and arrows are $\coprod_{G_1,G_2 \in Hom_{U-\mathfrak{Cat}}(J,C)} F_{(0)}(\lambda(G_1, G_2))$, with composition given by application of $F_{(1)}$ to the enriched composition arrows in A.

A.5 Example

In the case in which $C = U - \mathfrak{Cat}$ and $sk = Skel$, if F is the object functor, then the underlying category in the second part of the lemma would have as objects the functors and arrows weak natural transformations, for which $G_2(\phi) \circ f_{dom(\phi)} \cong f_{codom(\phi)} \circ G_1(\phi)$.

A.6 Lemma

(Induced Enrichments on Categories of Diagrams in $(A, \otimes) - \mathfrak{Cat}$, given $sk : A \to B$). If the amnesia is representable, and there is an isomorphism $\times_{U-\mathfrak{Set}} \circ (F \times F) \longrightarrow F \circ \otimes$ (i.e. the amnesia can made into a "strong" arrow in $\mathfrak{TCat}$), then $(A, \otimes) - \mathfrak{Cat}$, has a natural weak enrichment over itself.

A.7 Corollary

Auto-enrichment of $(\mathfrak{Ab}, \otimes) - \mathfrak{Cat}$.

A.8 Lemma

A full functor p whose domain is weakly enriched over some $(A, \otimes)$ induces a weak enrichment over the same on the codomain. ... ...

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