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Automata and cells in affine Weyl groups

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AUTOMATA AND CELLS IN AFFINE WEYL GROUPS

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ABSTRACT. Let $\widetilde{W}$ be an affine Weyl group, and let C be a left, right, or two-sided Kazhdan–Lusztig cell in $\widetilde{W}$. Let $\mathrm{Red}(C)$ be the set of all reduced expressions of elements of C, regarded as a formal language in the sense of the theory of computation. We show that $\mathrm{Red}(C)$ is a regular language. Hence the reduced expressions of the elements in any Kazhdan–Lusztig cell can be enumerated by a finite state automaton.

1. INTRODUCTION

1.1. Let W be a Coxeter group with generating set S. In their work on Coxeter groups and Hecke algebras, Kazhdan and Lusztig defined a partition of W into sets called *cells*. When W is a Weyl or affine Weyl group, it is known that cells have deep connections with many areas of algebra and geometry, such as singularities of Schubert varieties [20], representations of p-adic groups [22], characters of finite groups of Lie type [23], and the geometry of unipotent conjugacy classes in simple complex algebraic groups [4, 5].

The definition of cells is quite complicated (§2.2). It involves the construction of a subtle equivalence relation on W built from both easy and difficult combinatorial data. In particular from the definition it is not clear how "computable" cells are. For instance, it is highly nontrivial to decide whether two elements of W lie in the same cell or not, or to characterize all elements in a given cell. Nevertheless, in all known examples where cells have been explicitly computed, one sees that cells ultimately have a relatively simple geometric and combinatorial structure. We refer to [9, 16] for examples and further discussion of this phenomemon.

1.2. This paper addresses the following computational problem: given a cell C in a Coxeter group W, how can we encode the (typically) infinite amount of data represented by C with a finite structure? In this generality, this question was first considered by Casselman, who phrased an answer in terms of *finite state automata* (§2.4). More precisely, let $\mathrm{Red}(W)$ be the set of all reduced expressions of all elements of W, considered as a subset of the free monoid on the generating set S. We regard

$\mathrm{Red}(W)$ as a formal languange in the sense of the theory of computation. Given a cell C, let $\mathrm{Red}(C) \subset \mathrm{Red}(W)$ be the set of all reduced expressions of all $w \in C$. Then we have the following conjecture of Casselman:

1.3. **Conjecture.** *For any Coxeter group W and any cell $C \subset W$, there exists a finite state automaton accepting the language $\mathrm{Red}(C)$. That is, the language $\mathrm{Red}(C)$ is regular.*

The precise definitions of the terms in Conjecture 1.3 are given in §2.4; here we give an informal sense of what the conjecture means.

A finite state automaton is a simple theoretical model of a computer. It has finite memory and can complete only one task: acceptance/rejection of its input. More precisely, given a word $a_1 \cdots a_k$ on some alphabet, an automaton reads the word from left to right, and while doing so moves through finitely many memory states. After reading the word, the automaton decides based on which state it occupies whether or not to accept the word or throw it away. A language is called regular if one can find a finite state automaton accepting exactly the words in the language.

Thus Casselman's conjecture implies that given any cell C, there exists a simple machine $\mathscr{A}(C)$ that decides whether or not $w \in W$ lies in C simply by reading through a reduced expression $s_1 \cdots s_k$ for w. The finiteness of the automaton implies that the decision is only based on finitely many patterns appearing in the expression. One can also use $\mathscr{A}(C)$ to systematically list all reduced expressions of all elements of C.

We remark that work of Brink–Howlett [8] shows that the language $\mathrm{Red}(W)$ of all reduced expressions of all elements of W is regular, although this does not prove Conjecture 1.3: a sublanguage of a regular language need not be regular. Indeed, it is not even clear how one can use the tools underlying the fundamental results of Brink–Howlett to attack Conjecture 1.3. Moreover, Conjecture 1.3 does not specify what structures in W should be used to build the machines $\mathscr{A}(C)$.

Casselman's conjecture is trivially true for any finite Coxeter group, in particular for Weyl groups, since for such groups the language $\mathrm{Red}(W)$ is obviously finite. The first infinite example of Conjecture 1.3 follows from work of Shi [27] on Kazhdan–Lusztig cells and Eriksson [14] and Headley [17] on automata. More precisely, let $W = \widetilde{A}_n$, the affine Weyl group of type A. Shi showed that W can be partitioned into finitely many geometrically defined subsets, called *sign-type regions*, such that each left cell C of W is a union of finitely many such regions. Headley showed that the sign-type regions can be used as a set of states for an automaton $\mathscr{A}$ recognizing $\mathrm{Red}(W)$. Together these results imply Conjecture 1.3 for $\widetilde{A}_n$, since given C one can modify $\mathscr{A}$ to only accept the reduced expressions corresponding to elements of C.

1.4.   In this paper we prove Conjecture 1.3 when the Coxeter group is an affine Weyl group $\widetilde{W}$ (Theorem 4.6). The proof uses two ingredients.

The first is a family of finite state automata $\mathscr{A}_N$, $N \in \mathbb{Z}_{\geq 0}$, each of which recognizes $\mathtt{Red}(\widetilde{W})$. The construction generalizes work of Eriksson [14] and Headley [17]. Each $\mathscr{A}_N$ is built from the complement of a certain affine hyperplane arrangement $\mathscr{H}_N$.

The second is a result of Du [13], who proved that each left cell C of $\widetilde{W}$ can be represented as the union of a finite set of convex polyhedra of a certain type. We show that if $N \gg 0$, then we can write each of these polyhedra as a finite union of regions in the complement of $\mathscr{H}_N$. This allows us to define $\mathscr{A}(C)$ by identifying the set of states of $\mathscr{A}_N$ that correspond exactly to the reduced expressions of elements of C.

2. Background

2.1. In this section we recall background and standard notation. For more details and proofs we refer to [6, 18] for Coxeter groups, [6, 19] for Kazhdan–Lusztig cells, and [1] for automata and formal languages.

Let Φ be an irreducible, reduced root system, and let W be the associated Weyl group. Decompose Φ into a union of positive and negative roots $\Phi^+ \cup \Phi^-$. Let $\Delta \subset \Phi^+$ be the simple roots and let $S \subset W$ be the corresponding subset of generators in the presentation of W as a Coxeter group. For each $s \in S$, we write $\alpha_s \in \Delta$ for the associated simple root. Conversely, given a simple root α, we write $s_\alpha \in S$ for the associated generator.

We assume Φ spans a real vector space V equipped with a W-invariant inner product $(\ ,\ )$. The group W acts on V as usual: if $\alpha \in \Phi$, then we have the reflection

$$v \longmapsto v - \frac{2(v, \alpha)}{(\alpha, \alpha)}\alpha,$$

which maps W faithfully onto a subgroup of $\mathrm{GL}(V)$. We will write this reflection action as a *right* action: $v \mapsto v \cdot s_\alpha$. Let $\mathscr{C}^+$ be the positive Weyl chamber determined by Δ.

The root system Φ has a unique highest root $\widetilde{\alpha}$. By definition $\widetilde{\alpha}$ has the property that for any $\beta \in \Phi$, the difference $\widetilde{\alpha} - \beta$ can be written as a nonnegative linear combination of the simple roots. One also knows [7, IV, §1.8, Prop. 25(iv)] that for any positive root β, we have

$$\frac{2(\beta, \widetilde{\alpha})}{(\widetilde{\alpha}, \widetilde{\alpha})} \in \{0, 1\}.$$

Let $\alpha \in \Phi$ and $k \in \mathbb{Z}$. Let $H_{\alpha,k} \subset V$ be the affine hyperplane

(1) $$H_{\alpha,k} = \{v \in V \mid (\alpha, v) = k\},$$

and let $H^1_{\alpha,k}$ be the subset

(2) $$H^1_{\alpha,k} = \{v \in V \mid k \leq (\alpha, v) \leq k + 1\}.$$

Attached to the hyperplane $H_{\alpha,k}$ is the affine reflection $s_{\alpha,k}$, which acts on V by

$$s_{\alpha,k} \colon v \longmapsto v - 2((\alpha, v) - k)\alpha/(\alpha, \alpha).$$

The affine Weyl group $\widetilde{W}$ corresponding to Φ is the group generated by all the affine reflections $s_{\alpha,k}$. We can represent $\widetilde{W}$ as a finitely generated Coxeter group by using the generating set $\widetilde{S} = S \cup \{s_{\widetilde{\alpha},1}\}$, where we identify $s_\alpha \in S$ with $s_{\alpha,0}$.

Let $\mathscr{H}$ be the affine hyperplane arrangement consisting of all affine hyperplanes of the form (1). The connected components of $V \smallsetminus \mathscr{H}$ are called *alcoves*. There is a distinguished alcove A_0 defined by

$$A_0 = \{v \mid 0 < (\alpha, v) < 1 \quad \text{for all } \alpha \in \Phi^+\},$$

and $w \mapsto A_0 \cdot w$ gives a bijection between $\widetilde{W}$ and the set of alcoves. We often identify alcoves and elements of $\widetilde{W}$ under this bijection.

Any $w \in \widetilde{W}$ determines a function $b_w \colon \Phi^+ \to \mathbb{Z}$ as follows. The closure of the alcove $A_0 \cdot w$ can be uniquely written as the intersection of subsets of the form (2):

$$\overline{A_0 \cdot w} = \bigcap_{\alpha \in \Phi^+} H^1_{\alpha, k(\alpha)}.$$

We put $b_w(\alpha) = k(\alpha) \in \mathbb{Z}$.

An *expression* for $w \in \widetilde{W}$ is a representation of w as a product of elements of $\widetilde{S}$. An expression is *reduced* if it has minimal length among all expressions for w. We define the length $\ell(w)$ of w to be the length of a reduced expression for w.

Given an expression $s_1 \cdots s_N$, a *subexpression* is a (possibly empty) expression of the form $s_{i_1} \cdots s_{i_M}$, where $1 \leq i_1 < \cdots < i_M \leq N$. We endow $\widetilde{W}$ with a partial order by defining $u \leq w$ if an expression for u appears as a subexpression of a reduced expression for w.

2.2. Next we describe Kazhdan–Lusztig cells. Given $w \in \widetilde{W}$, we define the *left descent set* $L(w) \subset \widetilde{S}$ by

$$L(w) = \{s \in \widetilde{S} \mid \ell(sw) < \ell(w)\},$$

and analogously define the *right descent set* $R(w)$ by the condition $\ell(ws) < \ell(w)$.

Let $u, w \in \widetilde{W}$, and let $P_{u,w}(t) \in \mathbb{Z}[t]$ be the *Kazhdan–Lusztig polynomial* attached to the pair (u, w) [19]. We do not recall the definition here, but only mention the following properties:

(1) $P_{u,w} = 0$ unless $u \leq w$,
(2) $P_{u,u} = 1$, and
(3) $P_{u,w}$ has degree at most $d(u,w) := (\ell(w) - \ell(u) - 1)/2$.

We write u—w if $u < w$ and $\deg P_{u,w} = d(u,w)$. If $w < u$ we write u—w if w—u holds.

We are now ready to define the cells of $\widetilde{W}$. The *left W-graph* Γ_L of $\widetilde{W}$ is the directed graph with vertex set $\widetilde{W}$, and with an arrow from u to w if and only if u—w and $L(u) \not\subset L(w)$. We can similarly define the *right W-graph* Γ_R. Then the left and right cells of $\widetilde{W}$ are extracted from the graphs Γ_L, Γ_R as follows. Recall that given any directed graph, we say two vertices are in the same *strong connected component* if there exist directed paths from each vertex to the other.

2.3. **Definition.** The *left cells* (respectively, *right cells*) of $\widetilde{W}$ are the strong connected components of the graph Γ_L (resp., Γ_R). The elements $u, w \in \widetilde{W}$ are in the same *two-sided cell* if they are in the same left *or* right cell.

It is known that each affine Weyl group has only finitely many two-sided cells, and that each two-sided cell is a union of finitely many left cells [25]. We remark that in general the graphs Γ_L, Γ_R are extremely complicated. Figures 1–3 show the subgraph of Γ_L corresponding to a particular left cell of $\widetilde{G}_2$. The labels of the figure show the length difference $\ell = \ell(w) - \ell(u)$ of the words connected by edges in the graph; we have omitted the arrowheads for clarity.

2.4. Finally we discuss automata. Let A be a finite set, and let A^* be the free monoid generated by A under the operation of concatenation. Thus A^* consists of all sequences $a_1 \cdots a_k$, where $a_i \in A$, together with the empty sequence. A *formal language* L is any subset of A^*. We call A the *alphabet* for the language L. Among all formal languages, our interest lies in the regular languages, which are defined using finite state automata:

2.5. **Definition.** A *finite state automaton* $\mathscr{A}$ over the alphabet A consists of the following data:
   (1) a finite set Q, called the set of *states*,
   (2) a unique *initial state* $q_0 \in Q$,
   (3) a subset $F \subset Q$, called the set of *accepting states*, and
   (4) a function $t\colon Q \times A \to Q \cup \{\varnothing\}$, called the *transition function*.

A word $a_1 \cdots a_k \in A^*$ is said to be *accepted* by $\mathscr{A}$ if there exist states $q_0, \ldots, q_k$ such that $t(q_i, a_{i+1}) = q_{i+1}$ for $0 \leq i \leq k - 1$, and $q_k \in F$. A language L is *regular* if there exists a finite state automaton accepting exactly the words in L.

We may picture $\mathscr{A}$ as a decorated directed graph as follows. We label the vertices of the graph by Q, and circle the vertices corresponding to the final states. We put an edge from q to q' labelled by $a \in A$ if $t(q, a) = q'$. A path in this graph starting
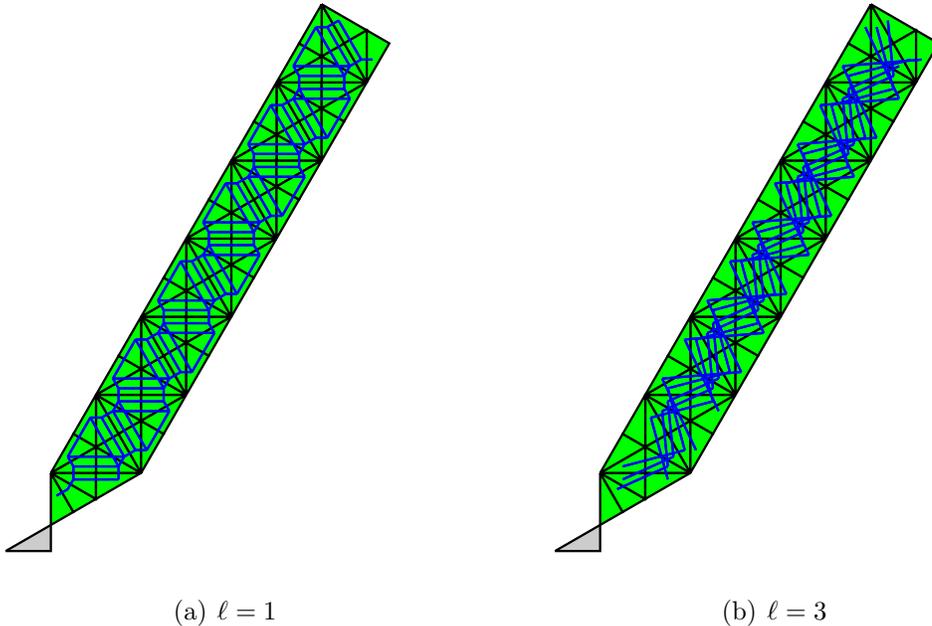
(a) $\ell = 1$ (b) $\ell = 3$

FIGURE 1.

at q_0 determines a word in A^* by concatenating the edge labels as one traverses the path. A word is accepted by $\mathscr{A}$ if and only if one can find a path building the word that ends in an accepting state.

3. THE AUTOMATON

3.1. Let $\mathrm{Red}(\widetilde{W})$ be the set of all reduced expressions for all elements $w \in \widetilde{W}$, regarded as a language on the alphabet $\widetilde{S}$. More generally, if U is any subset of $\widetilde{W}$, we let $\mathrm{Red}(U)$ be the set of all reduced expressions for all $w \in U$.

By work of Brink–Howlett [8], it is known that $\mathrm{Red}(\widetilde{W})$ is regular.[1] Our goal is to show the regularity of the language $\mathrm{Red}(C)$ for any cell of $\widetilde{W}$. To do this we generalize an automaton for affine Weyl groups first described by Eriksson [14] and Headley [17].

3.2. Let $\mathscr{H} \subset V$ be a finite affine hyperplane arrangement, and let $\mathscr{R}$ be the set of connected components of $V \smallsetminus \mathscr{H}$. We will always assume that $\mathscr{H}$ contains the hyperplanes $H_{\widetilde{\alpha},1}$ and $H_{\alpha,0}$, $\alpha \in \Delta$, which we denote by $\{H_s \mid s \in \widetilde{S}\}$. We say the set of regions $\mathscr{R}$ has *property* $(*)$ if it satisfies the following condition:

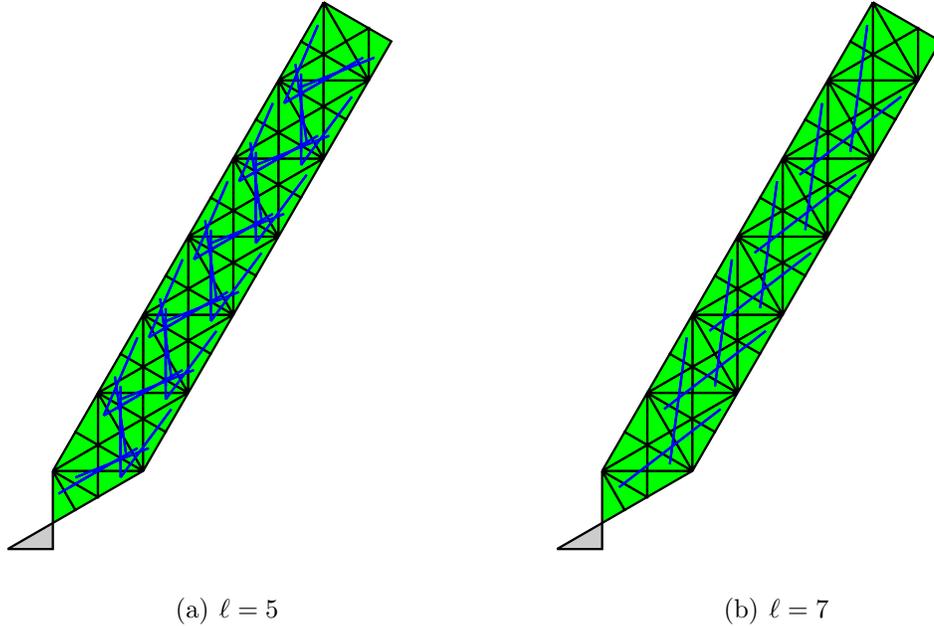[1]In fact [8] shows that the language of reduced expressions is regular for any Coxeter group.

(a) $\ell = 5$　　　　　　(b) $\ell = 7$

FIGURE 2.

> If $R \in \mathscr{R}$ and A_0 lie on the same side of H_s, then there is a unique $R' \in \mathscr{R}$ such that $R \cdot s \subset R'$.

The following result explains the connection between property $(*)$ and automata:

3.3. **Proposition.** *Suppose the set of connected components $\mathscr{R}$ of a finite affine hyperplane arrangement $\mathscr{H}$ satisfies $(*)$. Then there is a finite state automaton $\mathscr{A}$ accepting* $\mathrm{Red}(\widetilde{W})$ *with states given by $\mathscr{R}$.*

Proof. We define $\mathscr{A}$ by taking its states to be $\mathscr{R}$ and its initial state to be A_0; note that $A_0 \in \mathscr{R}$ since $\mathscr{H}$ contains $\{H_s \mid s \in \widetilde{S}\}$. We declare all states to be accepting. We define the transition function t by $t(R, s) = \varnothing$ unless A_0 and R lie on the same side of H_s, in which case we put $t(R, s) = R'$, where R' is the unique region with $R \cdot s \subset R'$.

The proof that $\mathscr{A}$ accepts $\mathrm{Red}(\widetilde{W})$ is essentially the same as that for Theorem V.6 in [17]. For the convenience of the reader, we give the details. First, it is clear that the unique expression for the identity in $\widetilde{W}$ is accepted. Also all expressions of length 1 are accepted, and such expressions are automatically reduced.

Now suppose that the reduced expression $w = s_k \cdots s_1$ is accepted and let $s \in \widetilde{S}$. We must show that $s_k \cdots s_1 s$ is accepted if and only if it is reduced. Recall that
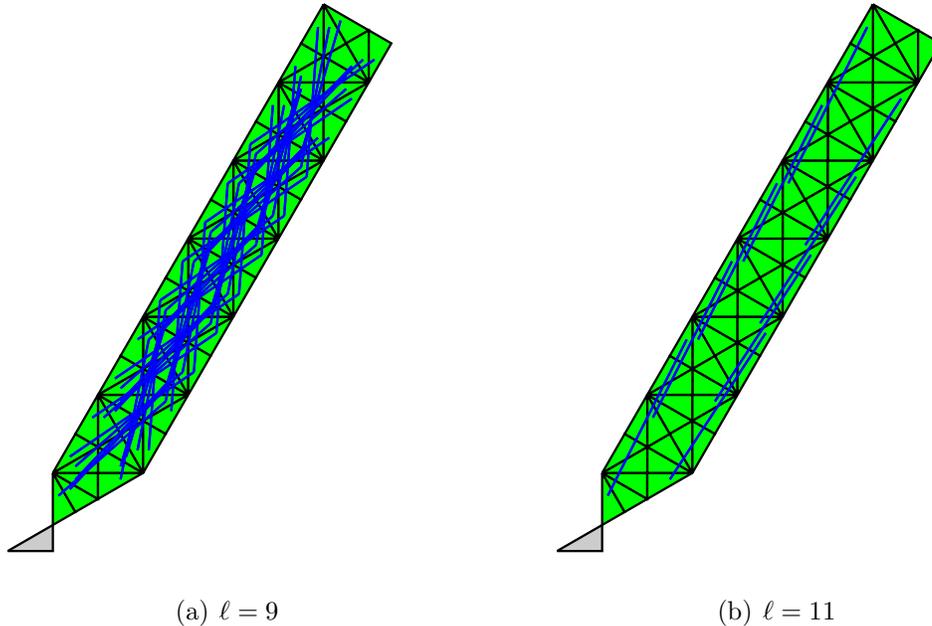
(a) $\ell = 9$ (b) $\ell = 11$

FIGURE 3.

$s_k \cdots s_1$ is reduced if and only if the hyperplanes

(3) $H_1, H_2 \cdot s_1, H_3 \cdot s_2 s_1, \ldots, H_k \cdot s_{k-1} \cdots s_1$

are distinct and separate A_0 from $A_0 \cdot w$ (cf. [18, §4.5]; here we abbreviate H_{s_i} by H_i). Thus if ws is reduced, then H_s must separate A_0 from $A_0 \cdot ws$. This means H_s does not separate A_0 from $A_0 \cdot w$, which implies $s_k \cdots s_1 s$ is accepted by $\mathscr{A}$.

Conversely, suppose $s_k \cdots s_1 s$ is accepted by $\mathscr{A}$. Then H_s does not separate A_0 from $A_0 \cdot w$, which means H_s separates A_0 from $A_0 \cdot ws$. Certainly the hyperplanes obtained by applying s (on the right) to (3) are distinct, and none equal H_s. Moreover if some hyperplane $H = H_j \cdot s_{j-1} \cdots s_1 s$ from this new list fails to separate A_0 from $A_0 \cdot ws$, then $H \cdot s$ fails to separate A_0 from $A_0 \cdot ws$, and thus cannot be part of (3). This completes the proof. $\square$

The automaton from Proposition 3.3 also satisfies the following property: if an expression $w = s_1 \cdots s_k$ is reduced and is accepted by the state R, then the alcove $A_0 \cdot w$ lies in the region R.

3.4.   Now let $N \geq 0$ be an integer, and let $\mathscr{H}_N$ be the arrangement

$$\mathscr{H}_N = \{H_{\alpha,k} \mid \alpha \in \Phi^+, k = -N, \cdots N+1\}.$$

Let $\mathscr{R}_N = V \smallsetminus \mathscr{H}_N$. Then we have the following theorem:

3.5. **Theorem.** *The set of regions $\mathscr{R}_N$ has the property $(*)$, and thus the automaton built from $\mathscr{R}_N$ using Proposition 3.3 accepts the language $\mathtt{Red}(\widetilde{W})$.*

Proof. The proof is a generalization of [17, Lemma V.5]. For each $\alpha \in \Phi^+$, and for $k = -N-1, \ldots, N+1$ define "root strips" by

$$R_\alpha^k = \begin{cases} \{v \mid N+1 < (\alpha, v)\} & k = N+1, \\ \{v \mid (\alpha, v) < -N\} & k = -N-1, \\ \{v \mid k < (\alpha, v) < k+1\} & \text{otherwise.} \end{cases}$$

The elements of $\mathscr{R}_N$ are just the connected components of all possible intersections of the R_α^k.

Let $R \in \mathscr{R}_N$, and suppose R and A_0 lie on the same side of H_s. We must show that $R \cdot s$ is contained in a unique element of $\mathscr{R}_N$. By the above discussion, we know that for each $\alpha \in \Phi^+$, we have $R \subset R_\alpha^k$ for some k depending on α. We must show that $R \cdot s \subset R_\beta^l$ for some $\beta \in \Phi^+$ and some l.

First assume s is not the affine reflection $s_{\widetilde{\alpha},1}$. Then either $\alpha = \alpha_s$ or $\alpha \neq \alpha_s$. In the first case, if $R \subset R_\alpha^k$ then we must have $k \geq 0$, and it is clear that $R \cdot s \subset R_\alpha^{-k-1}$. Note that the hypothesis that R and A_0 lie on the same side of H_s is essential, since $R_\alpha^{-N-1} \cdot s$ meets both R_α^N and R_α^{N+1}.

Next suppose $\alpha \neq \alpha_s$. Then if $k \neq N+1, -N-1$, we have

$$R_\alpha^k \cdot s = \{v \mid k < (\alpha \cdot s, v) < k+1\}$$
$$= R_{\alpha \cdot s}^k.$$

Moreover, $\alpha \cdot s \in \Phi^+ \smallsetminus \{\alpha_s\}$, since the only positive root s makes negative is α_s. Hence in this case $R \cdot s$ is taken into a unique region of $\mathscr{R}_N$ if s is not the affine reflection; if $k = N+1$ or $-N-1$ we argue similarly.

Finally assume s is the affine reflection, and suppose $R \subset R_\alpha^k$. Again there are two possibilities to consider. If α is the highest root $\widetilde{\alpha}$, then since R lies on the same side of H_s as A_0, we must have $k \leq 0$. Therefore $R \cdot s \subset R_\alpha^{-k+1}$. On the other hand, if α is not the highest root, then $c = 2(\alpha, \widetilde{\alpha})/(\widetilde{\alpha}, \widetilde{\alpha})$ either equals 0 or 1. If $c = 0$, then $s_{\widetilde{\alpha},1}$ stabilizes all the sets R_α^k. If $c = 1$, then $s_{\widetilde{\alpha},1}(\alpha) = \alpha - \widetilde{\alpha}$, which is negative. We have for $k \neq N+1$,

$$R_\alpha^k \cdot s = \{v \mid k < (\alpha - \widetilde{\alpha}, v) < k+1\}$$
$$= \{v \mid -k-1 < (\alpha', v) < -k\}$$
$$= R_{\alpha'}^{-k-1},$$

where $\alpha' \in \Phi^+$ is the root $\widetilde{\alpha} - \alpha$. If $k = N+1$, then $R_\alpha^{N+1} \cdot s \subset R_{\alpha'}^{-N-1}$. This completes the proof. $\qquad\square$

3.6. *Remark.* Recall that Φ is *simply-laced* if all roots have the same length. Suppose Φ is not simply-laced, and partition the positive roots Φ^+ into the long and short

roots $\Phi_l^+ \cup \Phi_s^+$. Choose a function $\nu\colon \Phi^+ \to \mathbb{Z}_{\geq 0}$ that is constant on Φ_l^+, Φ_s^+, and consider the affine arrangement $\mathscr{H}_\nu$ consisting of the hyperplanes

(4) $$\mathscr{H}_\nu = \{H_{\alpha,k} \mid k = -\nu(\alpha), \ldots, \nu(\alpha) + 1\}.$$

Then the proof of Theorem 3.5 actually shows that the set of regions of the complement $V \smallsetminus \mathscr{H}_\nu$ has property $(*)$ as well, and thus can be used as the set of states for an automaton recognizing $\mathrm{Red}(\widetilde{W})$. Indeed, the only observations one needs to make are that the action of W preserves root lengths, and that if $2(\alpha, \widetilde{\alpha})/(\widetilde{\alpha}, \widetilde{\alpha}) = 1$ then $\alpha - \widetilde{\alpha}$ has the same length as α.

3.7. *Remark.* The arrangement $\mathscr{H}_N$ is known in the combinatorics literature as the *extended Shi arrangement* [26]. The case $N = 0$ was studied by Shi [28], who called them *sign-type regions*. Figure 4 shows these regions and the alcoves for $\widetilde{A}_2$. Shi showed that the number of regions in $\mathscr{R}_0$ equals $(h + 1)^r$, where h is the Coxeter number of Φ and r is the rank of Φ, and also used sign-type regions to explicitly describe the Kazhdan–Lusztig cells for $\widetilde{A}_n$ [27]. For the extended Shi arrangement, Athanasiadis [2] showed that the number of regions in $\mathscr{R}_{N-1}$ is $(Nh + 1)^r$ if Φ is of classical type, i.e. type A, B, C, D.
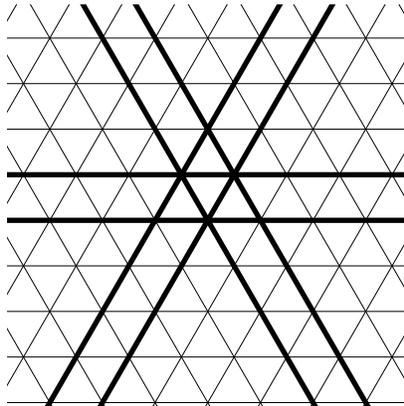

FIGURE 4. The regions in $\mathscr{R}_0$ for $\widetilde{A}_2$

4. KAZHDAN-LUSZTIG CELLS AND AUTOMATA

4.1. In this section we prove Theorem 4.6. The main idea of the proof is to show that we can choose N sufficiently large such that for any left cell C, we can find a finite set of regions $\{R_i\} \subset \mathscr{R}_N$ such that the alcoves in the union of the R_i are exactly those in C. The automaton accepting $\mathrm{Red}(C)$ is then given by modifying the automaton from Theorem 3.5 to make only the states corresponding to the R_i accepting.

Figure 5 shows an example for $\widetilde{W} = \widetilde{G}_2$. The dark lines are the hyperplanes in $\mathscr{H}_1$; each region of a given color is a two-sided cell, and the connected regions of a given color are the left cells. Note that we have not drawn the alcoves. It is clear from the figure that any left cell is a union of regions from $\mathscr{R}_1$. Note that $N = 1$ is the smallest value we can take for this to work; in particular left cells in $\widetilde{G}_2$ are not unions of sign-type regions.
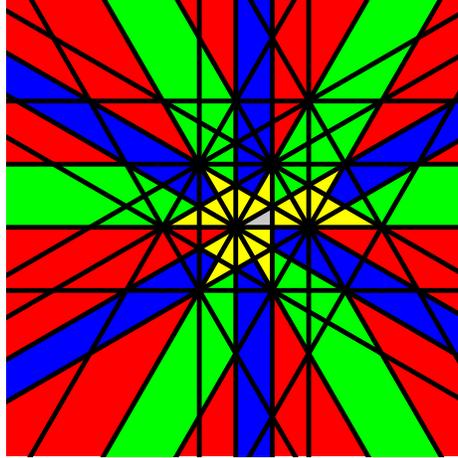

FIGURE 5. The left cells of $\widetilde{G}_2$ and the regions in $\mathscr{R}_1$

4.2.   The main tool we need to carry out the proof is a result of Du [13], who showed that left cells are finite unions of certain polyhedra in V. More precisely, let $\mathscr{P}$ be the set of all polyhedra in V of full dimension bounded by finitely many affine hyperplanes of the form $H_{\alpha,k}$, $\alpha \in \Phi^+$, $k \in \mathbb{Z}$. Then we have the following theorem:

4.3. **Theorem.** [13] *Let C be a left cell of $\widetilde{W}$. Identify C with the closure of the set of alcoves $\{A_0 \cdot w \mid w \in C\}$ in V. Then there exists a finite subset $\mathscr{P}(C) \subset \mathscr{P}$ such that*

$$C = \bigcup_{P \in \mathscr{P}(C)} P.$$

We give some indications of the proof of Theorem 4.3. Let $P \in \mathscr{P}$. Assume there exists a Weyl chamber $\mathscr{C}$ containing P. Let $\Delta' \subset \Phi$ be the simple system determined by $\mathscr{C}$ (so that all $\alpha \in \Delta'$ are nonnegative on $\mathscr{C}$). Then the dimension $\dim P$ of P is defined to be the cardinality of the set[2]

$$\{\alpha \in \Delta' \mid \alpha \text{ is unbounded on } P\}.$$

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[2]We remark that there is a typo in [13], in which "unbounded" is replaced by "bounded" in the definition.

Du defines a certain class of polyhedra in $\mathscr{P}$ called *special polyhedra* as follows. Recall (2) that we have defined the subsets $H^1_{\alpha,k}$. Fixing $\mathscr{C}$, we let $H^+_{\alpha,k}$ be the closure of the unique component of $V \smallsetminus H_{\alpha,k}$ that meets any translate of $\mathscr{C}$. Given Δ' as above we choose a function $b\colon \Delta' \to \mathbb{Z}_{\geq 0}$ and a subset $\Lambda \subset \Delta'$. Let $\overline{\Lambda} = \Delta' \smallsetminus \Lambda$. Then the special polyhedron $P(\Lambda, b)$ is defined by

$$P(\Lambda, b) = \Big(\bigcap_{\alpha \in \Lambda} H^1_{\alpha,b(\alpha)}\Big) \cap \Big(\bigcap_{\alpha \in \overline{\Lambda}} H^+_{\alpha,b(\alpha)}\Big).$$

Clearly any special polyhedron lies in $\mathscr{P}$, and so does any polyhedron built by forming intersections of special polyhedra with subsets of the form $H^1_{\alpha,k}$ and $H^+_{\alpha,k}$. Du proves the following properties of special polyhedra:

(1) For any special polyhedron P, we can find a subpolyhedron $P' \subset P$, also special, with $P \smallsetminus P'$ a finite union of special polyhedra P_i of lower dimension than that of P.

(2) The polyhedron P' can be chosen such that P' is a finite union of polyhedra $Q_j \in \mathscr{P}$ (not special in general), and such that each Q_j is contained in a single left cell. These Q_j are built using intersections of P with subsets of the form $H^1_{\alpha,k}$ and $H^+_{\alpha,k}$.

Finally Du completes the proof by induction and by choosing a finite collection of special polyhedra such that all alcoves will eventually be accounted for by the Q_j constructed in property (2). For this he uses the sign-type regions $\mathscr{R}_0$. Given any sign-type region X, he shows that there is a canonical special polyhedron $P(X) \supset X$ attached to X with $\dim X = \dim P(X)$, and such that $P(X) \smallsetminus X$ is a finite union of sign-type regions. The sign-type regions account for all alcoves, and thus so do the special polyhedra $P(X)$. Using induction on $\dim X$ and properties (1),(2) completes the proof.

The most subtle part of the proof of Theorem 4.3 is, given a special polyhedron P, the construction of P' and the polyhedra Q_j. For this Du constructs certain infinite sequences $y_1, y_2, \ldots, y_i, \ldots$ in special polyhedra that become left-equivalent for i sufficiently large. A key ingredient here is the boundedness of Lusztig's a-function for affine Weyl groups [24].

Returning to the general discussion, here is the connection between polyhedra from $\mathscr{P}$ and the regions $\mathscr{R}_N$:

4.4. Lemma. *Let $\{P_i\}$ be a finite subset of $\mathscr{P}$, and let P be the (not necessarily convex) union of the P_i. Then for N sufficiently large, we can find a finite set of regions $\mathscr{R}(P) \subset \mathscr{R}_N$ such that*

$$P = \bigcup_{R \in \mathscr{R}(P)} \overline{R}.$$

Proof. First we assume that P is a single polyhedron from $\mathscr{P}$. If P is bounded, then P is the closure of a finite union of alcoves. Since any finite collection of alcoves can appear as regions in $\mathscr{R}_N$ for $N \gg 0$, the statement follows.

If P is unbounded, then there are only finitely many positive roots α such that $(\alpha, v) \geq k(\alpha)$ is a defining inequality for P, where $k(\alpha) \in \mathbb{Z}$. Then we simply choose

$$N \gg \max_{\alpha \in \Phi^+} |k(\alpha)|,$$

and P can be written as a union of regions from $\mathscr{R}_N$.

Finally, if $P = \cup P_i$ is a union of polyhedra, then we compute N_i for each P_i as above and then choose $N \gg \max N_i$. This completes the proof. $\square$

4.5. We are now ready to prove our main theorem:

4.6. **Theorem.** *Let C be a left cell in $\widetilde{W}$, and let $\mathtt{Red}(C)$ be the language of all reduced expressions of $w \in C$. Then $\mathtt{Red}(C)$ is regular.*

Proof. By Theorem 4.3, the cell C is a union $P = \cup P_i$ of finitely many polyhedra from $\mathscr{P}$. By Lemma 4.4, we may find $N \gg 0$ such that each P_i is a union of the closures of regions of $\mathscr{R}_N$. Let $\mathscr{A}_N$ be the automaton from Theorem 3.5 constructed from $\mathscr{R}_N$. We define a finite state automaton $\mathscr{A}(C)$ by modifying $\mathscr{A}_N$ so that the only accepting states are those corresponding to the regions of $\mathscr{R}_N$ in P. By the comment after the proof of Proposition 3.3, an expression $w = s_1 \cdots s_k$ is accepted by $\mathscr{A}(C)$ if and only if it is reduced and $w \in C$. This completes the proof. $\square$

As remarked before, each two-sided cell contains only finitely many left cells [25]. Moreover, any right cell C_R is obtained by inverting some left cell C, which means $\mathtt{Red}(C_R)$ consists of all words in $\mathtt{Red}(C)$ reversed. Since both a finite union of regular languages and the reversal of a regular language are regular, we obtain the following corollary:

4.7. **Corollary.** *Let C be any cell in $\widetilde{W}$, including right and two-sided. Then $\mathtt{Red}(C)$ is regular.*

4.8. *Remark.* It is interesting to consider the minimal value of N needed to simultaneously show that all the cells of $\widetilde{W}$ give regular languages. A consequence of Shi's work [27] is that $N = 0$ suffices for type A, and Figure 5 shows that $N = 1$ is the minimal value needed for $\widetilde{G}_2$. Figure 5 also shows that the arrangement $\mathscr{H}_1$ contains extra lines not needed to decompose left cells into regions. For instance, the left- and rightmost vertical lines, as well as the highest and lowest horizontal lines, are not supporting boundary lines for any left cell, and thus are not needed to distinguish left cells from each other.

Thus one has the natural question of defining the smallest possible automaton needed to describe all the left cells of $\widetilde{W}$. Such an automaton would have theoretical value, since using it one could attempt to extend Shi's work for type A [27] to all

types, especially the classical types (cf. [3, 10–12]). The obvious idea is to eliminate some hyperplanes from $\mathscr{H}_N$, such as the aforementioned lines in Figure 5. However, one must be careful to ensure that the resulting set of regions still satisfies property $(*)$. For instance, in Figure 5 one could try eliminating the outermost vertical lines, since they are clearly not needed to distinguish left cells. But then to keep property $(*)$ one will be forced to remove other lines from the arrangement, and some of these lines are necessary to separate left cells.

Hence one must compromise: some extra hyperplanes can be deleted, but some must remain to preserve property $(*)$. Based on Remark 3.6 and examples, we propose the following conjecture:

4.9. **Conjecture.** *Define* $\nu\colon \Phi^+ \to \mathbb{Z}_{\geq 0}$ *by* $\nu(\alpha) = 0$ *if* α *is short and* $\nu(\alpha) = 1$ *if* α *is long. Let* $\mathscr{H}_\nu$ *be the affine hyperplane arrangement from* (4), *and let* $\mathscr{R}_\nu$ *be the connected components of the complement* $V \smallsetminus \mathscr{H}_\nu$. *Then any left cell of* $\widetilde{W}$ *is a union of regions from* $\mathscr{R}_\nu$.

We have checked Conjecture 4.9 for $\widetilde{C}_2$ (Figure 6(a)), $\widetilde{G}_2$ (Figure 6(b)), $\widetilde{C}_3$, and the canonical left cells of $\widetilde{B}_3$.

(a) $\widetilde{C}_2$ (b) $\widetilde{G}_2$
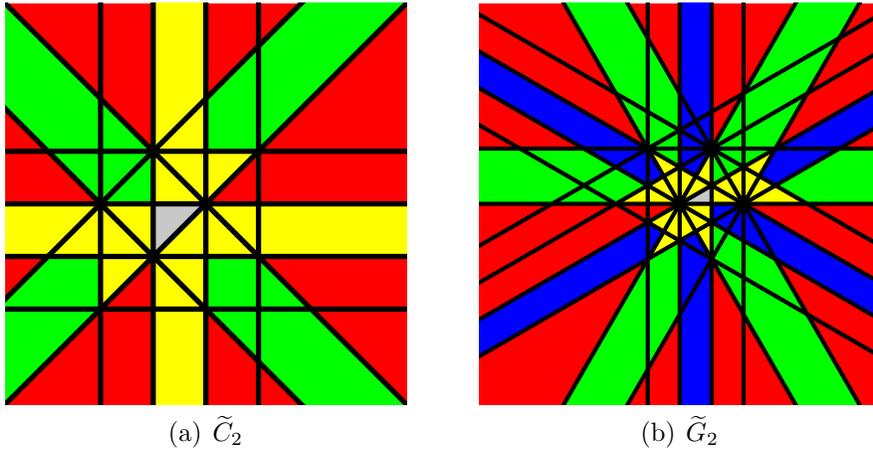
FIGURE 6.

4.10. *Remark.* One can ask if Theorem 4.6 and the stronger Conjecture 4.9 apply to the case of cells with *unequal parameters* [21]. There is some evidence that both are true, although this evidence is limited since there are fewer cases where one knows all the cells in complete detail. J. Guilhot [15] has recently given a conjectural description of the left cells in $\widetilde{G}_2$ for all choices of parameters, and has proved them in many cases. For all the examples in his conjectural description, Conjecture 4.9 holds.

4.11. *Remark.* It is natural to ask if the ideas in this paper can be used to prove Conjecture 1.3 for all Coxeter groups. For a general group, the analogue of the hyperplane arrangement $\mathscr{H}_0$ is the arrangement of *minimal* or *small* roots. This is a certain finite subset of the set of all root hyperplanes in the geometric realization of W that plays the decisive role in proving the regularity of $\mathtt{Red}(W)$. Then one could try to identify a large finite subset of hyperplanes containing the minimal ones that could be used to determine an analogous class of polyhedra $\mathscr{P}$. Unfortunately this naive generaliztion cannot work, since simple examples show that there exist left cells bounded by infinitely many root hyperplanes.

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